1 An online expectation minimization algorithm

Define $\mu = \alpha \exp(f)$, $\nu = \beta \exp(g)$, $x = (\mu, \nu)$. We will change variables without warning in the following. Define the Bregman divergence

$$\begin{aligned} D_\alpha(\mu, \mu_0) &= \langle \alpha,\, \exp(f_0 - f) - 1 - (f_0 - f) \rangle \\ D_\beta(\nu, \nu_0) &= \langle \beta,\, \exp(g_0 - g) - 1 - (g_0 - g) \rangle \\ D_{\alpha,\beta}(x, x_0) &= D_\alpha(\mu, \mu_0) + D_\beta(\nu, \nu_0) \end{aligned}$$

We want to solve the objective

$$\min_x \mathcal{F}(x) \triangleq \mathrm{KL}(\alpha, \mu) + \mathrm{KL}(\beta, \nu) + \langle \mu \otimes \nu,\, \exp(-C) \rangle$$

Define the prox objective

$$\begin{aligned} \mathcal{L}(x, x_t) &= 2F(x_t) + \langle \nabla \mathcal{F}(x_t),\, x - x_t \rangle + D_{\alpha,\beta}(x, x_t) \\ &= \mathbb{E}_{\hat\alpha \sim \alpha, \hat\beta \sim \alpha} \Big[2F(x_t) + \langle \nabla \mathcal{F}(x_t),\, x - x_t \rangle + D_{\hat\alpha,\hat\beta}(x, x_t) \Big] \end{aligned}$$

The Sinkhorn iterations then rewrites as

$$x_{t+1} = \operatorname*{argmin}_x \mathbb{E}_{\hat\alpha,\hat\beta} \mathcal{L}_{\hat\alpha,\hat\beta}(x, x_t)$$

and online Sinkhorn

$$x_{t+1} = (1 - \eta_t) x_t + \eta_t \operatorname*{argmin}_x \mathcal{L}_{\hat\alpha_t, \hat\beta_t}(x, x_t)$$

Probably useless ?

2 Variable mirror descent point of view

Consider the objective

$$\max_{f,g} \mathcal{F}(f, g) = \langle \alpha,\, f \rangle + \langle g, \beta,\, - \rangle \langle \alpha \otimes \beta,\, \exp(f \oplus g - C) \rangle + 1$$

The gradient reads

$$\nabla \mathcal{F}(f, g) = \Big(\alpha \big(1 - \exp(f - T_\beta(g))\big), \beta \big(1 - \exp(g - T_\alpha(f))\big) \Big) \in \mathcal{M}^+(\mathcal{X}^2)$$

Using the local Bregman divergence

$$\omega_t(f, g) = \langle \alpha,\, \exp(f_t - f) \rangle + \langle \beta,\, \exp(g_t - g) \rangle,$$

online Sinkhorn iterations rewrites as

$$\nabla \omega_t(f_{t+1}, g_{t+1}) = \nabla \omega_t(f_t, g_t) + \eta_t \tilde\nabla \mathcal{F}(f_t, g_t),$$

where

$$\tilde\nabla \mathcal{F}(f, g) = \Big(\hat\alpha_t \big(1 - \exp(f - T_\beta(g))\big), \hat\beta_t \big(1 - \exp(g - T_\alpha(f))\big) \Big) \in \mathcal{M}^+(\mathcal{X}^2)$$

is an unbiased estimate of $\nabla \mathcal{F}(f, g)$.