An Overview of Some Special Functions

Concepts and Use

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1 Dirac delta series

1.1 Impulse / The Dirac Delta

Named after the physicist Paul Dirac, the dirac delta function (which is really a distribution and not a function) is wildly used in physics, but it also plays a central role in elementary signal processing. The dirac delta can best be described as a thing that is infinite valued at the origo and zero everywhere else. This being hard to imagine and definitely puts the dirac delta in the category of discontinuos functions, one often employ a different definition of it:

$$\int_{\mathbb{S}} \delta(x-a)f(x)dx = f(a)$$

Where $\mathbb{S}$ is the support (relevant domain) of the function f and $a \in \mathbb{S}$.This property is very important and sometimes reffered to as the sifting property. One can formalize the use of this "function" by considering it the limit of a sequence of functions that are (close to) zero everywhere except from in an area around zero. Examples of function include: box functions, triangle functions and gaussians. Playing informally with this property we can do the following:

$$\int_{\mathbb{S}} \delta(x-a)f(x)dx = \int_{a+\epsilon}^{a-\epsilon} \delta(x-a)f(x)dx$$

This since the dirac delta is zero everywhere exept from when its argument is zero, in this case at $x = a$. Letting ϵ go to zero therby integrating over an ever decreasing area we can approximate it by assuming that f(x) is nearly constant locally and therefore extract it from the integral:

$$\int_{a+\epsilon}^{a-\epsilon} \delta(x-a)f(x)dx \approx f(a)\int_{a+\epsilon}^{a-\epsilon} \delta(x-a)dx = f(a) \implies \int_{a+\epsilon}^{a-\epsilon} \delta(x-a)dx = 1$$

1.2 Step function / Heaviside function

Another important and wildly used function is the step function:

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Where the $\frac{1}{2}$ is there due to convergence considerations and practicality. We can obtain an interesting relationship between the dirac delta and the step function by looking at the derivative of the step function. Because of its piecewise definition we have to take care around zero but otherwise we immediately see that:

$$\frac{dH}{dx} = 0, \qquad x \neq 0$$

Around zero we have a jump discontinuity and as such the derivative is undefined there. But knowing that the dirac delta satisfies the above relation it is tempting to set:

$$\frac{dH}{dx} = \delta(x)$$

One can see why by checking that the integral of the dirac function from a point below zero to any point above zero equals one. But taking the integral over an interval with strictly negative endpoints one gets zero.

1.3 Ramp function

The ramp function is defined as:

$$R(x) = xH(x)$$

From this we can quickly see that the derivative of the ramp is a step funcition. In total we then see that we have a chain of "special" functions starting with the dirac delta and ending with the ramp function. The ramp funciton is usefull for modelling a linear phenomena that is zero up to some point but then suddently "turns on".