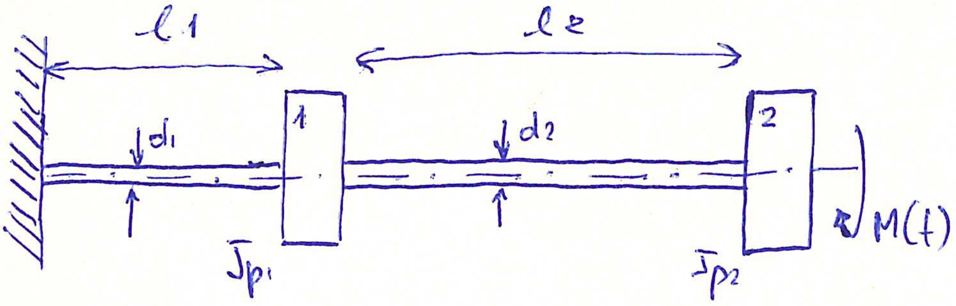
Ex. 1



$G = 7.7 \cdot 10^{10}$ N/m²

$l_1 = 0.5$ m

$l_2 = 1$ m

$d_1 = 12.8$ mm

$d_2 = 14.1$ mm

$J_{p1} = 6$ kg m²

$J_{p2} = 4$ kg m²

- Undamped system
- $M(t) = M_0 \operatorname{sen}(\lambda t) = 100 \operatorname{sen}(8t)$
- Bars with negligible inertia

a) COMPUTATION OF THE NATURAL FREQUENCIES AND EIGENVECTORS

- Computation of the torsional stiffness of the bars

Definition: $M_t = k_p \vartheta = \dfrac{G J_p}{l} \vartheta \Rightarrow k_p = \dfrac{G J_p}{l}$

$$J_{p1} = \frac{\pi d_1^4}{32} = 2.6 \cdot 10^{-9} \text{ m}^4$$

$$J_{p2} = \frac{\pi d_2^4}{32} = 3.9 \cdot 10^{-9} \text{ m}^4$$

$$k_{p1} = \frac{G J_{p1}}{l_1} = \frac{7.7 \cdot 10^{10} \cdot 2.6 \cdot 10^{-9}}{0.5} = 400.4 \ \frac{\text{Nm}}{\text{rad}}$$

$$k_{p2} = \frac{G J_{p2}}{l_2} = \frac{7.7 \cdot 10^{10} \cdot 3.9 \cdot 10^{-9}}{1} = 300.3 \ \frac{\text{Nm}}{\text{rad}}$$

- Equations of motion with Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right] - \frac{\partial \mathcal{L}}{\partial x_i} + \underbrace{\frac{\partial F}{\partial \dot{x}_i}}_{=0} = \frac{\partial \delta L}{\partial \delta x_i}$$

because the system is undamped

$$\mathcal{L} = T - U$$

Kinetic Energy T

$$T = \frac{1}{2} J_{p1} \dot{\theta}_1^2 + \frac{1}{2} J_{p2} \dot{\theta}_2^2$$

Potential Energy U

$$U = \frac{1}{2} k_{p1} \theta_1^2 + \frac{1}{2} k_{p2} (\theta_2 - \theta_1)^2 =$$

$$= \frac{1}{2} k_{p1} \theta_1^2 + \frac{1}{2} k_{p2} (\theta_2^2 + \theta_1^2 - 2\theta_1\theta_2) =$$

$$= \frac{1}{2} k_{p1} \theta_1^2 + \frac{1}{2} k_{p2} \theta_2^2 + \frac{1}{2} k_{p2} \theta_1^2 - k_{p2} \theta_1 \theta_2$$

External forces δL

$$\delta L = M(t)\, \delta\theta_2$$

(1) $$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}_1}\right] = J_{p1} \ddot{\theta}_1$$

$$\frac{\partial U}{\partial \theta_1} = k_{p1} \theta_1 + k_{p2} \theta_1 - k_{p2} \theta_2 \qquad \delta L = 0$$

(2) $$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}_2}\right] = J_{p2} \ddot{\theta}_2$$

$$\frac{\partial U}{\partial \theta_2} = k_{p2} \theta_2 - k_{p2} \theta_1$$

$$\frac{\partial \delta L}{\partial \delta\theta_2} = M(t)$$

Equations of motion

$$\begin{cases} J_{p1} \ddot{\theta}_1 + (k_{p1} + k_{p2}) \theta_1 - k_{p2} \theta_2 = 0 \\ J_{p2} \ddot{\theta}_2 + k_{p2} \theta_2 - k_{p2} \theta_1 = M(t) \end{cases}$$

Matrix form

$$\underbrace{\begin{bmatrix} J_{p1} & 0 \\ 0 & J_{p2}\end{bmatrix}}_{[M]}\begin{Bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{Bmatrix} + \underbrace{\begin{bmatrix} k_{p1}+k_{p2} & -k_{p2}\\ -k_{p2} & k_{p2}\end{bmatrix}}_{[K]}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ M(t)\end{Bmatrix}$$

$$[M]\{\ddot x\} + [K]\{x\} = \{0\}$$ Associated homogeneous problem

$$\{x\} = \{x_0\} e^{j\lambda t}$$

$$\{\dot x\} = \{x_0\} j\lambda e^{j\omega t} \qquad \Rightarrow \left(-\lambda^2[M] + [K]\right)\{x_0\} = \{0\}$$

$$\{\ddot x\} = -\{x_0\}\lambda^2 e^{j\omega t}$$

- Computation of the natural frequencies - WAY 1

$$\det\left[-\lambda^2[M] + [K]\right] = 0$$

$$-\lambda^2[M] + [K] = \begin{bmatrix} -\lambda^2 J_{p1} & 0 \\ 0 & -\lambda^2 J_{p2}\end{bmatrix} + \begin{bmatrix} k_{p1}+k_{p2} & -k_{p2}\\ -k_{p2} & k_{p2}\end{bmatrix} =$$

$$= \begin{bmatrix} -\lambda^2 J_{p1} + k_{p1} + k_{p2} & -k_{p2} \\ -k_{p2} & -\lambda^2 J_{p2} + k_{p2}\end{bmatrix}$$

$$\det\left[-\lambda^2[M] + [K]\right] = 0$$

$$(-\lambda^2 J_{p1} + k_{p1} + k_{p2})(-\lambda^2 J_{p2} + k_{p2}) - k_{p2}^2 = 0$$

$$\lambda^4 J_{p1}J_{p2} - \lambda^2 J_{p1}k_{p2} - \lambda^2 J_{p2}k_{p1} + k_{p1}k_{p2} - \lambda^2 J_{p2}k_{p2} + \cancel{k_{p2}^2} - \cancel{k_{p2}^2} = 0$$

$$J_{p1}J_{p2}\lambda^4 - (J_{p1}k_{p2} + J_{p2}k_{p1} + J_{p2}k_{p1})\lambda^2 + k_{p1}k_{p2} = 0$$

$$24\lambda^4 - 46.02\cdot 10^2\lambda^2 + 1.2\cdot 10^5 = 0$$

$$\lambda_{1,2}^2 = \frac{4602 \pm \sqrt{4602^2 - 4\cdot \overset{24}{\cancel{\text{[illegible]}}}\cdot 1.2\cdot 10^5}}{48} = \frac{4602 \pm \sqrt{9658404}}{48} =$$

$\lambda_1^2 = 160.68 \left(\frac{rad}{s}\right)^2 \Rightarrow \lambda_1 = 12.7 \frac{rad}{s}$

$\lambda_2^2 = 31.18 \left(\frac{rad}{s}\right)^2 \Rightarrow \lambda_2 = 5.6 \frac{rad}{s}$

Computation of the natural frequencies - way 2

The relevant eigenproblem

$\det\left(-\lambda^2[M] + [K]\right) = 0$

is reduced in the standard form:

1) $\det\left(K^{-1}M - \frac{1}{\lambda^2} I\right) = 0$

2) $\det\left(M^{-1}K - \lambda^2 I\right) = 0$

The first standard form is selected because the matrix M is diagonal

$$M = \begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix} \quad K = \begin{bmatrix} 700.7 & -300.3 \\ -300.3 & 300.3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 0.1667 & 0 \\ 0 & 0.25 \end{bmatrix} \quad D = M^{-1}K = \begin{bmatrix} 116.78 & -50.05 \\ -75.075 & 75.075 \end{bmatrix}$$

$$\det[D - \lambda^2 I] = 0 \Rightarrow \det\begin{bmatrix} 116.78 - \lambda^2 & -50.05 \\ -75.075 & 75.075 - \lambda^2 \end{bmatrix} = 0$$

$(116.78 - \lambda^2)(75.075 - \lambda^2) - (3757.5) = 0$

$\lambda^4 - 191.855\lambda^2 + 8767.3 - 3757.5 = 0$

$\lambda^4 - 191.855\lambda^2 + 5009.8 = 0$

$$\lambda_{1,2}^2 = \frac{191.855 \pm \sqrt{191.855^2 - 4 \cdot 5009.8}}{2} = \frac{191.855 \pm 129.49}{2}$$

$\lambda_1^2 = 160.72 \left(rad/s\right)^2 \Rightarrow \lambda_1 = 12.68\ rad/s$

$\lambda_2^2 = 31.23 \left(rad/s\right)^2 \Rightarrow \lambda_2 = 5.59\ rad/s$

(1.3)

$$\lambda_{1/2}^2 = \frac{191.855 \pm 147.577}{2}$$

$$\lambda_1^2 = 169.716 \Rightarrow \lambda_1 = 13.027 \text{ rad/s}$$

$$\lambda_2^2 = 22.139 \Rightarrow \lambda_2 = 4.705 \text{ rad/s}$$

- Computation of the eigenvectors

$$\left(-\lambda_n^2 \begin{bmatrix} J_{p1} & 0 \\ 0 & J_{p2} \end{bmatrix} + \begin{bmatrix} K_{p1}+K_{p2} & -K_{p2} \\ -K_{p2} & K_{p2} \end{bmatrix}\right) \begin{Bmatrix} \theta_{\phi 1} \\ \theta_{\phi 2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\theta_{\phi 2} = 1$$

$$-\lambda_n^2 J_{p2} - K_{p1}\theta_{\phi 1} + K_2 = 0$$

I) $-4.71^2 \cdot 4 - 300.3\,\theta_{\phi 1} + 300.3 = 0$

$$\theta_{\phi 1} = 0.58$$

II) $-12.68^2 \cdot 4 - 300.3\,\theta_{\phi 1} + 300.3 = 0$

$$\theta_{\phi 1} = 0.88$$

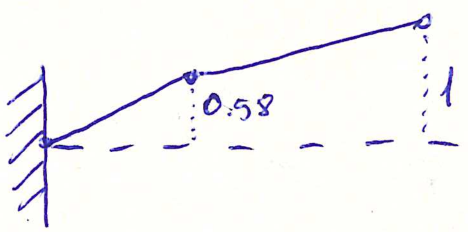
$$\{\theta_1\} = \begin{Bmatrix} 0.58 \\ 1 \end{Bmatrix}$$

$$\{\theta_2\} = \begin{Bmatrix} -0.88 \\ 1 \end{Bmatrix}$$

The eigenvectors represent the mode shapes at the natural frequencies

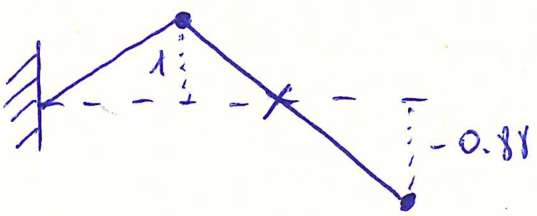
They have a physical interpretation

1) $\lambda_n = \lambda_{n1} = 5.6$ rad/s



When 1 moves of 0.58,
2 moves of 1.

2) $\lambda_n = \lambda_{n2} = 12.7$ rad/s



When 1 moves of 1
2 moves of -0.88

If we provide an impact to the structure, it will move with a combination of the two modes

In the 2° mode, there's a node that is not subject to any vibration. If we provide a torque on that node with a frequency equal to λ_2, we will not record any vibration, any energy is transferred.

· Computation of the response when $M(t) = M_0 \sin(\lambda t) = 100\sin(8t)$ is applied

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_{HOM} + \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_{PART.}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_{PART} = \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin \lambda t$$

$$\begin{Bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{Bmatrix}_{PART} = \lambda \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \cos \lambda t$$

$$\begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{Bmatrix}_{PART} = -\lambda^2 \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin \lambda t$$

By replacing in the eq. of motion (non homogeneous)

$$\underbrace{\left(-\lambda^2 \begin{bmatrix} J_{p1} & 0 \\ 0 & J_{p2} \end{bmatrix} + \begin{bmatrix} k_{p1}+k_{p2} & -k_{p2} \\ -k_{p2} & k_{p2} \end{bmatrix}\right)}_{K_{dyn}} \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin\lambda t = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix} \sin\lambda t$$

λ is the frequency of excitation ($\lambda = 8$ rad/s)

$$-\begin{bmatrix} 384 & 0 \\ 0 & 256 \end{bmatrix} + \begin{bmatrix} 700.7 & -300.3 \\ -300.3 & 300.3 \end{bmatrix} = \begin{bmatrix} 316.7 & -300.3 \\ -300.3 & 44.3 \end{bmatrix}$$

$$\begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{bmatrix} 316.7 & -300.3 \\ -300.3 & 44.3 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 100 \end{Bmatrix} = \begin{bmatrix} -0.394 \\ -0.416 \end{bmatrix}$$

* Computation of the maximum shear stress in the torsional bars using the configuration space and the state space approach

$$\tau_I = \frac{M_T}{J_{pI}} \frac{d_I}{2} = \frac{M_T}{\frac{\pi d_I^4}{32}} \frac{d_I}{2} = \frac{16\, M_T}{\pi d_I^3}$$

$$M_{T_1} = k_{p1}(\underbrace{\theta_{01} - \theta_{02}}_{0.394}) = k_{p1}\,\theta_{01} \qquad M_{T2} = k_{p2}(\underbrace{\theta_{02}}_{0.416} - \underbrace{\theta_{01}}_{0.394})$$

$$\tau_1 = \frac{k_{p1}\,\theta_{01} \cdot 16}{\pi d_1^3} \simeq \underline{390}\ \text{MPa}$$

$$\tau_2 = \frac{k_{p2}(\theta_{02} - \theta_{01}) \cdot 16}{\pi d_2^3} = 16\ \text{MPa}$$

Ex. 2

1. dof. $K = 10\ kN/m$, $m = 4\ kg$, $\delta = 0.3$

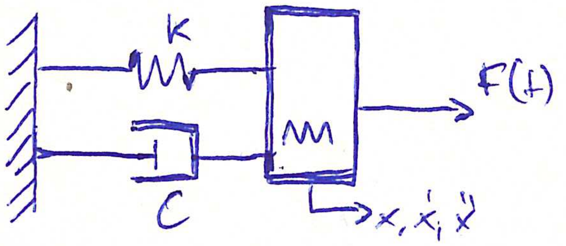
harmonic excitation $F(t) = F_0 \sin(\lambda t)$

$F_0 = 20N$

$\lambda_1 = 0.9\ \lambda_n$ $\qquad \lambda_n = \sqrt{\frac{K}{m}} = 50\ \frac{rad}{s}$

$\lambda_2 = 2.5\ \lambda_n$.

a) Viscous damping model



$m\ddot{x} + c\dot{x} + Kx = F(t)$

$m\ddot{x} + c\dot{x} + Kx = F_0 \sin(\lambda t)$

$x(t) = x_0 e^{j\lambda t}$

$\dot{x}(t) = j\lambda x_0 e^{j\lambda t}$

$\ddot{x}(t) = -\lambda^2 x_0 e^{j\lambda t}$

$-m\lambda^2 x_0 e^{j\lambda t} + j\lambda c x_0 e^{j\lambda t} + K x_0 e^{j\lambda t} = F_0 e^{j\lambda t}$

$-m\lambda^2 x_0 + j\lambda c x_0 + K x_0 = F_0$

$$x_0 = \frac{F_0}{-m\lambda^2 + j\lambda c + K} =$$

$$= \frac{F_0/K}{-\frac{m}{K}\lambda^2 + j\lambda\frac{c}{K} + 1}$$

$$\frac{m}{k} = \frac{1}{\lambda_n^2}$$

$$\frac{c\lambda}{k} = \lambda \frac{c}{\sqrt{k}\sqrt{k}} \frac{\sqrt{m}}{\sqrt{m}} \cdot \frac{2}{2} = \lambda \frac{c}{2\sqrt{km}} \frac{\sqrt{m}}{\sqrt{k}} = 2\lambda \frac{c}{C_{cr}} \frac{1}{\lambda_n} =$$

$$= 2\xi \frac{\lambda}{\lambda_n}$$

$$\Rightarrow x_0 = \frac{F_0/k}{-\left(\frac{\lambda}{\lambda_n}\right)^2 + 2j\xi\frac{\lambda}{\lambda_n} + 1} \qquad \frac{F_0}{k} = x_{st}$$

$$x_0 = \frac{x_{st}}{1 - \left(\frac{\lambda}{\lambda_n}\right)^2 + 2j\xi\frac{\lambda}{\lambda_n}} =$$

$$= x_{st} \cdot \frac{1 - \left(\frac{\lambda}{\lambda_n}\right)^2 - 2j\xi\frac{\lambda}{\lambda_n}}{\left[1 - \left(\frac{\lambda}{\lambda_n}\right)^2\right]^2 + \left[2\xi\frac{\lambda}{\lambda_n}\right]^2} \qquad (a+jb)(a-jb) = a^2+b^2$$

$$|x_0| = x_{st} \cdot \frac{\sqrt{\left(1-\left(\frac{\lambda}{\lambda_n}\right)^2\right)^2 + \left(2\xi\frac{\lambda}{\lambda_n}\right)^2}}{\left(\left(1-\frac{\lambda}{\lambda_n}\right)^2\right)^2 + \left(2\xi\frac{\lambda}{\lambda_n}\right)^2} =$$

$$= \frac{x_{st}}{\sqrt{\left(1-\left(\frac{\lambda}{\lambda_n}\right)^2\right)^2 + \left(2\xi\frac{\lambda}{\lambda_n}\right)^2}}$$

$$tg(\varphi) = -\frac{2\xi\frac{\lambda}{\lambda_n}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2}$$

$$\delta = \ln\left(\frac{x_i}{x_{i+1}}\right) = 2\pi\frac{\xi}{\sqrt{1-\xi^2}} \simeq 2\pi\xi$$

$$\xi = \frac{\delta}{2\pi} = \frac{0.3}{2\pi} = 0.0477$$

$$\lambda_1 = 0.9 \cdot 50 = 45 \frac{rad}{s}$$

$$|X_0| = \frac{x_{st}}{\sqrt{\left(1-\left(\frac{\lambda}{\lambda_n}\right)^2\right)^2 + \left(2\xi\frac{\lambda}{\lambda_n}\right)^2}} = 9.55 \cdot 10^{-3}\ m$$

$$\varphi = \operatorname{atan}\left(-\frac{2\xi\frac{\lambda}{\lambda_n}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2}\right)\frac{180}{\pi} = -24.3^\circ$$

$$H = \frac{x_{st}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2 + 2\xi j\frac{\lambda}{\lambda_n}}$$

$$Re(H) = 8.7 \cdot 10^{-3}\ m$$

$$Im(H) = 4 \cdot 10^{-3}\ m$$

$$\lambda_2 = 2.5 \cdot 50 = 125 \frac{rad}{s}$$

$$|X_0| = \frac{x_{st}}{\sqrt{\left(1-\left(\frac{\lambda}{\lambda_n}\right)^2\right)^2 + \left(2\xi\frac{\lambda}{\lambda_n}\right)^2}} = 3.8 \cdot 10^{-4}\ m$$

$$\varphi = \operatorname{atan}\left(-\frac{2\xi\frac{\lambda}{\lambda_n}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2}\right)\frac{180}{\pi} = -177.4^\circ$$

$$H = \frac{x_{st}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2 + 2\xi j\frac{\lambda}{\lambda_n}}$$

$$Re(H) = 3.8 \cdot 10^{-4}\ m$$

$$Im(H) = -1.73 \cdot 10^{-5}\ m$$

5) Hysteretic damping model

Vedi script Matlab

$$\left|\frac{x_0}{x_{st}}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\lambda}{\lambda_n}\right)^2\right]^2 + 2\xi\frac{\lambda}{\lambda_n}}}$$

$$x_0 = \frac{F_0}{k - m\lambda^2 + j\lambda c}$$

$$x_0 = \frac{F_0}{k - m\lambda^2 + j\eta\lambda c}$$

$$x_0 = \frac{F_0/k}{\frac{k-m\lambda^2}{k} + j\frac{\lambda c}{k}}$$

$$x_0 = \frac{F_0/k}{\frac{k-m\lambda^2}{k} + j\frac{\alpha}{k\pi}} \rightarrow \eta$$

$$= \frac{x_{st}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2 + j\eta}$$

$\eta = \frac{\alpha}{k\pi}$

hysteretic damping coefficient

$\eta = \delta/\pi$

$c = \frac{\alpha}{\pi\lambda}$

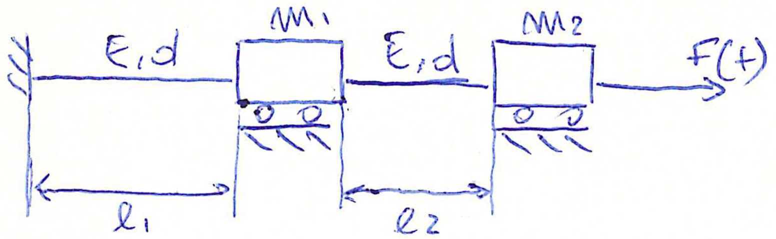
$$\eta = \frac{\delta}{\pi} = \frac{0.3}{\pi} = 0.0955$$

$$\varphi = \mathrm{atan}\left(-k\eta/(k-m\lambda^2)\right)/pi * 180$$

$$|\cdot| = \left|\frac{x_{st}}{1-\left(\frac{\lambda}{\lambda_n}\right)^2 + j\eta}\right|$$

UTORIAL 1

Ex. 3



$l_1 = 0.4$ m

$l_2 = 0.2$ m

$d = 0.02$ m

$E = 2.06 \cdot 10^{11}$ Pa

$m_1 = 50$ Kg

$m_2 = 20$ Kg

$\omega = 1700$ rad/s

$F_0 = 2000$ N

$F(t) = F_0 \sin(\omega t)$

Equivalent model

$$k_1 = \frac{E \cdot A}{l_1} = E \cdot \frac{\pi (d/2)^2}{l_1} = 162\, e^6 \text{ N/m}$$

$$k_2 = \frac{E \cdot A}{l_2} = E \cdot \frac{\pi (d/2)^2}{l_2} = 324\, e^6 \text{ N/m}$$

Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right] - \frac{\partial \mathcal{L}}{\partial x_i} + \overbrace{\frac{\partial F}{\partial \dot{x}_i}}^{=0 \,:\, \text{Undamped system}} = \frac{\partial \delta L}{\partial \delta x_i} \qquad \mathcal{L} = T - U$$

(T) $\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

(U) $\frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_2 x_1^2 - x_1 x_2 k_2$

(δL) F_2

(1) $\frac{d}{dt} \frac{\partial T}{\partial \dot{x}_1} = m_1 \ddot{x}_1$

$\frac{\partial U}{\partial x_1} = k_1 x_1 + k_2 x_1 - x_2 k_2 \qquad \partial L = 0$

(2) $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = m_2 \ddot{x}_1$

$\frac{\partial U}{\partial x_1} = k_2 x_2 - k_2 x_1$

$\frac{\partial \Omega}{\partial \delta x_1} = F_2$

Equations of motion

1) $m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$

2) $m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = F$

Matrix form

$M \ddot{x} + Kx = F$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 50 & 0 \\ 0 & 20 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4.8538 & -3.2358 \\ -3.2358 & 3.2358 \end{bmatrix} * 10^8 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Modal Analysis

$$D = M^{-1} K = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.05 \end{bmatrix} \begin{bmatrix} 4.8538 & -3.2358 \\ -3.2358 & 3.2358 \end{bmatrix} = \begin{bmatrix} 0.9708 & -0.6472 \\ -1.6179 & 1.6179 \end{bmatrix} 10$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}_{PART} = \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin \lambda t$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}_{PART} = \lambda \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin \lambda t$$

$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}_{PART} = -\lambda^2 \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin \lambda t$$

3.2

$$\left(-\lambda^2 [M] + [k]\right) \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \cancel{e^{i\lambda t}} = \begin{Bmatrix} 0 \\ F_0 \end{Bmatrix} \cancel{e^{i\lambda t}}$$

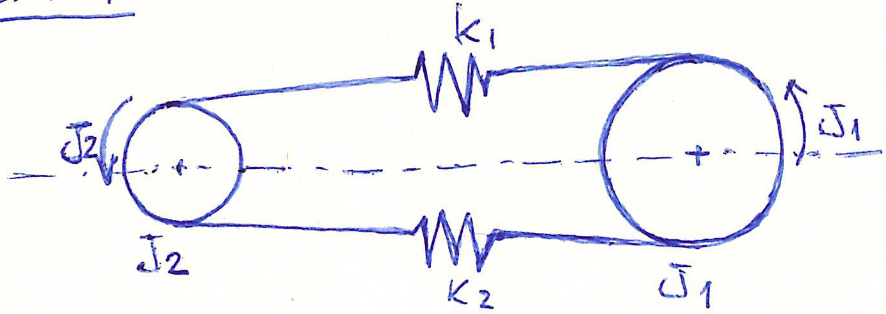
$\lambda = 1700$ rad/s

$$\begin{bmatrix} 3.4088 & -3.2358 \\ -3.2358 & 2.6578 \end{bmatrix} 10^8 \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2000 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{matrix} -4.587\ e\text{-}5 \text{ m} \\ -4.833\ e\text{-}5 \text{ m} \end{matrix}$$

TUTORIAL 1

Ex. 4



$J_1 = 0.2 \ \mathrm{Kgm^2}$

$J_2 = 0.15 \ \mathrm{Kgm^2}$

$R_1 = 0.07 \ \mathrm{m}$

$R_2 = 0.04 \ \mathrm{m}$

$L = 0.5 \ \mathrm{m}$

$E = 2.02 \cdot 10^9 \ \mathrm{N/m^2}$

$A = 4.26 \cdot 10^{-5} \ \mathrm{m^2}$

Equations of motion

1) $J_1 \ddot{\vartheta}_1 = -k_1 R_1 (\vartheta_1 R_1 - \vartheta_2 R_2) - k_2 R_1 (\vartheta_1 R_1 - \vartheta_2 R_2)$

$J_1 \ddot{\vartheta}_1 = -k_1 R_1^2 \vartheta_1 + k_1 R_1 R_2 \vartheta_2 - k_2 R_1^2 \vartheta_1 + k_2 R_1 R_2 \vartheta_2$

$J_1 \ddot{\vartheta}_1 = -(k_1 + k_2) R_1^2 \vartheta_1 + (k_1 + k_2) R_1 R_2 \vartheta_2$

2) $J_2 \ddot{\vartheta}_2 = -k_1 R_2 (\vartheta_2 R_2 - \vartheta_1 R_1) - k_2 R_1 (\vartheta_2 R_2 - \vartheta_1 R_2)$

$J_2 \ddot{\vartheta}_2 = -k_1 \vartheta_2 R_2^2 + k_1 \vartheta_1 R_1 R_2 - k_2 R_2^2 \vartheta_2 + k_2 R_1 R_2 \vartheta_1$

$J_2 \ddot{\vartheta}_2 = (k_1 + k_2) R_1 R_2 \vartheta_1 - (k_1 + k_2) R_2^2 \vartheta_2$

Lagrange (proposta per studenti)

Matrix form

$$M = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \qquad K = \begin{bmatrix} (k_1 + k_2) R_1^2 & -(k_1 + k_2) R_1 R_2 \\ -(k_1 + k_2) R_1 R_2 & (k_1 + k_2) R_2^2 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.15 \end{bmatrix} \qquad K = \begin{bmatrix} 1686.6 & -963.8 \\ -963.8 & 550.7 \end{bmatrix}$$

$$D = M^{-1} K = \begin{bmatrix} 5 & 0 \\ 0 & 6.67 \end{bmatrix} \begin{bmatrix} 1686.6 & -963.8 \\ -963.8 & 550.7 \end{bmatrix} = \begin{bmatrix} 8433.1 & -4818.9 \\ -6425.2 & 3671.6 \end{bmatrix}$$

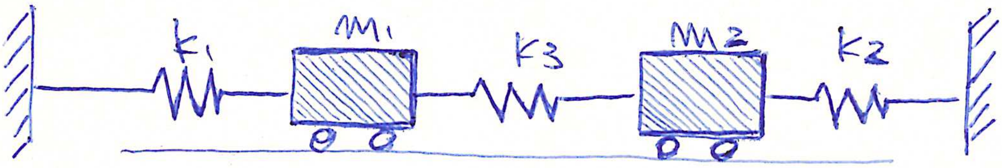
$$\det\left[D - \lambda^2 I\right] = 0$$

$$\begin{vmatrix} 8433.1 - \lambda^2 & -4818.9 \\ -963.8 & 3671.6 - \lambda^2 \end{vmatrix} = 0$$

$$\lambda_{1,2}^2 \begin{cases} \lambda_1^2 = 0 \ (rad/s)^2 \Rightarrow \lambda_1 = 0 \\ \lambda_2^2 = 12105 \ (rad/s)^2 \Rightarrow \lambda_2 = 110.02 \ rad/s = 17.5 \ Hz. \end{cases}$$

TUTORIAL 1

Ex. 5



SET 1

$k_1 = k_2 = 400000$ N/m

$k_3 = 800000$ N/m

$m_1 = m_2 = 2$ kg

SET 2

$k_1 = 400000$ N/m

$k_2 = 200000$ N/m

$k_3 = 800000$ N/m

$m_1 = 2$ kg

$m_2 = 2.5$ kg

Equations of motions

)
$$m_1 \ddot{x}_1 = -k_1 x_1 - k_3(x_1 - x_2)$$
$$m_1 \ddot{x}_1 = -k_1 x_1 - k_3 x_1 + k_3 x_2$$
$$m_1 \ddot{x}_1 + (k_1 + k_3) x_1 - k_3 x_2 = 0$$

)
$$m_2 \ddot{x}_2 = -k_2 x_2 - k_3(x_2 - x_1)$$
$$m_2 \ddot{x}_2 = -k_3 x_2 + k_3 x_1 - k_2 x_2$$
$$m_2 \ddot{x}_2 + (k_3 + k_2) x_2 - k_3 x_1 = 0$$

Lagrange

Undamped system

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right] - \frac{\partial \mathcal{L}}{\partial x_i} + \underbrace{\frac{\partial F}{\partial \dot{x}_i}}_{=0} = \frac{\partial \delta L}{\partial \delta x_i} \qquad \mathcal{L} = T - U$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 (x_1 - x_2)^2 + \frac{1}{2} k_2 x_2^2$$

$$= \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_1^2 + \frac{1}{2} k_3 x_2^2 - k_3 x_1 x_2 + \frac{1}{2} k_2 x_2^2$$

1) $m_1 \ddot{x}_1 + k_1 x_1 + k_3 x_1 - k_3 x_2 = 0$

$m_1 \ddot{x}_1 + (k_1 + k_3) x_1 - k_3 x_2 = 0$

2) $m_2 \ddot{x}_2 + k_3 x_2 + k_2 x_2 - k_3 x_1 = 0$

$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_3 x_1 = 0$

Matrix form

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [K] = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$D = M^{-1} K$$

$$|D - \lambda^2 I| = 0$$

$\cancel{D} = \cancel{\begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1200000 & -800000 \\ -800000 & 1000000 \end{bmatrix}}$ Computation of the natural frequencies and mode shapes

$$D1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1200000 & -800000 \\ -800000 & 1200000 \end{bmatrix} = \begin{bmatrix} 600000 & -400000 \\ -400000 & 600000 \end{bmatrix}$$

$$D2 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 1200000 & -800000 \\ -800000 & 1000000 \end{bmatrix} = \begin{bmatrix} 600000 & -400000 \\ -320000 & 400000 \end{bmatrix}$$

$$\lambda_{1,2\_1} \begin{cases} \lambda_1 = 447 \text{ rad/s} \\ \lambda_2 = 1000 \text{ rad/s} \end{cases} \qquad \lambda_{1,2\_2} \begin{cases} \lambda_1 = 358.5 \text{ rad/s} \\ \lambda_2 = 933.5 \text{ rad/s} \end{cases}$$

$$\phi\_1 = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \qquad \phi\_2 = \begin{bmatrix} 0.8274 & 0.6469 \\ -0.5616 & 0.7625 \end{bmatrix}$$

Computation of the modal matrices SET 1 5.2

$MM1\_1 = \phi_{1\_1}{}' M\_1 \, \phi_{1\_1} = 2$ Modal mass 1

$MM2\_1 = \phi_{2\_1} \, M\_1 \, \phi_{2\_1} = 2$ Modal mass 2

Normalized eigenvectors

$$\phi_{M_1} = \phi_{1,1} / \sqrt{MM_1} = \begin{Bmatrix} -0.5 \\ -0.5 \end{Bmatrix}$$

$$\phi_{M_2} = \phi_{2,1} / \sqrt{MM_2} = \begin{Bmatrix} -0.5 \\ 0.5 \end{Bmatrix}$$

$$\Phi = \begin{Bmatrix} -0.5 & -0.5 \\ -0.5 & 0.5 \end{Bmatrix}$$

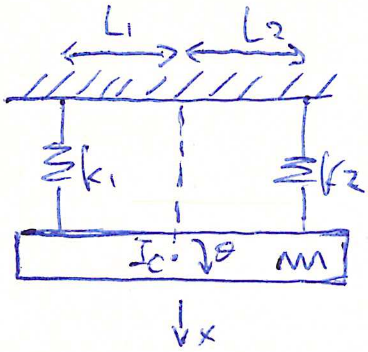
$$MM_1 = \phi^T M \phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$KM_1 = \phi^T M \phi = \begin{bmatrix} 200000 & 0 \\ 0 & 1000000 \end{bmatrix}$$

$$DM_1 = MM_1^{-1} KM_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 200000 & 0 \\ 0 & 1000000 \end{bmatrix} = \begin{bmatrix} 200000 & 0 \\ 0 & 1000000 \end{bmatrix}$$

$$\lambda_{M_{1,2}}^2 = \begin{cases} \lambda_{M_1}^2 = 200000 \, (\text{rad/s})^2 \Rightarrow \lambda_{M_1} = 447.2 \text{ rad/s} \\ \lambda_{M_2}^2 = 1000000 \, (\text{rad/s})^2 \Rightarrow \lambda_{M_2} = 1000 \text{ rad/s} \end{cases}$$

Ex. 7



$k_1 = k_2 = 20000\ N/m$

$m = 950\ kg$

$I_c = 1400\ kg\,m^2$

$L_1 = 1\ m$

$L_2 = 1.5\ m$

Equations of motion

1) $m\ddot{x} + k(x - L_1\theta) + k(x + L_2\theta) = 0$

$m\ddot{x} + 2kx + k(L_2 - L_1)\theta = 0$

2) $I_c\ddot{\theta} + kL_1(-x + L_1\theta) + kL_2(x + L_2\theta) = 0$

$I_c\ddot{\theta} + k(L_2 - L_1)x + k(L_1^2 + L_2^2)\theta = 0$

Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right] - \frac{\partial \mathcal{L}}{\partial x_i} + \frac{\partial F}{\partial \dot{x}_i} = \frac{\partial \delta L}{\partial \delta x_i} \qquad \mathcal{L} = T - U$$

(T) $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_c\dot{\theta}^2$

(U) $\frac{1}{2}k(x - L_1\theta)^2 + \frac{1}{2}k(x + L_2\theta)^2 =$

$= \frac{1}{2}kx^2 + \frac{1}{2}kL_1^2\theta^2 - kxL_1\theta + \frac{1}{2}kx^2 + \frac{1}{2}kL_2^2\theta^2 + kL_2x\theta$

(1) $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_1}\right] = m\ddot{x}$ $\qquad \frac{\partial \delta L}{\partial \delta x} = 0$

$\frac{\partial U}{\partial x_i} = kx - kL_1\theta + kx + kL_2\theta = 2kx + kL_2\theta - kL_1\theta$

(2) $\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{x}_2}\right] = I_c\ddot{\theta}$

$\frac{\partial U}{\partial x_2} = kL_1^2\theta - kxL_1 + kL_2^2\theta + kL_2x = k(L_1^2 + L_2^2)\theta + k(L_2 - L_1)x$

1) $m\ddot{x} + 2kx + k(L_2-L_1)\theta = 0$

2) $I_c\ddot{\theta} + k(L_1^2+L_2^2)\theta + k(L_2-L_1)x = 0$

Matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & I_c \end{bmatrix}}_{[M]} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \underbrace{\begin{bmatrix} 2k & k(L_2-L_1) \\ k(-L_1+L_2) & k(L_1^2+L_2^2) \end{bmatrix}}_{[k]} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[M]\{\ddot{x}\} + [k]\{x\} = \{0\}$$

$$D = M^{-1}k = \begin{bmatrix} 0.0011 & 0 \\ 0 & 0.0007 \end{bmatrix} \begin{bmatrix} 40000 & 10000 \\ +10000 & +65000 \end{bmatrix} = \begin{bmatrix} 42.1053 & 10.5263 \\ +7.1429 & +46.4286 \end{bmatrix}$$

$$\det[D - \lambda^2 I] = 0 \qquad \begin{vmatrix} 42.1053-\lambda^2 & 10.5263 \\ +7.1429 & +46.428-\lambda^2 \end{vmatrix} = 0$$

$$[(42.1-\lambda^2)(46.4-\lambda^2) - 74.55] = 0$$

$$\lambda^4 - 88.5\lambda^2 + 1953.4 - 74.55 = 0$$

$$\lambda^4 - 88.5\lambda^2 + 1878.9 = 0$$

$$\lambda_{1,2}^2 = \frac{88.5 \pm \sqrt{88.5^2 - 4 \cdot 1878.9}}{2} = \begin{cases} \lambda_1^2 = 35.35\ (rad/s)^2 \\ \lambda_2^2 = 53.145\ (rad/s)^2 \end{cases}$$

$\lambda_1 = 5.94$ rad/s

$\lambda_2 = 7.29$ rad/s

Computation of the eigenvectors

$$\left(-\lambda_n^2 \begin{bmatrix} m & 0 \\ 0 & I_c \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\right) \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\theta_{02} = 1$

$$-\lambda_n^2 I_c + k_{21}\theta_{01} + k_{22} = 0$$

(7-2)

1) $-5.94^2 \cdot 1400 + 10000\, \theta_{01} + 65000 = 0$

$\theta_{01} = -1.55$

2) $-7.3^2 \cdot 1400 + 10000\, \theta_{01} + 65000 = 0$

$\theta_{01} = +0.96$

$\{\theta_1\} = \begin{Bmatrix} -1.55 \\ 1 \end{Bmatrix} \quad \{\theta_2\} = \begin{Bmatrix} +0.96 \\ 1 \end{Bmatrix}$

Computation of the modal matrices

$\phi = [\theta_{01}, \theta_{02}] = \begin{bmatrix} -1.55 & 0.96 \\ 1 & 1 \end{bmatrix}$

Computation of the modal masses

$\theta_{0i}^T M \theta_{0i}$

$M_1 = \theta_{01}^T M \theta_{01} = \begin{bmatrix} -1.55 \\ 1 \end{bmatrix}^T \begin{bmatrix} 950 & 0 \\ 0 & 1400 \end{bmatrix} \begin{bmatrix} -1.55 \\ 1 \end{bmatrix} = 3693.4$

$M_2 = \theta_{02}^T M \theta_{02} = \begin{bmatrix} 0.96 \\ 1 \end{bmatrix} \begin{bmatrix} 950 & 0 \\ 0 & 1400 \end{bmatrix} \begin{bmatrix} 0.96 \\ 1 \end{bmatrix} = 2254.6$

Normalization of the eigenvectors

$\theta_{01\_norm} = \theta_{01} / \sqrt{M_1} = \begin{bmatrix} -0.0256 \\ 0.0165 \end{bmatrix}$

$\theta_{02\_norm} = \theta_{02} / \sqrt{M_2} = \begin{bmatrix} 0.02 \\ 0.021 \end{bmatrix}$

$\phi_{norm} = \begin{bmatrix} -0.0256 & 0.02 \\ 0.0165 & 0.021 \end{bmatrix}$

Computation of the modal matrices

$$MM = \phi_{norm}^T M \phi_{norm} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$KM = \phi_{norm}^T K \phi_{norm} = \begin{bmatrix} 35.33 & 0 \\ 0 & 52.20 \end{bmatrix}$$

Verification of the natural frequencies

$$DM = MM^{-1} KM = \begin{bmatrix} 35.33 & 0 \\ 0 & 55.2 \end{bmatrix}$$

$$\left| \det [DM - \lambda^2 I] \right| = 0$$

$$\left| \begin{bmatrix} 35.33 - \lambda^2 & 0 \\ 0 & 55.2 - \lambda^2 \end{bmatrix} \right| = 0 \qquad (35.33 - \lambda^2)(55.2 - \lambda^2) = 0$$

$$\lambda^4 - 90.53\lambda^2 + 1879.6 = 0$$

$$\Rightarrow \lambda_1 = 5.94 \text{ rad/s}$$

$$\lambda_2 = 7.99 \text{ rad/s}$$

TUTORIAL 1

Ex. 8

$q_1 = \begin{Bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{Bmatrix}$

Undamped system

$\bar{M} = 2\,kg$

$\bar{K} = 5000\ N/m$

$\bar{F} = 10 + \frac{(2U+P)}{3}$

$\lambda = 0.8\,\lambda_{n1}$

* The output ~~to~~ of a harmonic excitation of a vibrating structure is described only by the first mode.
* Compute the oscillation (amplitude) of the first degree of freedom.

$$x_0 = \frac{F_0/k}{1 - \left(\frac{\lambda}{\lambda_n}\right)^2 + \cancel{2j\xi\left(\frac{\lambda}{\lambda_n}\right)}^{=0}}$$ undamped system

Using the modal coordinates

$$\eta_{01} = \frac{\bar{F}_1/\bar{K}_1}{1 - \left(\frac{\lambda_1}{\lambda_n}\right)^2 + 0} = \frac{\left(10 + \frac{2U+P}{3}\right)\frac{1}{5000}}{1 - \left(\frac{4/5\,\lambda_n}{\lambda_n}\right)^2} =$$

$$= \frac{\left(10 + \frac{2U+P}{3}\right)\frac{1}{5000}}{\frac{9}{25}} = \frac{30 + 2U + P}{5400}$$

From $x = \phi\,\eta \Rightarrow x_{01} = q_1\,\eta_{01}$

$$x_{01} = \frac{1}{\sqrt{5}} \cdot \frac{30 + 2U + P}{5400}$$ If $U = 0$, $P = 1$.

$\Rightarrow x_{01} = 0.0026\ m$

TUTORIAL 1

Ex. 9

$q_1 = \begin{Bmatrix} 1 \\ 0.8165 \end{Bmatrix}$ $\quad q_2 = \begin{Bmatrix} 1 \\ -0.8165 \end{Bmatrix}$

$[k] = 10^3 \begin{bmatrix} 6 & -8 \\ -8 & 15 \end{bmatrix} N/m \quad [M] = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \quad \{F\} = 10^3 \left(1 + \frac{2U+P}{20}\right) \begin{Bmatrix} 5 \\ 8 \end{Bmatrix} N$

$\bar{F}_1 = q_1^T F$

$= \{1 \quad 0.8165\} \left[10^3 \left(1 + \frac{2U+P}{20}\right)\right] \begin{Bmatrix} 5 \\ 8 \end{Bmatrix} = \qquad U = 0 \quad P = 0$

$= 11532 \; N$

TUTORIAL 1

Ex. 10

$\{q_1\} = \left\{ \frac{1}{\sqrt{7}}, \frac{2}{\sqrt{7}} \right\}^T \quad \bar{M} = 3 \, kg \quad \bar{K} = 3000 \, N/m \quad F = \left(6 + \frac{2U+P}{3}\right) N$

$\lambda = 0.8 \lambda_{n1}$ The system is undamped

$\eta_{01} = \dfrac{\bar{F}_1 / \bar{K}_1}{1 - \left(\frac{\lambda}{\lambda_n}\right)^2} \quad \Rightarrow \quad x_{01} = q \eta = \dfrac{1}{\sqrt{7}} \cdot \dfrac{\bar{F}_1 / \bar{K}_1}{1 - \left(\frac{\lambda}{\lambda_n}\right)^2}$