Uncertainty principle $\Delta A \Delta B \geq |\langle\psi|\frac{1}{2i}[\hat{A},\hat{B}]|\psi\rangle|$, Variational Principle $E_{\text{gs}} \leq \langle H \rangle_\psi$ for all ψ
Simple harmonic oscillator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right) \ , \ a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right) \ , \ x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \ , \ p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^\dagger)$$

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) = \hbar\omega\left(N + \frac{1}{2}\right) \ , \ [a, a^\dagger] = 1 \ , \ [N, a] = -a \ , \ [N, a^\dagger] = a^\dagger$$

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \ , \ a|n\rangle = \sqrt{n}|n-1\rangle \ , \ \ a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Pauli matrices

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \ , \ \{\sigma_i, \sigma_j\} = 2\delta_{ij}I \ , \ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \ , \ (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b})I + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

Angular momentum

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k \ , \ [L_i, S] = 0 \Rightarrow [L^2, L_i] = 0 \ , \ [L_i, v_j] = i\hbar\epsilon_{ijk}v_k \Rightarrow \vec{L}\times\vec{v} + \vec{v}\times\vec{L} = 2i\hbar\vec{v}$$

$$J_\pm = J_x \pm iJ_y \ , \ J_x = \frac{1}{2}(J_+ + J_-) \ , \ J_y = \frac{1}{2i}(J_+ - J_-) \ , \ [J_z, J_\pm] = \pm\hbar J_\pm \ , \ [J_+, J_-] = 2\hbar J_z$$

$$J^2 = J_+J_- + J_z^2 - \hbar J_z = J_-J_+ + J_z^2 + \hbar J_z \ , \ [J^2, J_\pm] = 0$$

$$J_z|j,m\rangle = m\hbar|j,m\rangle \ , \ J^2|j,m\rangle = j(j+1)\hbar^2|j,m\rangle$$

$$J_\pm|j,m\rangle = \hbar^2\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle$$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|$$

Non-degenerate PT ($m \neq n$)

$$E_n^{(1)} = \langle n^{(0)}|\delta H|n^{(0)}\rangle = \delta H_{nn} \ , \ E_n^{(2)} = \sum_{m\neq n}\frac{\left|\langle m^{(0)}|\delta H|n^{(0)}\rangle\right|^2}{E_n^{(0)} - E_m^{(0)}} = \sum_{m\neq n}\frac{|\delta H_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$\langle m^{(0)}|n^{(1)}\rangle = \frac{\langle m^{(0)}|\delta H|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} = \frac{\delta H_{mn}}{E_n^{(0)} - E_m^{(0)}}$$

$$\langle m^{(0)}|n^{(2)}\rangle = \left(\sum_{l\neq n}\frac{\delta H_{ml}\delta H_{ln}}{(E_n^{(0)} - E_m^{(0)})(E_n^{(0)} - E_l^{(0)})}\right) - \frac{\delta H_{nn}\delta H_{mn}}{(E_n^{(0)} - E_m^{(0)})^2}$$

Degenerate PT, diagonalize δH in the degenerate subspaces of $H^{(0)}$.
FH Lemma

$$\left\langle\frac{\partial H}{\partial\lambda}\right\rangle_{\text{eigenstate}} = \frac{\partial E}{\partial\lambda}_{\text{eigenstate}}$$

Hydrogen atom

Hamiltonian: $H = \dfrac{p^2}{2m} - \dfrac{e^2}{r}$, $\psi_{n\ell m}(\vec{x}) = \dfrac{u_{n\ell}(r)}{r}Y_{\ell m}(\theta,\phi)$

radial equation: $\left(-\dfrac{\hbar^2}{2m}\dfrac{d^2}{dr^2} + V(r) + \dfrac{\hbar^2\ell(\ell+1)}{2mr^2}\right)u_{\nu\ell}(r) = E_{\nu\ell}u_{\nu\ell}(r)$, $u_{\nu\ell}(r) \sim r^{\ell+1}$ as $r \to 0$

spectrum: $E_n = -\dfrac{e^2}{2a_0n^2} = -\dfrac{1}{2}\alpha^2mc^2\dfrac{1}{n^2}$, where $a_0 = \dfrac{\hbar^2}{me^2}$, $\alpha = \dfrac{e^2}{\hbar c} \Rightarrow$ velocity scale $v \approx \alpha c$

spec. notation: $n^{2s+1}L_j$, e.g. $1S_{1/2}$, $3P_{1/2}$, $3P_{3/2}$, s can be omitted for single electrons

CSCO: $\text{CSCO}_1 = \{H, L^2, L_z, S_z\}$, $\text{CSCO}_2 = \{H, L^2, J^2, J_z\}$

Hydrogen atom fine structure, good basis $|n, \ell, j, m_j\rangle$

$$\delta H_{\text{fs}} = \delta H_{\text{rel}} + \delta H_{\text{so}} + \delta H_{\text{Darwin}} = -\frac{p^4}{8m^3c^3} + \frac{e^2}{2m^2c^2r^3}\vec{S}\cdot\vec{L} + \frac{\pi}{2}\frac{e^2\hbar^2}{m^2c^2}\delta^3(\vec{r})$$
$$E^{(1)}_{n\ell j m_j;\text{fs}} = -\frac{(E_n^{(0)})^2}{2mc^2}\left[\frac{4n}{j+\frac{1}{2}} - 3\right] = -\frac{\alpha^4(mc^2)}{2n^4}\left[\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right]$$

Zeeman and Stark effect

$$\delta H_Z = \frac{eB}{2mc}(L_z + 2S_z)\ ,\ \delta H_S = -e\phi$$

weak field: eigenstate $= |n, \ell, j, m_j\rangle$, first order correction is diagonal.
strong field: eigenstate $= |n, \ell, m_\ell, m_s\rangle$
WKB approximation

$$\psi(\vec{x},t) = \sqrt{\rho(\vec{x},t)}e^{iS(\vec{x},t)/\hbar}\ ,\ \vec{J} = \frac{\rho}{m}\nabla S \Rightarrow (S')^2 - i\hbar S'' = p^2(x) \text{ where } p^2(x) = 2m(E - V(x))$$

$$\text{validity: } |p| \gg \lambda\left|\frac{\mathrm{d}p}{\mathrm{d}x}\right| \Rightarrow T \gg \lambda\left|\frac{\mathrm{d}V}{\mathrm{d}x}\right|$$

$$\text{allowed: } \psi(x) = \frac{A}{\sqrt{p(x)}}\exp\left(\frac{i}{\hbar}\int_{x_0}^x p(x')\,\mathrm{d}x'\right) + \frac{B}{\sqrt{p(x)}}\exp\left(-\frac{i}{\hbar}\int_{x_0}^x p(x')\,\mathrm{d}x'\right)$$

$$\text{forbidden: } \psi(x) = \frac{C}{\sqrt{\kappa(x)}}\exp\left(\frac{1}{\hbar}\int_{x_0}^x \kappa(x')\,\mathrm{d}x'\right) + \frac{D}{\sqrt{\kappa(x)}}\exp\left(-\frac{1}{\hbar}\int_{x_0}^x \kappa(x')\,\mathrm{d}x'\right)$$

$$\text{tunneling: } T \sim \exp\left(-\frac{2}{\hbar}\int_a^b \kappa(x)\,\mathrm{d}x\right)$$

Airy functions

$$\mathrm{Ai}(u) = \frac{1}{\pi}\int_0^\infty \mathrm{d}k\cos\left(\frac{1}{3}k^3 + ku\right) \sim \begin{cases} \frac{1}{2\sqrt{\pi}}\frac{1}{u^{1/4}}\exp\left(-\frac{2}{3}u^{3/2}\right) & x \gg 1 \\ \frac{1}{\sqrt{\pi}}\frac{1}{|u|^{1/4}}\cos\left(\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right) & x \ll -1 \end{cases}$$

$$\mathrm{Bi}(u) = \frac{1}{\pi}\int_0^\infty \mathrm{d}k\left(e^{-k^3/3}e^{ku} + \sin\left(\frac{1}{3}k^3 + ku\right)\right) \sim \begin{cases} \frac{1}{\sqrt{\pi}}\frac{1}{u^{1/4}}\exp\left(\frac{2}{3}u^{3/2}\right) & x \gg 1 \\ -\frac{1}{\sqrt{\pi}}\frac{1}{|u|^{1/4}}\sin\left(\frac{2}{3}|u|^{3/2} - \frac{\pi}{4}\right) & x \ll -1 \end{cases}$$

Connection formulae

$$\text{L allowed, R forbidden: } \frac{2A}{\sqrt{k(x)}}\cos\left(\int_x^a k(x')\,\mathrm{d}x' - \frac{\pi}{4}\right) + \frac{B}{\sqrt{k(x)}}\sin\left(\int_x^a k(x')\,\mathrm{d}x' - \frac{\pi}{4}\right) \text{ for } x \ll a$$
$$\Leftrightarrow \frac{A}{\sqrt{\kappa(x)}}\exp\left(-\int_a^x \kappa(x')\,\mathrm{d}x'\right) - \frac{B}{\sqrt{\kappa(x)}}\exp\left(\int_a^x \kappa(x')\,\mathrm{d}x'\right) \text{ for } x \gg a$$
$$\text{R allowed, L forbidden: } \frac{A}{\sqrt{\kappa(x)}}\exp\left(-\int_x^b \kappa(x')\,\mathrm{d}x'\right) + \frac{B}{\sqrt{\kappa(x)}}\exp\left(\int_x^b \kappa(x')\,\mathrm{d}x'\right) \text{ for } x \ll b$$
$$\Leftrightarrow \frac{2A}{\sqrt{k(x)}}\cos\left(\int_b^x \kappa(x')\,\mathrm{d}x' - \frac{\pi}{4}\right) - \frac{B}{\sqrt{k(x)}}\sin\left(\int_b^x k(x')\,\mathrm{d}x' - \frac{\pi}{4}\right) \text{ for } x \gg b$$

Quantization with correction (EBK method)

$$\oint p_i\,\mathrm{d}q_i = \left(n_i + \frac{1}{4}\mu_i + \frac{1}{2}b_i\right)2\pi\hbar$$

where μ_i is the number of classical soft turning points and b_i is the number of hard walls.

Motion in EM field

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\ ,\ \vec{B} = \nabla \times \vec{A}$$
$$\Phi_B(S) = \int_S \vec{B}\cdot \mathrm{d}\vec{a} = \int_S (\nabla\times\vec{A})\cdot \mathrm{d}\vec{a} = \int_{\partial S} \vec{A}\cdot \mathrm{d}\vec{\ell}\ ,\ H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

Gauge transformations

$$\phi \to \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t}\ ,\ \vec{A} \to \vec{A} + \nabla\Lambda\ ,\ \psi \to e^{iq\Lambda/\hbar c}\psi = U\psi\ ,\ H = UHU^{-1}$$

Landau levels in Langau gauge $\vec{A} = (0, Bx, 0)$

$$\omega = \frac{qB}{mc}\ ,\ H_{k_y} = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{\hbar k_y}{m\omega}\right)^2\ ,\ E_{k_y,n_x} = \left(n_x + \frac{1}{2}\right)\hbar\omega$$
$$D = \frac{\Phi_B}{\Phi_0} \text{ where } \Phi_0 = \frac{2\pi\hbar c}{q} = 4.136\times 10^{-7}\ \mathrm{G}\cdot\mathrm{cm}^2$$

on a ring

$$E_n(\Phi) = \frac{\hbar^2}{2mb^2}\left(n - \frac{\Phi}{\Phi_0}\right)^2 \text{ periodic in } \Phi$$

TDPT, let $\tilde{O}(t) = e^{iH_0t/\hbar}Oe^{-iH_0t/\hbar}$ denotes operator O in the interaction picture

$$H = H_0 + V(t)\ ,\ |\psi(t)\rangle = e^{-iH_0t/\hbar}|\tilde{\psi}(t)\rangle\ ,\ \tilde{V}(t) = e^{iH_0t/\hbar}V(t)e^{-iH_0t/\hbar}$$
$$\Rightarrow i\hbar\frac{\partial}{\partial t}|\tilde{\psi}(t)\rangle = \tilde{V}(t)|\tilde{\psi}(t)\rangle$$
$$\Rightarrow |\tilde{\psi}(t)\rangle = |\tilde{\psi}(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t}\mathrm{d}t'\,\tilde{V}(t')|\tilde{\psi}(t_0)\rangle + \frac{1}{(i\hbar)^2}\int_{t_0}^{t}\mathrm{d}t'\int_{t_0}^{t'}\mathrm{d}t''\,\tilde{V}(t')\tilde{V}(t'')|\tilde{\psi}(t_0)\rangle + \cdots$$

for SHO with $H_0 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$

$$e^{iH_0t/\hbar}ae^{-iH_0t/\hbar} = ae^{-i\omega t}\ ,\ e^{iH_0t/\hbar}a^\dagger e^{-iH_0t/\hbar} = a^\dagger e^{i\omega t}$$

in the energy eigenstates of H_0

$$\frac{\mathrm{d}c_m}{\mathrm{d}t} = \frac{1}{i\hbar}\sum_{n\neq m}\langle m|V(t)|n\rangle e^{i(E_m-E_n)t/\hbar}\,c_n \Rightarrow c_m^{(1)} = \frac{1}{i\hbar}\sum_{n\neq m}\int_0^t V_{mn}(t')e^{i\omega_{mn}t'}\,c_n(0)$$

validity condition $|V_{fi}| \ll |\hbar\omega_{fi}|$.
Periodic perturbation

$$\delta H(t) = V\cos(\omega t) \Rightarrow P_{n\to m}(t) \approx \frac{|V_{mn}|^2t^2}{4\hbar^2}\frac{\sin^2((\omega_{mn}-\omega)t/2)}{\left[(\omega_{mn}-\omega)t/2\right]^2} \Rightarrow R_{n\to m} = \frac{\pi|V_{mn}|^2}{2\hbar^2}\delta(\omega_{mn}-\omega)$$

Density of states (with spin degeneracy, in d spatial dimensions)

$$\text{momentum: } \rho(k)\,\mathrm{d}k = (2s+1)\frac{L^d}{(2\pi)^d}k^{d-1}\,\mathrm{d}k\,\mathrm{d}\Omega_{d-1} = (2s+1)\frac{L^3}{(2\pi)^3}4\pi k^2\,\mathrm{d}k$$
$$\text{energy: } \rho(E)\,\mathrm{d}E = (2s+1)\frac{L^3}{(2\pi)^3}\frac{m}{\hbar^2}k\,\mathrm{d}\Omega\,\mathrm{d}E = (2s+1)\frac{L^3m(2mE)^{1/2}}{2\pi^2\hbar^3}\,\mathrm{d}E$$

Fermi's golden rule

$$\text{rate: } w_{i\to f} = \frac{P^{\text{TDPT}}_{i\to f}(t)}{t} \qquad \text{mean lifetime: } \tau = \frac{1}{\sum_{f\neq i} w_{i\to f}}$$

$$H = H_0 + V \qquad \Rightarrow w_{i\to f} = \frac{2\pi}{\hbar}|V_{fi}|^2 \rho(E_f)\big|_{E_f = E_i} \qquad \text{for } \left|\hbar \frac{\mathrm{d}w}{\mathrm{d}E}\right|_{E_f} \ll 1$$

$$H = H_0 + 2V\cos(\omega t) \qquad \Rightarrow w_{i\to f} = \frac{2\pi}{\hbar}|V_{fi}|^2 \rho(E_f)\big|_{E_f = E_i \pm \hbar\omega} \qquad \text{for } \left|\hbar \frac{\mathrm{d}w}{\mathrm{d}E}\right|_{E_f} \ll 1$$

Ionization

$$\vec{E}(t) = 2E_0\cos(\omega t)\,\hat{n}\ ,\ \delta H = -q\,\vec{r}\cdot\vec{E}(t) = -2q\,\vec{r}\cdot\hat{n}\,E_0\cos(\omega t) = -2\vec{d}\cdot\hat{n}\,E_0\cos(\omega t)$$

Rate equations, $E_b > E_a$

$$\frac{\mathrm{d}N_a}{\mathrm{d}t} = r_{\text{sp}}N_b + r_{\text{st}}(N_b - N_a) = AN_b + Bu(\omega_{ba})(N_b - N_a)\ ,\ \frac{N_b}{N_a} = e^{-\beta\hbar\omega_{ba}}$$
$$\Rightarrow \frac{r_{\text{st}}}{r_{\text{sp}}} = \frac{1}{e^{\beta\hbar\omega_0} - 1}$$

Einstein coefficients

$$A_{b\to a} = \frac{4}{3}\frac{\omega_{ba}^3}{\hbar c^3}|\vec{d}_{ab}|^2\ ,\ B_{b\to a} = B_{a\to b} = \frac{4\pi^2}{3\hbar^2}|\vec{d}_{ab}|^2\ ,\ A_{b\to a} = \frac{\hbar\omega_{ba}^3}{\pi^2 c^3}B_{b\to a}$$

Selection rules

$$\langle n', \ell', m'|\vec{r}|n, \ell, m\rangle \neq 0 \text{ only if } \Delta\ell = \pm 1 \text{ and } \Delta m = 0 \text{ (for } z),\ \pm 1 \text{ (for } x, y)$$

Adiabatic approximation

instantaneous: $H(t)|\psi(t)\rangle = E(t)|\psi(t)\rangle$

$$\text{equation: } i\hbar\dot{c}_k = (E_k - i\hbar\langle\psi_k|\dot{\psi}_k\rangle)c_k - i\hbar\sum_{n\neq k}\frac{\dot{H}_{kn}}{E_n - E_k}c_n$$

$$\text{phases: dynamical } \theta(t) = -\frac{1}{\hbar}\int_0^t E(t')\,\mathrm{d}t'\ ;\ \nu(t) = i\langle\psi(t)|\dot{\psi}(t)\rangle\ ,\ \text{geometrical } \gamma(t) = \int_0^t \nu(t')\,\mathrm{d}t'$$

ansatz: $|\Psi(t)\rangle \simeq c(0)e^{i\gamma(t)}e^{i\theta(t)}|\psi(t)\rangle$

transition: $c_m \sim \hbar\dot{H}_{mn}/\Delta^2\ ,\ P_m = O(1/T^2) \Rightarrow$ validity: $\hbar|\dot{H}_{mn}| \ll \min_t(E_m - E_n)^2$

Berry phase and all that

$$\text{phase, connection: } \gamma_n(\Gamma_{if}) = \int_{\Gamma_{if}} \mathrm{d}\vec{R}\cdot\vec{\mathcal{A}}_n\ ,\ \vec{\mathcal{A}}_n = i\langle\psi_n(\vec{R})|\nabla_{\vec{R}}|\psi_n(\vec{R})\rangle$$

$$\text{curvature: if } \vec{R}\in\mathbb{R}^3\ ,\ \gamma_n(\Gamma_{if}) = \int_S \mathrm{d}\vec{a}\cdot(\nabla\times\vec{\mathcal{A}}_n) = \int_S \mathrm{d}\vec{a}\cdot\vec{\mathcal{D}}_n$$

$$\text{gauge transformation: } |\psi\rangle \to e^{-i\beta(\vec{R})}|\psi\rangle\ ,\ \vec{\mathcal{A}} \to \vec{\mathcal{A}} + \nabla_{\vec{R}}\beta\ ,\ \gamma(t) \to \gamma(t) + \beta(t) - \beta(0)$$

Scattering

$$\text{currents: } \rho = \psi^*\psi\ ,\ \vec{j} = -\frac{i\hbar}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*) = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\nabla\psi)\ ,\ \frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

$$\text{cross section: } \frac{\mathrm{d}^2 N_{\text{scat}}}{\mathrm{d}\Omega\,\mathrm{d}t} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\frac{\mathrm{d}^2 N_{\text{inc}}}{\mathrm{d}A\,\mathrm{d}t}\ ;\ \psi(\vec{r}) \approx e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}\ ;\ \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\theta,\phi)|^2\ ,\ \mathrm{d}\sigma = \frac{\mathrm{d}w}{|j|}$$

Spherical wave expansion and optical theorem (ignoring ϕ dependence)

$$f_k(\theta) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,Y_{\ell,0}(\theta)e^{i\delta_\ell}\sin(\delta_\ell)\ ,\ \sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2(\delta_\ell) = \frac{4\pi}{k}\mathrm{Im}(f_k(0))$$

unitarity bound, scaling

$$\sigma_\ell \leq \frac{4\pi}{k^2}(2\ell+1)\ ,\ \delta_\ell \xrightarrow{k\to 0} -\frac{(ka)^{2\ell+1}}{(2\ell+1)!!(2\ell-1)!!}$$

$$\sigma_\ell \xrightarrow{k\to 0} \frac{4\pi a^2}{2\ell+1}\left(\frac{2^\ell \ell!}{(2\ell)!}\right)^4 (ka)^{4\ell}\ ,\ \text{terms with } ka \ll \ell \text{ are exponentially suppressed}$$

Born approximation

parameters: $\vec{q} = \vec{k}_f - \vec{k}_i\ ,\ q = 2k\sin(\theta/2)\ ,\ \hat{q}\cdot\hat{r} = \sin(\theta/2)$

amplitudes: $f_k^B(\theta,\phi) = -\dfrac{m}{2\pi\hbar^2}\displaystyle\int \mathrm{d}^3r\, e^{-i\vec{q}\cdot\vec{r}}V(\vec{r}) = -\frac{2m}{\hbar^2 q}\int_0^\infty \mathrm{d}r\, rV(r)\sin(qr)$ (for central potential)

validity: $|V| \ll \dfrac{\hbar^2}{ma^2}\cdot ka$

Symmetric and antisymmetric subspaces

$$V\otimes V \cong \mathrm{Sym}^2(V)\otimes \mathrm{Anti}^2(V)\ ,\ d^2 = \frac{d(d+1)}{2}+\frac{d(d-1)}{2}$$
$$\mathrm{Anti}^2(V\otimes W) \cong (\mathrm{Sym}^2(V)\otimes\mathrm{Anti}^2(W))\oplus(\mathrm{Anti}^2(V)\otimes\mathrm{Sym}^2(W))$$
$$\mathrm{Sym}^2(V\otimes W) \cong (\mathrm{Sym}^2(V)\otimes\mathrm{Sym}^2(W))\oplus(\mathrm{Anti}^2(V)\otimes\mathrm{Anti}^2(W))$$

e.g. addition of spin: $\frac{1}{2}\otimes\frac{1}{2}\cong 1\oplus 0$, 1 is symmetric and 0 is antisymmetric

subspaces: $|1,1\rangle = |{+}{+}\rangle\ ,\ |1,0\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle)\ ,\ |1,-1\rangle = |{-}{-}\rangle\ ,\ |0,0\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle - |{-}{+}\rangle)$

Identical particles

$\sigma \in S_N$, permutation P_σ

symmetrization $S = \dfrac{1}{N!}\displaystyle\sum_\sigma P_\sigma$, anti-symmetrization $A = \dfrac{1}{N!}\displaystyle\sum_\sigma (-1)^{\mathrm{sgn}(\sigma)}P_\sigma$

$S^\dagger = S\ ,\ S^2 = S\ ;\ A^\dagger = A\ ,\ A^2 = A\ ;\ AS = SA = 0$

Spin statistics theorem
Constructing antisymmetric wave functions

$$\text{Slater determinant: } \Psi(\vec{x}_1,...,\vec{x}_N) = \frac{1}{\sqrt{N!}}\det([\psi_i(\vec{x}_j)])$$

Constraining the Hamiltonian

$$PHP^{-1} = H\ ,\ \text{where } P \text{ is a permutation over identical particles}$$

Occupation number representation

$$|n_1, n_2, ...\rangle$$

Commutator formulae

$$[A, BC] = [A, B]C + B[A, C]\ ,\ [A, BCD] = [A, B]CD + B[A, C]D + BC[A, D]\ ,\ ...$$

Div grad curl and all that

$$\begin{aligned}
\boldsymbol{\nabla} f &= \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_i}\hat{\boldsymbol{e}}_i \\
\boldsymbol{\nabla} \cdot \mathbf{v} &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 v_1) + \frac{\partial}{\partial u_2}(h_3 h_1 v_2) + \frac{\partial}{\partial u_3}(h_1 h_2 v_3)\right) \\
\nabla^2 f &= \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) \\
\boldsymbol{\nabla} \times \mathbf{v} &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\boldsymbol{e}}_1 & h_2\hat{\boldsymbol{e}}_2 & h_3\hat{\boldsymbol{e}}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}
\end{aligned}$$