Classes of sensors for robots

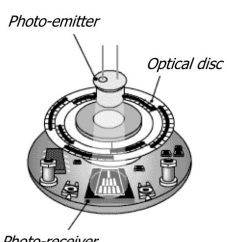
- proprioceptive sensors measure the internal state of the robot (position and velocity of joints, but also torque at joints or acceleration of links)
  - kinematic calibration, identification of dynamic parameters, control
- exteroceptive sensors measure/characterize robot interaction with the environment, enhancing its autonomy (forces/torques, proximity, vision, but also sensors for sound, smoke, humidity, …)
  - control of interaction with the environment, obstacle avoidance in the workspace, presence of objects to be grasped, …
  - mobile-base robots: localization in a map, navigation in unknown environments, …

Position sensors

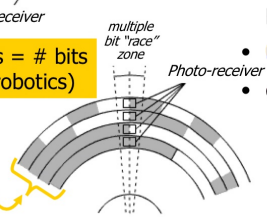
- provide an electrical signal proportional to the displacement (linear or angular) of a mechanical part with respect to a reference position
- linear displacements: potentiometers, linear variable-differential transformers (LVDT), inductosyns
- angular displacements: potentiometers, resolvers, syncros (all analog devices with A/D conversion), optical **encoders (digital)**, Hall sensors, ...

the most used in robotics, since also linear displacements are obtained through rotating motors and suitable transmissions

Absolute encoders

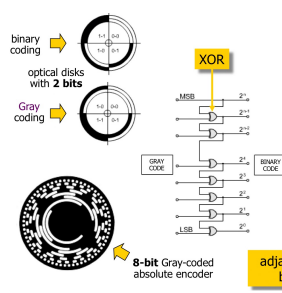


N_t = # tracks = # bits (min 12 in robotics)



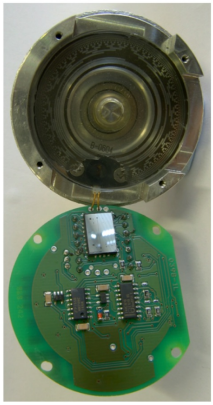
- rotating optical disk, with alternate transparent and opaque sectors on multiple concentric tracks
- (infrared) light beams are emitted by leds and sensed by photo-receivers
- light pulses are converted into electrical pulses, electronically processed and transmitted in output
- resolution = $360^\circ / 2^{N_t}$
- digital encoding of absolute position

  when the optical disk is rotating fast, the use of binary coding may lead to (large) reading errors, in correspondence to multiple transitions of bits



DECIMAL	BINARY	GRAY
0	0000	0000
1	0001	0001
2	0010	0011
3	0011	0010
4	0100	0110
5	0101	0111
6	0110	0101
7	0111	0100
8	1000	1100
9	1001	1101
10	1010	1111
11	1011	1110
12	1100	1010
13	1101	1011
14	1110	1001
15	1111	1000

adjacent codes differ by just one bit



13-bit absolute encoder opened: Gray-coded disk and electronics

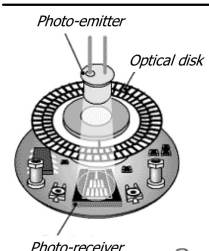
- ready to measure at start (no "homing")
- two modes for permanent operation
  - when switching off the drive, position parameters are saved on a flash memory (and brakes activated)
  - battery for the absolute encoder is always active, and measures position even when the drive is off
  - data memory > 20 years
- single-turn or multi-turn versions, e.g.
  - 13-bit single-turn has 2^{13} = 8192 steps per revolution (resolution = 0.044°)
  - 29-bit multi-turn has 8192 steps/revolution + counts up to 2^{16} = 65536 revolutions
- aluminum case with possible interface to field bus systems (e.g., CANopen or PROFIBUS)
- typical supply 5/28V DC @1.2 W

hollow shaft

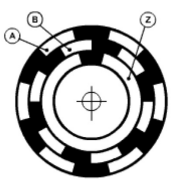
round flange

multi-turn

Incremental encoders



- optical rotating disk with three tracks, alternating transparent and opaque areas: measures incremental angular displacements by counting trains of N_e pulses ("counts") per turn (N_e = 100÷5000)



The three tracks on an optical disk (here N_e = 6)

- the two A and B tracks (channels) are in quadrature (phase shift of 90° electrical), allowing to detect the direction of rotation
- a third track Z is used to define the "0" reference position, with a reset of the counter (needs "homing" at start)
- some encoders provide as output also the three phases needed for the switching circuit of brushless motors

T1
AMO ♡

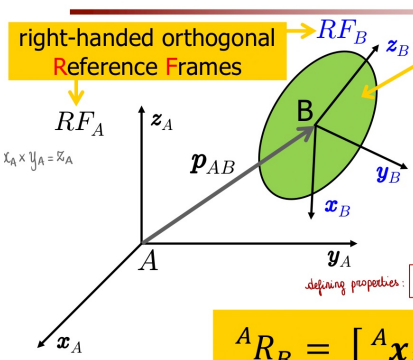
Position & orientation of rigid bodies 26.10.22

POSE

Position and orientation of a single rigid body

right-handed orthogonal Reference Frames → RF_B

rigid body ~ frame B



- position: ${}^A\boldsymbol{p}_{AB}$ (vector $\in \mathbb{R}^3$)

  frame in which I'm representing the vector

  from origin of frame A to origin of frame B

  Cartesian coordinates of vector $\overrightarrow{AB}$ expressed in RF_A

- orientation: ROTATION MATRIX

  orthonormal $\underset{N \times N}{3 \times 3}$ matrix /orthogonal

  defining properties: $(R^T = R^{-1} \Rightarrow R^T R = I)$, with det = +1

not all orthonormal matrices can be used $\left(\begin{cases} \det +1 \\ \det -1 \end{cases} \right)$

= det -1 (2 matrices)

$$ {}^A R_B = \left[{}^A\boldsymbol{x}_B \quad {}^A\boldsymbol{y}_B \quad {}^A\boldsymbol{z}_B \right] $$

- $\boldsymbol{x}_A\ \boldsymbol{y}_A\ \boldsymbol{z}_A$ ($\boldsymbol{x}_B\ \boldsymbol{y}_B\ \boldsymbol{z}_B$) are axis vectors (of unitary norm) of frame RF_A (RF_B)
- components in ${}^A R_B$ are the direction cosines of the axes of RF_B with respect to (w.r.t.) RF_A

matrix of directional cosines

scalar product:

$a \cdot b = a^T b = \|a\| \|b\| \cos\vartheta$

$a, b \in \mathbb{R}^3$

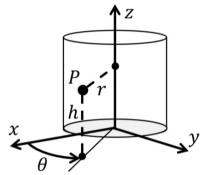
$(\vec{x} \cdot \vec{y} = xy \cos\alpha)$

Position

- for position representation, use of other coordinates than the Cartesian ones is possible, e.g., cylindrical or spherical
- direct transformation from cylindrical to Cartesian

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = h$$

is always well defined
(with $r > 0$ or $r \gtreqless 0$)



angle respect to a frame xy

- inverse transformation from Cartesian to cylindrical

$$x^2 + y^2 = r^2$$
$$\frac{y}{x} = \tan\theta \quad \Longrightarrow \quad r = \sqrt{x^2 + y^2}, \quad \theta = \operatorname{atan2}\{y, x\}, \quad h = z$$

assuming + ($r > 0$ only)

same angle, doesn't take into account the sign of x & y

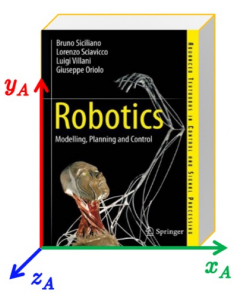
four-quadrant arc tangent

with a singularity for $x = y = 0$
z axis → θ undefined
MATLAB : atan2{0,0} → 0

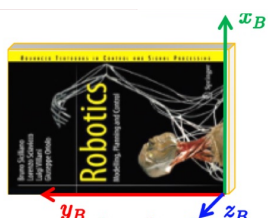
to take sign of x & y into account and know in which quadrant I am.

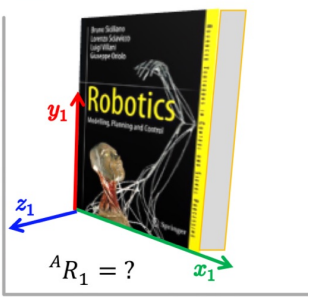
the sign of r has an impact on the inverse tranformation

Orientation



90° around z-axis





$$^{A}R_1 = ?$$

$$^{A}R_B = \begin{array}{c} \begin{array}{ccc} x_B & y_B & z_B \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} x_A \\ y_A \\ z_A \end{array}$$

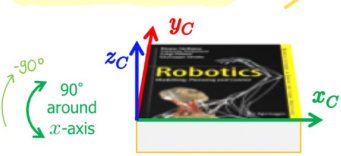
det = +1

$$^{B}R_A = \begin{array}{c} \begin{array}{ccc} x_A & y_A & z_A \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} = {}^{A}R_B^T$$

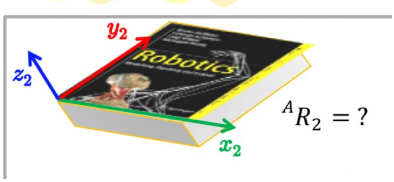
$$\boxed{^{A}R_A = {}^{A}R_B \, {}^{B}R_A = I}$$

$$^{B}R_C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = {}^{B}R_A \, {}^{A}R_C = {}^{A}R_B^T \, {}^{A}R_C$$

$$^{A}R_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$



-90°
90° around x-axis



$$^{A}R_2 = ?$$

frame A : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

^{A}frame B : $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

^{A}frame C : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

^{B}frame A : $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

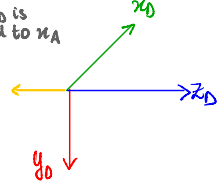
$= (^{A}\text{frame B})^T$

^{B}frame C : $\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

$= {}^{B}\text{frame A} \cdot {}^{A}\text{frame C} = (^{A}\text{frame B})^T \cdot {}^{A}\text{frame C}$

if ${}^{A}R_D = \begin{bmatrix} 0^{*} & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ $\Rightarrow$

*0 means x_D is orthogonal to x_A



NOT RIGHT HANDED! $\rightarrow$ Because $\det {}^{A}R_D = -1$!!

Rotation Matrix

${}^{A}R_B = [{}^{A}x_B \; {}^{A}y_B \; {}^{A}z_B]$: components : direction cosines of the axis x_B respect to x_A...

${}^{A}x_B$ (x_B projected on frame A)

scalar product

$$ {}^{A}R_B = \begin{bmatrix} \boldsymbol{x}_A{}^T\boldsymbol{x}_B & \boldsymbol{x}_A{}^T\boldsymbol{y}_B & \boldsymbol{x}_A{}^T\boldsymbol{z}_B \\ \boldsymbol{y}_A{}^T\boldsymbol{x}_B & \boldsymbol{y}_A{}^T\boldsymbol{y}_B & \boldsymbol{y}_A{}^T\boldsymbol{z}_B \\ \boldsymbol{z}_A{}^T\boldsymbol{x}_B & \boldsymbol{z}_A{}^T\boldsymbol{y}_B & \boldsymbol{z}_A{}^T\boldsymbol{z}_B \end{bmatrix} $$

direction cosine of $\boldsymbol{z}_B$ w.r.t. $\boldsymbol{x}_A$

$$ \boldsymbol{x}_A^T\boldsymbol{z}_B = \|\boldsymbol{x}_A\|\|\boldsymbol{z}_B\| \cos\beta = \cos\beta $$

orthonormal, with det = +1

*

rotation matrix · rotation matrix = rotation matrix!

orthonormal w. det +1!

it's the algebraic structure of a group $SO(3)$: 3x3 special orthonormal group, det +1

neutral element = I, $\rightarrow$ it has a neutral element: I (if I multiply a rotation matrix by I nothing changes)

inverse element = R^T

chain rule property

$$ {}^{k}R_i \, {}^{i}R_j = {}^{k}R_j \quad * $$

orientation of RF_i w.r.t. RF_k

orientation of RF_j w.r.t. RF_i

orientation of RF_j w.r.t. RF_k

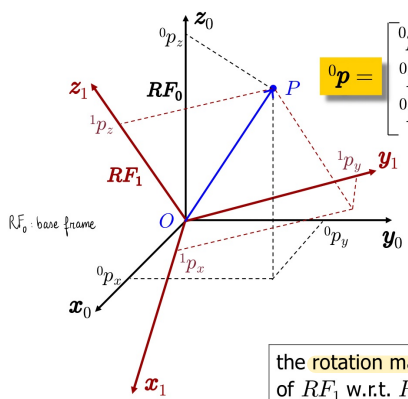
"moving axis"

this product is still going to give me a rot. matrix but I can't determine it that easily.

${}^{i}R_j \, {}^{k}R_i \neq {}^{k}R_j$ $\rightarrow$ order is important!

NOTE: in general, the **product** of rotation matrices does **not** commute!

Change of coordinates



RF_0: base frame

$$ {}^0\boldsymbol{p} = \begin{bmatrix} {}^0p_x \\ {}^0p_y \\ {}^0p_z \end{bmatrix} = {}^0p_x \, {}^0\boldsymbol{x}_0 + {}^0p_y \, {}^0\boldsymbol{y}_0 + {}^0p_z \, {}^0\boldsymbol{z}_0 $$

$(1 \;\; 0 \;\; 0)^T$ $(0 \;\; 1 \;\; 0)^T$ $(0 \;\; 0 \;\; 1)^T$

$$ = {}^1p_x \, {}^0\boldsymbol{x}_1 + {}^1p_y \, {}^0\boldsymbol{y}_1 + {}^1p_z \, {}^0\boldsymbol{z}_1 $$

$$ = \begin{bmatrix} {}^0\boldsymbol{x}_1 & {}^0\boldsymbol{y}_1 & {}^0\boldsymbol{z}_1 \end{bmatrix} \begin{bmatrix} {}^1p_x \\ {}^1p_y \\ {}^1p_z \end{bmatrix} $$

matrix · vector

$$ = {}^0R_1 \, {}^1\boldsymbol{p} $$

the rotation matrix ${}^0R_1$ (i.e., the orientation of RF_1 w.r.t. RF_0) represents also the change of coordinates of a vector from RF_1 to RF_0

given ${}^1\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

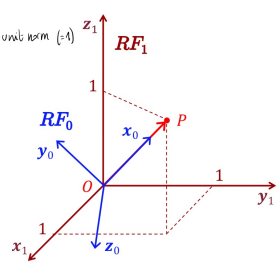
CHECK:
- columns/rows should have unit norm (=1)
- ortogonali
- det = +1

given ${}^0R_1 = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$

$$ {}^0\boldsymbol{p} = {}^0R_1 \, {}^1\boldsymbol{p} = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix} $$

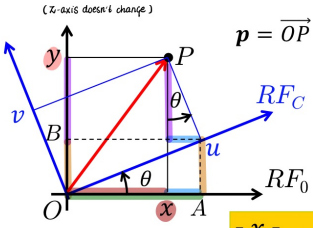
$$ \|\boldsymbol{p}\| = \|{}^0\boldsymbol{p}\| = \|{}^1\boldsymbol{p}\| = \sqrt{3} $$

... and where is RF_0 ?



- $\boldsymbol{x}_0$ is aligned with $\boldsymbol{p} = \overrightarrow{OP}$
- $\boldsymbol{z}_0$ is orthogonal to $\boldsymbol{y}_1$ ($\boldsymbol{z}_0^T \boldsymbol{y}_1 = 0$) and is positive on $\boldsymbol{x}_1$ ($\boldsymbol{z}_0^T \boldsymbol{x}_1 = 1/\sqrt{2}$)
- $\boldsymbol{y}_0$ completes a right-handed frame

Orientation of frames in a plane (elementary rotation around z-axis)



$\boldsymbol{p} = \overrightarrow{OP}$

$$x = OA - xA = u\cos\theta - v\sin\theta$$
$$y = OB + By = u\sin\theta + v\cos\theta$$
$$z = w$$

or...

$$\underbrace{{}^0\boldsymbol{p}}_{} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = R_z(\theta) \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

${}^0\boldsymbol{x}_C$ ${}^0\boldsymbol{y}_C$ ${}^0\boldsymbol{z}_C$ ${}^C\boldsymbol{p}$

$$R_z(-\theta) = R_z^T(\theta)$$

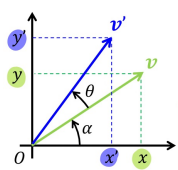
$\cos(-x) = \cos(x)$
$\sin(-x) = -\sin(x)$

similarly:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
(elementary rotation around x-axis)

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
(elementary rotation around y-axis)

Rotation of a vector around z



$$x = \|\boldsymbol{v}\|\cos\alpha$$
$$y = \|\boldsymbol{v}\|\sin\alpha$$

I rotate the vector

$$x' = \|\boldsymbol{v}\|\cos(\alpha+\theta) = \|\boldsymbol{v}\|(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$
$$= x\cos\theta - y\sin\theta$$
$$y' = \|\boldsymbol{v}\|\sin(\alpha+\theta) = \|\boldsymbol{v}\|(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$
$$= x\sin\theta + y\cos\theta$$
$$z' = z$$

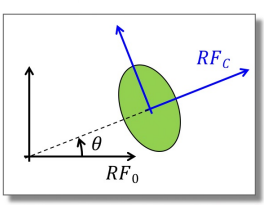
or...

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

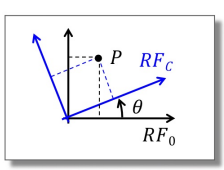
... same as before!

Different interpretations:

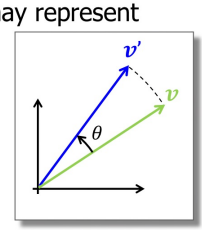
the same rotation matrix (e.g., $R_z(\theta)$) may represent



the orientation of a rigid body with respect to a reference frame RF_0
e.g., $[{}^0\boldsymbol{x}_C\ {}^0\boldsymbol{y}_C\ {}^0\boldsymbol{z}_C] = R_z(\theta)$



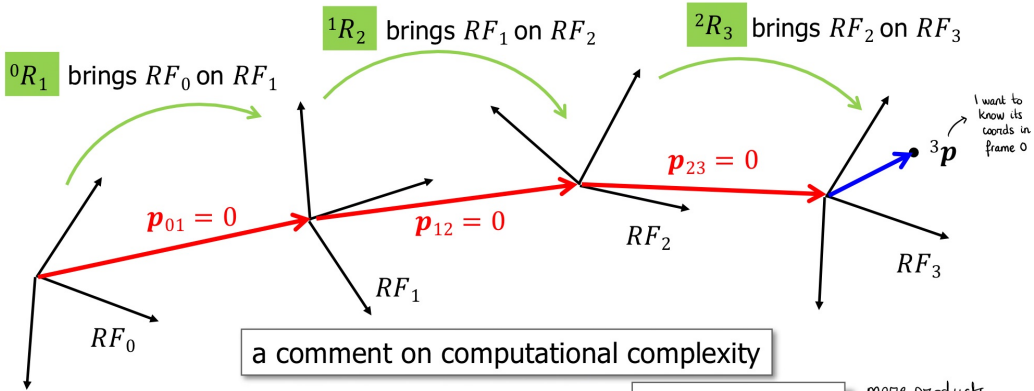
the change of coordinates from RF_C to RF_0
e.g., ${}^0\boldsymbol{p} = R_z(\theta)\ {}^C\boldsymbol{p}$



the rotation operator on vectors
e.g., $\boldsymbol{v}' = R_z(\theta)\,\boldsymbol{v}$

the rotation matrix ${}^0R_C$ is an operator superposing frame RF_0 to frame RF_C

Composition of rotations



a comment on computational complexity

$${}^0\boldsymbol{p} = ({}^0R_1\,{}^1R_2\,{}^2R_3)\,{}^3\boldsymbol{p} = {}^0R_3\,{}^3\boldsymbol{p}$$

63 products
42 summations

more products between matrices

compromise?

$${}^0\boldsymbol{p} = {}^0R_1\,({}^1R_2\,({}^2R_3\,{}^3\boldsymbol{p}))$$

27 products
18 summations

this one is better UNLESS the same operation should be applied not only to ${}^3p$ but to a multitude of vectors that I know in frame 3. In that case it's much better to compute the rotation matrix once & for all & apply it to the bunch of vectors.

${}^2\boldsymbol{p}$

${}^1\boldsymbol{p}$

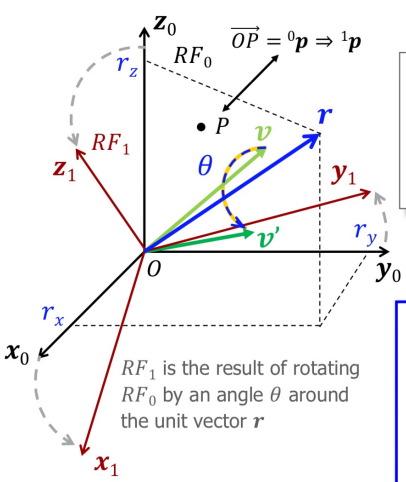
I express ${}^3p$ in frame 2 $\rightarrow {}^2p$ etc...

Lecture 28.10.22

Axis/angle (direct & inverse pb.)

Axis/angle representation

generalization of a rotation matrix around any unit vector



DATA

- axis $\boldsymbol{r}$ (unit vector in $\mathbb{R}^3$, $\|\boldsymbol{r}\| = 1$)
- angle θ, positive counterclockwise (as seen from an "observer" oriented like $\boldsymbol{r}$ with the head placed on the arrow, looking down to her/his feet)

DIRECT PROBLEM

parametrized by the given data!

find a rotation matrix $R(\theta, \boldsymbol{r})$

$$R(\theta, \boldsymbol{r}) = [{}^0\boldsymbol{x}_1\ {}^0\boldsymbol{y}_1\ {}^0\boldsymbol{z}_1]$$

such that

$${}^0\boldsymbol{p} = R(\theta, \boldsymbol{r})\,{}^1\boldsymbol{p} \qquad {}^0\boldsymbol{v}' = R(\theta, \boldsymbol{r})\,{}^0\boldsymbol{v}$$

: rotated around unit vector r

Inner & Outer products

<inner & outer products, dyadic expression, skew-sym matrix>

- (inner) row by column products between two 3×3 matrices

$$\underset{\text{inner product}}{C^TC} = \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

- dyadic expansion of a $n \times n$ matrix

$$\boldsymbol{e}_i = [0 \quad \dots \quad 1 \quad \dots \quad 0]^T, \qquad i = 1, \dots, n \qquad \Rightarrow \qquad A = \sum_{i,j=1}^{n} a_{ij} \boldsymbol{e}_i \boldsymbol{e}_j^T$$

- product of three $n \times n$ matrices using dyadic form

$$B = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 \dots & \boldsymbol{b}_{n-1} & \boldsymbol{b}_n \end{bmatrix} \qquad \Rightarrow \qquad B\,A\,B^T = \sum_{i,j=1}^{n} a_{ij} \boldsymbol{b}_i \boldsymbol{b}_j^T$$

- (outer) column by row products between two 3×3 matrices

$$\underset{\text{outer product}}{CC^T} = I \Rightarrow CC^T = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix}$$

$$= \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

Skew-symmetric matrices

- properties of a skew-symmetric matrix
  - a square matrix S is skew-symmetric iff $S^T = -S$
    $\Leftrightarrow s_{ij} = -s_{ji} \Rightarrow s_{ii} = 0$ (zeros on the diagonal) necessary but not sufficient
  - any square matrix A can be decomposed into its symmetric and skew-symmetric parts
    $A = \frac{A+A^T}{2} + \frac{A-A^T}{2} = A_{symm} + A_{skew}$
  - in quadratic forms the skew-symmetric part vanishes (only the symmetric part matters)
    $x^TAx = \frac{1}{2}[x^TAx + (x^TAx)^T] = \frac{1}{2}[x^TAx + x^TA^Tx] = x^T\frac{A+A^T}{2}x = x^TA_{symm}\,x$
  - canonical form of a 3×3 skew-symmetric matrix

also called vee map ∨
$v = S^\vee$

$$\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow S(\boldsymbol{v}) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix} \Rightarrow \boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

- expression of the vector product between two vectors $\in \mathbb{R}^3$

Sarrus rule for determinant of $\begin{bmatrix} n_x & n_y & n_z \\ s_x & s_y & s_z \\ \hat{\imath} & \hat{\jmath} & \hat{k} \end{bmatrix}$

$$\boldsymbol{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \boldsymbol{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \Rightarrow \boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \boldsymbol{n} \times \boldsymbol{s} = \begin{bmatrix} n_ys_z - s_yn_z \\ n_zs_x - s_zn_x \\ n_xs_y - s_xn_y \end{bmatrix} = S(\boldsymbol{n})\,\boldsymbol{s}$$

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = S(\boldsymbol{v}_1)\boldsymbol{v}_2 = -\boldsymbol{v}_2 \times \boldsymbol{v}_1 = -S(\boldsymbol{v}_2)\boldsymbol{v}_1 = S^T(\boldsymbol{v}_2)\boldsymbol{v}_1 \qquad S^T = -S$$

$$\boldsymbol{n} \times S = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix} \cdot \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix}$$

$$s \times n = S(s) \cdot n = S(-n) \cdot s$$

Notes: if the matrix is symmetric, all the eigen values are real!

if it is also positive definite, ($x^TAsymx \geq 0$, $0 \Leftrightarrow x=0$) the eigen values are real & >0

for a negative definite matrix → eigen values real & <0

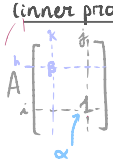
$$C^TC = \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} = \begin{bmatrix} \boldsymbol{n}^T\boldsymbol{n} & 0 & 0 \\ 0 & \boldsymbol{s}^T\boldsymbol{s} & 0 \\ 0 & 0 & \boldsymbol{r}^T\boldsymbol{r} \end{bmatrix} = I$$

(orthogonal)

(unit vectors)

row · column

(inner product)



$$B = A \cdot \begin{bmatrix} 0 \\ \boldsymbol{b}_j^T \\ 0 \end{bmatrix} \begin{matrix} \\ i \\ \\ \end{matrix} = \alpha \begin{bmatrix} \boldsymbol{a}_i \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_j^T \end{bmatrix}$$

column · row

outer product of 2 vectors (MATRIX)

$+\beta\, a_n b_k^T$

$$\rightarrow A \cdot B = A \cdot I \cdot B = \sum_{i=1}^{N} a_i b_i^T$$

= sum of dyads

$$A = A \cdot I = \sum_{i=1}^{N} a_i e_i^T \qquad e_i^T = [0 \dots 0\,1\,0 \dots 0]$$

i

= dyadic expansion of matrix A

$$\underset{\text{(outer product)}}{CC^T} = [\boldsymbol{n}\ \boldsymbol{s}\ \boldsymbol{r}] \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

: sum of 3 matrices, each of which is the outerproduct of one of the columns of the rotation matrix ⇒ identity

Prodotto riga per colonna

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

$$B = \begin{pmatrix} l & m \\ n & o \\ p & q \end{pmatrix}$$

$$AB = \begin{pmatrix} al + bn + cp & am + bo + cq \\ dl + en + fp & dm + eo + fq \end{pmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} l \\ n \\ p \end{bmatrix} = al + bn + cp \quad \text{scalare}$$

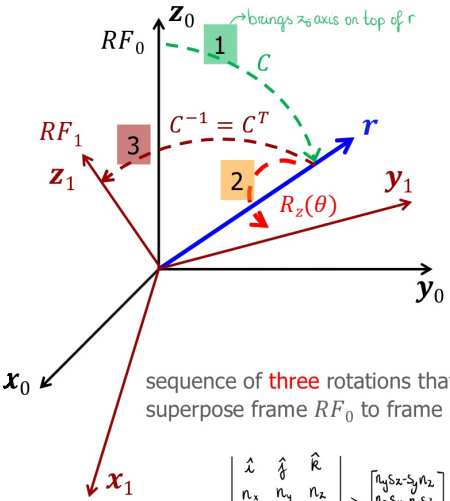
Prodotto esterno

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad [m \times 1] \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad [n \times 1]$$

$$u \otimes v = uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & & & \\ u_m v_1 & \dots & \dots & u_m v_n \end{bmatrix} \quad \text{matrice}$$

$$[m \times n]$$

Direct problem: → given the vector I want to find the rotation matrix



sequence of three rotations that superpose frame RF_0 to frame RF_1

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ n_x & n_y & n_z \\ s_x & s_y & s_z \end{vmatrix} \Rightarrow \begin{bmatrix} n_y s_z - s_y n_z \\ n_z s_x - n_x s_z \\ n_x s_y - n_y s_x \end{bmatrix} = r$$

$$\boxed{R(\theta, \boldsymbol{r}) = C\, R_z(\theta)\, C^T}$$

sequence of three rotations
(one of which is elementary)

around a mutable z-axis

$$C = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix}$$

data vector r

after the first rotation the z-axis coincides with $\boldsymbol{r}$

$\boldsymbol{n}$ and $\boldsymbol{s}$ are orthogonal unit vectors such that $\boldsymbol{n} \times \boldsymbol{s} = \boldsymbol{r}$

C moves the z_0 axis on top of the r vector. → The 3rd column expresses the directional cosines of the new z-axis (r) wrt the original one.

After I make this rotation I will have my z-axis on the original unit vector r. If I make a rotation of ϑ around this axis I will have an elementary rotation around the current z-axis: $R_z(\vartheta)$.

$$\begin{pmatrix} c\vartheta & -s\vartheta & \cdot \\ s\vartheta & c\vartheta & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

Now I have to express things in the original frame so I do the reverse operation → $C^{-1} = C^T$ (because C is a rotation matrix).

Axis/angle: Direct problem

solution

elementary rotation

$$\boxed{R(\theta, \boldsymbol{r}) = C\, R_z(\theta)\, C^T}$$

$$R(\theta, \boldsymbol{r}) = \begin{bmatrix} \boldsymbol{n} & \boldsymbol{s} & \boldsymbol{r} \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{s}^T \\ \boldsymbol{r}^T \end{bmatrix} \longleftrightarrow B\,A\,B^T = \sum_{i,j=1}^{n} a_{ij} \boldsymbol{b}_i \boldsymbol{b}_j^T$$

$$= \boldsymbol{r}\boldsymbol{r}^T + (\boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T)\, c\theta + (\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T)\, s\theta$$

in position 1,1 I have $c\vartheta$ that multiplies n & n^T
in position 2,2 $c\vartheta$ multiplies s & s^T
in position 3,3 1 multiplies r & r^T
in position 2,1 $s\vartheta$ multiplies s & n^T
in position 1,2 $-s\vartheta$ multiplies n & s^T

taking into account

$$CC^T = \boldsymbol{n}\boldsymbol{n}^T + \boldsymbol{s}\boldsymbol{s}^T + \boldsymbol{r}\boldsymbol{r}^T = I$$

$$\boldsymbol{s}\boldsymbol{n}^T - \boldsymbol{n}\boldsymbol{s}^T = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} = S(\boldsymbol{r})$$

depends only on $\boldsymbol{r}$ and θ !

$$\boxed{R(\theta, \boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)\, c\theta + S(\boldsymbol{r})\, s\theta}$$

I know r & ϑ but I still don't know s & n.

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \cdot (n_1 \;\; n_2 \;\; n_3) = \begin{pmatrix} \cancel{s_1 n_1} & s_1 n_2 & s_1 n_3 \\ s_2 n_1 & \cancel{s_2 n_2} & s_2 n_3 \\ s_3 n_1 & s_3 n_2 & \cancel{s_3 n_3} \end{pmatrix}$$

$$-$$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot (s_1 \;\; s_2 \;\; s_3) = \begin{pmatrix} \cancel{n_1 s_1} & n_1 s_2 & n_1 s_3 \\ n_2 s_1 & \cancel{n_2 s_2} & n_2 s_3 \\ n_3 s_1 & n_3 s_2 & \cancel{n_3 s_3} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & s_1 n_2 - n_1 s_2 & s_1 n_3 - n_1 s_3 \\ s_2 n_1 - n_2 s_1 & 0 & s_2 n_3 - n_2 s_3 \\ s_3 n_1 - n_3 s_1 & s_3 n_2 - n_3 s_2 & 0 \end{pmatrix} = S(r) = \begin{pmatrix} 0 & -r_z & r_y \\ & 0 & -r_x \\ \text{skew-sym} & & 0 \end{pmatrix}$$

$$r = n \times s = \begin{pmatrix} n_y s_z - s_y n_z \\ n_z s_x - n_x s_z \\ n_x s_y - n_y s_x \end{pmatrix} \begin{matrix} \leftarrow r_x \\ \leftarrow r_y \\ \leftarrow r_z \end{matrix}$$

developing computations...

$$R(\theta,\boldsymbol{r}) =$$

$$\begin{bmatrix} r_x^2(1-\cos\theta)+\cos\theta & r_xr_y(1-\cos\theta)-r_z\sin\theta & r_xr_z(1-\cos\theta)+r_y\sin\theta \\ r_xr_y(1-\cos\theta)+r_z\sin\theta & r_y^2(1-\cos\theta)+\cos\theta & r_yr_z(1-\cos\theta)-r_x\sin\theta \\ r_xr_z(1-\cos\theta)-r_y\sin\theta & r_yr_z(1-\cos\theta)+r_x\sin\theta & r_z^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

If I swap r w/ $-r$ & θ w/ $-\theta$

$rr^T \to - \cdot - = +$

$S(r) = \text{skewsym} = -S(r)$

but also θ has changed so $s\theta$ has the opposite sign. (while $c\theta = c(-\theta)$ even fnctn)

note that

$$\text{trace } R(\theta,\boldsymbol{r}) = \underbrace{(r_x^1+r_y^2+r_z^2)}_{1 \text{ (unit vector)}}(1-\cos\theta)+3\cos\theta = 1+2\cos\theta \to \text{indipendent on } r$$

$$R(\theta,\boldsymbol{r}) = R(-\theta,-\boldsymbol{r}) = R^T(-\theta,\boldsymbol{r})$$

rotation matrix : $R^{-1} = R^T$

$\hookrightarrow RR^{-1} = I$

$\begin{cases} \sin(-x) = -\sin x \\ \cos(-x) = \cos x \end{cases}$

→ This mapping between the axis-angle notation and the rotation matrices is not injective → 2 input data that goes into the same rotation ⇒ the inverse problem will give 2 solutions.

Example:

$$\boxed{R(\theta,\boldsymbol{r}) = \boldsymbol{r}\boldsymbol{r}^T + (I-\boldsymbol{r}\boldsymbol{r}^T)\,c\theta + S(\boldsymbol{r})\,s\theta}$$

$$\boldsymbol{r} = \begin{bmatrix}0\\0\\1\end{bmatrix} = \boldsymbol{z}_0$$

$$R(\theta,\boldsymbol{r}) = \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{rr^T} + \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{I-rr^T} c\theta + \underbrace{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}}_{S(r)} s\theta$$

r_z $-r_z$

$$= \begin{bmatrix}c\theta & -s\theta & 0\\ s\theta & c\theta & 0\\ 0 & 0 & 1\end{bmatrix} = R_z(\theta)$$

For $\hat{x}$ & $\hat{y}$:

$$r = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} \quad rr^T = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}\begin{bmatrix}1&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$(I-rr^T)c\theta = \begin{bmatrix}0&\cdot&\cdot\\\cdot&1&\cdot\\\cdot&\cdot&1\end{bmatrix}c\theta \quad S(r)\,s\theta = \begin{bmatrix}\cdot&\cdot&r_y\\\cdot&\cdot&-r_x\\-r_y&r_x&\cdot\end{bmatrix}s\theta$$

$$R = \begin{bmatrix}1&0&0\\0&c\theta&-s\theta\\0&s\theta&c\theta\end{bmatrix}$$

$c\theta$ $s\theta$ $-s\theta$ $c\theta$

Rodriguez Formula

$$\boldsymbol{v}' = R(\theta,\boldsymbol{r})\boldsymbol{v}$$

: coord. of a generic vector <u>after</u> the rot. of θ around r

$$\boxed{\boldsymbol{v}' = \boldsymbol{v}\cos\theta + (\boldsymbol{r}\times\boldsymbol{v})\sin\theta + (1-\cos\theta)(\boldsymbol{r}^T\boldsymbol{v})\,\boldsymbol{r}}$$

proof

$$\begin{aligned} R(\theta,\boldsymbol{r})\boldsymbol{v} &= (\boldsymbol{r}\boldsymbol{r}^T + (I-\boldsymbol{r}\boldsymbol{r}^T)\cos\theta + S(\boldsymbol{r})\sin\theta)\,\boldsymbol{v} \\ &= \boldsymbol{r}\boldsymbol{r}^T\boldsymbol{v}(1-\cos\theta) + \boldsymbol{v}\cos\theta + (\boldsymbol{r}\times\boldsymbol{v})\sin\theta \end{aligned}$$

Properties of $R(\theta,r)$

1. $R(\theta,\boldsymbol{r})\boldsymbol{r} = \boldsymbol{r}$ ($\boldsymbol{r}$ is the invariant axis in this rotation)
2. when $\boldsymbol{r}$ is one of the coordinate axes, R boils down to one of the known elementary rotation matrices
3. $(\theta,\boldsymbol{r}) \to R$ is not an injective map: $R(\theta,\boldsymbol{r}) = R(-\theta,-\boldsymbol{r})$
4. $\det(R) = +1 = \prod_i \lambda_i$ (eigenvalues)
5. $\text{tr}(R) = \text{tr}(\boldsymbol{r}\boldsymbol{r}^T) + (I-\boldsymbol{r}\,\boldsymbol{r}^T)c\theta = 1+2c\theta = \sum_i \lambda_i$

identities in green hold for any matrix

all the world rotates around r except r itself

⇒ r is an eigen vector $Ax = \lambda x$, $\lambda = 1$

(square)

for all matrices, the determinant is equal to the product of its eigenvalues.

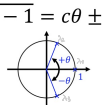
for all matrices, the trace is equal to the sum of the eigenvalues.

1. ⇒ $\lambda_1 = 1$ $\quad \lambda_1+\lambda_2+\lambda_3 = 1+2c\theta \quad \lambda_1\cdot\lambda_2\cdot\lambda_3 = 1 \quad \begin{cases}\lambda_2+\lambda_3 = 2c\theta\\ \lambda_2\cdot\lambda_3 = 1\end{cases}$

4. & 5. ⇒ $\lambda_2+\lambda_3 = 2\,c\theta$ ⇒ $\lambda^2 - 2\,c\theta\,\lambda + 1 = 0$

$$\Rightarrow \lambda_{2,3} = c\theta \pm \sqrt{c^2\theta - 1} = c\theta \pm i\,s\theta = e^{\pm i\theta}$$

all eigenvalues λ have unitary module ($\Leftarrow R$ orthonormal)



Inverse Problem

dataset : rotation matrices

9 numbers

GIVEN a rotation matrix $R = \{R_{ij}\}$,
FIND a unit vector $\boldsymbol{r}$ and an angle θ such that

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \quad \vartheta \in [\pi, \pi]$$

$$\boxed{R = \boldsymbol{r}\boldsymbol{r}^T + (I - \boldsymbol{r}\boldsymbol{r}^T)\cos\theta + S(\boldsymbol{r})\sin\theta = R(\theta, \boldsymbol{r})}$$

9 non linear eq. in 4 unknown

idk if there's a solution, and I don't have a direct method to apply to find it.

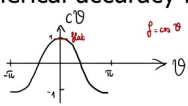
note first that $\mathrm{tr}(R) = R_{11} + R_{22} + R_{33} = 1 + 2\cos\theta$; so, one could solve

$$\boxed{\theta = \arccos \frac{R_{11} + R_{22} + R_{33} - 1}{2}}$$

HERE I CAN SOLVE IT IN CLOSED FORM, EXCEPT FOR SOME SINGULAR CASES.

but

- this formula provides only values in $[0, \pi]$ (thus, never negative angles θ)
  but I know the problem is not injective → I can generate the ∅ sol.
- loss of numerical accuracy for $\theta \to 0$ (sensitivity of $\cos\theta$ is low around 0)



$f = \cos\vartheta$

$\frac{df}{d\vartheta} = -\sin\vartheta$

↳ for $\theta \approx 0$ → $\cong 0$

& I'm using only 3 nbrs from the data

Axis/angle: Inverse problem

solution

$* \ (R - R^T)^T = R^T - R^{TT} = R^T - R = -(R - R^T)$ ✓ skewsym.

from the data

for $R(\theta, \boldsymbol{r})$:

$$\overbrace{R - R^T}^{* \text{ SKEWSYM}} = \begin{bmatrix} 0 & R_{12} - R_{21} & R_{13} - R_{31} \\ R_{21} - R_{12} & 0 & R_{23} - R_{32} \\ R_{31} - R_{13} & R_{32} - R_{23} & 0 \end{bmatrix} = 2\sin\theta \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

$$* \ 2\sin\vartheta \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{pmatrix}$$

it follows

since $\|\boldsymbol{r}\| = 1 \overset{*}{\Rightarrow} \sin\theta = \pm\frac{1}{2}\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}$ (*)

thus

$$\theta = \mathrm{atan2}\left\{\pm\overbrace{\sqrt{(R_{12} - R_{21})^2 + (R_{13} - R_{31})^2 + (R_{23} - R_{32})^2}}^{\sin}, \overbrace{R_{11} + R_{22} + R_{33} - 1}^{\cos}\right\} \quad (**)$$

see next slide

→ I get 2 solutions (±) (r, ϑ), $(-r, -\vartheta)$

$$\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \frac{1}{2\sin\theta} \begin{bmatrix} R_{32} - R_{23} \\ R_{13} - R_{31} \\ R_{21} - R_{12} \end{bmatrix}$$

can be used only if $\boxed{\sin\theta \neq 0}$

test is made on (*) using the data $\{R_{ij}\}$

atan2 function

- arctangent with output values "in the four quadrants"
  - two input arguments
  - takes values in $[-\pi, +\pi]$ it makes the ratio of the 1st argument over the 2nd one.
  - undefined only for $(0, 0)$
- uses the sign of both arguments to define the output quadrant
- based on arctan function with output values in $[-\pi/2, +\pi/2]$
- available in main languages (C++, Matlab, ...)

$(\sin\theta, \cos\vartheta)$

$$\mathrm{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & x > 0 \\ \pi + \arctan(\frac{y}{x}) & y \geq 0, x < 0 \\ -\pi + \arctan(\frac{y}{x}) & y < 0, x < 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ -\frac{\pi}{2} & y < 0, x = 0 \\ \text{undefined} & y = 0, x = 0 \end{cases}$$

full sensivity around 0

Singular cases (use when $\sin\theta = 0$)

no rotation! I can rotate by no angle around any axis → r undefined.

- if $\theta = 0$ from (**), there is no solution for $\boldsymbol{r}$ (rotation axis undefined)
- if $\theta = \pm\pi$ from (**), then set $\sin\theta = 0, \cos\theta = -1$ and solve

$$\Rightarrow \quad R = 2\boldsymbol{r}\boldsymbol{r}^T - I \qquad R(\theta, r) = rr^T - (I - rr^T)\overset{-1}{\cos\theta} + \cancel{S(r)\sin\theta}$$

$$\boxed{\boldsymbol{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \pm\sqrt{(R_{11}+1)/2} \\ \pm\sqrt{(R_{22}+1)/2} \\ \pm\sqrt{(R_{33}+1)/2} \end{bmatrix}}$$

(on the diagonal)

with $\boxed{\begin{aligned} r_x r_y &= R_{12}/2 \\ r_x r_z &= R_{13}/2 \\ r_y r_z &= R_{23}/2 \end{aligned}}$ (which I impose) $\Leftrightarrow$ used to resolve sign ambiguities $\Rightarrow$ two solutions of opposite sign

homework: write a code that determines the two solutions $(\theta, \boldsymbol{r})$

for $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$R = 2rr^T - I$

$\rightarrow R + I = 2rr^T$

$$rr^T = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}\begin{bmatrix} r_x & r_y & r_z \end{bmatrix} = \begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ r_y r_x & r_y^2 & r_y r_z \\ r_z r_x & r_z r_y & r_z^2 \end{bmatrix}$$

$$\begin{bmatrix} R_{11}+1 & R_{12} & R_{13} \\ R_{21} & R_{22}+1 & R_{23} \\ R_{31} & R_{32} & R_{33}+1 \end{bmatrix} = 2\begin{bmatrix} r_x^2 & r_x r_y & r_x r_z \\ r_y r_x & r_y^2 & r_y r_z \\ r_z r_x & r_z r_y & r_z^2 \end{bmatrix}$$

$$r_x^2 = \frac{R_{11}+1}{2}; \quad r_y^2 = \frac{R_{22}+1}{2}; \quad r_z^2 = \frac{R_{33}+1}{2}$$

$$\begin{cases} r_x = \pm\sqrt{(R_{11}+1)/2}, \ \cdots \\ r_x r_y = R_{12}/2; \quad r_x r_z = R_{13}/2; \quad r_y r_z = R_{23}/2 \end{cases}$$

→ I will find 2 solutions: rotation by π around r or by $-\pi$ around $-r$.

- check:
  - orthogonality: column 1 orth. to columns 2 & 3, column 2 orth. to column 3.
  - unitary vectors
  - det = 1
- off-diagonal terms subtracted are $\emptyset$ → $\sin\vartheta = \frac{1}{2} \pm \sqrt{(R_{12}-R_{21})^2 + (R_{13}-R_{31})^2 + (R_{23}-R_{32})^2} = \emptyset$! singular case

$$\cos\vartheta = \frac{R_{11}+R_{22}+R_{33}-1}{2} = -1$$

- $\left.\begin{cases} r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} 0 \\ \pm\sqrt{(-1/\sqrt{2}+1)/2} \\ \pm\sqrt{(1/\sqrt{2}+1)/2} \end{pmatrix} \\ r_y r_z = \left(-\frac{1}{\sqrt{2}}\right)/2 \end{cases}\right\}$ → 2 pairs: 2 solutions

r_y e r_z of opposite sign

Unit quaternion

set of 4 elements: 1 scalar & 1 3D vector

- to eliminate non-uniqueness and singular cases of the axis/angle $(\theta, \boldsymbol{r})$ representation, the unit quaternion can be used

$$Q = \{\eta, \boldsymbol{\epsilon}\} = \{\cos(\theta/2), \sin(\theta/2)\,\boldsymbol{r}\}$$

a scalar — 3-dim vector

- $\eta^2 + \|\epsilon\|^2 = 1$ (thus, "unit ...")
- $(\theta, \boldsymbol{r})$ and $(-\theta, -\boldsymbol{r})$ are associated to the same quaternion Q
- the rotation matrix R associated to a given quaternion Q is

$$R(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}$$

- no rotation is $Q = \{1, \mathbf{0}\}$, while the inverse rotation is $Q = \{\eta, -\boldsymbol{\epsilon}\}$
- unit quaternions are composed with special rules

$$Q_1 * Q_2 = \{\eta_1\eta_2 - \boldsymbol{\epsilon}_1^T\boldsymbol{\epsilon}_2, \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2\}$$