03OAIQW

Dynamic Design of Machines

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Mechanical Engineering

Nicola Amati (nicola.amati@polito.it)
Angelo Bonfitto (angelo.bonfitto@polito.it)
Salvatore Circosta (salvatore.circosta@polito.it)
Sara Luciani (sara.luciani@polito.it)

Tutorial 1 – Discrete linear systems

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Exercise 1

The undamped system presented in Figure 1 is characterized by two torsion bars with negligible inertia and two flywheels. The second flywheel is excited by a harmonic excitation that is equal to:

$M(t) = M_0 \sin(\omega t)$.

It is requested to study the dynamic behavior of the system computing:
a) the natural frequencies and the corresponding mode shapes;
b) the maximum shear stresses in the torsion bars using the configuration space and the state space approach.

Data: $G = 7.7 \cdot 10^{10}$ N/m^2, l_I = 0.5 m, l_{II} = 1.0 m, d_I = 12.8 mm, d_{II} = 14.1 mm, J_{p1} = 6 kgm^2, J_{p2} = 4 kgm^2, M_0 = 100 Nm, ω = 8 rad/s.

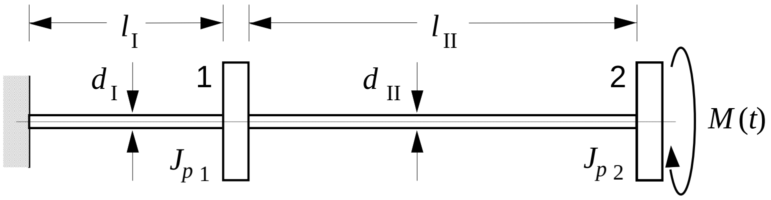

Figure 1. Exercise 1.

Solution

a) Computation of the natural frequencies and of the corresponding mode shapes

Given the torque equation:

$$M_t = K_p \theta$$

the torsional stiffness is equal to:

$$K_P = \frac{GJ_p}{l}$$

where J_p is the polar moment of inertia.

The polar moment of inertia and torsional stiffness of the two bars are:

$$\begin{cases} J_{P1} = \dfrac{\pi d_I^4}{32} = 2.63 \cdot 10^{-9} m^4 \\ J_{P2} = \dfrac{\pi d_{II}^4}{32} = 3.88 \cdot 10^{-9} m^4 \end{cases} \rightarrow \begin{cases} K_{P1} = \dfrac{GJ_{P1}}{l_I} = 405.02\ Nm/rad \\ K_{P2} = \dfrac{GJ_{P2}}{l_{II}} = 298.76 Nm/rad \end{cases}$$

The equations of motion are computed with the Lagrangian approach by computing the Lagrange equation $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_i$ with respect to the two degree of freedoms of the system $[\theta_1, \theta_2]$.

Considering that $\mathcal{L} = T - U$, where T is the kinetic energy and U is the potential energy, it is possible to compute:

The kinetic energy T

$$T = \frac{1}{2} J_{P1} \dot{\theta}_1^{\,2} + \frac{1}{2} J_{P2} \dot{\theta}_2^{\,2}$$

The potential energy U

$$U = \frac{1}{2} K_{P1} \theta_1^2 + \frac{1}{2} K_{P2} (\theta_2 - \theta_1)^2 = \frac{1}{2} K_{P1} \theta_1^2 + \frac{1}{2} K_{P2} \theta_1^2 + \frac{1}{2} K_{P2} \theta_2^2 - K_{P2} \theta_1 \theta_2$$

The virtual works

$$\frac{\partial \delta L}{\partial \delta x_i} \rightarrow \delta L = M(t) \delta \theta_2$$

Then, for each of the two degrees of freedom:

1) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = J_{P1} \ddot{\theta}_1$

$$\frac{\partial U}{\partial \theta_1} = K_{P1}\theta_1 + K_{P2}\theta_1 - K_{P2}\theta_2$$

$$\delta L = 0$$

2) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = J_{P2}\ddot{\theta}_2$

$$\frac{\partial U}{\partial \theta_2} = K_{P2}\theta_2 - K_{P2}\theta_1$$

$$\frac{\partial \delta L}{\partial \delta \theta_2} = M(t)$$

The equations of motion are:

$$\begin{cases} J_{P1}\ddot{\theta}_1 + (K_{P1} + K_{P2})\theta_1 - K_{P2}\theta_2 = 0 \\ J_{P2}\ddot{\theta}_2 - K_{P2}\theta_2 + K_{P2}\theta_1 = M(t) \end{cases}$$

In matrix form:

$$\begin{bmatrix} J_{P1} & 0 \\ 0 & J_{P2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} K_{P1} + K_{P2} & -K_{P2} \\ -K_{P2} & K_{P2} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M(t) \end{Bmatrix}$$

To compute the natural frequencies, the associated homogenous equation is considered:

$$[M]\{\ddot{\theta}\} + [K]\{\theta\} = \{0\}$$

Assuming a possible solution $\{\theta\} = \{\theta_0\}e^{i\lambda t}$ and substituting it in the previous equation:

$$(-\lambda^2[M] + [K])\{\theta_0\} = 0$$

Neglecting the trivial solution:

$$\det(-\lambda^2[M] + [K]) = 0$$

Then,

$$-\lambda^2[M] = \begin{bmatrix} -\lambda^2 J_{P1} & 0 \\ 0 & -\lambda^2 J_{P2} \end{bmatrix}$$

$$-\lambda^2[M] + [K] = \begin{bmatrix} -\lambda^2 J_{P1} + K_{P1} + K_{P2} & -K_{P2} \\ -K_{P2} & -\lambda^2 J_{P2} + K_{P2} \end{bmatrix}$$

$$\det(-\lambda^2[M] + [K]) = (-\lambda^2 J_{P2} + K_{P2})(-\lambda^2 J_{P1} + K_{P1} + K_{P2}) - K_{P2}^2$$

Imposing the determinant equal to zero the eigenvalues and then the natural frequencies are obtained:

$$\lambda_1^2 = 159.975 \rightarrow \lambda_1 = 12.65\ rad/s$$

$$\lambda_2^2 = 31.516 \rightarrow \lambda_2 = 5.614\ rad/s$$

To obtain the mode shapes corresponding to the natural frequencies, the eigenvectors have to be computed:

$$\left(-\lambda^2 \begin{bmatrix} J_{P1} & 0 \\ 0 & J_{P2} \end{bmatrix} + \begin{bmatrix} K_{P1} + K_{P2} & -K_{P2} \\ -K_{P2} & K_{P2} \end{bmatrix}\right) \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\lambda^2 J_{P1} + K_{P1} + K_{P2} & -K_{P2} \\ -K_{P2} & -\lambda^2 J_{P2} + K_{P2} \end{bmatrix} \begin{Bmatrix} \theta_{01} \\ \theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{cases} (-\lambda^2 J_{P1} + K_{P1} + K_{P2})\theta_{01} - K_{P2}\theta_{02} = 0 \\ -K_{P2}\theta_{01} + \left(-\lambda^2 J_{P2} + K_{p2}\right)\theta_{02} = 0 \end{cases}$$

Since the determinant $\det(-\lambda^2[M] + [K]) = 0$, the rank is less than the number of DoFs. To compute θ_{01} and θ_{02} one of the two equation is considered and, for this case, θ_{02} is imposed equal to 1.

$$\theta_{02} = 1 \rightarrow\ -K_{P2}\theta_{01} - \lambda^2 J_{P2} + K_{P2} = 0$$

$$\begin{cases} \lambda_1^2 = 159.975 \\ \lambda_2^2 = 31.516 \end{cases} \rightarrow \begin{cases} \theta_{01} = -1.14 \\ \theta_{02} = 0.578 \end{cases}$$

$$\{\theta_1\} = \begin{Bmatrix} -1.14 \\ 1 \end{Bmatrix}$$
$$\{\theta_2\} = \begin{Bmatrix} 0.58 \\ 1 \end{Bmatrix}$$

b) Computation of the maximum value of the shear stress

From the definition,

$$\tau_{max,j}(t) = \frac{k_j\left(\Phi_{z_{j+1}} - \Phi_{z_j}\right)}{W_j}$$

where Φ is the real deformation, and W_j is the torsional section module and is equal to:

$$W_j = \frac{J_P}{radius}$$

To compute the real deformation, the forced response is considered:

$$\begin{bmatrix} J_{P1} & 0 \\ 0 & J_{P2} \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} K_{P1}+K_{P2} & -K_{P2} \\ -K_{P2} & K_{P2} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M(t) \end{Bmatrix}$$

Since the forcing function is harmonic the solution is harmonic as well and it has the following form

$$\theta_1 = \Theta_1 \sin(\omega t)$$
$$\theta_2 = \Theta_2 \sin(\omega t)$$

where ω is the pulsation of the excitation.
Then,

$$\left(-\omega^2 \begin{bmatrix} J_{P1} & 0 \\ 0 & J_{P2} \end{bmatrix} + \begin{bmatrix} K_{P1}+K_{P2} & -K_{P2} \\ -K_{P2} & K_{P2} \end{bmatrix}\right) \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0 \end{Bmatrix}$$
$$\begin{cases} (-\omega^2 J_{P1} + K_{P1} + K_{P2})\Theta_1 - K_{P2}\Theta_2 = 0 \\ -K_{P2}\Theta_1 + (-\omega^2 J_{P2} + K_{P2})\Theta_2 = M_0 \end{cases}$$

The real deformations are computed as follows

$$\Theta_2 = \frac{(-\omega^2 J_{P1} + K_{P1} + K_{P2})\Theta_1}{K_{P2}} \rightarrow \Theta_2 = -423.08 \cdot 10^{-3}$$
$$-K_{P2}\Theta_1 + \frac{(-\omega^2 J_{P2} + K_{P2})(-\omega^2 J_{P1} + K_{P1} + K_{P2})}{K_{P2}}\Theta_1 = M_0 \rightarrow \Theta_1 = -395.27 \cdot 10^{-3}$$

Finally, the shear stresses are computed

$$\tau_1 = \frac{K_{P1}(-\Theta_1)}{J_{P1}} \cdot \frac{d_I}{2} = 389.59\ MPa$$
$$\tau_2 = \frac{K_{P2}(\Theta_2 - \Theta_1)}{J_{P2}} \cdot \frac{d_{II}}{2} = 15.096\ MPa$$

The same computation in the state space is obtained with the following representation:

$$\dot{z} = Az + Bu =$$

$$\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & -[M]^{-1}[K] \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \theta_1 \\ \theta_2 \end{Bmatrix} + \begin{bmatrix} -[M]^{-1} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ M(t) \end{Bmatrix}$$

and

$$y = Cz + Du =$$

$$\begin{Bmatrix}\tau_1\\ \tau_2\end{Bmatrix} = \begin{bmatrix} 0 & 0 & \dfrac{K_{P1}d_I}{2J_{P1}} & 0 \\ 0 & 0 & 0 & \dfrac{K_{P2d_{II}}}{2J_{P2}} \end{bmatrix} \begin{Bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \theta_1\\ \theta_2\end{Bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \begin{Bmatrix}0\\ 0\\ 0\\ M(t)\end{Bmatrix}$$

Exercise 2

The undamped system represented in Figure 2 is characterized by two bars having a diameter *d* (with negligible inertia) and two masses. A harmonic excitation $F(t) = F_0 \sin(\omega t)$ is applied on the mass 2.

It is requested to:

a) compute the amplitude of the oscillations by adopting the modal analysis,
b) compute the maximum stress in bar 1 and bar 2.
c) determine how the bar diameter should be modified to decrease the stress.

A design constraint imposes that $d_m \leq 1.25d$, where d_m is the modified diameter.

Data: l_1 = 0.4 m, l_2 = 0.2 m, d = 0.02 m, $E = 2.06 \cdot 10^{11}$ Pa, m_1 = 50 kg, m_2 = 20 kg, ω = 1700 rad/s, F_0 = 2000 N.

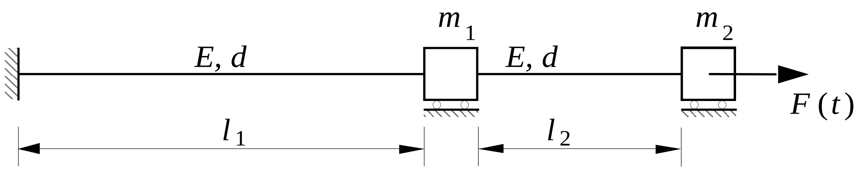

Figure 2. Exercise 2.

Solution

a) Computation of the amplitude of the oscillations by adopting the modal analysis

The stiffness of the bars is computed as:

$$K_1 = \frac{EA}{L_1} = \frac{E\left(\frac{\pi d_1^2}{4}\right)}{L_1} = 161.79 * 10^6 \frac{N}{m}$$

$$K_2 = \frac{EA}{L_2} = \frac{E\left(\frac{\pi d_2^2}{4}\right)}{L_2} = 323.58 * 10^6 \frac{N}{m}$$

To compute the oscillations, the Lagrange approach is adopted:

$$T = \frac{1}{2}\, m_1 (\dot{x_1})^2 + \frac{1}{2}\, m_2 (\dot{x_2})^2$$

$$U = \frac{1}{2}\, k_1 x_1^2 + \frac{1}{2}\, k_2 (x_2 - x_1)^2$$

$$\delta\, L = F\, \delta x_2$$

And applying $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_i, \mathcal{L} = T - U$, the equations of motion are obtained

$$1)\ m_1 \ddot{x_1} + k_1 x_1 + k_2 (x_2 - x_1)(-1) = 0$$

$$2)\ m_2\, \ddot{x_2} + k_2 (x_2 - x_1) = F(t)$$

In matrix form,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix}\ddot{x_1}\\ \ddot{x_2}\end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix}x_1\\ x_2\end{Bmatrix} = \begin{Bmatrix}0\\ F(t)\end{Bmatrix}$$

The dynamic matrix D of the configuration space:

$$D = M^{-1}K = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} = \begin{bmatrix} m_1(K_1 + K_2) & -m_1K_2 \\ -m_2K_2 & m_2K_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.02 & 0 \\ 0 & 0.05 \end{bmatrix}\begin{bmatrix} 4.8538 & -3.2358 \\ -3.2358 & 3.2358 \end{bmatrix} = \begin{bmatrix} 0.9708 & -0.06472 \\ -1.6179 & 1.6129 \end{bmatrix}$$

The natural frequencies are obtained as:

$$\det(D - \lambda^2 I) = 0 \rightarrow \det\left(\begin{bmatrix} m_1(K_1 + K_2) - \lambda^2 & -m_1K_2 \\ -m_2K_2 & m_2K_2 - \lambda^2 \end{bmatrix}\right) = 0$$

$$(m_2K_2 - \lambda^2)(m_1(K_1 + K_2) - \lambda^2) - m_1m_2K_1K_2 = 0$$

$$m_1m_2K_2(K_1 + K_2) - m_2K_2\lambda^2 - m_1(K_1 + K_2)\lambda^2 + \lambda^4 - m_1m_2K_1K_2 = 0$$

$$\lambda^4 - \lambda^2(m_1(K_1 + K_2) + m_2K_2) + m_1m_2K_2^2 = 0$$

Given the natural frequencies, the matrix Φ of the eigenvectors can be computed and, considering that:

$$x = \Phi\eta$$

then

$$\Phi^T M\Phi\ddot{\eta} + \Phi^T K\Phi\eta = \Phi^T F \rightarrow \bar{M}\ddot{\eta} + \bar{K}\eta = \bar{F}$$

Given the solution $\eta = \eta_0 e^{j\omega t}$:

$$(-\omega^2\bar{M} + \bar{K})\eta_0 = \bar{F}$$

with $\omega = 1700\frac{rad}{s}$.

Then,

$$\eta_{01} = -45.873 * 10^{-6}$$

$$\eta_{02} = -48.324 * 10^{-6}$$

b) Computation of the maximum stress in bar 1 and in bar 2

The bar has a uniform stress due to the normal load, then the stresses are computed as:

$$\sigma_1 = \frac{F}{A_1} = \frac{K_1x_{01}}{A_1} = 23.6MPa$$

$$\sigma_2 = \frac{F}{A_2} = \frac{K_2(x_{01} - x_{02})}{A_2} = 2.52\ MPa$$

c) Determination of how the bar diameter should be modified to decrease the stress. A design constraint imposes that $d_m \leq 1.25d$, where d_m is the modified diameter.

Some considerations on the dynamic behavior of the structure are needed. The natural frequencies are obtained from the eigenproblem

$$\det(-\omega^2[M] + [K]) = 0$$

and provide the results

$$\omega_1 = 1487\ \text{rad/s}$$

$$\omega_2 = 4866\ \text{rad/s}$$

Hence, the structure is working near the first natural frequency. This can be seen also from the frequency response in Figure 3:

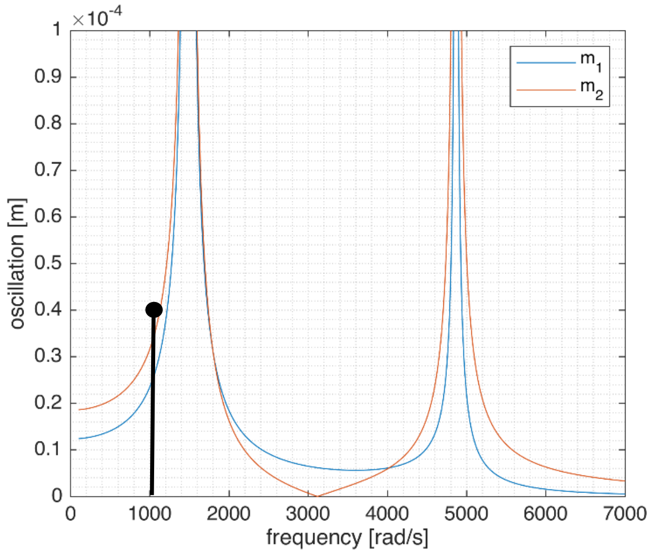

Figure 3. Frequency response of the system.

Since the designer can modify the bar diameter, the coefficient f_d is introduced so that

$$d_m = f_d d$$

The bars' stiffness is

$$K_{1m} = \frac{E(\frac{\pi d_m^2}{4})}{L_1} = f_d^2 K_1$$

$$K_{2m} = \frac{E(\frac{\pi d_m^2}{4})}{L_2} = f_d^2 K_2$$

By following the procedure adopted in a) and b) when coefficient f_d ranges from 0.1 to 2, the oscillation amplitude of the two masses is achieved

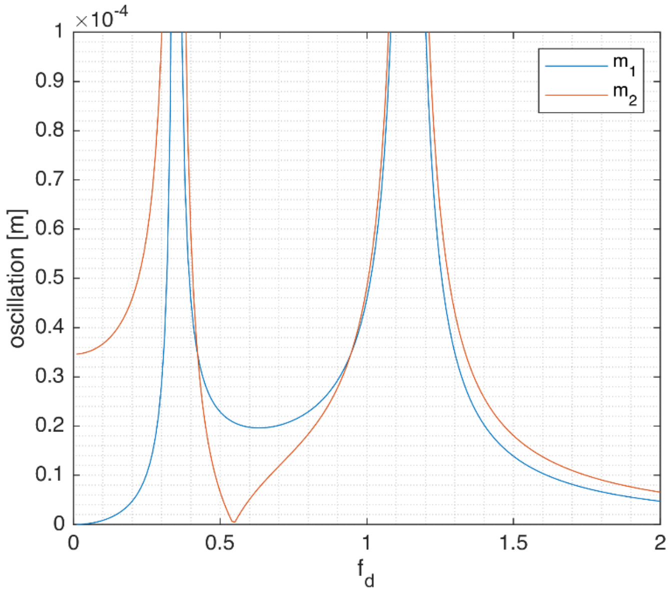

Figure 4

Note that $f_d = 1$ corresponds to the nominal diameter. A slightly larger diameter results in a stiffness increase that raises the first natural frequency. Consequently, the external force will excite the resonance thus yielding to very large displacement. By softening the structure, the first natural frequency decreases and goes far from the external force frequency. This results in a decrease of the amplitude of both masses. When the structure is further softened, the second natural frequency is decreased so that it is excited by the external force. Note that, as the coefficient f_d ideally tends to zero, the mass 1 becomes unperturbed since there would be no connection (the stiffness of bar 2 is null).

In the same way, the stress in the two bars can be computed:

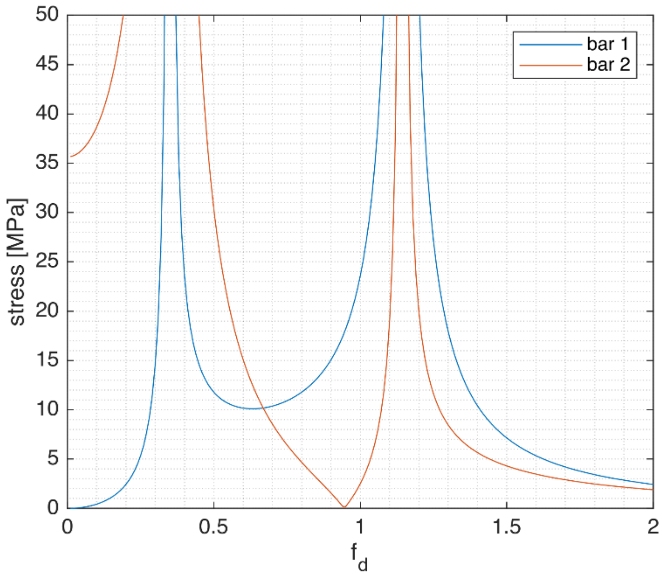

Figure 5

The stress is influenced by the oscillation amplitude and the bar cross section. Hence, the stress trend is slightly different from the oscillation amplitude plot.

The most convenient choice is the increase of the diameter in order to raise the first natural frequency above the external force frequency.

Nevertheless, the design constraint does not allow for this solution. Hence, the best trade of is to decrease the bar diameter by $f_d = 0.67$. A further diameter decrease would lead to very slight stress reduction in bar 1 at the cost of very large increase of the stress in bar 2. As result, the stress in bar 1 and bar 2 is 10 MPa.

Exercise 3

Consider the belt drive system represented in Figure 6.

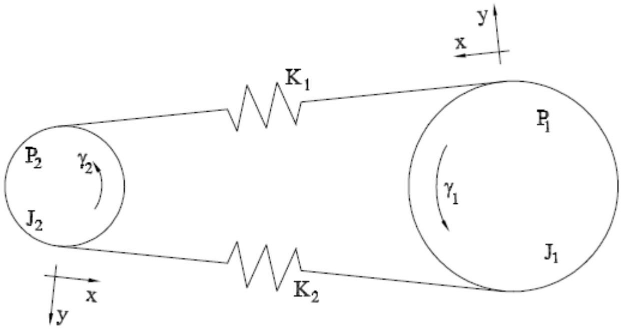

Figure 6. Exercise 3.

The Young modulus E of the belt in the axial direction is equal to $2.02 * 10^9$ N/m^2 while the transversal section area A is equal to $4.26 * 10^{-5}$ m^2. The main data are reported in Table 1.

Table 1. Data of Exercise 3.

$J_1 = 0.2$	Kgm^2
$J_2 = 0.15$	Kgm^2
$r_1 = 0.07$	m
$r_2 = 0.04$	m
$l = 0.5$	m

It is requested to compute the two natural frequencies of the systems.

Solution

To compute the equation of motion, the kinetic and potential energy are considered:

$$T = \frac{1}{2} J_1 \gamma_1^2 + \frac{1}{2} J_2 \gamma_2^2$$
$$U = \frac{1}{2} k_1 (\gamma_1 R_1 - \gamma_2 R_2)^2 + \frac{1}{2} k_2 (\gamma_2 R_2 - \gamma_1 R_1)^2$$

where

$$K_1 = \frac{EA}{L_1} = 172.104 * 10^3 \; Nm$$

The equations of motion are:

$$1)\; J_1 \ddot{\gamma_1} + k_1 R_1 (\gamma_1 R_1 - \gamma_2 R_2) - R_1 k_2 (\gamma_2 R_2 - \gamma_1 R_1) = 0$$
$$2)\; J_2 \ddot{\gamma_2} + k_2 R_2 (\gamma_2 R_2 - \gamma_1 R_1) - R_2 k_1 (\gamma_1 R_1 - \gamma_2 R_2) = 0$$

In matrix form,

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{Bmatrix} \ddot{\gamma_1} \\ \ddot{\gamma_2} \end{Bmatrix} + \begin{bmatrix} K_1 R_1^2 + K_2 R_1^2 & -K_1 R_1 R_2 - K_2 R_1 R_2 \\ -K_2 R_1 R_2 - K_1 R_2 R_1 & K_2 R_2^2 + K_1 R_2^2 \end{bmatrix} \begin{Bmatrix} \gamma_1 \\ \gamma_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$(-\omega^2 M + K) X_0 = 0$$

To compute the natural frequencies ω_n,

$$\det(-\omega_n^2 [M] + [K]) = 0$$

has to be solved:

$$det \begin{bmatrix} -\omega_n^2 J_1 + 2\, K\, R_1^2 & -2K\, R_1 R_2 \\ -2K\, R_1 R_2 & -\omega_n^2 J_2 + 2\, K\, R_2^2 \end{bmatrix} = 0$$

$$(-\omega_n^2 J_1 + 2\,K\,R_1^2)(-\omega_n^2 J_2 + 2\,K\,R_2^2) - 4K^2 R_1^2 R_2^2 = 0$$

$$\omega^4 J_1 J_2 - \omega^2\,J_1\,2\,K\,R_2^2 - \omega^2\,J_2\,2\,K\,R_1^2 + 4\,K\,R_1^2 R_2^2 - 4\,K\,R_1^2 R_2^2 = 0$$

$$\omega^2(\omega^2 J_1 J_2 - 2\,K\,J_1\,R_2^2 - 2\,K\,J_2\,R_2^2) = 0$$

The eigenvalues and the natural frequencies of the system are computed as:

$$\omega_1^2 = 0\,(rad/s)^2 \quad \omega_1 = 0\,Hz$$

$$\omega_2^2 = 12.1046 * 10^3\ \ (rad/s)^2 \quad \omega_2 = 17.5\,Hz$$

$$\omega_2 = 110.02\frac{rad}{s} \quad \omega_{2,Hz} = \frac{\omega_{2\,rad/s}}{2\,\pi}$$

Exercise 4

For the two degree of freedom system depicted in Figure 7

a) Determine the equations of dynamic equilibrium using the Langangian approach,
b) Arrange the equations in matrix form,
c) Find the natural frequencies and the mode shapes of the system,
d) Compute the modal matrices (modal mass and modal stiffness),
e) Mass-normalize the mode shapes,
f) Check the mass-orthogonality of the eigenvectors.

Make the calculation using the following two sets of data

$K_1 = K_2 = 400000$ N/m
$K_3 = 80000$ N/m
$m_1 = m_2 = 2$ kg
$K_1 = 400000$ N/m
$K_2 = 200000$ N/m
$K_3 = 80000$ N/m
$m_1 = 2$ kg
$m_2 = 2.5$ kg

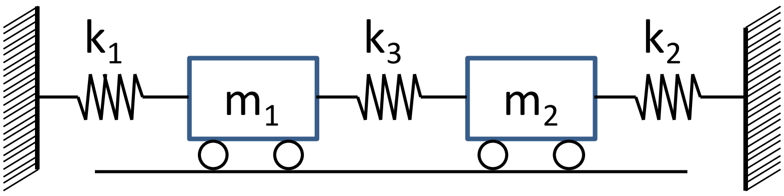

Figure 7. Exercise 4.

Solution

a) Determination of the equations of the dynamic equilibrium using the Lagrangian approach

To apply the Lagrangian approach, the kinetic and potential energy are computed:

$$T = \frac{1}{2}\, m_1 \dot{x_1}^2 + \frac{1}{2}\, m_2 \dot{x_2}^2$$
$$U = \frac{1}{2}\, k_1 x_1^2 + \frac{1}{2}\, k_3 (x_1 - x_2)^2 + \frac{1}{2}\, k_2 x_2^2$$

and then, applying $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x_i}}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_i, \mathcal{L} = T - U$ for the two DoFs (the motion related to the mass 1 and mass 2) the equations of motion are:

$$1)\; m_1 \ddot{x_1} + k_1 x_1 + k_3 (x_1 - x_2) = 0$$
$$2)\; m_2\, \ddot{x_2} + k_2 x_2 - k_3 (x_1 - x_2) = 0$$

b) Arrangement of the equation of motions in matrix form

Given the equations of motion, the matrix form is obtained as follows:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x_1} \\ \ddot{x_2} \end{Bmatrix} + \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

c) Find the natural frequencies and the mode shapes of the system

To compute the natural frequencies the associated homogenous equation is considered:

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

Assuming a possible solution $\{x\} = \{x_0\}e^{i\lambda t}$ and substituting it in the previous equation:

$$(-\omega^2[M] + [K])\{X_0\} = 0$$

The determinant is imposed equal to zero,

$$\det(-\omega_n^2[M] + [K]) = 0$$

$$det\begin{bmatrix} -\omega^2\, m_1 + k_1 + k_3 & -k_3 \\ -k_3 & -\omega^2 m_2 + k_2 + \ k_3 \end{bmatrix} = 0$$

$$(-\omega^2\, m_1 + k_1 + k_3)(-\omega^2 m_2 + k_2 + \ k_3) - k_3^2 = 0$$

$$\omega^4 m_1 m_2 - \ \omega^2 m_1 k_2 - \omega^2 m_1 k_3 - \omega^2 m_2 k_1 + k_1 k_2 + k_1 k_3 - \omega^2 m_2 k_3 + k_3 k_2 + \ k_3^2 - k_3^2 = 0$$

$$\omega^4 m_1 m_2 - \omega^2(m_1 k_2 + m_1 k_3 + m_2 k_1 + m_2 k_3) + k_1 k_2 + k_1 k_3 + k_3 k_2 = 0$$

Then,

$$\omega_1^2 = 1 * 10^6\ (rad/s)^2 \quad \omega_1 = 1000\ \ rad/s$$

$$\omega_2^2 = 200 * 10^3\ \ (rad/s)^2 \ \ \omega_2 = 447.213\ \ rad/s$$

To compute the eigenvectors:

$$\Theta_{01}(-\omega^2 m_1 + k_1 + k_3) - \Theta_{02} k_3 = 0$$

$$-\Theta_{01} k_3 + \Theta_{02}(-\omega^2 m_2 + k_2 + k_3) = 0$$

Since the system is not full rank (det = 0) the first DoF is chosen and I put $\Theta_{02} = 1$:

$$\omega = 1000, \qquad \Theta_{01} = -1$$

$$\omega = 447.213, \qquad \Theta_{01} = -1$$

The matrix of the eigenvectrors is then

$$\Phi = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

d) Computation of the modal matrices

The mass and stiffness modal matrices are obtained from the definition:

$$\overline{\mathrm{M}} = \Phi^{\mathrm{T}}\mathrm{M}\,\Phi = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\overline{\mathrm{K}} = \Phi^{\mathrm{T}}\mathrm{K}\,\Phi = \begin{bmatrix} 800000 & 0 \\ 0 & 400000 \end{bmatrix}$$

e) Mass-normalization of the mode shapes

The mass-normalization is obtained from the definition

$$\overline{\Theta_1} = \frac{\Theta_1}{\sqrt{\overline{M}_1}} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$\overline{\Theta_2} = \frac{\Theta_2}{\sqrt{\overline{\mathrm{M}}_2}} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

f) Checking of the mass-orthogonality of the eigenvectors

$$\Phi^{\mathrm{T}}\mathrm{M}\,\Phi = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 5

A dynamic system with three flywheels is depicted in Figure 5. Neglecting the inertia of the shafts, it is requested to

a) write the dynamic equations of equilibrium

b) determine natural frequencies and mode shapes of the system (torsional motion only)

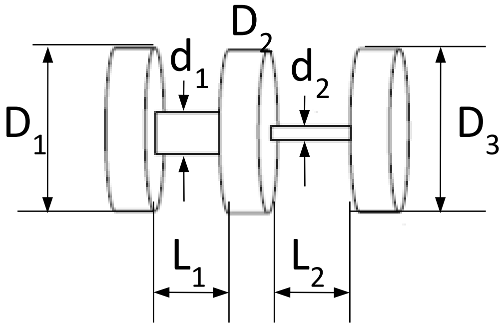

Figure 8. Exercise 5.

Data:

d_1 = 30 mm d_2 = 20 mm

L_1 = 100 mm L_2 = 80 mm

m_1 = 1 kg m_2 = 0.5 kg

m_3 = 2 kg

G = 7.7*10^{10} N/m^2

D_1 = 120 mm D_2 = 60 mm D_3 = 100 mm

Solution

a) Write the dynamic equations of equilibrium

The stiffness of the bar is:

$$J_{p1} = \frac{\pi d_1^2}{32} = 79.52 * 10^{-9} m^4$$

$$J_{p2} = \frac{\pi d_2^2}{32} = 15.708 * 10^{-9} m^4$$

$$J_{p3} = \frac{\pi d_3^2}{32} = \ m^4$$

Considering that $G = 7.7 * 10^{10}\ N/m^2$,

$$K_1 = G\, J_{p1}/L_1 = 61.23 * 10^3 N/m$$

$$K_2 = G\, J_{p2}/L_2 = 15.118 * 10^3 N/m$$

$$K_3 = G\, J_{p3}/L_3 = N/m$$

The kinetic energy is:

$$T = \frac{1}{2} J_1 \dot{\Theta}_1^2 + \frac{1}{2} J_2 \dot{\Theta}_2^2 + \frac{1}{2} J_3 \dot{\Theta}_3^2$$

Where,

$$J_1 = m_1 \frac{d_1^2}{4} = 225 * 10^{-6} kg\ m^2$$

$$J_2 = m_2 \frac{d_2^2}{4} = 50 * 10^{-6} kg\ m^2$$

$$J_3 = m_3 \frac{d_3^2}{4} = kg\ m^2$$

while the potential energy is:

$$U = \frac{1}{2}k_1(\Theta_2 - \Theta_1)^2 + \frac{1}{2}k_2(\Theta_3 - \Theta_2)^2$$

Then the equations of motion are obtained as:

1) $\mathrm{J}_1\ddot{\Theta}_1 - \mathrm{K}_1(\Theta_2 - \Theta_1) = 0$
2) $\mathrm{J}_2\ddot{\Theta}_2 + \mathrm{K}_1(\Theta_2 - \Theta_1) - \mathrm{K}_2(\Theta_3 - \Theta_2) = 0$
3) $\mathrm{J}_3\ddot{\theta}_3^2 + \mathrm{k}_2(\theta_3 - \theta_2) = 0$

In the matrix form:

$$\begin{bmatrix} j_1 & 0 & 0 \\ 0 & j_2 & 0 \\ 0 & 0 & j_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1^2 \\ \ddot{\theta}_2^2 \\ \ddot{\theta}_3^2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1 + k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \{0\}$$

Considering a solution of the type

$$\{\theta\} = \{\vartheta_0\}e^{i\omega t}$$
$$\{\ddot{\theta}\} = -\omega^2\{\theta_0\}e^{i\omega t}$$

The characteristic equation is derived:

$$(-[\boldsymbol{J}]\boldsymbol{\omega}^2 + [\boldsymbol{k}])\{\boldsymbol{\theta_0}\} = \{\mathbf{0}\}$$
$$det(-[\boldsymbol{J}]\boldsymbol{\omega}^2 + [\boldsymbol{k}]) = \{0\}$$

and then it is possible to compute the natural frequencies

$\omega_{n1} = \quad \left({}^{rad}/_{s}\right)$ $\qquad \omega_{n2} = \quad \left({}^{rad}/_{s}\right)$ $\qquad \omega_{n3} = \quad \left({}^{rad}/_{s}\right)$

Exercise 6

For the two degree of freedom depicted in the Figure 9

a) determine the equations of dynamic equilibrium (Lagrange approach).

b) Find natural frequencies and modal shapes of the system.

c) Use the eigenvectors to compute the modal matrices.

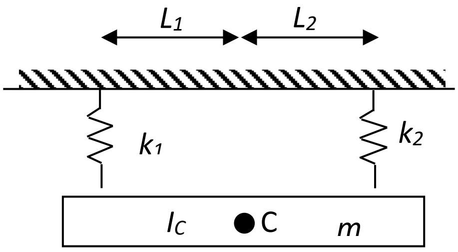

Figure 9. Exercise 6.

Data

$k_1 = k_2 = 20000$ N/m

m = 950 kg IC = 1400 kg m^2

$L_1 = 1.0$ m $L_2 = 1.5$ m

Solution

a) Determination of the equations of dynamic equilibrium (Lagrange approach)

The kinetic and potential energy are:

$$T = \frac{1}{2} I_c \dot{\Theta}^2 + \frac{1}{2} m\, \dot{x}^2$$

$$U = \frac{1}{2} k (x + L_1 A)^2 + \frac{1}{2} k (x - L_2 \Theta)^2$$

Therefore, applying $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) - \frac{\partial \mathcal{L}}{\partial x} = Q_i, \mathcal{L} = T - U$, the two equations of motion are obtained as follows

1) $I_c \ddot{\Theta} + K\, L_1 (x + L_1 \Theta) - K L_2 (x - L_2 \Theta) = 0$

2) $m\ddot{x} + K\, (x + L_1 \Theta) + k(x - L_2 \Theta) = 0$

In matrix form,

$$\begin{bmatrix} I_c & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\Theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} K\, L_1^2 + K\, L_2^2 & K(L_1 - L_2) \\ K(L_1 - L_2) & 2\,K \end{bmatrix} \begin{Bmatrix} x \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

b) Computation of the natural frequencies and modal shapes of the system

To compute the natural frequencies, the associated homogenous equation has be considered and then

$$\det(-\omega^2 [M] + [K]) = 0$$

then,

$$\det \begin{bmatrix} -\omega^2 I_c + \ K\, L_1^2 + K\, L_2^2 & K(L_1 - L_2) \\ K(L_1 - L_2) & -\omega^2 m + 2\,K \end{bmatrix} = 0$$

$$(-\omega^2 I_c + \ K\, L_1^2 + K\, L_2^2)(-\omega^2 m + 2\,K) - K^2 (L_1 - L_2)^2 = 0$$

$$\omega^4 I_c m - \omega^2 (2\,K\, I_c + m\, K\, (L_1^2 + L_2^2)) + 2\,K^2 L_1 L_2 + K^2 (L_1^2 + L_2^2) = 0$$

and the eigenvalues and natural frequencies are

$\omega_1^2 = 53.203 \qquad \omega_1 = 7.294$

$\omega_2^2 = 35.33 \qquad \omega_2 = 5.943$

To compute the modal shape:

$$\begin{bmatrix} -\omega^2 I_c + K\,L_1^2 + K\,L_2^2 & K(L_1 - L_2) \\ K(L_1 - L_2) & -\omega^2 m + 2\,K \end{bmatrix} \begin{Bmatrix} \Theta_{01} \\ \Theta_{02} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Considering the second equation,

$$\Theta_{01} K(L_1 - L_2) - \Theta_{02}(\omega^2 m + 2K) = 0$$

and imposing $\Theta_{02} = 1$:

$\omega = 7.294 \qquad \Theta_{01=} 0.96$

$\omega = 5.943 \qquad \Theta_{01=} - 1.55$

Therefore, the matrix of the eigenvectors is

$$\Phi = \begin{bmatrix} 0.96 & -1.55 \\ 1 & 1 \end{bmatrix}$$

c) Computation of the modal matrices

By definition, the modal mass matrix is obtained as: $\overline{M} = \Phi^T\,M\,\Phi$. The modal mass corresponding to the two modes are:

$M_1 = \theta_{01}^T M \theta_{01} = 3693.4$

$M_2 = \theta_{02}^T M \theta_{02} = 2254.6.$

Mass-normalizing the eigenvectors:

$$\theta_{01norm} = \frac{\theta_{01}}{\sqrt{M_1}} = \begin{bmatrix} -0.0256 \\ 0.0165 \end{bmatrix}$$

$$\theta_{02norm} = \frac{\theta_{02}}{\sqrt{M_2}} = \begin{bmatrix} 0.02 \\ 0.024 \end{bmatrix}$$

where the matrix of the normalized eigenvectors is:

$$\Phi_{norm} = \begin{bmatrix} -0.0256 & 0.02 \\ 0.0165 & 0.0211 \end{bmatrix}$$

Finally, the modal matrices are:

$$MM = \Phi_{norm}^T M \Phi_{norm} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$KM = \Phi_{norm}^T K \Phi_{norm} = \begin{bmatrix} 35.33 & 0 \\ 0 & 52.2 \end{bmatrix}$$

Exercise 7 U = 0 and P = 1

The output to a harmonic excitation of a vibrating structure is described only by the first mode:

$\{q\}_1 = \{1/\sqrt{5} \quad 2/\sqrt{5}\}^T$. Take into account that the modal mass is 2 kg, the modal stiffness is 5000 N/m, the modal force is equal to 10+(2U+P)/3 N. The excitation frequency is equal to $0.8\omega_1$, and the damping is negligible. Compute the amplitude of oscillation of the first degree of freedom.

U = last digit of your registration n°

P = second last digit of your registration n°

Solution

By definition

$$\{x\} = \Phi\{\eta\} \qquad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \Theta_{01} & \Theta_{11} \\ \Theta_{02} & \Theta_{12} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}$$

In the modal reference frame

$$[M]\{\ddot{\eta}\} + [K]\{\eta\} = \{F\}$$

Then,

$$\frac{\eta_1}{F} = \frac{1}{-\omega_2 m + K} \qquad \frac{x_{01}}{\Theta_{01}} = \frac{F}{-\omega_2 m + K}$$

By using U = 0 and P = 1

$$x_{01} = \Theta_{01}\frac{F}{-\omega_2 m + K} = \Theta_{01}\frac{F}{K(1 - \omega^2/\omega_n^2)} = 0.00257\, m$$

Exercise 8   U= 1 and P = 0,

A two degrees of freedom undamped system has the following modal shapes:

$\{q\}_1 = \{1 \quad 0.8165\}^T$, $\{q\}_2 = \{1 \quad -0.8165\}^T$.

The stiffness matrix, the mass matrix and the vector of the excitation forces are:

$$[K] = 10^3 \begin{bmatrix} 10 & -8 \\ -8 & 15 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{m}}, \quad [M] = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \mathrm{kg}, \quad \{F\} = 10^3 (1 + \frac{2U+P}{20}) \begin{Bmatrix} 5 \\ 8 \end{Bmatrix} \mathrm{N}$$

Compute the modal force corresponding to the first mode.

U= last digit of your registration n°

P= digit before the last of your registration n°

Solution

The modal force is

$$\bar{F} = \Phi^T F$$

Using U= 1 and P = 0,

$$\begin{bmatrix} 1 & 0.8165 \\ 1 & -0.8165 \end{bmatrix} * 20^3 \begin{Bmatrix} 5 \\ 8 \end{Bmatrix} = \begin{Bmatrix} 11.532 \\ -1.532 \end{Bmatrix} * 20^3$$

Exercise 9

Consider a vibrating system. The dynamic response is characterized using only the first mode $\{q\}_1 = \{1/\sqrt{7} \;\; 2/\sqrt{7}\}^T$. The modal mass is equal to 3 kg, the modal stiffness is equal to 3000 N/m, the modal force is equal to 10+(2*U*+*P*)/3 N. The excitation frequency is equal to $0.8 \cdot \omega_1$, the damping is negligible. Compute the amplitude of oscillation of the degree of freedom n° 1.

U= last digit of your registration n°
P= digit before the last of your registration n°

Solution

$$\frac{\eta_1}{F} = \frac{1}{K\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

$$x_{01} = q_{11}\frac{F}{K\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

$$x_{01} = 2.033 * 10^{-3} m$$

Exercise 10

Figure 10 shows the so-called quarter car model, one of the simplest models used to study the dynamic behavior of motor vehicle suspensions. The upper mass m5 simulates the part of the mass of the car body (the sprung mass) that can be considered supported by a given wheel, while the lower mass m2 simulates the wheel and all the parts that can be considered as rigidly connected with the unsprung mass. The two masses are connected by a spring-damper system simulating the suspension (k2,c2) and the silent block (elastometer, k3.1, k3.2,c3). The unsprung mass is connected to the ground with a second spring simulating the radial stiffness of the tire. The point at which the tire contacts the ground is assumed to move in a vertical direction with a given law y1(t), and it simulates the motion on uneven ground.

For the given quarter car model, it is requested to:

a) Determine the equations of dynamic equilibrium using the Lagrangian approach,
b) Arrange the equations in matrix form,
c) Find the natural frequencies and the mode shapes of the system,
d) Considering as input a harmonic excitation with amplitude y1 of 3 mm @ 5 Hz, compute the power dissipated in the elastomeric component.

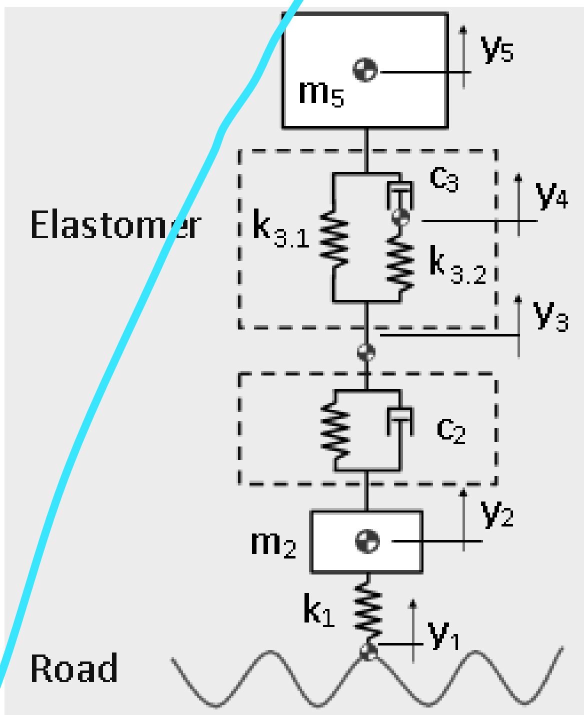

Figure 10. Exercise 10.

Data

Sprung mass	m5 = 400 kg;
Unsprung mass	m2 = 30 kg;
Spring stiffness	k2 = 24 kN/m
Spring damping	c2 = 1200 Ns/m;
Elastomer stiffness	k3.1 = 150 kN/m;
Elastomer stiffness	k3.2 = 90 kN/m;
Elastomer damping	c3 = 800 Ns/m;
Tire stiffness	k1 = 190 kN/m;

Solution

To compute the equation of motion using the Lagrangian approach it is necessary to determine:

The kinetic energy

$$T = \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2} m_5 \dot{y}_5^2$$

The potential energy

$$U = \frac{1}{2}k_1(y_2 - y_1)^2 + \frac{1}{2}k_2(y_3 - y_2)^2 + \frac{1}{2}k_{3.1}(y_5 - y_3)^2 + \frac{1}{2}k_{23.2}(y_4 - y_3)^2$$

The dissipative forces

$$\mathcal{F} = \frac{1}{2}C_2(\dot{y}_3 - \dot{y}_2)^2 + \frac{1}{2}C_3(\dot{y}_5 - \dot{y}_4)^2$$

The Lagrange equations related to the 4 DoFs are obtained from:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = \frac{\partial(\delta L)}{\partial(\delta q_i)}$$

for each degree of freedom:

1) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_2}\right) = m_2\ddot{y}_2^2$

$$\frac{\partial U}{\partial y_2} = k_1(y_2 - y_1) - k_2(y_3 - y_2)$$

$$\frac{\partial \mathcal{F}}{\partial \dot{y}_2} = -C_2(\dot{y}_3 - \dot{y}_2)$$

$$\frac{\partial(\delta L)}{\partial(\delta y_2)} = 0$$

The first equation is :

$$m_2\ddot{y}_2^2 + C_2\dot{y}_2 - C_2\dot{y}_3 + y_2(k_1 + k_2) - k_2y_3 = k_1y_1$$

2) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_3}\right) = 0$

$$\frac{\partial U}{\partial y_3} = k_2(y_3 - y_2) - k_{3.1}(y_5 - y_3) - k_{3.2}(y_4 - y_3)$$

$$\frac{\partial \mathcal{F}}{\partial \dot{y}_3} = C_2(\dot{y}_3 - \dot{y}_2)$$

$$\frac{\partial(\delta L)}{\partial(\delta y_3)} = 0$$

The second equation is:

$$-C_2\dot{y}_2 + C_2\dot{y}_3 - k_2y_2 + y_3(k_{3.1} + k_{3.2} + k_2) - k_{3.2}y_4 - k_{3.1}y_5 = 0$$

3) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_4}\right) = 0$

$$\frac{\partial U}{\partial y_4} = k_{3.2}(y_4 - y_3)$$

$$\frac{\partial \mathcal{F}}{\partial \dot{y}_4} = -C_3(\dot{y}_5 - \dot{y}_4)$$

$$\frac{\partial(\delta L)}{\partial(\delta y_4)} = 0$$

The third equation is:

$$C_3\dot{y}_4 - C_3\dot{y}_5 - k_{3.2}y_3 + k_{3.2}y_4 = 0$$

4) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_5}\right) = m_5\ddot{y}_5^2$

$$\frac{\partial U}{\partial y_5} = k_{3.1}(y_5 - y_3)$$

$$\frac{\partial \mathcal{F}}{\partial \dot{y}_5} = C_3(\dot{y}_5 - \dot{y}_4)$$

$$\frac{\partial(\delta L)}{\partial(\delta y_5)} = 0$$

The fourth equation is:

$$m_5\ddot{y}_5^2 - C_3\dot{y}_4 + C_3\dot{y}_5 - k_{3.1}y_3 + k_{3.1}y_5 = 0$$

In matrix form:

$$\begin{bmatrix} m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_5 \end{bmatrix} \begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \end{Bmatrix} + \begin{bmatrix} C_2 & -C_2 & 0 & 0 \\ -C_2 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & -C_3 \\ 0 & 0 & -C_3 & C_3 \end{bmatrix} \begin{Bmatrix} \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \end{Bmatrix} +$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_{3.1} + k_{3.2} & -k_{3.2} & -k_{3.1} \\ 0 & -k_{3.2} & k_{3.2} & 0 \\ 0 & -k_{3.1} & 0 & k_{3.1} \end{bmatrix} \begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} k_1 y_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The computation of the natural frequencies is obtained considering the following form of the solution:

$$\{y_i\} = \{y_0\}e^{i\omega t}$$
$$\{\dot{y}_i\} = i\omega\{y_0\}e^{i\omega t}$$
$$\{\ddot{y}_i\} = -\omega^2\{y_0\}e^{i\omega t}$$

Then,

$$\left([k] - [m]\omega_{ni}^2\right)\{y_0\} = \{0\}$$
$$\det\left([k] - [m]\omega_{ni}^2\right)\{y_0\} = \{0\}$$
$$\omega_{n1} = 0, \qquad \omega_{n2} = 0.2159, \qquad \omega_{n3} = 2.5611, \qquad \omega_{n4} = inf$$

The mode shapes are:

First mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

Second mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} -0.989 \\ 0.8757 \\ 0.8757 \\ -1 \end{Bmatrix}$$

Third mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.1315 \\ 0.1315 \\ -0.0074 \end{Bmatrix}$$

Fourth mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix}$$

Multiple choice problems

1) The Lagrange equations can be written
   [A] only for undamped systems,
   [B] only for damped and undamped systems subject to vibration
   [C] also for non conservative systems.

2) For a damped multi degree of freedom system it is not possible to obtain a complete modal decoupling. This sentence is
   [A] Always correct
   [B] Never correct
   [C] Not correct only in some cases.

3) An external excitation applied to a dynamic system is considered quasi static if
   [A] Its frequency is much lower with respect to the lowest natural frequency of the dynamic system.
   [B] Its frequency is equal or lower with respect to the lowest natural frequency of the dynamic system.
   [C] Its frequency is equal or higher with respect to the lowest natural frequency of the dynamic system.

4) A gyroscopic matrix present in the equations of motion of a mechanical system
   [A] Is symmetric
   [B] Is skew symmetric and may never cause instability.
   [C] Is skew symmetric and may cause instability.

5) The solution of the homogeneous equation describes
   [A] the behavior of the dynamic system that is free to move;
   [B] the behavior of the dynamic system after a step input;
   [C] the behavior of the dynamic system excited by a harmonic inputs.

Answers

1 [C], 2 [C], 3 [A], 4 [B], 5 [A]