Classical Mechanics Notes

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Contents

1 Newtonian Formalism

No one uses this, it's not high school.

2 Lagrangian Formalism

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

3 Hamiltonian Formalism

3.1 Hamilton's Equations

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}\ ,\ \ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}\ ,\ \ \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

3.2 Hamilton-Jacobi Equation

$$S \equiv \int \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)\,\mathrm{d}t = \int \sum_i p_i \dot{q}_i - \mathcal{H}(\mathbf{p}, \mathbf{q}, t)\,\mathrm{d}t = \int \sum_i p_i\,\mathrm{d}q_i - \mathcal{H}\,\mathrm{d}t$$
$$\Rightarrow \frac{\partial S}{\partial \mathbf{q}} = \mathbf{p}\ ,\ \ \frac{\partial S}{\partial t} = -\mathcal{H}\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right)$$

3.3 Canonical Transformation

Assume a transformation of a following form:

$$\mathrm{d}S = \sum_i p_i\,\mathrm{d}q_i - H\,\mathrm{d}t = \sum_i P_i\,\mathrm{d}Q_i - H'\,\mathrm{d}t + \mathrm{d}F$$

from which we define the type one canonical transformation:

$$\mathrm{d}F = \mathrm{d}F_1(\mathbf{q}, \mathbf{Q}, t) = \frac{\partial F_1}{\partial \mathbf{q}}\,\mathrm{d}\mathbf{q} + \frac{\partial F_1}{\partial \mathbf{Q}}\,\mathrm{d}\mathbf{Q} + \frac{\partial F_1}{\partial t}\,\mathrm{d}t$$

we have:

$$H' = H + \frac{\partial F_1}{\partial t}\ ,\ \ \mathbf{P} = -\frac{\partial F_1}{\partial \mathbf{Q}}\ ,\ \ \mathbf{p} = \frac{\partial F_1}{\partial \mathbf{q}}$$

We then perform Legendre transformation on the canonical variables to obtain the type two, three, and four canonical transformation:

$$F = F_2(\mathbf{q}, \mathbf{P}, t) - \sum_i P_i Q_i$$
$$F = F_3(\mathbf{p}, \mathbf{Q}, t) + \sum_i p_i q_i$$
$$F = F_4(\mathbf{p}, \mathbf{P}, t) + \sum_i p_i q_i - \sum_k P_k Q_k$$

3.4 Adiabatic Invariant

The quantity $J \equiv \oint p_q \cdot \mathrm{d}q$, is an invariant for slowly varying systems.

3.5 Poisson Brackets

We take the total derivative of some physical quantity $f(q, p, t)$:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial f}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t} \right) \\ &= \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial p_i} \right) \\ \{f, g\}_{\mathrm{P}} = \{f, g\} &= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_i} \right) \\ \frac{\mathrm{d}f}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \{f, \mathcal{H}\} \end{aligned}$$

4 Orbital Mechanics

4.1 Equations of Motion

Assume a central force potential $U(r)$, we know that:

$$\mathbf{F} = -\boldsymbol{\nabla} U = -\frac{\mathrm{d}U(r)}{\mathrm{d}r}\,\hat{\mathbf{r}}$$

$$\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t} = \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = 0$$

thus the angular momentum is conserved for any arbitrary central force. We can assume a 2-D motion since the direction of angular momentum is conserved.

Consider two particles, define $\mathbf{F}_{12}$ be the force acted on particle 1 by particle 2 and vice versa. Thus we have the equations of motion:

$$\begin{cases} m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{12} = -\boldsymbol{\nabla}_1 U \\ m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_{21} = -\boldsymbol{\nabla}_2 U \end{cases}$$

from basic calculus we know:

$$\boldsymbol{\nabla}_2 U = -\boldsymbol{\nabla}_1 U$$

We add up the two equations and we get:

$$m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 \equiv M\ddot{\mathbf{r}}_c = 0$$

if we divide both sides of the equations with the mass and subtract them, we obtain:

$$\mu\ddot{\mathbf{r}} = -\boldsymbol{\nabla} U \ , \ \ \mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

Since the angular momentum of the system is conserved, we can assume planer motion, from which we have:

$$\ddot{r} - r\dot{\theta}^2 = \frac{F}{\mu}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \Rightarrow \mu r^2\dot{\theta} \equiv \ell \ , \ \ \frac{\mathrm{d}\ell}{\mathrm{d}t} = 0$$

Perform change of variables:

$$u = 1/r \ , \ \ \dot{\theta} = \frac{\ell u^2}{\mu} \ , \ \ \frac{\mathrm{d}}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

$$\dot{r} = \frac{\ell u^2}{\mu}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{\mathrm{d}u}{\mathrm{d}\theta} \ , \ \ \ddot{r} = -\frac{\ell^2 u^2}{\mu^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2}$$

$$F = -\frac{\mathrm{d}U}{\mathrm{d}r} = u^2\frac{\mathrm{d}U}{\mathrm{d}u}$$

We have the modified equation of motion:

$$-\frac{\ell^2 u^2}{\mu^2}\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} - \frac{\ell^2}{\mu^2}u^3 = \frac{F}{\mu} \Rightarrow \frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{\mu}{\ell^2 u^2}F = -\frac{\mu}{\ell^2}\frac{\mathrm{d}U}{\mathrm{d}u}$$

On the other hand, we can start from the conservation of energy:

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

sub in $\ell = \mu r^2\dot{\theta}$, we have:

$$\dot{r}^2 = \frac{2}{\mu}(E - U(r)) - \frac{\ell^2}{\mu^2 r^2} = \frac{2}{\mu}(E - V(r)) \ , \ \ V(r) \equiv U(r) + \frac{\ell^2}{2\mu r^2}$$

then sub in:

$$\dot{r} = \dot{\theta}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\ell}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}$$

we arrive at the integral equation for the orbit:

$$\theta(r) = \pm\int \frac{\ell/r^2}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)}}\,\mathrm{d}r$$

4.2 Kepler's Problem

In Kepler's problem, the potential takes the form:

$$U(r) = -\frac{k}{r} \ , \ \ F(r) = -\frac{k}{r^2}\,\hat{\mathbf{r}}$$

sub this in the equations derived in the previous subsection we get:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\mu k}{\ell^2}$$

$$\theta(r) = \pm\int \frac{\ell/r^2}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{\ell^2}{2\mu r^2}\right)}}\,\mathrm{d}r$$

sub in $u = 1/r$:

$$\theta(u) = \pm \int \frac{\ell}{\sqrt{2\mu E + 2\mu k u - \ell^2 u^2}} \, \mathrm{d}u \overset{s \equiv \ell u}{=} \pm \int \frac{1}{\sqrt{2\mu E + \frac{2\mu k}{\ell} s - s^2}} \, \mathrm{d}s$$
$$= \sin^{-1} \left(\frac{s - \frac{\mu k}{\ell}}{\sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2}}} \right) + \theta_0$$

from which we derive the orbital equation:

$$r = \frac{p}{1 + \epsilon \cos\theta} \ , \ \ p \equiv \frac{\ell^2}{\mu k} \ , \ \ \epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

4.3 LRL Vector

4.4 Perihelion Precession of Mercury

According to General Relativity, the relativistic Lagrangian is:

$$\mathcal{L} = \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}$$

where $g_{\mu\nu}$ is the Schwarzschild metric:

$$\mathrm{d}s^2 = \left(1 - \frac{r_s}{r}\right) c^2 \, \mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \, \mathrm{d}\Omega^2$$

Applying appropriate approximations we arrive at the energy equation:

$$\frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \left(\frac{GMm}{r} + \frac{GM\ell^2}{mc^2 r^3} \right) = \frac{E^2}{2mc^2} - \frac{1}{2} mc^2$$

in which $\ell = mr^2\dot{\theta}$ is the quantity analogous to classical angular momentum. The full equation in terms of the radial velocity $\dot{r}$ thus reads:

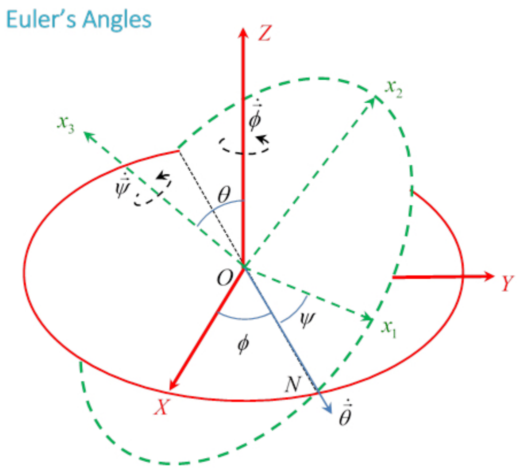
$$\frac{1}{2} m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \left(\frac{GMm}{r} + \frac{GM\ell^2}{mc^2 r^3} \right) = \frac{E^2}{2mc^2} - \frac{1}{2} mc^2$$

5 Rigid Body Motion

5.1 Euler Angles

To solve rigid body motion problems, one must find a systematic way to convert from the fixed reference frame to the principle axis of the object if interest, and Euler angles are the conventional way to do so.

The Euler angles ϕ, θ, ψ are defined as in the following figure:



From which we can derive the rotation matrix between reference frames:

$$\begin{pmatrix}\hat{\mathbf{e}}_1\\\hat{\mathbf{e}}_2\\\hat{\mathbf{e}}_3\end{pmatrix}=\begin{pmatrix}\cos\psi & \sin\psi & 0\\-\sin\psi & \cos\psi & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & \cos\theta & \sin\theta\\0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\-\sin\phi & \cos\phi & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}\\\hat{\mathbf{y}}\\\hat{\mathbf{z}}\end{pmatrix}$$

The total angular velocity is therefore:

$$\begin{aligned}\boldsymbol{\omega}&=\dot{\phi}\,\hat{\mathbf{z}}+\dot{\theta}(\cos\psi\,\hat{\mathbf{e}}_1-\sin\psi\,\hat{\mathbf{e}}_2)+\dot{\psi}\,\hat{\mathbf{e}}_3\\&=\dot{\phi}\,(\cos\theta\,\hat{\mathbf{e}}_3+\sin\theta(\sin\psi\,\hat{\mathbf{e}}_1+\cos\psi\,\hat{\mathbf{e}}_2))+\dot{\theta}(\cos\psi\,\hat{\mathbf{e}}_1-\sin\psi\,\hat{\mathbf{e}}_2)+\dot{\psi}\,\hat{\mathbf{e}}_3\\&=\hat{\mathbf{e}}_1\left(\dot{\phi}\sin\theta\sin\psi+\dot{\theta}\cos\psi\right)+\hat{\mathbf{e}}_2\left(\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi\right)+\hat{\mathbf{e}}_3\left(\dot{\phi}\cos\theta+\dot{\psi}\right)\end{aligned}$$

The total angular momentum and the rotational kinetic energy can be expressed in terms of the Euler angles:

$$\mathbf{L}=\hat{\mathbf{e}}_1I_1\left(\dot{\phi}\sin\theta\sin\psi+\dot{\theta}\cos\psi\right)+\hat{\mathbf{e}}_2I_2\left(\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi\right)+\hat{\mathbf{e}}_3I_3\left(\dot{\phi}\cos\theta+\dot{\psi}\right)$$

$$K=\frac{1}{2}\left(I_1\left(\dot{\phi}\sin\theta\sin\psi+\dot{\theta}\cos\psi\right)^2+I_2\left(\dot{\phi}\sin\theta\cos\psi-\dot{\theta}\sin\psi\right)^2+I_3\left(\dot{\phi}\cos\theta+\dot{\psi}\right)^2\right)$$

Especially if we are dealing with a symmetric top with $I_1=I_2\neq I_3$, we have:

$$K=\frac{1}{2}I_1\left(\dot{\phi}^2\sin^2\theta+\dot{\theta}^2\right)+\frac{1}{2}I_3\left(\dot{\phi}\cos\theta+\dot{\psi}\right)^2$$

and hence the frame we choose does not depend on ψ and does not have to rotate ψ, reducing the rotation matrices to:

$$\begin{pmatrix}\hat{\mathbf{e}}_1\\\hat{\mathbf{e}}_2\\\hat{\mathbf{e}}_3\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\0 & \cos\theta & \sin\theta\\0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\-\sin\phi & \cos\phi & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}\\\hat{\mathbf{y}}\\\hat{\mathbf{z}}\end{pmatrix}$$

also, the angular velocity of the body and the frame can be reduced to:

$$\begin{aligned}\boldsymbol{\omega}&=\hat{\mathbf{e}}_1\dot{\theta}+\hat{\mathbf{e}}_2\dot{\phi}\sin\theta+\hat{\mathbf{e}}_3\left(\dot{\phi}\cos\theta+\dot{\psi}\right)\\\boldsymbol{\omega}_f&=\hat{\mathbf{e}}_1\dot{\theta}+\hat{\mathbf{e}}_2\dot{\phi}\sin\theta+\hat{\mathbf{e}}_3\dot{\phi}\cos\theta\end{aligned}$$

5.2 Euler's Equation

$$\begin{aligned}
I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= \tau_1 \\
I_2\dot{\omega}_2 + \omega_3\omega_1(I_1 - I_3) &= \tau_2 \\
I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= \tau_3
\end{aligned}$$

6 Oscillations and Waves

6.1 Anharmonic Oscillator

7 Perturbation Theory

7.1 Elementary Methods

Consider the Hamiltonian:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

where $\mathcal{H}_0$ is a Hamiltonian we know the solution of the EOM to.

8 Classical Chaos

9 Lagrangian of Fields

9.1 From Particles to Continuous System

When discussing the motion of systems of particles, it is natural that we add up the action of each particle:

$$S = \sum S_i = \int \left(\sum \mathcal{L}_i\right) \mathrm{d}t$$

For a continuous system, we express the Lagrangian in terms of an integral of the Lagrangian density:

$$S = \iint \mathscr{L}\, \mathrm{d}^3x\, \mathrm{d}t = \int \mathscr{L}\, \mathrm{d}^4x$$

The variables of the Lagrangian density thus become a variable of position and time called field.

$$\mathscr{L} = \mathscr{L}\left(\phi, \frac{\partial\phi}{\partial t}, \boldsymbol{\nabla}\phi, \mathbf{r}, t\right) = \mathscr{L}(\phi, \partial_\mu\phi, x^\mu)\ , \quad \text{where } \phi = \phi(\mathbf{r}, t)$$

To find the EL equation of fields, we apply principle of least action:

$$
\begin{aligned}
\delta S &= \delta \int \mathscr{L}\,\mathrm{d}^4x \\
&= \int \frac{\partial \mathscr{L}}{\partial \phi}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\,\mathrm{d}^4x \\
&= \int \left(\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi\right)\mathrm{d}^4x \\
&= \int \left(\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi\,\mathrm{d}^4x = 0 \\
&\Rightarrow \frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} = 0
\end{aligned}
$$

Note that the third term in the integral vanishes because we can apply the 4-D divergence theorem and then assume boundary terms are 0 as in the 1-D EL equation.

9.2 Scalar Fields and Klein-Gordon Equation

The Lagrangian density describing scalar fields expressed in natural units is:

$$
\mathscr{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 - V(\phi)
$$

Applying EL equation yields the Klein-Gordon equation:

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} &= \frac{1}{2}\eta^{\mu\nu}(\partial_\nu\phi + \delta^\mu_\nu \partial_\mu \phi) = \eta^{\mu\nu}\partial_\nu\phi \\
\frac{\partial \mathscr{L}}{\partial \phi} &= -m^2\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} \\
&\Rightarrow \eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0
\end{aligned}
$$

9.3 Vibrating String

The Kinetic Energy Density of the string is:

$$
\mathrm{d}K = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2\mathrm{d}x
$$