# Fourier series

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## 1 Trigonometric series

The harmonic oscillatory motion is described by the equation

$$y = A\sin(\omega t + \varphi). \tag{1.1}$$

The function from the right hand side of the relation (1.1) is a periodic function of period $T = \frac{2\pi}{\omega}$. Consider in what follows only functions of period $2\pi$ and denote by $x$ their independent variable. Then relation (1.1) becomes

$$y = A\sin(x + \varphi). \tag{1.2}$$

Other functions with the same period are

$$y = A_k \sin(kx + \varphi_k), \quad k \in \mathbb{N},$$

and their sum

$$y = \sum_{k=0}^{n} A_k \sin(kx + \varphi_k), \quad n \in \mathbb{N},$$

called trigonometric polynomial of order $n$. Therefore, it is naturally to consider the problem of representation of a periodic function $f$, with period $T = 2\pi$, asa trigonometric polynomial, or the problem of expansion in a trigonometric series, i.e.

$$f(x)=\sum_{n=0}^{\infty} A_n \sin(nx+\varphi_n). \tag{1.3}$$

The general term of the series from relation (1.3), called the $n$th order harmonic of $f$, can be written as follows

$$A_n \sin(nx+\varphi_n) = a_n \cos nx + b_n \sin nx,$$

where

$$a_n = A_n \sin\varphi_n, \quad b_n = A_n \cos\varphi_n, \quad n \in \mathbb{N}.$$

**Definition 1** *A series of functions of the following form*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \tag{1.4}$$

*where* $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 1}$ *are sequences of real numbers is called* ***trigonometric series****.*

The sequence of partial sums of a trigonometric series $(T_n)_{n\geq 1}$, given by

$$T_n(x)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

is called **trigonometric polynomial**. The trigonometric polynomials are functions of period $T = 2\pi$, hence it suffices to study trigonometric series on an interval of length $2\pi$, for example $[-\pi,\pi]$.

**Remark 1** *The trigonometric series* (1.4) *is absolutely and uniformly convergent if the series of numbers*

$$\frac{|a_0|}{2}+\sum_{n=1}^{\infty}(|a_n|+|b_n|) \tag{1.5}$$

*is convergent, in view of Weierstrass test. But series* (1.4) *can converge without series* (1.5) *being convergent. Abel-Dirichlet test is suitable to investigate the convergence in this case.*

We are looking now for a relation between the sum and the coefficients of trigonometric series. The following result is useful for this aim.

**Lemma 1.1** *The following relation hold:*

$$\begin{aligned} \int_{-\pi}^{\pi} \sin mx \sin nx dx &= 0, \quad m,n \in \mathbb{N}, m \neq n; \\ \int_{-\pi}^{\pi} \cos mx \cos nx dx &= 0, \quad m,n \in \mathbb{N}, m \neq n; \\ \int_{-\pi}^{\pi} \sin mx \cos nx dx &= 0, \quad m,n \in \mathbb{N}; \\ \int_{-\pi}^{\pi} \cos^2 mx dx &= \int_{-\pi}^{\pi} \sin^2 mx dx = \pi, \quad m \in \mathbb{N}^*. \end{aligned} \tag{1.6}$$

**Proof.** We have

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x) dx$$
$$= \frac{1}{2(m-n)} \sin(m-n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(m+n)} \sin(m+n)x\Big|_{-\pi}^{\pi} = 0.$$

The proof of the next two relations follows analogously.

$$\begin{aligned} \int_{-\pi}^{\pi} \cos^2 mx dx &= \int_{-\pi}^{\pi} \frac{1+\cos 2mx}{2} dx = \frac{1}{2}\left(x + \frac{\sin 2mx}{2m}\right)\Big|_{-\pi}^{\pi} = \pi \\ \int_{-\pi}^{\pi} \sin^2 mx dx &= \int_{-\pi}^{\pi} \frac{1-\cos 2mx}{2} dx = \frac{1}{2}\left(x - \frac{\sin 2mx}{2m}\right)\Big|_{-\pi}^{\pi} = \pi \end{aligned}$$

■

A relation between the coefficients of a trigonometric series and its sum is given in the next theorem.

**Theorem 1** *If*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = f(x) \tag{1.7}$$

*uniformly on the interval* $[-\pi, \pi]$ *then:*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \quad n \geq 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \quad n \geq 1. \end{aligned} \tag{1.8}$$

**Proof.** We integrate the relation (1.7), taking account of the uniform convergence, and obtain

$$a_0\pi + \sum_{n=1}^{\infty}\left(a_0\int_{-\pi}^{\pi}\cos nx dx + b_n\int_{-\pi}^{\pi}\sin nx dx\right) = \int_{-\pi}^{\pi} f(x)dx.$$

Since

$$\int_{-\pi}^{\pi}\cos nx dx = \int_{-\pi}^{\pi}\sin nx dx = 0, \quad n \geq 1,$$

it follows

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx.$$

Now multiplying relation (1.7) by $\cos mx$, $m \geq 0$ the uniformly convergence is kept and integrating the relation on $[-\pi, \pi]$, one gets:

$$\begin{aligned} \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx dx &+ \sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos nx\cos mx dx + b_n\int_{-\pi}^{\pi}\sin nx\cos mx dx\right) \\ &= \int_{-\pi}^{\pi} f(x)\cos mx dx \end{aligned}$$

Taking account of the Lemma 1.1 it follows

$$\begin{aligned} a_m\int_{-\pi}^{\pi}\cos^2 mx dx &= \int_{-\pi}^{\pi} f(x)\cos mx dx \\ a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx, \ m \geq 1. \end{aligned}$$

The relation for $b_m$ follows analogously multiplying (1.7) by $\sin mx$ and integrating on $[-\pi, \pi]$. ■

## 2 Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period $T = 2\pi$, integrable on the interval $[-\pi, \pi]$.

**Definition 2** *The trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

*where the coefficients are given by the relations*

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad n \geq 0, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad n \geq 1, \end{aligned}$$

*is called* ***Fourier series*** *of $f$ and $a_n, b_n$ are called* ***Fourier coefficients*** *of $f$.*

**Remark 2** *1. If $f$ is an even, function then the Fourier coefficients of $f$ are given by the relations*

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx dx, \quad n \geq 0 \\ b_n &= 0, \quad n \geq 1 \end{aligned} \tag{2.1}$$

*while if $f$ is an odd, function then the Fourier coefficients of $f$ are given by*

$$a_n = 0, \quad n \geq 0 \tag{2.2}$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx dx, \quad n \geq 1. \tag{2.3}$$

**Proof.** The relation for Fourier coefficients follows taking account that if $f : [-a, a] \to \mathbb{R}$ ia an odd function then

$$\int_{-a}^{a} f(x) dx = 0$$

and if $f$ is an even function

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

■

For a Fourier series associated to a function $f$ we consider the following problems:

1) *Find the set of convergence of the series.*

2) *Under what conditions on $f$ the Fourier series is convergent on $[-\pi, \pi]$?*

3) *In case of convergence on $[-\pi, \pi]$ is the sum of the Fourier series the function $f$?*

4) *Under what conditions on $f$ the Fourier series is uniformly convergent on $[-\pi, \pi]$?*

5) *Is any trigonometric series a Fourier series?*

We will give some answers to the previous questions in what follows.

The following estimation for the Fourier coefficients holds.

**Theorem 2** *Suppose that $f$ has a continuous derivative of order $p$ on $[-\pi, \pi]$ and there exists $M \geq 0$ such that:*

$$|f^{(p)}(x)| \leq M, \quad \forall x \in \mathbb{R}.$$

*Then the Fourier coefficients of $f$ satisfy the inequalities*

$$|a_n| \leq \frac{2M}{n^p}, \quad |b_n| \leq \frac{2M}{n^p}, \quad n \geq 1.$$

**Proof.** Integrating by parts and taking into account that $f(-\pi) = f(\pi)$ we get

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\left(f(x)\frac{\sin nx}{x}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\frac{\sin nx}{n}dx\right) \\
&= -\frac{1}{\pi n}\int_{-\pi}^{\pi} f'(x)\sin nx dx = \frac{1}{\pi n^2}\int_{-\pi}^{\pi} f'(x)(\cos nx)'dx \\
&= \frac{1}{\pi n^2}\left(f'(x)\cos nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f''(x)(\cos nx)dx\right) \\
&= -\frac{1}{\pi n^2}\int_{-\pi}^{\pi} f''(x)(\cos nx)dx = \ldots .
\end{aligned}
$$

Generally $a_n$ is given by the relation

$$
a_n = \begin{cases} -\frac{1}{\pi n^p}\int_{-\pi}^{\pi} f^{(p)}(x)\cos nx dx, & \text{if} \quad p \text{ is even} \\ -\frac{1}{\pi n^p}\int_{-\pi}^{\pi} f^{(p)}(x)\sin nx dx, & \text{if} \quad p \text{ is odd} \end{cases}
$$

Therefore

$$
|a_n| \leq \frac{1}{\pi n^p}\int_{-\pi}^{\pi} |f^{(p)}(x)|dx \leq \frac{2M}{n^p}.
$$

The relation for $b_n$ follows analogously. ■

**Theorem 3 (Bessel's inequality)** *Let $f : [-\pi, \pi] \to \mathbb{R}$ be an integrable function and $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 1}$ be the Fourier coefficients of $f$. The following inequality holds:*

$$
\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx, \quad \forall n \geq 1.
$$

**Proof.** Let $(T_n)_{n\geq 1}$ be given by

$$
T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \quad x \in [-\pi, \pi], n \geq 1.
$$

We have

$$
\int_{-\pi}^{\pi}(f(x) - T_n(x))^2 dx = \int_{-\pi}^{\pi} f^2(x) - 2\int_{-\pi}^{\pi} f(x)T_n(x)dx + \int_{-\pi}^{\pi} T_n^2(x)dx.
$$

Forward

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)T_n(x)dx &= \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx \\
&+ \sum_{k=1}^{n}\left(a_k\int_{-\pi}^{\pi} f(x)\cos kx + b_k\int_{-\pi}^{\pi} f(x)\sin kxdx\right) \\
&= \frac{\pi a_0^2}{2} + \pi\sum_{k=1}^{n}(a_k^2+b_k^2)
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi} T_n^2(x)dx &= \int_{-\pi}^{\pi}\left(\frac{a_0^2}{4} + a_0\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx) + \left(\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right)^2\right)dx \\
&= \frac{\pi a_0^2}{2} + \int_{-\pi}^{\pi}\left(\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right)^2 dx \\
&= \frac{\pi a_0^2}{2} + \int_{-\pi}^{\pi}\left(\sum_{k=1}^{n}a_k\cos kx + \sum_{k=1}^{n}b_k\sin kx\right)^2 dx \\
&= \frac{\pi a_0^2}{2} + \sum_{k=1}^{n}a_k^2\int_{-\pi}^{\pi}\cos^2 kxdx + \sum_{k=1}^{n}b_k^2\int_{-\pi}^{\pi}\sin^2 kxdx \\
&+ 2\sum_{k<j}a_ka_j\int_{-\pi}^{\pi}\cos kx\cos jxdx + 2\sum_{k<j}b_kb_j\int_{-\pi}^{\pi}\sin kx\sin jxdx \\
&+ 2\sum_{k,j=1}^{n}a_kb_j\int_{-\pi}^{\pi}\cos kx\sin jxdx \\
&= \frac{\pi a_0^2}{2} + \pi\sum_{k=1}^{n}(a_k^2+b_k^2).
\end{aligned}$$

Hence

$$\int_{-\pi}^{\pi}(f(x)-T_n(x))^2dx = \int_{-\pi}^{\pi}f^2(x)dx - \pi\left(\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2)\right) \geq 0,$$

which leads to Bessel's inequality. ■

**Corollary 2.1** *Suppose that $f : [-\pi, \pi] \to \mathbb{R}$ is integrable on $[-\pi, \pi]$ and let $(a_n)_{n\geq 0}$, $(b_n)_{n\geq 1}$ be the Fourier coefficients of $f$. Then:*

1. *The series $\sum\limits_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges;*
2. $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$;
3. *If the Fourier series of $f$ is uniformly convergent to $f$ on $[-\pi, \pi]$, then* ***Parseval's equality*** *holds, i.e.:*

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx.$$

**Proof.**

1. The convergence of $\sum\limits_{n=1}^{\infty} (a_n^2 + b_n^2)$ follows from Bessel's inequality.
2. $\sum\limits_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty \Longrightarrow \lim\limits_{n\to\infty} (a_n^2 + b_n^2) = 0 \Longrightarrow \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0.$
3. Letting $n \to \infty$ in the relation

$$\int_{-\pi}^{\pi} (f(x) - T_n(x))^2 dx = \int_{-\pi}^{\pi} f^2(x) dx - \pi \left( \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right),$$

obtained in the proof of Theorem 3, one gets Parseval's equality.

■

**Remark 3** *The following series*

$$\sum_{n=1}^{\infty} \frac{\cos nx}{\sqrt{n}}$$

*is an example of a trigonometric series, which is not a Fourier series.*

**Proof.** Suppose that $\sum\limits_{n=1}^{\infty} \frac{\cos nx}{\sqrt{n}}$ is a Fourier series. Then there exists a function $f : [-\pi, \pi] \to \mathbb{R}$ such that the Fourier coefficients of $f$, $a_0 = 0$, $a_n = \frac{1}{\sqrt{n}}$, $b_n = 0$, $n \geq 1$, satisfy Bessel's inequality, i.e.:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)dx, \quad \forall n \geq 1,$$

which leads to contradiction, since

$$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = \infty.$$

■

One of the main problems in the theory of Fourier series is to find conditions for a given function $f$, such that its Fourier series converges to $f$. Many mathematicians tried to give an answer to this problem. Here we shall give some tests for convergence for Fourier series without proving them.

**Theorem 4 (Dirichlet)** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of period $2\pi$ with the property that there exists a partition of the interval $[-\pi, \pi]$, $-\pi = x_0 < x_1 < \ldots < x_n = \pi$ such that on every interval $(x_{k-1}, x_k)$, $1 \leq k \leq n$, $f$ is bounded and monotone. Then the Fourier series of $f$ is convergent on $\mathbb{R}$ and*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{f(x+0) + f(x-0)}{2}, \;\; x \in \mathbb{R}.$$

A function satisfying the conditions of Dirichlet theorem is called piecewise monotone.

For the uniform convergence of Fourier series we have the following results:

**Theorem 5** *If $f : [-\pi, \pi] \to \mathbb{R}$ is continuous, piecewise monotone on $[-\pi, \pi]$ and $f(\pi) = f(-\pi)$, then its Fourier series is uniformly convergent to $f$ on $[-\pi, \pi]$.*

**Theorem 6** *If $f : \mathbb{R} \to \mathbb{R}$ is a function of class $\mathcal{C}^1$, then its Fourier series is uniformly convergent to $f$ on $\mathbb{R}$.*

**Remark 4** *If $f:[0,\pi]\to\mathbb{R}$ satisfies the conditions of Dirichlet's theorem, then it can be expanded in a Fourier series of cosine, respectively sine, if we extend the function $f$ to an even function $f_1$, respectively to an odd function $f_2$ on the interval $[-\pi,\pi]$ :*

$$f_1(x)=\begin{cases} f(-x), & x\in[-\pi,0) \\ f(x), & x\in[0,\pi] \end{cases}$$

$$f_2(x)=\begin{cases} -f(-x), & x\in[-\pi,0) \\ 0, & x=0 \\ f(x), & x\in(0,\pi]. \end{cases}$$

*In this case the Fourier coefficients of $f$ can be calculated by using relations (2.1) for a series of cosine and respectively relations (2.2) for a series of sine.*

**Remark 5** *If $f:[a,b]\to\mathbb{R}$ is a function that satisfies the conditions of Dirichlet's theorem on $[a,b]$ it can be expanded in a Fourier series as follows.*

*First we introduce the function $\overline{f}:[a,b]\to\mathbb{R}$ defined by*

$$\overline{f}(x)=\begin{cases} f(x), & x\in(a,b] \\ f(b), & x=a \end{cases}$$

*and we extend $\overline{f}$ on $\mathbb{R}$ to a function of period $2T$,*

$$T=\frac{b-a}{2}.$$

*Now we consider the function $g:[-\pi,\pi]\to\mathbb{R}$ given by*

$$g(t)=f(\frac{T}{\pi}t),\quad t\in[-\pi,\pi].$$

*The following relation holds*

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{g(t+0)+g(t-0)}{2},$$

*where*

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt dt \quad n \geq 0, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sin nt dt \quad n \geq 1, \end{aligned}$$

*Finally for the function $f$ we get the expansion*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{T}x + b_n \sin\frac{n\pi}{T}x\right) = \frac{f(x+0)+f(x-0)}{2},\ x \in [a,b]$$

*and*

$$\begin{aligned} a_n &= \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi}{T}x dx, \quad n \geq 0, \\ b_n &= \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi}{T}x dx, \quad n \geq 1. \end{aligned}$$

**Example 2.1** *Expand in Fourier series the function $f : (-\pi, \pi] \to \mathbb{R}$ given by*

$$f(x) = x, \quad x \in (-\pi, \pi].$$

**Solution 1** *Since $f$ is an odd function $a_n = 0$ for every $n \in \mathbb{N}$. We have*

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx = -\frac{2}{\pi n}\int_0^{\pi} x(\cos nx)' dx \\ &= -\frac{2}{n\pi}\left(x\cos nx\Big|_0^{\pi} - \int_0^{\pi}\cos nx\, dx\right) = (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

*In view of Dirichlet's Theorem we obtain:*

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right) = \begin{cases} x, & x \in (-\pi, \pi) \\ 0, & x = \pm\pi. \end{cases}$$

Finally we present some examples of expansions in a Fourier series.

**Example 2.2** *Expand in Fourier series the function $f : \mathbb{R} \to \mathbb{R}$, of period $2\pi$, given by the relation*

$$f(x) = x^2, \quad x \in [-\pi, \pi],$$

*and prove the relation*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

**Solution 2** *Since $f$ is an even function, $b_n = 0$ for every $n \in \mathbb{N}^*$.*

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx \, dx = \frac{2}{\pi n} \left( x^2 \sin nx \Big|_0^{\pi} - 2 \int_0^{\pi} x \sin nx \, dx \right)$$

$$= \frac{4}{\pi n^2} \int_0^{\pi} x(\cos nx)' dx = \frac{4}{\pi n^2} \left( x \cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx \right) = (-1)^n \frac{4}{n^2}.$$

*By Dirichlet's theorem we get the expansion*

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad x \in [-\pi, \pi].$$

*For $x = \pi$ in the above relation we get*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

**Example 2.3** *Expand the function $f : [0, \pi] \to \mathbb{R}$, $f(x) = x$, in a Fourier series of cosine.*

**Solution 3** *We extend $f$ to an even function $\overline{f} : [-\pi, \pi] \to \mathbb{R}$,*

$$\overline{f}(x) = |x|, \quad x \in [-\pi, \pi].$$

*Then $b_n = 0$ for every $n \in \mathbb{N}^*$ and*

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\, dx = \frac{2}{\pi n^2}((-1)^n - 1), \quad n \geq 1.$$

*One gets the expansion*

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \;\; x \in [0,\pi].$$

**Example 2.4** *Expand in Fourier series the function* $f : \mathbb{R} \to \mathbb{R}$, *of period* $2\pi$, *defined by* $f(x) = e^{ax}$, $x \in (-\pi, \pi]$, $a \in \mathbb{R}\setminus\{0\}$. *Find the sum of the series* $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+a^2}$, $\;a \in \mathbb{R}$.

**Solution 4** *First we find the Fourier coefficients. We have*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi a}e^{ax}\Big|_{-\pi}^{\pi} = \frac{e^{a\pi} - e^{-a\pi}}{a\pi} = \frac{2}{a\pi}\mathrm{sh}a\pi.$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi a}\int_{-\pi}^{\pi}(e^{ax})'\cos nx\, dx \\
&= \frac{1}{\pi a}\left(e^{ax}\cos nx\Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} e^{ax}\sin nx\, dx\right) \\
&= \frac{1}{\pi a}\left((-1)^n(e^{a\pi} - e^{-a\pi}) + \frac{n}{a}\int_{-\pi}^{\pi}(e^{ax})'\sin nx\, dx\right) \\
&= \frac{1}{\pi a}\left(2(-1)^n\mathrm{sh}a\pi + \frac{n}{a}\left(e^{ax}\sin nx\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} e^{ax}\cos nx\, dx\right)\right) \\
&= \frac{1}{\pi a}\left(2(-1)^n\mathrm{sh}a\pi - \frac{n^2\pi}{a}a_n\right),
\end{aligned}$$

*consequently we get* $a_n = \frac{2(-1)^n a\cdot\mathrm{sh}a\pi}{(a^2+n^2)\pi}$, $n \geq 1$.

*Analogously, one gets* $b_n = \frac{2(-1)^{n+1}n\cdot\mathrm{sh}a\pi}{a^2+n^2}$, $n \geq 1$.
*By Dirichlet's theorem we get the expansion*

$$\frac{2\mathrm{sh}a\pi}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)\right) = \begin{cases} e^{ax}, & x \in (-\pi,\pi) \\ \mathrm{ch}a\pi, & x = \pm\pi \end{cases}$$

*For $x = \pi$ in the above relation we get*

$$\frac{2\text{sh}a\pi}{\pi}\left(\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}\right) = \text{ch}a\pi,$$

*therefore*

$$\sum_{n=1}^{\infty}\frac{1}{a^2+n^2} = \frac{\pi}{2a}\coth a\pi - \frac{1}{2a^2} \quad a \neq 0.$$

**Example 2.5** *Expand in Fourier series the function of period $2\pi$, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|,\ x \in [-\pi, \pi]$.*

*Find the sum of the series:* $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^2}$; $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$; $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^4}$; $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

**Solution 5** *Since $f$ is an even function $b_n = 0$ for every $n \geq 1$, and*

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,\ n \geq 0.$$

*We obtain $a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi$ and $a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \frac{2((-1)^n-1)}{\pi n^2}$, $n \geq 1$. Since $f$ is continuous on $\mathbb{R}$ we have*

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}, \end{aligned}$$

*consequently*

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2},\ x \in [-\pi, \pi].$$

*For $x = \pi$ in the above relation we get* $\sum\limits_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

*Taking into account the absolute convergence of the series we have*

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \left(\frac{1}{1^2}+\frac{1}{3^2}+\dots\right)+\left(\frac{1}{2^2}+\frac{1}{4^2}+\dots\right)$$
$$= \frac{\pi^2}{8}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

*therefore*

$$\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} \Leftrightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

*Parseval's equality*

$$\frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}({a_n}^2+{b_n}^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

*leads to*

$$\frac{\pi^2}{2}+\frac{16}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{2\pi^2}{3},$$

*hence*

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

*Now*

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}+\sum_{n=1}^{\infty}\frac{1}{(2n)^4}$$
$$= \frac{\pi^4}{96}+\frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4},$$

*therefore* $\sum\limits_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.