6) A $\omega \in \Omega^k(V)$ is decomposable iff $\omega = \phi_1 \wedge \dots \wedge \phi_k$ for some $\phi_i \in V^* = \Omega^1 V$. (1)

a) **Prop**: Every 2 form on an n-dimensional space with $n \leq 3$ is decomposable

Proof: For $n = 1,2$ is obvious. Say $\omega \in \Omega^2 V$ with $n = 3$ and ϕ_1, ϕ_2, ϕ_3 is a dual basis. Then $\omega = \omega_{12}\phi_1 \wedge \phi_2 + \omega_{13}\phi_1 \wedge \phi_3 + \omega_{23}\phi_2 \wedge \phi_3 = \phi_1 \wedge (\omega_{12}\phi_2 + \omega_{13}\phi_3) + \omega_{23}\phi_2 \wedge \phi_3 = (\phi_1 + \omega_{23}\phi_2) \wedge (\omega_{12}\phi_2 + \omega_{13}\phi_3)$ X

wrong, lets solve the general case later

b) **Prop**: The above affirmation is false for $n = 4$

Proof: Say $\phi_1, \dots, \phi_4$ is a dual basis, I claim $\omega = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$ isn't decomposable (that claims actually).

$$(a_1\phi_1 + a_2\phi_2 + a_3\phi_3 + a_4\phi_4) \wedge (b_1\phi_1 + b_2\phi_2 + b_3\phi_3 + b_4\phi_4) = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$$

$$\Leftrightarrow \sum_{1 \leq i < j \leq 4} (a_i b_j - a_j b_i)\phi_i \wedge \phi_j = \phi_1 \wedge \phi_2 + \phi_3 \wedge \phi_4$$

$$\Leftrightarrow \begin{cases} a_1 b_2 - a_2 b_1 = 1 \\ a_1 b_3 - a_3 b_1 = 0 \\ a_1 b_4 - a_4 b_1 = 0 \\ a_2 b_3 - a_3 b_2 = 0 \\ a_2 b_4 - a_4 b_2 = 0 \\ a_3 b_4 - a_4 b_3 = 1 \end{cases}$$

X absolute mess, lets try to develop a more general theory.

Let $(R, \oplus, \odot)$ be a ring and $(M, +)$ be an abelian group, then a (left) R-module is an ordered tuple $(R, \oplus, \odot, M, +, \cdot)$ where $\cdot : R \times M \longrightarrow M$ such that the following hold for $v, w \in M$ and $r, s \in R$:

1) $r \cdot (v + w) = r \cdot v + r \cdot w$
2) $(r \oplus s) v = r \cdot v + s \cdot v$] Distributivity
3) $(r \odot s) \cdot v = r \cdot (s \cdot v)$] associativity
4) $1 \cdot v = v$] respects neutral element

This is fairly standard in Algebra, when making a more complicated structure from simpler ones one essentially just sticks the simpler structures together and requires a few conditions for the operations to match. Rings do this with groups and monoids, modules with rings and groups and vector spaces with fields and groups.

Nonetheless the structure we are after are not modules, they are algebras (of a particular kind). An Algebra is an ordered tuple $(R, \oplus, \odot, M, +, \cdot, \tilde{\cdot})$ such that $(R, \oplus, \odot, M, +, \cdot)$ is a module and $(M, +, \tilde{\cdot})$ is a ring. We say that the Algebra is associative if $r\cdot(v \tilde{\cdot} w) = (r\cdot v)\tilde{\cdot}(w)$ $\forall r \in R, \forall v, w \in M$.

We say a ring A is graded if $A = A_1 \oplus A_2 \oplus \dots$ and $A_i A_j \subset A_{i+j}$ (for instance the ring of polynomials) with each A_i being closed additively. I'm gonna answer to a less general than usual definition for graduate Algebra. Simply, an algebra $(R, \oplus, \odot, M, +, \cdot, \tilde{\cdot})$ is graded if $(M, +, \tilde{\cdot})$ is a graded ring and

$$R\cdot M_i \subset M_i \quad \forall i$$

In this general context one can easily state what we are studying with decomposability, we are simply looking to know if $\underbrace{M_1 \tilde{\cdot} M_1 \tilde{\cdot} \dots \tilde{\cdot} M_1}_{k \text{ products}} = M_k$.

Although the statement is simple the generality makes studying it quite hard but some algebraic property may help us. I shall say that the particular structure we are interested in, the differential forms, has an even stronger structure, that of a differential graded algebra, which is a graded algebra with maps $d_i : M_i \to M_{i+1}$.

One thing I've realised is quite astonishing. A standard example of graded ring (not quite an Algebra) are the polynomials over a field $K[X]$ with the obvious operations ($\tilde{\cdot}$ is the convolution product). The statement that $\mathbb{C}[X]$ is made up of only decomposable elements is literally the Fundamental Theorem of Algebra. Clearly $\mathbb{R}[X]$ isn't decomposable (x^2+1 for instance). So decomposability is related to the FTA in the case of polynomial rings.

Through research I've found such a remarkable solution to this problem I can't contain my excitement, we are gonna go in a long journey to uncover it though. Say V is a vector space over $\mathbb{R}$, one wants to talk about choosing subspaces of V "smoothly", however this, at the moment, makes no sense. We would need to add a differential structure to the set of subspaces (of a fixed dimension) of V. The Grassmanian was made for precisely this reason.

Fix a base $v_1, \dots, v_m$ for V and fix a k, we will now define $Gr_k(V)$. Say $M_{m \times k}$ is the set of full rank $m \times k$ real matrices, clearly each $A \in M_{m \times k}$ has a subspace "induced" by it in some way, namely, if $A = (a_{ij})$ then $\text{span}\{\sum_{i=1}^{m} a_{i1} v_i, \dots, \sum_{i=1}^{m} a_{ik} v_i\} =: \text{sub}(A)$ is the associated subspace. Clearly though the association isn't one to one, for instance permuting the collumns of A leads to a matrix with the same associated subspace. Indeed

$$\text{sub}(A) = \text{sub}(A')$$
$$\iff A = A'M \quad , \exists M \in GL(k, \mathbb{R})$$

So let's say

$$A \sim A' \quad \text{if } A = A'M \ , \ \exists M \in GL(k, \mathbb{R})$$

and $Gr_k(V) = M_{m \times k}/\sim$. We note that $Gr_k(V)$ depends in no way on V besides its dimension and what field it's over so we may more accurately use the notation $Gr_{\mathbb{R}}(m, k)$, nonetheless I will stick with the old one. Lets proceed to the differential structure.

Let $U_{i_1, \dots, i_k}$, $1 \leq i_1 < \dots < i_k \leq m$ be the subset of $Gr_k(V)$ corresponding to the classes $[W]$; the matrix $W_{i_1, \dots, i_k}$ which is just W but removing the rows that aren't one of the i_j's is invertible. We need to check this is well defined. If $M \in GL(k, \mathbb{R})$, but this is (kinda) clear since

$$\det(WM)_{i_1, \dots, i_k} = \det(W_{i_1, \dots, i_k}) \cdot \det(M)$$

as can be seen by noticing that the i_j-th row of WM depends exclusively on M and on the i_j-th row of W. Then for $[W] \in U_{i_1, \dots, i_k}$ apply left operations by members of $GL(k, \mathbb{R})$ to obtain an $\hat{W}$; $[W] = [\hat{W}]$ and $\hat{W}_{i_1, \dots, i_k} = Id_{k \times k}$, for instance if $i_j = j$ $\forall j$ then $\hat{W}$ is of form

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \\ \hline & [W]_{i_1, \dots, i_k} & \end{bmatrix} \qquad (*)$$

where $[W]_{i_1, \dots, i_k}$ is an $(m-k) \times k$ matrix. Note that $[W]_{i_1, \dots, i_k}$ is completely determined from $[W]$ as a member of $U_{i_1, \dots, i_k}$ since if we multiplied $\hat{W}$ by any $B \in GL(k, \mathbb{R})$ it would lose the format $(*)$ unless $B = Id$. So define a chart whose domain is $U_{i_1, \dots, i_k}$ and its map is $x_{i_1, \dots, i_k}([W]) = [W]_{i_1, \dots, i_k}$.

We define these charts for all $1 \le i_1 < \ldots < i_k \le n$ to obtain $\binom{n}{k}$ charts, if they are C^∞-related we are done defining the smooth structure, the finitude of charts will make $Gr_k(V)$ 2-enumerable and we will still need to check Hausdorffness. To help with seeing C^∞-relatedness note that, using the previous page's notation, if $[W] \in U_{i_1,\ldots,i_k} \cap U_{j_1,\ldots,j_k}$ then one can obtain $\hat{W}$ and $\tilde{W}$; $[\hat{W}] = [\tilde{W}] = [W]$ via

$$\hat{W} = W \cdot (W_{i_1,\ldots,i_k})^{-1}$$
$$\tilde{W} = W \cdot (W_{j_1,\ldots,j_k})^{-1}$$

$$\Rightarrow \tilde{W} = \hat{W}(W_{i_1,\ldots,i_k}) \cdot (W_{j_1,\ldots,j_k})^{-1}$$

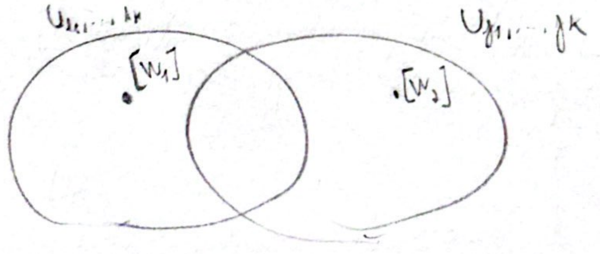
$$\Rightarrow x_{j_1,\ldots,j_k} = x_{i_1,\ldots,i_k} \cdot r_{i_1,\ldots,i_k} \cdot (I \circ r_{j_1,\ldots,j_k}) \qquad (1)$$

where $r_{i_1,\ldots,i_k} : M_{n\times k} \to M_{k\times k}$ is the obvious map as is $r_{j_1,\ldots,j_k}$ and I is the inversion map. Since $r_{i_1,\ldots,i_k} \cdot (I \circ r_{j_1,\ldots,j_k})$ is C^∞ (straightforward to prove I is C^∞ given a trick using determinants) this means $x_{j_1,\ldots,j_k}$ and $x_{i_1,\ldots,i_k}$ are C^∞ related and

$$A = \{(U_{i_1,\ldots,i_k}, x_{i_1,\ldots,i_k}) \mid 1 \le i_1 < \ldots < i_k \le n\}$$

is an atlas. Lets proceed to Hausdorffness.

Take $[W_1], [W_2] \in Gr_k(V)$, say $[W_1] \in U_{i_1,\ldots,i_k}$ and $[W_2] \in U_{j_1,\ldots,j_k}$ wlg if $[W_1] \in U_{j_1,\ldots,j_k}$, $[W_2] \in U_{i_1,\ldots,i_k}$ or $U_{i_1,\ldots,i_k} \cap U_{j_1,\ldots,j_k} = \emptyset$ we are done, so the only non-trivial case is the one drawn bellow



I talk about Hausdorffness latter

Ok, but what does the grassmanian have to do with differential forms? Well, note that if I get a dual base $\phi_1, \dots, \phi_n$ for V^*; pick k-differential forms $\omega_1, \dots \omega_k$ and get $\eta_1, \dots, \eta_k$; (6)

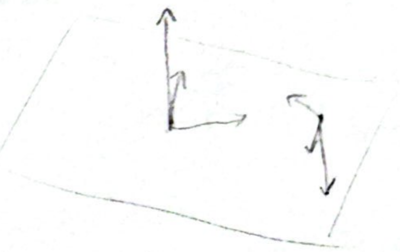
$$\eta_i = \sum_{j=1}^{k} a_i^j \omega_j$$

Then

$$\eta_1 \wedge \dots \wedge \eta_k = \sum_{\sigma \in S_k} \left(\prod_{i=1}^{n} a_{\sigma(i)}^{i} \right) \cdot \omega_1 \wedge \dots \wedge \omega_k \cdot \operatorname{sgn}(\sigma)$$

$$= \det(a_j^i) \cdot \omega_1 \wedge \dots \wedge \omega_k$$

hence $\eta_1 \wedge \dots \wedge \eta_k$ and $\omega_1 \wedge \dots \wedge \omega_k$ are multiples. With this we concluded that the wedge product of any k l.i. covectors in a k dimensional subspace of V^* generates the same k-form up to scaling by a non-zero constant. This in line with the cross product of any two vectors on a plane generating a vector perpendicular to it in $\mathbb{R}^3$.



We will show this leads to a problem with manifold dimensions differing when they should be the same if we assume every k-form is decomposable. First, the dimension of $Gr_k(V)$ is $k(n-k)$ (just the dimension of the matrix $[M]_{i_1, \dots, i_k}$). The dimension of $\Omega^k V$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, but we are actually interested in the dimension of

$$P(\Omega^k V) := \frac{\Omega^k V \setminus \{0\}}{\approx} \quad \text{where} \quad \omega \approx \omega' \text{ if } \omega = \lambda \omega', \ \exists \lambda \in \mathbb{R}^*$$

not sure why, but
I think without removing 0
this isn't a manifold

which is a sort of projective space made out of $\Omega^k V$ in the same way P^2 is made out of $\mathbb{R}^3$. I claim that $P(\Omega^k V)$ is a smooth manifold of dimension $\binom{n}{k} - 1$. In fact I will announce a more general version of this statement such that the notation becomes simpler and we look at only what matters

Lema 6.1. Let W be a vector space of dimension m over $\mathbb{R}$, define

$$P(W) = \frac{W \setminus \{0\}}{\approx} \quad \text{where } v \approx v' \text{ iff } v = \lambda v',\ \exists \lambda \in \mathbb{R}^{\times}$$

Then $P(W)$ is a smooth manifold of dimension $m-1$.

Proof: Firstly $W \setminus \{0\}$ is certainly a smooth manifold of dimension m since it is a subset of W which is itself a smooth manifold (all vector spaces over $\mathbb{R}$ are C^∞-manifolds). Fix an isomorphism $f: W \longrightarrow \mathbb{R}^m$ and set

$$\|\cdot\|_W : W \longrightarrow \mathbb{R}$$
$$v \longmapsto \|f(v)\|_{\mathbb{R}^m}$$

Then surely $\|\cdot\|_W$ is a norm. We will denote $\|\cdot\|_W$ by just $\|\cdot\|$. Let $SW := \{v \in W \mid \|v\| = 1\}$, clearly $f|_{SW} : SW \longrightarrow S^{m-1}$ may be used to give a smooth structure to SW by just composing charts of S^{m-1} with $f|_{SW}$. Then consider the group of actions in SW given by $\Gamma = \{e, I\}$ where $I(v) = -v$, this is a finite group and its action is properly discontinuous, hence $P(W) \cong \frac{SW}{\Gamma}$ is a smooth manifold by covering space theory and its dimension is the same as SW's, which is $m-1$. ▲

Obs: Loose proof, but I'm pretty sure all arguments hold since P^{m-1} itself is always a smooth manifold. Indeed I'm pretty sure $P^{m-1} \cong P(W)$.

Say then that every k-form of $\Omega^k V$ was decomposable and one could make a surjective map

$$\tilde{\xi} : (V^*)^k \longrightarrow \Omega^k V;$$

$$\tilde{\xi}(\omega_1, \dots, \omega_k) = \omega_1 \wedge \dots \wedge \omega_k$$

it is clearly C^∞. This would induce another C^∞ surjective map

$$\xi : Gr_k(V^*) \longrightarrow P(\Omega^k V);$$
$$\text{span}\{\omega_1, \dots, \omega_k\} \longmapsto [\tilde{\xi}(\omega_1, \dots, \omega_k)]$$

Admittedly we've define $Gr_k(V^*)$ as a quotient space of $n \times k$ matrices, but clearly this space is in bijection to the set of k-subspaces of V^* by letting the subspace be the one generated by the matrices columns. The argument at the beginning of page 6 shows that ξ is well defined.

Now, a surjective C^∞ map from an $k(m-k)$ dimensional manifold to a $\binom{m}{k}-1$ dimensional one is only possible if $k(m-k) \geq \binom{m}{k}-1$. Note that this is the case if $k \in \{1,2\}$ and $m \in \{1,2,3\}$ but isn't the case if $k=2$ and $m=4$ which explains the whole exercise. This also won't be the case for large k and m and this is what makes the Grassmanian interesting. Indeed for these small dimensions the Grassmanian is just a projective space, I didn't prove that $\tilde{\xi}$ is a diffeomorphism if it is surjective but I'm pretty sure it is. Nonetheless this is a digression

Well, it has been a while since I've done this first part but there are some holes missing in this story, namely:

1) ξ is always an embedding and is a diffeomorphism if $k(m-k)=\binom{m}{k}-1$
2) $Gr_k(V)$ is Hausdorff
3) Decomposability over graded algebras (maybe, but probably not)

<u>Lema 6.2</u>: $k(m-k) = \binom{m}{k}-1$ iff $k \in \{1, m-1, m\}$

<u>Proof</u>: ($\Leftarrow$) Obvious, just verify manually

($\Rightarrow$) The case $m=1$ is trivial. We can verify for $m \geq 4$

$$\binom{m}{m-2}-1 = \frac{m(m-1)}{2}-1 = \frac{m^2}{2}-\frac{m}{2}-1 > 2(m-2) = 2m-4$$

$$\Leftrightarrow \underbrace{m^2-5m+7>0}_{\text{holds for } m\geq 4}$$

and if $k(m-k) < \binom{m}{k}-1$ then it must be $k \notin \{1, m-1, m\}$ and so $k < m \leq \binom{m}{k-1}$ hence

$$k(m-k)+k < \binom{m}{k}+\binom{m}{k-1}-1$$

$$\Rightarrow k(m+1-k) < \binom{m+1}{k}-1$$

▲

<u>Obs</u>: My god, this was so much harder than expected.

Let's verify ξ is always an embedding. Fix a base $\phi_1, \dots, \phi_m$ for V^* and get the subspace generated by $\omega_1, \dots, \omega_k \in V^*$ where

$$\omega_i = \sum_{j=1}^{m} \omega_i^j \phi_j$$

and all ω_i are L.i, then one matrix for the subspace $span\{\omega_1, \dots, \omega_k\}$ is

$$\begin{bmatrix} \omega_1^1 & \cdots & \omega_k^1 \\ \omega_1^2 & & \omega_k^2 \\ \vdots & & \vdots \\ \omega_1^m & & \omega_k^m \end{bmatrix}$$

A chart for $\operatorname{Span}\{w_1, \ldots, w_k\}$ in $Gr_k(V)$ may be constructed, as we did before, but recall that this chart depends upon which k coordinates of the w_i form an invertible square matrix, let's suppose for now it is the first k of them, so that

$$W := \begin{bmatrix} w_1^1 & \cdots & w_k^1 \\ \vdots & & \vdots \\ w_1^n & & w_k^n \end{bmatrix} \sim \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & [W]_{1,\ldots,k} & \end{bmatrix}$$

notation I used before

This means $\det \begin{bmatrix} w_1^1 & \cdots & w_k^1 \\ \vdots & & \vdots \\ w_1^k & & w_k^k \end{bmatrix} \neq 0$, so it is not zero in a neighbourhood $U_{1,2,\ldots,k}$ of $[W]$ (in $Gr_k(V)$) and we may set as a chart for $[W]$ the map $(U_{1,2,\ldots,k}, x_{1,2,\ldots,k})$ where the map $x_{1,2,\ldots,k}$ is the obvious one into $\mathbb{R}^{(n-k)k}$ (I defined it before too)

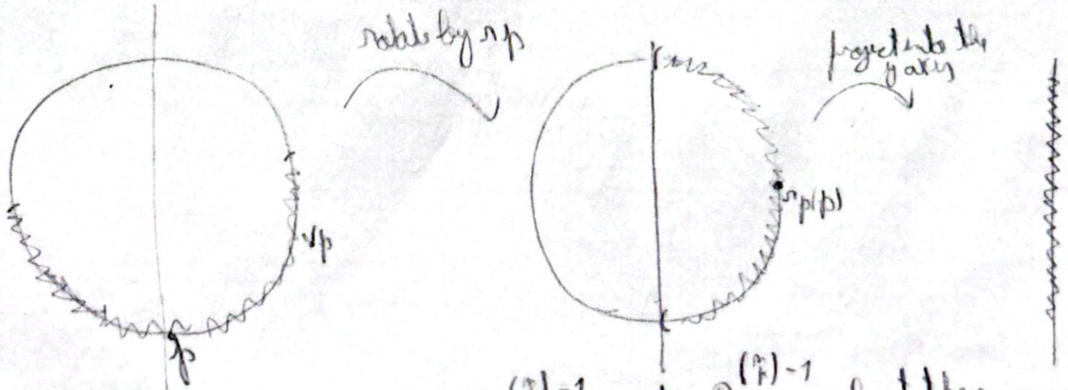
As for a chart of $P(\Lambda^k V)$ we are essentially looking for a chart of $P(\mathbb{R}^{\binom{n}{k}})$. We can use a chart of $S^{\binom{n}{k}-1}$ as a chart for $P(\mathbb{R}^{\binom{n}{k}})$ as long as this chart's domain doesn't contain antipodal points. This means the standard stereographic projection won't work. Thankfully the other standard chart for projective space does work though.

The process I will do now is an explicit construction using essentially the same idea as lemma 6.1 so the reader may skip it if he is convinced ξ is a C^∞ map. Fix an isomorphism of vector spaces

$$f : \Lambda^k V \longrightarrow \mathbb{R}^{\binom{n}{k}}$$

and set $S^{\binom{n}{k}-1}(V) := \{\eta \in \Lambda^k V \mid \|f(\eta)\| = 1\}$, then clearly $f| : S^{\binom{n}{k}-1}(V) \longrightarrow S^{\binom{n}{k}-1}$ is a bijection and one can define the smooth structure of $S^{\binom{n}{k}-1}(V)$ such that $f|$ is still a diffeomorphism. Fix a point $p \in S^{\binom{n}{k}-1}$, applying a rotation r_p if necessary we may assume $r_p(p) = (1, 0, \ldots, 0)$, then a chart for p is $(V_p := \{r_p^{-1}(x_1, \ldots, x_{\binom{n}{k}}) \mid x_1 > 0$ and $\sum x_i^2 = 1\}, P_p)$ where

$$P_p(r_p^{-1}(x_1, \ldots, x_m)) = (x_2, \ldots, x_m)$$



Now, obviously this is a chart into $(-1,1)^{\binom{n}{k}-1}$, not $\mathbb{R}^{\binom{n}{k}-1}$, but they are diffeomorphic so it doesn't matter.

As mentioned before, the major advantage of this chart is that it works for the projective space too. By that I mean that, if we set $P(\Omega^k V)$ as $S^{\binom{n}{k}-1}(V)/v\sim -v$ and put $v \approx -v$ for $v \in S^{\binom{n}{k}-1}$ then

$$\tilde{\beta l}: P(\Omega^k V) \longrightarrow S^{\binom{n}{k}-1}/\approx$$
$$[v]_\sim \longmapsto [\beta l(v)]_\approx$$

we define as being a diffeomorphism to define the smooth structure of $P(\Omega^k V)$ (it is the same as lemma 6.1 but explicitly) and if $q: S^{\binom{n}{k}-1} \longrightarrow S^{\binom{n}{k}-1}/\approx$ is the quotient map,

$$\tilde{P}_p : q(V_p) \longrightarrow \mathbb{R}^{\binom{n}{k}-1}$$
$$[v]_\approx \longmapsto P_p(v)$$

is well defined, is injective, and is indeed a diffeomorphism onto its image. We can then set as a chart for $\omega \in P(\Omega^k V)$ the pair $\left(\tilde{\beta l}^{-1}(q(V_{\tilde{\beta l}(\omega)})),\ \tilde{P}_{\beta l(\omega)} \circ \tilde{\beta l}\right)$. Although notation has become confusing, the diagram summarizes things well

$$\begin{tikzcd}
Gr_k(V) \supset U_{1,2,\dots,k} \arrow[rr, bend left, "\xi"] \arrow[d, "x_{1,2,\dots,k}"] & & \tilde{\beta l}^{-1}(q(V_{\tilde{\beta l}(\omega)})) \subset P(\Omega^k V) \arrow[d, "\tilde{\beta l}"] \\
\mathbb{R}^{(n-k)k} \arrow[rrd, dashed, "\Phi"] & & S^{\binom{n}{k}-1}/\approx \arrow[d, "\tilde{P}_{\beta l(\omega)}"] \\
& & \mathbb{R}^{\binom{n}{k}-1}
\end{tikzcd}$$

using isomorphism $\Omega^k V \to \mathbb{R}^{\binom{n}{k}}$ on quotients

the map $\Phi \to$

If we denote $\Phi_\omega := \tilde{P}_{\beta l(\omega)} \circ \tilde{\beta l} \circ \xi \circ x_1^{-1}$, then ξ is C^∞ iff Φ_ω is C^∞ by definition so now we can look at a map $\mathbb{R}^{(n-k)k} \to \mathbb{R}^{\binom{n}{k}-1}$ which will be (hopefully) easier to analyze. The map Φ_ω can be written explicitly as

$$\Phi_\omega\left(\begin{bmatrix} a_{11} & \cdots & a_{k1} \\ a_{(n-k)1} & & a_{(n-k)k} \end{bmatrix}\right) = \tilde{P}_{\beta l(\omega)} \circ \tilde{\beta l} \circ \xi \left(\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ a_{11} & \cdots & a_{1k} \\ a_{(n-k)1} & \cdots & a_{(n-k)k} \end{bmatrix}\right) =$$

$\{v_i\}$ is the base to which $\{\phi_i\}$ is a dual base

$$= \tilde{P}_{\beta l(\omega)} \circ \tilde{\beta l}\left(\left[\bigwedge_{i=1}^{k}\left(\phi_i + \sum_{j=1}^{n-k} a_{ji}\phi_{k+j}\right)\right]\right) = \tilde{P}_{\beta l(\omega)}\left(\left[\left(\bigwedge_{i=1}^{k}\left(\phi_i + \sum_{j=1}^{n-k} a_{ji}\phi_{k+j}\right)\right)(v_{i_1},\dots,v_{i_k})\right]_{i_1,\dots,i_k}\right)$$

a vector in $\mathbb{R}^{\binom{n}{k}}$ I'm indexing using $i_1,\dots,i_k$ to make things easier

$$= \left(\left(\bigwedge_{i=1}^{k}\left(\phi_i + \sum_{j=1}^{n-k} a_{ji}\phi_{k+j}\right)\right)(v_{i_1},\dots,v_{i_k}) / c\right)_{i_1,\dots,i_k \neq 1,\dots,k}$$

where

$$c = \operatorname{sgn}\left(\left(\bigwedge_{i=1}^{k}\varphi_i + \sum_{j=1}^{n}a_{ji}\cdot\varphi_{k+j}\right)(v_1,\ldots,v_k)\right)\cdot\sqrt{\sum_{1\le i_1<\ldots<i_k\le m}\left(\left(\bigwedge_{i=1}^{k}\varphi_i+\sum_{j=1}^{n}a_{ji}\cdot\varphi_{k+j}\right)(v_{i_1},\ldots,v_{i_k})\right)^2}$$

which is an absolutely disgusting formula even though I assumed no rotation was needed to simplify something. Regardless c is a C^∞ function of the a_{ji} given that the sgn is constant in a neighbourhood of ω and it is also not 0 in some neighbourhood of ω. Since also

$$(a_{ij}) \longrightarrow \bigwedge_{i=1}^{k}\left(\varphi_i + \sum_{j=1}^{n}a_{ji}\cdot\varphi_{k+j}\right)$$

is C^∞ then the whole map is C^∞ and so S is C^∞.

I will not write this explicitly, but the vectors $\frac{\partial \Phi_\omega}{\partial a_{ji}}$ are all Li hence by the inverse function theorem Φ_ω (hence S) is a diffeomorphism into its image locally. Despite this massive blablabla we've not even checked S is injective or surjective if $k\in\{1,n-1,n\}$! I guess looking at Φ_ω only complicated things since verifying $\{\frac{\partial\Phi_\omega}{\partial a_{ij}}\}$ is Li is horrible still. Still I think the excursion was worth, if for nothing else, for the fun of getting into hard details!

Let's start with injectivity then. If $\omega_1\wedge\ldots\wedge\omega_k = \lambda\cdot\eta_1\wedge\ldots\wedge\eta_k$ for $\lambda\neq 0$ then notice that they zero on the same vectors, that is to say, the $k-1$ form $v\lrcorner\,\omega_1\wedge\ldots\wedge\omega_k$ is the null form iff $v\lrcorner\,\eta_1\wedge\ldots\wedge\eta_1$ is also the null form. Say v_i is such that $\eta_i(v_i)=1$ for $i=1,\ldots,k$, then

$$\eta_i(v) = \eta_i(v)\cdot\prod_{j\neq i}\eta_j(v_j) = \frac{(\omega_1\wedge\ldots\wedge\omega_k)(v_1,\ldots,v_{i-1},v,v_{i+1},\ldots,v_k)}{\lambda}$$

$$= \sum_{\sigma\in S_k}\frac{\operatorname{sgn}(\sigma)}{k!\,\lambda}\cdot\left(\prod_{\substack{j=1\\ j\neq i}}^{k}\omega_{\sigma(j)}(v_j)\right)\cdot\omega_{\sigma(i)}(v)$$

$$= \sum_{j=1}^{k}\omega_j(v)\left(\sum_{\substack{\sigma\in S_k;\\ \sigma(i)=j}}\frac{\operatorname{sgn}(\sigma)}{k!\,\lambda}\cdot\prod_{\substack{j=1\\ j\neq i}}\omega_{\sigma(j)}(v_j)\right)$$

hence each v_i is contained in $\operatorname{span}\{\omega_1,\ldots,\omega_k\}$. The exact same argument but for the ω_i proves $\operatorname{span}\{\omega_1,\ldots,\omega_k\}\subset\operatorname{span}\{\eta_1,\ldots,\eta_k\}$ hence both spans are equal and S is injective.

For surjectivity, it is obvious if $k=1$ or $k=n$ (for $k=n$ all forms are multiples of the determinant)

If $k=m-1$ then notice that if $\eta=\sum_{i=1}^{m}\eta_i\,\phi_1\wedge\dots\wedge\hat{\phi}_i\wedge\dots\wedge\phi_m$, with $\phi_1,\dots,\phi_m$ a dual base then

$$\eta=\phi_1\wedge\sum_{i=2}^{m}\eta_i\cdot\phi_2\wedge\dots\wedge\hat{\phi}_i\wedge\dots\wedge\phi_m+\eta_1\cdot\phi_2\wedge\dots\wedge\phi_m \quad (*)$$

if η_2 is 0 ignore this next step and just don't divide by η_2 and remove $i=2$ from the summation

$$=\left(\phi_1+\frac{\eta_1}{\eta_2}\phi_2\right)\wedge\left(\sum_{i=2}^{m}\eta_i\cdot\phi_2\wedge\dots\wedge\hat{\phi}_i\wedge\dots\wedge\phi_m\right)$$

Then we repeat this process inductively, I will do one more step explicitly

$$=\left(\phi_1+\frac{\eta_1}{\eta_2}\phi_2\right)\wedge\left(\phi_2\wedge\sum_{i=3}^{m}\eta_i\phi_3\wedge\dots\wedge\hat{\phi}_i\wedge\dots\wedge\phi_m+\eta_2\phi_3\wedge\dots\wedge\phi_m\right)$$

$$=\left(\phi_1+\frac{\eta_1}{\eta_2}\phi_2\right)\wedge\left(\phi_2+\frac{\eta_2}{\eta_3}\phi_3\right)\wedge\left(\sum_{i=3}^{m}\eta_i\phi_3\wedge\dots\wedge\hat{\phi}_i\wedge\dots\wedge\phi_m\right)$$

again just ignoring $i=3$ in the summation and not dividing by η_3 if η_3 is 0.

Now, this is not enough to conclude ξ is an embedding, we still need to verify it is a homeomorphism into its image. For this (major) the simplest and most insightfull path is proving $Gr_k(V)$ is compact since continuous maps from compact spaces into Hausdorff spaces are closed (hence if invertible the inverse is continuous). To prove this I found a marvelous proof in Math Stack Exchange from "uncookedfalcon" that I will slightly modify to fit our needs. Consider the map

$$Inn:\left(S^{m-1}\right)^k\longrightarrow\mathbb{R}^{\frac{k(k-1)}{2}}$$
$$(v_1,\dots,v_k)\longmapsto(v_i\cdot v_j)_{i<j}$$
$$\|v_i\|=1\ \forall i$$

Clearly the domain is compact (by Tychonoff and Borel-Lebesgue if you will) hence any closed subset of it is compact. The map Inn is continuous, so $(I_{nn})^{-1}(0,0,\dots,0)$ is compact too. Let q be the projection map from the space of full rank $m\times k$ matrices to $Gr_k(V)$, the statement

$$\forall[W]\in Gr_k(V),\ \exists(v_1,\dots,v_k)\in I_{nn}^{-1}(0);\ q\left(\begin{bmatrix} v_1 & \cdots & v_k\end{bmatrix}\right)=[W]$$

is simply the statement that every finite-dimensional vector space with an inner product admits an orthonormal base, which is true by Gram-Schmidt. Hence $Gr_k(V)$ is the image by q of a compact set, hence is compact.

I won't go into too many details about why the Grassmanian is Hausdorff since this digression is becoming too long. We will use the lemma without proof:

<u>Lemma 6.3</u>: Let G be a compact topological group and X a Hausdorff space. If we define a continuous group action of G on X then X/G is Hausdorff.

Now we then need to express $Gr_k(V)$ as the quotient of a Hausdorff space by a compact group. There is a manifold called the Stiefel manifold defined as $V_k(V) := \{(v_1, \dots, v_k) \mid v_i \cdot v_j = \delta_{ij}\}$ and it turns out that $V_k(V)$ quotiented by $O(k)$ using the action of right matrix multiplication is the Grassmanian. Since $O(k)$ is compact this would then mean $Gr_k(V)$ is Hausdorff.