Recitation 5-6

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3 General Relativity

Source: Carroll Chapter 1.9, 4.6, 7.1-7.5. Note: only had time to cover parts 3.3-3.4.

3.1 The Stress-Energy Tensor

Review
The stress energy tensor $T^{\mu\nu}$ is a $(2,0)$ tensor that sources the Einstein's equations, in the same way that the four-current $J^\mu = (\rho, \mathbf{J})$ sources Maxwell's equations by introducing an inhomogeneity. It too, satisfies a conservation law: $\nabla_\mu T^{\mu\nu} = 0$. In this way, we can view $T^{\mu\nu}$ as a set of 4 individually conserved currents, $T^{\mu 0}, T^{\mu i}$. The former gives conservation of energy, and the latter 3 give conservation of momentum.

In class, we understood the component $T^{\mu\nu}$ as the flux of 4-momentum p^μ across a surface of constant x^ν. We enumerate its components:

- T^{00}: flux of energy in time, i.e. the rest-frame energy density ρ.
- $T^{0i} = T^{i0}$: flux of momentum in time, i.e. the momentum density.
- T^{ij}: momentum flux in space, i.e. stresses (forces between infinitesimal elements of matter) Equivalently, the ith component of force exerted per unit area in the jth direction. Diagonal terms give pressure p_i (not to be confused with momentum), off-diagonal terms give shear σ_{ij}

Example 1: Dust
Defined in flat spacetime as a collection of particles at rest with respect to each other.

$$T^{\mu\nu} = \rho U^\mu U^\nu$$

for U^μ the 4-velocity of the fluid. It is uniquely specified by its energy density ρ.

Example 2: Perfect fluid
Defined in flat spacetime as matter uniquely specified by its rest frame energy density ρ and isotropic pressure p. In its rest frame, isotropy implies that $T^{\mu\nu}$ is diagonal, with $T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$. A general covariant equation for this yields:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p g^{\mu\nu}$$

Writing out the conservation laws for a perfect fluid and expanding in the non-relativistic limit $|\mathbf{v}| \ll 1$, one finds:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p$$

which are the continuity and Euler equations respectively, from fluid dynamics.

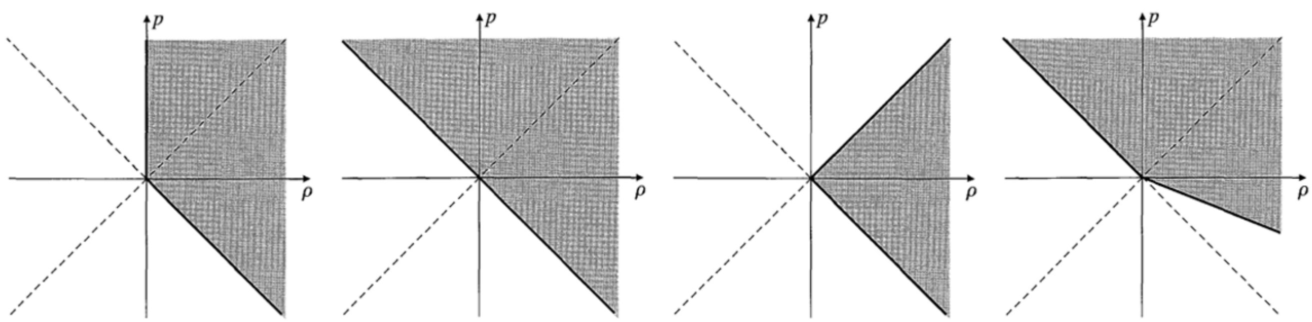
Energy Conditions
In the absence of any constraints on matter $T^{\mu\nu}$, it can be shown that any metric at all can solve Einstein's equations: for any $g_{\mu\nu}$, we can simply take $T_{\mu\nu} \propto G_{\mu\nu}$. Conservation is immediate from the Bianchi identity. This, of course, is unphysical. For instance, it includes spacetimes with closed timelike curves, which we don't believe should be possible.

This leads to the question of what sorts of energy-momenta are realistic. Energy conditions put constraints on $T^{\mu\nu}$. We list some commonly used conditions, along with what they mean for perfect fluids.

- Weak energy condition: $T_{\mu\nu}t^\mu t^\nu \geq 0$ for all timelike vectors t^μ.
  For a perfect fluid, an equivalent condition is that $T_{\mu\nu}U^\mu U^\nu \geq 0$ and $T_{\mu\nu}l^\mu l^\nu \geq 0$ for some null vector l^μ (by isotropy). This implies $\rho \geq 0$ and $\rho + p \geq 0$, i.e. the energy density is non-negative, and pressure can't be too negative.

- Null energy condition: $T_{\mu\nu}l^\mu l^\nu \geq 0$ for all null vectors l^μ.
  For a perfect fluid, this implies $\rho + p \geq 0$. The energy density may be negative, so long as there is a positive enough compensating pressure.

- Strong energy condition: $T_{\mu\nu}t^\mu t^\nu \geq \frac{1}{2}T^\rho{}_\rho t^\sigma t_\sigma$.
  For a perfect fluid, this implies $\rho + p \geq 0$ and $\rho + 3p \geq 0$. We will see that this condition implies gravity is attractive.



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3.2 Einstein's Equation

Einstein's equations are given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

In this section we will look at 2 properties of this equation: that gravity is attractive if we assume the SEC, and that there are fundamentally only 2 physical degrees of freedom in the metric components.

Gravity is Attractive no time oof

Degrees of Freedom
Here we wish to count gravitational degrees of freedom. There are 2 parts to this:

1. The number of physical metric components, i.e. those that cannot be fixed by some gauge transformation (by definition unphysical)
2. The number of independent Einstein's equations giving the dynamics of physical metric components

When all is said and done, we expect these to agree at 2 each. This is a very often very misunderstood procedure, so here I hope to clear things up.

Metric Components

The metric $g_{\mu\nu}$ is a symmetric tensor, with 10 independent components. However, these change under coordinate transformations $x^\mu \to x'^\mu$ as $g_{\mu\nu} \to \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}$. Coordinate transformations are the gauge transformations of GR; they are always unphysical, since nature does not care about which coordinate patch you work in. They account for 4 degrees of freedom, reducing the physical metric components down to 6.

But we are not done. It is a pithy but profound statement that "the gauge always hits twice." Let us see what this means in EM. Here we have the 4-potential A^μ, and the freedom to make gauge transformations $A_\mu \to A_\mu + \partial_\mu \lambda$. Suppose we work in Lorenz gauge, fixing $\partial_\mu A^\mu = 0$. However, we *still* have the freedom to make a 'harmonic' gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$ for $\partial^2 \lambda = 0$. One can use this to cut down another degree of freedom bringing us down from 4 to 2.

Next, let's see how this works in linearized GR. Using the trace-reversed metric perturbations, the gauge transformations $x^\mu \to x^\mu + \xi^\mu$ act as ($*$), for some vector field ξ^μ with 4 components. We can choose the Lorenz conditions $\partial^\mu h_{\mu\nu}$ fixing 4 degrees of freedom. However, we still have the freedom to make a gauge transformation satisfying $\partial^2 \xi^\mu = 0$. It is easy to check this is consistent with the Lorenz condition:

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu(\bar{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial_\rho \xi^\rho \eta_{\mu\nu}) = \partial^\mu \bar{h}_{\mu\nu} + \partial^2 \xi_\nu = 0 + 0 = 0$$

We can make any choice 4 harmonic functions x^μ, reducing our physical degrees of freedom down to 2. One may ask if there is any residual gauge freedom we fixed. It can be non-trivially shown using theorems from harmonic analysis about cohomology that that there is not. In linearized gravity these degrees of freedom correspond to the 2 polarizations of gravitational waves, which are very physical indeed.

The above was for linearized GR, but the very same is true in non-perturbative GR. The easiest way to see this is using the ADM formalism. Unfortunately this is outside the scope of the class, so I ask you to take my word for it. Again after exploiting all our gauge freedom, we are left with 2 physical degrees of freedom.

Einstein's Equations

Next we examine Einstein's equations. Since the Einstein and stress-energy tensors are both symmetric by design, $G_{\mu\nu} = 8\pi T_{\mu\nu}$ gives 10 equations. However, the differential Bianchi identity $\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} = 0$ implies $\nabla^\mu G_{\mu\nu} = 0$ (or equivalently, conservation laws $\nabla_\mu T^{\mu\nu} = 0$). These reduce the number of independent Einstein's equations to 6.

Again, we are not done. To proceed it helps to take a step back and ask what Einstein's equations are doing. They are differential equations relating first and second derivatives of $g_{\mu\nu}$, much like Newton's $m\ddot{x} = F$. Therefore, we can think of Einstein's equations as an equation of motion, or evolution equations for $g_{\mu\nu}$; if we know the initial conditions of $g_{\mu\nu}$ on some initial value slice Σ (e.g. $x^0 = 0$ in Minkowski spacetime), we can use Einstein's equations to solve for $g_{\mu\nu}$ everywhere else in the spacetime. The initial conditions we need to specify are $g_{\mu\nu}(\Sigma)$ and $\partial_t g_{\mu\nu}(\Sigma)$, in direct analogy with $x(0)$ and $x'(0)$ in Newtonian mechanics.

However, not all of Einstein's equations are true evolution equations. A true evolution equation would involve second derivatives with respect to x^0. In particular, it can be seen by tediously expanding in terms of partials and Christoffels that $G_{0\nu} = 8\pi T_{0\nu}$ does not contain any second derivatives $\partial_0^2 g_{\rho\sigma}$, only first derivatives $\partial_0 g_{\rho\sigma}$. Consequentially these 4 equations are constraints on the initial conditions, not actual evolution equations. Hence, if we are counting the number of independent 'evolution equations', there are only 2. This agrees with the number of physical metric components, as desired.

3.3 Linearized Gravity

Review
Linearized gravity is based on the weak field approximation, where we approximate the metric about a Minkowski background plus some small perturbation. That is:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

In this setting, we may treat the perturbation $h_{\mu\nu}$ as a field taking values on the Minkowski background. Using this to tediously compute Christoffel symbols and the Riemann tensor, we find the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} = \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu - \partial^2 h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\partial^2 h)$$

To simplify this further, we use our freedom in making the coordinate gauge-transformations:

$$h_{\mu\nu} \to h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$$

where ξ^μ is any vector field with $|\xi^\mu| \ll 1$, also valued in Minkowski space. Defining the trace-reversed metric perturbation as:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \tag{$*$}$$

we see that under the above gauge transformations it transforms as

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - (\partial\cdot\xi)\eta_{\mu\nu}$$

The Lorenz gauge defines ξ^μ as satisfying the differential equation

$$\partial^2\xi_\mu = -\partial^\lambda\bar{h}_{\lambda\mu}$$

Note that this can always be done: for each $\mu \in \{0,1,2,3\}$ this is just a (relativistic) Poisson equation with some source, for which existence and uniqueness of solutions has been established.
Making this gauge transformation has the effect of setting

$$\begin{aligned}\partial_\mu\bar{h}^{\mu\nu} \to \partial_\mu(\bar{h}^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu - (\partial\cdot\xi)\eta^{\mu\nu}) &= \partial_\mu\bar{h}^{\mu\nu} + \partial^2\xi^\nu + \partial^\nu(\partial\cdot\xi) - \partial^\nu(\partial\cdot\xi)\\ &= \partial_\mu\bar{h}^{\mu\nu} - \partial_\lambda\bar{h}^{\lambda\nu} = 0\end{aligned}$$

Therefore in this gauge $\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h$, and substituting this into the linearized Einstein equation it simplifies greatly to

$$G_{\mu\nu} = -\frac{1}{2}\partial^2\bar{h}_{\mu\nu}, \qquad \partial^2\bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}$$

This can be solved using a Greens function approach:

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G_N\int\frac{\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}T_{\mu\nu}(t_r,\mathbf{y}), \qquad t_r := t - |\mathbf{x}-\mathbf{y}|$$

In the case where $T_{\mu\nu}$ is time-independent, this simplifies to the regular Poisson equation from electrodynamics (for each component $\mu,\nu \in \{0,1,2,3\}$)

$$\vec{\nabla}^2\bar{h}_{\mu\nu} = -16\pi G_N T_{\mu\nu}, \qquad \bar{h}_{\mu\nu}(t,\mathbf{x}) = 4G_N\int\frac{\mathrm{d}^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}T_{\mu\nu}(\mathbf{y})$$

Example: The Lense-Thirring effect (Carroll 7.2)
Consider a thin spherical shell of matter of mass M and radius R, rotating slowly with angular velocity Ω. Relative to an inertial frame defined by the background Minkowski metric, compute the rotation of a freely-falling observer sitting at the center of the shell. In other words, compute the precession of the spatial components of a parallel-transported vector located at the center.

Step 1: Finding $T^{\mu\nu}$
First, we need the stress-energy tensor of a rotating shell with M, R, Ω. We define the vector $\mathbf{\Omega}$ of magnitude Ω, pointed in the direction of rotation. $T^{\mu\nu}$ is given by:

$$T^{\mu\nu} = \rho U^\mu U^\nu, \qquad \rho = \frac{M}{4\pi R^2}\delta(r-R)$$

where U^μ is the 4-velocity of a point on the shell. We can write its 3-velocity of said point as $\mathbf{v} = \mathbf{\Omega}\times\mathbf{r}$. This allows us to compute:

$$U^\mu = \gamma(\mathbf{v})\begin{pmatrix}1\\ \mathbf{v}\end{pmatrix} = \frac{1}{(1-(\mathbf{\Omega}\times\mathbf{r})^2)^{1/2}}\begin{pmatrix}1\\ \mathbf{\Omega}\times\mathbf{r}\end{pmatrix}$$

The stress-energy tensor is thus

$$\begin{aligned}T^{\mu\nu} &= \frac{M}{4\pi R^2}\frac{\delta(r-R)}{1-(\mathbf{\Omega}\times\mathbf{r})^2}\begin{pmatrix}1\\ \mathbf{\Omega}\times\mathbf{r}\end{pmatrix}\otimes\begin{pmatrix}1\\ \mathbf{\Omega}\times\mathbf{r}\end{pmatrix}\\ &= \frac{M}{4\pi R^2}\delta(r-R)\Big(1+R^2(\mathbf{\Omega}\times\hat{\mathbf{r}})^2+\mathcal{O}((\Omega R)^4)\Big)\begin{pmatrix}1 & (\mathbf{\Omega}\times\mathbf{r})^j\\ (\mathbf{\Omega}\times\mathbf{r})^i & (\mathbf{\Omega}\times\mathbf{r})^i(\mathbf{\Omega}\times\mathbf{r})^j\end{pmatrix}\end{aligned}$$

To leading non-trivial order in ΩR, we find:

$$T^{0\mu} = \frac{M}{4\pi R^2}\delta(r-R)\begin{pmatrix}1\\ \mathbf{\Omega}\times\mathbf{r}\end{pmatrix}, \qquad T^{ij} = 0$$

Step 2: Solving for $h_{\mu\nu}$
Next, we solve the linearized Einstein's equations for the perturbed metric. Using the form for T above, the only non-trivial components of Einstein's equation are

$$\partial^2\bar{h}_{0\nu} = -16\pi G_N T_{0\mu} = -\frac{4MG_N}{R^2}\delta(r-R)\begin{pmatrix}1\\ \mathbf{\Omega}\times\mathbf{r}\end{pmatrix}$$

We will only need the components $\nu = i$. Since these are off-diagonal, $\bar{h}_{0i} = h_{0i}$. Furthermore, noting that our system is time-independent we solve for $h_{\mu\nu}$ via Green's functions:

$$h_{0i}(t,\mathbf{r}) = 4G_N\int\frac{\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}T_{0i}(\mathbf{r}') = \frac{MG_N}{\pi R^2}\int\frac{\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\delta(r'-R)(\mathbf{\Omega}\times\mathbf{r})$$

This integral is done in Griffiths Problem 5.11, and works in a spherical coordinate system where the z'-axis is aligned with $\mathbf{r}$, and $\mathbf{\Omega}$ lies in the $x'z'$-plane, making an angle ψ with the z'-axis. We evaluate:

$$\begin{aligned}&\mathbf{\Omega} = \Omega(\sin\psi, 0, \cos\psi'), \quad \mathbf{r}' = r'(\sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'), \quad |\mathbf{r}-\mathbf{r}'| = \sqrt{r^2+r'^2-2rr'\cos\theta'}\\ &\mathbf{\Omega}\times\mathbf{r}' = r'\Omega(-\cos\psi\sin\theta'\sin\phi', \cos\psi\sin\theta'\cos\phi'-\sin\psi\cos\theta', \sin\psi\sin\theta'\sin\phi')\end{aligned}$$

The terms involving a single $\sin\phi$ or $\cos\phi$ vanish when integrated along $\int_0^{2\pi}\mathrm{d}\phi$, so we have left:

$$\begin{aligned}h_{0i}(t,\mathbf{r}) &= \frac{MG_N}{\pi R^2}\int\frac{R^2\sin\theta'\mathrm{d}\theta'}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}R\Omega(0,-2\pi\sin\psi\cos\theta', 0)\\ &= -2RMG_N\Omega\hat{\mathbf{y}}\sin\psi\int\frac{\sin\theta'\cos\theta'\mathrm{d}\theta'}{\sqrt{r^2+r'^2-2rr'\cos\theta'}}\\ &= -2RMG_N\Omega\hat{\mathbf{y}}\sin\psi\left(-\frac{1}{3R^2r^2}((R^2+r^2+Rr)|R-r|-(R^2+r^2-Rr)(R+r))\right)\end{aligned}$$

where in the third line we make the change of variables $u = \cos\theta'$ and use Mathematica. We are interested in the field inside the shell, for which $r < R$. This gives:

$$h_{0i}(t, \mathbf{r}) = -2RMG_N\Omega\hat{\mathbf{y}}\sin\psi\frac{2r}{3R^2} = \frac{4MG_N}{3R}(\mathbf{r}\times\mathbf{\Omega})_i = \frac{4MG_N}{3R}\epsilon_{ikl}r_k\Omega_l$$

Alternatively, you can verify for yourself that this solves the Poisson equation by computing $\partial^2 h_{0i}$.

Step 3: Parallel Transport

We are interested in what happens to an observer at the center of the shell. This is an inertial frame, where $U^\mu = (1, \vec{0})$. Let S^μ be a spatial vector in the observer frame, satisfying $S^\mu u_\mu = 0$, i.e. $S^0 = 0$. We wish to see how S^μ changes if we parallel transport it along U^μ, i.e. with respect to the observer time.

The parallel transport equation is given by

$$0 = U^\rho\nabla_\rho S^\mu = U^\rho(\partial_\rho S^\mu + \Gamma^\mu_{\rho\sigma}S^\sigma) = \frac{\mathrm{d}S^\mu}{\mathrm{d}t} + \Gamma^\mu_{0\sigma}S^\sigma$$

Taking $\mu = i$ yields

$$\frac{\mathrm{d}S^i}{\mathrm{d}t} + \Gamma^i_{0\sigma}S^\sigma = 0$$

The relevant Christoffel symbols for linearized gravity are:

$$\begin{aligned}
\Gamma^i_{j0} &= \frac{1}{2}(\partial_j h_{0i} - \partial_i h_{0j} + \partial_0 h_{ij}) = \frac{1}{2}(\partial_j h_{0i} - \partial_i h_{0j}) \\
&= \frac{4MG_N}{3R}\frac{1}{2}(\partial_j(\epsilon_{ikl}r_k\Omega_l) - \partial_i(\epsilon_{jkl}r_k\Omega_l)) = \frac{2MG_N}{3R}(\epsilon_{ikl}\delta_{jk}\Omega_l - \epsilon_{jkl}\delta_{ik}\Omega_l) = \frac{4MG_N}{3R}\epsilon_{ijl}\Omega_l \\
\Gamma^i_{jk} &= \frac{1}{2}(\partial_j h_{ki} + \partial_k h_{ji} - \partial_i h_{jk}) = 0
\end{aligned}$$

where we use that in our system $h_{ij} = 0$ from the Einstein's equations. Substituting this into the parallel transport equation yields:

$$\frac{\mathrm{d}S^i}{\mathrm{d}t} = -\Gamma^i_{0j}S^j = -\frac{4MG_N}{3R}\epsilon_{ijl}\Omega_l S_j = \frac{4MG_N}{3R}(\mathbf{\Omega}\times\mathbf{S})^i$$

This is our result. We see that $\mathbf{S}$ changes so long as it is not parallel to the rotation axis, $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$. This effect is called frame dragging, and is not found in Newtonian mechanics, where $\mathbf{S}$ would not change due to a rotating mass configuration.

For instance, suppose the sphere is rotating about the z-axis, and take the initial vector $\mathbf{S}(t = 0) = \hat{\mathbf{x}}$. Then, we have

$$\dot{S}_x = -\frac{4MG_N\Omega}{3R}S_y, \qquad \dot{S}_y = \frac{4MG_N\Omega}{3R}S_x$$

which can be solved to yield

$$\mathbf{S} = (\cos\omega t, \sin\omega t, 0), \qquad \omega := \frac{4MG_N\Omega}{3R}$$

where ω is the procession rate of $\mathbf{S}$ about the axis of rotation.

3.4 Gravitational Waves

Review

Solving for $h_{\mu\nu}$:

In solving for gravitational waves it is useful to introduce the transverse traceless gauge, in which:

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & s_{ij} \end{pmatrix}, \qquad s^i{}_i = 0, \qquad \partial^i s_{ij} = 0$$

We see that we have used our full gauge freedom in fixing 8 components of the metric; imposing the timelike components vanish fixes 4, tracelessness fixes 1, and s_{ij} transverse fixes 3. An equivalent way of saying this is that $h^{\mathrm{TT}}_{\mu\nu}$ has the following properties:

$$h^{\mathrm{TT}}_{0\nu} = 0, \qquad \eta^{\mu\nu} h^{\mathrm{TT}}_{\mu\nu} = 0, \qquad \partial^\mu h^{\mathrm{TT}}_{\mu\nu} = 0$$

Note that this is a special case of the Lorenz gauge $\partial^\mu h_{\mu\nu} = 0$. In particular this means the equation of motion reduces to:

$$\partial^2 h^{\mathrm{TT}}_{\mu\nu} = 0$$

Suppose the metric only depends on x^μ through some Lorentz scalar $k^\mu x_\mu$ where k^μ could a priori be any 4-vector. That is, we assume $h^{\mathrm{TT}}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu}(k \cdot x)$. Then, this equation implies $0 = k^2 = -(k^0)^2 + \mathbf{k}^2$. We therefore write:

$$k^\mu = (\omega, \mathbf{k}), \qquad \omega := (\mathbf{k}^2)^{1/2}$$

A basis of these functions are spanned by $A_{\mu\nu} e^{ik\cdot x}$. Since our differential equation is linear, we may without loss of generality work with the solutions:

$$h^{\mathrm{TT}}_{\mu\nu} = C_{\mu\nu} e^{ik\cdot x}, \qquad C_{0\mu} = 0, \qquad C^\mu{}_\mu = 0, \qquad k^\mu C_{\mu\nu} = 0$$

where the latter 3 conditions are demanded by the perturbation being spacelike, traceless, transverse. Without loss of generality, we may choose coordinates so that $\mathbf{z} \parallel \mathbf{k}$, so $k^\mu = (\omega, 0, 0, \omega)$. Then, these conditions give us

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where we have renamed $C_{11} = h_+$, $C_{12} = h_\times$. We see that we have 2 degrees of freedom, as expected.

Trajectories of Test Particles:

To gain some physical intuition of gravitational waves, we consider the motion of nearby test particles, in a locally inertial frame of any particular one. We examine the geodesic deviation equation in this setting:

$$\nabla_U \nabla_U S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma$$

Here U^μ are the 4-velocities (tangent vectors) along the timelike geodesics of the (locally) freefalling test particles, and S^μ is the deviation vector pointing from one geodesic to the next. If the test particles start with zero 3-velocity then we have $U^\mu = (1, 0, 0, 0)$ at all times to first order in $h^{TT}_{\mu\nu}$.

Using that in our gauge $R_{\mu 00\sigma} = \frac{1}{2}\partial_t^2 h^{\mathrm{TT}}_{\mu\sigma}$ (a routine calculation using Christoffel symbols), the above equation reduces (at leading order) to:

$$\frac{\partial^2}{\partial t^2} S_\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{\mu\sigma}$$

Substituting the metric solution we solved for yields:

$$\begin{pmatrix} \ddot{S}_1 \\ \ddot{S}_2 \\ \ddot{S}_3 \end{pmatrix} = -\frac{\omega^2}{2} e^{ik\cdot x} \begin{pmatrix} h_+ & h_\times & \\ h_\times & -h_+ & \\ & & 0 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

Using that $\dot{S}^i(t=0) = 0$, we see immediately that $S_3(t) = S_3(0)$. That is, geodesics are not distorted in the direction $\mathbf{k}$ of the gravitational wave.

It is instructive to solve this ODE explicitly for the cases where h_+ and h_- are individually zero. Introducing $\Delta S^\alpha(t) = S^\alpha(t) - S^\alpha(0)$ for $\alpha = 1, 2$, we have:

1. $h_\times = 0$:

$$\begin{pmatrix} \Delta S^1(t) \\ \Delta S^2(t) \end{pmatrix} = \frac{\omega^2}{2} h_+ e^{ik\cdot x} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} S^1(0) \\ S^2(0) \end{pmatrix}$$

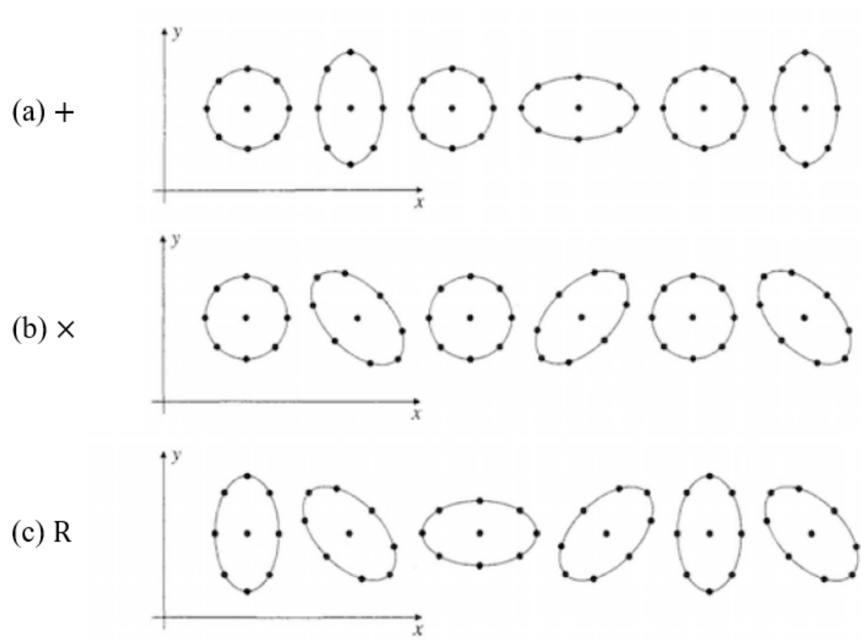
   We see that geodesics along x are stretched further apart while those along y are brought closer together, coupled with a sinusoidal factor oscillating with time. A ring of test particles thus oscillates in a $+$ pattern.

2. $h_\times = 1$:

$$\begin{pmatrix} \Delta S^1(t) \\ \Delta S^2(t) \end{pmatrix} = \frac{\omega^2}{2} h_\times e^{ik\cdot x} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} S^1(0) \\ S^2(0) \end{pmatrix}$$

   Now geodesics along $x + y$ direction are stretched further apart while those in the $x - y$ direction are brought closer together. A ring of test particles oscillates in an x pattern.

Similar to EM waves, these form a basis of polarizations for gravitational waves. Note that these effects are suppressed by the metric components $h_+, h_\times \ll 1$, since we are working in the weak field approximation. Furthermore, distortion is proportional to ω^2: waves with larger ω (energy) therefore distort spacetime more, as expected.



Instead of choosing (h_+, h_-) as a basis of gravitational modes, we can instead define right/left circularly polarized modes by superimposing h_+ and $h_\times$, offset by a phase of $\pm\pi/2$. That is:

$$R: \frac{h_R}{\sqrt{2}}(1, i), \qquad L: \frac{h_L}{\sqrt{2}}(1, -i), \qquad \text{(in the } (h_+, h_\times) \text{ basis)}$$

Now, geodesics deviate as:

$$\begin{pmatrix} \Delta S^1(t) \\ \Delta S^2(t) \end{pmatrix} = \frac{\omega^2}{2} h_{R/L} e^{ik\cdot x} \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} S^1(0) \\ S^2(0) \end{pmatrix}$$

A ring of test particles rotates about the z-axis consistent with a right/left-handed coordinate system.

Example: Detecting gravitational waves
We can detect gravitational waves by monitoring the distance between 2 free-falling masses. Suppose the masses are length L away, with one at the origin. We fire a laser from one mass to the other. We put a mirror at the other mass so it bounces back, returning some time later. Suppose the metric is given by:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 + A\cos\omega(t-z))\mathrm{d}x^2 + (1 - A\cos\omega(t-z))\mathrm{d}y^2 + \mathrm{d}z^2$$

Find the detector orientation that registers the largest response. Compute this change in arrival time.

From the line element, we find $h_{\mu\nu} \propto \mathrm{diag}(0, A, -A, 0)$. This corresponds to the h_+ mode of a gravitational wave directed along $\mathbf{z}$. The weak field approximation assumes $|A| \ll 1$. The wave only distorts space in the plane perpendicular, i.e. along $\mathbf{x}$ and $\mathbf{y}$. Owing to the h_+ mode the directions along which neighboring geodesics change the most are $\mathbf{x}$ and $\mathbf{y}$. Without loss of generality, we orient our system such that the other mass is at $(x, y, z) = (L, 0, 0)$.

Next we must compute geodesics. Since we are firing a laser along the $\mathbf{x}$-direction we want null geodesics, where $\mathrm{d}y = \mathrm{d}z = 0$ and $z = 0$ (the common z-coordinate for both masses). These are given by:

$$0 = \mathrm{d}s^2 = -\mathrm{d}t^2 + (1 + A\cos\omega t)\mathrm{d}x^2 + 0 + 0$$

or equivalently to first order in the weak-field approximation ($|A| \ll 1$)

$$\mathrm{d}x = \frac{1}{\sqrt{1 + A\cos\omega t}}\mathrm{d}t \approx \left(1 - \frac{1}{2}A\cos\omega t\right)\mathrm{d}t$$

Now we integrate x from 0 to $2L$ (since we take a round trip). This will correspond to some time $t_f = 2L + \delta t$, where $2L$ is the time it takes light to travel to and back in the absence of a gravitational wave, and δt is of order A. Therefore:

$$\begin{aligned} 2L = \int_0^L \mathrm{dx} = \int_0^{2L+\delta t} \mathrm{d}t \left(1 - \frac{1}{2}A\cos\omega t\right) &= 2L + \delta t - \frac{A}{2\omega}\sin\omega(2L + \delta t) \\ &= 2L + \delta t - \frac{A}{2\omega}\sin\omega(2L) \end{aligned}$$

In the last line we expand the second term to first order in A, and δt is of order A. We see that the leading order $2L$s cancel out on both sides, and get

$$\delta t = \frac{A}{2\omega}\sin(2\omega L)$$

This is the change in time it takes for a laser to bounce back to the observer in the presence of a gravitational wave. In particular, note that $\delta t = 0$ for $\omega = \frac{n\pi}{2L}$. Waves of this frequency will be invisible to the observer.
Lastly, recalling that $\sin x/x \leq 1$ for $x \geq 0$ with equality only at $x = 0$, we see that

$$\delta t = AL\frac{\sin 2\omega L}{2\omega L} \leq AL = (\delta t)_{\max}$$

with equality as $\omega \to 0$, i.e. a infinitely 'soft' wave of nearing zero energy.