Recitation 3-4

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2 Riemannian Geometry

Source: Carroll Chapter 3.2-3.3, 3.6-3.7, Appendix I. A more detailed treatment of these concepts can be found there.

2.1 The Covariant Derivative

In flat space, the partial derivative ∂_μ is used to compare quantities (scalars, vectors, tensors, etc.) at two different points $p, q \in \mathbb{R}^k$. This is a very useful thing to do if we wish to do calculus on manifolds. Unfortunately, the partial derivative does not transform tensorially. So motivates the need for a tensorial 'covariant derivative' ∇_μ. This will allow us to generalize flat space physics to be curved spacetime, for instance

$$\partial_\mu T^{\mu\nu} = 0 \to \nabla_\mu T^{\mu\nu} = 0$$

In flat space, the partial derivative ∂_μ maps (k, l) tensor fields to $(k, l+1)$ tensor fields, in a way that obeys linearity and the Leibniz rule on tensor product. We would therefore like our covariant derivative to have the same properties:

$$\begin{aligned} &\text{1) Linearity:} && \nabla(T + S) = \nabla T + \nabla S \\ &\text{2) Leibniz rule:} && \nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S) \end{aligned}$$

It can be shown (Wald) that the Leibniz rule demands that ∇ can be written as a partial derivative, plus a linear transformation. That is, a covariant derivative of a tensor is obtained by first taking the partial derivative, the applying a correction to make the result covariant.

For vectors, this means:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$$

The correction can be thought of as the following: for each component ν, V^ν gets assigned a correction $(\Gamma^\nu)_{\mu\lambda} V^\lambda$, qhwew $(\Gamma^\nu)_{\mu\lambda}$ (for fixed ν) acts a matrix multiplication (linear transformation). It should be emphasized that $\Gamma^\nu_{\mu\lambda}$ is **not** a tensor even though we give it 'tensor indices', these are merely some coefficients. Indeed, they are purposefully constructed to be non-tensorial, in such a way that the combination in the above equation transforms as a tensor.

For one-forms, we have a similar statement:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu\nu} \omega_\lambda$$

Covariant Derivatives of Vectors, Covectors, Tensors

At this stage, there is no reason that $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ is related to $\Gamma^{\alpha}_{\beta\gamma}$. To relate these, and to make any further progress, we introduce 2 new properties:

$$\begin{aligned} &\text{3) The identity is constant:} \qquad && \nabla_\mu \delta^\nu_\rho = 0 \\ &\text{4) Reduces to } \partial_\mu \text{ on scalars:} \qquad && \nabla_\mu \phi = \partial_\mu \phi \end{aligned}$$

Note: (3) implies ∇_μ commutes with contractions, e.g. $\nabla_\mu T^\lambda{}_{\lambda\rho} = \nabla_\mu(\delta^\nu_\lambda T^\lambda{}_{\nu\rho}) = \delta^\nu_\lambda \nabla_\mu T^\lambda{}_{\nu\rho} = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$

This allows us to relate the covariant derivatives on vectors and 1-forms by computing:

$$\begin{aligned} \nabla_\mu(\omega_\lambda V^\lambda) &= (\nabla_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\nabla_\mu V^\lambda) = (\partial_\mu \omega_\lambda)V^\lambda + \tilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) + \omega_\lambda \Gamma^\lambda_{\mu\rho}V^\rho \\ &= \partial_\mu(\omega_\lambda V^\lambda) = (\partial_\mu \omega_\lambda)V^\lambda + \omega_\lambda(\partial_\mu V^\lambda) \end{aligned}$$

In the first equality we used properties (3) and (2). In the second line we used property (4). Comparing the right hand sides of both lines, we see that $\tilde{\Gamma}^\sigma_{\mu\lambda}\omega_\sigma V^\lambda + \omega_\lambda \Gamma^\lambda_{\mu\rho}V^\rho = 0$, or after relabelling some indices $(\tilde{\Gamma}^\sigma_{\mu\lambda} - \Gamma^\sigma_{\mu\lambda})\omega_\sigma V^\lambda = 0$. But since ω and V are arbitrary 1-forms/vectors, we arrive at the result:

$$\tilde{\Gamma}^\sigma_{\mu\lambda} = -\Gamma^\sigma_{\mu\lambda}$$

Therefore:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda, \qquad \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu}\omega_\lambda \tag{1}$$

This can be generalized to (k,l) tensors using property (2), for which we have the result:

$$\begin{aligned} \nabla_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} = \partial_\sigma T^{\mu_1\cdots\mu_k}{}_{\nu_1\cdots\nu_l} &+ \Gamma^{\mu_1}_{\sigma\lambda}T^{\lambda\mu_2\cdots\mu_k}1_{\nu_1\cdots\nu_l} + \Gamma^{\mu_2}_{\sigma\lambda}T^{\mu_1\lambda\cdots\mu_k}{}_{\nu_1\cdots\nu_l} + \cdots \\ &- \Gamma^\lambda_{\sigma\nu_1}T^{\mu_1\cdots\mu_k}1_{\lambda\nu_2\cdots\nu_l} - \Gamma^\lambda_{\sigma\nu_2}T^{\mu_1\cdots\mu_k}{}_{\nu_1\lambda\cdots\nu_l} - \cdots \end{aligned} \tag{2}$$

That is, for each upper index we introduce a term with a single $+\Gamma$, and for each lower index a term with a single $-\Gamma$.

The Christoffel Connection

So far, we've seen that to specify a covariant derivative, we need to specify a set of $4^3 = 64$ coefficients $\Gamma^\lambda_{\mu\nu}$ with various properties. This is called a connection, and there are various connections we can put on a differentiable manifold (it is called a connection because it will allow us to do parallel transport, relating vectors at a tangent space T_pM to another tangent space T_qM). However, if we are given a metric $g_{\mu\nu}$, this allows us to define a unique connection satisfying the following 2 properties:

$$\begin{aligned} &\text{5) Torsion free:} \qquad && \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \\ &\text{6) Metric compatibility:} \qquad && \nabla_\rho g_{\mu\nu} = 0 \end{aligned}$$

Note that (6) implies ∇_μ commutes with raising and lowering indices.

Proposition: there is a unique connection $\Gamma^\lambda_{\mu\nu}$ satisfying (5) and (6).

Proof: We will show both existence and uniqueness by deriving a manifestly unique expression for $\Gamma^\lambda_{\mu\nu}$ in terms of the metric. We start by explicitly writing out the equation for metric compatiblity for 3 different permutations of indices:

$$\begin{aligned} 0 &= \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu}g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu}g_{\mu\lambda} \\ 0 &= \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\rho\nu}g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho}g_{\nu\lambda} \\ 0 &= \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho}g_{\lambda\mu} - \Gamma^\lambda_{\nu\rho}g_{\rho\lambda} \end{aligned}$$

Now, we subtract the second and third equations from the first and use the torsion-free property (5):

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0$$

Finally we multiply both sides by $g^{\sigma\rho}$ to obtain:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{3}$$

This is the formula for the Christoffel connection.

2.2 Parallel Transport, Geodesics

Once we have the covariant derivative, we can use it to 'move' tensors from one point to another. This is the notion of parallel transport: a tensor T is parallel transported along a path $x^\mu(\lambda)$ if its covariant derivative along the path vanishes:

$$\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} \nabla_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} = 0 \tag{4}$$

Note that this is a well-defined tensor equation. The intuition here is that we are 'keeping a tensor constant' while moving it along the path by setting the 'directional covariant derivative' of the tensor, $\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\nabla_\mu$ to be zero.

For a vector, this means that:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} V^\mu + \Gamma^\mu_{\sigma\rho} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} V^\rho = 0$$

A geodesic is a curve along which the tangent vector is itself parallel-transported. For a curve x^λ, the tangent vector is $\mathrm{d}x^\mu/\mathrm{d}\lambda$, and the parallel transport equation gives:

$$0 = \frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} \tag{5}$$

This is the geodesic equation. It can be shown using the calculus of variations that these correspond to shortest distance paths between 2 points.

In flat space, all the connection coefficients vanish $\Gamma^\lambda_{\mu\nu} = 0$, so the geodesic equation reduces to $(x^\mu)'' = 0$. The solutions are precisely the straight lines, $x^\mu(\lambda) = a^\mu \lambda + b^\mu$.

Example:
Find the geodesics on the hyperbolic plane $\mathbb{H}_2$, given by Poincaré metric:

$$\mathrm{d}s^2 = \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$$

Use this to compute the distance between 2 points on a geodesic.

<u>Solution</u>:
The metric and inverse metric are:

$$g_{\mu\nu} = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad g^{\mu\nu} = y^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We first need to compute the Christoffel symbols in terms of the metric

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\alpha}(\partial_\rho g_{\alpha\sigma} + \partial_\sigma g_{\alpha\rho} - \partial_\alpha g_{\rho\sigma})$$

When the metric is diagonal (as ours is), we can simplify:

$$\begin{aligned}\Gamma^{\mu}_{\rho\sigma} &= \frac{1}{2} g^{\mu\mu}(\partial_\rho g_{\mu\sigma} + \partial_\sigma g_{\mu\rho} - \partial_\mu g_{\rho\sigma}) \\ &= \frac{1}{2} g^{\mu\mu}(\delta^{\sigma}_{\mu}\partial_\rho g_{\sigma\sigma} + \delta^{\rho}_{\mu}\partial_\sigma g_{\rho\rho} - \delta^{\rho}_{\sigma}\partial_\mu g_{\rho\rho})\end{aligned}$$

where now there is no more implicit sum over repeated indices.

$\mathbb{H}_2$ is a 2-dimensional manifold, so naïvely we need to compute $2^3 = 8$ separate Christoffel symbols. However $\Gamma^{\mu}_{\rho\sigma} = \Gamma^{\mu}_{\sigma\rho}$, so we only need to find 6: Γ^{x}_{xx}, Γ^{x}_{xy}, Γ^{x}_{yy}, Γ^{y}_{xx}, Γ^{y}_{xy}, Γ^{y}_{yy}. Note further that $g_{\mu\nu}$ depends on only y, so all x-derivatives will vanish. We compute:

$$\begin{aligned}\Gamma^{x}_{xx} &= \frac{1}{2} g^{xx}(\partial_x g_{xx} + \partial_x g_{xx} - \partial_x g_{xx}) = \frac{1}{2} g^{xx}\partial_x g_{xx} = 0 \\ \Gamma^{x}_{yy} &= \frac{1}{2} g^{xx}(0 + 0 - \partial_x g_{yy}) = -\frac{1}{2} g^{xx}\partial_x g_{yy} = 0 \\ \Gamma^{x}_{xy} &= \frac{1}{2} g^{xx}(0 + \partial_y g_{xx} - 0) = \frac{1}{2} g^{xx}\partial_y g_{xx} = \frac{1}{2} y^2 \frac{-2}{y^3} = -\frac{1}{y} \\ \Gamma^{y}_{yy} &= \frac{1}{2} g^{yy}(\partial_y g_{yy} + \partial_y g_{yy} - \partial_y g_{yy}) = \frac{1}{2} g^{yy}\partial_y g_{yy} = -\frac{1}{y} \\ \Gamma^{y}_{xx} &= \frac{1}{2} g^{yy}(0 + 0 - \partial_y g_{xx}) = -\frac{1}{2} g^{yy}\partial_y g_{xx} = \frac{1}{y} \\ \Gamma^{y}_{xy} &= \frac{1}{2} g^{yy}(\partial_x g_{yy} + 0 - 0) = \frac{1}{2} g^{yy}\partial_x g_{yy} = 0\end{aligned}$$

To summarize:

$$\Gamma^{x}_{\mu\nu} = \frac{1}{y}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^{y}_{\mu\nu} = \frac{1}{y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now, we can substitute these into the geodesic equation:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda} = 0$$

Taking $\mu = x$ and $\mu = y$ we have 2 such equations, given by:

$$\begin{aligned}0 &= x'' + (\Gamma^{x}_{xx}(x')^2 + 2\Gamma^{x}_{xy}(x'y') + \Gamma^{x}_{yy}(y')^2) = x'' - \frac{2}{y}x'y' \\ 0 &= y'' + (\Gamma^{y}_{xx}(x')^2 + 2\Gamma^{y}_{xy}(x'y') + \Gamma^{y}_{yy}(y')^2) = y'' + \frac{1}{y}((x')^2 - (y')^2)\end{aligned}$$

A trick here is to compute the following quantity:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{yy'}{x'}\right) = \frac{yy''x' + (y')^2x' - yy'x''}{(x')^2} = -\frac{y}{x'}\frac{1}{y}((x')^2 - (y')^2) + \frac{(y')^2}{x'} - \frac{yy'}{(x')^2}\frac{2}{y}x'y' = -x'$$

where in the second equality we use both differential equations to substitute x'' and y''. Therefore, we know $(yy'/x' + x)' = 0$, so $yy'/x' + x = x_0$ for $x_0 \in \mathbb{R}$. We rewrite this equation in the following way:

$$0 = yy' + (x - c)' = \frac{1}{2}(y^2)' + \frac{1}{2}((x - x_0)^2)' = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}(y^2 + (x - x_0)^2)$$

Finally, this gives the equation

$$(x - x_0)^2 + y^2 = r^2, \qquad x_0, r \in \mathbb{R}$$

Therefore we see that the geodesics in $\mathbb{H}^2$ are precisely the (half)-circles with centers on the x-axis, or equivalently the circles intersecting the x-axis at right angles.

One caveat: the above derivation assumed that $x' \neq 0$ everywhere, i.e. our curves are not $x = c$. However, these also solve the geodesic equations, so any vertical line is also a geodesic.

We plot some geodesics below.

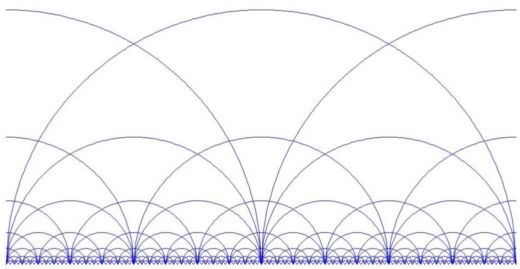

Figure 1: Some geodesics in $\mathbb{H}^2$

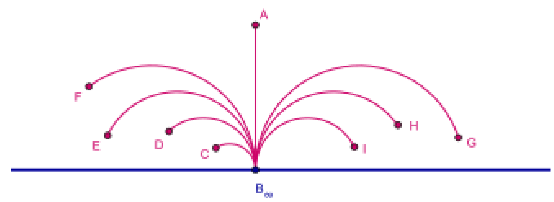

Figure 2: Geodesics in $\mathbb{H}^2$ emanating from a point on its boundary

Now we find the length for 2 points on a geodesic. First we do this for straight line geodesics $x = x_0$. Here $\mathrm{d}y = 0$, and we compute:

$$l[y_1, y_2] = \int \mathrm{d}s = \int_{y_1}^{y_2} \frac{\mathrm{d}y}{y} = \ln\left|\frac{y_2}{y_1}\right|$$

Next, consider geodesics to be circles $(x - x_0)^2 + y^2 = r^2$. Let us parameterize this by:

$$x = t, y = \sqrt{r^2 - (t - x_0)^2}, \qquad t \in [x_1, x_2]$$

The length is:

$$\begin{aligned} l[x_1, x_2] &= \int \mathrm{d}s = \int \frac{1}{y}\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_{x_1}^{x_2} \frac{\mathrm{d}t}{y(t)} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \\ &= \int_{x_1}^{x_2} \mathrm{d}t \frac{r}{r^2 - (t - x_0)^2} = \frac{1}{2}\ln\left|\frac{(r + x_0 - x_1)(r - x_0 + x_2)}{(r - x_0 + x_1)(r + x_0 - x_2)}\right| \end{aligned}$$

In some cases this can be simplified: for instance if the points have the same y-coordinate and $d := |x_2 - x_1|$, then the length is:

$$l_y(d) = \log\frac{d + \sqrt{d^2 + 4y^2}}{-d + \sqrt{d^2 + 4y^2}} = 2\,\mathrm{arcsinh}\left(\frac{d}{2y}\right)$$

From these formulae, we see that the boundary $\{y = 0\} = \partial\mathbb{H}_+$ is 'infinitely far away' from any point in the interior.

2.3 Curvature

The Riemann Curvature Tensor

In Euclidean (or Minkowski space), flatness manifests in different ways. For instance, parallel transport along a closed loop leaves vectors invariant, covariant derivatives of tensors commute, and initially parallel geodesics remain parallel.

None of these are generally true in a curved spacetime, and the failure of these to hold are due to curvature, quantified by the Riemann curvature tensor $R^{\rho}{}_{\sigma\mu\nu}$. There are various ways of defining/deriving this object.

One way is the failure of covariant derivatives to commute:

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma \tag{6}$$

By expanding out the left-hand side explicitly and using that $\Gamma_{\mu\nu} = \Gamma_{\nu\mu}$, it is tedious but straightforwards to show that:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \tag{7}$$

Another way to define $R^\rho{}_{\sigma\mu\nu}$ is to measure the change in a vector upon parallel transporting it in an infinitesimal loop. This is the way you showed in class, but based on some of your questions in lecture and on Piazza I'll show this in a more mathematically sophisticated way.

The Parallel Propagator

By now we've seen the definition for parallel transport for a vector V^μ along a path $x^\mu(\lambda)$:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\partial_\mu V^\nu + \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\Gamma^\nu_{\mu\sigma}V^\sigma = 0$$

This is a differential equation for the transported vector $V^\mu(\lambda)$ that can be formally solved from which the Riemann tensor will fall out. This will be our task in this section.

First, we note that solving this differential equation amounts to finding a matrix $P^\mu{}_\rho(\lambda, \lambda_0)$ relating the the initial vector $V^\mu(\lambda_0)$ to its value later down the path:

$$V^\mu(\lambda) = P^\mu{}_\rho(\lambda, \lambda_0)V^\rho(\lambda_0) \tag{$*$}$$

We call $P^\mu{}_\rho(\lambda, \lambda_0)$ the parallel propagator, and will generally depend on the path $x^\mu(\lambda)$ taken if our manifold is not flat.

Next we define the object $A^\mu{}_\rho(\lambda) = -\Gamma^\mu_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}$, using which we rewrite the parallel transport equation as:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}V^\mu(\lambda) = A^\mu{}_\rho V^\rho(\lambda) \qquad \implies \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda}P^\mu{}_\rho(\lambda, \lambda_0) = A^\mu{}_\sigma(\lambda)P^\sigma{}_\rho(\lambda, \lambda_0) \tag{$\dagger$}$$

The second equation here follows from substituting in ($*$), and using that the resulting equation holds for any $V^\rho(\lambda_0)$ to eliminate it from both sides.

We now have a differential equation for the parallel propagator, which may seem more complicated than what we started out with. However, it can be solved in the following way. First, we integrate both sides from λ_0 to λ:

$$P^\mu{}_\rho(\lambda, \lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^{\lambda} \mathrm{d}\eta A^\mu{}_\sigma(\eta)P^\sigma{}_\rho(\eta, \lambda_0)$$

The Kronecker-delta follows from the initial condition at $\lambda = \lambda_0$: $P^\mu{}_\rho(\lambda_0, \lambda_0) = \delta^\mu_\nu$, since initially the vector starts unchanged at all.

This equation can be solved by iteration, taking the RHS of (†) and substituting it into itself repeatedly:

$$P^\mu{}_\rho(\lambda,\lambda_0) = \delta^\mu_\rho + \int_{\lambda_0}^{\lambda} \mathrm{d}\eta A^\mu{}_\rho(\eta) + \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta} \mathrm{d}\eta\mathrm{d}\eta' A^\mu{}_\sigma(\eta)A^\sigma{}_\rho(\eta') + \cdots \tag{‡}$$

The nth term in this series is an integral over an n-dimensional right-triangle:

$$\int_{\lambda_0}^{\lambda} \mathrm{d}\eta_1 A(\eta_1), \qquad \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_2} \mathrm{d}\eta_1\mathrm{d}\eta_2 A(\eta_2)A(\eta_1), \qquad \int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_3}\int_{\lambda_0}^{\eta_2} \mathrm{d}^3\eta A(\eta_3)A(\eta_2)A(\eta_1), \cdots$$

where we are now viewing the integrands as matrices, omitting the indices. It would be simpler if we could integrate over an n-cube. There are two obstructions to this which must be dealt with:

- There are $n!$ such right-triangles in an n-cube, so we would have to multiply by $1/n!$ to compensate.
- The order of matrix multiplication matters. At nth order the integrand is $A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)$, such that $\eta_n \geq \eta_{n-1} \geq \cdots \geq \eta_1$. To accomodate for this we define the path-ordering symbol $\mathcal{P}$. That is, $\mathcal{P}[A(\eta_n)A(\eta_{n-1})\cdots A(\eta_1)]$ is defined as the product of n matrices $A(\eta_i)$, ordered such that the largest η_i is on the left, decreasing towards the right.

Taking both of these into account, the nth order term of (‡) is given by:

$$\int_{\lambda_0}^{\lambda}\int_{\lambda_0}^{\eta_n}\cdots\int_{\lambda_0}^{\eta_2} \mathrm{d}^n\eta A(\eta_n)\cdots A(\eta_1) = \frac{1}{n!}\int_{[\lambda_0,\lambda]^n} \mathrm{d}^n\eta\mathcal{P}[A(\eta_n)\cdots A(\eta_1)]$$

Finally, we can write (‡) in matrix form as:

$$P(\lambda,\lambda_0) = \mathbb{1} + \sum_{n=1}^{\infty}\frac{1}{n!}\int_{[\lambda_0,\lambda]^n} \mathrm{d}^n\eta\mathcal{P}[A(\eta_n)\cdots A(\eta_1)] = \mathcal{P}\mathrm{exp}\left(\int_{\lambda_0}^{\lambda} A(\eta)\mathrm{d}\eta\right)$$

or more explicitly, substituting back $A^\mu{}_\rho(\lambda) = -\Gamma^\mu_{\sigma\rho}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\lambda}$:

$$P^\mu{}_\nu(\lambda,\lambda_0) = \mathcal{P}\mathrm{exp}\left(-\int_{\lambda_0}^{\lambda}\Gamma^\mu_{\sigma\nu}\frac{\mathrm{d}x^\sigma}{\mathrm{d}\eta}\mathrm{d}\eta\right)$$

This formula is known as a Dyson series. It also arises from solving the Schrödinger's equation for a general time-dependent Hamiltonian $H(t)$, there the analog of the parallel-propagator is the time evolution operator $U(t,t_0)$. A special case of this formula arises if the path is a loop, i.e. the starting and ending points are the same. This transports the initial vector back to the same tangent space, and is thus just a Lorentz transformation on the original tangent space. The transformation is known as the holonomy of a loop.

To derive the Riemann curvature tensor, we can expand the Dyson series for a path γ starting at x, forming an infinitesimal square with sides a^μ and b^μ. We have:

$$\begin{aligned}(P-\mathbb{1}) &= \int_\gamma \mathrm{d}\eta A(\eta) + \frac{1}{2}\mathcal{P}\left(\int_\gamma \mathrm{d}\eta A(\eta)\int_\gamma \mathrm{d}\eta' A(\eta')\right) \\ &= -\int_\gamma \Gamma_\lambda \mathrm{d}x^\lambda + \frac{1}{2}\mathcal{P}\left(\int_\gamma \Gamma_\lambda\mathrm{d}x^\lambda\int_\gamma\Gamma_\lambda\mathrm{d}x^\lambda\right)\end{aligned}$$

Where we view $\Gamma_\lambda = \Gamma^{\cdot}{}_{\lambda\cdot}$ as a matrix with 2 indices implicit.

First we expand the first term to second order. On each infinitesimal segment of the path, we can approximate the integral by Γ_λ(midpoint) in the direction of the path, multiplied by the length of the path. Therefore:

$$
\begin{aligned}
-\int_\gamma \Gamma_\lambda \mathrm{d}x^\lambda &= -\left[(a\cdot\Gamma)(x+\tfrac{a}{2}) + (b\cdot\Gamma)(x+a+\tfrac{b}{2}) + (-a\cdot\Gamma)(x+\tfrac{a}{2}+b) + (-b\cdot\Gamma)(x+\tfrac{b}{2})\right] \\
&= -\big[a\cdot(\Gamma(x) + (\tfrac{a}{2}\cdot\partial)\Gamma(x)) + b\cdot(\Gamma(x) + (a\cdot\partial)\Gamma(x) + (\tfrac{b}{2}\cdot\partial)\Gamma(x)) \\
&\qquad - a\cdot(\Gamma(x) + (\tfrac{a}{2}\cdot\partial)\Gamma(x) + (b\cdot\partial)\Gamma(x)) - b\cdot(\Gamma(x) + (\tfrac{b}{2}\cdot\partial)\Gamma(x))\big] \\
&= -(a\cdot\partial)(b\cdot\Gamma) + (b\cdot\partial)(a\cdot\Gamma) = -(\partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu)a^\mu b^\nu
\end{aligned}
$$

Now we expand the second term to second order. We define the points 1, 2, 3, 4 as the midpoints of these segments, $1 = x+\frac{a}{2}$, $2 = x+a+\frac{b}{2}$, $3 = x+b+\frac{a}{2}$, $4 = x+\frac{b}{2}$.

$$
\begin{aligned}
&\frac{1}{2}\mathcal{P}\left(\int_\gamma \Gamma_\lambda \mathrm{d}x^\lambda \int_\gamma \Gamma_\lambda \mathrm{d}x^\lambda\right) \\
&= \frac{1}{2}\mathcal{P}\left[(a\cdot\Gamma(1) + b\cdot\Gamma(2) - a\cdot\Gamma(3) - b\cdot\Gamma(4))(a\cdot\Gamma(1) + b\cdot\Gamma(2) - a\cdot\Gamma(3) - b\cdot\Gamma(4))\right] \\
&= \tfrac{1}{2}\big[a\cdot\Gamma(1)\ a\cdot\Gamma(1) + b\cdot\Gamma(2)\ a\cdot\Gamma(1) - a\cdot\Gamma(3)\ a\cdot\Gamma(1) - b\cdot\Gamma(4)\ a\cdot\Gamma(1) \\
&\qquad + b\cdot\Gamma(2)\ a\cdot\Gamma(1) + b\cdot\Gamma(2)\ b\cdot\Gamma(2) - a\cdot\Gamma(3)\ b\cdot\Gamma(2) - b\cdot\Gamma(4)\ b\cdot\Gamma(2) \\
&\qquad - a\cdot\Gamma(3)\ a\cdot\Gamma(1) - a\cdot\Gamma(3)\ b\cdot\Gamma(2) + a\cdot\Gamma(3)\ a\cdot\Gamma(3) + b\cdot\Gamma(4)\ a\cdot\Gamma(3) \\
&\qquad - b\cdot\Gamma(4)\ b\cdot\Gamma(1) - b\cdot\Gamma(4)\ b\cdot\Gamma(2) + b\cdot\Gamma(4)\ b\cdot\Gamma(3) + b\cdot\Gamma(4)\ b\cdot\Gamma(4)\big] \\
&= \frac{1}{2}[2b\cdot\Gamma(x)a\cdot\Gamma(x) - 2a\cdot\Gamma(x)b\cdot\Gamma(x)] = -(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu)a^\mu b^\nu
\end{aligned}
$$

In the last line, we use that points 1, 2, 3, 4 are all equal to x to zeroth order, so derivative contributions to the above equation will be third order, and thus safely ignored.

Putting both pieces together, we have:

$$
(P - \mathbb{1}) = -(\partial_\mu\Gamma_\nu - \partial_\nu\Gamma_\mu)a^\mu b^\nu - (\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu)a^\mu b^\nu
$$

Restoring indices and acting this on a vector V^σ, we have:

$$
\begin{aligned}
\Delta V^\rho = V(\lambda_f)^\rho - V(\lambda_0)^\rho = (P-\mathbb{1})^\rho{}_\sigma V^\sigma &= -(\partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\sigma} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\sigma})a^\mu b^\nu \\
&= -R^\rho{}_{\sigma\mu\nu}a^\mu b^\nu V^\sigma
\end{aligned}
$$

This is precisely our desired result (with a minus sign in this convention, which can be reversed by flipping a and b): the change of a vector V^ρ upon transporting in an infinitesimal loop with sides a, b is given by the Riemann curvature tensor.

Properties of the Riemann Tensor
The Riemann curvature enjoys the following properties:

- Symmetries: $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \quad R_{\rho\sigma\mu\nu} = R_{\rho\sigma[\nu\mu]}, \quad R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \quad R_{\rho[\sigma\mu\nu]} = 0$
- Bianchi identity: $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$

We can also define the (symmetric) Ricci tensor $R_{\mu\nu} := R^{\lambda}{}_{\mu\lambda\nu}$ and the Ricci scalar $R := R^{\mu}{}_{\mu}$.

With these symmetries, let's show that the Riemann tensor has 20 independent components. Here we'll work in d dimensions, and at the end of the day take $d = 4$. We do this in two steps:

1. First, we use that $R_{\rho\sigma\mu\nu}$ is antisymmetric in the first 2 indices, antisymmetric in the last 2 indices, and symmetric under the interchange of these pairs. We can therefore think of it as a symmetric matrix $R_{[\mu\nu][\mu\nu]}$, where the pairs $\rho\sigma$ and $\mu\nu$ are thought of as individual indices.
   A $m \times m$ symmetric matrix has $\frac{1}{2}m(m+1)$ independent components, while an antisymmetric $n \times n$ antisymmetric matrix has $\frac{1}{2}m(m-1)$. Therefore, using these symmetries reduces the number of independent components to:
$$\frac{1}{2}\left[\frac{1}{2}d(d-1)\right]\left[\frac{1}{2}d(d-1)+1\right] = \frac{1}{8}(d^4 - 2d^3 + 3d^2 - 2d)$$

2. It remains to use the 4th symmetry $R_{\rho[\sigma\mu\nu]} = 0$. In addition to the first three symmetries, it can be shown that including this constraint is equivalent to including the condition $R_{[\rho\sigma\mu\nu]} = 0$. That is, in the presence of the first 3 symmetries:
$$R_{\rho[\sigma\mu\nu]} = 0 \Longleftrightarrow R_{[\rho\sigma\mu\nu]} = 0$$
   A totally antisymmetric 4-tensor has $d(d-1)(d-2)(d-3)/4!$ terms, all of which vanish. Therefore, in total we are left with
$$\frac{1}{8}(d^4 - 2d^3 + 3d^2 - 2d) - \frac{1}{24}d(d-1)(d-2)(d-3) = \frac{1}{12}d^2(d^2-1)$$
   independent components of the Riemann curvature tensor. Taking $n = 1, 2, 3, 4$ gives respectively 0, 1, 6, 20, the last of which is our desired result.

What does the number of independent components mean in low dimensions?

- $d = 1$. In 1 dimension the Riemann tensor has zero independent components at all, meaning that there is no notion of curvature. This makes sense; the covariant derivative has only 1 component, so $[\nabla_\mu, \nabla_\nu]V^\rho = [\nabla_1, \nabla_1]V^\rho = 0$.
- $d = 2$. In 2 dimensions there is 1 component (reduced from a naïve $2^4 = 16$). Therefore, knowing the Ricci scalar is as good as knowing the entire Riemann curvature tensor. More explicitly, it can be shown by using symmetries that
$$R_{\rho\sigma\mu\nu} = \frac{R}{2}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

Example: compute the Riemann curvature tensor for the hyperbolic plane $\mathbb{H}_2$.
Recall that the metric and Christoffel symbols are given by:
$$g_{\mu\nu} = \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma^x_{\mu\nu} = \frac{1}{y}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^y_{\mu\nu} = \frac{1}{y}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We compute:

$$
\begin{aligned}
R = R_{\mu\nu} g^{\mu\nu} = g^{\mu\nu} g^{\rho\sigma} R_{\rho\mu\sigma\nu} &= g^{xx} g^{xx} R_{xxxx} + g^{xy} g^{xy} R_{xyxy} + g^{yx} g^{yx} R_{yxyx} + g^{yy} g^{yy} R_{yyyy} \\
&= 2 g^{xx} g^{yy} R_{xyxy}
\end{aligned}
$$

In the third equality we use that the metric is diagonal. In the last, we use the symmetries of the Riemann tensor, namely $R_{xxxx} = R_{yyyy} = 0$ (antisymmetry in indices 12), and $R_{xyxy} = R_{yxyx}$ (antisymmetry twice, in indices 12, and 34). It remains to compute R_{xyxy}:

$$
\begin{aligned}
R_{\lambda\sigma\mu\nu} &= g_{\lambda\rho} R^{\rho}{}_{\sigma\mu\nu} = g_{\lambda\rho}(\partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\alpha}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\rho}_{\nu\alpha}\Gamma^{\alpha}_{\mu\sigma}) \\
R_{xyxy} &= g_{x\rho}(\partial_x \Gamma^{\rho}_{yy} - \partial_y \Gamma^{\rho}_{xy} + \Gamma^{\rho}_{x\alpha}\Gamma^{\alpha}_{yy} - \Gamma^{\rho}_{y\alpha}\Gamma^{\alpha}_{xy}) \\
&= g_{xx}(\partial_x \Gamma^{x}_{yy} - \partial_y \Gamma^{x}_{xy} + \Gamma^{x}_{x\alpha}\Gamma^{\alpha}_{yy} - \Gamma^{x}_{y\alpha}\Gamma^{\alpha}_{xy}) \\
&= g_{xx}(0 - \partial_y \Gamma^{x}_{xy} + \Gamma^{x}_{xy}\Gamma^{y}_{yy} - \Gamma^{x}_{yx}\Gamma^{x}_{xy}) \\
&= \frac{1}{y^2}\left[-\partial_y \frac{-1}{y} + \frac{-1}{y}\frac{-1}{y} - \frac{-1}{y}\frac{-1}{y}\right] = \frac{1}{y^2}\frac{-1}{y^2} = -\frac{1}{y^4}
\end{aligned}
$$

In the third line we use that the metric is diagonal, i.e. $g_{xy} = 0$. In the next line we use that $\Gamma^{x}_{xx} = \Gamma^{x}_{yy} = 0$. Finally, we have:

$$
\begin{aligned}
R &= 2 g^{xx} g^{yy} R_{xyxy} = 2 y^2 y^2 \frac{-1}{y^4} = -2 \\
R_{\rho\sigma\mu\nu} &= \frac{R}{2}(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}) = -(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})
\end{aligned}
$$

We see that the Ricci scalar is constant and negative everywhere across our manifold, i.e. $\mathbb{H}_2$ is a uniformly hyperbolic space.