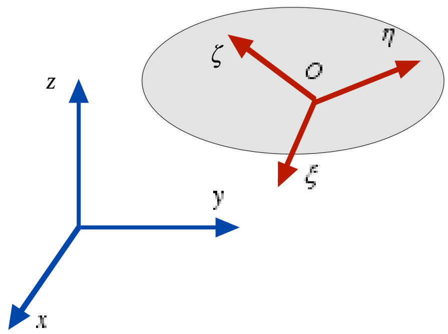
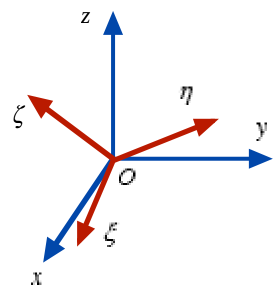
Lesson 8-A: Euler Angles

Reference Frames

- In order to concentrate on the rotational coordinates of a body, we eliminate the translational coordinates by allowing the two reference frames ξ–η–ζ and x-y-z to coincide at the origins

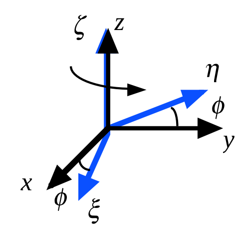




Planar Rotation in Space

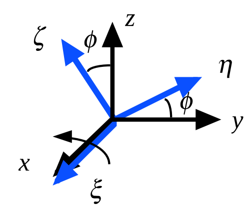
- Three planar rotations:
  - Assume that we perform a planar rotation in space, e.g., we perform a planar rotation in the x-y plane (ξ–η plane) by rotating about the z-axis (or ζ–axis). The transformation matrix for this rotation is

$$\mathbf{A} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



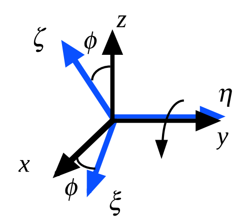
  - Rotation about x-axis (or ξ-axis)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$



  - Rotation about y-axis (or η-axis)

$$\mathbf{A} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$



Note the signs for the "sinϕ" terms!

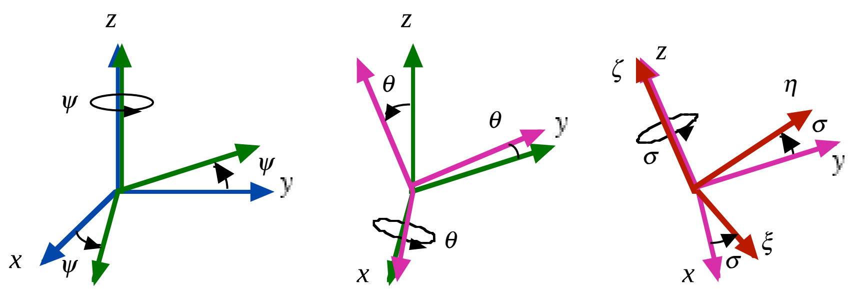
Euler Angles

- Euler angles are the most commonly used rotational coordinates
- There are many different conventions of Euler angles
- All conventions are the result of three consecutive rotations about three different axes
- Depending on the choice of rotational axes, different conventions are found

- The most common convention is the *z-x-z* convention (initially defined for gyroscopes)
- Another common convention is the *x-y-z* also known as the Bryant angles
- There is an inherent problem associate with any of these conventions known as the *singularity* problem

z-x-z Convention of Euler Angles

- In the *z-x-z* convention, the *x-y-z* frame is rotated three times: first about the *z*-axis by an angle ψ; then about the new *x*-axis by an angle θ; then about the newest *z*-axis by an angle σ. If the three rotational angles are chosen correctly, the rotated frame will coincide with the ξ–η–ζ frame.



- The transformation matrix is found by considering three planar transformation matrices

$$\mathbf{D} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix **A** is the product of these three planar transformation matrices

$$\mathbf{A} = \mathbf{DCB} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} c\psi\, c\sigma - s\psi\, c\theta\, s\sigma & -c\psi\, s\sigma - s\psi\, c\theta\, c\sigma & s\psi\, s\theta \\ s\psi\, c\sigma + c\psi\, c\theta\, s\sigma & -s\psi\, s\sigma + c\psi\, c\theta\, c\sigma & -c\psi\, s\theta \\ s\theta\, s\sigma & s\theta\, c\sigma & c\theta \end{bmatrix}$$

where: $c \equiv \cos$ and $s \equiv \sin$

- Note that the resulting transformation matrix is highly nonlinear in terms of the three angles
- This process does not tell us how to chose the value for each angle! If the angles are not chosen correctly, following the rotations, the *x-y-z* frame will not coincide with the ξ–η–ζ frame!
- Furthermore, there is a problem of "singularity" that we need to be aware of!

Inverse Problem

- Assume that the values of the nine direction cosines; i.e., all the nine elements of the transformation matrix, are known. How do we determine the three Euler angles?

- We equate some of the direction cosines with the entries of the transformation matrix **A**:

$$\begin{bmatrix} c\psi\, c\sigma - s\psi\, c\theta\, s\sigma & -c\psi\, s\sigma - s\psi\, c\theta\, c\sigma & s\psi\, s\theta \\ s\psi\, c\sigma + c\psi\, c\theta\, s\sigma & -s\psi\, s\sigma + c\psi\, c\theta\, c\sigma & -c\psi\, s\theta \\ s\theta\, s\sigma & s\theta\, c\sigma & c\theta \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then we can write

$$\cos\theta = a_{33} \qquad \cos\psi = \frac{-a_{23}}{\sin\theta} \qquad \cos\sigma = \frac{a_{32}}{\sin\theta}$$

$$\sin\psi = \frac{a_{13}}{\sin\theta} \qquad \sin\sigma = \frac{a_{31}}{\sin\theta}$$

-- The formula $\cos\theta = a_{33}$ provides two answers for θ; use one of them! (Does it make any difference which one we choose?)

-- Formulas for $\cos\psi$ and $\sin\psi$ provide a single value for ψ

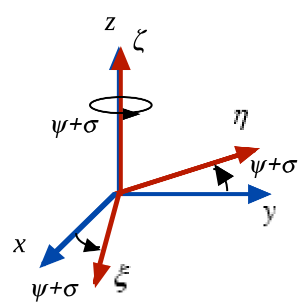
-- Formulas for $\cos\sigma$ and $\sin\sigma$ provide a single value for σ

-- The three angles can be determined only if $\sin\theta \neq 0$

Singularity

- When $\sin\theta = 0$ we fail to compute the other two Euler angles
- This occurs when $\theta = 0, \pi, 2\pi, \ldots$
- The transformation matrix finds the following form when $\sin\theta = 0$:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$$



- Graphically, the axes of the first and third rotations coincide; i.e., we have rotated about the original *z*-axis twice. Therefore, the first and the third angles can be combined as one rotation!