Sequences of real and complex numbers

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This chapter, as the title suggests, is dedicated to the study of sequences of real and complex numbers. The first sections contain preliminary notions, conventional notations and a brief overview on primary notions necessary to make this book as self-contained as possible.

1 Real numbers

Let X, Y be two nonempty sets. A *triple* $\rho = (X, Y; \mathcal{R})$, where $\mathcal{R} \subseteq X \times Y$, is called a *binary relation.* The set X is called the *domain* of the relation ρ, Y is the *codomain* or the *range* of ρ while $\mathcal{R}$ represents the *graph* of relation ρ. If $(x, y) \in \mathcal{R}$ we say that x is ρ related to y (or x is in relation ρ with y) and we write $x\rho y$.

The notion of binary relation, frequently used in many branches of mathematics and computer sciences, is in fact a generalization of the notion of function. A *function* $f = (X, Y; \mathcal{R})$ is a relation with the property that for every $x \in X$ there exists a unique $y \in Y$ such that xfy (xfy is denoted usually by $y = f(x)$.).

Definition 1 *A binary relation $\rho = (X, Y; \mathcal{R})$ is called:*

- reflexive, *if $x\rho x$ for all $x \in X$;*
- symmetric, *if $x\rho y \Longrightarrow y\rho x$;*
- antisymmetric, *if $x\rho y$ and $y\rho x \Longrightarrow x = y$;*
- transitive, *if $x\rho y$ and $y\rho z \Longrightarrow x\rho z$;*

- equivalence, *if it is reflexive, symmetric and transitive;*
- order relation, *if it is reflexive, antisymmetric and transitive;*
- totally order relation, *if it an order relation and for all* $x, y \in X$ *we have* $x\rho y$ *or* $y\rho x$.

Example 1.1 *Let* $\mathcal{P}(X)$ *be the collection of all subsets of a given set* X. *Then the equality relation,* " $=$ ", *and the inclusion relation,* " $\subseteq$ ", *are an equivalence respectively an order relation on* $\mathcal{P}(X)$.

A set X endowed with an order relation is called an *ordered set*. An order relation is usually denoted by " $\leq$ ".

Definition 2 *Let* $(X, \leq)$ *be an ordered set and* A *a nonempty subset of* X. *An element* $x \in X$ *is called an* upper bound *(a* lower bound*) of* A *if*

$$a \leq x \quad (x \leq a) \quad \text{ for every } a \in A.$$

A set possessing a lower and an upper bound is called a bounded set.

The numbers a and b are called an lower (upper) bound of the the sequence $(a_n)_{n\geq 1}$

Definition 3 *Let* A *be a bounded subset of an ordered set* $(X, \leq)$. *The greatest lower bound (the smallest upper bound) of* A *(if there exist) is called the* infimum *(*supremum*) of* A *and are denoted by* $\inf A (\sup A)$.

If $\inf A$ *or* $\sup A$ *belong to* A, *they are called the* minimum *respectively the* maximum *of* A *and are denoted by* $\min A$ *and* $\max A$.

Example 1.2 *Consider the set* $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. *Then* $\inf A = 0$ *and* $\sup A = 1$.

Definition 4 *Let* $f : X \to Y$ *be a function,* $A \subseteq X$, $B \subseteq Y$. *The sets*

$$f(A) = \{f(x) | x \in A\}$$
$$f^{-1}(B) = \{x \in X | f(x) \in B\}$$

are called the image *of* A, *respectively the* counterimage *of* A *through the function* f. *The set* $f(X)$ *is called the* range *of* f.

Definition 5 *A set* $\mathbb{R}$ *endowed with the algebraic operations* "$+ : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$", "$\cdot : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$," *and a totally order relation* " $\leq$ " *satisfying the following properties:*

a) $(\mathbb{R}, +, \cdot)$ *is a commutative field;*

b)
- *If* $x, y \in \mathbb{R}$, $x \leq y$, *then* $x + z \leq y + z$ *for every* $z \in \mathbb{R}$.
- *If* $x, y \in \mathbb{R}$, $x \leq y$, *then* $xz \leq yz$ *for every* $z \geq 0$.

c) Every nonempty and upper bounded subset of $\mathbb{R}$ *has a supremum;*

is called a set of real numbers.

Remark 1.1 1) *The first two conditions represent the algebraic part of Definition 5. The last condition, called the* axiom of upper bound of Cantor-Dedekind, *is the starting point for the basic results of mathematical analysis.*

2) *Some example of sets satisfying the conditions of Definition 5 are the set of infinite decimal fractions and the set of points of an ordered axe. One can prove that every two sets satisfying the conditions of Definition 5 are isomorphic.*

If the set $\mathbb{R}$ is given, then one defines the set $\mathbb{N}$ of *natural numbers*, the set $\mathbb{Z}$ of *integers* and the set $\mathbb{Q}$ of *rational numbers* by:

$$\begin{aligned}
\mathbb{N} &= \{0, 1, 2, \ldots\} \\
\mathbb{Z} &= \{x \in \mathbb{R} | x \in \mathbb{N} \text{ or} - x \in \mathbb{N}\} \\
\mathbb{Q} &= \{xy^{-1} | x, y \in \mathbb{Z}, y \neq 0\}.
\end{aligned}$$

One defines also the sets

$$\begin{aligned}
\mathbb{N}^* &= \mathbb{N} \setminus \{0\} \\
\mathbb{R}_+ &= \{x \in \mathbb{R} | 0 \leq x\} \\
\mathbb{R}_+^* &= \{x \in \mathbb{R} | 0 < x\} \\
\mathbb{R}_- &= \{x \in \mathbb{R} | x \leq 0\} \\
\mathbb{R}_-^* &= \{x \in \mathbb{R} | x < 0\},
\end{aligned}$$

where $x < y$ means $x \leq y$ and $x \neq y$.

The following characterizations of the supremum and the infimum of a subset of $\mathbb{R}$ hold.

Proposition 1.1 *Let A be a nonempty subset of $\mathbb{R}$. Then $a = \sup A$ if and only if the following conditions are fulfilled:*

1. *$x \leq a$ for every $x \in A$;*

2. *for every $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that $a - \varepsilon < x_\varepsilon$.*

Proposition 1.2 *Let A be a nonempty subset of $\mathbb{R}$. Then $a = \inf A$ if and only if the following conditions are fulfilled:*

1. *$a \leq x$ for every $x \in A$;*

2. *for every $\varepsilon > 0$, there exists $x_\varepsilon \in A$ such that $x_\varepsilon < a + \varepsilon$.*

The proof of the previous propositions follows easily from the definition of the supremum and infimum of a set.

Proposition 1.3 *Every nonempty lower bounded subset of $\mathbb{R}$ possesses an infimum.*

Proposition 1.4 *(Archimedes's axiom)*

For every $x, y \in \mathbb{R}_+^*$ there exists $n \in \mathbb{N}$ such that $nx > y$.

Proposition 1.5 *For every $x \in \mathbb{R}$ there exists a unique $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.*

The number n from Proposition 1.5 is called the *integer part* of x and is denoted by $[x]$.

The *absolute value* of a real number x is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

For $a, b \in \mathbb{R}$, $a < b$ one defines the following bounded intervals:

$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} | a < x < b\} \\
[a, b] &= \{x \in \mathbb{R} | a \leq x \leq b\} \\
[a, b) &= \{x \in \mathbb{R} | a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} | a < x \leq b\}
\end{aligned}$$

The sets

$$\begin{aligned}
(a, +\infty) &= \{x \in \mathbb{R} | a < x\} \\
[a, +\infty) &= \{x \in \mathbb{R} | a \leq x\} \\
(-\infty, a) &= \{x \in \mathbb{R} | x < a\} \\
(-\infty, a] &= \{x \in \mathbb{R} | x \leq a\} \\
(-\infty, +\infty) &= \mathbb{R},
\end{aligned}$$

are called *unbounded intervals.*

Definition 6 *A set $V \subseteq \mathbb{R}$ is called a* neighborhood *of $x \in \mathbb{R}$ if there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq V$.*

By $\mathcal{V}(x)$ we denote the collection of all neighborhoods of $x \in \mathbb{R}$.

Definition 7 *Let A be a subset of $\mathbb{R}$. An element $a \in A$ is called an* interior point *of A if there exists $V \in \mathcal{V}(a)$ such that $V \subseteq A$.*

Definition 8 *An element $x \in \mathbb{R}$ is called an* cluster point (accumulation point) *of a set $A \subseteq \mathbb{R}$ if for every $V \in \mathcal{V}(x)$ we have $V \cap (A \setminus \{x\}) \neq \emptyset$.*

We denote by A' the set of all cluster points of A and by $\operatorname{int} A$ the interior of the set A. By $\overline{\mathbb{R}}$ we denote the set defined by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, where $-\infty$ and $+\infty$ are two elements satisfying the following properties of order:

$$-\infty < +\infty, \quad -\infty < x, \quad x < +\infty, \text{ for every } x \in \mathbb{R}.$$

We extend the operations " + " and " $\cdot$ " from $\mathbb{R}$ to $\overline{\mathbb{R}}$ setting by definition:

$$
\begin{aligned}
x + (+\infty) = (+\infty) + x = +\infty, &\quad x \in \mathbb{R} \cup \{+\infty\} \\
x + (-\infty) = (-\infty) + x = -\infty, &\quad x \in \mathbb{R} \cup \{-\infty\} \\
x \cdot (+\infty) = (+\infty) \cdot x = +\infty, &\quad x \in \mathbb{R}_+^* \cup \{+\infty\} \\
x \cdot (-\infty) = (-\infty) \cdot x = -\infty, &\quad x \in \mathbb{R}_+^* \cup \{-\infty\} \\
\frac{x}{+\infty} = \frac{x}{-\infty} = 0, &\quad x \in \mathbb{R}.
\end{aligned}
$$

Definition 9 *A set $V \subseteq \mathbb{R}$ is called a neighbourhood of $+\infty$ (or $-\infty$) if there exists $a \in \mathbb{R}$ with the property $(a, +\infty) \subseteq V$ (or $(-\infty, a) \subseteq V$).*

The families of all neighbourhood of $+\infty$ and $-\infty$ are denoted by $\mathcal{V}(+\infty)$ and $\mathcal{V}(-\infty)$.

2 Sequences of real numbers

Definition 10 *A function $f : \mathbb{N}^* \to \mathbb{R}$ is called* sequence *of real numbers.*

Usually a sequence is denoted by $(a_n)_{n\geq 1}$ or (a_n), where $a_n = f(n)$, $n \in \mathbb{N}^*$.

Definition 11 *A sequence $(a_n)_{n\geq 1}$ is called* monotonically increasing (decreasing) *if*

$$a_n \leq a_{n+1} \quad (a_n \geq a_{n+1}) \text{ for all } n \geq 1.$$

Definition 12 *A sequence $(a_n)_{n\geq 1}$ is said to be* bounded *if there exists $a, b \in \mathbb{R}$ with the property*

$$a \leq a_n \leq b \text{ for all } n \geq 1.$$

A sequence which is not bounded is said to be unbounded.

Definition 13 *An element $a \in \overline{\mathbb{R}}$ is called the* limit *of a sequence $(a_n)_{n\geq 1}$ if for every $V \in \mathcal{V}(a)$ there exists $n_V \in \mathbb{N}^*$ such that $a_n \in V$ for every $n \geq n_V$.*

If a sequence has a limit, then it is unique. We denote usually $a = \lim\limits_{n\to\infty} a_n$ ($a = \lim a_n$) or $a_n \to a$.

A sequence having a finite limit is said to be *convergent*. If a sequence is not convergent it is called *divergent*. The following theorem is a characterization of convergent sequences.

Theorem 1 *A sequence $(a_n)_{n\geq 1}$ is convergent if and only if there exists $a \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ with the property that for all $n \geq n_\varepsilon$ we have $|a_n - a| < \varepsilon$.*

Proof. Necessity Suppose that there exists $\lim\limits_{n\to\infty} a_n = a$, $a \in \mathbb{R}$ and take $\varepsilon > 0$. Let $V = (a - \varepsilon, a + \varepsilon)$ be a neighbourhood of a. Then

$$\exists n_\varepsilon \in \mathbb{N} \text{ such that } a_n \in V, \forall n \geq n_\varepsilon \iff |a_n - a| < \varepsilon, \forall n \geq n_\varepsilon.$$

Sufficiency Let V be a neighbourhood of a. There exists $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon) \subseteq V$. If follows $a_n \in V$ for all $n \geq n_\varepsilon$, hence $\lim\limits_{n\to\infty} a_n = a$. ∎

The next result, known as Weierstrass theorem, is frequently used in the study of the convergence of a sequence.

Theorem 2 *Every monotonic and bounded sequence of real numbers is convergent.*

Proof. Suppose that $(a_n)_{n\geq 1}$ is an increasing and upper bounded sequence. Then the set $A = \{a_n | n \geq 1\}$ is upper bounded, hence there exists $a = \sup A$, $a \in \mathbb{R}$. Then for all $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $a - \varepsilon < a_n$. Since $(a_n)_{n\geq 1}$ is monotonically increasing it follows $0 < a - a_n < \varepsilon$ for all $a \geq n_\varepsilon$. Hence $(a_n)_{n\geq 1}$ is convergent to a. ∎

Theorem 3 *(Principle of nested intervals) Let $(I_n)_{n\geq 1}$, $I_n = [a_n, b_n]$, be a sequence of closed nested intervals, $I_{n+1} \subseteq I_n$, $\forall n \geq 1$. Then $\cap_{n\geq 1} I_n \neq \emptyset$. If $\lim\limits_{n\to\infty} (b_n - a_n) = 0$, then $\cap_{n\geq 1} I_n$ is a singleton*

Proof. Define $A = \{a_n | n \in \mathbb{N}^*\}$, $B = \{b_n | n \in \mathbb{N}^*\}$. The relation $I_{n+1} \subseteq I_n$, $n \geq 1$, implies that $(a_n)_{n\geq 1}$is increasing and $(b_n)_{n\geq 1}$ is decreasing. Since b_1

is an upper bound of A and a_1 is a lower bound of B there exists $\sup A = a$ $\inf B = b$, $a, b \in \mathbb{R}$. By the relation

$$a_n \leq a \leq b \leq b_n, \quad n \geq 1,$$

it follows $[a, b] \subseteq \cap_{n\geq 1} I_n$. If $b_n - a_n \to 0$, clearly $a = b$. ■

Definition 14 *Let $(a_n)_{n\geq 1}$ be a sequence and $(n_k)_{k\geq 1}$ a strictly increasing sequence of positive integers. Then $(a_{n_k})_{k\geq 1}$ is called a* subsequence *of $(a_n)_{n\geq 1}$.*

Example 2.1 *$(a_{2n})_{n\geq 1}$, $(a_{2n-1})_{n\geq 1}$ are subsequences of the sequence $(a_n)_{n\geq 1}$.*

Corollary 2.1 (Cesàro lemma) *A bounded sequence of real numbers contains at least a convergent subsequence.*

Proof. Let $(x_n)_{n\geq 1}$ be a bounded sequence in $\mathbb{R}$. There exists $a, b \in \mathbb{R}$ such that $a \leq x_n \leq b$, $\forall n \geq 1$. Let $c = \frac{a+b}{2}$. From the intervals $[a, c]$, $[c, b]$ we choose the interval which contains an infinite number of terms of $(x_n)_{n\geq 1}$ and we denote it by $I_1 = [a_1, b_1]$. The set $\{n \in \mathbb{N}^* | x_n \in [a_1, b_1]\}$ is infinite and $b_1 - a_1 = \frac{b-a}{2}$. Let $c_1 = \frac{a_1+b_1}{2}$. From the intervals $[a_1, c_1]$, $[c_1, b_1]$ we choose the interval which contains an infinite number of terms of $(x_n)_{n\geq 1}$ and we denote it by $I_2 = [a_2, b_2]$. We have $b_2 - a_2 = \frac{b-a}{2^2}$.

Denoting $I_0 = [a, b]$ one obtains a sequence $I_n = [a_n, b_n]$, $n \geq 0$, with the property that $I_{n+1} \subseteq I_n$ and $b_n - a_n = \frac{b-a}{2^n}$, $n \geq 0$. Taking account of Theorem 3 it follows $\cap_{n\geq 0} I_n = \{x\}$, $x \in [a, b]$. Now we choose from every interval I_k a term $x_{n_k} \in I_k$, $k \geq 0$. We have $|x_{n_k} - x| \leq \frac{b-a}{2^k}$, $k \geq 0$, hence $\lim\limits_{k\to\infty} x_{n_k} = x$. ■

Cesaro lemma is also called as Bolzano-Weierstrass Theorem.

Definition 15 *A sequence $(a_n)_{n\geq 1}$ is said to be* fundamental *or* Cauchy sequence *if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that for all $m, n \geq n_\varepsilon$ we have $|a_m - a_n| < \varepsilon$.*

The above definition is equivalent to the next one:

A sequence $(a_n)_{n\geq 1}$ is *fundamental* if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ such that $\forall n \geq n_\varepsilon$, $\forall p \in \mathbb{N}$ $|a_{n+p} - a_n| < \varepsilon$.

Remark 2.1 *The notion of Cauchy sequence was introduced to obtain a characterization convergent sequences without using their limit.*

Theorem 4 *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. Necessity Suppose that $(a_n)_{n\geq 1}$ is convergent and let $\lim\limits_{n\to\infty} a_n = a$, $a \in \mathbb{R}$.

Let $\varepsilon > 0$ be a fixed number. Then there exists $n_\varepsilon \in \mathbb{N}^*$ such that for all $n \geq n_\varepsilon$ $|a_n - a| < \frac{\varepsilon}{2}$. Now, for all $m, n \geq n_\varepsilon$ we have

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

therefore $(a_n)_{n\geq 1}$ is a Cauchy sequence.

Sufficiency Suppose $(a_n)_{n\geq 1}$ is a Cauchy sequence.

1) We prove first that $(a_n)_{n\geq 1}$ is bounded. For $\varepsilon = 1$ in Definition 15 follows that $\exists n_1 \in \mathbb{N}$ such that $|a_n - a_{n_1}| < 1, \quad n \geq n_1$ Then

$$|a_n| \leq |a_n - a_{n_1}| + |a_{n_1}| < 1 + |a_{n_1}|, \quad \forall n \geq n_1,$$

hence $(a_n)_{n\geq 1}$ is a bounded sequence.

2) We prove that $(a_n)_{n\geq 1}$ is a convergent sequence. From the boundedness of $(a_n)_{n\geq 1}$ it follows that there exists a convergent subsequence $(a_{n_k})_{k\geq 1}$, $\lim\limits_{n\to\infty} a_n = a$. Let $\varepsilon > 0$ be a positive number. By the convergence of $(a_{n_k})_{k\geq 1}$ we find $k_\varepsilon \in \mathbb{N}$ with the property

$$|a_{n_k} - a| < \frac{\varepsilon}{2}, \quad \forall k \geq k_\varepsilon. \tag{2.1}$$

Since $(a_n)_{n\geq 1}$ is a Cauchy sequence it follows that there exists $n_\varepsilon \in \mathbb{N}^*$ with the property

$$|a_m - a_n| < \frac{\varepsilon}{2}, \quad \forall m, n \geq n_\varepsilon. \tag{2.2}$$

Now take $N_\varepsilon = \max\{n_\varepsilon, n_{k_\varepsilon}\}$. For every $n \geq N_\varepsilon$ we have in view of the relations (2.1) and (2.2)

$$|a_n - a| \leq |a_n - a_{n_{k_\varepsilon}}| + |a_{n_{k_\varepsilon}} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $(a_n)_{n\geq 1}$ is a convergent sequence.

■

Example 2.2 *Prove that* $(a_n)_{n\geq 1}$,

$$a_n = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \cdots + \frac{\sin n}{n^2}$$

is a convergent sequence.

Solution. We prove that $(a_n)_{n\geq 1}$ is a Cauchy sequence. For every $n, p \in \mathbb{N}^*$ we have:

$$\begin{aligned}
|a_{n+p} - a_n| &= \left|\frac{\sin(n+1)}{(n+1)^2} + \frac{\sin(n+2)}{(n+2)^2} + \cdots + \frac{\sin(n+p)}{(n+p)^2}\right| \\
&\leq \left|\frac{\sin(n+1)}{(n+1)^2}\right| + \left|\frac{\sin(n+2)}{(n+2)^2}\right| + \cdots + \left|\frac{\sin(n+p)}{(n+p)^2}\right| \\
&\leq \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+p)^2} \\
&\leq \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(n+p-1)(n+p)} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{n+p-1} - \frac{1}{n+p}\right) \\
&= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n}.
\end{aligned}$$

For every $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}^*$ such that $\frac{1}{n} < \varepsilon$, for all $n \geq n_\varepsilon$. Thus

$$|a_{n+p} - a_n| \leq \varepsilon, \quad \forall n \geq n_\varepsilon, p \geq 1.$$

Since $(a_n)_{n\geq 1}$ is a fundamental sequence it is convergent. ■

Example 2.3 *Prove that* $\lim\limits_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = \infty$.

Solution. Define $(a_n)_{n\geq 1}$, by

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Since $(a_n)_{n\geq 1}$ is monotonically increasing there exists $\lim a_n \in \overline{\mathbb{R}}$. We prove that $(a_n)_{n\geq 1}$ is not a Cauchy sequence. For every $n \geq 1$ we have

$$\begin{aligned} |a_{2n} - a_n| &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\ &\geq \underbrace{\frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}}_{\text{n times}} = \frac{1}{2} \end{aligned}$$

Since $(a_n)_{n\geq 1}$ is divergent and its limit exists, we conclude that $\lim\limits_{n\to\infty} a_n = +\infty$. ∎

3 The upper limit and the lower limit of a sequence

Definition 16 *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. We call $a \in \overline{\mathbb{R}}$ a* limit point *of the sequence $(a_n)_{n\geq 1}$ if every neighbourhood of a contains infinitely many terms of $(a_n)_{n\geq 1}$.*

For a sequence $(a_n)_{n\geq 1}$ we denote by $L(a_n)$ the set of its limit points.

Example 3.1 *a) For $(a_n)_{n\geq 1}$, $a_n = (-1)^n$, $L(a_n) = \{-1, 1\}$.*

b) The sequence $(a_n)_{n\geq 1}$, $a_n = \sin \frac{n\pi}{2}$, has $L(a_n) = \{-1, 0, 1\}$.

Theorem 5 *The element $a \in \overline{\mathbb{R}}$ is a limit point of $(a_n)_{n\geq 1}$ if and only if there exists a subsequence $(a_{n_k})_{k\geq 1}$ of $(a_n)_{n\geq 1}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$.*

Proof. Necessity Suppose that $a \in L(a_n)$ and $a \in \mathbb{R}$. Let $V_k = \left(a - \frac{1}{k}, a + \frac{1}{k}\right)$, $k \in \mathbb{N}^*$, be a sequence of neighbourhood of a. Every V_k contains infinitely many terms of (a_n). Let $a_{n_1} \in V_1$, $a_{n_2} \in V_2, \ldots$, with $n_1 < n_2 < \ldots$. We have

$$|a_{n_k} - a| < \frac{1}{k}, \quad k \geq 1,$$

hence $\lim\limits_{k\to\infty} a_{n_k} = a$. Analogously follows the existence of $(a_{n_k})_{k\geq 1}$ for $a \in \{-\infty, +\infty\}$.
Sufficiency

Suppose that $(a_{n_k})_{k\geq 1}$ is a subsequence of $(a_n)_{n\geq 1}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$. Then every neighbourhood of a contains infinitely many elements of $(a_n)_{n\geq 1}$, thus $a \in L(a_n)$. ■

Definition 17 *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Then* $\sup L(a_n)$ *is called the* upper limit *of* $(a_n)_{n\geq 1}$ *and* $\inf L(a_n)$ *is called the* lower limit *of* $(a_n)_{n\geq 1}$.

We denote the upper limit and the lower limit of $(a_n)_{n\geq 1}$ by

$$\overline{\lim} a_n, \quad \text{respectively} \quad \underline{\lim} a_n,$$

or by

$$\limsup a_n, \quad \text{respectively} \quad \liminf a_n,$$

Example 3.2 *a) For the sequence $(a_n)_{n\geq 1}$, defined by*

$$a_n := 1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots,$$

we get $L(a_n) = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \cup \{0\}$ therefore $\overline{\lim} a_n = 1$ and $\underline{\lim} a_n = 0$.

b) It can be proved that the sequence $(a_n)_{n\geq 1}$, $a_n = \sin n$, has $L(a_n) = [-1, 1]$, $\overline{\lim} a_n = 1$ and $\underline{\lim} a_n = -1$.

Theorem 6 *A sequence $(a_n)_{n\geq 1}$ has a limit if and only if*

$$\overline{\lim} a_n = \underline{\lim} a_n.$$

4 Some remarkable sequences

Proposition 4.1 *(the number e) The sequence $(e_n)_{n\geq 1}$,*

$$e_n = \left(1 + \frac{1}{n}\right)^n,$$

is monotonically increasing and bounded above. By definition

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Proof. Define $(e'_n)_{n\geq 1}$, $e'_n = \left(1+\frac{1}{n}\right)^{n+1}$. We prove that $(e'_n)_{n\geq 1}$ is monotonically decreasing. We have

$$\begin{aligned}
\frac{e'_n}{e'_{n+1}} &= \left(\frac{n+1}{n}\right)^{n+1}\left(\frac{n+1}{n+2}\right)^{n+2} = \left(\frac{(n+1)^2}{n^2+2n}\right)^{n+1}\cdot\frac{n+1}{n+2} \\
&= \left(1+\frac{1}{n^2+2n}\right)^{n+1}\cdot\frac{n+1}{n+2} > \left(1+(n+1)\frac{1}{n^2+2n}\right)\cdot\frac{n+1}{n+2} \\
&= \frac{n(n+2)^2+1}{n(n+2)^2} > 1, \quad \forall n \geq 1,
\end{aligned}$$

taking account of Bernoulli's inequality:

$$(1+t)^n > 1+nt, \quad t \in (-1,\infty)\setminus\{0\}, n \in \mathbb{N}^*.$$

We prove that $(e_n)_{n\geq 1}$ is an increasing sequence.

$$\begin{aligned}
\frac{e_{n+1}}{e_n} &= \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n}{n+1}\right)^{n} = \left(\frac{(n+2)}{n+1}\right)^{n+1}\frac{n+1}{n} \\
&= \left(\frac{n^2+2n}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} = \left(1-\frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} \\
&> \left(1-(n+1)\frac{1}{(n+1)^2}\right)\frac{n+1}{n} = 1, \quad \forall n \geq 1,
\end{aligned}$$

By the relation

$$e_1 < e_2 < \cdots < e_{n+1} < \cdots < e'_{n+1} < e'_n < \cdots < e'_2 < e'_1$$

follows that (e'_n), (e_n) are convergent sequences and by

$$0 < e'_n - e_n = \left(1+\frac{1}{n}\right)^n\frac{1}{n} < \frac{e_n}{n} \leq \frac{e_1}{n}, \quad n \geq 1,$$

one gets

$$\lim e_n = \lim e'_n := e.$$

■

Corollary 4.1 *The following inequality holds*

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}, \ \forall n \in \mathbb{N}^*.$$

Remark 4.1 *The sequence* $(E_n)_{n\geq 1}$,

$$E_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

is monotonically increasing and $\lim\limits_{n\to\infty} E_n = e$ $(e \notin \mathbb{Q}, e \backsimeq 2,71828\ldots)$.

Proposition 4.2 *(Euler's constant) The sequence* $(\gamma_n)_{n\geq 1}$,

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

is monotonically decreasing and bounded below. We denote

$$\lim_{n\to\infty} \gamma_n := \gamma$$

(γ *is called Euler's constant and* $\gamma = 0,577\ldots$*).*

Proof. From Corollary 4.1 follows

$$\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}, \quad \forall n \geq 1. \tag{4.1}$$

Summing the previous inequalities one gets

$$\sum_{k=1}^{n} \frac{1}{k+1} < \ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}, \quad \forall n \geq 1. \tag{4.2}$$

We have

$$\gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \ln(n+1) + \ln n$$

and taking account of (4.1) follows $\gamma_{n+1} < \gamma_n$, $n \geq 1$.

Now, in view of inequality (4.2) we get

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n > \ln(n+1) - \ln n > 0, \quad n \geq 1.$$

■

The following theorems are often used for finding the limit of a sequence.

Theorem 7 (Sandwich Theorem) *Let* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ *be sequences of real numbers with the property that*

$$a_n \leq b_n \leq c_n \text{ for all } n \geq n_0.$$

If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l, l \in \overline{\mathbb{R}},$$

then $\lim_{n\to\infty} b_n = l$.

Theorem 8 (Stolz-Cesaro I) *Let* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *be two sequences of real numbers satisfying the conditions:*
1) $(b_n)_{n\geq 1}$ *is strictly monotone and unbounded*;
2) *there exists* $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l, l \in \overline{\mathbb{R}}$;

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = l$.

Theorem 9 (Stolz-Cesaro II) *Let* $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ *be two sequences of real numbers satisfying the conditions:*
1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$;
2) $(b_n)_{n\geq 1}$ *is strictly monotone*;
3) *there exists* $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l, l \in \overline{\mathbb{R}}$;

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = l$.

Proof. We prove only Theorem 9. Suppose that $(b_n)_{n\geq 1}$ is monotonically increasing and $l \in \mathbb{R}$. Let ε be a positive number. There exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$

$$\left|\frac{a_{n+1}-a_n}{b_{n+1}-b_n} - l\right| < \varepsilon$$

or

$$(b_{n+1}-b_n)(l-\varepsilon) < a_{n+1}-a_n < (b_{n+1}-b_n)(l+\varepsilon), \quad n \geq n_\varepsilon.$$

Adding the previous relations from n to $n+p-1$ follows

$$(l-\varepsilon)\sum_{k=n}^{n+p-1}(b_{k+1}-b_k) < \sum_{k=n}^{n+p-1}(a_{k+1}-a_k) < (l+\varepsilon)\sum_{k=n}^{n+p-1}(b_{k+1}-b_k)$$

$$(l-\varepsilon)(b_{n+p}-b_n) < a_{n+p}-a_n < (l+\varepsilon)(b_{n+p}-b_n)$$

$$-\varepsilon < \frac{a_{n+p}-a_n}{b_{n+p}-b_n} - l < \varepsilon,$$

for all $n \geq n_\varepsilon, p \geq 1$. Letting $p \to \infty$ in the previous relation one gets

$$-\varepsilon \leq \frac{a_n}{b_n} - l \leq \varepsilon, \quad n \geq n_\varepsilon,$$

hence

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

■

Corollary 4.2 *Let $(a_n)_{n\geq 1}$, be a sequence of positive numbers with the property that there exists $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, $l \in \overline{\mathbb{R}}$. Then $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = l$.*

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} e^{\ln \sqrt[n]{a_n}} = e^{\lim\limits_{n\to\infty} \frac{\ln a_n}{n}} \\
&= e^{\lim\limits_{n\to\infty} \frac{\ln a_{n+1} - \ln a_n}{n+1-n}} = e^{\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}} = e^{\ln l} = l.
\end{aligned}$$

■

5 Sequences of complex numbers

A sequence of complex numbers $(z_n)_{n\geq 1}$ has the form

$$z_n = a_n + ib_n, \quad n \geq 1,$$

where $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences of real numbers, i.e., $a_n = \Re z_n$, $b_n = \Im z_n$. Many notions and theorems carry over from real to complex sequences.

Definition 18 *A sequence $(z_n)_{n\geq 1}$ of complex numbers is said to be* convergent *to $z \in \mathbb{C}$ if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that $|z_n - z| < \varepsilon$ for every $n \geq n_\varepsilon$.*

Definition 19 *A sequence $(z_n)_{n\geq 1}$ of complex numbers is said to be* fundamental *(or* Cauchy sequence*) if for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}^*$ such that for all $m, n \geq n_\varepsilon$ we have $|z_m - z_n| < \varepsilon$.*

Theorem 10 *A sequence* $(z_n)_{n\geq 1}$ *of complex numbers converges to* $z \in \mathbb{C}$ *if and only if the real sequences* $(\Re z_n)_{n\geq 1}$ *and* $(\Im z_n)_{n\geq 1}$ *converge to* $\Re z$ *and* $\Im z$ *respectively.*

Proof. Consider $z_n - a_n + ib_n$, $n \geq 1$, $z = a + bi$, $a, b, a_n, b_n \in \mathbb{R}$.
Necessity

Suppose that $z_n \to z$ and let $\varepsilon > 0$ be given. Then $\exists n_\varepsilon \in \mathbb{N}^*$ such that

$$|z_n - z| < \varepsilon \quad \forall n \geq n_\varepsilon.$$

We have

$$\begin{aligned} |a_n - a| &< |z_n - z| < \varepsilon, \\ |b_n - b| &< |z_n - z| < \varepsilon, \quad \forall n \geq n_\varepsilon, \end{aligned}$$

thus $a_n \to a$, $b_n \to b$.

Sufficiency Suppose $a_n \to a$, $b_n \to b$ and prove that $z_n \to z$. Let $\varepsilon > 0$ be given. Then there exist n'_ε $n''_\varepsilon \in \mathbb{N}^*$ such that

$$\begin{aligned} |a_n - a| &< \frac{\varepsilon}{2}, \quad \forall n \geq n'_\varepsilon, \\ |b_n - b| &< \frac{\varepsilon}{2}, \quad \forall n \geq n''_\varepsilon. \end{aligned}$$

Taking $n_\varepsilon = \max\{n'_\varepsilon, n''_\varepsilon\}$ it follows

$$|z_n - z| < |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n \geq n_\varepsilon,$$

hence $\lim\limits_{n\to\infty} z_n = z$. ■

As for real sequences we can prove.

Theorem 11 *A sequence of complex numbers is convergent if and only if is a Cauchy sequence.*

Since the complex field $\mathbb{C}$ is not an ordered field, all notions and proprieties where the order is involved do not make sense for complex sequences or they need modifications. The sandwich theorem does not hold; there is no notion of monotone sequence, upper and lower limit. Nevertheless the notion of boundedness, subsequence and limit points do hold in the complex case too.