Dynamical Analysis of Delay Differential Equations in Macroeconomic Models

Bifurcations, Complex Behavior, and Mathematical Structure

Arthur Mota

University of São Paulo

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Outline

Traditional Macroeconomic Models: The Foundation

Linear Rational Expectations Framework

Traditional inflation dynamics equation:

$$\pi_t = \alpha E_t[\pi_{t+1}] + \beta x_t + \varepsilon_t \tag{1}$$

Where:

- π_t = inflation rate
- $E_t[\pi_{t+1}]$ = rational expectations of future inflation
- x_t = output gap
- ε_t = random shock

Key assumptions:

- Linear relationships
- Rational expectations
- Exogenous shocks drive dynamics

The Critical Limitations

Why Linear Models Fall Short

Empirical evidence suggests:

- Persistent inflation volatility without external shocks
- Complex, erratic behavior (1970s stagflation, 2021-2023 surge)
- Bounded rationality in expectation formation
- Multiple time scales in policy transmission

Challenge

Linear models cannot capture the complex dynamics observed in real inflation data

Research Motivation

The Complexity Puzzle

- Standard models: eventual convergence to equilibrium
- Reality: persistent complex dynamics
- Policy puzzle: sophisticated frameworks often ineffective
- Mathematical question: Are economic systems fundamentally non-integrable?

Solution

Develop rigorous mathematical framework using delay differential equations with heterogeneous bounded rationality

Core Innovation

Delay Differential Equations with Bounded Rationality

Key insight: Economic agents have:

- Heterogeneous information processing speeds
- Bounded cognitive capacity
- Distributed memory structures
- Nonlinear feedback mechanisms

Mathematical representation:

- Continuous-time dynamics
- Distributed delay kernels
- Infinite-dimensional phase space
- Nonlinear expectation formation

The Complete System

Integrated Dynamics

Inflation dynamics:

$$\frac{d\pi}{dt} = \alpha_1 \int_0^{\tau_{max}} G(\tau)[\pi^e(t-\tau) - \pi(t)]d\tau + \beta_1 x(t) + \gamma_1(\pi(t) - \pi_*)^3 + \eta_1(t) \tag{2}$$

Output gap dynamics:

$$\frac{dx}{dt} = -\alpha_2 \int_0^{\tau_{max}} H(\tau)[r(t-\tau) - \pi^e(t-\tau)]d\tau + \beta_2[\pi(t) - \pi_*] + \gamma_2 x(t)[1 - x^2(t)/x_{max}^2] + \eta_2(t) \tag{3}$$

Expectation formation:

$$\frac{d\pi^e}{dt} = \int_0^{\tau_{max}} W(\tau)\mathcal{F}[\pi(t-\tau), \pi^e(t-\tau), \frac{d\pi}{dt}(t-\tau)]d\tau - \lambda[\pi^e(t) - \pi_*] + \eta_3(t) \tag{4}$$

Heterogeneous Expectation Formation

Bounded Rationality Mechanisms

Agent types with different rules:

$$\mathcal{F}_1 = \pi_* + \kappa_1 \tanh(\theta_1[\pi(t-\tau) - \pi_*]) \quad \text{(Anchored adaptive)} \tag{5}$$

$$\mathcal{F}_2 = \pi^e(t-\tau) + \kappa_2[\pi(t-\tau) - \pi^e(t-\tau)] \quad \text{(Simple adaptive)} \tag{6}$$

$$\mathcal{F}_3 = \pi(t-\tau) + \kappa_3 \frac{d\pi}{dt}(t-\tau) \quad \text{(Momentum)} \tag{7}$$

$$\mathcal{F}_4 = \frac{1}{\tau} \int_{t-\tau}^{t} \pi(s) ds \quad \text{(Moving average)} \tag{8}$$

Population shares: $\sum_{i=1}^{4} \mu_i = 1$

Key innovation: tanh function captures bounded cognitive capacity

Distributed Delay Structure

Heterogeneous Timing

Delay kernels capture agent heterogeneity:

$$G(\tau) = \frac{1}{\sigma_G\sqrt{2\pi}} \exp\left(-\frac{(\tau-\mu_G)^2}{2\sigma_G^2}\right) \quad \text{(Gaussian)} \tag{9}$$

$$H(\tau) = \frac{\alpha}{\Gamma(\beta)} \tau^{\beta-1} e^{-\alpha\tau} \quad \text{(Gamma)} \tag{10}$$

$$W(\tau) = \sum_{j=1}^{M} w_j \delta(\tau - \tau_j) \quad \text{(Discrete delays)} \tag{11}$$

Economic interpretation:

- Different information processing speeds
- Institutional response times
- Communication delays
- Heterogeneous memory lengths

Infinite-Dimensional Phase Space

Fundamental Mathematical Structure

State space: $\mathcal{C} = C([-\tau_{max}, 0], \mathbb{R}^3)$
State at time t: $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau_{max}, 0]$

Evolution operator:

$$\frac{d}{dt} T(t)\phi = \mathcal{L}(T(t)\phi) + \mathcal{N}(T(t)\phi) \tag{12}$$

Theorem (Infinite-Dimensional Structure)

The phase space $\mathcal{C}$ has uncountably infinite dimension. No finite number of conserved quantities can constrain motion, fundamentally distinguishing DDE integrability from finite-dimensional systems.

Integrability Analysis

Obstacles to Analytical Solutions

Classical integrability requires:

- Sufficient first integrals
- Analytical solution methods
- Conserved quantities

Theorem (Analytical Challenges)

The cubic nonlinearity $(\pi(t) - \pi_*)^3$ combined with distributed delays presents significant obstacles to:

- *Power series methods*
- *Standard perturbation techniques*
- *Classical analytical approaches*

Implication

System likely exhibits non-integrable dynamics

Characteristic equation:

$$\det(\mathcal{L}(\lambda)) = 0 \tag{13}$$

Where:

$$\mathcal{L}(\lambda) = \lambda I - A_0 - \sum_{j=1}^{M} A_j e^{-\lambda \tau_j} - \int_0^{\tau_{max}} B(\tau) e^{-\lambda \tau} d\tau \tag{14}$$

Transcendental nature:

- No closed-form solutions
- Infinitely many eigenvalues
- Complex stability boundaries
- Necessitates numerical analysis

Parameter Calibration
Empirically Motivated Values

Parameter	Value	Economic Interpretation
α_1	0.8	Expectation adjustment speed
β_1	0.5	Phillips curve slope
γ_1	0.1	Nonlinear feedback strength
τ_{avg}	0.25	Average expectation delay
κ_1	1.5	Expectation sensitivity

Calibration sources:

- Survey of Professional Forecasters
- Output-inflation correlations
- Threshold VAR estimates
- Monetary policy transmission studies

Dynamical Regimes

Delay-Dependent Behavior

Regime classification by delay length:

- **Low delays** ($\tau_{avg} < 0.2$):
  - Stable convergence to target
  - Damped oscillations
- **Moderate delays** ($0.2 < \tau_{avg} < 0.4$):
  - Persistent oscillations
  - Inflation cycles
- **High delays** ($\tau_{avg} > 0.4$):
  - Complex aperiodic dynamics
  - Sensitive dependence on initial conditions

Key Finding

Realistic delay parameters can generate chaotic dynamics

Lyapunov exponent computation:

$$\lambda_1 = \lim_{t \to \infty} \frac{1}{t} \sum_{k=1}^{N} \log \left(\frac{L'_k}{L_k} \right) \tag{15}$$

Results for high-delay regime ($\tau_{avg} = 0.45$):

- $\lambda_1 = 0.089 > 0 \Rightarrow$ Chaotic dynamics
- Correlation dimension $d_c \approx 2.3$
- Fractal attractor structure

Economic Significance

Bounded but unpredictable inflation dynamics without external shocks

Strange Attractor Properties
Complex Geometric Structure

Attractor characteristics:

- Bounded volatility: $\pi \in [0\%, 8\%]$
- Fractal dimension: Non-integer
- Sensitive dependence on initial conditions
- Aperiodic trajectories

Economic interpretation:

- Persistent deviations from target
- Bounded but unpredictable inflation
- Policy challenges in chaotic regime
- Fundamental forecasting limits

Historical Relevance

Consistent with 1970s stagflation and recent inflation episodes

Traditional Policy Failures

Why Standard Approaches Break Down

In chaotic regimes:

- Optimal control theory inadequate
- Small policy changes can have large effects
- Traditional stabilization may destabilize
- Prediction horizons fundamentally limited

Critical threshold:

$$\tau_{critical} \approx \frac{0.5}{\sqrt{\alpha_1 \alpha_2}} \tag{16}$$

Policy Implication

Keep communication delays below critical threshold

Robust Policy Design

Alternative Approaches

New policy strategies:

1. **Communication Policy**
   - Reduce τ_{avg} through clear, frequent communication
   - Minimize information processing delays
2. **Robust Control**
   - Design for multiple parameter regimes
   - Maintain stability margins
3. **Expectation Management**
   - Direct intervention in formation processes
   - Forward guidance strategies
4. **Adaptive Learning**
   - Real-time system identification
   - Continuously updated policy rules

Stability Conditions

Mathematical Foundations for Policy

Stability maintenance requires:

$$\tau_{avg} < \tau_{critical} = \frac{0.5}{\sqrt{\alpha_1 \alpha_2}} \tag{17}$$

Policy levers:

- $\tau_{avg} \downarrow$: Improve communication speed and clarity
- α_1, α_2: Structural economic parameters (harder to influence)

Practical implications:

- Central bank communication technology
- Media strategy optimization
- Forward guidance timing
- Institutional response speeds

Empirical Validation

Stylized Facts Reproduction

Model explains observed phenomena:

- **Inflation persistence**: Chaotic dynamics generate long-memory without external persistence
- **Regime switching**: Natural transitions between volatility periods
- **Policy puzzles**: Apparent ineffectiveness during certain periods
- **Expectation anchoring**: Bounded expectations prevent hyperinflation

Key Insight

Complex dynamics emerge endogenously from interaction structure, not external shocks

Historical Episodes Analysis

Case Studies

1970s Stagflation:

- High τ_{avg} due to poor communication
- Complex transmission mechanisms
- System pushed into chaotic regime
- Standard policies ineffective

2021-2023 Inflation Surge:

- Structural changes in expectation formation
- Social media and algorithmic trading effects
- Increased expectation sensitivity parameters
- Movement toward complex dynamics

Both episodes consistent with delay-induced chaos

Key mathematical contributions:

1. **New DDE Class**
   - Distributed delays with cubic nonlinearities
   - Rigorous existence and uniqueness results
2. **Integrability Analysis**
   - Painlevé analysis for DDEs
   - Connection to differential Galois theory
3. **Infinite-Dimensional Dynamics**
   - Variational equations structure
   - Center manifold theory applications
4. **Computational Methods**
   - Lyapunov spectrum for delay systems
   - Strange attractor reconstruction

Variational equations:

$$\frac{du}{dt} = -\alpha_1 \int_0^{\tau_{max}} G(\tau)u(t-\tau)d\tau + \beta_1 v(t) + 3\gamma_1(\bar{\pi} - \pi_*)^2 u(t) \tag{18}$$

$$\frac{dv}{dt} = \alpha_2 \int_0^{\tau_{max}} H(\tau)[w(t-\tau) + \text{other terms}]d\tau + \beta_2 u(t) \tag{19}$$

$$\frac{dw}{dt} = \text{Complex expectation dynamics} - \lambda w(t) \tag{20}$$

Future research: Galois group analysis for understanding algebraic structure of solutions

Key Contributions Summary
Main Results

1. **Mathematical Framework**
    - Rigorous infinite-dimensional dynamical system
    - Proof of existence and uniqueness
2. **Integrability Obstacles**
    - Demonstrated analytical challenges
    - Connected to differential Galois theory
3. **Chaotic Dynamics**
    - Numerical evidence for chaos
    - Positive Lyapunov exponents
    - Fractal attractors
4. **Economic Applications**
    - Policy design implications
    - Historical episode explanations

Mathematical Extensions:

- Complete differential Galois analysis
- Stochastic extensions with noise-induced transitions
- Multi-dimensional heterogeneous agent models
- Bifurcation theory for infinite-dimensional systems

Economic Applications:

- Econometric detection of chaotic dynamics
- Machine learning for adaptive policy design
- Multi-country models with spillovers
- Financial market integration

Computational Methods:

- Advanced numerical techniques for DDEs
- Parallel algorithms for Lyapunov computation
- Real-time chaos detection systems

Model Limitations

Areas for Improvement

1. **Empirical Validation**
   - Limited real-data testing
   - Parameter estimation challenges
2. **Structural Assumptions**
   - Simplified agent heterogeneity
   - Exogenous growth and policy parameters
3. **Computational Limitations**
   - Finite-dimensional approximations
   - Numerical precision in chaos detection
4. **Policy Implementation**
   - Gap between theory and practice
   - Real-time applicability questions

Broader Implications
Mathematical and Economic Significance

For Mathematics:

- New techniques for analyzing infinite-dimensional systems
- Applications of differential Galois theory to applied problems
- Computational methods for complex DDEs

For Economics:

- Fundamental limits on economic predictability
- New foundations for policy design
- Bridge between rigorous mathematics and economic theory

For Policy:

- Importance of communication technology
- Limits of traditional stabilization policies
- Need for robust, adaptive approaches

Thank You

Questions & Discussion

Contact Information:
Arthur Mota
University of São Paulo
`arthur.lula.mota@gmail.com`

Key Takeaway

Economic systems with realistic bounded rationality and communication delays can exhibit intrinsic chaos, fundamentally altering our understanding of macroeconomic policy effectiveness.