



01OAIQD – Dynamic Design of Machines

Academic year 2020-2021

Discrete linear systems

Emission date: October 16, 2020

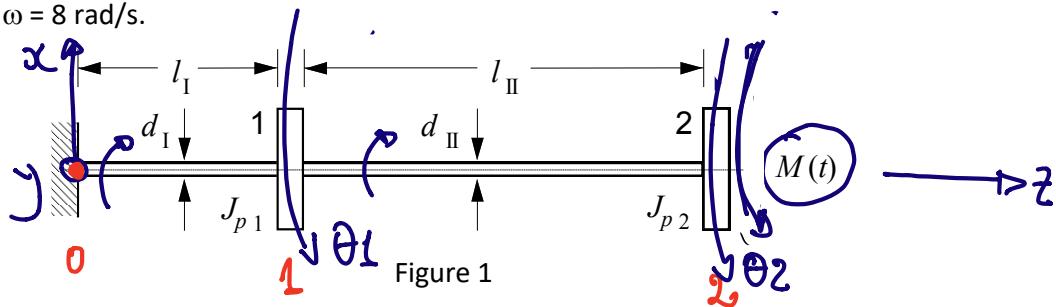
GROUP:

EXERCISE 1

The undamped system presented in Figure 1 is characterized by two torsion bars (negligible inertia) and two flywheels. The second one is excited by a harmonic excitation $M(t) = M_0 \sin(\omega t)$. It is requested to study the dynamic behavior of the system computing:

- a) the natural frequencies and the corresponding mode shapes;
- b) the potential and kinetic energy associated to the different subsystems for each mode.
- c) the maximum shear stresses in the torsion bars using the configuration space and the state space approach.

Data: $G = 7.7 \cdot 10^{10} \text{ N/m}^2$, $l_I = 0.5 \text{ m}$, $l_{II} = 1.0 \text{ m}$, $d_I = 12.8 \text{ mm}$, $d_{II} = 14.1 \text{ mm}$, $J_{p1} = 6 \text{ kgm}^2$, $J_{p2} = 4 \text{ kgm}^2$, $M_0 = 100 \text{ Nm}$, $\omega = 8 \text{ rad/s}$.



$$\omega_{n1} = 5.6 \text{ rad/s}, \omega_{n2} = 12.7 \text{ rad/s}, \{\theta\}_1 = \{0.58 1\}^T, \{\theta\}_2 = \{1 -0.87\}^T, \tau_I = 390 \text{ MPa}, \tau_{II} = 16 \text{ MPa}.$$

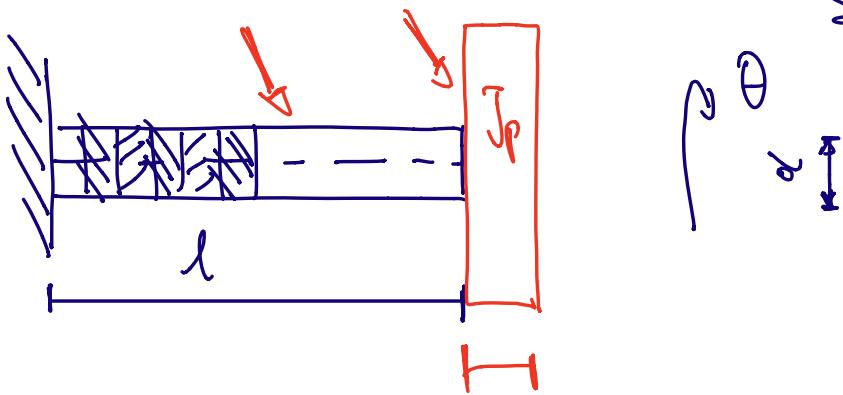
EXERCISE 2

The undamped system represented in Figure 2 is characterized by two bars having a diameter d (with negligible inertia) and two masses. A harmonic excitation $F(t) = F_0 \sin(\omega t)$ is applied on the mass 2.

It is requested to:

- compute the amplitude of the oscillations by adopting the modal analysis,
- compute the maximum stress in bar 1 and bar 2.

SoluN om Ex.1.



$$L = T - U$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} + \frac{\partial \tilde{F}}{\partial \dot{x}_i} = \frac{\partial (\delta L)}{\partial \delta x_i}$$

$\tilde{F} = 0$ because we do not have damping

T

U

δL

$$T = \frac{1}{2} J_{p1} \dot{\theta}_1^2 + \frac{1}{2} J_{p2} \dot{\theta}_2^2$$

$$\bullet \bar{U} = \frac{1}{2} k_I (\theta_1 - \theta_0)^2 + \frac{1}{2} k_{II} (\theta_2 - \theta_1)^2$$

$$k_I = \frac{G I_{pI}}{l_I} \quad I_{pI} = \frac{\pi d_I^4}{32} = 2.6 \cdot 10^{-9} \text{ m}^4$$

$$k_{II} = \frac{G I_{pII}}{l_{II}} \quad I_{pII} = \frac{\pi d_{II}^4}{32} = 3.9 \cdot 10^{-9} \text{ m}^4$$

$$k_I = 400 \cdot 4 \frac{\text{N} \cdot \text{m}}{\text{rad}}$$

$$k_{II} = 300 \cdot 3 \frac{\text{N} \cdot \text{m}}{\text{rad}}$$

T not depending on θ

$V \parallel \parallel \parallel \dot{\theta}$

10

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_i} \right) + \frac{\partial U}{\partial \theta_i} + \frac{\partial \tilde{L}}{\partial \dot{\theta}_i} = \frac{\partial \delta L}{\partial \delta \theta_i}$$

$$\theta_i = \theta_1 \text{ or } \theta_i = \theta_2$$

$$T = \frac{1}{2} J_{p1} \dot{\theta}_1^2 + \frac{1}{2} J_{p2} \dot{\theta}_2^2$$

$$U = \frac{1}{2} k_I (\theta_1 - \theta_0)^2 + \frac{1}{2} k_{II} (\theta_2 - \theta_1)^2$$

$$\boxed{\theta_1} \quad \frac{\partial T}{\partial \dot{\theta}_1} = J_{p1} \cdot \dot{\theta}_1 ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_1} \right) = J_{p1} \ddot{\theta}_1$$

$$\frac{\partial U}{\partial \theta_1} = k_I \theta_1 + k_{II} (\theta_2 - \theta_1) (-1)$$

$$\boxed{\delta L = H_t \cdot \delta \theta_2} ; \quad \frac{\partial \delta L}{\partial \delta \theta_1} = 0$$

1st equation :

$$J_{P_1} \ddot{\theta}_1 + k_{II}(\theta_2 - \theta_1) (-1) = 0$$

$$J_{P_1} \ddot{\theta}_1 + k_I \theta_1 + k_{II} \theta_1 - k_{III} \theta_2 = 0$$

$$\boxed{J_{P_1} \ddot{\theta}_1 + (k_I + k_{II}) \theta_1 - k_{III} \theta_2 = 0}$$

$$\boxed{\theta_2} \quad \frac{\partial T}{\partial \dot{\theta}_2} = J_{P_2} \dot{\theta}_2 \quad ; \quad \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}_2} \right) = J_{P_2} \ddot{\theta}_2$$

$$\frac{\partial U}{\partial \theta_2} = k_{II} (\theta_2 - \theta_1)$$

$$\frac{\partial \delta L}{\partial \delta \theta_2} = \frac{\partial (H_t \cdot \delta \theta_2)}{\partial \delta \theta_2} = H_t$$

$$\boxed{J_{P_2} \ddot{\theta}_2 - k_{II} \theta_1 + k_{III} \theta_2 = H_t}$$

$$\begin{array}{|c|c|} \hline J_{P_1} & 0 \\ \hline 0 & J_{P_2} \\ \hline \end{array} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{array}{|c|c|} \hline k_I + k_{II} & -k_{II} \\ \hline -k_{II} & k_{II} \\ \hline \end{array} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[M] \quad \{\ddot{\theta}\} \quad [k] \quad \{\dot{\theta}\} = \{F\}$$

$$[M]\{\ddot{\theta}\} + [k]\{\dot{\theta}\} = \{0\}$$

$$\{\ddot{\theta}\} = \{\theta_0\} e^{i\omega t}; \{\dot{\theta}\} = +i\omega \{\theta_0\} e^{i\omega t}$$

$$\ddot{\theta} = -\omega^2 \{\theta_0\} e^{i\omega t}$$

$\{\theta_0\}$ and ω are the values that we have to compute

$$\left(-\omega^2 \begin{bmatrix} J_{P_1} & 0 \\ 0 & J_{P_2} \end{bmatrix} + \begin{bmatrix} k_I + k_{II} & -k_{II} \\ -k_{II} & k_{II} \end{bmatrix} \right) \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\det \left(-\omega^2 \begin{bmatrix} J_{p_1} & 0 \\ 0 & J_{p_2} \end{bmatrix} + \begin{bmatrix} k_I + k_{II} & -k_{II} \\ -k_{II} & k_{II} \end{bmatrix} \right) = 0$$

$$\begin{vmatrix} -\omega^2 J_{p_1} + k_I + k_{II} & -k_{II} \\ -k_{II} & k_{II} - J_{p_2} \omega^2 \end{vmatrix} = 0$$

$$(-\omega^2 J_{p_1} + k_I + k_{II})(k_{II} - J_{p_2} \omega^2) - k_{II}^2 = 0$$

$$(J_{p_1} \cdot J_{p_2}) \omega^4 - (J_{p_1} \cdot k_{II} + J_{p_2} k_I + J_{p_2} k_{II}) \omega^2 + k_I \cdot k_{II} = 0$$

$$24 \omega^4 - 46.02 \cdot \omega^2 + 1.2024 \cdot 10^5 = 0$$

$$ax^4 + bx^2 + c = 0$$

$$* \quad \omega_{2/2}^2 = \begin{cases} 31.18 \left(\frac{\text{rad}}{\Delta} \right)^2 \Rightarrow \omega_{m1} = 5.6 \frac{\text{rad}}{\Delta} \\ 160.68 \left(\frac{\text{rad}}{\Delta} \right)^2 \Rightarrow \omega_{m2} = 12.7 \frac{\text{rad}}{\Delta} \end{cases}$$

$$\begin{bmatrix} -\omega^2 J_{p1} + k_I + k_{II} & -k_{II} \\ -k_{II} & -\omega^2 J_{p2} + k_{II} \end{bmatrix} \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\theta_{02} = 1$$

$$\boxed{(-\omega^2 J_{p1} + k_I + k_{II}) \theta_{01} - k_{II} = 0}$$

$$\boxed{-k_{II} \theta_{01} - \omega^2 J_{p2} + k_{II} = 0}$$

$$\theta_{01} = \frac{-\omega^2 J_{p2} + k_{II}}{k_{II}} = 1 - \omega^2 \frac{J_{p2}}{k_{II}}$$

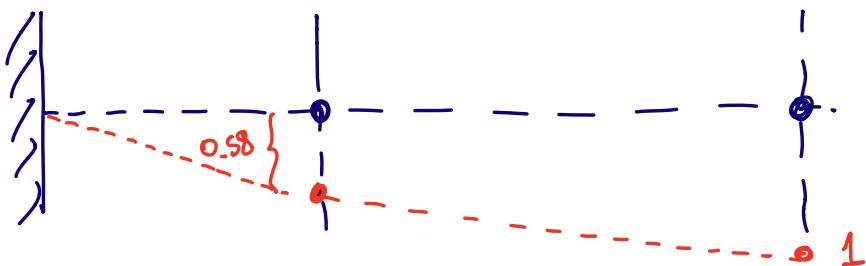
$$\theta_{01} = 1 - 1.332 \cdot 10^{-2} \omega^2$$

$$\omega = \omega_{m1} = 5.6 \frac{\text{rad}}{\text{s}} \Rightarrow \theta_{01} = 0.58$$

$$\omega = \omega_{m2} = 12.7 \frac{\text{rad}}{\text{s}} \Rightarrow \theta_{01} = -0.88$$

$$\omega_{m1} \Rightarrow \begin{Bmatrix} 0.58 \\ 1 \end{Bmatrix}; \quad \omega_{m2} \Rightarrow \begin{Bmatrix} -0.88 \\ 1 \end{Bmatrix}$$

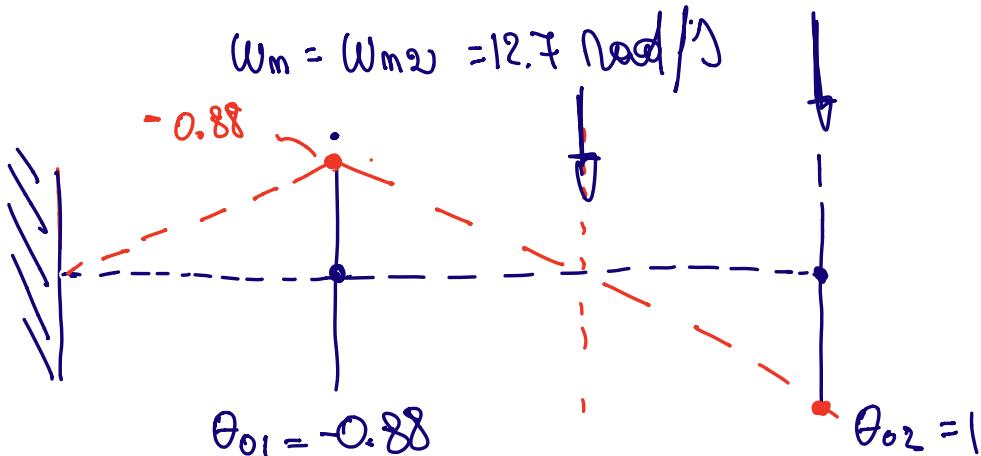
$$\omega = \omega_{m1} = 5.6 \frac{\text{rad}}{\text{s}}$$



$$\theta_{01} = 0.58$$

$$\theta_{02} = 1$$

$$\omega_m = \omega_{m2} = 12.7 \frac{\text{rad}}{\text{s}}$$



MODAL ANALYSIS

$$\text{mode 1} \Leftrightarrow \omega_{m1} = 5.6 \frac{\text{rad}}{\text{s}} \Rightarrow \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0.58 \\ 1 \end{Bmatrix}$$

$$U_1 = \frac{1}{2} \theta_{01}^2 k_I + \frac{1}{2} k_{II} (\theta_{02} - \theta_{01})^2$$

$$U_{11} = \frac{\frac{1}{2} \theta_{01}^2 k_I}{U_1} = 0.72$$

$$U_{12} = \frac{\frac{1}{2} k_{II} (\theta_{02} - \theta_{01})^2}{U_1} = 0.28$$

$$T_1 = \frac{1}{2} J_{p1} (\omega_{m1} \cdot \theta_{01})^2 + \frac{1}{2} (\omega_{m1} \theta_{02})^2$$

$$T_{11} = \frac{\frac{1}{2} J_{p1} (\omega_{m1} \theta_{01})^2}{T_1} = 0.3354$$

$$T_{12} = \frac{\frac{1}{2} J_{p2} (\omega_{m1} \theta_{02})^2}{T_1} = 0.6646$$

mode 2 $\Rightarrow \overset{\text{oon}}{\omega_{m2} = 12.7 \frac{\text{rad}}{\text{s}}} ; \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} -0.88 \\ 1 \end{Bmatrix}$

$$V_{21} = 0.274$$

$$V_{22} = 0.726$$

$\overset{\text{oon}}{\text{FORCED RESPONSE}}$

$$\begin{bmatrix} J_{p1} & 0 \\ 0 & J_{p2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_I + k_{II} & -k_{II} \\ -k_{III} & k_{II} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \text{Momentum} \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \text{sim } \omega t$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \cos \omega t (\omega)$$

$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = - \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \sin \omega t (\omega^2)$$

$$\left(-\omega^2 \begin{array}{|c|c|} \hline J_{P1} & 0 \\ \hline 0 & J_{P2} \\ \hline \end{array} + \begin{array}{|c|c|} \hline k_I + k_{II} & -k_{II} \\ \hline -k_{II} & k_{II} \\ \hline \end{array} \right) \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} \cancel{\sin \omega t} =$$

$$= \begin{Bmatrix} 0 \\ H_0 \end{Bmatrix} \cancel{\sin \omega t}$$

$$\begin{array}{|c|c|} \hline 316.7 & -300.3 \\ \hline -300.3 & 111.7 \\ \hline \end{array} \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \end{Bmatrix}$$

$$\theta_{01} = -0.394 \text{ rad}$$

$$\theta_{02} = -0.416 \text{ rad}$$

$$\ddot{\gamma}_I = \frac{H_t}{I_{p1}} \frac{d\gamma_I}{2} = \frac{k_I(\theta_1 - \theta_0)}{I_{p1}} \frac{d\gamma_I}{2} = 390 \text{ Hz}$$

$$\ddot{\gamma}_{II} = \frac{H_t}{I_{p2}} \frac{d\gamma_{II}}{2} = \frac{k_{II}(\theta_2 - \theta_1)}{I_{p2}} \frac{d\gamma_{II}}{2} = 16 \text{ Hz}$$

STATE SPACE

$$[H] \{ \ddot{\theta} \} + [k] \{ \theta \} = \{ \underbrace{0}_{H_t} \} \text{ input}$$

$\begin{cases} \dot{\gamma}_I \\ \dot{\gamma}_{II} \end{cases}$ } OUTPUT

$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} = - \begin{bmatrix} H \end{bmatrix}^{-1} \begin{bmatrix} K \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} + \begin{bmatrix} H \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ H(t) \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_5 \end{Bmatrix} = \boxed{\begin{array}{|c|c|} \hline \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & - \begin{bmatrix} H \end{bmatrix}^{-1} \begin{bmatrix} K \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \hline \end{array}} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{Bmatrix} + \begin{bmatrix} \begin{bmatrix} H \end{bmatrix}^{-1} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{Bmatrix} 0 \\ H(t) \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{x} \\ \ddot{z} \end{Bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} \begin{Bmatrix} z \\ w \end{Bmatrix} + \begin{bmatrix} A \\ B \end{bmatrix} \begin{Bmatrix} w \\ u \end{Bmatrix}$$

(4×1) (4×4) (4×1) $\underbrace{(4 \times 2) \quad (2 \times 1)}_{(4 \times 1)}$

$$\ddot{x}_I = \frac{H_I}{I_{pI}} \frac{dx_I}{2} = \frac{k_I(\theta_1 - \theta_0)}{I_{pI}} \frac{dx_I}{2}$$

$$\tilde{I}_I = \frac{k_I \cdot d_I}{2 I_{pI}} \cdot \theta_1$$

$$\tilde{I}_{II} = \frac{M_t}{I_{pII}} \frac{d_{II}}{2} = \frac{k_{II}(\theta_2 - \theta_1)}{I_{pII}} \frac{d_{II}}{2} =$$

$$= -\frac{k_{II} d_{II}}{2 I_{pII}} \theta_1 + \frac{k_{II} d_{II}}{2 I_{pII}} \theta_2$$

$$\{y\} = [C]\{z\} + [D]\{u\}$$

↓

$$\begin{pmatrix} \tilde{I}_I \\ \tilde{I}_{II} \end{pmatrix} = u \begin{pmatrix} 0 & 0 & \frac{k_I \cdot d_I}{2 I_{pI}} & 0 \\ 0 & 0 & \frac{k_{II} d_{II}}{2 I_{pII}} & \frac{k_{II} d_{II}}{2 I_{pII}} \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \theta_1 \\ \theta_2 \end{pmatrix} +$$

[G]

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ M_t \end{pmatrix}$$

↑