1 An online expectation minimization algorithm

Define $\mu = \alpha \exp(f)$, $\nu = \beta \exp(g)$, $x = (\mu, \nu)$. We will change variables without warning in the following. Define the Bregman divergence

$$\begin{aligned}
D_\alpha(\mu, \mu_0) &= \langle \alpha,\, \exp(f_0 - f) - 1 - (f_0 - f) \rangle \\
D_\beta(\nu, \nu_0) &= \langle \beta,\, \exp(g_0 - g) - 1 - (g_0 - g) \rangle \\
D_{\alpha,\beta}(x, x_0) &= D_\alpha(\mu, \mu_0) + D_\beta(\nu, \nu_0)
\end{aligned}$$

We want to solve the objective

$$\min_x \mathcal{F}(x) \triangleq \mathrm{KL}(\alpha, \mu) + \mathrm{KL}(\beta, \nu) + \langle \mu \otimes \nu,\, \exp(-C) \rangle - 1$$

Define the prox objective

$$\begin{aligned}
\mathcal{L}(x, x_t) &= 2\mathcal{F}(x_t) + \langle \nabla \mathcal{F}(x_t),\, x - x_t \rangle + D_{\alpha,\beta}(x, x_t) \\
&= \mathbb{E}_{\hat\alpha \sim \alpha, \hat\beta \sim \alpha} \Big[2F(x_t) + \langle \nabla \mathcal{F}(x_t),\, x - x_t \rangle + D_{\hat\alpha,\hat\beta}(x, x_t) \Big]
\end{aligned}$$

The Sinkhorn iterations then rewrites as

$$x_{t+1} = \operatorname*{argmin}_x \mathbb{E}_{\hat\alpha,\hat\beta} \mathcal{L}_{\hat\alpha,\hat\beta}(x, x_t)$$

and online Sinkhorn

$$x_{t+1} = (1 - \eta_t) x_t + \eta_t \operatorname*{argmin}_x \mathcal{L}_{\hat\alpha_t, \hat\beta_t}(x, x_t)$$

Probably useless ?

2 Variable mirror descent point of view

Consider the objective

$$\max_{f,g} \mathcal{F}(f, g) = \langle \alpha,\, f \rangle + \langle g, \beta,\, - \rangle \langle \alpha \otimes \beta,\, \exp(f \oplus g - C) \rangle + 1$$

The gradient reads

$$\nabla \mathcal{F}(f, g) = \Big(\alpha \big(1 - \exp(f - T_\beta(g)) \big), \beta \big(1 - \exp(g - T_\alpha(f)) \big) \Big) \in \mathcal{M}^+(\mathcal{X}^2)$$

Using the local Bregman divergence

$$\omega_t(f, g) = \langle \alpha,\, \exp(f_t - f) \rangle + \langle \beta,\, \exp(g_t - g) \rangle,$$

online Sinkhorn iterations rewrites as

$$\nabla \omega_t(f_{t+1}, g_{t+1}) = \nabla \omega_t(f_t, g_t) + \eta_t \tilde\nabla \mathcal{F}(f_t, g_t),$$

where

$$\tilde\nabla \mathcal{F}(f, g) = \Big(\hat\alpha_t \big(1 - \exp(f - T_\beta(g)) \big), \hat\beta_t \big(1 - \exp(g - T_\alpha(f)) \big) \Big) \in \mathcal{M}^+(\mathcal{X}^2)$$

is an unbiased estimate of $\nabla \mathcal{F}(f, g)$.

3 An EM point of view

$$f_t, g_t = \operatorname*{argmax}_{f,g} Q_t^\star((f,g),(f_t,g_t)) \triangleq \mathbb{E}_{Y\sim\beta}\left[\mathbb{E}_{X\sim\alpha}\left[f(X) - e^{f(X)+g_t(Y)-C(X,Y)}\right]\right] + \mathbb{E}_{X\sim\alpha}\left[\mathbb{E}_{Y\sim\beta}\left[g(Y) - e^{f_t(X)+g(Y)-C(X,Y)}\right]\right]$$

We now define the approximate functions

$$\begin{aligned}
Q_t((f,g),(f_t,g_t)) &= \mathbb{E}_{Y\sim\hat\beta_t}\left[\mathbb{E}_{X\sim\alpha}\left[f(X) - e^{f(X)+g_t(Y)-C(X,Y)}\right]\right] \\
&+ \mathbb{E}_{X\sim\hat\alpha_t}\left[\mathbb{E}_{Y\sim\beta}\left[g(Y) - e^{f_t(X)+g(Y)-C(X,Y)}\right]\right] \\
&= \mathbb{E}_{X\sim\alpha}[f(X)] + \mathbb{E}_{X\sim\alpha}\left[\sum_{i=n_t}^{n_{t+1}} b_i e^{f(X)+g_t(y_i)-C(X,y_i)}\right] \\
&+ \mathbb{E}_{Y\sim\beta}[g(Y)] + \mathbb{E}_{Y\sim\beta}\left[\sum_{i=n_t}^{n_{t+1}} a_i e^{g(Y)+f_t(x_i)-C(x_i,Y)}\right]
\end{aligned}$$

Running the iterations

$$f_t, g_t = \operatorname*{argmax}_{f,g} Q_t((f,g),(f_t,g_t))$$

amounts to set

$$f_t(\cdot) = -\log \sum_{i=n_t}^{n_{t+1}} b_i e^{g_t(y_i)-C(\cdot,y_i)} \quad g_t(\cdot) = -\log \sum_{i=n_t}^{n_{t+1}} a_i e^{f_t(x_i)-C(x_i,\cdot)},$$

which is the randomized Sinkhorn algorithm. Setting

$$\bar{Q}_{t+1} = (1-\eta_t)\bar{Q}_t + \eta_t Q_t$$

and running the iterations

$$f_t, g_t = \operatorname*{argmin}_{f,g} \bar{Q}_t((f,g),(f_t,g_t))$$

gives online Sinkhorn:

$$f_t(\cdot) = -\log \sum_{i=1}^{n_{t+1}} e^{q_i-C(\cdot,y_i)} \quad g_t(\cdot) = -\log \sum_{i=1}^{n_{t+1}} e^{p_i-C(x_i,\cdot)},$$

with the update rule on q_i, p_i as : see paper. Every function Q_t is parametrized by $(p_i, q_i, x_i, y_i)_{i=(n_t,n_{t+1}]}$, and $\bar{Q}_t$ by $(p_i, q_i, x_i, y_i)_{i=(0,n_{t+1}]}$. Thus the parametrization of f_t, g_t is encoded using an argmax trick, and we recover the structure of a stochastic expectation-maximization algorithm (less the probabilistic point of view).