Recitation 1-2

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Logistics

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For questions, feel free to use Piazza or email me directly. I'll try to get back to you within 12 hours.

1 Differential Geometry

One theme of mathematics is to start with an object, and to give it some hierarchy of structures, where each new structure we gives builds off the existing ones. Differential geometry is no exception.

- At the most primitive level we can start with a set M, which is just a collection of points.
- Next we can give it a notion of open sets, i.e. a topology, which defines things like limits and continuity. If M further looks locally like $\mathbb{R}^k$ in a notion we will make precise, we call it a topological manifold.
- If the manifold is smooth, it a differentiable manifold. These are the natural objects on which to do calculus. We can define objects such as derivatives, tangent spaces, tensors, and so on.
- If we further endow M with a metric $g_{\mu\nu}$, it becomes a Riemannian manifold. Among other things, this allows us to relate tangent spaces at different points. We can define objects such as geodesics, curvature, and so on.
- Kähler manifold (complex structure), Calabi-Yau manifold ('flatness' condition), etc.

In the first week, we will focus on the third point. Next week will focus on the fourth.

1.1 Manifolds

Manifolds are natural places to do geometry (topology, calculus, etc.). We know already how to do calculus on $\mathbb{R}^k$. However, this can be generalized to some region $M \subseteq \mathbb{R}^n$, provided it only looks locally like some Euclidean $\mathbb{R}^k$.

Definition: Topological manifold
A k-dimensional topological manifold $M \subseteq \mathbb{R}^n$ is given by a collection of open sets $M = \cup_i U_i$ and

coordinate charts $\varphi_i : U_i \to \mathbb{R}^k$. We require φ_i's are invertible ($\varphi_i^{-1} : \varphi_i(U_i) \to U_i$ is well-defined) and continuous. The set (U_i, φ_i) constitute an atlas for M.

Intuition: the charts φ_i make precise what it means for a manifold to be locally $\mathbb{R}^k$. For any point $p \in M$, some neighborhood of p can be identified with a neighborhood of $\mathbb{R}^k$, in a way that is bijective and continuous (points near p stay stay near $\varphi(p)$ under the map). The charts literally provide local 'coordinates' on M.

Definition: Differentiable manifold
Given any 2 coordinate charts (U, φ), (V, ψ) with $U \cap V$ non-trivial, we can define the 'transition functions':

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V), \qquad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

If these functions, as maps from (subregions of) $\mathbb{R}^k \to \mathbb{R}^k$ are all differentiable, then M is also a differentiable manifold.

Intuition: This should be thought of as a consistency condition that makes notions of differentiability independent of the coordinate chart used.

Example: S^2
Consider the set $M = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}$ We will show this is a manifold by putting coordinates on it. One example is latitude/longitude. But instead we will consider the following 'stereographic' coordinate chart:

$$\varphi_S : M \backslash \{(0, 0, 1)\} \to \mathbb{R}^2, \quad \varphi_S(x, y, z) = \left(\frac{2x}{1 - z}, \frac{2y}{1 - z} \right)$$

This maps the sphere, excluding the north pole, to the plane. To see where a point p gets mapped to, draw a line between the north pole and p, and extrapolate it to the plane $z = -1$; the x and y coordinates ot which the line intersects the plane are precisely the coordinates of φ_S. We see that a region about the south pole gets mapped to a region about the origin.
However, because this does not cover the north pole, this is not enough to show that S^2 is a manifold. To do this, we use a similar coordinate chart:

$$\varphi_N : M \backslash \{(0, 0, -1)\} \to \mathbb{R}^2, \quad \varphi_N(x, y, z) = \left(\frac{2x}{1 + z}, \frac{2y}{1 + z} \right)$$

which maps a region of the north pole to $\mathbb{R}^2$. Together, φ_N and φ_S cover the entire manifold. Furthermore one can show that the transition map is:

$$(\varphi_N \circ \varphi_S^{-1})(u, v) = \left(\frac{4u}{u^2 + v^2}, \frac{4v}{u^2 + v^2} \right)$$

Which is smooth, i.e. differentiable. Therefore, we have shown that S^2 is a differentiable manifold.

Other examples: $\mathbb{R}^k$, higher dimensional spheres S^n, torus T^2, etc.
Non examples:

- 2 intersecting cones $C = \{(x, y, z) \in \mathbb{R}^3, z^2 = x^2 + y^2\}$
  Note, however, that the single cone $C_+ = \{(x, y, z) \in \mathbb{R}^3, z = \sqrt{x^2 + y^2}\}$ is a differentiable manifold
- 'Hawaiian earring' $\mathbb{H} = \bigcup_{n=1}^{\infty} \{(x - 1/n)^2 + y^2 = (1/n)^2\}$
- The disk $\mathbb{D}^2 = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 < 1\}$. The interior is a 2-manifold, while the boundary is a 1-manifold. This however, falls into a generalized definition of a 'manifold with boundary'

1.2 Vectors and Tangent Spaces

Next, I hope to give some intuition on tangent spaces, by defining them in several ways, and explaining why they are all equivalent. Consider a manifold M, with a point $p \in M$. The tangent space is denoted T_pM, defined as the following:

Definition 1: space of tangent vectors to curves through p
This should be familiar from lecture. For a curve $\gamma : \mathbb{R} \to M$ with $\gamma(\lambda = 0) = p$, the tangent vector (at $\lambda = 0$) is given by:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial x^i}{\partial \lambda}\frac{\partial}{\partial x^i} = \frac{\partial \gamma(\lambda)^i}{\partial \lambda}\frac{\partial}{\partial x^i}$$

This is precisely a directional derivative, where the direction is given by the 'tangent' vector to curve. The tangent space is the vector space spanned by all of the tangent vectors.

Proposition: T_pM is a vector space.
From this definition, the tangent space is a vector space because for any 2 curves $\gamma_1(\lambda_1)$ and $\gamma_2(\lambda_2)$, we can produce another curve:

$$\gamma : \mathbb{R} \to M, \qquad \gamma(\lambda) = \varphi^{-1}(\varphi(\gamma(c_1\lambda) + \varphi(\gamma(c_2\lambda)))$$

This defines the tangent vector:

$$c_1\frac{\mathrm{d}}{\mathrm{d}\lambda_1} + c_2\frac{\mathrm{d}}{\mathrm{d}\lambda_2} := \left(c_1\frac{\partial \gamma_1(\lambda)^i}{\partial \lambda} + c_2\frac{\partial \gamma_2(\lambda)^i}{\partial \lambda}\right)\frac{\partial}{\partial x^i}$$

Therefore, linear combinations of tangent vectors are also tangent vectors. The zero vector is given by the curve $\gamma(\lambda) = p \ \forall \lambda \in \mathbb{R}$.

Definition 2: space of 'derivations' (derivatives) at p
Let $C(M)$ denote the space of continuous functions from M to $\mathbb{R}$. A tangent vector v_p at p is a linear map $v_p : C(M) \to \mathbb{R}$ obeying the Leibniz rule:

$$v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$$

The intuition is that v_p behaves as a differential operator of $C(M)$ at the point p; we know derivatives are linear and obey the product rule. The tangent space T_pM is the set of all such tangent vectors.

Equivalence of Definitions 1 and 2:
How does this relate to our first definition? We've seen that a curve $\gamma(\lambda)$ defines the object $v_p = \mathrm{d}/\mathrm{d}\lambda|_{\lambda=0}$. This acts on continuous functions $f : M \to \mathbb{R}$ as:

$$v_p(f) = \frac{\mathrm{d}}{\mathrm{d}\lambda}f(\gamma(t))|_{\lambda=0}$$

Because it is an actual derivative, is also satisfies the Leibniz rule. Conversely, it can be shown that any tangent vector as per our new definition can be constructed in this way.

Definition 3: image of the derivative $(D\varphi^{-1})_{ij}$ at p
For any coordinate chart φ mapping a neighborhood of p to $\mathbb{R}^k$, we can define the matrix of partial derivatives (at p):

$$(D\varphi^{-1})_i{}^j = \begin{pmatrix} \frac{\partial(\varphi^{-1})^1}{\partial x^1} & \frac{\partial(\varphi^{-1})^1}{\partial x^2} & \cdots & \frac{\partial(\varphi^{-1})^1}{\partial x^k} \\ \frac{\partial(\varphi^{-1})^2}{\partial x^1} & \frac{\partial(\varphi^{-1})^2}{\partial x^2} & \cdots & \frac{\partial(\varphi^{-1})^2}{\partial x^k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\varphi^{-1})^n}{\partial x^1} & \frac{\partial(\varphi^{-1})^n}{\partial x^2} & \cdots & \frac{\partial(\varphi^{-1})^n}{\partial x^k} \end{pmatrix}$$

Here i runs from $1, \ldots, n$ and j runs from $1, \ldots k$. This is a $n \times k$ matrix mapping vectors from $\mathbb{R}^k$ to vectors in $\mathbb{R}^n$. The tangent space $T_pM = \text{Image}(D\varphi^{-1}|_p)_{ij}$

Equivalence of Definitions 1 and 3:
This definition presupposes a choice of coordinates. To see why this is equivalent to the definition using curves through p, consider the curves given by:

$$\gamma : \mathbb{R} \to M, \qquad \gamma(\lambda) := \varphi^{-1}(\lambda, x^2, \ldots, x^k), \qquad x\text{'s fixed}$$

In the language of our first definition, this defines a tangent vector which we can write in the $\partial/\partial x^i$ basis:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial \gamma(\lambda)^i}{\partial \lambda}\frac{\partial}{\partial x^i} = \frac{\partial(\varphi^{-1})^i}{\partial x^1}\frac{\partial}{\partial x^i}$$

Similarly, we can define curves by instead varying the jth coordinate:

$$\gamma : \mathbb{R} \to M, \qquad \gamma(\lambda) := \varphi^{-1}(x^1, \ldots, x^{j-1}, \lambda, x^{j+1}, x^k), \qquad x\text{'s fixed}$$

which define the tangent vectors

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial(\varphi^{-1})^i}{\partial x^j}\frac{\partial}{\partial x^i}$$

obtained by mapping $e^j = (\overbrace{0, \ldots, 1}^{j}, \ldots, 0)$ to the vector $\dfrac{\partial(\varphi^{-1})^i}{\partial x^j}$. But this is precisely what the matrix $(D\varphi^{-1})_i{}^j$ does! The span of these vectors gives the tangent space T_pM, therefore $T_pM = \text{Image}(D\varphi^{-1}|_p)_{ij}$.

Remark: The derivative maps one tangent space to another
Very generally, we can consider the derivative of a function between manifolds, $f : M \to N$:

$$(Df)_i{}^j \to \frac{\partial f(x)^j}{\partial x^i}$$

where x^j are coordinates on M. Then, $Df : T_pM \to T_{f(p)}N$ is a linear map (matrix) between tangent spaces
Example: $\varphi^{-1} : \mathbb{R}^k \to M$. Then $D\varphi^{-1} : (T_{\varphi(p)}\mathbb{R}^k \equiv \mathbb{R}^k) \to T_pM$, as we previously saw.

What's the point in having so many definitions of the same object? Definition 1 in terms of curves through p is very intuitive, and it is easy to see why this aligns with our intuitive understanding of a tangent space. On the downside, it is quite difficult to use this to actually compute a tangent space on some manifold, as that would involve parameterizing a large space of paths. Definition 2 as 'derivations' is more abstract, but is extremely useful in the language of differential geometry, and proofs. Definition 3 is my favorite, and although it may be a bit unintuitive, it provides the best way (i.e. an algorithmic way) to actually compute tangent spaces of arbitrary manifolds.

Example: Find the tangent space to S^2 at the point $p = (1, 0, 0)$.
<u>Method 1</u>:
By parameterizing paths through p. Generally, this is hard to do directly. However, given a coordinate chart φ mapping some neighborhood of p to $\mathbb{R}^k$, one can use the paths we previously saw:

$$\gamma(\lambda) = \varphi^{-1}(x^1, \ldots, x^{i-1}, \lambda, x^{i+1}, x^k), \qquad x\text{'s fixed}$$

For each, we can compute $\mathrm{d}/\mathrm{d}\lambda|_{\lambda=0}$, using the chain rule to express it in terms of our coordinate basis. The span of these give us our tangent space. I will not do this explicitly in the interests of time, but it is excellent practice to do so, using the stereographic coordinate chart we saw earlier, as well as polar coordinates. Instead, I will do what is essentially the same calculation in another way.

Method 2:
The inverse of the stereographic projection based at the south pole is:

$$\varphi_S^{-1} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \varphi_S^{-1}(u_1, u_2) = (x_1, x_2, x_3) = \left(\frac{2u_1}{1+u_1^2+u_2^2}, \frac{2u_2}{1+u_1^2+u_2^2}, \frac{-1+u_1^2+u_2^2}{1+u_1^2+u_2^2} \right)$$

Based on this, we compute the matrix of partial derivatives:

$$(D\varphi_S^{-1})_i{}^j = \frac{\partial(\varphi_S^{-1})_i}{\partial u_j} = \frac{2}{(1+u_1^2+u_2^2)^2} \begin{pmatrix} 1-u_1^2+u_2^2 & -2u_1u_2 \\ -2u_1u_2 & 1+u_1^2-u_2^2 \\ 2u_1 & 2u_2 \end{pmatrix}$$

The point that gets sent to $(x, y, z) = (1, 0, 0)$ is $(u_1, u_2) = (1, 0)$. Evaluating the matrix of partials at this point gives us:

$$(D\varphi_S^{-1}|_{(u_1,u_2)=(1,0)})_i{}^j = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It remains to evaluate the image of this matrix. The vector e_1 gets sent to $(0, 0, 1)$, and e_2 gets sent to $(0, 1, 0)$. Therefore, the tangent space is:

$$T_{p=(1,0,0)}M = (1, 0, 0) + \operatorname{span}((0, 0, 1) + (0, 1, 0)) = \{(1, 0, 0) + (0, \lambda_1, \lambda_2)\} = \{(1, \lambda_1, \lambda_2)\}$$

Here $\lambda_{1,2} \in \mathbb{R}$ are just parameters. But this is just a parameterization of the plane $x = 1$, which matches with intuition.

1.3 Covectors and Tensors

Tensors on General Vector Spaces

For any vector space V, we can define the dual space V^* as the space of linear maps from V to $\mathbb{R}$. An element of V^* is called a co-vector (or when V is a tangent space, a 1-form). To specify any $\omega \in V$ it is sufficient to specify their action on a basis e_i, since

$$\omega(v^i e_i) = v^i \omega(e_i)$$

A basis of 1-forms is given by the 'coordinate co-vectors':

$$\omega^j(e_i) = \delta^j_i, \qquad \omega^j(v) = v^i \omega^j(e_i) = v^i \delta^j_i = v^j$$

We see that all this does is pick out the jth coordinate of a vector in the e_i basis. This makes sense, as choosing some coordinate is a way of mapping vectors to numbers.
A general co-vector can be written in this basis as:

$$\omega = \alpha_j \omega^j, \qquad \omega(v) = \alpha_j \omega^j(v^i e_i) = \alpha_j v^i \delta^j_i = \alpha_i v^i$$

We see that co-vectors map vectors to numbers. Similarly, we can view vectors as mapping co-vectors to numbers:

$$v : V^* \to \mathbb{R}, \qquad v(\omega) := \omega(v)$$

More generally, we can define (k,l)-tensors as objects mapping k-copies of V^* and l-copies of V to numbers:

$$T : \overbrace{V^* \otimes \cdots \otimes V^*}^{k} \otimes \overbrace{V \otimes \cdots \otimes V}^{l} \to \mathbb{R}, \qquad T(\omega_1, \ldots, \omega_k, v_1, \ldots, v_l) \in \mathbb{R}$$

We require T be linear in each argument, i.e.

$$T(\ldots, c_1 v_1 + c_2 v_2, \ldots) = c_1 T(\ldots, v_1, \ldots) + c_2 T(\ldots, v_2, \ldots)$$
$$T(\ldots, c_1 \omega_1 + c_2 \omega_2, \ldots) = c_1 T(\ldots, \omega_1, \ldots) + c_2 T(\ldots, \omega_2, \ldots)$$

In this language, co-vectors are $(0,1)$-tensors, and vectors are $(1,0)$-tensors. Due to multilinearity, a (k,l) tensor can be specified if we know how they act on all the basis of vectors/covectors:

$$\begin{aligned} T(\omega_1, \ldots, \omega_k, v_1, \ldots, v_l) &= \alpha_{1i_1} \cdots \alpha_{ki_k} v_1^{j_1} \cdots v_l^{j_l} T(\omega^{i_1}, \ldots, \omega^{i_k}, e_{j_1}, \ldots e_{j_l}) \\ &= \alpha_{1i_1} \cdots \alpha_{ki_k} v_1^{j_1}, \cdots, v_l^{j_l} T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l} \end{aligned}$$

A note on using upper/lower indices: upper indices are conventionally used for components of vectors, and lower indices are conventionally used for components of covectors. The components of a (k,l) tensor has k upper indices and l lower indices. The ordering of indices is important, for instance generally $T^i{}_j \neq T^j{}_i$.

Tensors on Manifolds

Now, all we have to do is apply this to when our vector space is a tangent space T_pM. Vectors are spanned by coordinate derivatives $\partial_i := \partial/\partial x^i$. That is, any tangent vector v_p can be written:

$$v_p = v^i \left(\frac{\partial}{\partial x^i}\right)_p, \qquad v^1, \ldots, v^k \in \mathbb{R}$$

Co-vectors/1-forms are spanned by 'coordinate co-vectors', which will denote as $\mathrm{d}x^i$. That is, any 1-form $\omega \in T_pM^*$ can be written

$$\omega = \alpha_i \mathrm{d}x^i, \qquad \omega(v) = \alpha_i v^j \mathrm{d}x^i \left(\frac{\partial}{\partial x_j}\right) = \alpha_i v^j \delta^i_j = \alpha_i v^i$$

Vectors and covectors on a manifold can be given in a basis (coordinate) independent way. We've seen how a vector can be specified by a curve $\gamma(\lambda)$, written as $\mathrm{d}/\mathrm{d}\lambda$. On the other hand, covectors can be written as $\mathrm{d}f$ for some function f. You've all seen this before, a more suggestive way of writing this is

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i$$

from which we can see that this is really just a gradient 1-form. In this language, a 1-form acts on vectors (and vice versa) as:

$$\mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}f}{\mathrm{d}\lambda} = \frac{\mathrm{d}f(\gamma(\lambda))}{\mathrm{d}\lambda}$$

This can be verified using coordinates:

$$\mathrm{d}f\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\right) = \frac{\partial f}{\partial x^i}\frac{\partial x^j}{\partial \lambda}\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i}\frac{\partial x^j}{\partial \lambda}\delta^i_j = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \lambda} = \frac{\mathrm{d}f}{\mathrm{d}\lambda}$$

A tensor is a map

$$T : \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k} \otimes \underbrace{T_pM^* \otimes \cdots \otimes T_pM^*}_{l} \to \mathbb{R}$$

It acts on covectors and vectors as:

$$\begin{aligned} T(\omega_1, \ldots, \omega_k, v_1, \ldots, v_l) &= \alpha_{1i_1} \cdots \alpha_{ki_k} v_1^{j_1} \cdots v_l^{j_l} T(\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_k}, \partial_{j_1}, \ldots \partial_{j_l}) \\ &= \alpha_{1i_1} \cdots \alpha_{ki_k} v_1^{j_1}, \cdots, v_l^{j_l} T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l} \end{aligned}$$

Vectors vs. Vector Fields

Lastly, a note on vectors(/covectors/tensors) and vector(/covector/tensor) fields. A vector is formally based on the tangent space at a point p. However, say that for each point p, we have a corresponding $v_p \in T_pM$. This constitutes a vector field, which maps to each $p \in M$ a vector v_p. We may denote the vector field as a function $v(p)$. In basis notation, we may write:

$$v(p) = v^i(p)\left(\frac{\partial}{\partial x_i}\right)_p$$

Here, the components will depend on the point p, and the basis elements are derivatives taken at the point p. Similarly, for a covector field:

$$\omega(p) = \alpha_i(p)\mathrm{d}x^i_p$$

One can discuss tensor fields in the very same way.

Operations on Tensors

Let $\mathbb{T}^{(k,l)}$ denote the space of (k,l) tensors. In the following we list some common operations on tensors.

1. Tensor product: $\otimes$: $(\mathbb{T}^{k,l}, \mathbb{T}^{k',l'}) \to \mathbb{T}^{k_1+k_2, l_1+l_2}$
   In component form, $(T \otimes S)^{i_1 \cdots i_{k+k'}}{}_{j_1 \cdots j_{l+l'}} = T^{i_1 \cdots i_k}{}_{j_1 \cdots j_l} S^{i_{k+1} \cdots i_{k+k'}}{}_{j_{l+1} \cdots j_{l+l'}}$

2. Contraction: Contr.$(m,n) : \mathbb{T}^{k,l} \to \mathbb{T}^{k-1,l-1}$
   where we choose to contract the mth upper index with the nth lower index.
   In component form, the contracted tensor is $\sum_\alpha T^{i_1\cdots i_{m-1}\alpha i_{m+1}\cdots i_k}{}_{j_1\cdots j_{n-1}\alpha j_{n+1}\cdots j_l}$
   where we have written the implicit sum over the α, and used the same letter T to denote the contracted tensor.

3. Raising/lowering: this requires a metric $(0,2)$ tensor g_{ij} and its inverse, a $(0,2)$ tensor g^{ij}.
   Then, we can define: $\sharp(m) : \mathbb{T}^{k,l} \to \mathbb{T}^{k+1,l-1}$, raising the mth lower index
   $\flat(n) : \mathbb{T}^{k,l} \to \mathbb{T}^{k-1,l+1}$, lowering the nth upper index
   In component form, these are given by: $T^{i_1\cdots i_{m-1}}{}_\alpha{}^{i_{m+1}\cdots i_k}{}_{j_1\cdots j_l} = g_{\alpha i_m} T^{i_1\cdots i_k}{}_{j_1\cdots j_l}$
   $T^{i_1\cdots i_l}{}_{j_1\cdots j_{n-1}}{}^\alpha{}_{j_{n+1}\cdots j_l} = g^{\alpha j_n} T^{i_1\cdots i_l}{}_{j_1\cdots j_l}$

1.4 Integration on Manifolds

Differential Forms
Let M be a n-dimensional manifold. A k-form is a totally antisymmetric $(0,k)$-tensor. This means that it changes sign when any 2 indices are exchanged, e.g. $T_{ijk} = -T_{jik} = T_{jki}$. In particular, if any index is repeated the tensor component vanishes. Ocan check that $\dim\Lambda_p^k = \binom{n}{k}$ for $0 \leq k \leq n$, else $\dim\Lambda_p^k = 0$.

Next we'll want to extend the notion of a tensor product to forms. The problem is that the tensor product of a k-form with an l-form will in general not be antisymmetric in all its indices. To amend this, we define the wedge product:

$$\wedge : \Lambda^k \times \Lambda^l \to \Lambda^{k+l}, \qquad (\omega \wedge \eta)_{i_1\cdots i_k j_1\cdots j_l} = \frac{(k+l)!}{k!l!}\omega_{[i_1\cdots i_k}\eta_{j_1\cdots j_l]}$$

where we define the antisymmetrization:

$$T_{[i_1\cdots i_k]} = \sum_{\sigma\in S_k} (-1)^{\mathrm{sign}(\sigma)} T_{\sigma(i_1)\cdots\sigma(i_k)}$$

A basis of k forms at a point is given by $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, where none of the indices are the same (else the wedge product would make it zero). Since on an n-dimensional manifold each index runs from $1 \to n$, there are $\binom{n}{k}$ ways of choosing such a basis, thus $\dim\Lambda_p^k = \binom{n}{k}$.

Integration
Differential forms are the natural object to integrate on a manifold. Intuitively, they provide a notion of k-volume (length for $k=1$, area for $k=2$, etc.). To see why, let us examine the properties of a volume.

- A volume element should assign to each (infinitesimal) region an (infinitesimal) real number corresponding to the volume of said region. It is simplest to work with parallelepipeds (k-dimensional parallelegrams); in a general curved manifold we may have no idea what a (rectangular) parallelepiped means, but in the infinitesimal limit any manifold is locally flat (locally $\mathbb{R}^n$).

- A k-dimensional parallelepiped based at a point p is specified by a collection of k vectors in T_pM. A k-volume element, then, should map k vectors to the real numbers. Furthermore, if we scalar multiply any vector by λ we change it's length by a vactor of λ, which also changes the volume by the same factor. But this is precisely what a $(0,k)$ tensor does: it maps k vectors to a number, linear in each argument.

- There is one more feature we need to define a bona fide volume. Namely, volumes should be oriented. That is, if 2 vectors are interchanged, the voume should be of the same magnitude but opposite sign. This is easiest to see in 1 and 2 dimensions. This implies, for instaance, that the volume vanishes when 2 vectors are collinear. Therefore, our volume $(0,k)$-tensor should be totally antisymmetric, which is a k-form.

Now, we are ready to define the integral of a k-form vector field ω over a k-dimensional manifold M. For simplicity, let us suppose that M can be parameterized by a single coordinate chart, $\varphi : M \to \mathbb{R}^k$. Without loss of generality, we may write ω in the coordinate basis as:

$$\omega = \alpha(x^1, \dots, x^k)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$$

Then, we define the integral of ω over M as:

$$\int_M \omega = \int_{\varphi(M)} \alpha(x^1, \dots, x^k)\mathrm{d}x^1 \cdots \mathrm{d}x^k$$

Where the right hand side is now a standard Riemann integral in plain old $\mathbb{R}^k$, which we know how to do. One can check that this definition is coordinate independent, following from the transformation law of tensors.

If we are only given the structure of a manifold M, there is no natural choice of volume element, as some special k-form ω we can integrate. This is remedied if we have a metric g_{ij}, which allows us to raise and lower tensor indices. Then, we can define a special volume element by:

$$\omega^{i_1 \cdots i_k}\omega_{i_1 \cdots i_k} = (-1)^s n!$$

where s is the number of minus signs in the signature of g_{ij}. Using some algebra (which I will not do) it can be shown that this uniquely specifies:

$$\omega = \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k$$

Therefore, the natural volume of an k-dimensional manifold M is given by:

$$\mathrm{vol}(M) = \int_M \sqrt{|g|}\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^k = \int_{\varphi(M)} \sqrt{|g|}\mathrm{d}x^1 \cdots \mathrm{d}x^k$$

Example: find the volume of S^2, using stereographic coordinates from last recitation.
We view the sphere S^2 as embedded in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3), $x_1^2 + x_2^2 + x_3^2 = 1$. The metric in $\mathbb{R}^3$ is just the flat Euclidean metric $g_{ij}^{\text{Eucl.}} = \mathrm{diag}(1, 1, 1)$. We wish to find the metric this induces on S^2, in stereographic (u_1, u_2) coordinates. To do this we use the the tensor transformation property:

$$g_{ab}^{\text{stereog.}} = \frac{\partial x^i}{\partial u^a}\frac{\partial x^j}{\partial u^b} g_{ij}^{\text{Eucl.}}$$

where we recall that (x_1, x_2, x_3) and (u_1, u_2) are related by the stereographic coordinate chart,

$$\varphi_S^{-1} : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \varphi_S^{-1}(u_1, u_2) = (x_1, x_2, x_3) = \left(\frac{2u_1}{1 + u_1^2 + u_2^2}, \frac{2u_2}{1 + u_1^2 + u_2^2}, \frac{-1 + u_1^2 + u_2^2}{1 + u_1^2 + u_2^2}\right)$$

Computing this for $i, j = (1, 2, 3)$ and $(a, b) = (1, 2)$, we obtain:

$$g_{ab}^{\text{stereog.}} = \frac{4}{(1 + u_1^2 + u_2^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, the naturally volume element on S^2 in these coordinates is:

$$\omega = \sqrt{\det g}\ \mathrm{d}u \wedge \mathrm{d}v = \frac{4}{(1 + u_1^2 + u_2^2)^2}\mathrm{d}u \wedge \mathrm{d}v$$

Finally, we are ready to compute the S^2 volume:

$$\mathrm{vol}(S^2) = \int_{S^2} \omega = \int_{\varphi(S^2)=\mathbb{R}^2} \frac{4}{(1+u_1^2+u_2^2)^2} \mathrm{d}u\mathrm{d}v$$

where we used that the stereographic projection maps S^2 (excluding the north pole) to the entire real plane $\mathbb{R}^2$. You may be worried that our coordinate chart doesn't cover the north pole. However, this is only a single point, which has zero area (more formally, measure zero), therefore there is no harm in ignoring it.

To compute this integral (now just a regular integral on $\mathbb{R}^2$ instead of on a manifold S^2) we can use polar coordinate $(u_1, u_2) = (r\cos\theta, r\sin\theta)$

$$\mathrm{vol}(S^2) = \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} r\mathrm{d}r \frac{4}{(1+r^2)^2} = 8\pi \int_0^{\infty} \frac{r}{(1+r^2)^2} = 8\pi \cdot \frac{1}{2} = 4\pi$$

This is the area of a sphere with radius 1, as desired.

Example: find the length of the path on S^2 determined by $\theta = \phi$, $0 \le \phi \le \pi$, in polar coordinates.

Ignoring the 2 endpoints (of zero length), the path γ traced out is a 1-manifold on S^2. To put coordinates on γ we can use the coordinate $t = \theta = \phi$, running from $0 \le t \le \pi$. That is, the coordinate chart is $\varphi^{-1}(t) = (t, t)$ mapping from $t \in [0, \pi]$ to polar coordinates (θ, ϕ).

In polar coordinates, the metric on S^2 (of unit radius) is given by $g_{ij}^{\text{polar}} = \mathrm{diag}(1, \sin^2\theta)$. To find the metric this induces on the manifold γ, we again simply use our formula from above:

$$g_{ab}^{\text{path}} = \frac{\partial x^i}{\partial u^a}\frac{\partial x^j}{\partial u^b} g_{ij}^{\text{polar}}$$

Here i runs from 1 to 2, with $x^1 = \theta$, $x^2 = \phi$. Because γ is just a 1-dimensional manifold a, b can only be 1, with $u^1 = t$. Computing $\frac{\partial\theta}{\partial t} = 1$ and $\frac{\partial\phi}{\partial t} = 1$ on γ (since the path is defined by $t = \theta = \phi$), we have:

$$g_{ab}^{\text{path}} = \left(1 + \sin^2 t\right)$$

This is just a 1×1 matrix, with determinant $\det g^{\text{path}} = \sqrt{1+\sin^2 t}$. The '1-volume' form is thus

$$\omega = \sqrt{1+\sin^2 t}\mathrm{d}t$$

and the length of the path γ is therefore:

$$\int_\gamma \omega = \int_{\varphi(\gamma)=[0,\pi]} \sqrt{1+\sin^2 t}\mathrm{d}t = 2E(-1) \approx 3.8202$$

where $E(x)$ is an elliptic integral.

A Review

A.1 Index Notation

Index notation (Einstein summation convention): whenever an index is repeated twice, an implicit sum is performed over it. The best way to become familiar with this is to go through examples.

Point of caution: when using index notation, never use the same index in more than 1 sum. For instance $a_i b_i c_j d_j = \sum_{i,j} a_i b_i c_j d_j$ is well defined, but $a_i b_i c_i d_i$ is not. This seems obvious, but it is very easy to accidentally do. If you do this, you will be prone to error in further algebra,

Example: dot-product in $\mathbb{R}^n$, $v \cdot w = v_i w_i$. Here i runs from $1 \to n$.
Example: cross-product in $\mathbb{R}^3$, $(v \times w)_k = \epsilon_{ijk} v^i w^j$. Here $i, j, k \in \{1, 2, 3\}$, and ϵ_{ijk} is the Levi-Civita:

$$\begin{cases} \epsilon_{ijk} = 0, & \text{if any 2 indices are the same} \\ \epsilon_{ijk} = 1, & \text{if } ijk \text{ is an even permutation of 123} \\ \epsilon_{ijk} = -1, & \text{if } ijk \text{ is an odd permutation of 123} \end{cases}$$

Example: dot-product of cross-products: $(a \times b) \cdot (c \times d)$
We simplify:

$$\begin{aligned} (a \times b) \cdot (c \times d) &= (\epsilon_{ijk} a_i b_j)(\epsilon_{klm} c_l d_m) = \epsilon_{ijk} \epsilon_{klm} a_i b_j c_l d_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_i b_j c_l d_m \\ &= a_i c_i b_j d_j - a_i d_i b_j c_j = a \cdot c b \cdot d - a \cdot d b \cdot c \end{aligned}$$

where in the second line we have used the identity (it is a good exercise to check this):

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$