Taylor's formula. Taylor series

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One of the basic problems in mathematics is to approximate a function belonging to a given class by another function which is more simple or by a sequence of functions. By Taylor's formula a function of class C^n is approximated by a polynomial function.

1 Taylor's formula

The next result called Taylor's Theorem was given by the English mathematician Brook Taylor in 1715. It seems that a version of this result was obtained in 1671 by the Scottish mathematician James Gregory. As a consequence we can obtain simple formulae for the evaluation of many functions such as the exponential function, logarithmic function and trigonometric functions.

Theorem 1 *Let I be an open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function possessing a derivative of order $(n+1)$ on I. Then for every x, $x_0 \in I$, $x \neq x_0$, there exists a point c between x_0 and x such that*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Proof.

Suppose $x_0 < x$. Let $F : [x_0, x] \to \mathbb{R}$, be defined by

$$F(t) = f(t) + \frac{f'(t)}{1!}(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n + A(x-t)^{n+1},$$

where the real number A is chosen such that F satisfies the conditions of Rolle's theorem on $[x_0, x]$. Then F must satisfy the condition

$$F(x) = F(x_0)$$

or equivalently
(1.1)
$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + A(x - x_0)^{n+1}.$$

Now we have to prove that there exists $c \in (x_0, x)$ such that

$$A = \frac{f^{(n+1)}(c)}{(n+1)!}.$$

By Rolle's theorem follows that there exists $c \in (x_0, x)$ satisfying

$$F'(c) = 0.$$

Consequently, we have

$$\begin{aligned} F'(t) &= f'(t) + f''(t)(x - t) - f'(t) + \frac{f'''(t)}{2!}(x - t)^2 - \frac{2f''(t)}{2!}(x - t) \\ &+ \cdots + \frac{f^{(n+1)}(t)}{n!}(x - t)^n - \frac{f^{(n)}(t)}{n!}n(x - t)^{n-1} - A(n + 1)(x - t)^n, \\ F'(t) &= \frac{f^{(n+1)}(t)}{n!}(x - t)^n - A(n + 1)(x - t)^n. \end{aligned}$$

Since $F'(c) = 0$ we get $F'(c) = \frac{f^{(n+1)}(c)}{n!}(x - c)^n - A(n + 1)(x - c)^n = 0$, which leads to $A = \frac{f^{(n+1)}(c)}{(n+1)!}$. ■

Next, we present another form of Taylor's formula with the reminder in integral form.

Theorem 2 *Let I be an open interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a function possessing a continuous derivative of order $(n + 1)$ on I. Then for every x, $x_0 \in I$, $x \neq x_0$, the following relation holds:*

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n dt.$$

Proof. We start by using the relation

$$f(x) = f(x_0) - \int_{x_0}^{x} f'(t)(x-t)' dt.$$

Now, using the formula of integration by parts, $\int_{x_0}^{x} uv' = uv|_{x_0}^{x} - \int_{x_0}^{x} u'v$ with $u(t) = f'(t)$ and $v(t) = x - t$ we get

$$\begin{aligned}
f(x) &= f(x_0) - f'(t)(x-t)|_{x_0}^{x} + \int_{x_0}^{x} f''(t)(x-t)dt \\
&= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) - \frac{1}{2!}\int_{x_0}^{x} f''(t)\left((x-t)^2\right)' dt \\
&= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) - \frac{1}{2!} f''(t)(x-t)^2|_{x_0}^{x} + \frac{1}{2!}\int_{x_0}^{x} f'''(t)(x-t)^2 dt, \\
&= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2| + \frac{1}{2!}\int_{x_0}^{x} f'''(t)(x-t)^2 dt,
\end{aligned}$$

and so on. Finally, we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt.$$

■

Remark 1 *1. The polynomial*

$$T_n f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the ***Taylor's polynomial of*** n^{th} ***degree*** *of f at x_0.*

$T_n(x)$ satisfies the following relations:

$$T_n^{(k)} f(x_0) = f^{(k)}(x_0), \quad \forall k = 0, \ldots, n.$$

2. The last term from Taylor's formula

$$R_n f(x) := \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1},$$

is called the ***reminder of*** n^{th} ***order****.*

3. The point c is called the ***intermediary point*** *and it can be written as*

$$c = x_0 + \theta(x - x_0), \quad 0 < \theta < 1.$$

Corollary 1.1 (Maclaurin) *If $f : I \to \mathbb{R}$ satisfies the conditions of Theorem1 and $0 \in I$, then there exists c between 0 and x such that*

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Proof. We take $x_0 = 0$ in Taylor's formula. ■

We give some expansions of elementary functions using Maclaurin formula

Corollary 1.2 *The following relations hold:*

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + e^{\theta x}\frac{x^{n+1}}{(n+1)!}, \; 0 < \theta < 1, x \in \mathbb{R};$
2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + \frac{(-1)^n x^{2n}}{(2n)!}\sin\theta x, \; 0 < \theta < 1, x \in \mathbb{R};$
3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}\sin\theta x, 0 < \theta < 1, x \in \mathbb{R};$
4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n x^{n+1}}{(n+1)(1+\theta x)^{n+1}}, \; 0 < \theta < 1, x > -1;$
5. $(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n +$
$+ \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}(1+\theta x)^{\alpha-n-1}x^{n+1}, \; |x| < 1, \; 0 < \theta < 1, \alpha \in \mathbb{R}.$

Proof. The relations 1. – 5. follow from Maclaurin formula.

The last relation is called the binomial expansion. Clearly, if $\alpha \in \mathbb{N}^*$ we get Newton's binomial formula. ■

We give now an application of Taylor's formula for finding the local extrema of a function.

Theorem 3 *Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a function possessing a continuous derivative of order n, $n \geq 2$, and $x_0 \in I$ such that*

$$f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0.$$

Then:

1. *If n is an even number, x_0 is a local extremum of f as follows:*

   i) $f^{(n)}(x_0) > 0 \Longrightarrow x_0$ *is a point of local minimum;*

   ii) $f^{(n)}(x_0) < 0 \Longrightarrow x_0$ *is a point of local maximum.*

2. *If n is an odd number, x_0 is not a point of local extremum of f.*

Proof. By the continuity of $f^{(n)}$ and $f^{(n)}(x_0) \neq 0$ it follows that there exists $V = (x_0 - \varepsilon, x_0 + \varepsilon)$, $V \subseteq I$, such that $f^{(n)}(x) > 0$ or $f^{(n)}(x) < 0$ for every $x \in V$.

Now for $x \in V$ we have

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n-1)(x_0)}}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x - x_0)^n \\ &= f(x_0) + \frac{f^{(n)}(c)}{n!}(x - x_0)^n, \quad c = x_0 + \theta(x - x_0), 0 < \theta < 1. \end{aligned}$$

Then:

1. *i)* If $f^{(n)}(x_0) > 0 \Longrightarrow f^{(n)}(c) > 0$ and $(x - x_0)^n \geq 0 \Longrightarrow f(x) \geq f(x_0)$, $\forall x \in V$, hence x_0 is a point of local minimum of f.

   ii) If $f^{(n)}(x_0) < 0 \Longrightarrow f^{(n)}(c) < 0$ and $(x - x_0)^n \geq 0 \Longrightarrow f(x) \leq f(x_0)$, $\forall x \in V$, hence x_0 is a point of local maximum of f.

2. If n is an odd number $(x - x_0)^n$ changes the sign at x_0, $f^{(n)}(c)$ has constant sign on V, thus x_0 is not a point of local extremum of f.

■

2 Taylor series

Definition 1 *A function $f : I \to \mathbb{R}$, is said to be of class C^k, $k \in \mathbb{N}^*$, on I if all the derivatives of f up to the order k exists and they are continuous.*

We denote by

$$\begin{aligned} C(I) &= \{f : I \to \mathbb{R} |\text{ f continuous on I }\}, \\ C^k(I) &= \{f : I \to \mathbb{R} |\text{ f of class } C^k \text{ on I }\}, \\ C^\infty(I) &= \{f : I \to \mathbb{R} | f \text{ is of class } C^k \text{ on I for every } k \in \mathbb{N}\}. \end{aligned}$$

Let I be an open interval in $\mathbb{R}$, $f \in C^\infty(I)$ and $x_0 \in I$.

Definition 2 *The power series given by*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots \tag{2.1}$$

is called Taylor's series of f at the point x_0.

For $x_0 = 0$ the corresponding series is called Maclaurin's series.

The case when the Taylor's series of the function f at x_0 is convergent in a neighborhood of x_0 to that very function is of particular importance. In this case the following relation holds:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x), \quad x \in (x_0-\delta, x_0+\delta), \tag{2.2}$$

where $(x_0-\delta, x_0+\delta) \subseteq I$. The function f is said to be expanded into Taylor's series in powers of $x - x_0$.

Remark 2 *1. It turns out that not every differentiable function can be represented as a Taylor series (i.e., the relation* (2.2) *does not hold). This was demonstrated by the famous mathematician Augustin Louis Cauchy, who gave the counter-example presented in what follows.*

Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

One proves that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$, therefore the Taylor series of f at $x_0 = 0$ is given by

$$\sum_{n=0}^{\infty} \frac{0}{n!}x^n = 0, \quad \forall x \in \mathbb{R}.$$

Hence there is no neighborhood of $x_0 = 0$ such that the equality (2.2) *holds.*

2. *A function $f : I \to \mathbb{R}$, which can be expanded in Taylor series at any point of the interval I is called* analytic *on I (i.e., the relation* (2.2) *holds for every point $x_0 \in I$ and for some positive δ).*

Theorem 4 *The relation* (2.2) *holds if and only if the sequence $(R_n(x))_{n\geq 1}$ of the reminder in Taylor's formula converges to zero for all $x \in (x_0-\delta, x_0+\delta)$*

Proof. For every $n \geq 1$ and every $x \in (x_0 - \delta, x_0 + \delta)$ we have:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x) \tag{2.3}$$

in view of Taylor's formula. Then by (2.3) it follows that the equality (2.2) holds if and only if

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for all } x \in (x_0 - \delta, x_0 + \delta).$$

■

Corollary 2.1 *Suppose that there exists $M > 0$ such that*

$$|f^{(n)}(x)| \leq M^n, \quad \forall n \geq 1, x \in (x_0 - \delta, x_0 + \delta). \tag{2.4}$$

Then f can be expanded in Taylor's series at the point x_0 on the interval $(x_0 - \delta, x_0 + \delta)$.

Proof. For every $x \in (x_0 - \delta, x_0 + \delta)$ and every $n \in \mathbb{N}^*$ we have

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

where $c = x_0 + \theta(x - x_0)$, $0 < \theta < 1$. Then

$$|R_n(x)| \leq M^{n+1}\frac{|x - x_0|^{n+1}}{(n+1)!}$$

and since $\lim\limits_{n\to\infty} M^{n+1}\frac{|x-x_0|^{n+1}}{(n+1)!} = 0$, according to ratio test for sequences, it follows $\lim\limits_{n\to\infty} R_n(x) = 0$. ■

Remark 3 *Clearly, Corollary 2.1 remains true if in relation* (2.4) M^n *is replaced by* M.

Finally we give the expansion in Taylor's series of some elementary functions.

Theorem 5 *The following relations hold:*

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots,\ x \in \mathbb{R};$
2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,\ x \in \mathbb{R};$
3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,\ x \in \mathbb{R};$
4) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots,\ |x| < 1;$
5) $(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots,\ |x| < 1,\ \alpha \in \mathbb{R}.$

Proof.

1) Let $a > 0$ and $f : (-a, a) \to \mathbb{R}$, $f(x) = e^x$. Then $f^{(n)}(x) = e^x$ and $|f^{(n)}(x| \le e^a$, for all $n \in \mathbb{N}$, $x \in (-a, a)$. We get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in (-a, a)$$

and since a is arbitrary the relation holds on $\mathbb{R}$.

The relations 2), 3) follow analogously.

4) For $|t| < 1$ we have the expansion in geometric series

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots$$

By integration on $[0, x]$, $|x| < 1$, the previous relation leads to

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

5) The radius of convergence of the binomial series

$$1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$$

is

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\cdot\frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n-1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right| = 1. \end{aligned}$$

Denote by $f(x)$ its sum for $x\in(-1,1)$. We have:

$$xf'(x) = \frac{\alpha}{1!} + \frac{\alpha(\alpha-1)}{2!}x + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(n-1)!}x^n + \cdots$$

Summing the last equalities it follows

$$\begin{aligned} (1+x)f'(x) &= \alpha + \left(\frac{\alpha(\alpha-1)}{1!} + \alpha\right)x + \left(\frac{\alpha(\alpha-1)(\alpha-2)}{2!} + \frac{\alpha(\alpha-1)}{1!}\right)x^2 + \cdots \\ &= \alpha\left(1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!} + \cdots\right) = \alpha f(x). \end{aligned}$$

The relation $(1+x)f'(x) = \alpha f(x)$ is equivalent to

$$(1+x)^\alpha f'(x) - \alpha(1+x)^{\alpha-1}f(x) = 0, \quad x\in(-1,1),$$

or

$$\left(\frac{f(x)}{(1+x)^\alpha}\right)' = 0, \quad x\in(-1,1).$$

Then there exists $C\in\mathbb{R}$ such that

$$f(x) = C(1+x)^\alpha, \quad x\in(-1,1).$$

By the last relation one gets $C = f(0) = 1$, hence

$$f(x) = (1+x)^\alpha.$$

■

Example 2.1 *Expand in power series of x the functions:*

1. $f:\mathbb{R}\setminus\{2,3\}\to\mathbb{R}$, $f(x) = \frac{5-2x}{x^2-5x+6}$;

2. $f : (-1,1) \to \mathbb{R}$, $f(x) = \arcsin x$;

3. $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$, $f(x) = \frac{e^{-x}}{1+x}$.

Solution 1 *1.*

$$f(x) = \frac{5-2x}{x^2-5x+6} = \frac{1}{2-x} + \frac{1}{3-x}, \quad x \in \mathbb{R} \setminus \{2,3\}.$$

We have

$$\begin{aligned} \frac{1}{2-x} &= \frac{1}{2}\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots\right), \quad |x| < 2, \\ \frac{1}{3-x} &= \frac{1}{3}\frac{1}{1-\frac{x}{3}} = \frac{1}{3}\left(1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots\right), \quad |x| < 3, \end{aligned}$$

and

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{3^{n+1}}\right) x^n, \quad |x| < 2.$$

2. *First we expand in power series of x the derivative of f using the binomial series*

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} \\ &= 1 + \frac{-\frac{1}{2}}{1!}(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x^2)^2 + \ldots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(-x^2)^n + \ldots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2^n (n!)} x^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^{2n}, \quad |x| < 1. \end{aligned}$$

By integration on $[0,x]$, $|x| < 1$, it follows

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1$$

3. The Cauchy product of the expansions

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \dots, \; x \in \mathbb{R}$$

and

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad |x| < 1,$$

leads to

$$\begin{aligned} f(x) &= \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \dots\right)(1 - x + x^2 - x^3 + \cdots) \\ &= 1 - \left(1 + \frac{1}{1!}\right)x + \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)x^2 - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right)x^n, \; |x| < 1. \end{aligned}$$