8.325 FINAL PROJECT

Introduction to Finite Temperature QFT

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1 Introduction

In the active research on quark gluon plasmas from heavy ion collisions and their respective implications on early universe cosmology, quantum fields and their interactions with a thermal background have to be studied in great detail. A few notable examples include studies on hard probes in heavy ion collisions and the extrapolation of the QCD phase diagram. To study these systems, formulations of finite temperature QFT are developed, well-known examples include the Matsubara formalism, also called the imaginary time formalism, and the Schwinger-Keldysh formalism, also called the real time formalism.

This paper briefly summarizes the perturbative aspects of the imaginary time formalism of finite temperature QFT as presented in [1–5]. In particular, the convention of [1] is adapted and used throughout. The paper aims to develop a basic understanding of finite temperature QFT, and serves to bridge familiar concepts and results from statistical mechanics with their field theory counterparts.

2 Imaginary Time Formalism

2.1 KMS condition

Recall the usual partition function and density operator of a state in thermal equilibrium with temperature β^{-1}

$$Z = \mathrm{Tr}\big(e^{-\beta H}\big)\ ,\ \rho = \frac{1}{Z}e^{-\beta H} \tag{2.1}$$

An operator has the expectation value

$$\langle A(t)\rangle_\beta = \frac{1}{Z}\,\mathrm{Tr}\big(Ae^{-\beta H}\big) \tag{2.2}$$

The above intuition gives rise to the Kubo–Martin–Schwinger (KMS) condition. For operators $A(t)$ and $B(t)$ acting on a system in thermal equilibrium, their correlation function satisfy

$$\begin{aligned}\langle A(t)B(t')\rangle_\beta &= \frac{1}{Z}\,\mathrm{Tr}\big(A(t)B(t')e^{-\beta H}\big)\\ &= \frac{1}{Z}\,\mathrm{Tr}\big(A(t)e^{\beta H}e^{-\beta H}B(t')e^{-\beta H}\big)\\ &= \frac{1}{Z}\,\mathrm{Tr}\big(B(t')e^{-\beta H}A(t)e^{\beta H}e^{-\beta H}\big) = \langle B(t')A(t+i\beta)\rangle_\beta\end{aligned} \tag{2.3}$$

where we inserted an identity operator, used the cyclic property of the trace operation, and analytically continued the operator time evolution in Heisenberg picture.

2.2 Boundary conditions

Inspired by the KMS condition, in a finite temperature field theory, the partition function of a thermal state is given by Wick rotating to imaginary time $t = i\tau$ and imposing a periodic boundary condition $\tau \sim \tau + \beta$

$$Z = \int \mathcal{D}\Phi \exp\bigg(\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{d-1}x\,\mathcal{L}_E\bigg) \tag{2.4}$$

where Φ satisfies the periodic boundary condition $\Phi(0,\mathbf{x}) = \Phi(\beta,\mathbf{x})$ if it is bosonic and the anti-periodic condition $\Phi(0,\mathbf{x}) = -\Phi(\beta,\mathbf{x})$ if it is fermionic. Here we adapt the convention in which the Euclidean Lagrangian and the Minkowski Lagrangian are related by $\mathcal{L}_E = \mathcal{L}_M(t \to i\tau)$. However, Faddeev-Popov ghosts, which appear from quantizing non-Abelian gauge fields, obey periodic boundary condition despite being anti-commuting Grassmann variables [5].

The (anti-)periodicity is not a Lorentz invariant condition, as a frame must be chosen in order to define the temperature, still, translational symmetry in time and space, as well as rotational symmetry in space are preserved after the periodic identification. These symmetries will serve as useful constraints on the propagator.

From the periodicity conditions, fields can be expanded in the following form

$$\phi(\tau, \mathbf{x}) = \sum_n e^{i\omega_n^+ \tau} \phi_n(\mathbf{x}) = \sqrt{\beta V} \sum_n \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} e^{i\omega_n^+ \tau} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{n,\mathbf{k}} \tag{2.5}$$

$$= \sqrt{\frac{\beta}{V}} \sum_{n,\mathbf{k}} e^{i\omega_n^+ \tau} e^{i\mathbf{k}\cdot\mathbf{x}} \phi_{n,\mathbf{k}} \tag{2.6}$$

$$\psi^\alpha(\tau, \mathbf{x}) = \sum_n e^{i\omega_n^- \tau} \psi_n(\mathbf{x}) = \sqrt{V} \sum_n \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} e^{i\omega_n^- \tau} e^{i\mathbf{k}\cdot\mathbf{x}} \psi^\alpha_{n,\mathbf{k}} \tag{2.7}$$

$$= \frac{1}{\sqrt{V}} \sum_{n,\mathbf{k}} e^{i\omega_n^- \tau} e^{i\mathbf{k}\cdot\mathbf{x}} \psi^\alpha_{n,\mathbf{k}} \tag{2.8}$$

with

$$\omega_n^+ = \frac{2n\pi}{\beta} \tag{2.9}$$

$$\omega_n^- = \frac{(2n+1)\pi}{\beta} \tag{2.10}$$

called the Matsubara frequencies. In the above mode expansions, the normalization convention for the Fourier amplitudes $\phi_{n,\mathbf{k}}$ and $\psi^\alpha_{n,\mathbf{k}}$ is chosen such that these amplitudes have mass dimension 0 and are therefore independent of β and V, the Fourier amplitudes for gauge bosons and higher spin particles should also be normalized under the same convention. On top of that, the following sum and integrals are used interchangeably in this paper

$$V \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \longleftrightarrow \sum_{\mathbf{k}} \tag{2.11}$$

3 Real Scalar Theory

In this section ω_n denotes the bosonic Matsubara frequencies ω_n^+.

3.1 Partition function in the free theory

Consider the free real scalar field with action

$$S_E = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{d-1}x\, \mathcal{L}_E = -\frac{1}{2} \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{d-1}x \left((\partial_\tau \phi)^2 + (\partial_i \phi)^2 + m^2 \phi^2 \right) \tag{3.1}$$

$$= -\frac{1}{2}\beta^2 V \sum_n \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} |\phi_{n,\mathbf{k}}|^2 \left(\omega_n^2 + |\mathbf{k}|^2 + m^2 \right) \tag{3.2}$$

$$= -\frac{1}{2}\beta^2 \sum_{n,\mathbf{k}} |\phi_{n,\mathbf{k}}|^2 \left(\omega_n^2 + |\mathbf{k}|^2 + m^2 \right) \tag{3.3}$$

The partition function can be evaluated [1]

$$Z = \int \mathcal{D}\phi \, e^{S_E} = \mathcal{N} \prod_{n,\mathbf{k}} \int (\mathrm{d}\phi_{n,\mathbf{k}}) \exp\left(-\frac{1}{2}\beta^2 |\phi_{n,\mathbf{k}}|^2 \left(\omega_n^2 + |\mathbf{k}|^2 + m^2 \right) \right) \tag{3.4}$$

$$= \mathcal{N} \prod_{n,\mathbf{k}} \left(\beta^2 \left(\omega_n^2 + |\mathbf{k}|^2 + m^2 \right) \right)^{-1/2} \tag{3.5}$$

$$\ln Z = -\frac{1}{2} \sum_{n,\mathbf{k}} \ln \left(\beta^2 \left(\omega_n^2 + |\mathbf{k}|^2 + m^2 \right) \right) + \ln \mathcal{N} \tag{3.6}$$

$$= V \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \left[-\frac{1}{2}\beta\omega_{\mathbf{k}} - \ln\left(1 - e^{-\beta\omega_{\mathbf{k}}}\right) \right] + \text{constant term} \tag{3.7}$$

where $\omega_{\mathbf{k}} = \sqrt{m^2 + |\mathbf{k}|^2}$, and $\mathcal{N}$ represent generic normalization constants independent of V and β, owing to the choice that $\phi_{n,\mathbf{k}}$ is dimensionless. (3.7) reproduces the formula for Helmholtz free energy of non-interacting bosons found in any statistical mechanics textbook

$$F = -T \ln Z = V \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \left[\frac{1}{2}\omega_{\mathbf{k}} + T \ln\left(1 - e^{-\beta\omega_{\mathbf{k}}}\right) \right] \tag{3.8}$$

The first term can be interpreted as the zero point energy of the bosonic modes, and the second term is the usual free energy distribution obtained using the Bose-Einstein distribution.

When evaluating the internal energy and pressure, the zero point contributions should be subtracted, as only energy level difference and pressure difference can be measured in a theory in which gravity is absent. The internal energy and the pressure thus read

$$E = -\frac{\partial \ln Z}{\partial \beta} - E_0 = V \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{\omega}{e^{\beta\omega} - 1} \tag{3.9}$$

$$P = -\frac{\partial F}{\partial V} - P_0 = T \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \ln\left(1 - e^{-\beta\omega_{\mathbf{k}}}\right) \tag{3.10}$$

In particular, take $d = 4$, in the high temperature, high energy limit, the pressure reduces to a familiar result

$$P = T \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{e^{-n\beta\omega_{\mathbf{k}}}}{n} \tag{3.11}$$

$$\sim \frac{T}{2\pi^2} \cdot T^3 \int_0^{\infty} \mathrm{d}(\beta k)\, (\beta k)^2 \sum_{n=1}^{\infty} \frac{e^{-n(\beta k)}}{n} \tag{3.12}$$

$$= \frac{T^4}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\mathrm{d}^2}{\mathrm{d}n^2} \int_0^{\infty} \mathrm{d}x\, e^{-nx} = \frac{T^4}{2\pi^2} \sum_{n=1}^{\infty} \frac{2}{n^4} = \frac{\pi^2}{90} T^4 \tag{3.13}$$

[1] For formulae applied to evaluate the discrete sums in n, see appendix.

3.2 Propagator in the free theory

Define the finite temperature propagator of a real scalar theory as

$$D(\tau_1, \mathbf{x}_1; \tau_2, \mathbf{x_2}) = \langle \phi(\tau_1, \mathbf{x}_1)\phi(\tau_2, \mathbf{x}_2)\rangle = D(\tau_1 - \tau_2, |\mathbf{x}_1 - \mathbf{x}_1|) \tag{3.14}$$

By translational and rotational symmetry, the propagator only depends on the difference in imaginary time and the distance. The momentum space propagator can be found through applying mode expansion and imposing translational symmetry conditions

$$D(\omega_n, \mathbf{k}) = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{d-1}x\, e^{-i(\omega_n \tau + \mathbf{k}\cdot\mathbf{x})} D(\tau, \mathbf{x}) = \beta^2 \langle \phi_{n,\mathbf{k}}\, \phi_{-n,-\mathbf{k}}\rangle \tag{3.15}$$

In the free theory, given the Gaussian integral

$$\frac{\int \mathrm{d}x\, x^2 e^{-\frac{1}{2}ax^2}}{\int \mathrm{d}x\, e^{-\frac{1}{2}ax^2}} = \frac{1}{\int \mathrm{d}x\, e^{-\frac{1}{2}ax^2}} \left(-2\frac{\partial}{\partial a} \int \mathrm{d}x\, e^{-\frac{1}{2}ax^2}\right) = \frac{1}{a} \tag{3.16}$$

the momentum space propagator can be identified as

$$D_0(\omega_n, \mathbf{k}) = \beta^2 \langle \phi_{n,\mathbf{k}}\, \phi_{-n,-\mathbf{k}}\rangle = \frac{1}{\omega_n^2 + |\mathbf{k}|^2 + m^2} \tag{3.17}$$

Wick's theorem in the free theory can also be proven using the same constraints as those that produce (3.15), and (3.16) is also useful when constructing the propagator in an interacting theory.

3.3 $\lambda\phi^4$ theory and Feynman rules

For an interacting theory that introduces an extra term S_I in the action, the partition function can be separated into two components

$$Z = \int \mathcal{D}\phi\, e^{S_0 + S_I} = \int \mathcal{D}\phi\, e^{S_0} \left(1 + S_I + \frac{1}{2}S_I^2 ...\right) = Z_0 Z_I \tag{3.18}$$

$$\ln Z = \ln Z_0 + \ln Z_I \tag{3.19}$$

where

$$Z_I = \left\langle e^{S_I} \right\rangle_0 \tag{3.20}$$

Diagrammatically speaking, $\ln Z_I$ is the sum of connected vacuum bubble diagrams with appropriate combinatorial factors.

To produce the finite temperature Feynman rules is merely a matter of replacing a few components of the usual Feynman rules in Euclidean field theory. For the canonical example, consider the $\lambda\phi^4$ theory with

$$\mathcal{L}_I = -\lambda\phi^4 \tag{3.21}$$

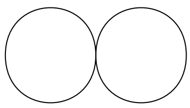
The finite temperature Feynman rules are given by the following steps

- Draw out connected bubble diagrams up to order of interest.
- Calculate the combinatoric factor, that is, how many different ways the legs can be contracted.
- Give each $\phi\phi\phi\phi$ vertex a factor of $-\lambda$.
- Give each scalar line a propagator

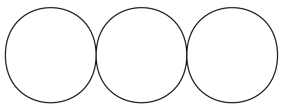
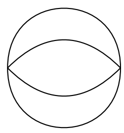
$$\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} D_0(\omega_n, \mathbf{k}) \tag{3.22}$$

- Impose momentum conservation at each vertex. By virtue of (3.15), when contracting the fields, incoming and outgoing lines should be paired up, such that the lines in each pair carry the opposite momentum. That is, if the momenta of lines on a four point vertex are labeled by $\mathbf{k}_{i=1,2,3,4}$, then there are three allowed pairings: $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 = 0$, $\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4 = 0$, and $\mathbf{k}_1 + \mathbf{k}_4 = \mathbf{k}_2 + \mathbf{k}_3 = 0$. Overall, after integrating all the delta functions, there should always be one factor of βV left over.

Following the above prescription, the first order interacting correction to the partition function is given by

$$\ln Z_1 = 3$$

$$= -3\lambda\beta V \left(\frac{1}{\beta}\sum_n \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} D_0(\omega_n, \mathbf{k})\right)^2 \tag{3.23}$$

And the second order correction is given by

$$\ln Z_2 = 36$$

$$+\, 12$$

$$\tag{3.24}$$

3.4 Self energy

The scalar self energy is defined via

$$D(\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + |\mathbf{k}|^2 + m^2 + \Pi(\omega_n, \mathbf{k})} = \frac{D_0}{1 + \Pi D_0} \tag{3.25}$$

By expanding the action into the quadratic term and the interaction term

$$e^S = e^{S_0} e^{S_I} = e^{-\frac{1}{2}\beta^2 \sum_{n,\mathbf{k}} \phi_{-n,-\mathbf{k}} D_0^{-1}(\omega_n,\mathbf{k}) \phi_{n,\mathbf{k}}} e^{S_I} \tag{3.26}$$

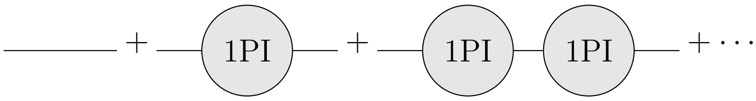
the general form of the propagator is manifest

$$D = \frac{-2}{Z}\frac{\delta Z}{\delta D_0^{-1}} = 2D_0^2 \frac{\delta \ln Z}{\delta D_0} = \frac{-2}{Z}\frac{\delta Z}{\delta D_0^{-1}} = D_0 + 2D_0^2 \frac{\delta \ln Z_I}{\delta D_0} \tag{3.27}$$

To calculate D perturbatively, expand (3.25)

$$D = D_0 \left(1 + (-\Pi D_0) + (-\Pi D_0)^2 ...\right) \tag{3.28}$$

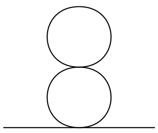
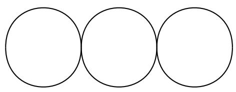
diagrammatically this is the familiar expansion in number of 1 PI components

$D =$  (3.29)

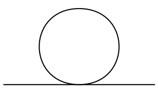
from which the form of Π can be determined

$$\Pi = -2 \left(\frac{\delta \ln Z_I}{\delta D_0}\right)_{1\ \mathrm{PI}} \tag{3.30}$$

To operator with (3.30), one starts with the vacuum bubble diagrams, and removes one edge in a way that leaves the remaining diagram 1 PI. The combinatorial factor of a diagram is determined by multiplying the factor of the bubble diagram it started with, the number of ways to remove one edge that gives the diagram of interest, and the final extra -2 from the formula. For instance, there is a two loop contribution

$\Pi_2 \supset -144$  $= -2\frac{\delta}{\delta D_0}\Big(36$  $\Big)_{1\ \mathrm{PI}}$ (3.31)

The lowest order contribution of Π is

$\Pi_1 = -12$  $= 12\frac{\lambda}{\beta}\sum_n \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega_n^2 + \omega_{\mathbf{k}}^2}$ (3.32)

$$= 12\lambda \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega_{\mathbf{k}}} \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1} + \Pi_1^{\mathrm{vac}} \tag{3.33}$$

where Π_1^{vac} is an infinitely large term independent of β. It should be renormalized by introducing a counterterm δm^2 in the Lagrangian. In $d = 4$, expanding the renormalized self energy in the $\beta = T^{-1} \to 0$ and $m \to 0$ limit yields

$$\Pi_1^{\mathrm{ren}} \sim 12\lambda \int \mathrm{d}k\, \frac{4\pi^2 k^2}{8\pi^3} \frac{1}{k} \frac{1}{e^{\beta k} - 1} = \frac{\lambda}{\beta^2} \frac{6}{\pi^2} \int_0^\infty \mathrm{d}(\beta k) \frac{(\beta k)}{e^{(\beta k)} - 1} = \lambda T^2 \tag{3.34}$$

which suggests that the scalar acquires a finite-temperature correction on its pole mass.

4 Further Comments

This section briefly goes through and comments on content beyond the real scalar theory that is only stated but not derived in the presentation.

4.1 Dirac fermions

Now let ω_n denote the fermionic Matsubara frequencies. Analogous to (3.3), the action of a free Dirac fermion can be calculated by mode expansion

$$S = i\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{d-1}x\, \bar{\psi}(\gamma^0\partial_\tau + i\gamma^i\partial_i - m)\psi \tag{4.1}$$

$$S_E = -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^{d-1}x\, \bar{\psi}(\gamma^0\partial_\tau + i\gamma^i\partial_i - m)\psi \tag{4.2}$$

$$= -\beta \sum_{n,\mathbf{k}} \bar{\psi}_{n,\mathbf{k}}(i\omega_n\gamma^0 - \gamma^i k_i - m)\psi_{n,\mathbf{k}} \tag{4.3}$$

The partition function is thus given by

$$\ln Z = \sum_{n,k} \ln\det\big(-\beta(i\omega_n\gamma^0 - \gamma^i k_i - m)\big) \tag{4.4}$$

$$= \sum_{n,k} \ln\det\left(-\beta\begin{pmatrix}(i\omega_n - m) & \boldsymbol{\sigma}\cdot\mathbf{k} \\ -\boldsymbol{\sigma}\cdot\mathbf{k} & (-i\omega_n - m)\end{pmatrix}\right) \tag{4.5}$$

$$= 2\sum_{n,k} \ln\big(\beta^2(\omega_n^2 + m^2 + |\mathbf{k}|^2)\big) \tag{4.6}$$

$$= 4V\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left(\frac{1}{2}\beta\omega_{\mathbf{k}} + \ln\big(1 + e^{-\beta\omega_{\mathbf{k}}}\big)\right) \tag{4.7}$$

In which we evaluate the determinant using the Dirac representation.

If a finite chemical potential μ is included in the calculation, one would yield [1]

$$\ln Z = 2V\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\left(\frac{1}{2}\beta\omega_{\mathbf{k}} + \ln\big(1 + e^{-\beta(\omega_{\mathbf{k}}-\mu)}\big) + \frac{1}{2}\beta\omega_{\mathbf{k}} + \ln\big(1 + e^{-\beta(\omega_{\mathbf{k}}+\mu)}\big)\right) \tag{4.8}$$

There is a factor of two resulting from the two spin degrees of freedom, this partition function takes both the particle with chemical potential $+\mu$ and antiparticle with chemical potential $-\mu$ into account.

4.2 Gauge bosons and ghosts

We then proceed to calculate the partition function of Gauge bosons under finite temperature with the Faddeev-Popov method [4]. The free part of the Lagrangian after the introduction of ghosts consists of two parts

$$\mathcal{L}_{\text{gauge, free}} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 \tag{4.9}$$

$$\mathcal{L}_{\text{ghost, free}} = -\bar{c}^a\delta^{ab}\partial^2 c^b \tag{4.10}$$

Take photons for example, there is only 1 Lie algebra index, the vector boson thus gives 4 degrees of freedom

$$\ln Z \supset 4 \cdot -\frac{1}{2} \sum_{n,\mathbf{k}} \ln\left(\beta^2\left(\omega_n^2 + \omega_{\mathbf{k}}^2\right)\right) \tag{4.11}$$

and the ghosts cancel two degrees of freedom

$$\ln Z \supset (-2) \cdot -\frac{1}{2} \sum_{n,\mathbf{k}} \ln\left(\beta^2\left(\omega_n^2 + \omega_{\mathbf{k}}^2\right)\right) \tag{4.12}$$

the resulting partition function is left with two degrees of freedom

$$\ln Z = 2 \cdot -\frac{1}{2} \sum_{n,\mathbf{k}} \ln\left(\beta^2\left(\omega_n^2 + \omega_{\mathbf{k}}^2\right)\right) = 2V \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \left[-\frac{1}{2}\beta\omega_{\mathbf{k}} - \ln\left(1 - e^{-\beta\omega_{\mathbf{k}}}\right)\right] \tag{4.13}$$

The same result can be obtained by observing that physical photons have two helicity degrees of freedom. In particular, the radiation pressure and energy density of photons can be calculated rather simply

$$P_\gamma = 2P_\phi = \frac{\pi^2}{45}T^4 \Rightarrow u_\gamma = 3P_\gamma = \frac{\pi^2}{15}T^4 \tag{4.14}$$

this is the familiar Stefan–Boltzmann law of black body radiation.

Additionally, in QED under temperature T, photon modes gain a thermal mass

$$m_{\text{th}}^2 \propto e^2 T^2 \tag{4.15}$$

as the pole of the propagator is shifted by self interactions. One important implication of the photon thermal mass is Debye screening of charges under finite temperature.

5 Conclusion

In this paper it is shown that through the imaginary time formalism of finite temperature QFT, many familiar results can be obtained and generalized systematically. In particular, formulae in classical and quantum statistical mechanics are reproduced by modifying the usual Feynman rules. The periodicity conditions convert integrals in the time direction into discrete sums that diverge, which requires treatment by renormalization.

Acknowledgments

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A Useful Math Formulae

In general, one can apply residue theorem to evaluate sums of functions of the Matsubara frequencies. In this paper, we do not explain how these sums are obtained, as the literature provided all contain detailed explanations. Though we do make heavy use of the following identities

$$\ln\left(b^2+a^2\right)=\int_1^a \frac{2x}{b^2+x^2}\,\mathrm{d}x-\ln\left(1+b^2\right) \tag{A.1}$$

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^2+x^2}=\frac{\pi\coth(\pi x)}{x}\Rightarrow\sum_{n\in\mathbb{Z}}\frac{1}{(2\pi n)^2+x^2}=\frac{\coth(x/2)}{2x} \tag{A.2}$$

$$\sum_{n\in\mathbb{Z}}\frac{1}{(2n+1)^2+x^2}=\frac{\pi\tanh(\pi x/2)}{2x}\Rightarrow\sum_{n\in\mathbb{Z}}\frac{1}{(2n+1)^2\pi^2+x^2}=\frac{\tanh(x/2)}{2x} \tag{A.3}$$

(A.1) is used to isolate the temperature dependent part of the partition function from the divergent ground state degeneracy, (A.2) and (A.3) are used to evaluate propagators in the Feynman diagrams. For example, the detailed steps progressing from (3.6) to (3.7) are given as follows

$$-\frac{1}{2}\sum_{n,\mathbf{k}}\ln\left(\beta^2\left(\omega_n^2+|\mathbf{k}|^2+m^2\right)\right)=-\frac{1}{2}\sum_{n,\mathbf{k}}\ln\left((2\pi n)^2+(\beta\omega_{\mathbf{k}})^2\right) \tag{A.4}$$

$$=-\frac{1}{2}\sum_{n,\mathbf{k}}\left(\int_1^{\beta\omega_{\mathbf{k}}}\frac{2x}{(2\pi n)^2+x^2}\,\mathrm{d}x-\ln\left(1+(2\pi n)^2\right)\right)=-\frac{1}{2}\sum_{\mathbf{k}}\int_1^{\beta\omega_{\mathbf{k}}}\coth(x/2)\,\mathrm{d}x+\mathrm{C} \tag{A.5}$$

$$=-\frac{1}{2}\sum_{\mathbf{k}}\int_1^{\beta\omega_{\mathbf{k}}}1+\frac{2}{e^x-1}\,\mathrm{d}x+\mathrm{C}=\sum_{\mathbf{k}}\left(-\frac{1}{2}\beta\omega_{\mathbf{k}}-\ln\left(1-e^{-\beta\omega_{\mathbf{k}}}\right)\right)+\mathrm{C} \tag{A.6}$$

where C denote constants independent on T and V.

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