

Evolutionary Large-Scale Global Optimization

An Introduction

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Outline

- 1 Introduction: Large Scale Global Optimization
- 2 Approaches to Large-Scale Optimization
- 3 Variable Interaction: Definitions and Importance
- 4 Interaction Learning: Exploiting Modularity
- 5 Conclusion
- 6 Questions

Optimization

$$\min f(\mathbf{x}), \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n \quad (1)$$

$$s.t. : \mathbf{g}(\mathbf{x}) \leq 0 \quad (2)$$

$$\mathbf{h}(\mathbf{x}) = 0 \quad (3)$$

Can be converted to unconstrained optimization using:

- Penalty method;
- Lagrangian;
- Augmented Lagrangian.

Our focus is unconstrained optimization. We must learn how to walk before we can run.

Large Scale Global Optimization (LSGO)

How large is large?

- The notation of large-scale is not fix.
- Changes over time.
- Differs from problem to problem.
- **The dimension at which existing methods start to fail.**

State-of-the-art (EC)

- Binary: ≈ 1 billion $[a]$.
- Integer (linear): ≈ 1 billion $[b]$, $[c]$.
- Real: ≈ 1000 -5000.

[a] Kumara Sastry, David E Goldberg, and Xavier Llorca. "Towards billion-bit optimization via a parallel estimation of distribution algorithm". In: *Genetic and Evolutionary Computation Conference*. ACM. 2007, pp. 577–584.

[b] Kalyanmoy Deb and Christie Myburgh. "Breaking the Billion-Variable Barrier in Real-World Optimization Using a Customized Evolutionary Algorithm". In: *Genetic and Evolutionary Computation Conference*. ACM. 2016, pp. 653–660.

[c] Kalyanmoy Deb and Christie Myburgh. "A population-based fast algorithm for a billion-dimensional resource allocation problem with integer variables". In: *European Journal of Operational Research* 261.2 (2017), pp. 460–474.

Large Scale Global Optimization: Applications

Why large-scale optimization is important?

- Growing applications in various fields.
 - ▶ Target shape design optimization [a].
 - ▶ Satellite layout design [b].
 - ▶ Parameter estimation in large scale biological systems [c].
 - ▶ Seismic waveform inversion [d].
 - ▶ Parameter calibration of water distribution systems [e].
 - ▶ Vehicle routing [f].

[a] Zhenyu Yang et al. "Target shape design optimization by evolving B-splines with cooperative coevolution". In: *Applied Soft Computing* 48 (Nov. 2016), pp. 672–682.

[b] Hong-Fei Teng et al. "A dual-system variable-grain cooperative coevolutionary algorithm: satellite-module layout design". In: *IEEE transactions on evolutionary computation* 14.3 (Dec. 2010), pp. 438–455.

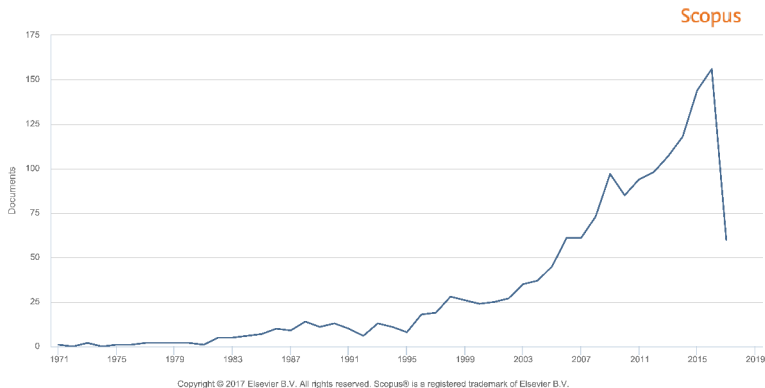
[c] Shuhei Kimura et al. "Inference of S-system models of genetic networks using a cooperative coevolutionary algorithm". In: *Bioinformatics* 21.7 (Apr. 2005), pp. 1154–1163.

[d] Chao Wang and Jinghui Gao. "High-dimensional waveform inversion with cooperative coevolutionary differential evolution algorithm". In: *IEEE Geoscience and Remote Sensing Letters* 9.2 (Mar. 2012), pp. 297–301.

[e] Yu Wang et al. "Two-stage based ensemble optimization framework for large-scale global optimization". In: *European Journal of Operational Research* 228.2 (2013), pp. 308–320.

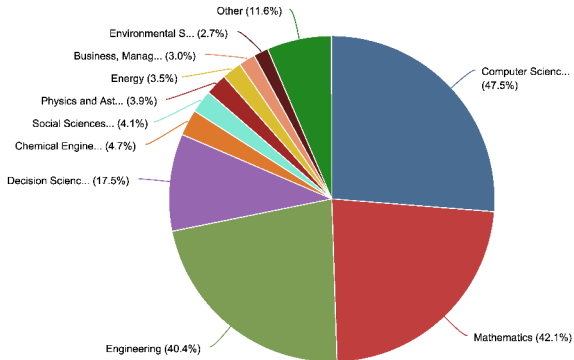
[f] Yi Mei, Xiaodong Li, and Xin Yao. "Cooperative coevolution with route distance grouping for large-scale capacitated arc routing problems". In: *IEEE Transactions on Evolutionary Computation* 18.3 (2014), pp. 435–449.

Large Scale Global Optimization: Research



Large Scale Global Optimization: Research

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The Challenge of Large Scale Optimization

Why is it difficult?

- Exponential growth in the size of search space (**curse of dimensionality**).

Research Goal

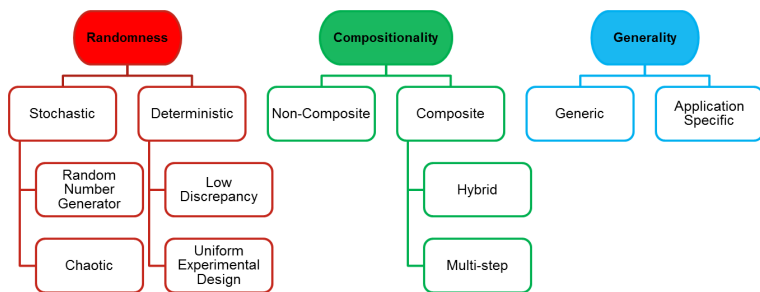
- Improving search quality (get to the optimal point).
- Improving search efficiency (get there fast).

Large Scale Global Optimization: Evolutionary Approaches

- 1 Initialization
- 2 Sampling and Variation Operators
- 3 Approximation and Surrogate Modeling
- 4 Local Search and Memetic Algorithms
- 5 Decomposition and Divide-and-Conquer
- 6 Parallelization (GPU, CPU)
- 7 Hybridization

Initialization Methods

- Study the importance of initialization methods [1] in large-scale optimization.



[1] Borhan Kazimipour, Xiaodong Li, and A Kai Qin. "A review of population initialization techniques for evolutionary algorithms". In: *Evolutionary Computation (CEC), 2014 IEEE Congress on*. IEEE. 2014, pp. 2585–2592.

Initialization Methods

- Inconclusive evidence for or against initialization methods:
 - ▶ Uniform design works worse than RNG, while good-lattice point and opposition-based methods perform better [1].
 - ▶ Another study showed that population size has a more significant effect than the initialization [2].
 - ▶ Achieving uniformity is difficult in high-dimensional spaces [3].
 - ▶ Yet another study suggest comparing average performances may not reveal the effect of initialization [4].
- Shortcomings:
 - ▶ It is difficult to isolate the effect of initialization.
 - ▶ Different effect on different algorithms (mostly tested on DE).
 - ▶ Numerous parameters to study.

[1] Borhan Kazimipour, Xiaodong Li, and A Kai Qin. "Initialization methods for large scale global optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2013, pp. 2750–2757.

[2] Borhan Kazimipour, Xiaodong Li, and A Kai Qin. "Effects of population initialization on differential evolution for large scale optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2014, pp. 2404–2411.

[3] Borhan Kazimipour, Xiaodong Li, and A Kai Qin. "Why advanced population initialization techniques perform poorly in high dimension?" In: *SEAL*. 2014, pp. 479–490.

[4] Eduardo Segredo et al. "On the comparison of initialisation strategies in differential evolution for large scale optimisation". In: *Optimization Letters* (2017), pp. 1–14.

Sampling and Variation Operators

- Opposition-based sampling [1]
- Center-based sampling [2].
- Quantum-behaved particle swarm [3].
- Competitive Swarm Optimizer [4].
- Social learning PSO [5].
- Mutation operators [6], [7].

[1] Hui Wang, Zhijian Wu, and Shahryar Rahnamayan. "Enhanced opposition-based differential evolution for solving high-dimensional continuous optimization problems". In: *Soft Computing* 15.11 (2011), pp. 2127–2140.

[2] Sedigheh Mahdavi, Shahryar Rahnamayan, and Kalyanmoy Deb. "Center-based initialization of cooperative co-evolutionary algorithm for large-scale optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2016, pp. 3557–3565.

[3] Deyu Tang et al. "A quantum-behaved particle swarm optimization with memetic algorithm and memory for continuous non-linear large scale problems". In: *Information Sciences* 289 (2014), pp. 162–189.

[4] Ran Cheng and Yaochu Jin. "A competitive swarm optimizer for large scale optimization". In: *IEEE Transactions on Cybernetics* 45.2 (2015), pp. 191–204.

[5] Ran Cheng and Yaochu Jin. "A social learning particle swarm optimization algorithm for scalable optimization". In: *Information Sciences* 291 (2015), pp. 43–60.

[6] Hongwei Ge et al. "Cooperative differential evolution with fast variable interdependence learning and cross-cluster mutation". In: *Applied Soft Computing* 36 (2015), pp. 300–314.

[7] Ali Wagdy Mohamed and Abdulaziz S Almazayad. "Differential Evolution with Novel Mutation and Adaptive Crossover Strategies for Solving Large Scale Global Optimization Problems". In: *Applied Computational Intelligence and Soft Computing* 2017 (2017).

Approximation Methods and Surrogate Modeling

- High-Dimensional Model Representation (HDMR) [1].
- Radial Basis Functions [2].
- Kriging and Gradient-Enhanced Kriging Metamodels [3].
- Piecewise Polynomial (Spline) [4].

[1] Enying Li, Hu Wang, and Fan Ye. "Two-level Multi-surrogate Assisted Optimization method for high dimensional nonlinear problems". In: *Applied Soft Computing* 46 (2016), pp. 26–36.

[2] Rommel G Regis. "Evolutionary programming for high-dimensional constrained expensive black-box optimization using radial basis functions". In: *IEEE Transactions on Evolutionary Computation* 18.3 (2014), pp. 326–347.

[3] Selvakumar Ulaganathan et al. "A hybrid sequential sampling based metamodeling approach for high dimensional problems". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2016, pp. 1917–1923.

[4] Zhenyu Yang et al. "Target shape design optimization by evolving B-splines with cooperative coevolution". In: *Applied Soft Computing* 48 (Nov. 2016), pp. 672–682.

Local Search and Memetic Algorithms

- Multiple Trajectory Search (MTS) [1].
- Memetic algorithm with local search chaining [2].
 - ▶ MA-SW-Chains [3].
 - ▶ MA-SSW-Chains [4].
- Multiple offspring sampling (MOS) [5], [6].

[1] Lin-Yu Tseng and Chun Chen. "Multiple trajectory search for large scale global optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2008, pp. 3052–3059.

[2] Daniel Molina, Manuel Lozano, and Francisco Herrera. "Memetic algorithm with local search chaining for large scale continuous optimization problems". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2009, pp. 830–837.

[3] Daniel Molina, Manuel Lozano, and Francisco Herrera. "MA-SW-Chains: Memetic algorithm based on local search chains for large scale continuous global optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2010, pp. 1–8.

[4] Daniel Molina et al. "Memetic algorithms based on local search chains for large scale continuous optimisation problems: MA-SSW-Chains". In: *Soft Computing* 15.11 (2011), pp. 2201–2220.

[5] Antonio LaTorre, Santiago Muelas, and José-María Peña. "Multiple offspring sampling in large scale global optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2012, pp. 1–8.

[6] Antonio LaTorre, Santiago Muelas, and José-María Peña. "A MOS-based dynamic memetic differential evolution algorithm for continuous optimization: a scalability test". In: *Soft Computing* 15.11 (2011), pp. 2187–2199.

Parallelization

- Algorithms capable of parallelization [1], [2].
- GPU [3], [4].
- CPU/OpenMP [5].

[1] Jing Tang, Meng Hiot Lim, and Yew Soon Ong. "Diversity-adaptive parallel memetic algorithm for solving large scale combinatorial optimization problems". In: *Soft Computing* 11.9 (2007), pp. 873–888.

[2] Hui Wang, Shahryar Rahnamayan, and Zhijian Wu. "Parallel differential evolution with self-adapting control parameters and generalized opposition-based learning for solving high-dimensional optimization problems". In: *Journal of Parallel and Distributed Computing* 73.1 (2013), pp. 62–73.

[3] Kumara Sastry, David E Goldberg, and Xavier Llorca. "Towards billion-bit optimization via a parallel estimation of distribution algorithm". In: *Genetic and Evolutionary Computation Conference*. ACM. 2007, pp. 577–584.

[4] Alberto Cano and Carlos García-Martínez. "100 Million dimensions large-scale global optimization using distributed GPU computing". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2016, pp. 3566–3573.

[5] AJ Umbarkar. "OpenMP Genetic Algorithm for Continuous Nonlinear Large-Scale Optimization Problems". In: *International Conference on Soft Computing for Problem Solving*. Springer. 2016, pp. 203–214.

Hybridization (The best of both worlds)

- Rationale: benefiting from unique features of different optimizers.
 - ▶ EDA+DE: [1].
 - ▶ PSO+ABC: [2].
 - ▶ Different DE variants: JADE+SaNSDE [3].
 - ▶ PSO+ACO [4].
 - ▶ Minimum Population Search+CMA-ES [5].

[1] Yu Wang, Bin Li, and Thomas Weise. "Estimation of distribution and differential evolution cooperation for large scale economic load dispatch optimization of power systems". In: *Information Sciences* 180.12 (2010), pp. 2405–2420.

[2] LN Vitorino, SF Ribeiro, and Carmelo JA Bastos-Filho. "A hybrid swarm intelligence optimizer based on particles and artificial bees for high-dimensional search spaces". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2012, pp. 1–6.

[3] Sishi Ye et al. "A hybrid adaptive coevolutionary differential evolution algorithm for large-scale optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2014, pp. 1277–1284.

[4] Wu Deng et al. "A novel two-stage hybrid swarm intelligence optimization algorithm and application". In: *Soft Computing* 16.10 (2012), pp. 1707–1722.

[5] Antonio Bolufé-Röhler, Sonia Fiol-González, and Stephen Chen. "A minimum population search hybrid for large scale global optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2015, pp. 1958–1965.

Decomposition Methods

- Divide-and-conquer
- Dimensionality reduction

Variable Interaction, Linkage, Epistasis

What is variable interaction?

- Genetics: two genes are said to interact with each other if they collectively represent a feature at the phenotype level.
- The extent to which the fitness of one gene can be suppressed by another gene.
- The extent to which the value taken by one gene activates or deactivates the effect of another gene.

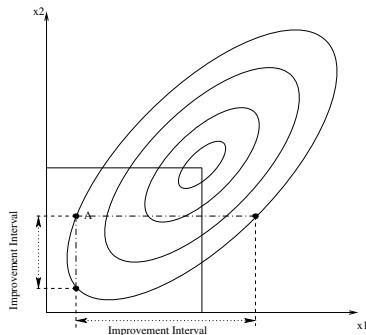
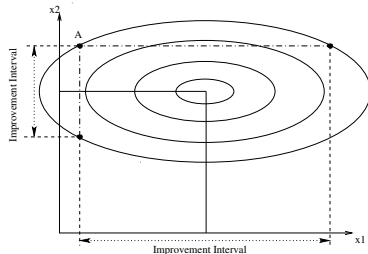
Why variable interaction?

- The effectiveness of optimization algorithms is affected by how much they respect variable interaction.
- Also applies to classic mathematical programming methods.

Variable Interaction, Linkage, Epistasis

Illustrative Example

- $f(x, y) = x^2 + \lambda_1 y^2$
- $g(x, y) = x^2 + \lambda_1 y^2 + \lambda_2 xy$



Definitions

Variable Interaction

A variable x_i is separable or does not interact with any other variable iff:

$$\arg \min_{\mathbf{x}} f(\mathbf{x}) = \left(\arg \min_{x_i} f(\mathbf{x}), \arg \min_{\forall x_j, j \neq i} f(\mathbf{x}) \right),$$

where $\mathbf{x} = (x_1, \dots, x_n)^\top$ is a decision vector of n dimensions.

Partial Separability

A function $f(\mathbf{x})$ is partially separable with m independent subcomponents iff:

$$\arg \min_{\mathbf{x}} f(\mathbf{x}) = \left(\arg \min_{\mathbf{x}_1} f(\mathbf{x}_1, \dots), \dots, \arg \min_{\mathbf{x}_m} f(\dots, \mathbf{x}_m) \right),$$

$\mathbf{x}_1, \dots, \mathbf{x}_m$ are disjoint sub-vectors of \mathbf{x} , and $2 \leq m \leq n$.

Note: a function is fully separable if sub-vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are 1-dimensional (i.e., $m = n$).

Definitions

Full Nonseparability

A function $f(\mathbf{x})$ is fully non-separable if every pair of its decision variables interact with each other.

Additive Separability

A function is *partially additively separable* if it has the following general form:

$$f(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}_i) ,$$

where \mathbf{x}_i are mutually exclusive decision vectors of f_i , $\mathbf{x} = (x_1, \dots, x_n)^\top$ is a global decision vector of n dimensions, and m is the number of independent subcomponents.

Effect of Variable Interaction

Sampling and Variation Operators:

- GAs: inversion operator to promote tight linkage [1].
 - ▶ Increasing the likelihood of placing **linked genes close to each other to avoid disruption by crossover.**
 - ▶ Rotation of the landscape has a detrimental effect on GA [2].
- The need for rotationally invariance:
 - ▶ Model Building Methods:
 - ★ Estimation of Distribution Algorithms and Evolutionary Strategies: Covariance Matrix Adaptation.
 - ★ Bayesian Optimization: Bayesian Networks.
 - ▶ DE's crossover is not rotationally invariant.
 - ▶ PSO is also affected by rotation [3].

[1] David E Goldberg, Robert Lingle, et al. "Alleles, loci, and the traveling salesman problem". In: *International Conference on Genetic Algorithms and Their Applications*. Vol. 154. 1985, pp. 154–159.

[2] Ralf Salomon. "Re-evaluating genetic algorithm performance under coordinate rotation of benchmark functions. A survey of some theoretical and practical aspects of genetic algorithms". In: *BioSystems* 39.3 (1996), pp. 263–278.

[3] Daniel N Wilke, Schalk Kok, and Albert A Groenwold. "Comparison of linear and classical velocity update rules in particle swarm optimization: Notes on scale and frame invariance". In: *International journal for numerical methods in engineering* 70.8 (2007), pp. 985–1008.

Effect of Variable Interaction

- ① Approximation and Surrogate Modelling:
 - ▶ Should be able to capture variable interaction.
 - ▶ Second order terms of HDMR.
- ② Local Search and Memetic Algorithms:
 - ▶ What subset of variables should be optimized in each iteration of local search?
 - ▶ Coordinate-wise search may not be effective. Memetics perform well on separable functions! A coincidence?!
- ③ Decomposition and Divide-and-Conquer:
 - ▶ Interacting variables should be placed in the same component.

Linkage Learning and Exploiting Modularity

- Implicit Methods:

- ▶ In EC:

- ★ Estimation of Distribution Algorithms
 - ★ Bayesian Optimization: BOA, hBOA, Linkage Trees
 - ★ Adaptive Encoding, CMA-ES

- ▶ Classic Optimization:

- ★ Quasi-Newton Methods: Approximation of the Hessian.

- Explicit Methods:

- ▶ In EC:

- ★ Random Grouping
 - ★ Statistical Correlation-Based Methods
 - ★ Delta Grouping
 - ★ Meta Modelling
 - ★ Monotonicity Checking
 - ★ Differential Grouping

- ▶ Classic Optimization

- ★ Block Coordinate Descent
 - ★ Adaptive Coordinate Descent

Implicit Methods

- Scaling Up EDAs:
 - ▶ Model Complexity Control [1].
 - ▶ Random Matrix Projection [2].
 - ▶ Use of mutual information [3].
 - ▶ Cauchy-EDA [4].
- Scaling up CMA-ES:
 - ▶ CC-CMA-ES [5].
 - ▶ LM-CMA [6].

[1] Weishan Dong et al. "Scaling up estimation of distribution algorithms for continuous optimization". In: *IEEE Transactions on Evolutionary Computation* 17.6 (2013), pp. 797–822.

[2] Ata Kabán, Jakramate Bootkrajang, and Robert John Durrant. "Toward large-scale continuous EDA: A random matrix theory perspective". In: *Evolutionary Computation* 24.2 (2016), pp. 255–291.

[3] Qi Xu, Momodou L Sanyang, and Ata Kabán. "Large scale continuous EDA using mutual information". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2016, pp. 3718–3725.

[4] Momodou L Sanyang, Robert J Durrant, and Ata Kabán. "How effective is Cauchy-EDA in high dimensions?" In: *IEEE Congress on Evolutionary Computation*. IEEE. 2016, pp. 3409–3416.

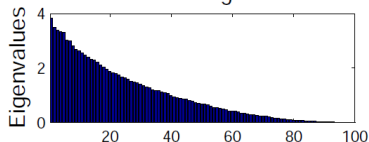
[5] Jinpeng Liu and Ke Tang. "Scaling up covariance matrix adaptation evolution strategy using cooperative coevolution". In: *International Conference on Intelligent Data Engineering and Automated Learning*. Springer. 2013, pp. 350–357.

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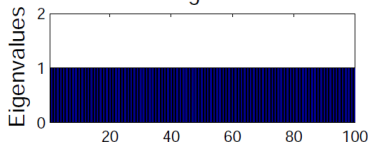
Scalability issues of EDAs

- Accurate estimation requires a large sample size which grows exponentially with the dimensionality of the problem [1].
- A small sample results in poor estimation of the eigenvalues [2].
- The cost of sampling from a multi-dimensional Gaussian distribution increases cubically with the problem size [3].

Maximum Likelihood eigenvalue estimates



True eigenvalues

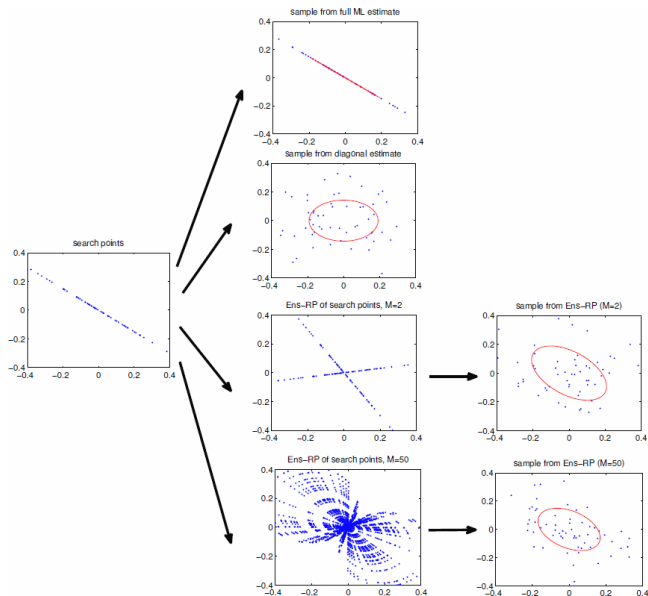


[1] Jerome Friedman, Trevor Hastie, and Robert Tibshirani. *The elements of statistical learning*. Vol. 1. Springer series in statistics Springer, Berlin, 2001.

[2] Roman Vershynin. "Introduction to the non-asymptotic analysis of random matrices". In: *arXiv preprint arXiv:1011.3027* (2010).

[3] Weishan Dong and Xin Yao. "Unified eigen analysis on multivariate Gaussian based estimation of distribution algorithms". In: *Information Sciences* 178.15 (2008), pp. 3000–3023.

Random Projection EDA



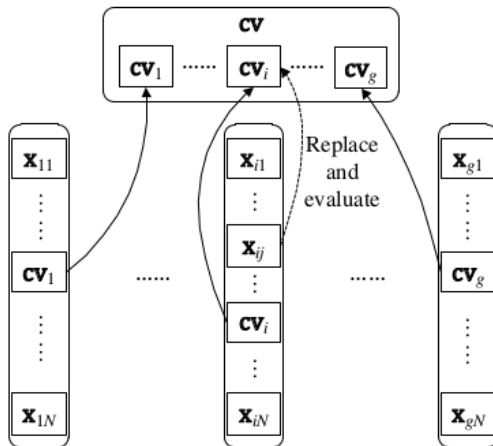
Explicit Methods



- A large problem can be subdivided into smaller and simpler problems.
- Dates back to René Descartes (*Discourse on Method*).
- Has been widely used in many areas:
 - ▶ Computer Science: Sorting algorithms (quick sort, merge sort)
 - ▶ Optimization: Large-scale linear programs (Dantzig)
 - ▶ Politics: Divide and rule (In *Perpetual Peace* by Immanuel Kant: *Divide et impera* is the third political maxims.)

Acknowledgement: the above image is obtained from: <http://draininbrain.blogspot.com.au/>

Decomposition in EAs: Cooperative Co-evolution [1]



[1] Mitchell A. Potter and Kenneth A. De Jong. "A cooperative coevolutionary approach to function optimization". In: *Proc. Int. Conf. Parallel Problem Solving from Nature*. Vol. 2. 1994, pp. 249–257.

CC is a Framework

CC as a scalability agent:

- CC is not an optimizer.
- Requires a component optimizer.
- CC **coordinates** how the component optimizer is applied to components.
- A scalability agent.

Challenges of CC

Main Questions

- 1 How to decompose the problem?
- 2 How to allocated resources?
- 3 How to coordinate?

The Decomposition Challenge

How to decompose?

- There are many possibilities.
- Which decomposition is the best?

Optimal decomposition

- It is governed by the **interaction structure** of decision variables.
- An optimal decomposition is the one that minimizes the interaction between components.

Survey of Decomposition Methods

- Uninformed Decomposition [1]
 - ▶ n 1-dimensional components (the original CC)
 - ▶ k s -dimensional components.
 - Random Grouping [2]
 - Statistical Correlation-Based Methods
 - Delta Grouping [3]
 - Meta Modelling [4]
 - Monotonicity Checking [5]
 - Differential Grouping [6]
-

[1] F. van den Bergh and Andries P Engelbrecht. "A cooperative approach to particle swarm optimization". In: *IEEE Transactions on Evolutionary Computation* 2.3 (June 2004), pp. 225–239.

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[4] Sedigheh Mahdavi, Mohammad Ebrahim Shiri, and Shahryar Rahnamayan. "Cooperative co-evolution with a new decomposition method for large-scale optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2014, pp. 1285–1292.

[5] Wenxiang Chen et al. "Large-scale global optimization using cooperative coevolution with variable interaction learning". In: *Parallel Problem Solving from Nature*. Springer. 2010, pp. 300–309.

[6] Mohammad Nabi Omidvar et al. "Cooperative co-evolution with differential grouping for large scale optimization". In: *IEEE Transactions on Evolutionary Computation* 18.3 (2014), pp. 378–393.

Illustrative Example (Canonical CC)

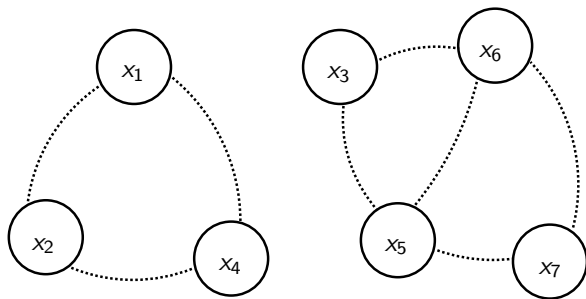


Figure: Variable interaction of a hypothetical function.

- n 1-dimensional components:

- ▶ $C_1: \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}$
- ▶ $C_2: \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}$
- ▶ ...
- ▶ $C_c: \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_6\}, \{x_7\}$

Illustrative Example (fixed k s -dimensional)

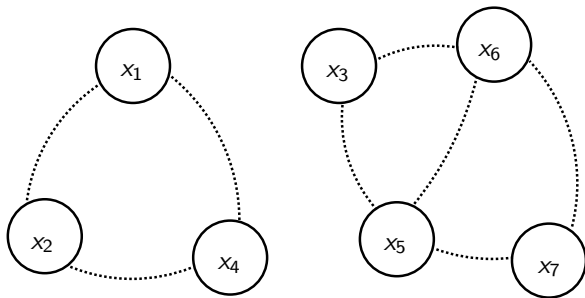


Figure: Variable interaction of a hypothetical function.

- k s -dimensional ($k = 2, s = 4$):
 - ▶ $C_1: \{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7\}$
 - ▶ $C_2: \{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7\}$
 - ▶ ...
 - ▶ $C_c: \{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7\}$

Illustrative Example (Random Grouping)

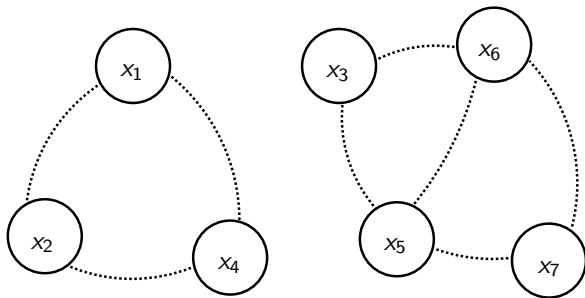


Figure: Variable interaction of a hypothetical function.

- Random Grouping ($k = 2, s = 4$):

- ▶ C_1 : $\{x_2, x_3, x_6, x_5\}, \{x_7, x_1, x_4\}$
- ▶ C_2 : $\{x_3, x_4, x_1, x_2\}, \{x_6, x_7, x_5\}$
- ▶ ...
- ▶ C_c : $\{x_1, x_5, x_6, x_7\}, \{x_2, x_4, x_3\}$

Random Grouping

Theorem

Given N cycles, the probability of assigning v interacting variables x_1, x_2, \dots, x_v into one subcomponent for at least k cycles is:

$$P(X \geq k) = \sum_{r=k}^N \binom{N}{r} \left(\frac{1}{m^{v-1}} \right)^r \left(1 - \frac{1}{m^{v-1}} \right)^{N-r} \quad (4)$$

where N is the number of cycles, v is the total number of interacting variables, m is the number of subcomponents, and the random variable X is the number of times that v interacting variables are grouped in one subcomponent.

Random Grouping

Example

Given $n = 1000$, $m = 10$, $N = 50$ and $v = 4$, we have:

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \left(1 - \frac{1}{10^3}\right)^{50} = 0.0488$$

which means that over 50 cycles, the probability of assigning 4 interacting variables into one subcomponent for at least 1 cycle is only 0.0488. As we can see this probability is very small, and it will be even less if there are more interacting variables.

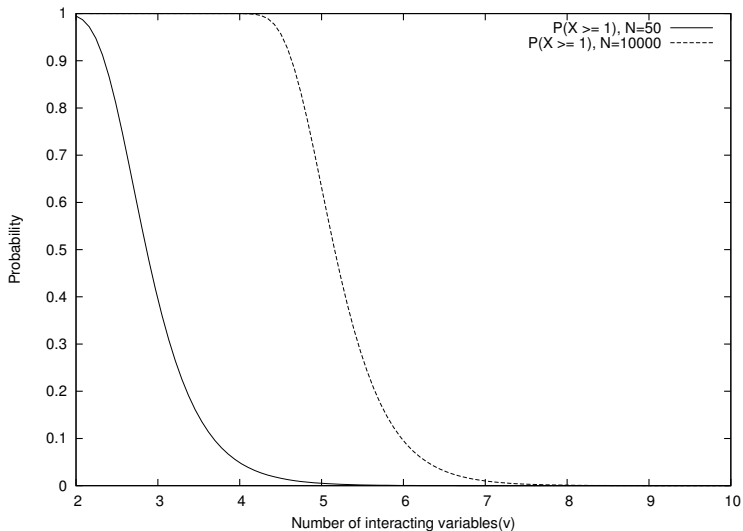


Figure: Increasing v , the number of interacting variables will significantly decrease the probability of grouping them in one subcomponent, given $n = 1000$ and $m = 10$.

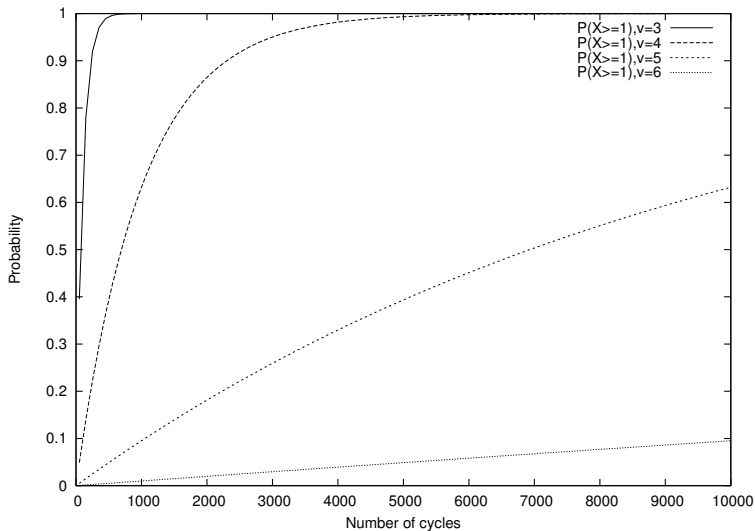


Figure: Increasing N , the number of cycle increases the probability of grouping v number of interacting variables in one subcomponent.

Illustrative Example (Informed with Fixed Groups)

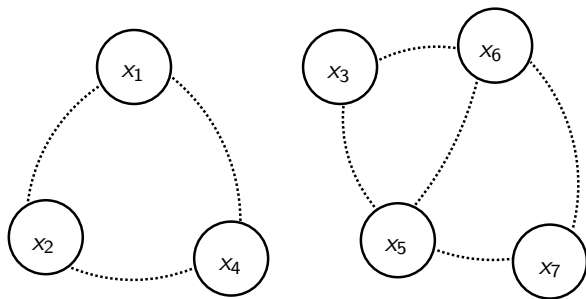
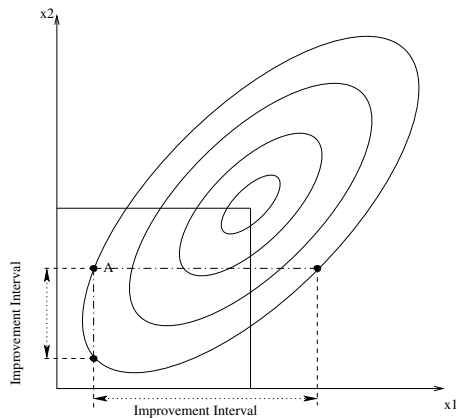
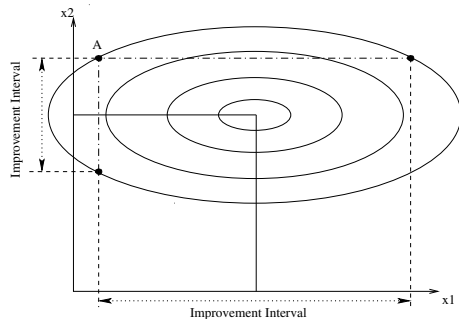


Figure: Variable interaction of a hypothetical function.

- Delta Grouping ($k = 2, s = 4$):

- ▶ $C_1: \{x_1, x_5, x_2, x_4\}, \{x_3, x_6, x_7\}$
- ▶ $C_2: \{x_3, x_5, x_6, x_7\}, \{x_1, x_2, x_4\}$
- ▶ ...
- ▶ $C_c: \{x_3, x_6, x_1, x_4\}, \{x_2, x_5, x_7\}$

Delta Grouping



Infomred Decompositions with Fixed Groups

- Adaptive Variable Partitioning [1].
- Delta Grouping [2].
- Min/Max-Variance Decomposition (MiVD/MaVD) [3].
 - ▶ Sorts the dimensions based on the diagonal elements of the covariance matrix in CMA-ES.
- Fitness Difference Partitioning [4], [5], [6].

[1] Tapabrata Ray and Xin Yao. "A cooperative coevolutionary algorithm with correlation based adaptive variable partitioning". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2009, pp. 983–989.

[2] Mohammad Nabi Omidvar, Xiaodong Li, and Xin Yao. "Cooperative co-evolution with delta grouping for large scale non-separable function optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2010, pp. 1–8.

[3] Jinpeng Liu and Ke Tang. "Scaling up covariance matrix adaptation evolution strategy using cooperative coevolution". In: *International Conference on Intelligent Data Engineering and Automated Learning*. Springer. 2013, pp. 350–357.

[4] Eman Sayed, Daryl Essam, and Ruhul Sarker. "Dependency identification technique for large scale optimization problems". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2012, pp. 1–8.

[5] Eman Sayed et al. "Decomposition-based evolutionary algorithm for large scale constrained problems". In: *Information Sciences* 316 (2015), pp. 457–486.

[6] Adan E Aguilar-Justo and Efrén Mezura-Montes. "Towards an improvement of variable interaction identification for large-scale constrained problems". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2016, pp. 4167–4174.

Infomred Decompositions with Variable Groups

- Multilevel Grouping: MLCC [1], MLSoft [2].
- Adaptive Variable Partitioning 2 [3].
- 4CDE [4].
- Fuzzy Clustering [5].

[1] Zhenyu Yang, Ke Tang, and Xin Yao. "Multilevel cooperative coevolution for large scale optimization". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2008, pp. 1663–1670.

[2] Mohammad Nabi Omidvar, Yi Mei, and Xiaodong Li. "Effective decomposition of large-scale separable continuous functions for cooperative co-evolutionary algorithms". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2014, pp. 1305–1312.

[3] Hemant Kumar Singh and Tapabrata Ray. "Divide and conquer in coevolution: A difficult balancing act". In: *Agent-Based Evolutionary Search*. Springer, 2010, pp. 117–138.

[4] Yazmin Rojas and Ricardo Landa. "Towards the use of statistical information and differential evolution for large scale global optimization". In: *International Conference on Electrical Engineering Computing Science and Automatic Control*. IEEE. 2011, pp. 1–6.

[5] Jianchao Fan, Jun Wang, and Min Han. "Cooperative coevolution for large-scale optimization based on kernel fuzzy clustering and variable trust region methods". In: *IEEE Transactions on Fuzzy Systems* 22.4 (2014), pp. 829–839.

Illustrative Example (Exact Methods)

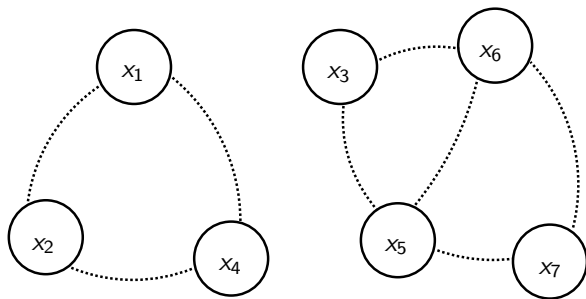


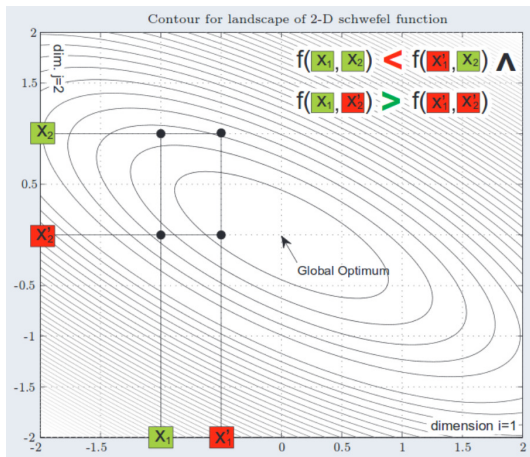
Figure: Variable interaction of a hypothetical function.

- Differential Grouping and Variable Interaction Learning:

- ▶ $C_1: \{x_1, x_2, x_4\}, \{x_3, x_5, x_6, x_7\}$
- ▶ $C_2: \{x_1, x_2, x_4\}, \{x_3, x_5, x_6, x_7\}$
- ▶ ...
- ▶ $C_c: \{x_1, x_2, x_4\}, \{x_3, x_5, x_6, x_7\}$

Monotonicity Check

$$\exists \mathbf{x}, \mathbf{x}'_i, \mathbf{x}'_j : f(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_j, \dots, \mathbf{x}_n) < f(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_j, \dots, \mathbf{x}_n) \wedge \\ f(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_n) > f(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}'_j, \dots, \mathbf{x}_n)$$



Monotonicity Check (Algorithms)

- Linkage Identification by Non-Monotonicity Detection [1]
- Adaptive Coevolutionary Learning [2]
- Variable Interaction Learning [3]
- Variable Interdependence Learning [4]
- Fast Variable Interdependence [5]

[1] Masaharu Munetomo and David E Goldberg. "Linkage identification by non-monotonicity detection for overlapping functions". In: *Evolutionary Computation* 7.4 (1999), pp. 377–398.

[2] Karsten Weicker and Nicole Weicker. "On the improvement of coevolutionary optimizers by learning variable interdependencies". In: *IEEE Congress on Evolutionary Computation*. Vol. 3. IEEE. 1999, pp. 1627–1632.

[3] Wenxiang Chen et al. "Large-scale global optimization using cooperative coevolution with variable interaction learning". In: *Parallel Problem Solving from Nature*. Springer. 2010, pp. 300–309.

[4] Liang Sun et al. "A cooperative particle swarm optimizer with statistical variable interdependence learning". In: *Information Sciences* 186.1 (2012), pp. 20–39.

[5] Hongwei Ge et al. "Cooperative differential evolution with fast variable interdependence learning and cross-cluster mutation". In: *Applied Soft Computing* 36 (2015), pp. 300–314.

Differential Grouping [1]

Theorem

Let $f(\mathbf{x})$ be an additively separable function. $\forall a, b_1 \neq b_2, \delta \in \mathbb{R}, \delta \neq 0$, if the following condition holds

$$\Delta_{\delta, x_p}[f](\mathbf{x})|_{x_p=a, x_q=b_1} \neq \Delta_{\delta, x_p}[f](\mathbf{x})|_{x_p=a, x_q=b_2}, \quad (5)$$

then x_p and x_q are non-separable, where

$$\Delta_{\delta, x_p}[f](\mathbf{x}) = f(\dots, x_p + \delta, \dots) - f(\dots, x_p, \dots), \quad (6)$$

refers to the forward difference of f with respect to variable x_p with interval δ .

[1] Mohammad Nabi Omidvar et al. "Cooperative co-evolution with differential grouping for large scale optimization". In: *IEEE Transactions on Evolutionary Computation* 18.3 (2014), pp. 378–393.

Separability $\Rightarrow \Delta_1 = \Delta_2$

Assuming:

$$f(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x}_i)$$

We prove that:

$$\text{Separability} \Rightarrow \Delta_1 = \Delta_2$$

By contraposition ($P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$):

$$\Delta_1 \neq \Delta_2 \Rightarrow \text{non-separability}$$

or

$$|\Delta_1 - \Delta_2| > \epsilon \Rightarrow \text{non-separability}$$

Deductive Reasoning

Strong Syllogism



$A \Rightarrow B$
 A is true
 $\therefore B$ is true



$A \Rightarrow B$
 B is false
 $\therefore A$ is false

Weak Syllogism



$A \Rightarrow B$
 A is false
 $\therefore B$ is less plausible



$A \Rightarrow B$
 B is true
 $\therefore A$ is more plausible

Deductive Reasoning - Example

Strong Syllogism



Rain \Rightarrow Cloud

It is rainy

\therefore It is cloudy



Rain \Rightarrow Cloud

It is not cloudy

\therefore It is not rainy

Weak Syllogism



Rain \Rightarrow Cloud

It is not rainy

\therefore Cloud becomes less likely



Rain \Rightarrow Cloud

It is cloudy

\therefore Rain becomes more likely

The Differential Grouping Algorithm

Detecting Non-separable Variables

$$|\Delta_1 - \Delta_2| > \epsilon \Rightarrow \text{non-separability}$$

Detecting Separable Variables

$$|\Delta_1 - \Delta_2| \leq \epsilon \Rightarrow \text{Separability (more plausible)}$$

Example

Consider the non-separable objective function $f(x_1, x_2) = x_1^2 + \lambda x_1 x_2 + x_2^2$, $\lambda \neq 0$.

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = 2x_1 + \lambda x_2.$$

This clearly shows that the change in the global objective function with respect to x_1 is a function of x_1 and x_2 . By applying the Theorem:

$$\begin{aligned}\Delta_{\delta, x_1}[f] &= [(x_1 + \delta)^2 + \lambda(x_1 + \delta)x_2 + x_2^2] - [x_1^2 + \lambda x_1 x_2 + x_2^2] \\ &= \boxed{\delta^2 + 2\delta x_1 + \lambda x_2 \delta.}\end{aligned}$$

Differential Grouping vs CCVIL

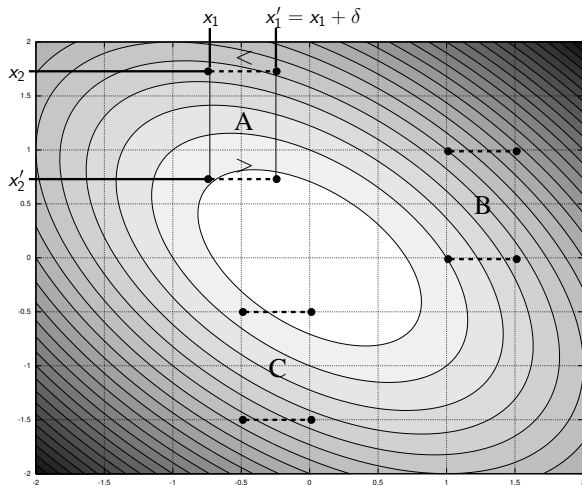


Figure: Detection of interacting variables using differential grouping and CCVIL on different regions of a 2D Schwefel Problem 1.2.

Differential Grouping Family of Algorithms

- Linkage Identification by Non-linearity Check (LINC, LINC-R) [1]
- Differential Grouping (DG) [2]
- Global Differential Grouping (GDG) [3]
- Improved Differential Grouping (IDG) [4]
- eXtended Differential Grouping (XDG) [5]
- Graph-based Differential Grouping (gDG) [6]
- Fast Interaction Identification [7]

-
- [1] Masaru Tezuka, Masaharu Munetomo, and Kiyoshi Akama. "Linkage identification by nonlinearity check for real-coded genetic algorithms". In: *Genetic and Evolutionary Computation—GECCO 2004*. Springer. 2004, pp. 222–233.
- [2] Mohammad Nabi Omidvar et al. "Cooperative co-evolution with differential grouping for large scale optimization". In: *IEEE Transactions on Evolutionary Computation* 18.3 (2014), pp. 378–393.
- [3] Yi Mei et al. "Competitive Divide-and-Conquer Algorithm for Unconstrained Large Scale Black-Box Optimization". In: *ACM Transaction on Mathematical Software* 42.2 (June 2015), p. 13.
- [4] Mohammad Nabi Omidvar et al. *IDG: A Faster and More Accurate Differential Grouping Algorithm*. Technical Report CSR-15-04. University of Birmingham, School of Computer Science, Sept. 2015.
- [5] Yuan Sun, Michael Kirley, and Saman Kumara Halgamuge. "Extended differential grouping for large scale global optimization with direct and indirect variable interactions". In: *Genetic and Evolutionary Computation Conference*. ACM. 2015, pp. 313–320.
- [6] Yingbiao Ling, Haijian Li, and Bin Cao. "Cooperative co-evolution with graph-based differential grouping for large scale global optimization". In: *International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery*. IEEE. 2016, pp. 95–102.
- [7] Xiao-Min Hu et al. "Cooperation coevolution with fast interdependency identification for large scale optimization". In: *Information Sciences* 381 (2017), pp. 142–160.

Shortcomings of Differential Grouping

- Cannot detect the overlapping functions.
- Slow if all interactions are to be checked.
- Requires a threshold parameter (ϵ).
- Can be sensitive to the choice of the threshold parameter (ϵ).

Algorithm 1: DG2

 $(\Lambda, \mathbf{F}, \check{f}, f_{\text{base}}, \Gamma) = \text{ISM}(f, n, \bar{\mathbf{x}}, \underline{\mathbf{x}});$ $\Theta = \text{DSM}(\Lambda, \mathbf{F}, \check{f}, f_{\text{base}}, n);$ $(k, \mathbf{y}_1, \dots, \mathbf{y}_k) = \text{ConnComp}(\Theta) ;$ $\mathbf{x}_{\text{sep}} = \{\}, g = 0;$ **for** $i = 1 \rightarrow k$ **do** **if** $|\mathbf{y}_i| = 1$ **then** $\mathbf{x}_{\text{sep}} = \mathbf{x}_{\text{sep}} \cup \mathbf{y}_i;$ **else** $g = g + 1, \mathbf{x}_g = \mathbf{y}_i;$

Differential Grouping 2

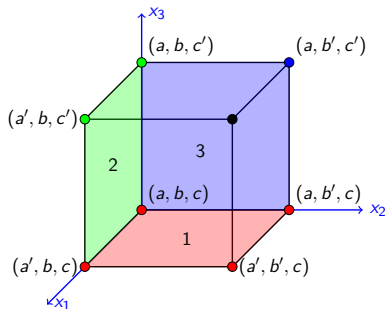


Figure: Geometric representation of point generation in DG2 for a 3D function.

$$x_1 \leftrightarrow x_2: \Delta^{(1)} = f(a', b, c) - f(a, b, c), \Delta^{(2)} = f(a', b', c) - f(a, b', c)$$

$$x_1 \leftrightarrow x_3: \Delta^{(1)} = f(a', b, c) - f(a, b, c), \Delta^{(2)} = f(a', b, c') - f(a, b, c')$$

$$x_2 \leftrightarrow x_3: \Delta^{(1)} = f(a, b', c) - f(a, b, c), \Delta^{(2)} = f(a, b', c') - f(a, b, c')$$

Algorithm 2: ISM

```
 $\Lambda = \mathbf{0}_{n \times n};$   
 $\mathbf{F}_{n \times n} = \text{NaN}_{n \times n};$  // matrix of all NaNs  
 $\check{\mathbf{f}}_{n \times 1} = \text{NaN}_{n \times 1};$  // vector of all NaNs  
 $\mathbf{x}^{(1)} = \underline{\mathbf{x}}, f_{\text{base}} = f(\mathbf{x}^{(1)}), \Gamma = 1;$   
 $\mathbf{m} = \frac{1}{2}(\overline{\mathbf{x}} + \underline{\mathbf{x}});$   
for  $i = 1 \rightarrow n - 1$  do  
  if  $\neg \text{isnan}(\check{\mathbf{f}}_i)$  then  
     $\mathbf{x}^{(2)} = \mathbf{x}^{(1)}, \mathbf{x}_i^{(2)} = \mathbf{m}_i;$   
     $\check{\mathbf{f}}_i = f(\mathbf{x}^{(2)}), \Gamma = \Gamma + 1;$   
  for  $j = i + 1 \rightarrow n$  do  
    if  $\neg \text{isnan}(\check{\mathbf{f}}_j)$  then  
       $\mathbf{x}^{(3)} = \mathbf{x}^{(1)}, \mathbf{x}_j^{(3)} = \mathbf{m}_j;$   
       $\check{\mathbf{f}}_j = f(\mathbf{x}^{(3)}), \Gamma = \Gamma + 1;$   
     $\mathbf{x}^{(4)} = \mathbf{x}^{(1)}, \mathbf{x}_i^{(4)} = \mathbf{m}_i, \mathbf{x}_j^{(4)} = \mathbf{m}_j;$   
     $\mathbf{F}_{ij} = f(\mathbf{x}^{(4)}), \Gamma = \Gamma + 1;$   
     $\Delta^{(1)} = \check{\mathbf{f}}_i - f(\mathbf{x}^{(1)});$   
     $\Delta^{(2)} = \mathbf{F}_{ij} - \check{\mathbf{f}}_j;$   
     $\Lambda_{ij} = |\Delta^{(1)} - \Delta^{(2)}|;$ 
```

DG2: Accuracy

$$n = \pm s \times \beta^{e-p},$$

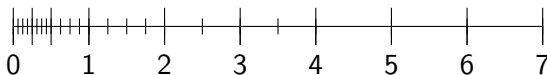


Figure: Non-uniform distribution of floating-point numbers for a hypothetical system ($\beta = 2$, $e_{\min} = -1$, $e_{\max} = 3$, and $p = 3$). The vertical bars denote all the representable numbers in this system.

Theorem

If $x \in \mathbb{R}$ lies in the range of \mathbb{F} , then

$$fl(x) = x(1 + \delta), \quad |\delta| < \mu_M,$$

where μ_M is called the unit roundoff, which is equal to $\frac{1}{2}\beta^{1-p}$.

DG2: Accuracy

Theorem

Given a floating-point number system that satisfies IEEE 754 such that $|\delta_i| < \mu_M$. We have:

$$\prod_{i=1}^k (1 + \delta_i)^{e_i} = 1 + \theta_k, \quad (7)$$

where

$$|\theta_k| \leq \frac{\mu_M}{1 - n\mu_M} := \gamma_k, \quad e_i = \pm 1,$$

provided that $k\mu_M < 1$.

DG2: Accuracy

$$\hat{\Delta}_1 = f(\mathbf{x}) \ominus f(\mathbf{x}') = (f(\mathbf{x}) - f(\mathbf{x}'))(1 + \delta_1) = \Delta^{(1)}(1 + \delta_1),$$

$$\hat{\Delta}_2 = f(\mathbf{y}) \ominus f(\mathbf{y}') = (f(\mathbf{y}) - f(\mathbf{y}'))(1 + \delta_2) = \Delta^{(2)}(1 + \delta_2),$$

$$\begin{aligned}\hat{\lambda} &= |\hat{\Delta}_1 \ominus \hat{\Delta}_2| = |\hat{\Delta}_1 - \hat{\Delta}_2|(1 + \delta_3) \\ &= |f(\mathbf{x})(1 + \delta_1)(1 + \delta_3) - f(\mathbf{x}')(1 + \delta_1)(1 + \delta_3) \\ &\quad - f(\mathbf{y})(1 + \delta_2)(1 + \delta_3) + f(\mathbf{y}')(1 + \delta_2)(1 + \delta_3)|.\end{aligned}$$

DG2: Accuracy

$$\begin{aligned} |\lambda - \hat{\lambda}| &\leq \gamma_2 \left| (f(\mathbf{x}) - f(\mathbf{x}')) - (f(\mathbf{y}) - f(\mathbf{y}')) \right| \\ &= \gamma_2 \left| (f(\mathbf{x}) + f(\mathbf{y}')) - (f(\mathbf{y}) + f(\mathbf{x}')) \right| \\ &\leq \gamma_2 \cdot \max \left\{ (f(\mathbf{x}) + f(\mathbf{y}')), (f(\mathbf{y}) + f(\mathbf{x}')) \right\} := e_{\text{inf}}. \end{aligned} \tag{8}$$

Equation (8) is based on the assumption that the codomain of f is non-negative, i.e., $f : \mathbb{R} \rightarrow \mathbb{R}_0^+$. A more general form for $f : \mathbb{R} \rightarrow \mathbb{R}$ is as follows:

$$e_{\text{inf}} = \gamma_2 (|f(\mathbf{x})| + |f(\mathbf{y}')| + |f(\mathbf{y})| + |f(\mathbf{x}')|). \tag{9}$$

DG2: Accuracy

$$|f(\cdot) - \hat{f}(\cdot)| \leq \gamma_{\sqrt{\phi}} f(\cdot) := e_{\text{sup}}. \quad (10)$$

$$e_{\text{sup}} = \gamma_{\sqrt{n}} \max\{f(\mathbf{x}), f(\mathbf{x}'), f(\mathbf{y}), f(\mathbf{y}')\} \quad (11)$$

$$\epsilon = \frac{\eta_0}{\eta_0 + \eta_1} e_{\text{inf}} + \frac{\eta_1}{\eta_0 + \eta_1} e_{\text{sup}}, \quad (12)$$

Algorithm 3: $\Theta = \text{DSM}(\Lambda, \mathbf{F}, \check{\mathbf{f}}, f_{\text{base},n})$

 $\Theta = \text{NaN}_{n \times n};$ $\eta_1 = \eta_2 = 0;$ **for** $i = 1 \rightarrow n - 1$ **do** **for** $j = i + 1 \rightarrow n$ **do** $f_{\max} = \max\{f_{\text{base}}, \mathbf{F}_{ij}, \check{\mathbf{f}}_i, \check{\mathbf{f}}_j\};$ $e_{\inf} = \gamma_2 \cdot \max\{f_{\text{base}} + \mathbf{F}_{ij}, \check{\mathbf{f}}_i + \check{\mathbf{f}}_j\};$ $e_{\sup} = \gamma_{\sqrt{n}} \cdot f_{\max};$ **if** $\Lambda_{ij} < e_{\inf}$ **then** $\Theta_{i,j} = 0; \eta_0 = \eta_0 + 1;$ **else if** $\Lambda_{ij} > e_{\sup}$ **then** $\Theta_{i,j} = 1; \eta_1 = \eta_1 + 1;$ **for** $i = 1 \rightarrow n - 1$ **do** **for** $j = i + 1 \rightarrow n$ **do** $f_{\max} = \max\{f_{\text{base}}, \mathbf{F}_{ij}, \check{\mathbf{f}}_i, \check{\mathbf{f}}_j\};$ $e_{\inf} = \gamma_2 \cdot \max\{f_{\text{base}} + \mathbf{F}_{ij}, \check{\mathbf{f}}_i + \check{\mathbf{f}}_j\};$ $e_{\sup} = \gamma_{\sqrt{n}} \cdot f_{\max};$ **if** $\Theta_{i,j} \neq \text{NaN}$ **then** $\epsilon = \frac{\eta_0}{\eta_0 + \eta_1} \cdot e_{\inf} + \frac{\eta_1}{\eta_0 + \eta_1} \cdot e_{\sup};$ **if** $\Lambda_{ij} > \epsilon$ **then** $\Theta_{i,j} = 1;$ **else** $\Theta_{i,j} = 0;$

Direct/Indirect Interactions

Indirect Interactions

In an objective function $f(\mathbf{x})$, decision variables x_i and x_j interact directly (denoted by $x_i \leftrightarrow x_j$) if

$$\exists \mathbf{a} : \left. \frac{\partial f}{\partial x_i \partial x_j} \right|_{\mathbf{x}=\mathbf{a}} \neq 0,$$

decision variables x_i and x_j interact *indirectly* if

$$\frac{\partial f}{\partial x_i \partial x_j} = 0,$$

and there exists a set of decision variables $\{x_{k1}, \dots, x_{ks}\}$ such that $x_i \leftrightarrow x_{k1}, \dots, x_{ks} \leftrightarrow x_j$.

Efficiency vs Accuracy

Saving budget at the expense of missing overlaps:

- eXtended Differential Grouping [1].
- Fast Interdependency Identification [2].



Figure: The interaction structures represented by the two graphs cannot be distinguished by XDG.

[1] Yuan Sun, Michael Kirley, and Saman Kumara Halgamuge. "Extended differential grouping for large scale global optimization with direct and indirect variable interactions". In: *Genetic and Evolutionary Computation Conference*. ACM. 2015, pp. 313–320.

[2] Xiao-Min Hu et al. "Cooperation coevolution with fast interdependency identification for large scale optimization". In: *Information Sciences* 381 (2017), pp. 142–160.

Benchmark Suites

- CEC'2005 Benchmark Suite (non-modular)
- CEC'2008 LSGO Benchmark Suite (non-modular)
- CEC'2010 LSGO Benchmark Suite
- CEC'2013 LSGO Benchmark Suite

Challenges of CC

Main Questions

- 1 How to decompose the problem?
- 2 How to allocated resources?
- 3 How to coordinate?

The Imbalance Problem

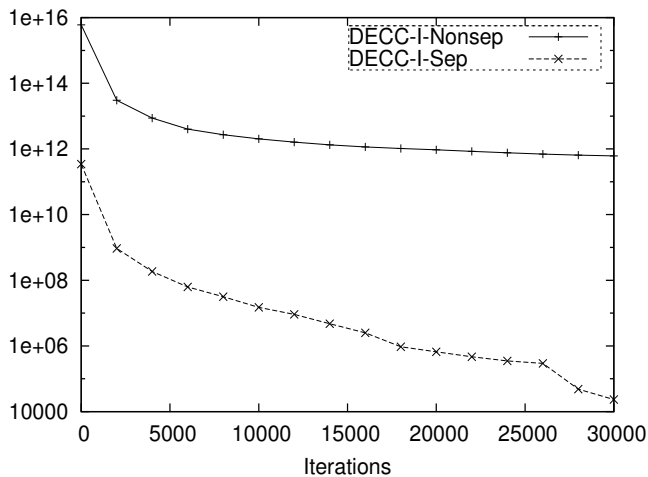
- Non-uniform contribution of components.

Imbalanced Functions

$$f(\mathbf{x}) = \sum_{i=1}^m w_i f_i(\mathbf{x}_i), \quad (13)$$

$$w_i = 10^{s\mathcal{N}(0,1)},$$

The Imbalance Problem (2)



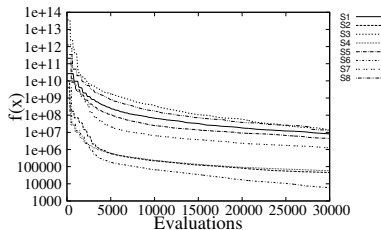
Contribution-Based Cooperative Co-evolution (CBCC)

Types of CC

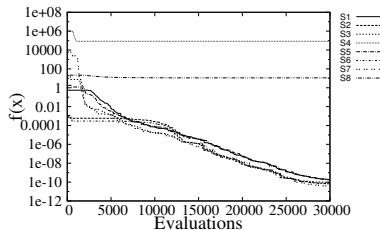
- CC: **round-robin** optimization of components.
- CBCC: favors components with a **higher contribution**.
 - ▶ Quantifies the contribution of components.
 - ▶ Optimizes the one with the highest contribution.

How to Quantify the Contribution

- For quantification of contributions a relatively accurate decomposition is needed.
- Changes in the objective value while other components are kept constant.



(a) Round-Robin CC



(b) Contribution-Based CC

Contribution-Aware Algorithms

- Contribution-Based Cooperative Co-evolution (CBCC) [1], [2].
- Incremental Cooperative Coevolution [3]
- Multilevel Framework for LSGO [4]

[1] Mohammad Nabi Omidvar, Xiaodong Li, and Xin Yao. "Smart use of computational resources based on contribution for cooperative co-evolutionary algorithms". In: *Proc. of Genetic and Evolutionary Computation Conference*. ACM, 2011, pp. 1115–1122.

[2] Mohammad Nabi Omidvar et al. "CBCC3 – A Contribution-Based Cooperative Co-evolutionary Algorithm with Improved Exploration/Exploitation Balance". In: *Proc. IEEE Congr. Evolutionary Computation*. 2016, pp. 3541–3548.

[3] Sedigheh Mahdavi, Shahryar Rahnamayan, and Mohammad Ebrahim Shiri. "Incremental cooperative coevolution for large-scale global optimization". In: *Soft Computing* (2016), pp. 1–20.

[4] Sedigheh Mahdavi, Shahryar Rahnamayan, and Mohammad Ebrahim Shiri. "Multilevel framework for large-scale global optimization". In: *Soft Computing* (2016), pp. 1–30.

Large-Scale Multiobjective Optimization

Large-scale multiobjective optimization is growing popularity:

- Development of a benchmark [1].
- Exploiting modularity using CC [2], [3], [4].
- Analysis of the existing benchmarks [5].

[1] Ran Cheng et al. "Test problems for large-scale multiobjective and many-objective optimization". In: *IEEE Transactions on Cybernetics* (2016).

[2] Luis Miguel Antonio and Carlos A Coello Coello. "Use of cooperative coevolution for solving large scale multiobjective optimization problems". In: *IEEE Congress on Evolutionary Computation*. IEEE. 2013, pp. 2758–2765.

[3] Luis Miguel Antonio and Carlos A Coello Coello. "Decomposition-Based Approach for Solving Large Scale Multi-objective Problems". In: *Parallel Problem Solving from Nature*. Springer. 2016, pp. 525–534.

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Analysis of ZDT

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{matrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6$

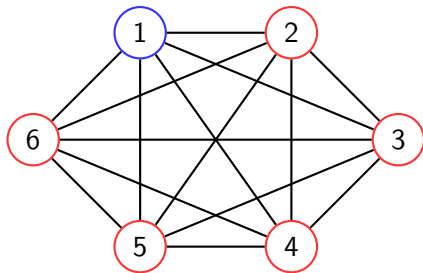


Figure: Variable interaction structures of the f_2 function of ZDT test suite.

Analysis of DTLZ1-DTLZ4

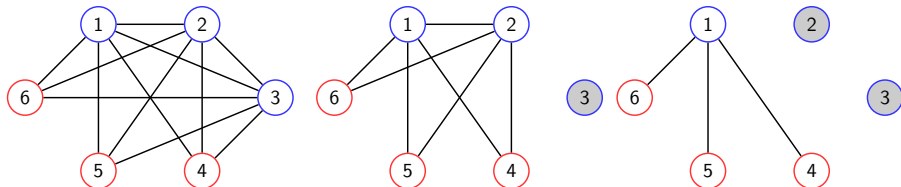


Figure: Variable interaction graphs of DTLZ1 to DTLZ4 .

Proposition 1

For DTLZ1 to DTLZ4, $\forall f_i, i \in \{1, \dots, m\}$, we divide the corresponding decision variables into two non-overlapping sets: $\mathbf{x}_I = (x_1, \dots, x_\ell)^T$, $\ell = m - 1$ for $i \in \{1, 2\}$ while $\ell = m - i + 1$ for $i \in \{3, \dots, m\}$; and $\mathbf{x}_{II} = (x_m, \dots, x_n)^T$. All members of \mathbf{x}_I not only interact with each other, but also interact with those of \mathbf{x}_{II} ; all members of \mathbf{x}_{II} are independent from each other.

Analysis of DTLZ5-DTLZ7

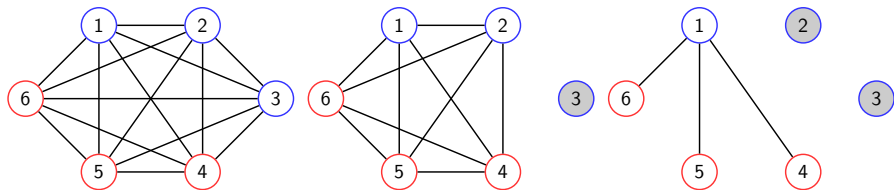


Figure: Variable interaction graphs of DTLZ5 and DTLZ6.

Proposition 2

For DTLZ5 and DTLZ6, $\forall f_i, i \in \{1, \dots, m\}$, we divide the corresponding decision variables into two non-overlapping sets: $\mathbf{x}_I = (x_1, \dots, x_\ell)^T$, $\ell = m - 1$ for $i \in \{1, 2\}$ while $\ell = m - i + 1$ for $i \in \{3, \dots, m\}$; and $\mathbf{x}_{II} = (x_m, \dots, x_n)^T$. For f_i , where $i \in \{1, \dots, m - 1\}$, all members of \mathbf{x}_I and \mathbf{x}_{II} interact with each other; for f_m , we have the same interaction structure as DTLZ1-DTLZ4.

Proposition 3

All objective functions of DTLZ7 are fully separable.

Some Future Directions (I)

- What if the components have overlap?
- Differential group is off-line and can be time-consuming. Is there a more efficient method?
- Do we need to get 100% accurate grouping? What is the relationship between grouping accuracy and optimality achieved by a CC algorithm?

Some Future Directions (II)

- CC for combinatorial optimization, e.g.,
 - ▶ Y. Mei, X. Li and X. Yao, “Cooperative Co-evolution with Route Distance Grouping for Large-Scale Capacitated Arc Routing Problems,” IEEE Transactions on Evolutionary Computation, 18(3):435-449, June 2014.
- However, every combinatorial optimization problem has its own characteristics. We need to investigate CC for other combinatorial optimization problems.

Some Future Directions (III)

- Learning variable interdependencies is a strength of estimation of distribution algorithms (EDAs), e.g.,
 - ▶ W. Dong, T. Chen, P. Tino and X. Yao, “Scaling Up Estimation of Distribution Algorithms for Continuous Optimization,” IEEE Transactions on Evolutionary Computation, 17(6):797-822, December 2013.
 - ▶ A. Kaban, J. Bootkrajang and R.J. Durrant. “Towards Large Scale Continuous EDA: A Random Matrix Theory Perspective.” Evolutionary Computation
- Interestingly, few work exists on scaling up EDAs.

LSGO Resources

- There is an IEEE Computational Intelligence Society (CIS) Task Force on LSGO:
- Upcoming LSGO Tutorials
 - ▶ July 2017 GECCO (Berlin, Germany).
 - ▶ November 2017 SEAL (Shenzhen, China).
- LSGO Repository: <http://www.cercia.ac.uk/projects/lsgo>

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Questions

Thanks for your attention!

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