Recitation 7

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3.4 Properties of Einstein's Equation

Einstein's equations are given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

In this section we will look at 2 properties of this equation: that gravity is attractive if we assume the SEC, and that there are fundamentally only 2 physical degrees of freedom in the metric components.

Gravity is Attractive no time oof

Degrees of Freedom
Here we wish to count gravitational degrees of freedom. There are 2 parts to this:

1. The number of physical metric components, i.e. those that cannot be fixed by some gauge transformation (by definition unphysical)
2. The number of independent Einstein's equations giving the dynamics of physical metric components

When all is said and done, we expect these to agree at 2 each. This is a very often very misunderstood procedure, so here I hope to clear things up.

Metric Components
The metric $g_{\mu\nu}$ is a symmetric tensor, with 10 independent components. However, these change under coordinate transformations $x^\mu \to x'^\mu$ as $g_{\mu\nu} \to \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}g_{\rho\sigma}$. Coordinate transformations are the gauge transformations of GR; they are always unphysical, since nature does not care about which coordinate patch you work in. They account for 4 degrees of freedom, reducing the physical metric components down to 6.

But we are not done. It is a pithy but profound statement that "the gauge always hits twice." Let us see what this means in EM. Here we have the 4-potential A^μ, and the freedom to make gauge transformations $A_\mu \to A_\mu + \partial_\mu\lambda$. Suppose we work in Lorenz gauge, fixing $\partial_\mu A^\mu = 0$. However, we *still* have the freedom to make a 'harmonic' gauge transformation $A_\mu \to A_\mu + \partial_\mu\lambda$ for $\partial^2\lambda = 0$. One can use this to cut down another degree of freedom bringing us down from 4 to 2.

Next, let's see how this works in linearized GR. Using the trace-reversed metric perturbations, the gauge transformations $x^\mu \to x^\mu + \xi^\mu$ act as (see last week's recitation)

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - (\partial\cdot\xi)\eta_{\mu\nu}$$

for some vector field ξ^μ with 4 components. We can choose the Lorenz conditions $\partial^\mu h_{\mu\nu}$ fixing 4 degrees of freedom. However, we still have the freedom to make a gauge transformation satisfying $\partial^2\xi^\mu = 0$. It is easy to check this is consistent with the Lorenz condition:

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu(\bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\rho \xi^\rho \eta_{\mu\nu}) = \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \xi_\nu = 0 + 0 = 0$$

We can make any choice 4 harmonic functions x^μ, reducing our physical degrees of freedom down to 2. One may ask if there is any residual gauge freedom we fixed. It can be non-trivially shown using theorems from harmonic analysis about cohomology that that there is not. In linearized gravity these degrees of freedom correspond to the 2 polarizations of gravitational waves, which are very physical indeed.

The above was for linearized GR, but the very same is true in non-perturbative GR. The easiest way to see this is using the ADM formalism. Unfortunately this is outside the scope of the class, so I ask you to take my word for it. Again after exploiting all our gauge freedom, we are left with 2 physical degrees of freedom.

Einstein's Equations

Next we examine Einstein's equations. Since the Einstein and stress-energy tensors are both symmetric by design, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ gives 10 equations. However, the differential Bianchi identity $\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0$ implies $\nabla^\mu G_{\mu\nu} = 0$ (or equivalently, conservation laws $\nabla_\mu T^{\mu\nu} = 0$). These reduce the number of independent Einstein's equations to 6.

Again, we are not done. To proceed it helps to take a step back and ask what Einstein's equations are doing. They are differential equations relating first and second derivatives of $g_{\mu\nu}$, much like Newton's $m\ddot{x} = F$. Therefore, we can think of Einstein's equations as an equation of motion, or evolution equations for $g_{\mu\nu}$; if we know the initial conditions of $g_{\mu\nu}$ on some initial value slice Σ (e.g. $x^0 = 0$ in Minkowski spacetime), we can use Einstein's equations to solve for $g_{\mu\nu}$ everywhere else in the spacetime. The initial conditions we need to specify are $g_{\mu\nu}(\Sigma)$ and $\partial_t g_{\mu\nu}(\Sigma)$, in direct analogy with $x(0)$ and $x'(0)$ in Newtonian mechanics.

However, not all of Einstein's equations are true evolution equations. A true evolution equation would involve second derivatives with respect to x^0. In particular, it can be seen by tediously expanding in terms of partials and Christoffels that $G_{0\nu} = 8\pi T_{0\nu}$ does not contain any second derivatives $\partial_0^2 g_{\rho\sigma}$, only first derivatives $\partial_0 g_{\rho\sigma}$. Consequentially these 4 equations are constraints on the initial conditions, not actual evolution equations. Hence, if we are counting the number of independent 'evolution equations', there are only 2. This agrees with the number of physical metric components, as desired.

3.5 Cosmology

Review

Robertson-Walker Universes

The two assumptions of cosmology are that on a large enough scale, the universe is spatially homogeneous and isotropic; that is, the same at all points, and the same in all spatial directions. That is, our spacetime is $\mathbb{R} \times \Sigma$ where Σ is a maximally symmetric spatial 3-manifold. The metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\gamma_{ij}(u)\mathrm{d}u^i \mathrm{d}u^j$$

Using symmetry principles, it can be shown that a general such metric can be written in FLRW form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \mathrm{d}\Omega\right]$$

Here $k \in \{+1, 0, -1\}$ determines whether the curvature is (uniformly) positive/flat/negative. These correspond to the spatial slices Σ being spheres/planes/hyperboloids, for this reason these cases are called open/flat/closed universes. The scale factor $a(t)$ tells us how big each spacelike slice is at a given t. Such coordinates are comoving; only a comoving observer will see that the universe looks isotropic.

The Stress-Energy

We can couple this metric to a matter source that is also homogenous and isotropic; a natural choice is a perfect fluid, given by:

$$T^{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$$

In comoving coordinates $U^\mu = (1, 0, 0, 0)$. It is often helpful to introduce an equation of state, which relates thermodynamic state variables such as ρ and p. A simple and commonly used one is $p = w\rho$, where w is a t-independent constant. The conservation of energy equation then allows us to solve for ρ:

$$0 = \nabla_\mu T^\mu{}_0 = -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \qquad \Longrightarrow \qquad \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \qquad \Longrightarrow \qquad \rho \propto a^{-3(1+w)}$$

Matter obeys $w = 0$ ($p = 0$), for which $\rho_M \propto a^{-3}$. Radiation obeys $w = 1/3$ with $\rho_R \propto a^{-4}$. The vacuum energy can also be treated as a fluid, with $w = -1$ and $\rho_\Lambda \propto a^0$. If the energy of the universe ρ is mostly due to one of these sources, we say it is matter/radiation/vacuum dominated.

The Friedmann Equation

Now we consider Einstein's equation. In class you showed that only the $\mu\nu = 00$ and $\mu\nu = ii$ components are non-trivial. They are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

In class you discussed how the second equation gives information pertaining to stress-energy conservation. The first equation is Friedmann's equation, and dictates the evolution of the scale factor $a(t)$. It will be the focus of the rest of our study.

Before we proceed, we introduce some terminology commonplace in cosmology. The first is the Hubble parameter $H = \dot{a}/a$, characterizing the rate of expansion relative to the universe's current size. We also define the density parameter Ω and critical density ρ_c:

$$\Omega := \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_c}, \qquad \rho_c := \frac{3H^2}{8\pi G}$$

This is useful because we can write the Friedmann equation as:

$$\frac{\rho}{\rho_c} - 1 = \Omega - 1 = \frac{k}{H^2 a^2}$$

Therefore, the sign of k (i.e. whether is universe has open/flat/closed) is determined by the sign of $\Omega - 1$. We summarize in the following table:

ρ	Ω	k	Universe
$< \rho_c$	< 1	< 0	open
$= \rho_c$	$= 1$	$= 0$	flat
$> \rho_c$	> 1	> 0	closed

Evolution of the Scale Factor

Our next goal is to solve the Friedmann equation for $a(t)$. We can assume that the energy density $\rho = \sum_i \rho_i$ comes from multiple sources, where each source evolves as a power law: $\rho_i = \rho_{i0} a^{-n_i}$ (i.e. $w_i = n_i/3 - 1$). It is also convenient to treat the contribution of spatial curvature as a (fictitious) energy density: $\rho_k := -\frac{3k}{8\pi G}a^{-2}$. Then, Friedmann's equation becomes:

$$H^2 = \frac{8\pi G}{3}\sum_{i,k} \rho_i$$

Dividing both sides by H^2 we see $\Omega_c = 1 - \sum_i \Omega_i$.

To make further progress, we may simplify again, assuming there is only one kind of energy density $\rho \propto a^{-n}$ (again $w = n/3 - 1$). There are 2 cases.

1. $n > 0$. Friedmann's equation gives:

$$(\dot{a}/a)^2 \propto a^{-n} \qquad \Longrightarrow \qquad a \propto t^{2/n}$$

For instance, a matter dominated universe has $a(t) \propto t^{2/3}$, and a radiation dominated universe has $a(t) \propto t^{1/2}$. All of these have a singularity $a = 0$, at $t = 0$, which is called a Big Bang. At this point $\rho \propto a^{-n} = t^{-2} \to \infty$, so the energy density becomes arbitrarily dense, and we expect classical GR to spectacularly break down. For $n > 2$, we can draw the spacetime conformal (Penrose) diagram as follows:

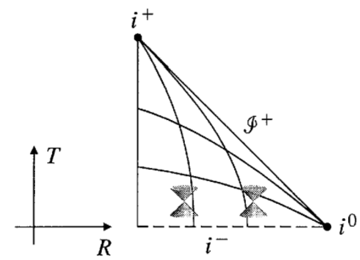

Figure 1: Conformal diagram for FLRW universe with $a(t) \propto t^{2/n}$ for $n > 2$. The dashed line represents the singularity at $t = 0$.

2. $n = 0$. This happens for the vacuum dominated case $w = -1$, and gives an exception to the Big Bang scenario. Here Friedmann's equation tells us $(\dot{a}/a)^2 \propto 1$, with solution $a \propto e^{\alpha t}$. The scale factor expands now as an exponential instead of a power law, allowing us to continue back indefinitely to $t \to -\infty$. The metric is:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{\alpha t}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$$

This is one way of writing coordinates for de Sitter space, which is a maximally symmetric space of positive curvature and scale factor $a \propto \cosh t$. This may seem contradictory, since in assuming our universe is vacuum dominated we took $k = 0$ and found $e^{\alpha t}$. However, the seeming disparity can be solved by making a change of coordinates.

To work with more realistic models, we can choose expanding cosmologies with both matter and a cosmological constant, Ω_M, Ω_Λ, with k fixed by $\Omega_k = 1 - \Omega_M - \Omega_\Lambda$. As these universes expand the relative influences of matter, curvature, and vacuum are altered since different densities evolve at different rates:

$$\Omega_\Lambda \propto \Omega_k a^2 \propto \Omega_m a^3$$

If $a \to 0$ in the past in the case of a big bang, curvature and vacuum will be negligible. If $a \to \infty$ in the future in the case of a big rip, curvature and matter will be negligible, and the universe will asymptote to de Sitter. It is also possible that $a(t)$ attains some maximum value before decreasing to zero, these are recollapse or big brunch cosmologies. We plot some examples below.

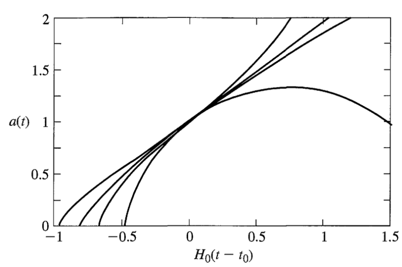

Figure 2: Expansion histories for different $(\Omega_m, \Omega_\Lambda)$. From top to bottom these correspond to (0.3, 0.7), (0.3, 0), (1, 0), (4,0).