1 Online Sinkhorn

The Sinkhorn objective rewrites

$$\max_{f,g\in\mathcal{C}(\mathcal{X})} \langle f,\, \alpha\rangle + \langle g,\, \beta\rangle - \varepsilon\langle \alpha\otimes\beta,\, \exp(-\frac{f\oplus g - C}{\varepsilon})\rangle$$

We perform the following change of variable $\mu = \alpha e^{f/\varepsilon}$, $\nu = \beta e^{g/\varepsilon}$, to obtain the equivalent problem, in $\mathcal{M}^+(\mathcal{X})$

$$\min_{\mu,\nu\in\mathcal{M}^+(\mathcal{X})} \mathrm{KL}(\alpha|\mu) + \mathrm{KL}(\beta|\mu) + \varepsilon\langle \mu\otimes\nu,\, \exp(-\frac{C}{\varepsilon})\rangle \triangleq f(\mu,\nu).$$

The problem is not jointly convex, but convex in μ and ν. We may approach this problem from a game point of view of finding a local Nash equilibrium $(\mu^\star,\nu^\star)$ such that

$$\begin{aligned}\mu^\star &= \operatorname*{argmin}_{\mu\in V(\mu^\star)} \mathcal{F}(\mu,\nu^\star) \\ \nu^\star &= \operatorname*{argmin}_{\nu\in V(\nu^\star)} \mathcal{F}(\mu^\star,\nu),\end{aligned} \tag{1}$$

where V are open sets. Such a formalism is useful as results on mirror descent convergence in multi-agent setting exist for this problem. To solve (1), we need to define distance generating functions to move back and forth from μ and ν and their dual form. We define

$$\begin{aligned}\omega_\alpha(\mu) &\triangleq \mathrm{KL}(\alpha|\mu) \\ \omega_\beta(\nu) &\triangleq \mathrm{KL}(\beta|\nu)\end{aligned}$$

, associated the the mirror maps

$$\begin{aligned}\nabla_\mu\omega_\alpha(\mu) &= -\frac{\mathrm{d}\alpha}{\mathrm{d}\mu} \qquad \big(= -\exp(-f/\varepsilon)\big), \\ \nabla_\nu\omega_\beta(\nu) &= -\frac{\mathrm{d}\beta}{\mathrm{d}\nu} \qquad \big(= -\exp(-g/\varepsilon)\big),\end{aligned}$$

with inverse

$$\begin{aligned}\nabla_\mu\omega^\star_\alpha(p) &= -\frac{\alpha}{p}, \\ \nabla_\nu\omega^\star_\beta(q) &= -\frac{\beta}{q}.\end{aligned}$$

Algorithm. Let us consider the simple simultaneous mirror descent setting, where we build the sequence of iterate $(\mu_t,\nu_t)_t$. It is easy to shows that if we start from $\mu_0 \gg \alpha$ and $\nu_0 \gg \beta$, the iterates will remain absolutely continuous with respect to α and β. We will therefore write $\mu_t = \alpha e^{f_t/\varepsilon}$, $\nu_t = \beta e^{g_t/\varepsilon}$. The mirror descent iterations rewrite (for μ)

$$\mu_{t+1} = \frac{\alpha}{e^{-f_t/\varepsilon} + \eta\nabla_\mu\mathcal{F}(\mu_t,\nu_t)},$$

with $\nabla_\mu\mathcal{F}(\mu_t,\nu_t) = -\exp(-\frac{f_t}{\varepsilon}) + \varepsilon\int_y \exp(\frac{g(y)-C(\cdot,y)}{\varepsilon})\mathrm{d}\beta(y)$. We therefore have the following update rules

$$\begin{aligned}\exp(-\frac{f_{t+1}}{\varepsilon}) &= (1-\eta)\exp(-\frac{f_t}{\varepsilon}) + \eta\mathbb{E}_\beta[\varepsilon\exp(\frac{g_t(y)-C(\cdot,y)}{\varepsilon})], \\ \exp(-\frac{g_{t+1}}{\varepsilon}) &= (1-\eta)\exp(-\frac{g_t}{\varepsilon}) + \eta\mathbb{E}_\alpha[\varepsilon\exp(\frac{f_t(x)-C(x,\cdot)}{\varepsilon})].\end{aligned}$$

Assuming we sample $\hat\beta_t = \sum_{i=1}^n b_{i,t}\delta_{y_{i,t}}$ and $\hat\alpha_t = \sum_{i=1}^n a_{i,t}\delta_{x_{i,t}}$, we can approximate the expectations above, and expect, with decreasing step-sizes to achieve convergence.

Some variants (more likely to converge better) may be considered. The alternated variant writes

$$\exp(-\frac{f_{t+1}}{\varepsilon}) = (1-\eta)\exp(-\frac{f_t}{\varepsilon}) + \eta\mathbb{E}_\beta[\varepsilon\exp(\frac{g_t(y)-C(\cdot,y)}{\varepsilon})],$$
$$\exp(-\frac{g_{t+1}}{\varepsilon}) = (1-\eta)\exp(-\frac{g_t}{\varepsilon}) + \eta\mathbb{E}_\alpha[\varepsilon\exp(\frac{f_{t+1}(x)-C(x,\cdot)}{\varepsilon})],$$

and the extrapolated version

$$\exp(-\frac{f_{t+1/2}}{\varepsilon}) = (1-\eta)\exp(-\frac{f_t}{\varepsilon}) + \eta\mathbb{E}_\beta[\varepsilon\exp(\frac{g_t(y)-C(\cdot,y)}{\varepsilon})],$$
$$\exp(-\frac{g_{t+1/2}}{\varepsilon}) = (1-\eta)\exp(-\frac{g_t}{\varepsilon}) + \eta\mathbb{E}_\alpha[\varepsilon\exp(\frac{f_t(x)-C(x,\cdot)}{\varepsilon})],$$
$$\exp(-\frac{f_{t+1}}{\varepsilon}) = \exp(-\frac{f_t}{\varepsilon}) - \eta\exp(-\frac{g_{t+1/2}}{\varepsilon}) + \eta\mathbb{E}_\beta[\varepsilon\exp(\frac{g_{t+1/2}(y)-C(\cdot,y)}{\varepsilon})],$$
$$\exp(-\frac{g_{t+1}}{\varepsilon}) = \exp(-\frac{g_t}{\varepsilon}) - \eta\exp(-\frac{g_{t+1/2}}{\varepsilon}) + \eta\mathbb{E}_\alpha[\varepsilon\exp(\frac{f_{t+1/2}(x)-C(x,\cdot)}{\varepsilon})].$$

Computations. In the simple simultaneous case, we can track f_t in memory by the following representation

$$f_t(\cdot) = -\varepsilon\log\sum_{s=0}^t w_{t,s}\sum_{j=1}^n b_{s,j}\exp\Big(g_{s-1}(y_{s,j}) - \frac{C(\cdot,y_{s,j})}{\varepsilon}\Big)$$
$$g_t(\cdot) = -\varepsilon\log\sum_{s=0}^t w_{t,s}\sum_{i=1}^n a_{s,i}\exp\Big(f_{s-1}(x_{s,i}) - \frac{C(x_{s,i},\cdot)}{\varepsilon}\Big),$$

with $w_{t,s} = \eta(1-\eta)^{t-s}$ for $1 \leqslant s \leqslant t$, $w_{t,0} = (1-\eta)^t$, and we set $g_{-1} = f_{-1} = 0$. The weights are a bit more complex is η depends on time.

The alternated version sets

$$g_t(\cdot) = -\varepsilon\log\sum_{s=0}^t w_{t,s}\sum_{i=1}^n a_{s,i}\exp\Big(f_s(x_{s,i}) - \frac{C(x_{s,i},\cdot)}{\varepsilon}\Big),$$

Setting $q_{t,s,i} = w_{t,s}b_{s,j}\exp(g_{s-1}(y_{s,j}))$ and $p_{t,s,i} = w_{t,s}b_{s,j}\exp(f_{s-1}(x_{s,j})/\varepsilon)$, we can derive simple update rules for p and q:

$$p_{t,t,i} = \eta b_{t,j}\exp(f_{t-1}(x_{t,j})), \qquad \forall\, s < t, \quad p_{t,s,i} = (1-\eta)p_{t-1,s,i}$$

2 Analysis

Bregman divergence associated to φ (désolé j'ai changé de notation).

$$d_\varphi(f_2|f_1) = \langle\alpha, \exp(\frac{f_2-f_1}{\varepsilon}) - 1 - \frac{f_2-f_1}{\varepsilon}\rangle \geqslant \langle\alpha, (\frac{f_2-f_1}{2\varepsilon})^2\rangle$$

For convergence of MD on $\min f(x)$ with mirror map φ, we need to show, according to Gabriel, Jalal and Kelvin

$$\mu d_\varphi(x_2|x_1) \leqslant d_f(x_2|x_1) \leqslant L d_\varphi(x_2|x_1).$$

Can we use that here ? Beware that we are in an alternated setting

2.1 Sketch of proof

See proof of Th2 in Ya Ping's paper.

2.1.1 General proof

By using the dual iteration and the three point property (normally holds by def of D_α and D_β):

$$\begin{aligned}\langle \mu_t - \mu, -\nabla_\mu \mathcal{F}(\mu_t, \nu_t)\rangle &= \frac{1}{\eta}\langle \mu_t - \mu, \nabla_\mu w_\alpha(\mu_{t+1}) - \nabla_\mu w_\alpha(\mu_t)\rangle \\ &= \frac{1}{\eta}[D_{w_\alpha}(\mu, \mu_t) - D_{w_\alpha}(\mu, \mu_{t+1}) + D_{w_\alpha}(\mu_t, \mu_{t+1})]\end{aligned}$$

Suppose we can show (TO DO):

$$D_{w_\alpha}(\mu_t, \mu_{t+1}) \leq \eta^2 M^2$$

Then we have:

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\langle \mu_t - \mu, -\nabla_\mu \mathcal{F}(\mu_t, \nu_t)\rangle &= \sum_{t=1}^{T}\frac{1}{\eta}[D_{w_\alpha}(\mu, \mu_t) - D_{w_\alpha}(\mu, \mu_{t+1}) + D_{w_\alpha}(\mu_t, \mu_{t+1})] \\ &\leq \frac{D_{w_\alpha}(\mu, \mu_1)}{\eta} + \eta M^2 T\end{aligned}$$

Similarly:

$$\frac{1}{T}\sum_{t=1}^{T}\langle \nu_t - \nu, -\nabla_\nu \mathcal{F}(\mu_t, \nu_t)\rangle \leq \frac{D_{w_\beta}(\mu, \mu_1)}{\eta} + \eta M^2 T$$

Summing up the two previous equations and replacing (μ, ν) by $(\mu*, \nu*)$, we get:

$$\frac{1}{T}\sum_{t=1}^{T}\langle \mu_t - \mu*, -\nabla_\mu \mathcal{F}(\mu_t, \nu_t)\rangle + \langle \nu_t - \nu*, -\nabla_\nu \mathcal{F}(\mu_t, \nu_t)\rangle \leq \frac{D_0}{\eta} + 2\eta M^2 T$$

where $D_0 = D_{w_\alpha}(\mu^*, \mu_1) + D_{w_\beta}(\nu^*, \nu_1)$.

Then, by optimality of μ^* and convexity of $\mathcal{F}$:

$$\mathcal{F}(\mu_t, \nu_t) - \mathcal{F}(\mu^*, \nu^*) \leq \mathcal{F}(\mu_t, \nu_t) - \mathcal{F}(\mu^*, \nu_t) \leq \langle \mu^* - \mu_t, \nabla_\mu \mathcal{F}(\mu_t, \nu_t)\rangle = \langle \mu_t - \mu^*, -\nabla_\mu \mathcal{F}(\mu_t, \nu_t)\rangle$$

Hence:

$$\frac{1}{T}\sum_{t=1}^{T}\mathcal{F}(\mu_t, \nu_t) - \mathcal{F}(\mu^*, \nu^*) \leq \frac{D_0}{\eta} + 2\eta^2 M^2 T$$

2.1.2 Tricky part

Now let's try to prove Equation 2.1.1. What we need is

- $D_{w_\alpha}(\mu, \nu) = D_{w_\alpha^*}(\nabla_\mu w_\alpha(\mu), \nabla_\mu w_\alpha(\nu))$ (eq A.13 in Ya Ping's paper)
- *relative smoothness of* w_α^* *wrt* $\|.\|_\infty$ (eq A.11 and A.12 in Ya Ping's paper)

If it's true:

$$\begin{aligned}D_{w_\alpha}(\mu_t, \mu_{t+1}) = D_{w_\alpha^*}(\nabla_\mu w_\alpha(\mu_t), \nabla_\mu w_\alpha(\mu_{t+1})) &\leq \|\nabla_\mu w_\alpha(\mu) - \nabla_\mu w_\alpha(\nu)\|_\infty^2 \\ &= \|\exp(-\frac{f_{t+1}}{\varepsilon}) - \exp(-\frac{f_t}{\varepsilon})\|_\infty^2 = \eta^2 \|\nabla \mathcal{F}_\mu(\mu_t, \nu_t)\|_\infty^2\end{aligned}$$

and

$$\|\nabla \mathcal{F}_\mu(\mu_t, \nu_t)\|_\infty^2 \leq ?$$

WIP

$$D_{w_\alpha}(\mu_t, \mu_{t+1}) \geqslant \| \log \frac{\mathrm{d}\mu_{t+1}}{\mathrm{d}\mu_t} \|_\alpha^2$$

We can show

$$\begin{aligned} D_{\omega_\alpha}(\mu_t|\mu_{t+1}) = \eta^2 \langle \alpha, \, 1 - \exp(\frac{f_t - \hat{f}_{t+1}}{\varepsilon}) \rangle, = \eta^2 (1 - \langle \alpha, \, \nabla_\mu \mathcal{F}(\mu_t, \nu_t) \rangle) \\ \hat{f}_{t+1}(\cdot) = -\varepsilon \log \int_y \exp(\frac{g_t(y) - C(\cdot, y)}{\varepsilon}) \mathrm{d}\beta(y). \end{aligned}$$

Avec Sinkhorn sans bruit $f_t - \hat{f}_{t+1}$ va rester tranquille.

3 Proof of convergence

We want to solve

$$\min_{\mu \in \mathcal{M}^+(\mathcal{X}), \nu \in \mathcal{M}^+(\mathcal{X})} F(\mu, \nu) \triangleq \mathrm{KL}(\alpha|\mu) + \mathrm{KL}(\beta|\nu) + \langle \mu \otimes \nu, \, \exp(-C) \rangle$$

Let's write $x = (\mu, \nu)$ and $F(\mu, \nu) = F(x)$ the objective. We define the iterates $x_t = (\mu_t, \nu_t)$, $x_{t+1/2} = (\mu_{t+1}, \nu_t)$, $x_t = (\mu_{t+1}, \nu_{t+1})$. First note that we have

$$D_{F(\mu, \cdot)} = D_{\omega_\alpha(\cdot)} \qquad D_{F(\cdot, \nu)} = D_{w_\beta(\cdot)},$$

so that at every iteration, we perform a mirror step with a function that is both 1-relatively smooth and 1-relatively strongly convex.

Let ν be fixed, and let us define $F_\nu(\cdot) = F(\cdot, \nu)$. From the smoothness of $F_\nu(\cdot)$ and from its convexity we have, for all $\mu_x, \mu_y, \mu_z \gg \alpha$,

$$\begin{aligned} F(\mu_x, \nu) &\leqslant F(\mu_y, \nu) + \langle \nabla_\mu F(\mu_y, \nu), \, \mu_x - \mu_y \rangle + L D_{\omega_\alpha}(\mu_x, \mu_y), \\ F(\mu_y, \nu) &\leqslant F(\mu_z, \nu) + \langle \nabla_\mu F(\mu_y, \nu), \, \mu_y - \mu_z \rangle \end{aligned}$$

Combining both, we obtain

$$\begin{aligned} F(\mu_x, \nu) \leqslant F(\mu_z, \nu) + \langle \nabla_\mu F(\mu_y, \nu), \, \mu_x - \mu_z \rangle + L D_{\omega_\alpha}(\mu_x, \mu_y) \\ \langle \nabla F(\mu_y, \nu), \, \mu_x - \mu_z \rangle \geqslant F(\mu_x, \nu) - F(\mu_z, \nu) - L D_{\omega_\alpha}(\mu_x, \mu_y). \end{aligned}$$

We now use the three point propery:

$$D_{\omega_\alpha}(\mu_z, \mu_y) - D_{\omega_\alpha}(\mu_z, \mu_x) - D_{\omega_\alpha}(\mu_x, \mu_y) = \langle \nabla \omega_\alpha(\mu_x) - \nabla \omega_\alpha(\mu_y), \, \mu_z - \mu_x \rangle,$$

replacing $\mu_y = \mu_k, \mu_x = \mu_{k+1}, \nu = \nu_k$, we obtain from the definition of the gradient update

$$\begin{aligned} D_{\omega_\alpha}(\mu_z, \mu_k) - D_{\omega_\alpha}(\mu_z, \mu_{k+1}) - D_{\omega_\alpha}(\mu_{k+1}, \mu_k) &= \eta_k \langle \nabla_\mu F(\mu_k, \nu_k), \, \mu_{k+1} - \mu_z \rangle + \eta_k \langle \xi_k, \, \mu_{k+1} - \mu_z \rangle \\ &\geqslant \eta_k (F(\mu_{k+1}, \nu_k) - F(\mu_z, \nu_k))) - L \eta_k D_{\omega_\alpha}(\mu_{k+1}, \mu_k) \\ &\quad + \eta_k \langle \xi_k, \, \mu_{k+1} - \mu_z \rangle. \end{aligned}$$

Hence, mimicking the derivation for ν

$$\begin{aligned} \eta_k (F(\mu_{k+1}, \nu_{k+1}) - F(\mu_{k+1}, \nu_z)) \leqslant D_{\omega_\beta}(\nu_z, \nu_k) - D_{\omega_\beta}(\nu_z, \nu_{k+1}) - (1 - \eta_k L) D_{\omega_\beta}(\nu_{k+1}, \nu_k). \\ \eta_k (F(\mu_{k+1}, \nu_k) - F(\mu_z, \nu_k)) \leqslant D_{\omega_\alpha}(\mu_z, \mu_k) - D_{\omega_\alpha}(\mu_z, \mu_{k+1}) - (1 - \eta_k L) D_{\omega_\alpha}(\mu_{k+1}, \mu_k) \end{aligned}$$

Setting $\mu_z = \mu_k$, $\nu_z = \nu_k$, we obtain a descent lemma.

$$F(\mu_{k+1}, \nu_{k+1}) \leqslant F(\mu_{k+1}, \nu_k) \leqslant F(\mu_k, \nu_k),$$

which ensures almost sure convergence of $F(\mu_k, \nu_k)$ to $F^\star$ using constant step-sizes (gradient is zero for $k \to \infty$).

We further have, replacing $\mu_z = \mu^\star, \nu_z = \nu^\star$

$$F(\mu_{k+1}, \nu_{k+1}) - F^\star - \frac{\sum_{k=1}^k \eta_k (F(\mu_{k+1}, \nu^\star) + F(\mu^\star, \nu_k) - 2F^\star))}{2\sum_{k=1}^k \eta_k} \leqslant \frac{D_{\omega_\alpha}(\mu^\star, \mu_1) + D_{\omega_\beta}(\nu^\star, \nu_1)}{2\sum_{k=1}^K \eta_k}$$

Now note that

$$(F(\mu_{k+1}, \nu^\star) + F(\mu^\star, \nu_k) - 2F^\star) = D_{\omega_\alpha}(T(g_k, \beta), f^\star) + D_{\omega_\beta}(T(f_{k+1}, \alpha), g^\star)$$

We may show the contractance of the soft c-tranform for the following metric

$$\varphi(f, g) = \min_{f^\star, g^\star \in \mathcal{S}} D_{\omega_\alpha}(f, f^\star) + D_{\omega_\beta}(g, g^\star).$$

Namely, if

$$\varphi(T(f, \alpha), T(g, \beta)) \leqslant \varphi(f, g).$$

It is then easy to show (convexity argument) that

$$\varphi(f_{t+1}, g_{t+1}) \leqslant (1-\eta)\varphi(f_t, g_t) + \eta\varphi(T(f_t, \alpha), T(g_t, \beta))$$

What can be shown is unfortunately

$$D_{\omega_\alpha}(T(g, \beta), f^\star) + D_{\omega_\beta}(T(f, \alpha), g^\star). \leqslant D_{\omega_\alpha}(f^\star, f) + D_{\omega_\beta}(g^\star, g).$$

Simultaneous gradient descent. We have

$$\begin{aligned}
\eta_k(F(\mu_k, \nu_{k+1}) - F(\mu_k, \nu_z)) &\leqslant D_{\omega_\beta}(\nu_z, \nu_k) - D_{\omega_\beta}(\nu_z, \nu_{k+1}) - (1-\eta_k L) D_{\omega_\beta}(\nu_{k+1}, \nu_k). \\
\eta_k(F(\mu_{k+1}, \nu_k) - F(\mu_z, \nu_k)) &\leqslant D_{\omega_\alpha}(\mu_z, \mu_k) - D_{\omega_\alpha}(\mu_z, \mu_{k+1}) - (1-\eta_k L) D_{\omega_\alpha}(\mu_{k+1}, \mu_k)
\end{aligned}$$

Therefore

$$\begin{aligned}
\eta_k \big(F(\mu_k, \nu_{k+1}) &+ F(\mu_{k+1}, \nu_k) - 2F(\mu_z, \nu_z) \\
&-(F(\mu_k, \nu_z) + F(\mu_z, \nu_k) - 2F(\mu_z, \nu_z) \big) \leqslant D_{\omega_\beta}(\nu_z, \nu_k) + D_{\omega_\alpha}(\mu_z, \mu_k) \\
&\qquad - \big(D_{\omega_\alpha}(\mu_z, \mu_{k+1}) + D_{\omega_\beta}(\nu_z, \nu_{k+1}) \big) \\
&\qquad - (1-\eta_k) \big(D_{\omega_\beta}(\nu_{k+1}, \nu_k) + D_{\omega_\alpha}(\mu_{k+1}, \mu_k) \big)
\end{aligned}$$

Let's observe that, for all $(\mu^\star, \nu^\star) \in \mathcal{S}$, (μ, ν)

$$\begin{aligned}
D_{\omega_\beta}(\nu^\star, \nu) + D_{\omega_\alpha}(\mu^\star, \mu) &= \mathrm{KL}(\alpha|\mu) + \mathrm{KL}(\beta|\nu) + \big(\langle \mu \otimes \nu^\star, \exp(-C) \rangle + \langle \mu^\star \otimes \nu, \exp(-C) \rangle - 2 \big) - F(\mu^\star, \nu^\star) \\
&= F(\mu, \nu^\star) + F(\mu^\star, \nu) - 2F(\mu^\star, \nu^\star)
\end{aligned}$$

We take $(\mu_z, \nu_z) = (\mu^\star, \nu^\star)$, that optimizes

$$\begin{aligned}
G(\mu_k, \nu_k) &\triangleq \min_{\mu^\star, \nu^\star} D_{\omega_\beta}(\nu^\star, \nu_k) + D_{\omega_\alpha}(\mu^\star, \mu_k) \\
&= \mathrm{KL}(\alpha|\mu) + \mathrm{KL}(\beta|\nu) + 2\big(\sqrt{\langle \mu^\star \otimes \nu, \exp(-C) \rangle \langle \mu \otimes \nu^\star, \exp(-C) \rangle} - 1) - F^\star
\end{aligned}$$

Then

$$\begin{aligned}
\eta_k \big(F(\mu_k, \nu_{k+1}) + F(\mu_{k+1}, \nu_k) - 2F^\star) &\leqslant (1+\eta_k) G(\mu_k, \nu_k) - G(\mu_{k+1}, \nu_{k+1}) \\
&\quad - (1-\eta_k) \big(D_{\omega_\beta}(\nu_{k+1}, \nu_k) + D_{\omega_\alpha}(\mu_{k+1}, \mu_k) \big)
\end{aligned}$$

Note that, using $\sqrt{ab} \geqslant \frac{a+b}{2}$

$$F(\mu_k, \nu_{k+1}) + F(\mu_{k+1}, \nu_k) - G(\mu_k, \nu_k) - 2F^\star \geqslant \mathrm{KL}(\alpha|\mu_{k+1}) + \mathrm{KL}(\beta|\nu_{k+1}) - F^\star + 2u_k,$$

where

$$u_k \triangleq \frac{\langle \mu_k \otimes (\nu_{k+1} - \nu^\star),\, \exp(-C)\rangle + \langle (\mu_{k+1} - \mu^\star) \otimes \nu_k,\, \exp(-C)\rangle}{2}$$

Now let's observe that the harmonic mean is always smaller than the arithmetic mean:

$$\frac{\mathrm{d}\mu_{k+1}}{\mathrm{d}\alpha} = \frac{1}{(1-\eta_k)\frac{1}{\frac{\mathrm{d}\mu_k}{\mathrm{d}\alpha}} + \eta_k \frac{1}{\frac{\mathrm{d}\alpha \exp(T(g_k,\beta))}{\mathrm{d}\alpha}}} \leqslant (1-\eta_k)\frac{\mathrm{d}\mu_k}{\mathrm{d}\alpha} + \eta_k \frac{\mathrm{d}\alpha \exp(T(g_k,\beta))}{\mathrm{d}\alpha},$$

hence

$$\begin{aligned} \mu_{k+1} &\leqslant (1-\eta_k)\mu_k + \eta_k T(\nu_k, \beta) \\ \nu_{k+1} &\leqslant (1-\eta_k)\nu_k + \eta_k T(\mu_k, \beta) \end{aligned}$$

Therefore, from the definition of the c-transform

$$u_{k+1} \leqslant (1-\eta_k) u_k$$

Finally

$$\eta_k(\mathrm{KL}(\alpha|\mu_{k+1}) + \mathrm{KL}(\beta|\nu_{k+1}) - F^\star) \leqslant G(\mu_k, \nu_k) - G(\mu_{k+1}, \nu_{k+1}) - 2\eta_k \mu_k,$$

hence convergence !

4 Proofs

4.1 A simple "local" convergence proof in the non-noisy case

Remark that

$$\begin{aligned} f_{k+1} &= -\log(\exp(-f_k)(1-\eta_k) + \eta_k \exp(-T(g_k, \beta))) \\ g_{k+1} &= -\log(\exp(-g_k)(1-\eta_k) + \eta_k \exp(-T(f_{k+1}, \alpha))) \end{aligned}$$

Let $(f^\star, g^\star)$ be a coupl of solution. There exists $x \in \mathcal{X}$ such that $\|f_{k+1} - f^\star\|_{\mathrm{var}} = |f_{k+1}(x) - f^\star(x)|$. For this x, using the convexity of $-\log(x)$ (more or less the mirror map),

$$\begin{aligned} \|f_{k+1} - f^\star\|_{\mathrm{var}} &= |-\log\big((1-\eta_k)\exp(f^\star - f_k(x)) + \eta_k \exp(f^\star - T(g_k,\beta)(x)\big)| \\ &\leqslant (1-\eta_k)|f_k(x) - f^\star(x)| + \eta_k |T(g_k,\beta)(x) - f^\star(x)| \\ &\leqslant (1-\eta_k)\|f_k - f^\star\|_{\mathrm{var}} + \eta_k \|T(g_k,\beta) - T(g^\star,\beta)\|_{\mathrm{var}} \\ &\leqslant (1-\eta_k)\|f_k - f^\star\|_{\mathrm{var}} + \eta_k \kappa \|g_k - g^\star\|_{\mathrm{var}} \end{aligned}$$

Similarly

$$\|g_{k+1} - f^\star\|_{\mathrm{var}} \leqslant (1-\eta_k)\|g_k - g^\star\|_{\mathrm{var}} + \eta_k \kappa \|f_{k+1} - f^\star\|_{\mathrm{var}}$$

Therefore

$$\|g_{k+1} - g^\star\|_{\mathrm{var}} + \|f_{k+1} - f^\star\|_{\mathrm{var}} \leqslant (1 - \eta_k + \kappa^2 \eta_k^2))(\|f_k - f^\star\|_{\mathrm{var}} + \|g_k - g^\star\|_{\mathrm{var}}),$$

and we still have convergence as long as $\sum \eta_k = \infty$. This shows that

$$\frac{f_t + T(g_t, \beta)}{2}, \frac{g_t + T(f_t, \alpha)}{2} \to f^\star, g^\star.$$

4.2 An adapted mirror descent convergence proof in the non-noisy case

We want to solve

$$\min_{\mu\in\mathcal{M}^+(\mathcal{X}),\nu\in\mathcal{M}^+(\mathcal{X})} F(\mu,\nu) \triangleq \mathrm{KL}(\alpha|\mu)+\mathrm{KL}(\beta|\nu)+\langle\mu\otimes\nu,\,\exp(-C)\rangle$$

First note that we have

$$D_{F(\mu,\cdot)}=D_{\omega_\alpha(\cdot)} \qquad D_{F(\cdot,\nu)}=D_{w_\beta(\cdot)},$$

so that at every iteration, we perform a mirror step with a function that is 1-relatively smooth.

Let ν be fixed, and let us define $F_\nu(\cdot)=F(\cdot,\nu)$. From the relative smoothness of $F_\nu(\cdot)$ and from its convexity we have, for all $\mu_x,\mu_y,\mu_z\gg\alpha$,

$$\begin{aligned}
F(\mu_x,\nu)&\leqslant F(\mu_y,\nu)+\langle\nabla_\mu F(\mu_y,\nu),\,\mu_x-\mu_y\rangle+D_{\omega_\alpha}(\mu_x,\mu_y),\\
F(\mu_y,\nu)&\leqslant F(\mu_z,\nu)+\langle\nabla_\mu F(\mu_y,\nu),\,\mu_y-\mu_z\rangle
\end{aligned}$$

Combining both, we obtain

$$\begin{aligned}
F(\mu_x,\nu)\leqslant F(\mu_z,\nu)+\langle\nabla_\mu F(\mu_y,\nu),\,\mu_x-\mu_z\rangle+D_{\omega_\alpha}(\mu_x,\mu_y)\\
\langle\nabla F(\mu_y,\nu),\,\mu_x-\mu_z\rangle\geqslant F(\mu_x,\nu)-F(\mu_z,\nu)-D_{\omega_\alpha}(\mu_x,\mu_y).
\end{aligned}$$

We now use the three point propery:

$$D_{\omega_\alpha}(\mu_z,\mu_y)-D_{\omega_\alpha}(\mu_z,\mu_x)-D_{\omega_\alpha}(\mu_x,\mu_y)=\langle\nabla\omega_\alpha(\mu_x)-\nabla\omega_\alpha(\mu_y),\,\mu_z-\mu_x\rangle,$$

replacing $\mu_y=\mu_k,\mu_x=\mu_{k+1},\nu=\nu_k$, we obtain

$$\begin{aligned}
D_{\omega_\alpha}(\mu_z,\mu_k)-D_{\omega_\alpha}(\mu_z,\mu_{k+1})-D_{\omega_\alpha}(\mu_{k+1},\mu_k)&=\eta_k\langle\nabla_\mu F(\mu_k,\nu_k),\,\mu_{k+1}-\mu_z\rangle\\
&\geqslant\eta_k(F(\mu_{k+1},\nu_k)-F(\mu_z,\nu_k)))-\eta_k D_{\omega_\alpha}(\mu_{k+1},\mu_k).
\end{aligned}$$

Hence, mimicking the derivation for ν,

$$\begin{aligned}
\eta_k(F(\mu_{k+1},\nu_k)-F(\mu_z,\nu_k))&\leqslant D_{\omega_\alpha}(\mu_z,\mu_k)-D_{\omega_\alpha}(\mu_z,\mu_{k+1})-(1-\eta_k)D_{\omega_\alpha}(\mu_{k+1},\mu_k) \qquad (2)\\
\eta_k(F(\mu_{k+1},\nu_{k+1})-F(\mu_{k+1},\nu_z))&\leqslant D_{\omega_\beta}(\nu_z,\nu_k)-D_{\omega_\beta}(\nu_z,\nu_{k+1})-(1-\eta_k)D_{\omega_\beta}(\nu_{k+1},\nu_k)
\end{aligned}$$

Setting $\mu_z=\mu_k$, $\nu_z=\nu_k$, we obtain a descent lemma.

$$F(\mu_{k+1},\nu_{k+1})\leqslant F(\mu_{k+1},\nu_k)\leqslant F(\mu_k,\nu_k),$$

Summing both equations of (2), we obtain

$$\begin{aligned}
&\eta_k(F(\mu_{k+1},\nu_k)+F(\mu_{k+1},\nu_{k+1})-(F(\mu_z,\nu_k)+F(\mu_{k+1},\nu_z)))\\
&\leqslant D_{\omega_\alpha}(\mu_z,\mu_k)+D_{\omega_\beta}(\nu_z,\nu_k)-(D_{\omega_\alpha}(\mu_z,\mu_{k+1})+D_{\omega_\beta}(\nu_z,\nu_{k+1}))
\end{aligned}$$

For $k\in\mathbb{N}$, we set $(\mu_z,\nu_z)=(\mu_k^\star,\nu_k^\star)$, such that

$$(\mu_k^\star,\nu_k^\star)\triangleq\operatorname*{argmin}_{\mu^\star,\nu^\star} D_{\omega_\beta}(\nu^\star,\nu_k)+D_{\omega_\alpha}(\mu^\star,\mu_k),$$

and define

$$\begin{aligned}
G(\mu_k,\nu_k)&\triangleq\min_{\mu^\star,\nu^\star} D_{\omega_\beta}(\nu^\star,\nu_k)+D_{\omega_\alpha}(\mu^\star,\mu_k)\\
&=\mathrm{KL}(\alpha|\mu_k)+\mathrm{KL}(\beta|\nu_k)+2\big(\sqrt{\langle\mu^\star\otimes\nu,\,\exp(-C)\rangle\langle\mu\otimes\nu^\star,\,\exp(-C)\rangle}-1\big)-F^\star.
\end{aligned}$$

We obtain

$$\eta_k((F(\mu_{k+1},\nu_{k+1})-F^\star\leqslant G(\mu_k,\nu_k)-G(\mu_{k+1},\nu_{k+1})+\eta_k w_k+\eta_k z_k,$$

where

$$\begin{aligned}
z_k&=1-\langle\mu_{k+1}\otimes\nu_k,\,\exp(-C)\rangle\\
w_k&=\langle\mu_k^\star\otimes\nu_k,\,\exp(-C)\rangle+\langle\mu_{k+1}\otimes\nu_k^\star,\,\exp(-C)\rangle-2\\
&=(\langle\mu_{k+1}\otimes\nu^\star,\,\exp(-C)\rangle\langle\mu^\star\otimes\nu_k,\,\exp(-C)\rangle)^{1/2}\left(\left(\frac{\langle\mu_k\otimes\nu^\star,\,\exp(-C)\rangle}{\langle\mu_{k+1}\otimes\nu^\star,\,\exp(-C)\rangle}\right)^{1/2}+\left(\frac{\langle\mu_{k+1}\otimes\nu^\star,\,\exp(-C)\rangle}{\langle\mu_k\otimes\nu^\star,\,\exp(-C)\rangle}\right)^{1/2}\right)-2
\end{aligned}$$

The last term is quite ugly, due to the alternated nature of the algorithm.

Simultaenous updates. In the non alternated version:

$$
\begin{aligned}
&\eta_k(F(\mu_{k+1}, \nu_k) + F(\mu_{k+1}, \nu_k) - (F(\mu_z, \nu_k) + F(\mu_k, \nu_z))) \\
&\leqslant D_{\omega_\alpha}(\mu_z, \mu_k) + D_{\omega_\beta}(\nu_z, \nu_k) - (D_{\omega_\alpha}(\mu_z, \mu_{k+1}) + D_{\omega_\beta}(\nu_z, \nu_{k+1}))
\end{aligned}
$$