01OAIQD – Dynamic Design of Machines

Academic year 2019-2020

Discrete linear systems

TUTORIAL 3

EXERCISE 1

The beam sketched in figure 1, is constrained so that at the end A all the degrees of freedom are locked while at the end B it is allowed only the translation in the vertical direction. The stricture is loaded with a force P, lumped in the point B.
It is requested to

- assemble the stiffness matrix;
- compute the reaction forces.

Data

- Section: rectangular with base b = 10 mm and height h = 20 mm
- Material: iron steel, E = 210.000 MPa
- Length: L = 1 m
- Load: P = 100 kN

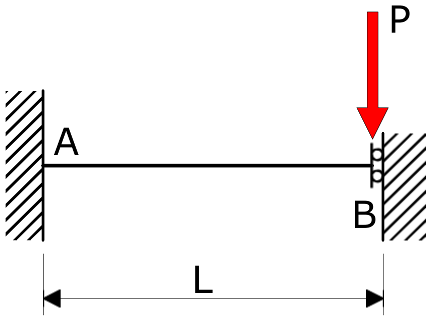
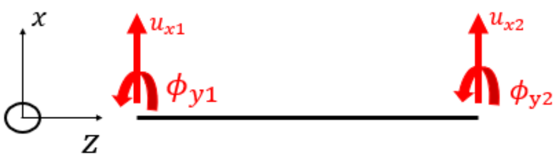
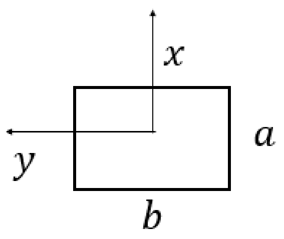


Figure 1

SOLUTION

It is requested to assemble the stiffness matrix and to compute the reaction force





$$[K] = E\frac{I}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$\begin{Bmatrix} F_{x1} \\ M_{y1} \\ F_{x2} \\ M_{y2} \end{Bmatrix} = [K] \begin{Bmatrix} u_{x1} \\ \phi_{y1} \\ u_{x2} \\ \phi_{y2} \end{Bmatrix}$$

The boundary conditions impose that $u_{x1} = 0, \quad \phi_{y1} = 0 \quad and \quad \phi_{y2} = 0$. So,

$F_{x1} = -12\, u_{x2} \frac{EI}{l^3}$

$M_{y1} = 6l\, u_{x2} \frac{EI}{l^3}$

$F_{x2} = 12\, u_{x2} \frac{EI}{l^3}$

$M_{y2} = -6l\, u_{x2} \frac{EI}{l^3}$

$P = \frac{EI}{l^3}\, 12\, u_{x2} \longrightarrow u_{x2} = \frac{l^3}{12\, EI}\, P = -5.95m$

$I = \frac{1}{12}\, b\, a^3 = 6.66 * 10^{-9}$

So,

$F_{x1} = -12\, \frac{l^3}{12\, EI}\, P \frac{EI}{l^3} = 12\, P$

$M_{y1} = 6l\, \frac{l^3}{12\, EI}\, P \frac{EI}{l^3} = -\frac{1}{2}\, P\, l$

$M_{y2} = -6l\, \frac{l^3}{12\, EI}\, P \frac{EI}{l^3} = \frac{1}{2}\, P\, l$

EXERCISE 2

The beam sketched in figure 2, is constrained so that at the end A all the degrees of freedom are locked. The structure is loaded with a lumped vertical load applied in B.
It is requested to define the stiffness matrix and compute the reaction forces in A and the defection in B.

Data

- Section: rectangular with base b = 10 mm and height h = 20 mm
- Material: iron steel, E = 210.000 MPa
- Length: L = 1 m
- Load: P = 10 kN
- Stiffness of the elastic member: k = 10000 N/m

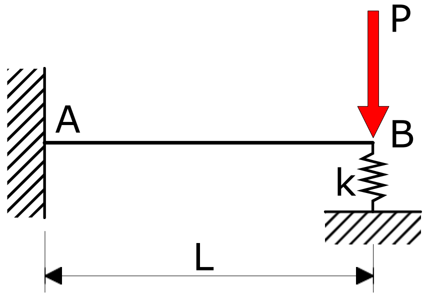
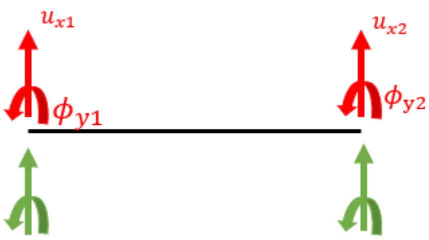


Figure 2

SOLUTION

Define the stiffness matrix and compute the reaction force in A and the deflection in B



$$\begin{Bmatrix} F_{x1} \\ M_{y1} \\ F_{x2} \\ M_{y2} \end{Bmatrix} = E\,\frac{I}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \phi_{y1} \\ u_{x2} \\ \phi_{y2} \end{Bmatrix}$$

Applying the boundary conditions: $u_{x1} = \ \phi_{y1} = 0$

1) $F_{x1} = \frac{EI}{l^3}(-12\,u_{x2} + 6l\,\phi_{y2})$

2) $M_{y1} = \frac{EI}{l^3}\left(6l\,u_{x2} + 2l^2\phi_{y2}\right)$

3) $F_{x2} = \frac{EI}{l^3}\left(12\,u_{x2} - 6l\,\phi_{y2}\right)$ $\qquad F_{x2} = -P - K\,u_B$

4) $M_{y2} = \frac{EI}{l^3}\left(-6l\,u_{x2} + 4\,l^2\phi_{y2}\right) = 0$ rotational DOF is allowed in B, while translational DOF is constrained

u_{x2} *and* ϕ_{y2} can be obtained by equations 3) and 4):

$$\begin{cases} -P - K\,u_{x2} = \dfrac{EI}{l^3}\left(12\,u_{x2} - 6l\,\phi_{y2}\right) \\ u_{x2} = \dfrac{4l^2\phi_{y2}}{6l} = \dfrac{2}{3}\,l\,\phi_{y2} \end{cases}$$

$$-P - K\,\frac{2}{3}\,l\,\phi_{y2} - \frac{EI}{l^3}\left(12\,\frac{2}{3}\,l\,\phi_{y2} - 6l\,\phi_{y2}\right) = 0$$

$$\phi_{y2}\left(K\frac{2}{3}l + E\frac{I}{l^2}\,8 - E\,I\frac{6}{l^2}\right) = -P \longrightarrow \phi_{y2} = -1.05\,rad$$

$$u_{x2} = \frac{2}{3}l\,\phi_{y2} - 704.43 + 10^{-3}m$$

EXERCISE 3

The structures in figures 3a, 3b, 3c similar from the geometrical point of view, have the same loads but different constraints. It is requested to solve the structures.

Data

Beam AC: base b = 15 mm; height h = 30 mm
Beam BC: base b = 10 mm; height h = 20 mm
Geometry: $\alpha = \beta = 45°$
Material: Iron Steel, E = 210.000 MPa
Length: L = 1 m
External load: P = 10000 N

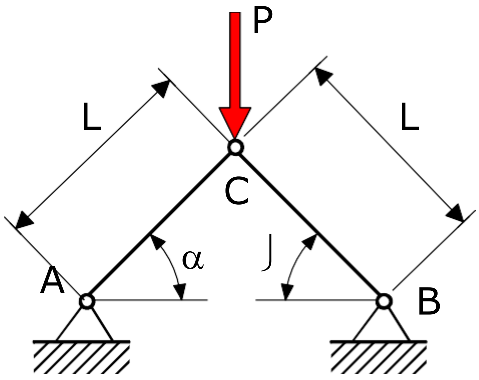


Figure 3a

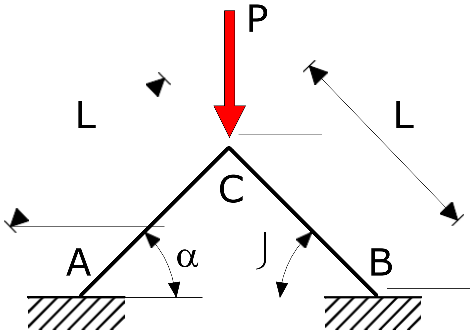


Figure 3c

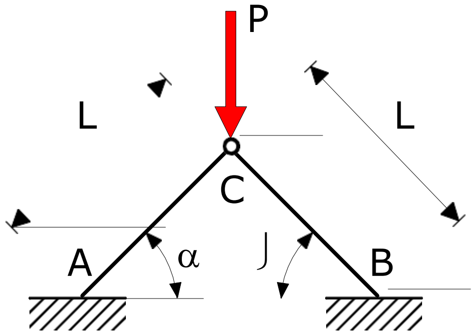


Figure 3b

SOLUTION in the dedicated pdf file

EXERCISE 4

Compute the reaction forces of the structure in figure 4.

Data

- Section: rectangular with base b = 10 mm and height h = 20 mm
- Material: iron steel, E = 210.000 MPa
- Length L= 1 m

External load q = 10000 N/m

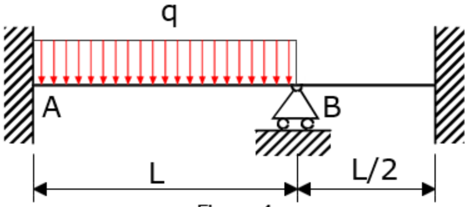
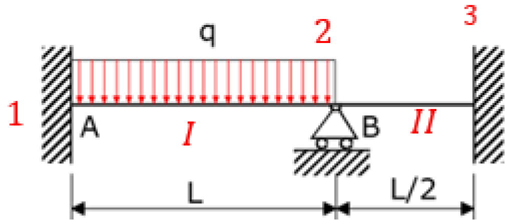
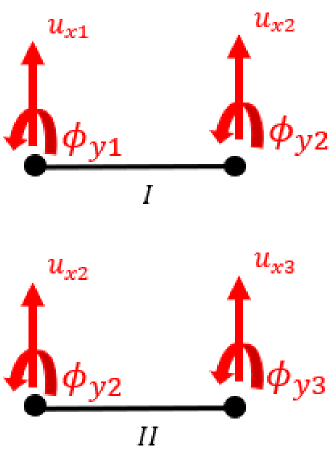


Figure 4

SOLUTION





The inertia is:

$$I = \frac{1}{12} b\, h^3 = 6.66 * 10^{-9}$$

I:

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \phi_{y1} \\ u_{x2} \\ \phi_{y2} \end{Bmatrix} \text{ where l = L}$$

II:

$$[K] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \begin{Bmatrix} u_{x2} \\ \phi_{y2} \\ u_{x3} \\ \phi_{y3} \end{Bmatrix} \text{ where l = L/2}$$

I

$$
\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & 6l & 2l^2 & 0 & 0 \\ -12 & -6l & 12+12 & -6l-6l & -12l & 6l \\ 6l & 2l^2 & -6l-6l & 4l^2+4l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \phi_{y1} \\ u_{x2} \\ \phi_{y2} \\ u_{x3} \\ \phi_{y3} \end{Bmatrix}
$$

II

$$
\{\boldsymbol{F}\} = \begin{Bmatrix} -q\frac{l}{2} \\ -\frac{ql^2}{12} \\ -q\frac{l}{2} \\ \frac{ql^2}{12} \\ 0 \\ 0 \end{Bmatrix} \rightarrow \begin{cases} -12\frac{EI}{l^3}\,u_{x1} = -\frac{ql}{2} \\ -6\frac{EI}{l^3}\,l\,\phi_{y1} = -\frac{ql^2}{12} \\ 24\frac{EI}{l^3}u_{x2} = -\frac{ql}{2} \\ \phi_{y2} = 0 \\ u_{x3} = \phi_{y3} = 0 \end{cases}
$$

EXERCISE 5

Compute the reaction forces and the vertical displacement in point C.
Data

- beam AC: area A_1 = 3000 mm^2,
- moment of inertia = 3.5 10^7 mm^4 ,
- beam BC: area A_2 = 615 mm^2
- Geometry: □ = 45°
- Material: iron steel, E = 210.000 MPa
- Length: L = 1 m
- External load: q = 570 N/mm

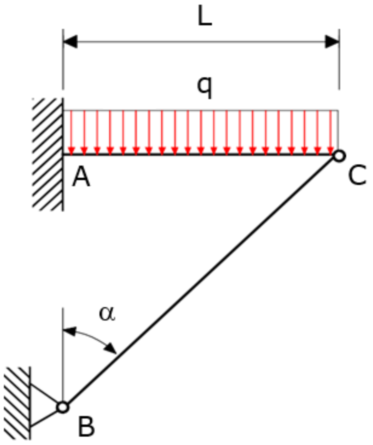
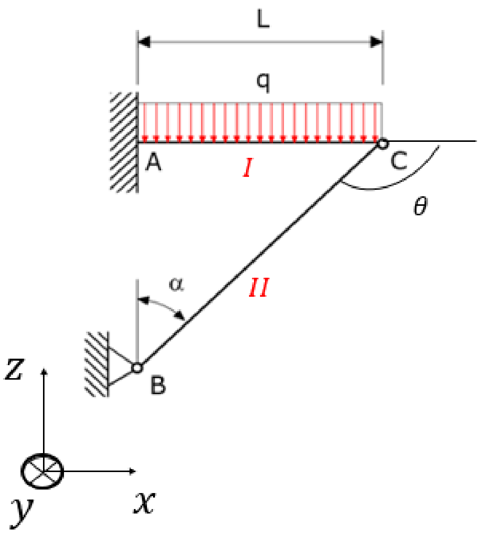
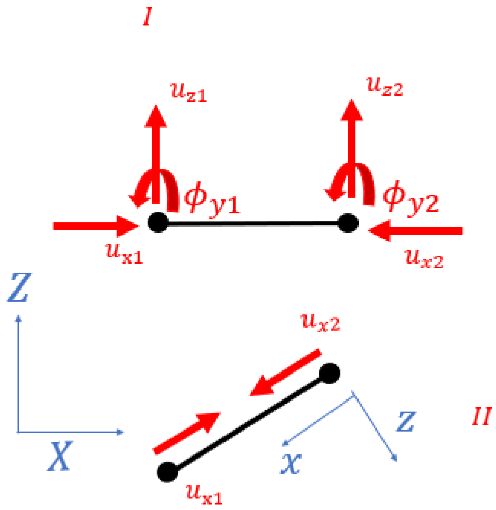


Figure 5

SOLUTION





The first element stiffness matrix is:

$$
[k] = E \begin{bmatrix} \frac{A}{L} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & 12\frac{I}{L^3} & \frac{6I}{L^2} & 0 & -12\frac{I}{L^3} & \frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & 4\frac{I}{L} & 0 & -\frac{6I}{L^2} & \frac{2I}{L} \\ \frac{A}{L} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & -12\frac{I}{L^3} & -\frac{6I}{L^2} & 0 & 12\frac{I}{L^3} & -\frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{2I}{L} & 0 & -\frac{6I}{L^2} & \frac{4I}{L} \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{z1} \\ \phi_{y1} \\ u_{x2} \\ u_{z2} \\ \phi_{y2} \end{Bmatrix}
$$

The second element is like a truss with particular orientation, so we just have the force along the axial direction then:

$$[k] = [R]^T[k][R] = \frac{EA}{L}\begin{bmatrix} cos^2\theta & cos\theta sin\theta & -cos^2\theta & -cos\theta sin\theta \\ cos\theta sin\theta & sin^2\theta & -cos\theta sin\theta & -sin^2\theta \\ -cos^2\theta & -cos\theta sin\theta & cos^2\theta & cos\theta sin\theta \\ -cos\theta sin\theta & -sin^2\theta & cos\theta sin\theta & sin^2\theta \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \end{Bmatrix}$$

So, the overall [K] will be:

$$\begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 & 0 & 0 \\ 0 & 12\frac{EI}{L^3} & 6\frac{EL}{L^2} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & 0 \\ 0 & 6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 0 \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L}+\frac{EA}{L}\cos^2\theta & \frac{EA}{L}\sin\theta\cos\theta & 0 & -\frac{EA}{L}\cos^2\theta & -\frac{EA}{L}\sin\theta\cos\theta \\ 0 & -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & \frac{EA}{L}\sin\theta\cos\theta & 12\frac{EI}{L^3}+\frac{EA}{L}\sin^2\theta & -6\frac{EI}{L^2} & -\frac{EA}{L}\sin\theta\cos\theta & -\frac{EA}{L}\sin^2\theta \\ 0 & 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & 0 \\ 0 & & & -\frac{EA}{L}\cos^2\theta & -\frac{EA}{L}\sin\theta\cos\theta & 0 & \frac{EA}{L}\cos^2\theta & \frac{EA}{L}\sin\theta\cos\theta \\ 0 & & & -\frac{EA}{L}\sin\theta\cos\theta & -\frac{EA}{L}\sin^2\theta & 0 & \frac{EA}{L}\sin\theta\cos\theta & \frac{EA}{L}\sin^2\theta \end{bmatrix}$$

$$[K]\begin{Bmatrix} u_{x1} \\ u_{z1} \\ \phi_{y1} \\ u_{x2} \\ u_{z2} \\ \phi_{y2} \\ u_{x3} \\ u_{z3} \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{z1} \\ M_{y1} \\ F_{x2} \\ F_{z2} \\ M_{y2} \\ F_{x3} \\ F_{z3} \end{Bmatrix}$$

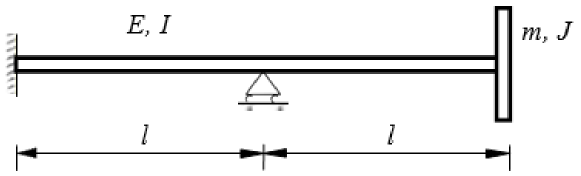
$$\{\boldsymbol{F}\} = \begin{Bmatrix} -q\frac{L}{2} \\ 0 \\ -\frac{qL^2}{12} \\ -q\frac{L}{2} \\ 0 \\ \frac{qL^2}{12} \\ 0 \\ 0 \end{Bmatrix}$$

EXERCISE 6

Compute the natural frequencies and the corresponding mode shapes of the systems presented in the Figure. It is characterized by a beam with negligible inertia and a rigid body with mass m and moment of inertia J.

Data: E = 2.0•10^5 N/mm^2, l = 0.1 m, I = 4.91•10^{-10} m^4, m = 5 kg, J = 0.05 kgm^2

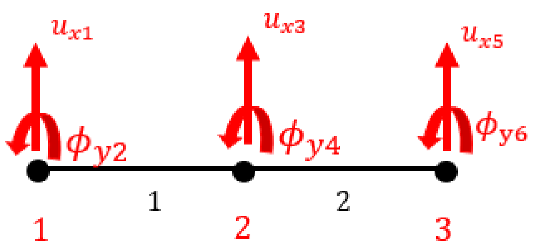
$$[K] = E\frac{I}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



ξ_1 = 106 rad/s, ξ_2 = 452 rad/s.

SOLUTION

Compute the natural frequencies and the corresponding mode shapes



DOF / ELEM	1	2	3	4
1	1	2	3	4
2	3	4	5	6

$$[K_{el}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

k_{11}^1	k_{12}^1	k_{13}^1	k_{14}^1	0	0
k_{21}^1	k_{22}^1	k_{23}^1	k_{24}^1	0	0
k_{31}^1	k_{32}^1	$k_{33}^1 + k_{11}^2$	$k_{34}^1 + k_{12}^2$	k_{13}^2	k_{14}^2
k_{41}^1	k_{42}^1	$k_{43}^1 + k_{21}^2$	$k_{44}^1 + k_{22}^2$	k_{23}^2	k_{24}^2
0	0	k_{31}^2	k_{32}^2	k_{33}^2	k_{34}^2
0	0	k_{41}^2	k_{42}^2	k_{43}^2	k_{44}^2

I cancel the first 3 rows and columns because $u_{x1} = \phi_{y2} = u_{x3} = 0$

The mass matrix is:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}\begin{Bmatrix} \phi_{y4} \\ u_{x5} \\ \phi_{y6} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\begin{Bmatrix} \phi_{y4} \\ u_{x5} \\ \phi_{y6} \end{Bmatrix} = 0$$

$$\det(-\omega^2[M] + [K]) = 0$$

$$\det\left(-\begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{bmatrix}\omega^2 + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}\right) = 0$$

$k_{11} = k'_{44} + k''_{22} = 3.92 * 10^3 + 3.92 * 10^3 = 7.856 * 10^3$

$k_{12} = k''_{23} = k_{21} = -58.92 * 10^3$

$k_{13} = k^2_{24} = k_{31} = 1.964 * 10^3$

$k_{22} = k_{33}$
$k_{33} = k_{44}$
$k_{23} = k_{34} = k_{32}$

Thus,

$$\det\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & -m\omega^2 + k_{22} & k_{23} \\ k_{31} & k_{32} & -J\,\omega^2 + k_{33} \end{bmatrix} =$$

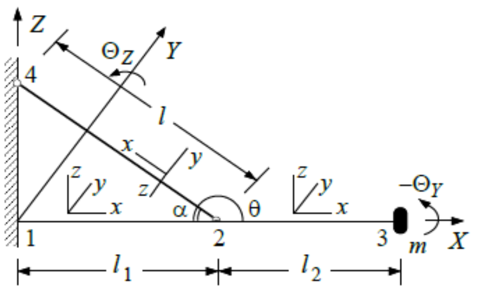
$$= k_1 \det\begin{bmatrix} -m\omega^2 + k_{22} & k_{23} \\ k_{23} & -J\omega^2 + k_{33} \end{bmatrix} - k_{12}\det\begin{bmatrix} k_{21} & k_{23} \\ k_{31} & -J\omega^2 + k_{33} \end{bmatrix} + k_{13} det\begin{bmatrix} k_{21} & -J\omega^2 + k_{22} \\ k_{31} & k_{32} \end{bmatrix}$$

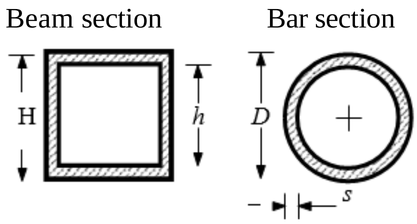
Solving the equation, the natural frequencies are:

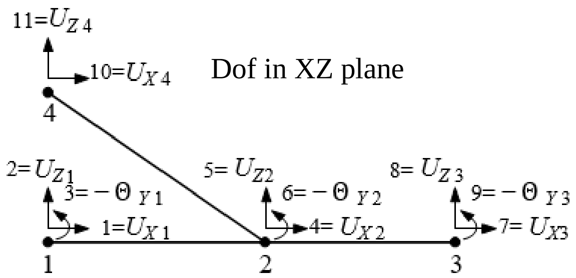
$$\begin{cases} \omega_1^2 = 1.6959 * 10^9 \\ \omega_2^2 = 26.627 * 10^8 \end{cases}$$

EXERCISE 7

Consider the dynamic system sketched in the figure. It consists of an arm (1-3) and a and a stiffening bar (2-4). Both are constrained on the left side. The arm (1-3) supports the mass *m* ant the right end. The material is aluminium alloy 2024 T6 (ASM 4112 standard).







Working scheme

Two beam elements (1-2 and 2-3) and one bar

Element (2-4), axial and flexural vibrations coupled

System's data

Parameter	Symbol	SI units	Value
Gravity acceleration	g	m/s^2	9.81
Instrument mass	m	kg	15
Young's elastic modulus	E	N/m^2	$7.2 \bullet 10^{10}$
Ultimate strength	R_m	N/m^2	$328 \bullet 10^{6}$
Elastic strength	$R_{p0.2}$	N/m^2	$216 \bullet 10^{6}$
Fatigue limit	σ_{D-1}	N/m^2	$115 \bullet 10^{6}$
Beam length (element 1–2)	l_1	m	0.300
Beam length (element 2–3)	l_2	m	0.123
Beams external height	H	m	0.100
Beams internal height	h	m	0.094
Bar length (element 2–4)	$l = l_1/\cos\alpha$	m	$l = l_1/\cos(20°)$
Bar orientation	Θ	degree	160
Bar external diameter	D	m	0.035
Bar radial thickness	s	m	0.001

Excitation

- motion of the constraints (nodes 1 and 4) with harmonic oscillation (simultaneous along *X*,*Y* and *Z* axes);
- acceleration of the constraints (nodes 1 and 4) along *Y* axis, due to a shock.

Harmonic oscillation simultaneous along *X* and *Z*	
Frequency Z_e (Hz)	Intensity
8.5 35 50	ampl. 10 mm acc. 3g acc. 1g

Table 1: typology of excitations

It is required to:

- Perform the dynamic analysis of the free behavior of the system;
- Perform the dynamic analysis of the forced behavior of the system (harmonic excitations as explained in table 1). It is suggested to plot the maximum displacement at the end of the arm.
- Make the resistance verification of the structure with respect of the MIL-STD 810C standard as summarized in table 2. It is suggested to plot the maximum stress in the frequency range.

Free response	Forced response	
	Harmonic oscillation simultaneous along X and Z	
Z_{n1}>max(Z_e)=50 Hz	max(U_{X3}, U_{Z3})≤3 mm $\sigma \leq R_m/1.575$ $\sigma \leq R_{p0.2}/1.155$ $\sigma \leq \sigma_{D-1}$	

Table 2: Verification (MIL-STD 810C)

APPENDIX

Geometrical characteristics of the structure

The characateristics of the 2 beams are the same, since they differ only for their length:

$$A_T = H^2 - h^2 = 0.1^2 - 0.094^2 = 1.164 * 10^{-3} m^2$$

$$I = \frac{H^4}{12} - \frac{h^4}{12} = \frac{1}{12}(0.1^4 - 0.094^4) = 1.8271 * 10^{-6} m^4$$

The characteristics of the bar are:

$$l = \frac{l_1}{cos\alpha} = \frac{0.3}{\cos(20)} = 3.1925 * 10^{-1}\, m$$

$$A_A = \frac{\pi}{4}(D^2 - (d-2s)^2) = \frac{\pi}{4}(0.035^2 - 0.033^2) = 1.0681 * 10^{-4} m^2$$

Elements' stiffness matrix

Beams 1-2 and 2-3 in the XZ plane

$$[K_{XZ}]_{1-2,2-3} = E\begin{bmatrix} \frac{A_T}{l_{1,2}} & 0 & 0 & -\frac{A_T}{l_{1,2}} & 0 & 0 \\ 0 & \frac{12I}{l_{1,2}^3} & 6\frac{I}{l_{1,2}^2} & 0 & -\frac{12I}{l_{1,2}^3} & 6\frac{I}{l_{1,2}^2} \\ 0 & 6\frac{I}{l_{1,2}^2} & 4\frac{I}{l_{1,2}} & 0 & -6\frac{I}{l_{1,2}^2} & 2\frac{I}{l_{1,2}} \\ -\frac{A_T}{l_{1,2}} & 0 & 0 & \frac{A_T}{l_{1,2}} & 0 & 0 \\ 0 & -\frac{12I}{l_{1,2}^3} & -6\frac{I}{l_{1,2}^2} & 0 & \frac{12I}{l_{1,2}^3} & -6\frac{I}{l_{1,2}^2} \\ 0 & 6\frac{I}{l_{1,2}^2} & 2\frac{I}{l_{1,2}} & 0 & -6\frac{I}{l_{1,2}^2} & \frac{4I}{l_{1,2}} \end{bmatrix}$$

Bar 2-4 in the XZ plane

$$[K_{XZ}]_{2-4} = \frac{EA}{l}\begin{bmatrix} \cos^2\theta & cos\theta\, sin\theta & -\cos^2\theta & -cos\theta\, sin\theta \\ cos\theta\, sin\theta & \sin^2\theta & -cos\theta\, sin\theta & -\sin^2\theta \\ -\cos^2\theta & -cos\theta\, sin\theta & \cos^2\theta & cos\theta\, sin\theta \\ -cos\theta\, sin\theta & -\sin^2\theta & cos\theta\, sin\theta & \sin^2\theta \end{bmatrix}$$

SOLUTION (complete solution in the dedicated folder)

$$\underbrace{\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[K]_{III}} \begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$\underbrace{\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{[K]_{III}\ (local\ ref)} \begin{Bmatrix} u_{x1} \\ u_{z1} \\ u_{x2} \\ u_{z2} \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ 0 \\ F_{x2} \\ 0 \end{Bmatrix}$$

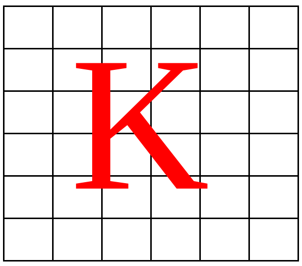
$$[k_{III}]_{global} = [R]^T[k_{III}][R]$$

Development of the map:

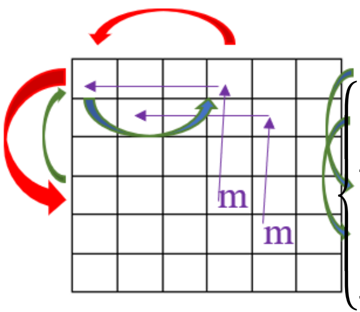
GL/L	u_{x1} 1	u_{z1} 2	$-\theta_{y1}$ 3	u_{x2} 4	u_{z2} 5	$-\theta_{y2}$ 6	u_{x3} 7	u_{z3} 8	$-\theta_{y3}$ 9	u_{x4} 10	u_{z4} 11
Elem I	1	2	3	4	5	6					
Elem II				1	2	3	4	5	6		
Elem III				1	2					3	4

$k_{II\,GL} = k_{11_I}$
$k_{23\,GL} = k_{23_I}$
$k_{34\,GL} = k_{34_I}$
$k_{44\,GL} = k_{44_I} + k_{11_{II}} + k_{11_{III}}$
$k_{45\,GL} = k_{45_I} + k_{12_I} + k_{12_{III}}$

$$\begin{bmatrix} m & & \\ & m & \\ & & \end{bmatrix}\begin{Bmatrix} u_{x3} \\ u_{z3} \\ -\theta_{y3} \end{Bmatrix}$$



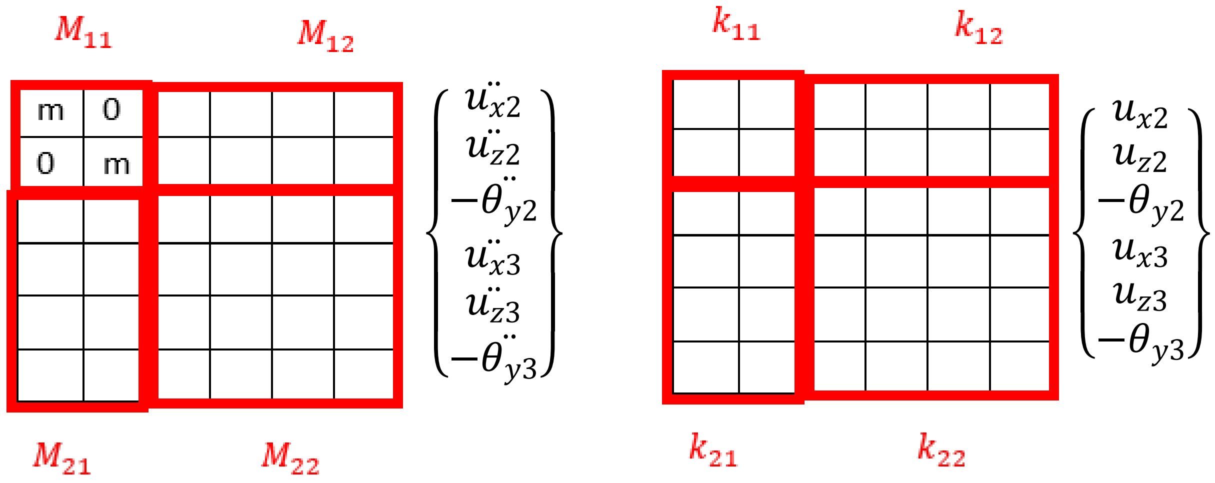
$$\begin{Bmatrix} u_{x2} \\ u_{z2} \\ -\theta_{y2} \\ u_{x3} \\ u_{z3} \\ -\theta_{y3} \end{Bmatrix}$$



$$\begin{Bmatrix} \ddot{u}_{x2} \\ \ddot{u}_{z2} \\ -\ddot{\theta}_{y2} \\ \ddot{u}_{x3} \\ \ddot{u}_{z3} \\ -\ddot{\theta}_{y3} \end{Bmatrix}$$

$$\{q\} = \begin{Bmatrix} u_{x2} \\ u_{z2} \\ -\theta_{y2} \\ u_{x3} \\ u_{z3} \\ -\theta_{y3} \end{Bmatrix}$$

$$\{q_m\} = \begin{Bmatrix} u_{x3} \\ u_{z3} \end{Bmatrix} \quad \{q_s\} = \begin{Bmatrix} -\theta_{y2} \\ u_{x2} \\ u_{z2} \\ -\theta_{y3} \end{Bmatrix} \quad \{q\} = \begin{Bmatrix} \{q_m\} \\ \{q_s\} \end{Bmatrix} = \begin{Bmatrix} u_{x2} \\ u_{z2} \\ -\theta_{y2} \\ u_{x3} \\ u_{z3} \\ -\theta_{y3} \end{Bmatrix}$$



$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_{x3} \\ \ddot{u}_{z3} \end{Bmatrix} + \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{Bmatrix} u_{x3} \\ u_{z3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[M_{11}]\{\ddot{q}_m\} + [k_{cond}]\{q_m\} = \{0\}$$

$$[k_{cond}] = \underbrace{[k_{11}]}_{(1x1)} - \underbrace{\underbrace{[k_{12}]}_{(2x4)} \underbrace{[k_{22}]^{-1}}_{(4x4)} \underbrace{[k_{21}]}_{(4x2)}}_{(2x2)}$$

$$\omega_1 \longrightarrow \underbrace{\{\phi_{m1}\}}_{(2x1)} \qquad \omega_2 \longrightarrow \underbrace{\{\phi_{m2}\}}_{(2x1)}$$

$$\begin{cases} \omega_{n1} = 673.10 \ (\frac{rad}{s}) \\ \omega_{n2} = 3814.60 \ (\frac{rad}{s}) \end{cases}$$

$$\{\phi_s\} = -[k_{22}]^{-1}[k_{21}]\{\phi_m\}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} q_m \\ q_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[k_{21}]\{q_m\} + [k_{22}]\{q_s\} = \{0\}$$

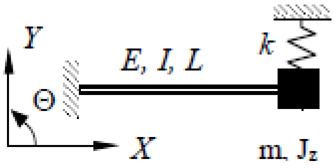
$$\{q_s\} = -[k_{22}]^{-1}[k_{21}]\{q_m\}$$

Then the corresponding mode shapes are :

$$\begin{Bmatrix} u_{x2} \\ u_{x2} \\ -\theta_{y2} \\ u_{x3} \\ u_{z3} \\ -\theta_{y3} \end{Bmatrix} = \begin{Bmatrix} 0.1754 \\ 0.0085 \\ -0.0677 \\ 0.2581 \\ -.0072 \\ -.01583 \end{Bmatrix} \qquad \begin{Bmatrix} u_{x2} \\ u_{x2} \\ -\theta_{y2} \\ u_{x3} \\ u_{z3} \\ -\theta_{y3} \end{Bmatrix} = \begin{Bmatrix} 0.0071 \\ 0.1453 \\ 0.8496 \\ 0.0072 \\ 0.2581 \\ 0.9505 \end{Bmatrix}$$

EXERCISE 8

Compute the first bending natural frequency of the system represented in the figure. The inertia contribution of the beam having a lenght L is negligible.
Data: (E = 2*10^5 MPa, L = 1000 mm, I_z = 7.5*10^5 mm^4, k = 100 N/mm, m = 20 [1+(U+2P)/30] kg, J_z=0.2
Kg*m^2).



$$[\boldsymbol{K}] = \begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

ξ_1 = ... Hz

SOLUTION

$$[k]=\begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 12\frac{EI}{L^3}+\boxed{k} & -6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}\begin{Bmatrix} u_{y1} \\ \varphi_{z1} \\ u_{y2} \\ \varphi_{z2} \end{Bmatrix}$$

Where point 1 is locked so we can neglect corresponding row and column.

$$[M]=\begin{bmatrix} m & 0 \\ 0 & J_z \end{bmatrix}\begin{Bmatrix} \ddot{u}_{y2} \\ \ddot{\varphi}_{z2} \end{Bmatrix}$$

$$[k]=\begin{bmatrix} 19 & -9 \\ -9 & 6 \end{bmatrix}10^5$$

U=P=0

$$[M]=\begin{bmatrix} 20 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\{u\} = \{u_0\}e^{i\omega t} \qquad \{\ddot{u}\} = -\omega^2\{u_0\}e^{i\omega t}$$
$$\{\varphi\} = \{\varphi_0\}e^{i\omega t} \qquad \{\ddot{\varphi}\} = -\omega^2\{\varphi_0\}e^{i\omega t}$$

$$det([k] - [M]\omega^2) = 0$$

$$\omega_{n1} = 164 \left({}^{rad}/_{s} \right)$$ [redacted]

$$\omega_{n2} = 1752 \left({}^{rad}/_{s} \right)$$ [redacted]