Sequences and series of functions

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This chapter contains notions and results concerning sequences and series of functions, which play an important role in approximation of functions. As a particular case we study power series.

The notion of uniform convergence was initially considered by Stokes and Siedel (1847) who emphasized its role in preserving continuity. Power series appeared first in Newton's paper (1665) who believed that every function can be expanded in power series. Later on, near 1750, D. Bernoulli introduced the idea of representing a function by a functional series of sine and cosine. The relation between the sum and the coefficients of a trigonometric series was given by J. Fourier in 1805, but he could not prove the convergence of the series. This problem was solved by Dirichlet in 1829.

1 Sequences of functions

Definition 1 *A sequence of functions* $(f_n)_{n\geq 1}$, $f_n : A \to \mathbb{R}$ *is said to be* convergent *or* pointwise convergent *to the function* $f : A \to \mathbb{R}$ *on* A *if*

$$f(x) = \lim_{n\to\infty} f_n(x),$$

for every $x \in A$.

We denote the pointwise convergence by

$$f_n \to f \quad or \quad f_n \xrightarrow{PC} f.$$

Example 1.1 *a) Let* $(f_n)_{n\geq 1}$, $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$. *We have*

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$$

Hence $f_n \xrightarrow{PC} f,\ f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1. \end{cases}$

b) Let $(f_n)_{n\geq 1},\ f_n : \mathbb{R} \to \mathbb{R}$, *be given by the relation*

$$f_n(x) = \frac{x + x^2 e^{nx}}{1 + e^{nx}}, \quad x \in \mathbb{R}, n \in \mathbb{N}^*.$$

Then

$$\lim_{n\to\infty} f_n(x) = \begin{cases} x, & x < 0 \\ 0, & x = 0 \\ x^2, & x > 0. \end{cases}$$

Hence $f_n \xrightarrow{PC} f,\ f(x) = \begin{cases} x, & x < 0 \\ 0, & x = 0 \\ x^2, & x > 0. \end{cases}$

Remark 1 *Pointwise convergence does not preserve generally the properties of the terms of a sequence of functions.*

For instance the terms of the sequence of the previous Example 1.1 a) are continuous functions, but its limit is not continuous. The functions f_n *from Example 1.1 b) are differentiable but its limit is not differentiable at* $x = 0$. *This is the reason why a new notion of convergence is introduced in the theory of sequences of functions.*

Definition 2 *A sequence of functions* $(f_n)_{n\geq 1},\ f_n : A \to \mathbb{R}$ *is said to be* uniformly convergent *to the function* $f : A \to \mathbb{R}$ *on* A *if for every* $\varepsilon > 0$, *there exists* $n_\varepsilon \in \mathbb{N}$ *such that for all* $n \geq n_\varepsilon$ *and all* $x \in A$

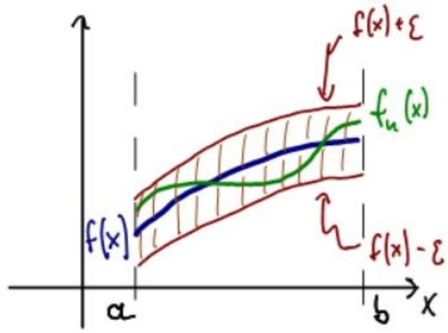
$$|f_n(x) - f(x)| < \varepsilon.$$

We denote the uniform convergence by

$$f_n \rightrightarrows f \quad \text{or} \quad f_n \xrightarrow{\text{UC}} f.$$

Geometric representation of uniform convergence.
The relation $|f_n(x) - f(x)| < \varepsilon$, $x \in A$, $n \geq n_\varepsilon$ can be rewritten as

$$f(x) - \varepsilon \leq f_n(x) \leq f(x) + \varepsilon, \quad x \in A, n \geq n_\varepsilon.$$



The geometric representation is given above.

The graph of f_n is located in the region bounded by the graphs of the functions $f - \varepsilon$ and $f + \varepsilon$ for sufficiently large n.

Remark 2 *Every uniformly convergent sequence of functions is pointwise convergent, but the converse of this statement is not generally true.*

Indeed, the sequence from Example 1.1 a) is not uniformly convergent on $[0, 1]$, but it is pointwise convergent.

Proof. Suppose contrary. Then for every $\varepsilon = \frac{1}{4}$ there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon, \quad n \geq n_0, x \in [0, 1].$$

It follows

$$x^n < \frac{1}{4}, \quad n \geq n_0, x \in [0, 1).$$

For $x = 1 - \frac{1}{n}$, $n \geq n_0$, one gets

$$\left(1 - \frac{1}{n}\right)^n < \frac{1}{4}, \quad n \in \mathbb{N}^*.$$

Letting $n \to \infty$ in the previous relation we obtain $\frac{1}{e} \leq \frac{1}{4}$, a contradiction ■

We give in what follows some properties of uniformly convergent sequences of functions.

Theorem 1 (A test of uniform convergence) *Let* $(f_n)_{n\geq 1}$, $f_n : A \to \mathbb{R}$ *be a sequence of functions and* $f : A \to \mathbb{R}$. *If there exists a sequence of positive numbers* $(a_n)_{n\geq 1}$ *such that:*

1. $\lim_{n\to\infty} a_n = 0$;

2. $|f_n(x) - f(x)| \le a_n$, *for all* $n \ge n_0$ *and all* $x \in A$.

Then $f_n \xrightarrow{UC} f$ *on* A.

Proof. Let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \ge n_\varepsilon$ we get $a_n < \varepsilon$, since $\lim_{n\to\infty} a_n = 0$. It follows

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \ge n_\varepsilon, \forall x \in A,$$

hence $f_n \xrightarrow{\text{UC}} f$. $\blacksquare$

Example 1.2 *Test the uniform convergence of the sequence of functions* $(f_n)_{n\ge1}$, $f_n : \mathbb{R} \to \mathbb{R}$, *given by*

$$f_n(x) = \frac{\cos x + 2n \sin x}{n+1}, \quad x \in \mathbb{R}.$$

Solution 1 *We get*

$$\lim_{n\to\infty} f_n(x) = 2\sin x, \quad x \in \mathbb{R}.$$

We prove that $f_n \xrightarrow{UC} f, f(x) = 2\sin x, x \in \mathbb{R}$. *We have*

$$|f_n(x) - f(x)| = \frac{|\cos x - 2\sin x|}{n+1} \le \frac{3}{n+1}, \quad n \ge 1, x \in \mathbb{R}.$$

Since $\lim_{n\to\infty} \frac{3}{n+1} = 0$, *it follows that* $f_n \xrightarrow{UC} f$ *on* $\mathbb{R}$, *according to Theorem 5.*

Another test of uniform convergence is contained in the following theorem.

Theorem 2 (Cauchy's criterion) *A sequence of functions* $(f_n)_{n\ge1}$, $f_n : A \to \mathbb{R}$ *is uniformly convergent on* A *if and only if for every* $\varepsilon > 0$ *there exists* $n_\varepsilon \in \mathbb{N}$ *such that for all* $m, n \ge n_\varepsilon$ *and all* $x \in A$

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Proof. Necessity. Suppose that $f_n \xrightarrow{\text{UC}} f$ on A and take $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \quad \forall n \geq n_\varepsilon, x \in A.$$

Then for all $m, n \in \mathbb{N}$, $m, n \geq n_\varepsilon$ and all $x \in A$ we get

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Sufficiency. Suppose that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|f_m(x) - f_n(x)| < \frac{\varepsilon}{2}, \quad \forall m, n \geq n_\varepsilon, x \in A, \tag{1.1}$$

and prove that there exists $f : A \to A$, $f_n \xrightarrow{\text{UC}} f$ on A.

Fix $x \in A$. Then the sequence of numbers $(f_n(x))_{n\geq 1}$ is fundamental on $\mathbb{R}$, hence it is convergent. Denote

$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in A.$$

Letting $m \to \infty$ in the relation (1.1) we obtain

$$|f(x) - f_n(x)| \leq \frac{\varepsilon}{2} < \varepsilon, \quad \forall n \geq n_\varepsilon,$$

therefore $f_n \xrightarrow{\text{UC}} f$ on A. ■

Remark 3 *The theorem 2 can be reformulated as follows.*

Let $(f_n)_{n\geq 1}$, $f_n : A \to \mathbb{R}$, be a sequence of functions. Then:

"$(f_n)_{n\geq 1}$ uniformly convergent $\iff$ $(f_n)_{n\geq 1}$ uniformly fundamental."

We show in the next theorem that pointwise convergence under some appropriate conditions, leads to uniform convergence. The result is knows as the Monotone Convergence Theorem.

Theorem 3 (Dini) *Let $(f_n)_{n\geq 1}$, $f_n : [a, b] \to \mathbb{R}$, be a sequence of continuous functions and $f : [a, b] \to \mathbb{R}$ a continuous function such that:*

1. $f_1(x) \le f_2(x) \le f_3(x) \le \ldots, \forall x \in [a,b]$;

2. $f_n \xrightarrow{PC} f$ *on* $[a,b]$.

Then $f_n \xrightarrow{UC} f$ *on* $[a,b]$.

We present some properties of uniformly convergent sequences of functions, concerning preservation of boundedness, continuity, differentiability, etc.

Theorem 4 (Preservation of boundedness) *Let* $(f_n)_{n\ge1}$, $f_n : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, *be a sequence of bounded functions,* $f_n \xrightarrow{UC} f$ *on* A. *Then* f *is a bounded function.*

Proof. By $f_n \xrightarrow{\text{UC}} f$ on A it follows for $\varepsilon = 1$, that there exists $n_1 \in \mathbb{N}$ such that for every $n \ge n_1$

$$|f_n(x) - f(x)| \le 1, \quad \forall x \in A.$$

On the other hand there exists $M \ge 0$ such that

$$|f_{n_1}(x)| \le M, \quad \forall x \in A,$$

since f_{n_1} is a bounded function. We get

$$\begin{aligned} |f(x)| &\le |f(x) - f_{n_1}(x) + f_{n_1}(x)| \le |f(x) - f_{n_1}(x)| + |f_{n_1}(x)| \\ &\le 1 + M, \quad \forall x \in A. \end{aligned}$$

■

Theorem 5 (Preservation of continuity) *Let* $(f_n)_{n\ge1}$, $f_n : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, *be a sequence of continuous functions,* $f_n \xrightarrow{UC} f$ *on* A. *Then the function* f *is continuous on* A.

Proof. Let $x_0 \in A$ and let $\varepsilon > 0$. By the uniform convergence of $(f_n)_{n\ge1}$ it follows that there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \quad \forall x \in A.$$

The continuity of f_{n_ε} at x_0 implies the existence of $\delta > 0$ such that for $x \in A$, $|x - x_0| < \delta$, we get

$$|f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x_0)| < \frac{\varepsilon}{3}.$$

Now for all $x \in A$, $|x - x_0| < \delta$, we have:

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_{n_\varepsilon}(x)| + |f_{n_\varepsilon}(x) - f_{n_\varepsilon}(x_0)| + |f_{n_\varepsilon}(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

hence f is continuous at x_0. ■

Theorem 6 *Let $(f_n)_{n\geq 1}$, $f_n : [a,b] \to \mathbb{R}$, be a sequence of continuous functions, $f_n \xrightarrow{UC} f$ on $[a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. First we remark that f is continuous on $[a,b]$ in view of Theorem 5, therefore f is integrable on $[a,b]$.

Let ε be a positive number. By $f_n \xrightarrow{\text{UC}} f$ it follows that there exists $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$:

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}, \quad \forall x \in [a,b].$$

We have:

$$\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leq \int_a^b |f_n(x)dx - f(x)|dx \leq \int_a^b \frac{\varepsilon}{b-a} dx = \varepsilon,$$

for all $n \geq n_\varepsilon$. The theorem is proved.

■

Remark 4 *Theorem 6 also holds if we suppose that $(f_n)_{n\geq 1}$ is a sequence of integrable functions.*

Theorem 7 (Preservation of differentiability) *Let* $(f_n)_{n\geq 1}$, $f_n : [a,b] \to \mathbb{R}$ *be a sequence of differentiable functions such that:*

1. $f_n \xrightarrow{PC} f$ *on* $[a,b]$;

2. $f_n' \xrightarrow{UC} g$ *on* $[a,b]$.

Then f *is differentiable on* $[a,b]$ *and* $f' = g$.

Proof. Fix $x_0 \in [a,b]$ and $\varepsilon > 0$. By Condition 2. it follows that there exists $n_\varepsilon \in \mathbb{N}$ such that

$$|f_n'(x) - f_m'(x)| < \frac{\varepsilon}{3}, \quad \forall m, n \geq n_\varepsilon, \forall x \in [a,b]. \tag{1.2}$$

Take $n_0 > n_\varepsilon$. Since $f_m' \xrightarrow{\text{UC}} g$, follows $f_{n_0}' - f_m' \xrightarrow{\text{UC}} f_{n_0}' - g$, and letting $m \to \infty$, $n = n_0$ in (1.2) we obtain

$$|f_{n_0}'(x) - g(x)| \leq \frac{\varepsilon}{3}, \quad \forall x \in [a,b]. \tag{1.3}$$

Since f_{n_0} is differentiable at x_0, there exists $U \in \mathcal{V}(x_0)$ satisfying

$$\left| \frac{f_{n_0}(x) - f_{n_0}(x_0)}{x - x_0} - f_{n_0}'(x_0) \right| < \frac{\varepsilon}{3}, \quad \forall x \in U \cap [a,b],\ x \neq x_0. \tag{1.4}$$

Now for $m, n \geq n_\varepsilon$ one gets

$$\begin{aligned} \left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right| &= \left| \frac{(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))}{x - x_0} \right| \\ &= |f_n'(c) - f_m'(c)| < \frac{\varepsilon}{3}, \end{aligned}$$

where c is the intermediary point from Lagrange theorem applied to the function $f_n - f_m$ on $[x_0, x]$. Letting $m \to \infty$, $n = n_0$ in the previous inequality it follows:

$$\left| \frac{f_{n_0}(x) - f_{n_0}(x_0)}{x - x_0} - \frac{f(x) - f(x_0)}{x - x_0} \right| \leq \frac{\varepsilon}{3}, \quad x \in [a,b],\ x \neq x_0. \tag{1.5}$$

For $x \in U \cap [a,b]$, $x \neq x_0$, we get

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - g(x_0)\right| \leq \left|\frac{f(x)-f(x_0)}{x-x_0} - \frac{f_{n_0}(x)-f_{n_0}(x_0)}{x-x_0}\right|$$
$$+\left|\frac{f_{n_0}(x)-f_{n_0}(x_0)}{x-x_0} - f'_{n_0}(x_0)\right| + |f'_{n_0}(x_0) - g(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since ε is an arbitray positive number we have $f'(x_0) = g(x_0)$. ■

2 Series of functions

The results obtained for sequences of functions may be transferred to series of functions by means of the sequence of partial sums.

Let $(f_n)_{n\geq 1}$, $f_n : A \to \mathbb{R}$ be a sequence of functions. Then a series of functions of general term f_n has the form

$$\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots .$$

Define the sequence $(s_n)_{n\geq 1}$, of partial sums of $\sum\limits_{n=1}^{\infty} f_n$ by

$$s_n = f_1 + f_2 + \ldots + f_n, \quad n \geq 1.$$

Definition 3 *The series of functions* $\sum\limits_{n=1}^{\infty} f_n$ *is said to be pointwise (uniformly) convergent if the sequence of its partial sums* $(s_n)_{n\geq 1}$, *is pointwise (uniformly) convergent.*

If there exists $\lim\limits_{n\to\infty} s_n = f$, then f is called the sum of $\sum\limits_{n=1}^{\infty} f_n$. We denote

$$\sum_{n=1}^{\infty} f_n \overset{\text{PC}}{=} f \quad \text{and} \quad \sum_{n=1}^{\infty} f_n \overset{\text{UC}}{=} f,$$

for pointwise and uniformly convergence, respectively.

For deciding the uniform convergence of a series of functions we frequently use the next result.

Theorem 8 (Weierstrass) *Let* $\sum_{n=1}^{\infty} f_n,\ f_n : A \to \mathbb{R}$, *be a series of functions and* $(a_n)_{n\geq 1}$ *a sequence of positive numbers such that:*

1. $|f_n(x)| \leq a_n,\ \forall n \geq n_0,\ \forall x \in A$;

2. $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Then $\sum_{n=1}^{\infty} f_n$ *is uniformly and absolutely convergent on* A.

Proof.

Fix $\varepsilon > 0$. By the convergence of $\sum_{n=1}^{\infty} a_n$ it follows that there exists $n_\varepsilon \in \mathbb{N}$ such that

$$a_{n+1} + \ldots + a_{n+p} < \varepsilon, \quad \forall n \geq n_\varepsilon, \forall p \geq 1.$$

Taking account of 1. we obtain

$$\begin{aligned} |f_{n+1}(x) + \cdots + f_{n+p}(x)| &\leq |f_{n+1}(x)| + \cdots + |f_{n+p}(x)| \\ &\leq a_{n+1} + \cdots + a_{n+p} \leq \varepsilon \end{aligned}$$

for all $n \geq \max\{n_\varepsilon, n_0\}$, $p \geq 1$ and $x \in A$.

Therefore the sequence of partial sums of the series $\sum_{n=1}^{\infty} f_n$ is uniformly fundamental. The series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent in view of Cauchy's criterion.

■

Example 2.1 *Prove that the series* $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+x^2}$ *is uniformly and absolutely convergent on* $\mathbb{R}$.

Solution 2 *Take* $f_n(x) = \frac{\cos nx}{n^2+x^2}$, *for all* $x \in \mathbb{R}$, $n \in \mathbb{N}^*$. *We have*

$$|f_n(x)| \leq \frac{1}{n^2 + x^2} \leq \frac{1}{n^2}, \quad x \in \mathbb{R},\ n \geq 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent, the conclusion follows according to Weierstrass test.*

The properties of uniformly convergent sequences of functions are valid for series of functions as follows:

Theorem 9 *Let $(f_n)_{n\geq 1}$, $f_n : A \to \mathbb{R}$, $n \geq 1$, be a sequence of bounded functions such that $\sum\limits_{n=1}^{\infty} f_n = f$ uniformly on A. Then f is a bounded function.*

Proof. The result follows from Theorem 4 since $(s_n)_{n\geq 1}$, $s_n = f_1 + f_2 + \cdots + f_n$ is a sequence of bounded functions and $s_n \xrightarrow{\text{UC}} f$ on A. ■

Theorem 10 *Let $(f_n)_{n\geq 1}$, $f_n : A \to \mathbb{R}$, $n \geq 1$, $A \subseteq \mathbb{R}$, be a sequence of continuous functions such that $\sum\limits_{n=1}^{\infty} f_n = f$ uniformly on A. Then f is a continuous function.*

Proof. The result is a simple consequence of Theorem 5. ■

Theorem 11 *Let $(f_n)_{n\geq 1}$, $f_n : [a, b] \to \mathbb{R}$, $n \geq 1$, be a sequence of differentiable functions such that:*

1. $\sum\limits_{n=1}^{\infty} f_n = f$ *pointwise on* $[a, b]$;

2. $\sum\limits_{n=1}^{\infty} f'_n = g$ *uniformly on* $[a, b]$.

Then f is differentiable and $f' = g$.

Proof. The proof follows from Theorem 7 ■

So, under the conditions of previous theorem the series $\sum\limits_{n=1}^{\infty} f_n$ can be termwise differentiated, i.e.,

$$\left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f'_n.$$

Theorem 12 *Let $(f_n)_{n\geq 1}$, $f_n : [a,b] \to \mathbb{R}$, $n \geq 1$, be a sequence of integrable functions with the property that $\sum\limits_{n=1}^{\infty} f_n = f$, uniformly on $[a,b]$. Then f is integrable on $[a,b]$ and*

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x) dx.$$

Proof. The proof follows from Theorem 6. $\blacksquare$

So, we conclude that under the conditions of Theorem 12 the series $\sum\limits_{n=1}^{\infty} f_n$ can be termwise integrated.

Finally we present a consequence of Dini's Theorem.

Theorem 13 *Let $(f_n)_{n\geq 1}$, $f_n : [a,b] \to [0,\infty)$, $n \geq 1$, be a sequence of continuous nonnegative functions and $f : [a,b] \to \mathbb{R}$ a continuous function such that $\sum\limits_{n=1}^{\infty} f_n = f$ pointwise on $[a,b]$. Then $\sum\limits_{n=1}^{\infty} f_n$ is uniformly convergent to f on $[a,b]$.*

Proof. Let $(s_n)_{n\geq 1}$, $s_n = f_1 + f_2 + \cdots + f_n$, $n \geq 1$. Then $(s_n)_{n\geq 1}$ is increasing, so the conclusion follows from Dini's Theorem. $\blacksquare$

3 Power series

Power series are particular series of functions, their sequence of partial sums is polynomial. They play an important role in the definition of elementary functions and in approximation.

Definition 4 *Let $(a_n)_{n\geq 0}$ be a sequence of real numbers and $x_0 \in \mathbb{R}$. A series of functions of the form*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + ...$$

is called power series with center x_0.

Denoting $x - x_0 = t$ we obtain a power series with center 0:

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \ldots,$$

therefore in what follows we consider only power series with center zero.

A result concerning the convergence of power series is given in the next theorem.

Theorem 14 (First Abel's theorem) *For every power series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *there exists* $R \in [0, \infty) \cup \{\infty\}$ *such that:*

1. *The series converges absolutely for* $|x| < R$*;*
2. *The series diverges for* $|x| > R$*;*
3. *The series is absolutely and uniformly convergent for* $|x| \leq r < R$*.*

Proof. If the series $\sum\limits_{n=0}^{\infty} a_n x^n$ converges only for $x = 0$ the conclusions of the theorem are true with $R = 0$. We prove that if the series converges for $x_0 \neq 0$, then it converges for $x \in \mathbb{R}$, $|x| < |x_0|$. By the convergence of the series $\sum\limits_{n=0}^{\infty} a_n x_0^n$ it follows $\lim\limits_{n\to\infty} a_n x_0^n = 0$, so there exists $M > 0$ such that

$$|a_n x_0^n| \leq M, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Let $x \in \mathbb{R}$, $|x| < |x_0|$. We have:

$$|a_n x^n| \;=\; \left| a_n x_0^n \frac{x^n}{x_0^n} \right| = |a_n x_0^n| \left| \frac{x}{x_0} \right|^n \leq M \left| \frac{x}{x_0} \right|^n, \forall n \geq 0. \tag{3.2}$$

By the convergence of the geometric series $\sum\limits_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$ and (3.2) it follows the convergence of $\sum\limits_{n=0}^{\infty} |a_n x^n|$, according to the first comparison test for series with positive terms. Take now

$$R = \sup\{|x_0| : \sum_{n=0}^{\infty} a_n x_0^n \text{ is convergent}\}.$$

1. For $x \in \mathbb{R}$, $|x| < R$ there exists $x_0 \in \mathbb{R}$, $|x| < |x_0| < R$, and since $\sum_{n=0}^{\infty} a_n x_0^n$ is absolutely convergent $\Longrightarrow \sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

2. Let $x \in \mathbb{R}$, $|x| > R$ and suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges. Then $R \geq |x|$, which is a contradiction. So $\sum_{n=0}^{\infty} a_n x^n$ diverges.

3. For $x \in \mathbb{R}$, $|x| \leq r < R$, since the series of numbers $\sum_{n=0}^{\infty} a_n r^n$ is absolutely convergent and

$$|a_n x^n| \leq |a_n r^n|, \quad \forall n \geq 0,$$

it follows that the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly and absolutely convergent, in view of Weierstrass criterion.

■

Definition 5 *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series. The unique element* $R \in [0, \infty) \cup \{\infty\}$, *defined by Theorem 14 is called the radius of convergence of the series.*

The set

$$C = \{x_0 \in \mathbb{R} | \sum_{n=0}^{\infty} a_n x_0^n \in \mathbb{R}\}$$

is called the set of convergence of the series.

The following relation holds:

$$(-R, R) \subseteq C \subseteq [-R, R].$$

A formula for the evaluation of the radius of convergence is given below.

Theorem 15 (Cauchy-Hadamard) *Let* $\sum\limits_{n=0}^{\infty} a_n x^n$ *be a power series of radius of convergence* R *such that* $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$ *exists. Then:*

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Proof. Denote $\rho = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$, $\rho \in [0, \infty) \cup \{\infty\}$and let $x \in \mathbb{R}$ be a fixed number. We apply Cauchy's root test to the series of positive numbers $\sum\limits_{n=0}^{\infty} |a_n x^n|$. We have:

$$\lim_{n\to\infty} \sqrt[n]{|a_n x^n|} = |x| \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x|\rho.$$

For $|x|\rho < 1 \iff |x| < \frac{1}{\rho}$, the series is convergent. For $|x|\rho > 1 \iff |x| > \frac{1}{\rho}$, the series is divergent.

It follows $R = \frac{1}{\rho}$, in view of Abel's first theorem. ■

A general formula for the evaluation of the radius of convergence is given below.

Remark 5 *A stronger result can be found in []. For every power series* $\sum\limits_{n=0}^{\infty} a_n x^n$ *the radius of convergence is given by the relation*

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}}.$$

Remark 6 *Let* $\sum\limits_{n=0}^{\infty} a_n x^n$ *be a power series with the property that* $\lim\limits_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ *exists. Then*

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

Proof. In view of a consequence of Stolz-Cesaro lemma

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

■

For the convergence of a power series for $|x| = R$ we have the following result.

Theorem 16 (Second Abel's theorem) *Let* $\sum\limits_{n=0}^{\infty} a_n x^n$ *be a power series. If the series converges for* $x = R$ *or* $x = -R$*, then its sum is continuous at* R *or* $-R$*, respectively.*

Example 3.1 *Find the set of convergence of the following series:*

a) $\sum\limits_{n=1}^{\infty} n^n x^n$;

b) $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$;

c) $\sum\limits_{n=0}^{\infty} \frac{x^n}{2^n(n+1)}$.

Solution 3 *a)* $a_n = n^n$, $n \geq 1$,

$$R = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{a_n}} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

so the set of convergence is $C = \{0\}$.

b) $a_n = \frac{1}{n!}$, $n \geq 0$,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} (n+1) = +\infty,$$

so $C = \mathbb{R}$.

c) $a_n = \frac{1}{2^n(n+1)}$, $n \geq 0$,

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{2(n+2)}{n+1} = 2.$$

For $x = 2$ *we get the series* $\sum\limits_{n=0}^{\infty} \frac{1}{n+1} = +\infty$.

For $x = -2$ *we get the series* $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n+1}$*, which is convergent according to Leibnitz test. Consequently,* $C = [-2, 2)$.

The properties of power series follow from the properties of series of functions.

Theorem 17 *Let* $(a_n)_{n\geq 0}$ *be a sequence of real numbers and suppose that*

$$\sum_{n=0}^{\infty} a_n x^n = S(x), \quad x \in (-R, R).$$

Then:

1. *S is differentiable on* $(-R, R)$ *and*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} n a_n x^{n-1} = S'(x), \quad x \in (-R, R).$$

2. *For every* $[a, b] \subset (-R, R)$ *the following relation holds*

$$\int_a^b \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \int_a^b a_n x^n dx = \int_a^b S(x) dx.$$

Example 3.2 *Find the set of convergence and the sum of the power series*

$$\sum_{n=0}^{\infty} (-1)^n n^2 x^n.$$

Solution 4 $R = \lim\limits_{n\to\infty} \left|\frac{(-1)^n n^2}{(-1)^{n+1}(n+1)^2}\right| = 1$. *For* $x = \pm 1$ *the series diverges, therefore the set of convergence is* $(-1, 1)$. *We have:*

$$\sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}, \quad x \in (-1, 1). \tag{3.3}$$

By differentiation the above relation becomes

$$\sum_{n=0}^{\infty} (-1)^n n x^{n-1} = \frac{-1}{(1+x)^2}, \quad x \in (-1, 1). \tag{3.4}$$

Now multiplying relation (3.4) by x, a new differentiation leads to

$$\sum_{n=0}^{\infty}(-1)^n n^2 x^{n-1} = \frac{x-1}{(1+x)^3}, \quad x \in (-1,1), \tag{3.5}$$

which leads to

$$\sum_{n=0}^{\infty}(-1)^n n^2 x^{n} = \frac{x(x-1)}{(1+x)^3}, \quad x \in (-1,1).$$

Example 3.3 *Find the sum of the series of numbers*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}.$$

Solution 5 *The series converges in view of Leibniz test for alternating series. We, consider the power series*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}x^{n+1}.$$

We obtain $R=1$ and denote by S its sum on $(-1,1)$, i.e.,

$$S(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n(n+1)}x^{n+1}, \quad x \in (-1,1). \tag{3.6}$$

By differentiation one obtains

$$S'(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{n}, \quad x \in (-1,1) \tag{3.7}$$

$$S''(x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1} = \frac{1}{1+x}, \quad x \in (-1,1). \tag{3.8}$$

Integrating the previous relation it follows

$$S'(x) = \ln(x+1) + \mathcal{C}, \quad x \in (-1,1). \tag{3.9}$$

For $x = 0$ in (3.7) and (3.9) one gets $\mathcal{C} = 0$. Consequently by (3.9) it follows

$$(3.10) \qquad \begin{aligned} S(x) &= \int \ln(x+1)dx = x\ln(x+1) - \int \frac{x}{x+1}dx \\ &= (1+x)\ln(x+1) - x + \mathcal{C}. \end{aligned}$$

Putting $x = 0$ in (3.6) and (3.10) one gets $\mathcal{C} = 0$. Since the power series (3.6) is convergent for $x = 1$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} = \lim_{x \nearrow 1} S(x) = 2\ln 2 - 1,$$

in view of Abel's second theorem.

4 Taylor series

Let I be an open interval in $\mathbb{R}$, $f \in C^{\infty}(I)$ and $x_0 \in I$.

Definition 6 *The power series given by*

$$(4.1) \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

is called Taylor's series of f at the point x_0 (or in powers of $x - x_0$).

For $x_0 = 0$ the corresponding series is sometimes called Maclaurin's series.

The case when Taylor's series of the function f at x_0 is convergent in a neighborhood of x_0 to that very function is of particular importance. In this case

$$(4.2) \qquad \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x), \quad x \in (x_0 - \delta, x_0 + \delta),$$

where $(x_0 - \delta, x_0 + \delta) \subseteq I$. The function f is said to be expanded into Taylor's series in powers of $x - x_0$. Generally the relation (4.2) does not hold. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

One proves that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, therefore the Taylor series of f at $x_0 = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0, \quad \forall x \in \mathbb{R}.$$

Hence there is no neighborhood of $x_0 = 0$ such that the equality (4.2) holds.

Theorem 18 *The following relations hold:*

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots,\ x \in \mathbb{R};$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,\ x \in \mathbb{R};$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots,\ x \in \mathbb{R};$

4) $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots,\ |x| < 1;$

5) $(1 + x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots,\ |x| < 1,\ \alpha \in \mathbb{R}.$