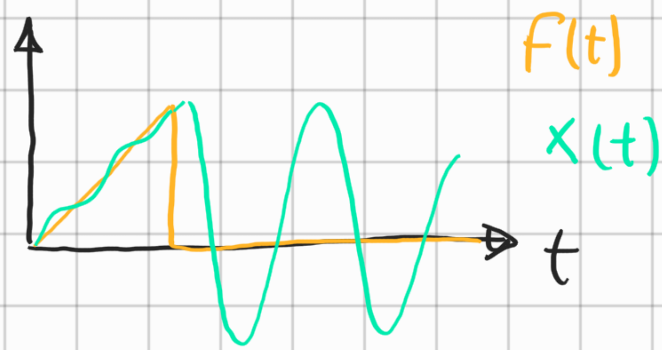
Ex 5

$m = 2\,kg \qquad k = 100\,\frac{N}{m} \qquad c = 2\,\frac{Ns}{m}$

→ Undamped response

We expect this response



Hence, the impulse response is

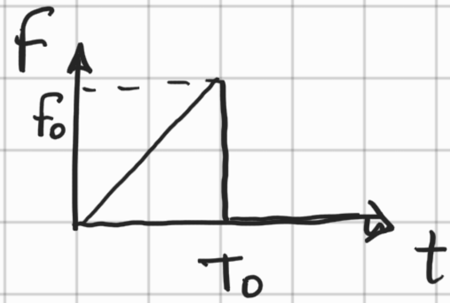
$$g(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

We can solve the problem by means of the convolution integral

$$x_1(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau = \int_0^t F(t-\tau)\, g(\tau)\, d\tau$$

we use this

Our force is



$$f = \begin{cases} \frac{f_0}{T_0}\, t & \text{if } t \le T_0 \\ 0 & \text{otherwise} \end{cases}$$

Let's compute the response at $t < T_0$

- $F(t-\tau) = \frac{f_0}{T_0}(t-\tau)$
- $g(\tau) = \frac{1}{m\omega_n} \sin \omega_n \tau$

$$\Longrightarrow x_1(t) = \underbrace{\boxed{\frac{F_0}{T_0} \cdot \frac{1}{m\omega_n}}}_{A} \int_0^t (t-\tau) \sin \omega_n \tau \, d\tau$$

$$\Longrightarrow x_1(t) = A \cdot \left[t \underbrace{\int_0^t \sin \omega_n \tau \, d\tau}_{①} - \underbrace{\int_0^t \tau \sin \omega_n \tau \, d\tau}_{②} \right]$$

①

$$\int_0^t \sin \omega_n \tau \, d\tau = -\frac{1}{\omega_n} \cos \omega_n \tau \Big|_0^t = -\frac{1}{\omega_n} \cos \omega_n t + \frac{1}{\omega_n}$$

② is solved by parts

$$\hookrightarrow \int u\,dv = uv - \int v\,du$$

in our case

$u = \tau \longrightarrow du = d\tau$

$dv = \sin \omega_m \tau \longrightarrow v = -\frac{1}{\omega_m} \cos \omega_m \tau$

$$\Rightarrow \int_0^t \tau \sin \omega_m \tau \, d\tau = \left[-\frac{\tau}{\omega_m} \cos \omega_m \tau \right]_0^t + \frac{1}{\omega_m} \int_0^t \cos \omega_m \tau \, d\tau =$$

$$= -\frac{t}{\omega_m} \cos \omega_m t + \left[\frac{1}{\omega_m^2} \sin \omega_m \tau \right]_0^t$$

$$\Rightarrow \int_0^t \tau \sin \omega_m \tau \, d\tau = -\frac{t}{\omega_m} \cos \omega_m t + \frac{1}{\omega_m^2} \sin \omega_m t$$

By substituting into the equation

$$\Rightarrow x_1(t) = \cancel{-\frac{At}{\omega_m} \cos \omega_m t} + \frac{At}{\omega_m} + \cancel{\frac{At}{\omega_m} \cos \omega_m t} - \frac{A}{\omega_m^2} \sin \omega_m t$$

$$\Rightarrow \boxed{x_1(t) = \frac{A}{\omega_m} t - \frac{A}{\omega_m^2} \sin \omega_m t \quad , t \in [0; T_0]}$$

$$\dot{x}_1(t) = \frac{A}{\omega_n} - \frac{A}{\omega_n}\cos\omega_n t$$

$$\omega_n = \sqrt{K/m} = 22{,}36 \text{ rad/s}$$

$$A = \frac{F_0}{T_0}\frac{1}{m\omega_n} = 0{,}268 \ \frac{m}{s^2}$$

$$\Rightarrow \begin{cases} x_1(T_0) \\ \dot{x}_1(T_0) \end{cases}$$

When $t > T_0$, the force is null $\Rightarrow$ free response

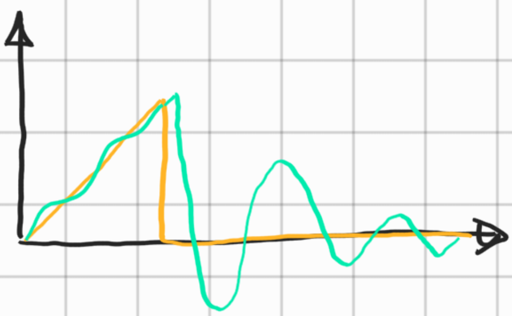
$$\boxed{x_2(t-T_0) = x_1(T_0)\cos[\omega_n(t-T_0)] + \frac{\dot{x}_1(T_0)}{\omega_n}\sin[\omega_n(t-T_0)] \qquad t > T_0}$$

$\rightarrow$ Damped response

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

We expect this response

$$\xi = \frac{c}{2\sqrt{km}}$$



Hence, the impulse response is

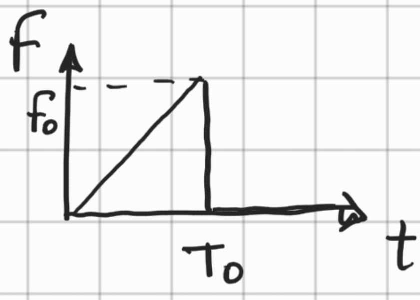
$$g(t) = \frac{1}{m\omega_d} e^{-\xi\omega_n t} \sin \omega_d t$$

We can solve the problem by means of the convolution integral

$$x_1(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau = \int_0^t F(t-\tau)\, g(\tau)\, d\tau$$

we use this

Our force is



$$f = \begin{cases} \dfrac{f_0}{T_0}\, t & \text{if } t \leq T_0 \\ 0 & \text{otherwise} \end{cases}$$

Let's compute the response at $t < T_0$

- $F(t-\tau) = \dfrac{f_0}{T_0}(t-\tau)$ • $g(\tau) = \dfrac{1}{m\omega_d} e^{-\xi\omega_n\tau} \sin\omega_d\tau$

$$\Rightarrow x_1(t) = \underbrace{\boxed{\frac{F_0}{T_0}\cdot\frac{1}{m\omega_d}}}_{A} \int_0^t (t-\tau)\, e^{-\xi\omega_n\tau} \sin\omega_d\tau \, d\tau$$

$$\Rightarrow x_1(t) = A\cdot\left[t \underbrace{\int_0^t e^{-\xi\omega_n\tau} \sin\omega_d\tau \, d\tau}_{1} - \underbrace{\int_0^t e^{-\xi\omega_n\tau}\, \tau \sin\omega_d\tau \, d\tau}_{2} \right]$$

$(*)$

these integrals are solved by parts $\rightarrow \int u\,dv = uv - \int v\,du$

Let's solve the indefinite integral and then evaluate the solution at the bounds of the integration interval

① $\int e^{-\xi\omega_n\tau} \sin\omega_n\tau \, d\tau$

$u = e^{-\xi\omega_n\tau} \longrightarrow du = -\xi\omega_n e^{-\xi\omega_n\tau} d\tau$

$dv = \sin\omega_d\tau \, d\tau \longrightarrow v = -\frac{1}{\omega_d}\cos\omega_d\tau$

$$\Rightarrow \int e^{-\xi\omega_n\tau}\sin\omega_d\tau \, d\tau = -\frac{e^{-\xi\omega_n\tau}}{\omega_d}\cos\omega_d\tau -$$

$$-\xi\frac{\omega_n}{\omega_d}\int e^{-\xi\omega_n\tau}\cos\omega_d\tau \, d\tau$$

by parts

$u = e^{-\xi\omega_n\tau} \longrightarrow du = -\xi\omega_n e^{-\xi\omega_n\tau}$

$dv = \cos\omega_d\tau \, d\tau \longrightarrow v = \frac{1}{\omega_d}\sin\omega_d\tau$

$$\Rightarrow \underline{\int e^{-\xi\omega_n\tau}\sin\omega_d\tau \, d\tau} = -\frac{e^{-\xi\omega_n\tau}}{\omega_d}\cos\omega_d\tau -$$

$$-\frac{\xi\omega_n}{\omega_d^2}e^{-\xi\omega_n\tau}\sin\omega_d\tau - \left(\xi\frac{\omega_n}{\omega_d}\right)^2 \underline{\int e^{-\xi\omega_n\tau}\sin\omega_d\tau \, d\tau}$$

Let's group the highlighted terms and manipulate the equation

$$\Rightarrow \boxed{\int e^{-\xi\omega_n\tau}\sin\omega_d\tau\, d\tau = -\frac{e^{-\xi\omega_n\tau}}{\omega_d}\left(\cos\omega_d\tau + \frac{\xi\omega_n}{\omega_d}\sin\omega_d\tau\right)\cdot \frac{1}{\left(1+\frac{\xi\omega_n}{\omega_d}\right)^2} = I(\tau)}$$

let's define this variable

this is the solution

$$\int_0^t e^{-\xi\omega_n\tau}\sin\omega_d\tau\, d\tau = \left[-\frac{e^{-\xi\omega_n\tau}}{\omega_d\left(1+\xi\frac{\omega_n}{\omega_d}\right)^2}\cos\omega_d\tau + \right.$$

$$\left. + \xi\frac{\omega_n}{\omega_d}\sin\omega_d\tau\right]_{\tau=0}^{\tau=t} = I(\tau)\Big|_{\tau=0}^{\tau=t}$$

(1)

② $\int e^{-\xi\omega_n\tau}\,\tau\,\sin\omega_d\tau\,d\tau \longrightarrow$ by parts

$u = \tau \longrightarrow du = d\tau$

$dv = e^{-\xi\omega_n\tau}\sin\omega_d\tau\,d\tau \longrightarrow v = \underbrace{\int e^{-\xi\omega_n\tau}\sin\omega_d\tau\,d\tau}$

this is $I \longrightarrow v = I$

$$\Rightarrow \int e^{-\xi\omega_n\tau}\,\tau\,\sin\omega_d\tau\,d\tau = \tau I - \underbrace{\int I\,d\tau}$$

$$\int I\,d\tau = -\frac{1}{\omega_d\left(1+\xi\frac{\omega_n}{\omega_d}\right)^2}\int e^{-\xi\omega_n\tau}\cos\omega_d\tau\,d\tau -$$

$$-\frac{\xi\,\omega_n/\omega_d}{\omega_d\left(1+\xi\frac{\omega_n}{\omega_d}\right)^2}\underbrace{\int e^{-\xi\omega_n\tau}\sin\omega_d\tau\,d\tau}$$

this is I

$$\Rightarrow \int I\,d\tau = -\frac{1}{\omega_d\left(1+\xi\frac{\omega_n}{\omega_d}\right)^2}\underbrace{\int e^{-\xi\omega_n\tau}\cos\omega_d\tau\,d\tau} - \frac{\xi\omega_n/\omega_d}{\omega_d\left(1+\xi\frac{\omega_n}{\omega_d}\right)^2}\cdot I$$

Let's solve this integral (by parts)

$u = e^{-\xi \omega_n \tau} \longrightarrow du = -\xi \omega_n e^{-\xi \omega_n \tau} d\tau$

$dv = \cos \omega_d \tau \, d\tau \longrightarrow v = \frac{1}{\omega_d} \sin \omega_d \tau$

$$\Rightarrow \int e^{-\xi \omega_n \tau} \cos \omega_d \tau \, d\tau = \frac{e^{-\xi \omega_n \tau}}{\omega_d} \sin \omega_d \tau +$$

$$+ \xi \frac{\omega_n}{\omega_d} \underbrace{\int e^{-\xi \omega_n \tau} \sin \omega_d \tau \, d\tau}_{\text{this is } I}$$

$$\Rightarrow \int e^{-\xi \omega_n \tau} \cos \omega_d \tau \, d\tau = \frac{e^{-\xi \omega_n \tau}}{\omega_d} \sin \omega_d \tau + \xi \frac{\omega_n}{\omega_d} \cdot I$$

Therefore:

$$\boxed{\int I \, d\tau = -\frac{1}{\omega_d \left(1 + \xi \frac{\omega_n}{\omega_d}\right)^2} \left[\frac{e^{-\xi \omega_n \tau}}{\omega_d} \sin \omega_d \tau + \xi \frac{\omega_n}{\omega_d} I \right] - \frac{\xi^{\omega_n / \omega_d}}{\omega_d \left(1 + \xi \frac{\omega_n}{\omega_d}\right)^2} \cdot I = H(\tau)}$$

$\leftarrow$ let's define this variable

So ② is

$$\int_0^t e^{-\zeta \omega_n \tau} \tau \sin \omega_d \tau \, d\tau = \Big[\tau\, I(\tau) - H(\tau) \Big]_{\tau=0}^{\tau=t}$$

Hence, by substituting ① and ② into ⊛ we get the solution $x_1(t)$, $t \in [0, T_0]$

Now we can proceed as we've done for the undamped system case:

we compute the first derivative $\dot{x}_1(t)$ to get the initial conditions of solution x_2. So:

$$x_1(t) \longrightarrow \dot{x}_1(t) \longrightarrow \begin{cases} x_1(T_0) \\ \dot{x}_1(T_0) \end{cases}$$

Then, we compute the response at $t > T_0$

When $t > T_0$, the force is null $\Rightarrow$ free response

$$\boxed{\begin{aligned} x_2(t-T_0) &= e^{-\xi\omega_n t}\left\{x_1(T_0)\cos[\omega_d(t-T_0)] + \frac{1}{\sqrt{1-\xi^2}}\cdot\right.\\ &\left.\cdot\left[\frac{\dot{x}_1(T_0)}{\omega_n} + \xi\, x_1(T_0)\right]\sin[\omega_d(t-T_0)]\right\} \end{aligned}}$$

This solution holds at $t > T_0$