Series of real numbers

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By *series*, roughly speaking, one defines the notion of an infinite sum. The first results concerning series was given Before Christ. Archimede considered the geometric series and calculated its sum. For a long time, the idea that such a potentially infinite summation could produce a finite result was considered paradoxical. This paradox was solved using the concept of a limit during the 17th century only. Many speculations concerning series appeared before a rigourous definition. For instance in the infinite sum

$$1 + (-1) + 1 + (-1) + \cdots$$

taking

$$(1 - 1) + (1 - 1) + \cdots = 0$$

while taking

$$1 + (-1 + 1) + (-1 + 1) + \cdots = 1,$$

which leads to a contradiction. Gauss, Bolzano and Cauchy gave the rigourous definition of series which is used nowadays.

Let $(a_n)_{n\geq 1}$ be a sequence of numbers. Define the sequence $(s_n)_{n\geq 1}$ by the relation

$$s_n = a_1 + a_2 + \cdots + a_n, \quad n \geq 1.$$

Definition 1 *The pair of sequences $((a_n), (s_n))$ is called* series *with the* general term a_n*.*

A series of general term a_n is denoted either by

$$\sum_{n=1}^{\infty} a_n \quad \text{or by} \quad \sum_{n\geq 1} a_n \quad \text{or simply by} \quad \sum a_n.$$

The sequence $(s_n)_{n\geq 1}$ is called the *the sequence of partial sums of the series* $\sum\limits_{n=1}^{\infty} a_n$.

Definition 2 *A series* $\sum\limits_{n=1}^{\infty} a_n$ *is said to be* convergent *if* $(s_n)_{n\geq 1}$ *is convergent and its limit is called the* sum *of the series. A series is said to be* divergent *if* $(s_n)_{n\geq 1}$ *is divergent.*

For a convergent series $\sum\limits_{n=1}^{\infty} a_n$ with sum $s = \lim\limits_{n\to\infty} s_n$ we write

$$\sum_{n=1}^{\infty} a_n = s.$$

Example 1 *1)* ***The geometric series*** $\sum\limits_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots$, $q \in \mathbb{R}$, *is convergent if and only if* $q \in (-1, 1)$. *The following relation holds:*

$$\sum_{n=0}^{\infty} q^n = \begin{cases} \frac{1}{1-q}, & \textit{if} \ \ q \in (-1,1) \\ +\infty, & \textit{if} \ \ q \in [1, \infty) \end{cases}$$

If $q \leq -1$, *then the geometric series is divergent.*

Proof. *We get*

$$s_n = 1 + q + \ldots + q^n = \begin{cases} \frac{1-q^{n+1}}{1-q}, & q \neq 1 \\ n+1, & q = 1, \end{cases}$$

hence

$$\lim_{n\to\infty} s_n = \begin{cases} +\infty, & q \leq 1 \\ \frac{1}{1-q}, & q \in (-1,1) \\ \nexists, & q \leq -1 \end{cases}$$

■

2) ***The harmonic series*** $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ *is divergent and* $\sum\limits_{n=1}^{\infty} \frac{1}{n} = +\infty$.

Proof. *The sequence of partial sums of the harmonic series given by*

$$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, \quad n \geq 1,$$

is divergent, $\lim_{n\to\infty} s_n = \infty$. *(see Example...sequences)* ■

3) ***The generalized harmonic series*** $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$, *is convergent if and only if* $\alpha > 1$. *For* $\alpha > 1$ *we denote*

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha}.$$

The function $\zeta : (1, \infty) \to \mathbb{R}$ *is called* ***Riemann's Zeta function****. The following relations hold*

$$\zeta(2) = \frac{\pi^2}{6} \text{ (Euler)}, \quad \zeta(4) = \frac{\pi^4}{90}.$$

Next we enumerate some properties of convergent series.

Theorem 1 *Suppose that* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *are convergent series and* $\sum_{n=1}^{\infty} a_n = a$, $\sum_{n=1}^{\infty} b_n = b$, $a, b \in \mathbb{R}$. *Then:*

$$\sum_{n=1}^{\infty} (a_n + b_n) = a + b; \tag{0.1}$$

$$\sum_{n=1}^{\infty} (\lambda a_n) = \lambda a, \lambda \in \mathbb{R}; \tag{0.2}$$

$$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_n = 0 \tag{0.3}$$

The sequence $(r_n)_{n\geq 1}$, $r_n = \sum_{k=n+1}^{\infty} a_k$ *is called the* reminder of order n *of the series* $\sum_{n=1}^{\infty} a_n$.

Theorem 2 (A necessary condition of convergence) *If* $\sum_{n=1}^{\infty} a_n$ *is a convergent series then* $\lim_{n\to\infty} a_n = 0$.

Proof. We have $\lim_{n\to\infty} s_n = s$, $s \in \mathbb{R}$ and $a_n = s_n - s_{n-1}$, $n > 1$. If follows

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

■

Remark 1 *The previous condition is not sufficient for the convergence of a series. (see the harmonic series)*

A general test of convergence for series is given in the next theorem.

Theorem 3 (Cauchy's convergence criterion) *A series* $\sum_{n=1}^{\infty} a_n$ *is a convergent if and only if for every* $\varepsilon > 0$, *there exists* $n_\varepsilon \in \mathbb{N}$ *such that*

$$|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon, \quad \forall n \geq n_\varepsilon, p \geq 1.$$

Proof. The series $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ $(s_n)_{n\geq 1}$ is convergent $\iff$ $(s_n)_{n\geq 1}$ is a Cauchy sequence $\iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that $|s_{n+p} - s_n| < \varepsilon$, for all $n \geq n_\varepsilon, p \geq 1$. $\iff \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that $|a_{n+1} + a_{n+2} + \cdots + a_{n+p}| < \varepsilon$, for all $n \geq n_\varepsilon, p \geq 1$. ■

Definition 3 *A series* $\sum_{n=1}^{\infty} a_n$ *is called* absolutely convergent *if the series* $\sum_{n=1}^{\infty} |a_n|$ *is convergent.*

Theorem 4 *Every absolutely convergent series is convergent.*

Proof. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series and let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| < \varepsilon, \quad \forall n \geq n_\varepsilon, p \geq 1.$$

On the other hand

$$|a_{n+1}+a_{n+2}+\cdots+a_{n+p}| \leq |a_{n+1}|+|a_{n+2}|+\cdots+|a_{n+p}|, \quad \forall n \geq n_\varepsilon, p \geq 1,$$

hence the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. ■

Definition 4 *A convergent series* $\sum\limits_{n=1}^{\infty} a_n$ *which is not absolutely convergent is called a* semiconvergent *series.*

Generally is difficult to use Cauchy's test of convergence in practice but the Cauchy's convergence criterion leads to sufficient conditions for the convergence of a series, called convergence tests.

Theorem 5 (Abel-Dirichlet Test) *Let* $(a_n)_{n\geq 1}$ *be a monotonically decreasing sequence with* $\lim\limits_{n\to\infty} a_n = 0$ *and* $(b_n)_{n\geq 1}$ *a sequence such that the series* $\sum\limits_{n=1}^{\infty} b_n$*, has the sequence of partial sums bounded. Then the series* $\sum\limits_{n=1}^{\infty} a_n b_n$ *is convergent.*

Proof. Let $(s_n)_{n\geq 1}$, $(t_n)_{n\geq 1}$ be given by

$$\begin{aligned} s_n &= = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ t_n &= = b_1 + b_2 + \cdots + b_n, \quad n \geq 1. \end{aligned}$$

Using the hypothesis it follows that $\exists M > 0$ such that $|t_n| \leq M$ for all $n \geq 1$. We have

$$\begin{aligned} |s_{n+p} - s_n| &= |a_{n+1}b_{n+1} + a_{n+2}b_{n+2} + \cdots + a_{n+p}b_{n+p}| \\ &= |a_{n+1}(t_{n+1}-t_n) + a_{n+2}(t_{n+2}-t_{n+1}) + \cdots + a_{n+p}(t_{n+p}-t_{n+p-1})| \\ &= |-t_n a_{n+1} + t_{n+1}(a_{n+1}-a_{n+2}) + \cdots + t_{n+p-1}(a_{n+p-1}-a_{n+p}) + t_{n+p}a_{n+p}| \\ &\leq |-t_n||a_{n+1}| + |t_{n+1}||a_{n+1}-a_{n+2}| + \cdots + |t_{n+p-1}||a_{n+p-1}-a_{n+p}| + |t_{n+p}||a_{n+p}| \\ &\leq M(a_{n+1} + a_{n+1} - a_{n+2} + \cdots + a_{n+p-1} - a_{n+p} + a_{n+p}) = 2Ma_{n+1}, \end{aligned}$$

since $(t_n)_{n\geq 1}$ is bounded and $|a_{n+k}-a_{n+k+1}| = a_{n+k}-a_{n+k+1}$, $1 \leq k \leq p-1$, in view of monotonicity of $(a_n)_{n\geq 1}$. By the inequality

$$|s_{n+p} - s_n| \leq 2Ma_{n+1}, \quad n \geq p \geq 1,$$

we obtain for every $p \geq 1$

$$|s_{n+p} - s_n| \xrightarrow{n\to\infty} 0,$$

thus the series $\sum\limits_{n=1}^{\infty} a_n b_n$ is convergent. ■

Example 2 *Test for convergence the series* $\sum\limits_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$,

Solution 1 *Denote* $a_n = \frac{1}{\sqrt{n}}$, $b_n = \sin n$, $t_n = b_1 + \cdots + b_n$, $n \geq 1$. *The sequence* $(a_n)_{n\geq 1}$ *is monotonically decreasing to zero. Taking account of the trigonometric identity*

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{\sin \frac{nx}{2} \sin \frac{(n+1)x}{2}}{\sin \frac{nx}{2}},$$

which holds for every $x \in \mathbb{R} \setminus \{2k\pi | k \in \mathbb{Z}\}$ *it follows*

$$|t_n| \leq \frac{1}{\sin \frac{1}{2}}, \quad \forall n \geq 1.$$

The conditions of Abel-Dirichlet test are satisfied, therefore the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{\sqrt{n}}$ *is convergent.*

Corollary 0.1 *Let* $(a_n)_{n\geq 1}$ *be a decreasing sequence with* $\lim\limits_{n\to\infty} a_n = 0$. *Then the* alternate series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

is convergent.

Proof. Take $b_n = (-1)^{n-1}$, $n \geq 1$. The series $\sum\limits_{n=1}^{\infty} (-1)^{n-1}$ has a bounded sequence of partial sums, $a_n \searrow 0$, hence the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent in view of Abel-Dirichlet test. ■

1 Series with positive terms

We consider in what follows series of the form $\sum\limits_{n=1}^{\infty} a_n$, where $a_n > 0$ for every $n \in \mathbb{N}^*$. The study of the convergence of series with positive terms is based on the next result.

Lemma 1.1 *A series with positive terms is convergent if and only if its sequence of partial sums is bounded.*

Proof. Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms and $s_n = a_1 + a_2 + \cdots + a_n$, $n \geq 1$. The sequence $(s_n)_{n\geq 1}$ is monotonically increasing. Then there exists $\lim s_n = s$, $s \in \mathbb{R}$, if and only if $(s_n)_{n\geq 1}$ is bounded. ■

In what follows we present some convergence test for series with positive terms.

Theorem 6 (First comparison test) *Let $\sum\limits_{n=1}^{\infty} a_n$, $\sum\limits_{n=1}^{\infty} b_n$ be series with positive terms such that $a_n \leq b_n$, for all $n \geq n_0$. Then*

i) If $\sum\limits_{n=1}^{\infty} b_n$ convergent $\Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ convergent;

ii) If $\sum\limits_{n=1}^{\infty} a_n$ divergent $\Longrightarrow \sum\limits_{n=1}^{\infty} b_n$ divergent.

Proof. Define $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + t_n$, $n \geq 1$. We may suppose without loss of generality $n_0 = 1$, since if we eliminate a finite number of terms in a series its convergence is unchanged. Then

$$s_n \leq t_n, \quad \forall n \geq 1.$$

i) $\sum\limits_{n=1}^{\infty} b_n$ convergent $\Longrightarrow$ $(t_n)_{n\geq 1}$ bounded $\Longrightarrow$ $(s_n)_{n\geq 1}$ bounded $\Longrightarrow$ $\sum\limits_{n=1}^{\infty} a_n$ convergent;

ii) $\sum\limits_{n=1}^{\infty} a_n$ divergent $\Longrightarrow \lim\limits_{n\to\infty} s_n = +\infty \Longrightarrow \lim\limits_{n\to\infty} t_n = +\infty \Longrightarrow \sum\limits_{n=1}^{\infty} b_n$ divergent.

■

Theorem 7 (Second comparison test) *Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be series with positive terms such that* $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$, *for all* $n \geq n_0$. *Then*

i) *If* $\sum_{n=1}^{\infty} b_n$ *convergent* $\Longrightarrow \sum_{n=1}^{\infty} a_n$ *convergent;*

ii) *If* $\sum_{n=1}^{\infty} a_n$ *divergent* $\Longrightarrow \sum_{n=1}^{\infty} b_n$ *divergent.*

Proof. We may suppose without loss of generality $n_0 = 1$. Then, we have

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \Longleftrightarrow \frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n}, \quad n \geq 1.$$

Consequently, $\frac{a_n}{b_n} \leq \frac{a_1}{b_1}$ for every $n \geq 1$, which leads to

$$a_n \leq \frac{a_1}{b_1} b_n, \quad n \geq 1.$$

The conclusion of the theorem follows in view of the first comparison test.
■

Theorem 8 (Third comparison test) *Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be series with positive terms with the property that there exists*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = l, \quad l \in [0, \infty).$$

Then

1) *If* $l \neq 0$ *the series are simultaneously convergent or divergent.*

2) *If* $l = 0$ *we have:*

   i) $\sum_{n=1}^{\infty} b_n$ *convergent* $\Longrightarrow \sum_{n=1}^{\infty} a_n$ *convergent;*

   ii) $\sum_{n=1}^{\infty} a_n$ *divergent* $\Longrightarrow \sum_{n=1}^{\infty} b_n$ *divergent.*

Proof. $\lim_{n\to\infty} \frac{a_n}{b_n} = l \iff \forall\varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$

$$l - \varepsilon < \frac{a_n}{b_n} < l + \varepsilon.$$

1) If $l \neq 0$ take ε such that $l - \varepsilon > 0$. Then for $n \geq n_\varepsilon$

$$(l - \varepsilon)b_n < a_n < (l + \varepsilon)b_n$$

and the conclusion follows from the first comparison test.

2) If $l = 0$ then for a fixed ε one gets

$$a_n < \varepsilon b_n, \quad n \geq n_\varepsilon$$

and the conclusion follows from the first comparison test.

■

Example 3 *The series* $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ *is divergent since*

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Theorem 9 (Cauchy's condensation test) *Let* $(a_n)_{n\geq 1}$ *be a decreasing sequence of positive numbers. Then* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} 2^n a_{2^n}$ *are simultaneously convergent or divergent.*

Proof. Define

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n, \quad n \geq 1, \\ t_n &= = a_1 + 2a_2 + \cdots + 2^n a_{2^n}, \quad n \geq 0. \end{aligned}$$

For $n \leq 2^k$ we have

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} = t_k. \end{aligned}$$

On the other hand, if $n \geq 2^k$

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_n) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_n + \cdots + 2^{k-1}a_{2^k} = \frac{1}{2}t_k. \end{aligned}$$

Then the boundedness of $(s_n)_{n\geq 1}$ is equivalent with the boundedness of $(t_n)_{n\geq 1}$ which leads to the conclusion. ■

Corollary 1.1 *The* generalized harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \frac{1}{1^\alpha} + \frac{1}{2^\alpha} + \cdots + \frac{1}{n^\alpha} + \cdots, \alpha \in \mathbb{R},$$

converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.

Proof. Put $a_n = \frac{1}{n^\alpha}$, $n \geq 1$. If $\alpha \leq 0 \Longrightarrow \lim a_n \neq 0 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ is divergent. For $\alpha > 0$, $(a_n)_{n\geq 1}$ is decreasing thus the generalized harmonic series is convergent if and only if the series

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^n \frac{1}{2^{n\alpha}} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^n$$

is convergent. The geometric series $\sum\limits_{n=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^n$ is convergent if and only if $\frac{1}{2^{\alpha-1}} < 1 \Longleftrightarrow \alpha > 1$. ■

Corollary 1.2 *Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms such that there exists $\alpha \in \mathbb{R}$ with the property*

$$\lim_{n\to\infty} n^\alpha a_n = l, \quad l \in [0, \infty).$$

Then:

- *If $\alpha > 1 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ is convergent.*

- *If* $\alpha \leq 1, l \neq 0 \Longrightarrow \sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. The relation $\lim_{n\to\infty} n^{\alpha} a_n = l$ is equivalent to

$$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n^{\alpha}}} = l,$$

and the result follows from the third comparison test. ■

Example 4 *Test for convergence the series* $\sum_{n=2}^{\infty} (\sqrt[n]{a} - 1)$, $a > 1$.

Solution 2 *We have*

$$\lim_{n\to\infty} n(\sqrt[n]{a} - 1) = \lim_{n\to\infty} \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} = \ln a,$$

so the series in divergent in view of Corollary 1.2 with $\alpha = 1$.

Theorem 10 (Cauchy's root test) *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with positive terms.*

1) *If there exists* $q \in [0, 1)$ *and* $n_0 \in \mathbb{N}$ *such that*

$$\sqrt[n]{a_n} \leq q, \quad \forall n \geq n_0,$$

then $\sum_{n=1}^{\infty} a_n$ *is convergent.*

2) *If the relation* $\sqrt[n]{a_n} > 1$ *holds for infinitely many numbers* n, *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof.

1) By $\sqrt[n]{a_n} \leq q$, for all $n \geq n_0$, we obtain $a_n \leq q^n$, $\forall n \geq n_0$, and since the geometric series $\sum_{n=1}^{\infty} q^n$ is convergent it follows that $\sum_{n=1}^{\infty} a_n$ is convergent, in view of the first comparison test.

2) If $\sqrt[n]{a_n} > 1$ for infinitely many n we obtain $a_n > 1$ for infinitely many n. Then $\lim\limits_{n\to\infty} a_n \neq 0$, hence $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

■

In practice we use the following Corollary which is easier to apply.

Corollary 1.3 *Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms such that there exists*
$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = l$, $l \in \overline{\mathbb{R}}$. *Then:*

i) If $l < 1 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ is convergent;

ii) If $l > 1 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ is divergent;

iii) If $l = 1$ the test fails (is not efficient).

Proof. The points i) and ii) follow from Theorem 10. For iii) one can give examples of convergent and divergent series with $l = 1$. The series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent, while the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and in both cases $l = 1$. ■

Example 5 *Test for convergence the series $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.*

Solution 3 *$a_n = \left(\frac{n}{n+1}\right)^{n^2}$. We have*

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1,$$

the series is convergent in view of Cauchy's root test.

Theorem 11 (D'Alembert ratio test) *Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms.*

1) If there exists $q \in [0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \leq q, \quad \forall n \geq n_0,$$

then $\sum\limits_{n=1}^{\infty} a_n$ is convergent.

2) If there exists $n_0 \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \geq 1$ for all $n \geq n_0$, then $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Proof.

1) We may suppose that $n_0 = 1$. We have:

$$a_n \leq q a_{n-1} \leq q^2 a_{n-2} \leq \cdots \leq q^{n-1} a_1, \quad n \geq n_0,$$

and the conclusion follows applying the first comparison test.

2) The inequality $a_{n+1} \geq a_n$, for all $n \geq n_0$ implies $\lim\limits_{n\to\infty} a_n \neq 0 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ divergent.

■

In practice we use frequently the following test.

Corollary 1.4 *Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with positive terms such that there exists $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, $l \in \overline{\mathbb{R}}$. Then:*

i) If $l < 1 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ is convergent;

ii) If $l > 1 \Longrightarrow \sum\limits_{n=1}^{\infty} a_n$ is divergent;

iii) If $l = 1$ the test fails (is not efficient).

Example 6 *Test for convergence the series:*

1. $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$

2. $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$

Solution 4 *1.* $a_n = \frac{n^2}{2^n}$, $n \geq 1$. *We have*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} < 1,$$

hence the series is convergent in view of the ratio test.

2. $(a_n)_{n\geq 1}$, $a_{2n-1} = \frac{1}{2^n}$, $a_{2n} = \frac{1}{3^n}$. *We have*

$$\begin{aligned} \frac{a_{2n-1}}{a_{2n}} &= \left(\frac{3}{2}\right)^n > 1 \\ \frac{a_{2n+2}}{a_{2n+1}} &= \left(\frac{2}{3}\right)^n < 1, \end{aligned}$$

therefore the ratio test cannot be applied. But $\sqrt[n]{a_n} \leq \frac{1}{2}$, $\forall n \geq 1$, *hence taking account of Cauchy's root test the series is convergent.*

Remark 2 *The Cauchy's root test is powerful than D'Alembert's ratio test.*

Theorem 12 (Raabe-Duhamel test) *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a series with positive terms.*

1) If there exists $q > 1$ *and* $n_0 \in \mathbb{N}^*$ *such that*

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q, \quad \forall n \geq n_0,$$

then $\sum\limits_{n=1}^{\infty} a_n$ *is convergent.*

2) If there exists $n_0 \in \mathbb{N}$ *such that* $n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1$ *for all* $n \geq n_0$ *then the series* $\sum\limits_{n=1}^{\infty} a_n$ *is divergent.*

Proof.

1) We may suppose without loss of generality that $n_0 = 1$. By the relation

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q, \quad n \geq 1,$$

we obtain

$$na_n - (n+1)a_{n+1} \geq (q-1)a_{n+1}, \quad n \geq 1.$$

Adding the previous relations one gets

$$\begin{aligned}
\sum_{k=1}^{n-1}(ka_k - (k+1)a_{k+1}) &\geq (q-1)\sum_{k=1}^{n-1} a_{k+1} \Longleftrightarrow \\
a_1 - na_n &\geq (q-1)(a_2 + a_3 + \cdots + a_n) \Longleftrightarrow \\
a_2 + a_3 + \cdots + a_n &\leq \frac{a_1 - na_n}{q-1} \leq \frac{a_1}{q-1}, \quad n \geq 1.
\end{aligned}$$

Consequently, the sequence of partial sums of the series is bounded, therefore the series is convergent.

2) The relation $n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1$, $n \geq 1$ is equivalent to

$$na_n \leq (n+1)a_{n+1}, \quad n \geq 1,$$

or

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} \leq \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$ the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent in view of the second comparison test.

■

Corollary 1.5 *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a series with positive terms such that there exists*
$\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = l$, $l \in \overline{\mathbb{R}}$. *Then:*

i) If $l > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ *is convergent;*

ii) If $l < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ *is divergent;*

iii) If $l = 1$ *the test fails (is not efficient).*

Example 7 *Test for convergence the following series*

$$\sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{n!}, \quad a > 0.$$

Solution 5 *Let* $a_n = \frac{a(a+1)\cdots(a+n-1)}{n!}$, $n \geq 1$. *The ratio test fails since*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a+n}{n+1} = 1.$$

Applying Raabe-Duhamel test we have

$$\lim_{n\to\infty} n\left(\frac{a_{n+1}}{a_n} - 1\right) = \lim_{n\to\infty} n\left(\frac{a+n}{n+1} - 1\right) = 1 - a.$$

Since $1 - a < 1 \Longleftrightarrow a > 0$, *it follows that the series is divergent.*

Theorem 13 (Cauchy's integral test) *Let* $f : [1, \infty) \to \mathbb{R}_+$ *be a decreasing function. Then the series* $\sum_{n=1}^{\infty} f(n)$ *is convergent if and only if the sequence* $(F(n))_{n\geq 1}$ *given by*

$$F(n) = \int_1^n f(x)dx$$

is convergent.

Proof. Put $s_n = f(1) + f(2) + \cdots + f(n), n \geq 1$. For every $k \in \mathbb{N}^*$ the following relation holds

$$f(k+1) \leq f(x) \leq f(k), \quad x \in [k, k+1].$$

By integration on the interval $[k, k+1]$ one gets

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k).$$

Adding the previous inequalities from 1 to $n-1$ it follows

$$f(2) + f(3) + \cdots + f(n) \leq \int_1^n f(x)dx \leq f(1) + f(2) + \cdots + f(n-1),$$

or equivalently

$$s_n - f(1) \leq F(n) \leq s_{n-1}, \quad n > 1.$$

Hence $(s_n)_{n\geq 1}$ is convergent if and only if the sequence $(F(n))_{n\geq 1}$ is convergent. ■

2 Some properties of series

It is well known that in a finite sum we might change the order of terms and we obtain the same sum. For infinite sums this property does not hold, as we have seen at the beginning of the chapter. In the next theorem we prove that for absolutely convergent series a change of the order of terms does not change the sum of the series.

Theorem 14 *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be an absolutely convergent series of sum* s*. Then for every bijection* $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ *the series* $\sum\limits_{n=1}^{\infty} a_{\sigma(n)}$ *is absolutely convergent and its sum remains* s*.*

Proof. Denote $b_n = a_{\sigma(n)}$, $s_n = a_1 + \cdots + a_n$, $t_n = b_1 + \cdots + b_n$, $n \geq 1$. Let $\varepsilon > 0$ then there exists n_0 depending on ε such that for all $n \geq n_0$ and all $p \geq 1$

$$|a_{n+1}| + |a_{n+2}| + \cdots + |a_{n+p}| < \frac{\varepsilon}{2} \text{ and } |s - s_{n_0}| < \frac{\varepsilon}{2}.$$

We choose $n_1 \in \mathbb{N}^*$ with the property that for $m \geq n_1$, t_m contains the terms $a_1, a_2, \ldots, a_{n_0}$ and let $a_{n_0+k_1}, \ldots, a_{n_0+k_m}$ be the other terms of t_m. It follows

$$|a_{n_0+k_1} + \cdots + a_{n_0+k_m}| < |a_{n_0+k_1}| + \cdots + |a_{n_0+k_m}| < \frac{\varepsilon}{2},$$

hence

$$|t_m - s_{n_0}| < \frac{\varepsilon}{2}, \quad \forall m \geq n_1.$$

We have

$$|s - t_m| < |s - s_{n_0}| + |t_m - s_{n_0}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $m \geq n_1$, therefore $\lim\limits_{m\to\infty} t_m = s$ ■

Corollary 2.1 *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be series with positive terms. Then for every bijection* $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ *we have* $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_{\sigma(n)}$.

One can prove the following surprising result showing the difference between infinite and finite sums (see...).

Theorem 15 (Riemann) *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a semiconvergent series. Then for every* $s \in \overline{\mathbb{R}}$ *there exists a bijection* $\sigma : \mathbb{N}^* \to \mathbb{N}^*$ *such that* $\sum\limits_{n=1}^{\infty} a_{\sigma(n)} = s$.

Example 8 *Consider the semiconvergent series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

Changing the order of the terms we get the following series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots .$$

In order to find the sum of the new series it is easier to calculate the partial *sums* s_{3n}, s_{3n+1} *and* s_{3n+2}. *Therefore we get*

$$\begin{aligned}
s_{3n} &= \left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)\\
&= \left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{4n-2}+\frac{1}{4n}\right)\\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)\\
&= \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n}\right)-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\\
&= \frac{1}{2}\left(\gamma_{2n}+\ln 2n\right)-\frac{1}{2}\left(\gamma_n+\ln n\right)\\
&= \frac{1}{2}\gamma_{2n}-\frac{1}{2}\gamma_n+\frac{1}{2}\ln 2.
\end{aligned}$$

Consequently, $\lim\limits_{n\to\infty} s_{3n} = \frac{1}{2}\ln 2$. *Since* $s_{3n+1} = s_{3n} + \frac{1}{2n+1}$ *and* $s_{3n+2} = s_{3n} + \frac{1}{2n+1} - \frac{1}{4n+2}$ *we have* $\lim\limits_{n\to\infty} s_{3n+1} = \lim\limits_{n\to\infty} s_{3n+2} = \frac{1}{2}\ln 2$.

We conclude that $\lim\limits_{n\to\infty} s_n = \frac{1}{2}\ln 2$, *which is the sum of the new series.*

Definition 5 *Let* $\sum\limits_{n=0}^{\infty} a_n$, $\sum\limits_{n=0}^{\infty} b_n$ *be two series of real numbers. The series* $\sum\limits_{n=0}^{\infty} c_n$, *where* $c_n = \sum\limits_{k=0}^{n} a_k b_{n-k}$, $n \geq 0$, *is called the* Cauchy product *of the given series.*

Generally the Cauchy product of two convergent series is not necessary a convergent series.

Theorem 16 (Mertens) *Let* $\sum\limits_{n=0}^{\infty} a_n$ *be an absolutely convergent series and* $\sum\limits_{n=0}^{\infty} b_n$ *be a convergent series with*

$$\sum_{n=0}^{\infty} a_n = a, \quad \sum_{n=0}^{\infty} b_n = b.$$

Then the Cauchy product $\sum\limits_{n=0}^{\infty} c_n$ *of* $\sum\limits_{n=0}^{\infty} a_n$ *and* $\sum\limits_{n=0}^{\infty} b_n$ *is convergent and*

$$\sum_{n=0}^{\infty} c_n = ab.$$

Proof. Put

$$A_n = \sum_{k=0}^{n} a_k, B_n = \sum_{k=0}^{n} b_k, C_n = \sum_{k=0}^{n} c_k, \beta_n = B_n - b, \quad n \geq 0.$$

Then

$$\begin{aligned} C_n &= = a_0b_0 + (a_0b_1 + a_1b_0) + \cdots + (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0) \\ &= a_0B_n + a_1B_{n-1} + \cdots + a_nB_0 \\ &= a_0(\beta_n + b) + a_1(\beta_{n-1} + b) + \cdots + a_n(\beta_0 + b) \\ &= A_nb + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0. \end{aligned}$$

Denoting

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0,$$

in order to show that $C_n \to ab$ it is sufficient to prove that $\lim \gamma_n = 0$, since $A_nb \to ab$. Denote $\alpha := \sum\limits_{n\to\infty} |a_n|$ and let $\varepsilon > 0$ be given. From $\sum\limits_{n=0}^{\infty} b_n = b$ it follows $\lim\limits_{n\to\infty} \beta_n = 0$. Hence we can choose $n_0 \in \mathbb{N}$ such that $|\beta_n| < \varepsilon$ for $n \geq n_0$. Consequently,

$$\begin{aligned} |\gamma_n| &\leq |a_n\beta_0 + \cdots + a_{n-n_0}\beta_{n_0}| + |a_{n-n_0+1}\beta_{n_0+1} + \cdots + a_0\beta_n| \\ &\leq |a_n\beta_0 + \cdots + a_{n-n_0}\beta_{n_0}| + \varepsilon\alpha. \end{aligned}$$

Keeping n_0 fixed and letting $n \to \infty$ we get

$$\overline{\lim_{n\to\infty}} |\gamma_n| \leq \varepsilon\alpha.$$

Since ε is arbitrary we obtain $\lim\limits_{n\to\infty} \gamma_n = 0$. ∎

3 Solved problems

Ex. 1 *Prove that*

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+3)} = \frac{7}{36}.$$

Solution 6

$$\begin{aligned} a_n &= \frac{1}{n(n+1)(n+3)} = \frac{n+2}{n(n+1)(n+2)(n+3)} = \frac{(n+3)-1}{n(n+1)(n+2)(n+3)} \\ &= \frac{1}{n(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)(n+3)}. \end{aligned}$$

The partial sum s_n *is given by*

$$\begin{aligned} s_n &= \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right) - \frac{1}{3}\sum_{k=1}^{n}\left(\frac{1}{k(k+1)(k+2)} - \frac{1}{(k+1)(k+2)(k+3)}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right) - \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right) \end{aligned}$$

Consequently $\lim\limits_{n\to\infty} s_n = \frac{1}{4} - \frac{1}{18} = \frac{7}{36}$.

Ex. 2 *Test for convergence:*

a) $\sum\limits_{n=1}^{\infty} \frac{(n!)(2n)!}{(3n)!} a^n,\ a > 0;$

b) $\sum\limits_{n=2}^{\infty} \left(\sqrt[n]{a} - \sqrt[n]{b} - 2\right),\ a, b > 0.$

Solution 7 *a)* $a_n = \frac{(n!)(2n)!}{(3n)!} a^n,\ a > 0$. *Using the ratio test we get*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!(2n+2)!a^{n+1}}{(3n+3)!} \cdot \frac{(3n)!}{n!(2n)!a^n} = \lim_{n\to\infty} \frac{a2(n+1)(2n+2)}{3(3n+1)(3n+2)} = \frac{4a}{27}.$$

For $a < \frac{27}{4}$ *the series converges while for* $a > \frac{27}{4}$ *the series is divergent.*
For $a = \frac{27}{4}$, *the ratio test fails and we apply Raabe-Duhamel test.*

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{2(3n+1)(3n+2)}{9(n+1)(2n+1)}-1\right)=\lim_{n\to\infty}\frac{-9n-5}{2(n+1)(2n+1)}=-\frac{9}{4}<1,$$

consequently the series is divergent.

b)

$$\begin{aligned}\lim_{n\to\infty} na_n &= \lim_{n\to\infty} n\left(a^{\frac{1}{n}}-1+b^{\frac{1}{n}}-1\right)\\ &= \lim_{n\to\infty} n\left(\frac{a^{\frac{1}{n}-1}}{\frac{1}{n}}+\frac{b^{\frac{1}{n}-1}}{\frac{1}{n}}\right)=\ln(ab)\end{aligned}$$

If $\ln(ab)\neq 0 \iff ab\neq 1$, *then the series is divergent. If* $ab=1$, *then* $\ln\frac{1}{a}$ *and the series becomes*

$$\sum_{n=2}^{\infty}\frac{(\sqrt[n]{a}-1)^2}{\sqrt[n]{a}}.$$

We have $\lim\limits_{n\to\infty} n^2\frac{(\sqrt[n]{a}-1)^2}{\sqrt[n]{a}}=(\ln a)^2$, *so the series is convergent.*

Ex. 3 *Consider the series* $\sum\limits_{n=2}^{\infty}\frac{a^n}{n^b(\log n)^c}$, *where* $a,b,c\in\mathbb{R}$. *Find* a,b,c *such that the series:*

i) *converges absolutely;*

ii) *converges but not absolutely;*

iii) *diverges.*

(Berkley, Sp 97)

Solution 8 *i) Let* $a_n=\frac{a^n}{n^b(\log n)^c}$, $n\geq 2$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|a|,$$

so the series converges absolutely for $|a|<1$ *and it is not absolutely convergent for* $|a|>1$. *On the other hand if* $|a|>1$, *then clearly* $a_n\nrightarrow 0$, *so the series is divergent.*

ii) For $a = 1$ we get $a_n = \frac{1}{n^b(\log n)^c}$, $n \geq 2$. If $b > 0$ then $\lim\limits_{n\to\infty} a_n = +\infty$ therefore the series $\sum\limits_{n=2} \infty a_n$ is divergent.

If $b = 0$ we get $a_n = \frac{1}{(\log n)^c}$. Clearly if $c \leq 0$ then $a_n \nrightarrow 0$, so the series is divergent. For $c > 0$, according to Cauchy's condensation test the series has the same nature as

$$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} 2^n \frac{1}{(n \log 2)^c} = +\infty,$$

consequently $\sum\limits_{n=2}^{\infty} a_n = +\infty$.

If $b > 0$ it follows that $(a_n)_{n\geq 2}$ is decreasing for sufficiently large n. Then

$$\sum_{n=2}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} \frac{1}{2^{(b-1)n} n^c}.$$

Let $b_n = \frac{1}{2^{(b-1)n} n^c}$. Then $\lim\limits_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{1}{2^{b-1}}$, therefore $\sum\limits_{n=2}^{\infty} b_n$ is convergent for $b > 1$ and divergent for $b < 1$. For $b = 1$ the series becomes $\sum\limits_{n=2}^{\infty} b_n = \sum\limits_{n=2}^{\infty} \frac{1}{n^c}$ and the series converges for $c > 1$ and diverges for $c \leq 1$. The conclusion follows from Cauchy's condensation test. For $x = -1$, according to Leibnitz test the series $\sum\limits_{n=2}^{\infty} a_n$ is convergent $\Longleftrightarrow \lim\limits_{n\to\infty} a_n = 0 \Longleftrightarrow b > 0$, or $b = 0, c > 0$.

4 Proposed problems