Topic B
Heat flux identification

This study[1] deals with the identification of a heat flux boundary condition for a transient heat transfer problem in a 1D medium.

1: Forward problem

One considers the 1D time-dependent heat equation giving the temperature field $T(x,t)$, resulting from the diffusion of an initial temperature field T_0 in a medium located between the planes of equations $x = 0$ and $x = L$ with heat flux boundary conditions:

$$\begin{gathered}
\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t),\ 0 < x < L,\ 0 \leq t \leq T \\
\frac{\partial T}{\partial x}(0,t) = 0,\ 0 \leq t \leq T \\
\frac{\partial T}{\partial x}(L,t) = q(t),\ 0 \leq t \leq T \\
T(x,0) = T_0,\ 0 \leq x \leq L
\end{gathered}$$

where α is the thermal diffusivity coefficient of the medium, T_0 is a constant temperature assumed to be known, and $q(t)$ is an imposed time-dependent heat flux at $x = L$, verifying $q(0) = 0$.

Adopting a space semi-discretization with N 1D linear elements of identical sizes $h = L/N$, the previous equations can be recast in matrix form as:

$$\mathbb{M}\dot{\Theta}(t) + \mathbb{K}\Theta(t) = \big(0\ \ldots\ 0\ q(t)\big)^{\mathrm{T}}$$

where $\Theta(t) = \big(\theta_0(t)\ \theta_1(t)\ \ldots\ \theta_N(t)\big)^{\mathrm{T}}$ is the vector of nodal relative temperatures ($T_0 + \theta_k$ corresponding to the temperature at node k), and $\mathbb{M}$, $\mathbb{K}$ are $(N+1) \times (N+1)$-matrices given by:

$$\mathbb{M} = \frac{h}{6}\begin{bmatrix} 2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & 4 & 1 & \ddots & & \vdots \\ 0 & 1 & 4 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & 4 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & 2 \end{bmatrix} \text{ and } \mathbb{K} = \frac{\alpha}{h}\begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}$$

This matrix system can be then solved using Euler or Crank-Nicholson schemes with a constant time step $\Delta t = T/M$.

[1] Study initially proposed by Marc Bonnet

2: Proposed identification problem

The aim is to identify the heat flux condition at $x = L$ as $\mathbf{p} = \big(q(\Delta t)\ q(2\Delta t)\ \dots\ q(M\Delta t)\big)^{\mathrm{T}}$, given the time-dependent measured temperatures observed using $N-1$ thermocouples placed at all the inner nodes ($0 < k < N$): $\mathbf{d} = \big(\dots\ \ \theta_k(\Delta t)\ \dots\ \theta_k(M\Delta t = T)\ \dots\big)^{\mathrm{T}}$, where the time series associated with the k-th thermocouple is concatenated with the time series of the other thermocouples.

The following questions should be seen as guidelines rather than an exhaustive list of points to address.

2.1: Build numerically the $(N-1)M \times M$ matrix $\mathbb{G}$ such that the inverse problem consists in finding $\mathbf{p}$ satisfying $\mathbb{G}\mathbf{p} = \mathbf{d}$, and study its properties. Hint: the component G_{kl} of the matrix $\mathbb{G}$ is such that $G_{kl} = \theta_j^{(i)}(l\Delta t)$, where $k = M(j-1) + i$ (for $1 \leq i \leq M$ and $0 < j < N$) and $\Theta^{(i)}(t)$ is the solution of the forward problem for a given flux $(0\ \dots\ 0\ 1\ 0\ \dots\ 0)^{\mathrm{T}}$ (with the 1 at the i-th position).

2.2: Using synthetic data created with the forward problem, propose several methods to solve the inverse problem and compare the associated results.

2.3: Study the influence on the identification results of adding some noise to the synthetic data (test several noise levels). Analyse how some regularization techniques could improve the results.

2.4: Study how/if the choice of the time steps to build the matrix $\mathbb{G}$ (for both synthetic data and forward simulation) modifies the identification results.