General Stability under actions

Let X be a set. We define an associative operation

$$\cdot : X \times X \longrightarrow X$$
$$(x,y) \longmapsto x\cdot y$$

which is just any function. Then $(X,\cdot)$ is called a grupoid, usually this is such a weak structure nothing of really usefull comes out of it. Here we use semigroups as a base for our other definitions. Jumping quite a few steps on the algebra ladder we define a monoid as being a semigroup with unity, that is, $\exists 1 \in X;\ 1\cdot x = x\cdot 1 = x \ \forall x \in X$.

Note that given any set S, the set of functions from S to itself is a monoid under function composition. We define a semigroup morphism to be a function between semigroups $(X,\cdot),(\tilde{X},\tilde{\cdot})$

$$\varphi : X \longrightarrow \tilde{X}$$

such that $\varphi(x\cdot y) = \varphi(x)\tilde{\cdot}\varphi(y)\ \forall x,y\in X$. If in addition $(X,\cdot)$ and $(\tilde{X},\tilde{\cdot})$ are monoids this same φ will be called a monoid morphism.

With these definitions we can give the crucial definition for us here. Let $End(S)$ we be the set of functions from S to itself, we've already remarked it is a monoid, then if $(M,\cdot)$ is a monoid, a monoid action of M on S is simply a monoid morphism

$$\psi : M \longrightarrow End(S)$$

and this will allow us to formulate stability under actions. First, there are two crucial examples of actions we lay out here. If ψ is the action we commention $\psi_x := \psi(x)$ for simplicity of notation.

Example 1: Translation: Let M act on itself and put $\psi_x(y) = x\cdot y$.

Example 2: Conjugation: If M is in addition a group, that is, $\forall x\in M, \exists y \in M;$ $x\cdot y = y\cdot x = 1$ then we can let M act on itself by saying $\psi_x(y) = xyx^{-1}$.

We note that in the second example each ψ_x is actually a monoid isomorphism (morphism that is a bijection) with inverse $\psi_{x^{-1}}$.

If $(M,\cdot)$ is a monoid and $M' \subset M$, if $M'\cdot M' := \{m\cdot m' \mid m,m' \in M'\} \subset M'$ then we say that $(M',\cdot|_{M'})$ is a submonoid ($\cdot|_{M'}$ has its codomain restricted as well, we will, however, ignore such minutia and just write $(M',\cdot)$). We will say the monoid $(M',\cdot)$ is stable under an action $\psi : X \longrightarrow End(M)$ (End seeing M as a set) if $\psi_x(M') \subset M'$ $\forall x \in X$.

With such general formulations done, we shall have many examples of basic Algebra as particular cases.

Groups: Let $(G, \cdot)$ be a group, we can define a monoid action of the multiplicative monoid (group too) $\{-1, 1\}$ on G by saying $\psi_1 = Id$ and ψ_{-1} is the inversion map $\psi_{-1}(x) = x^{-1}$. Then a subgroup on G is a submonoid that is stable by this action. A normal subgroup is a subgroup that is stable by conjugation.

Rings: A ring $(R, +, \cdot)$ is a trio such that $(R, +)$ is a group with the abelian property $x + y = y + x$ (usually "+" means commutative operation in algebra), $(R, \cdot)$ is a monoid and we have the following compatibility conditions between the additive and multiplicative structures

$$1)\ x\cdot(y+z) = xy + xz \qquad \forall x, y, z \in R$$
$$2)\ (y+z)\cdot x = yx + zx$$

The analogous concept of normal subgroups for rings is that of an ideal. An ideal $I \subset R$ is a subset such that $(I, +)$ is a subgroup of $(R, +)$ and I is stable by translations of $(R, \cdot)$. Said differently an ideal is a submonoid I of R that is stable by the inversion action of $\{-1, 1\}$, stable by conjugation of $(R, +)$ and stable by translation of $(R, \cdot)$.

Modules: A module (left module to be precise, but we will always say "module" to refer to "left module") $(V, +, R, \oplus, \odot, \cdot)$ is a quintuple such that $(V, +)$ is an abelian group, $(R, \oplus, \odot)$ is a ring and

$$\cdot : R \times V \longrightarrow V$$

is a function such that the following holds:

$$1)\ a\cdot(v + w) = a\cdot v + a\cdot w$$
$$2)\ (a \oplus b)\cdot v = a\cdot v + b\cdot v$$
$$3)\ (a \odot b)\cdot v = a\cdot(b \cdot v)$$

One may see (1) and (2) as compatibility conditions and (3) as an associativity condition. Then a submodule $V' \subset V$ is one such that $(V', +)$ is a subgroup of $(V, +)$ and V' is stable under the action induced by R given by

$$\psi : R \longrightarrow End(V) \qquad (End\ seeing\ V\ as\ a\ set)$$

such that $\psi_a(v) = a\cdot v$.

We leave as final remarks a few considerations. The first is that the definition of module is precisely what one would imagine by a "ring action" on an abelian group even though this is not an usual definition at all. Let me explain what I mean more precisely. If $(V,+)$ is an abelian group, we may make the set of bijections of V into itself a ring by saying $(f+g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(g(x))$. We will still call this ring $End(V)$ even though the maps there need not be morphisms. Then a ring action of $(R, \oplus, \odot)$ on $(V,+)$ is simply a ring morphism

$$\Psi : R \longrightarrow End(V)$$

where by ring morphism we mean a monoid morphism in each of the multiplicative and additive structures, so we may say a module is instead a triple $(V, +, \Psi)$.

I hope with this short note I was able to convey a bit of unified nature of algebraic structures. This is, as I believe (as a mere undergrad) the characteristical feature of Algebra that sets it apart from other areas of mathematics. Here the unifying glue was monoid actions but this is merely a drop of water amidst a sea of examples of concepts that provide unity in Algebra.