6) A $\omega \in \Omega^k(V)$ is decomposable iff $\omega = \phi_1 \wedge \dots \wedge \phi_k$ for some $\phi_i \in V^* = \Omega^1 V$.

a) Prop: Every 2 form on an n-dimensional space with $n \leq 3$ is decomposable

Proof: For $n=1,2$ is obvious. Say $\omega \in \Omega^2 V$ with $n=3$ and ϕ_1, ϕ_2, ϕ_3 is a dual basis. Then $\omega = \omega_{12}\phi_1 \wedge \phi_2 + \omega_{13}\phi_1 \wedge \phi_3 + \omega_{23}\phi_2 \wedge \phi_3 = \phi_1 \wedge (\omega_{12}\phi_2 + \omega_{13}\phi_3) + \omega_{23}\phi_2 \wedge \phi_3 = (\phi_1 + \omega_{23}\phi_2) \wedge (\omega_{12}\phi_2 + \omega_{13}\phi_3)$ ∎

b) Prop: The above affirmation is false for $n=4$

Proof: Say $\phi_1, \dots, \phi_4$ is a dual basis, I claim $\omega = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$ isn't decomposable (I think claims actually).

$$(a_1\phi_1 + a_2\phi_2 + a_3\phi + a_4\phi_4) \wedge (b_1\phi_1 + b_2\phi_2 + b_3\phi_3 + b_4\phi_4) = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$$

$$\Leftrightarrow \sum_{1 \leq i < j \leq 4} (a_i b_j - a_j b_i)\phi_i \wedge \phi_j = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$$

$$\Leftrightarrow \begin{cases} a_1 b_2 - a_2 b_1 = 1 \\ a_1 b_3 - a_3 b_1 = 0 \\ a_1 b_4 - a_4 b_1 = 0 \\ a_2 b_3 - a_3 b_2 = 0 \\ a_2 b_4 - a_4 b_2 = 0 \\ a_3 b_4 - a_4 b_3 = 1 \end{cases}$$

~ absolute mess, lets try to develop a more general theory.

Let $(R, \oplus, \odot)$ be a ring and $(M, +)$ be an abelian group, then a (left) R-module is an ordered tuple $(R, \oplus, \odot, M, +, \cdot)$ where $\cdot : R \times M \to M$ such that the following hold for $v, w \in M$ and $r, s \in R$:

1) $r \cdot (v + w) = r \cdot v + r \cdot w$
2) $(r \oplus s) v = r \cdot v + s \cdot v$] Distributivity
3) $(r \odot s) \cdot v = r \cdot (s \cdot v)$] associativity
4) $1 \cdot v = v$] respects neutral element

This is fairly standard in Algebra, when making a more complicated structure from simpler ones one essentially just sticks the simpler structures together and requires a few conditions for the operations to match. Rings do this with groups and monoids, modules with rings and groups and vector spaces with fields and groups.

Nonetheless the structure we are after are not modules, they are algebras (of a particular kind). An Algebra is an ordered tuple $(R, \oplus, \odot, M, +, \cdot, \tilde{\cdot})$ such that $(R, \oplus, \odot, M, +, \cdot)$ is a module and $(M, +, \tilde{\cdot})$ is a ring. We say that the Algebra is associative if $r\cdot(v \tilde{\cdot} w) = (r\cdot v)\tilde{\cdot}(w)$ $\forall r \in R, \forall v, w \in M$.

We say a ring A is graded if $A = A_1 \oplus A_2 \oplus \cdots$ and $A_i A_j \subset A_{i+j}$ (for instance the ring of polynomials) with each A_i being closed additively. I'm gonna answer to a less general than usual definition for graduate Algebra. Simply, an algebra $(R, \oplus, \odot, M, +, \cdot, \tilde{\cdot})$ is graded if $(M, +, \tilde{\cdot})$ is a graded ring and

$$R \cdot M_i \subset M_i \qquad \forall i$$

In this general context one can easily state what we are studying with decomposability, we are simply looking to know if $\underbrace{M_1 \tilde{\cdot} M_1 \tilde{\cdot} \cdots \tilde{\cdot} M_1}_{k-1 \text{ products}} = M_k$

Although the statement is simple the generality makes studying it quite hard but some algebraic property may help us. I shall say that the particular structure we are interested in, the differential forms, has an even stronger structure, that of a differential graded algebra, which is a graded algebra with maps $d_i : M_i \to M_{i+1}$.

One thing I've realised is quite astonishing. A standard example of graded ring (not quite an Algebra) are the polynomials over a field $K[X]$ with the obvious operations ($\tilde{\cdot}$ is the convolution product). The statement that $\mathbb{C}[X]$ is made up of only decomposable elements is literally the Fundamental Theorem of Algebra. Clearly $\mathbb{R}[X]$ isn't decomposable (x^2+1 for instance). So decomposability is related to the FTA in the case of polynomial rings.

Through research I've found such a remarkable solution to this problem I can't contain my excitement, we are gonna go on a long journey to uncover it though. Say V is a vector space over $\mathbb{R}$, one wants to talk about choosing subspaces of V "smoothly", however this, at the moment, makes no sense. We would need to add a differential structure to the set of subspaces (of a fixed dimension) of V. The Grassmanian was made for precisely this reason.

Fix a base $v_1,\dots,v_m$ for V and fix k, we will now define $Gr_k(V)$. Say $M_{m\times k}$ is the set of full rank $m\times k$ real matrices, clearly each $A\in M_{m\times k}$ has a subspace "induced" by it in some way, namely, if $A=(a_{ij})$ then $\mathrm{span}\{\sum_{i=1}^{m} a_{i,1}v_i,\dots,\sum_{i=1}^{m} a_{i,k}v_i\} =: \mathrm{sub}(A)$ is the associated subspace. Clearly though the association isn't one to one, for instance permuting the collumns of A leads to a matrix with the same associated subspace. Indeed

$$\mathrm{sub}(A)=\mathrm{sub}(A')$$
$$\iff A = A'M \quad, \exists M\in GL(k,\mathbb{R})$$

So let's say

$$A\sim A' \quad \text{if } A=A'M \ , \exists M\in GL(k,\mathbb{R})$$

and $Gr_k(V)=M_{m\times k}/\sim$. We note that $Gr_k(V)$ depends in no way on V besides its dimension and what field it is over so we may more accurately use the notation $Gr_{\mathbb{R}}(m,k)$, nonetheless I will stick with the old one. Lets proceed to the differential structure.

Let $U_{i_1,\dots,i_k}$, $1\le i_1<\dots<i_k\le m$ be the subset of $Gr_k(V)$ corresponding to the classes $[W]$; the matrix $W_{i_1,\dots,i_k}$ which is just W but removing the rows that aren't one of the i_j's is invertible. We need to check this is well defined. If $M\in GL(k,\mathbb{R})$, but this is (kinda) clear since

$$\det(WM)_{i_1,\dots,i_k} = \det(W_{i_1,\dots,i_k})\cdot\det(M)$$

as can be seen by noticing that the i_j-th row of WM depends exclusively on M and on the i_j-th row of W. Then for $[W]\in U_{i_1,\dots,i_k}$ apply left operations by members of $GL(k,\mathbb{R})$ to obtain an $\hat{W}$; $[W]=[\hat{W}]$ and $\hat{W}_{i_1,\dots,i_k} = Id_{k\times k}$, for instance if $i_j=j$ $\forall j$ then $\hat{W}$ is of form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \\ \hline & [W]_{i_1,\dots,i_k} & \end{bmatrix} \qquad (*)$$

where $[W]_{i_1,\dots,i_k}$ is an $(m-k)\times k$ matrix. Note that $[W]_{i_1,\dots,i_k}$ is completely determined from $[W]$ as a member of $U_{i_1,\dots,i_k}$ since if we multiplied $\hat{W}$ by any $B\in GL(k,\mathbb{R})$ it would lose the format $(*)$ unless $B = Id$. So define a chart whose domain is $U_{i_1,\dots,i_k}$ and its map is $x_{i_1,\dots,i_k}([W])=[W]_{i_1,\dots,i_k}$.

We define these charts for all $1 \le i_1 < \ldots < i_k \le n$ to obtain $\binom{n}{k}$ charts, if they are C^∞-related we are done defining the smooth structure, the finitude of charts will make $Gr_k(V)$ 2-enumerable and we will still need to check Hausdorffness. To help with seeing C^∞-relatedness note that, using the previous page's notation, if $[W] \in U_{i_1,\ldots,i_k} \cap U_{j_1,\ldots,j_k}$ then one can obtain $\hat{W}$ and $\tilde{W}$; $[\hat{W}] = [\tilde{W}] = [W]$ via

$$\hat{W} = W \cdot (W_{i_1,\ldots,i_k})^{-1}$$
$$\tilde{W} = W \cdot (W_{j_1,\ldots,j_k})^{-1}$$

$$\Rightarrow \tilde{W} = \hat{W}(W_{i_1,\ldots,i_k}) \cdot (W_{j_1,\ldots,j_k})^{-1}$$

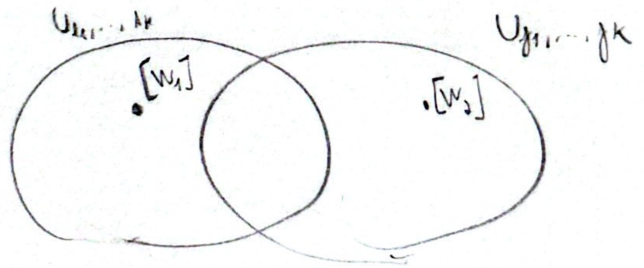
$$\Rightarrow \mathcal{X}_{j_1,\ldots,j_k} = \mathcal{X}_{i_1,\ldots,i_k} \cdot r_{i_1,\ldots,i_k} \cdot (I \circ r_{j_1,\ldots,j_k}) \qquad (1)$$

where $r_{i_1,\ldots,i_k} : M_{n\times k} \longrightarrow M_{k\times k}$ is the obvious map as is $r_{j_1,\ldots,j_k}$ and I is the inversion map. Since $r_{i_1,\ldots,i_k} \cdot (I \circ r_{j_1,\ldots,j_k})$ is C^∞ (straightforward to prove I is C^∞ given a trick using determinants) this means $\mathcal{X}_{j_1,\ldots,j_k}$ and $\mathcal{X}_{i_1,\ldots,i_k}$ are C^∞ related and

$$\mathcal{A} = \{(U_{i_1,\ldots,i_k}, \mathcal{X}_{i_1,\ldots,i_k}) \mid 1 \le i_1 < \ldots < i_k \le n\}$$

is an atlas. Lets proceed to Hausdorffness.

Take $[W_1], [W_2] \in Gr_k(V)$, say $[W_1] \in U_{i_1,\ldots,i_k}$ and $[W_2] \in U_{j_1,\ldots,j_k}$ w.l.g if $[W_1] \in U_{j_1,\ldots,j_k}$, $[W_2] \in U_{i_1,\ldots,i_k}$ or $U_{i_1,\ldots,i_k} \cap U_{j_1,\ldots,j_k} = \emptyset$ we are done, so the only non-trivial case is the one drawn bellow



Ok, but what does the grassmanian have to do with differential forms? Well, note that if I get a dual base $\phi_1, \dots, \phi_n$ for V^*, I put k-differential forms $\omega_1, \dots, \omega_k$ and get $\eta_1, \dots, \eta_k$;

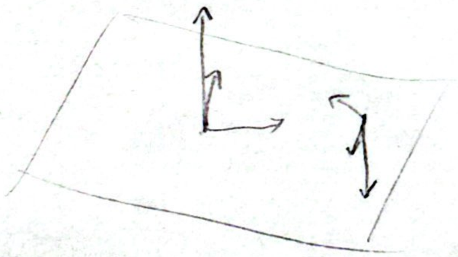
$$\eta_i = \sum_{j=1}^{k} a_i^j \omega_j$$

Then

$$\eta_1 \wedge \dots \wedge \eta_k = \sum_{\sigma \in S_k} \left(\prod_{i=1}^{n} a^i_{\sigma(i)} \right) \cdot \omega_1 \wedge \dots \wedge \omega_k \cdot \operatorname{sgn}(\sigma)$$

$$= \det(a_j^i) \cdot \omega_1 \wedge \dots \wedge \omega_k$$

hence $\eta_1 \wedge \dots \wedge \eta_k$ and $\omega_1 \wedge \dots \wedge \omega_k$ are multiples. With this we conclude that the wedge product of any k L.I. covectors in a k dimensional subspace of V^* generates the same k-form up to scaling by a non-zero constant. This in line with the cross product of any two vectors on a plane generating a vector perpendicular to it in $\mathbb{R}^3$.



We will show this leads to a problem with manifold dimensions differing when they should be the same if we assume every k-form is decomposable. First the dimension of $Gr_k(V)$ is $k(n-k)$ (just the dimension of the matrix $[W]_{i_1, \dots, i_k}$). The dimension of $\Omega^k V$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, but we are actually interested in the dimension of

$$P(\Omega^k V) := \frac{\Omega^k V \setminus \{0\}}{\approx} \quad \text{where} \quad \omega \approx \omega' \text{ if } \omega = \lambda \omega', \ \exists \lambda \in \mathbb{R}^*$$

not sure why, but
I think without removing 0
this isn't a manifold.

which is a sort of projective space made out of $\Omega^k V$ in the same way P^2 is made out of $\mathbb{R}^3$. I claim that $P(\Omega^k V)$ is a smooth manifold of dimension $\binom{n}{k} - 1$. In fact I will announce a more general version of this statement such that the notation becomes simpler and we look at only what matters.

Lemma 6.1. Let W be a vector space of dimension m over $\mathbb{R}$, define

$$P(W) := \frac{W \setminus \{0\}}{\approx} \quad \text{where } v \approx v' \text{ iff } v = \lambda v',\ \exists \lambda \in \mathbb{R}^{\times}$$

Then $P(W)$ is a smooth manifold of dimension $m-1$.

Proof: Firstly $W \setminus \{0\}$ is certainly a smooth manifold of dimension m since it is a subset of W which is itself a smooth manifold (all vector spaces over $\mathbb{R}$ are C^∞-manifolds). Fix an isomorphism $f: W \longrightarrow \mathbb{R}^m$ and set

$$\begin{aligned} \|\cdot\|_W : W &\longrightarrow \mathbb{R} \\ v &\longmapsto \|f(v)\|_{\mathbb{R}^m} \end{aligned}$$

Then surely $\|\cdot\|_W$ is a norm. We will denote $\|\cdot\|_W$ by just $\|\cdot\|$. Let $SW := \{v \in W : \|v\| = 1\}$, clearly $f|_{SW} : SW \longrightarrow S^{m-1}$ may be used to give a smooth structure to SW by just composing charts of S^{m-1} with $f|_{SW}$. Then consider the group of actions in SW given by $\Gamma = \{e, I\}$ where $I(v) = -v$, this is a finite group and its action is properly discontinuous hence $P(W) \cong \frac{SW}{\Gamma}$ is a smooth manifold by covering space theory and its dimension is the same as SW's, which is $m-1$. ▲

Obs: Loose proof, but I'm pretty sure all arguments hold since P^{m-1} itself is always a smooth manifold. Indeed I'm pretty sure $P^{m-1} \cong P(W)$.

Say then that every k-form of $\Omega^k V$ was decomposable and one could make a surjective map

$$\tilde{\xi} : (V^*)^k \longrightarrow \Omega^k V;$$

$$\tilde{\xi}(\omega_1, \dots, \omega_k) = \omega_1 \wedge \dots \wedge \omega_k$$

it is clearly C^∞. This would induce another C^∞ surjective map

$$\begin{aligned} \xi : Gr_k(V^*) &\longrightarrow P(\Omega^k V); \\ \operatorname{span}\{\underbrace{\omega_1, \dots, \omega_k}_{L_i}\} &\longmapsto [\tilde{\xi}(\omega_1, \dots, \omega_k)] \end{aligned}$$

Admittedly we've define $Gr_k(V^*)$ as a quotient space of $n \times k$ matrices, but clearly this space is in bijection to the set of k-subspaces of V^* by letting the subspace be the one generated by the matrices columns. The argument at the beginning of page 6 shows that ξ is well defined.

Now, a surjective C^∞ map from an $k(n-k)$ dimensional manifold to a $\binom{n}{k}-1$ dimensional one is only possible if $k(n-k) \geqslant \binom{n}{k}-1$. Note that this is the case if $k \in \{1,2\}$ and $n \in \{1,2,3\}$ but isn't the case if $k=2$ and $n=4$ which explains the whole exercise. This also won't be the case for large k and n and this is what makes the Grassmanian interesting. Indeed for these small dimensions the Grassmanian is just a projective space. I didn't prove that $\tilde{S}$ is a diffeomorphism if it is surjective but I'm pretty sure it is. Nonetheless this is a digression