Pondering The Schwarzschild Solution

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1 Introduction and method

In this report we will in short develop the Schwarzschild solution from the field equations of general relativity. To do this we will first carry out a general discussion about the assumptions we make during the derivation. Then we will take a look at the implementation of these assumptions in the form of a special metric: the static isotropic metric. The properties of this metric embody what we seek to model, but to actually make it into a solution we will have to use it as an ansatz and solve the equations with it. Having done this we move on to a quick discussion about the singularity that appear in the solution and whether this is an actual singularity or only a coordinate one induced by our choice of system. The end goal is indeed to present a derivation of the metric.

2 Assumptions and Overview

The entire goal of our derivation is to model a curved spacetime caused by a massive, static, uncharged and spherically symmetric body. These assumptions put major limitations on the form of the metric and therefore on the solutions of the EFEs. Listing our assumtions:

Spherically symetric : This allows us to limit the form of our metric to one that is isotropic; the same when measured from all directions.

Static This allows us to disregard all dependence of the metric on the rotation of the massive body and time as well.

Uncharged This allows us to disregard all fields associated with charges and currents, ie. electromagnetic fields.

3 The Static Isotroic Metric

Looking at a general metric $g^{\mu\nu}$ we can use the above assumptions and already set some of the components to zero. We will do this by looking at the coordinate transformations that the metric has to be invariant under to be spherically symmetric and static. We use a standard (physics convention) spherical coordinate system (r, θ, ϕ), where ϕ is the azimuthal angle.

Azimuthal components: ϕ A simple way to see that these components have to be zero is to rotate our coordinate system π radians around, or in other words flip the sign of the azimuthal component. We leave the rest of the coordinate system the same:

$$\begin{aligned} x'^3 &= -x^3 \\ \frac{\partial x^3}{\partial x'^3} &= -1 \\ g'_{3\nu} &= \frac{\partial x^3}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\alpha\beta} \\ &= -\frac{\partial x^\nu}{\partial x'^\beta} g_{3\beta}, \qquad \nu \neq 3 \\ &= -g_{3\nu} \end{aligned}$$

Our assumtion was that the component shouldn't change under this transformation, so the only choice of $g_{3\nu}$ that works is $g_{3\nu} = 0, \nu \neq 3$.

Polar components: θ Using the same logic as above for a transformation $\theta \rightarrow -\theta$, we obtain: $g_{2\nu} = 0, \nu \neq 2$.

Time components: t As well as the above we have already assumed that our metric components be indepentent of t in our coordinate basis. By the same arguments as for the ϕ and θ components we get: $g_{0\nu} = 0, \nu \neq 0$

Arranging all this in a matrix we see that:

$$[g]_X = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}$$

That is, we have diagonalised the metric.

Using differential forms for the coordinate basis and dropping the baracket we write this as:

$$g(r) = g_{00}dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\phi^2$$

Because of our assumption of spherical symmetry we can further restrict this. An easy way of seeing this is that on a hypersurface of constant r and t we would need the metric to look excactly like the normal two-sphere metric (up to scale). Accordingly we require $g_{22} = r^2$ and $g_{33} = r^2 \sin^2\theta$. On top of this it is easy to see that the two remaining components must be independent of ϕ and θ (t as well by assumtption). Another point is that since this metric can be made (locally) flat in a local interial system, we know its signature (Minkowski). Sylvester's theorem then tells us that our diagnnoal (curvilinear) metric has the same number of negative diagonal entries. Thus, upon renaming and adding signs, we finally obtain:

$$g(r) = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

4 Christoffel symbols and the Ricci tensor

To solve the EFEs we are dependent on knowing the components of the Ricci tensor, hence also the Christoffel symbols. One could use a CAS for doing this but I did it by hand to get develop a feeling for the process. The $\Gamma^\lambda_{\mu\nu}$ are found from our metric above by using the folowing formula for the Levi-Cività connection:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})$$

The calculations yield the following nonzero components:

$$\begin{aligned}
\Gamma^r_{rr} &= \frac{A'}{2A} & \qquad \Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta \\
\Gamma^r_{\theta\theta} &= -\frac{r}{A} & \Gamma^\theta_{r\theta} &= \frac{1}{r} \\
\Gamma^r_{\phi\phi} &= \frac{r\sin^2/\theta}{A} & \Gamma^\phi_{r\phi} &= \frac{1}{r} \\
\Gamma^r_{tt} &= \frac{B'}{2A} & \Gamma^\phi_{\phi\theta} &= \cot\theta
\end{aligned}$$

$$\Gamma^t_{tr} = \frac{B'}{2B}$$

As mentioned above we need the Ricci tensor components to solve the EFEs. In fact, since we are about to solve the vacuum equations without cosmological constant, the equations we are going to have to solve are:

$$R_{\mu\nu} = 0$$

So we realise that knowing the Ricci tensor is a must for this application. The Ricci tensor is a two-tensor that is the contraction of the Riemann curvature tensor with itself on the first and third comonents:

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$$

Using then the formula for the Riemann curvature and applying this contraction we obtain:

$$R_{\mu\nu} = \frac{\partial\Gamma^{\sigma}_{\mu\sigma}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\sigma}_{\mu\nu}}{\partial x^{\sigma}} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\sigma\lambda}$$

Using all of this for the actual computations by inserting the Christoffel symbols and its derivatives. We obtain:

$$\begin{aligned}
R_{rr} &= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}(\frac{A'}{A} + \frac{B'}{B}) - \frac{1}{r}\frac{A'}{A} \\
R_{\theta\theta} &= \frac{r}{2A}(\frac{B'}{B} - \frac{A'}{A}) + \frac{1}{A} - 1 \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta \\
R_{tt} &= -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}(\frac{A'}{A} + \frac{B'}{B}) - \frac{1}{r}\frac{B'}{A} \\
R_{\mu\nu} &= 0, \qquad \mu \neq \nu
\end{aligned}$$

5 The Schwarzschild Solution

Finally we are in the position to use the results from the section above. As mentioned the equations we're trying to solve are the EFEs in a vacuum:

$$R_{\mu\nu} = 0$$

Taking a look at the components above we see that $R_{\mu\nu}$ is zero already, $R_{\phi\phi}$ contains $R_{\theta\theta}$, so setting $R_{\theta\theta}, R_{tt}, R_{rr}$ to zero will solve the equations. Lookint at the terms we see that R_{tt} and R_{rr} share the first and middle terms exept from a difference in dividing by A and B. This gives us the idea of cross dividing these two by B and A to be able to add them and hopefully remove some terms:

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = 0 = -\frac{1}{rA}(\frac{A'}{A} + \frac{B'}{B}) = -\frac{1}{rA}\frac{d}{dr}(\ln A + \ln B)$$

Giving us that $\ln A + \ln B = C$. Knowing that this metric is supposed to look like the Minkowskian metric far from the massive object, we need $A, B -> 1$ as $r -> \infty$. This gives that C has to be zero, and by playing with the logarithms we get:

$$A = \frac{1}{B}$$

Looking now at the two other components and inserting the above result for A we can simplify the expressions greatly:

$$R_{rr} = \frac{B''}{2B} + \frac{1}{r}\frac{B'}{B} = 0$$
$$R_{\theta\theta} = rB' + B - 1 = 0$$

Looking carefully it's now possible to see the relationship that reveals itself between these two. Playing with the rr component:

$$2Br * R_r r = rB'' + 2B' = \frac{d}{dr}(rB' + B + D) = R'_{\theta\theta}, \qquad D = -1$$

So it suffices to set $R_{\theta\theta} = 0$:

$$R_{\theta\theta} = rB' + B - 1 = 0 \implies \frac{d}{dr}(rB) = 1 \implies rB = r + E$$

We can gain insight in the value of this constant by utilising that in the Newtonian limit, the $g_{tt} = -B$ component of the metric approaches $-1 - 2\phi$. Inserting the normal expression for the Newtonian potential gives: $B = 1 - \frac{2MG}{r}$. Thus we see that $E = -2MG$. We finally obtain for A,B and g:

$$B = 1 - \frac{2MG}{r}$$
$$A = (1 - \frac{2MG}{r})^{-1}$$
$$g(r) = (\frac{2MG}{r} - 1)dt^2 + \frac{r}{r - 2MG}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

6 The Non-Singular Singularites

The Schwarzschild solution from above has two singularities: one at $r = 0$ and one at $r_s = 2MG$ or in geometrisised units $r_s = \frac{2MG}{c^2}$. At the time of the discovery of the EFEs and the Schwarzschild metric there was a particularily wild discussion about the nature of the Schwarzschild singularity at r_s. The consensus reached was that the singularity indeed was a coordinate singularity, an artefact of the system we use to model the situation. In most cases like for our sun or for the earth, the Schwarzschild radius is well within the borders of the bodies themselves. Since we have solved the EFEs for empty space, our solutions are only solutions outside these borders (where there is empty space), inside we would have to solve the full EFEs; a much more daunting task!

References

[Weinberg, 1972] Weinberg, S. (1972). *Gravitation and cosmology*. Wiley.