03OAIQW

Dynamic Design of Machines

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Mechanical Engineering

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Tutorial 2 – Reduction tehcniques

Jefcott rotor

11/12/2020

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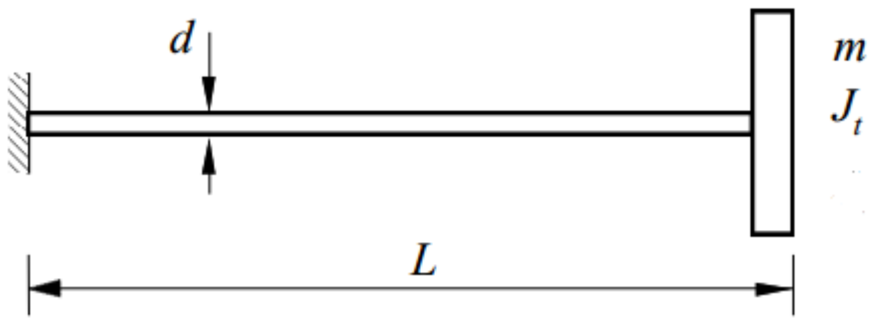
Exercise 1

Compute the first bending natural frequency of the structure shown in the figure, which is composed by a cantilever shaft (whose mass can be neglected) having a flywheel of mass m and inertia Jt attached to its end.
It is requested to compare the results obtained studying the full system with those obtained using the Guyan reduction.



Data:
$E = 1011$ N/m^2,
$\nu = 0.3$,
$L = 1$ m,
$d = 0.03$ m,
$m_I = 10$ kg,
$m_{II} = 100$ kg
$J_t = 1$ kgm^2,

The stiffness matrix of the beam element (due to the slenderness of the shaft we refer to the Euler formulation) is:

$$[K] = E\frac{I}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \varphi_{y1} \\ u_{x2} \\ \varphi_{x2} \end{Bmatrix}$$

μ_{1I}=43.87 rad/s, μ'_{1I}=44.13 rad/s (Guyan), μ_{1II}=15.273 rad/s, μ'_{1II}=15.274 rad/s (Guyan)

Solution

1) Compute the first bending natural frequency of the structure

The beam element with the corresponding degrees of freedom is considered

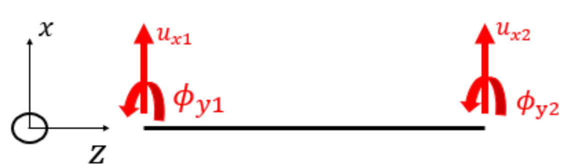


Figure 1. Exercise 1.

Since the Node 1 is locked , u_{x1}, φ_{y1} are equal to zero and the corresponding rows and column in the stiffness matrix K are neglected.

$$[K] = E\frac{I}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \varphi_{y1} \\ u_{x2} \\ \varphi_{x2} \end{Bmatrix}$$

The solution of the problem is given by:

$$[M]\begin{Bmatrix} \ddot{u}_{x2} \\ \ddot{\varphi}_{y2} \end{Bmatrix} + [K]\begin{Bmatrix} u_{x2} \\ \varphi_{y2} \end{Bmatrix} = \{0\}$$

where

$$I = \frac{\pi d^4}{64} = 39.76 * 10^{-9}\ [m]$$

Therefore, the stiffness matrix is equal to:

$$[K] = E\frac{I}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} = \begin{bmatrix} 95376 & -47688 \\ -47688 & 31792 \end{bmatrix}$$

and the mass matrix is:

$$[M] = \begin{bmatrix} m & 0 \\ 0 & J_t \end{bmatrix}$$

The general solution is of the following type:

$$\{u_x\} = u_{x0}e^{i\omega t}\ \ ,\{\dot{u_x}\} = i\omega u_{x0}e^{i\omega t},\ \ \{\ddot{u_x}\} = -\omega^2 u_{x0}e^{i\omega t}$$

$$\{\varphi_y\} = \varphi_{y0}e^{i\omega t}\ \ ,\{\dot{u_x}\} = i\omega \varphi_{y0}e^{i\omega t},\ \ \{\ddot{u_x}\} = -\omega^2 \varphi_{y0}e^{i\omega t}$$

the characteristic equation is obtained:

$$([K] - [M]\omega^2)\begin{Bmatrix} u_{x0} \\ \varphi_{y0} \end{Bmatrix}e^{i\omega t} = \{0\}$$

The first bending natural frequency can be obtained solving the previous equation:

$$det([K] - [M]\omega^2) = \{0\}$$

$$det\left(\begin{bmatrix} 95376 & -47688 \\ -47688 & 31792 \end{bmatrix} - \begin{bmatrix} m\omega^2 & 0 \\ 0 & J_t\omega^2 \end{bmatrix}\right) = \{0\}$$

$$(-m\omega^2 + 95376)(-J\omega^2 + 31792) - (47688)^2 = 0$$

If m = 10 [kg],

$$\omega_n = 43.76 \text{ (rad/s)}$$

If m = 100 [kg],

$$\omega_n = 15.273 \text{ (rad/s)}$$

2) Compare the results obtained studying the full system with those obtained using the Guyan reduction.

The Guyan reduction is an approximated method and we define master and slave degree to decrease the order of our matrixes.

$u_{x2}: master$

$\varphi_{y2}: slave$

$$[M]_{cond}\{\ddot{q}_m\} + [K]_{cond}\{q_m\} = \{0\}$$

$$[K]_{cond} = K_{11} - K_{12}[K_{22}]^{-1}[K_{21}]$$

$$[M]_{cond} = M_{11} - M_{12}[K_{22}]^{-1}[K_{21}] - ([M_{12}][K_{22}]^{-1}[K_{21}])^T + K_{12}[K_{22}]^{-1}[M_{21}][K_{22}]^{-1}[K_{21}]$$

K_{11} K_{12}

$$[K] = E\frac{I}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}$$

K_{21} K_{22}

M_{11} M_{12}

$$[M] = \begin{bmatrix} m & 0 \\ 0 & J_t \end{bmatrix}$$

M_{21} M_{22}

$$[K]_{cond} = 12\frac{EI}{L^3} + 6\frac{EI}{L^2}\frac{L}{4EI}\left(-6\frac{EI}{L^2}\right) = 3\frac{EI}{L^3} = 2.385 * 10^4\ [{}^{N}/_{m}]$$

$$[M]_{cond} = m + \left(\frac{[K_{12}]}{[K_{22}]}\right)(J_t)[K_{22}][k_{21}]$$

If m = 10 [kg],

$$[M]_{cond} = 12.25\ [Kg]\ so\ \omega_{n1} = \sqrt{\frac{23.85 * 10^3}{12.25}} = 44.129\left(\frac{rad}{s}\right)$$

Which is higher than exact value.

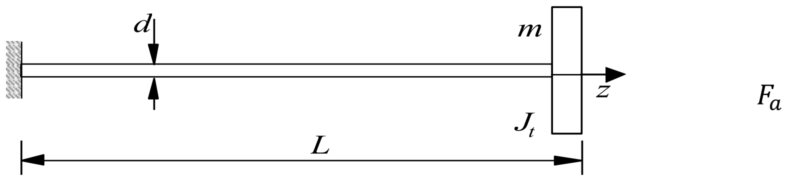
If m = 100 [kg],

$$[M]_{cond} = 102.23\ [Kg]\ so\ \omega_n = \sqrt{\frac{23.85 * 10^3}{102.23}} = 15.274(rad/s)$$

here by approximation, the same natural frequency is obtained.

Exercise 2

The rotor shown in the figure is composed by a flywheel of mass *m* and moments of inertia *Jt* and *Jp* attached to the end of a shaft with negligible mass.



Assuming to orientate the structure vertically and to consider the effect of gravity, it requested to calculate the critical speed in the following three cases:

a) *Z* axis coincident with respect to the gravitational field (mass below the constraint)

b) *Z* axis non coinciding with gravitational field (mass above the constraint)

c) Absence of gravity (orientation is indifferent)

Data:

$E = 2 \times 10^{11}$ N/m^2,
$L = 0.3$ m,
$d = 2.5$ mm,
$m = 0.5$ kg,
$J_t = 0.0\ 1$ kgm^2
$J_p = 0.0\ 2$ kgm^2

The stiffness matrix for the beam element (considering the slenderness, the definition is referred to the Euler formulation) and geometrical matrix.

$$[K] = E\frac{I}{L^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$[K_g] = \frac{F_a}{30L}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix}$$

Results:

ν_a=13.9 rad/s , ν_b=9.1 rad/s, ν_c=11.8 rad/s.

Solution

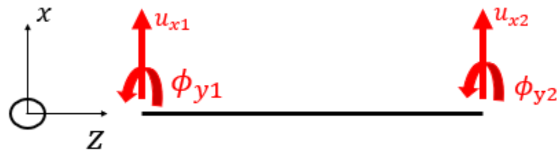


Figure 2. Exercise 2.

$$[M]\{\ddot{q}\} - i\Omega[G]\{\dot{q}\} + [K]\{q\} = \{0\}$$

where:

$$[M] = \begin{bmatrix} m & 0 \\ 0 & J_t \end{bmatrix}$$

$$[G] = \begin{bmatrix} 0 & 0 \\ 0 & J_p \end{bmatrix}$$

This additional term is because of coupling between axial and bending force.

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} \underbrace{\pm \frac{F_a}{30L}\begin{bmatrix} 36 & -3L \\ -3L & 4L^2 \end{bmatrix}}_{k_g}$$

$\boxed{+}$ When Z axis and gravitational field are coincident. (the mass is below the constrain)

$\boxed{-}$ When Z axis and gravitational field are not coincident. (the mass is above the constrain)

By having $J_p > J_t$ there is coupling in two different plane and by solving the equations in complex coordinate, the gyroscopic matrix becomes symmetric so:

$$\begin{cases} J_p = J_t & Disk\ rotor \\ J_p < J_t & long(slender)rotor \\ J_p < J_t & there\ is\ coupling\ effect \end{cases}$$

The complex coordinates are:

$$\{q\} = \begin{Bmatrix} z \\ \phi \end{Bmatrix}$$

$$z = u_x + iu_y$$

$$\phi = \phi_y - i\phi_x$$

and the solution has the following form:

$$\{Z\} = z_0 e^{i\omega t} \quad ,\{\dot{z}\} = i\omega z_0 e^{i\omega t}, \quad \{\ddot{z}\} = -\omega^2 z_0 e^{i\omega t}$$

$$\{\phi\} = \phi_0 e^{i\omega t} \quad ,\{\dot{z}\} = i\omega \phi_0 e^{i\omega t}, \quad \{\ddot{z}\} = -\omega^2 \phi_0 e^{i\omega t}$$

By substituting the previous expression to the general equation we reach:

$$(-\omega^2[M] + \omega\Omega[G] + [K])\begin{Bmatrix} z_0 \\ \phi_0 \end{Bmatrix} e^{i\omega t} = \{0\}$$

Thus,

$$det(-\omega^2[M] + \omega\Omega[G] + [K]) = 0 \quad at\ critical\ speed\ \Omega_{rotation} = \Omega_{whirl}\ so:$$

$$det\left(\Omega_{cr}{}^2([G] - [M]) + [K]\right) = \{0\}$$

where

$$[M] = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$[G] = \begin{bmatrix} 0 & 0 \\ 0 & 0.02 \end{bmatrix}$$

Considering,

$$I = \frac{\pi d^4}{64} = 1.92 * 10^{-12}[m^4]$$

$$[K] = \begin{bmatrix} 170.67 & -25.60 \\ -25.60 & 5.12 \end{bmatrix} \pm \begin{bmatrix} 19.62 & -0.50 \\ -0.50 & 0.20 \end{bmatrix}$$

1) Calculate the critical speed when both axis are coincide

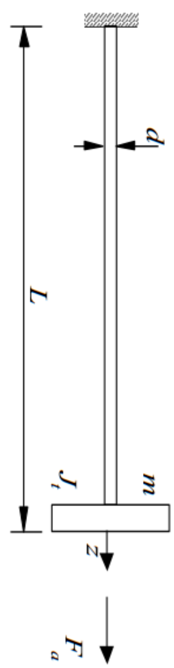


Figure 3. Exercise 2 - Configuration A

Considering the stiffness matrix:

Axes are coincident

$$[K] = \begin{bmatrix} 170.67 & -25.60 \\ -25.60 & 5.12 \end{bmatrix} + \begin{bmatrix} 19.62 & -0.50 \\ -0.50 & 0.20 \end{bmatrix} = \begin{bmatrix} 190.29 & -26.10 \\ -26.10 & 5.32 \end{bmatrix}$$

Solving the equation:

$$det\left(-\Omega_{cr}{}^{2}\begin{bmatrix} m & 0 \\ 0 & J_t - J_p \end{bmatrix}) + \begin{bmatrix} 190.29 & -26.10 \\ -26.10 & 5.32 \end{bmatrix}\right) = \{0\}$$

$$-0.005\omega^4 - 0.7571\omega^2 + 331.13 = 0$$

$$\omega_n = 13.9(rad/s)$$

2) Calculate the critical speed when axes are in opposite direction:

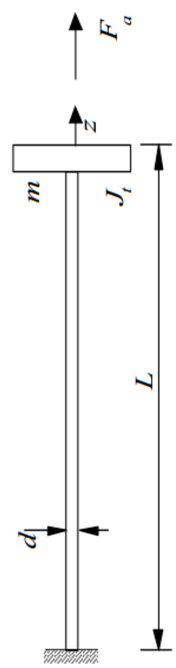


Figure 4. Exercise 2 - Configuration B

The stiffness matrix is:

$$[K] = \begin{bmatrix} 170.67 & -25.60 \\ -25.60 & 5.12 \end{bmatrix} - \begin{bmatrix} 19.62 & -0.50 \\ -0.50 & 0.20 \end{bmatrix} = \begin{bmatrix} 151.05 & -25.10 \\ -25.10 & 4.92 \end{bmatrix}$$

Solving the equation:

Axes are in opposite direction

$$det\left(\begin{bmatrix} 190.29 - m\Omega_{cr}{}^{2} & -26.10 \\ -26.10 & 5.32 - (J_t - J_p)\Omega_{cr}{}^{2} \end{bmatrix}\right) = \{0\}$$

$$-0.005\omega^4 - 0.9495\omega^2 + 113.156 = 0$$

$\omega_n = 9.1(rad/s)$

3) Calculate the critical speed in the absence of gravity

In this last case, there is no additional k_g term and, so [K] is equal to:

$$[K] = \frac{EI}{L^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 170.67 & -25.60 \\ -25.60 & 5.12 \end{bmatrix}$$

Solving the equation:

$$-0.005\omega^4 - 0.8533\omega^2 + 218.32 = 0 \rightarrow \omega_n = 11.8(rad/s)$$

Generally, by coupling axial and bending movement (coincide or opposite axes), the natural frequency of the system increases when axes are coincident since shaft the shaft is more rigid because the effect of coupling is only on the stiffness matrix. On the other hand, when axes are in opposite direction, the stiffness of the system becomes minimum a lower natural frequency is obtained.

Exercise 3

A rotor with mass m can be modeled as a Jeffcott rotor having a deformable shaft on stiff supports with distance L between the supports. The shaft has a full circular section area with stiffness $k = 48EI/L^3 (I = \pi d^4/64)$. Given that the mass is fixed to the rotor with eccentricity ϵ and that at rated speed the radius of the orbit is equal to z_0=-1.2ϵ, calculate:

a) The critical speed ν_{cr};
b) The diameter of the shaft d;
c) The maximum bending stress on the shaft and the reactions on the supports at rated speed ν.

Data:
$E = 2*10^{11}$ N/m^2,
L = 0.5 m,
m = 15 kg,
ϵ = 100 μm,
ν = 10000 rpm.

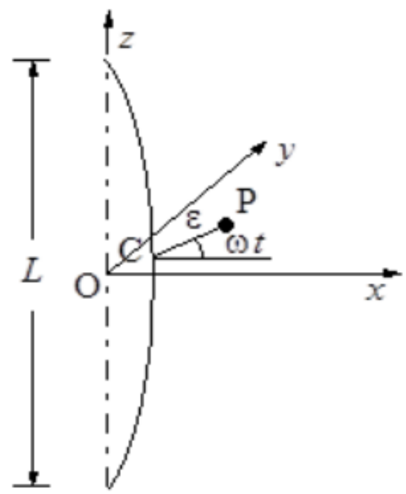


Figure 5. Exercise 3.

Solution

1) Calculate the critical speed ν_{cr}

Starting with the homogeneous equation:

$$[M]\{\ddot{Z}\} + [K]\{Z\} = \{0\}$$

$$\{Z\} = z_0 e^{i\omega t} \quad , \{\dot{z}\} = i\omega z_0 e^{i\omega t} , \quad \{\ddot{z}\} = -\omega^2 z_0 e^{i\omega t}$$

$$([K] - [M]\omega^2) z_0 e^{i\omega t} = \{0\}$$

$$\omega_n = \sqrt{\frac{K}{M}}$$

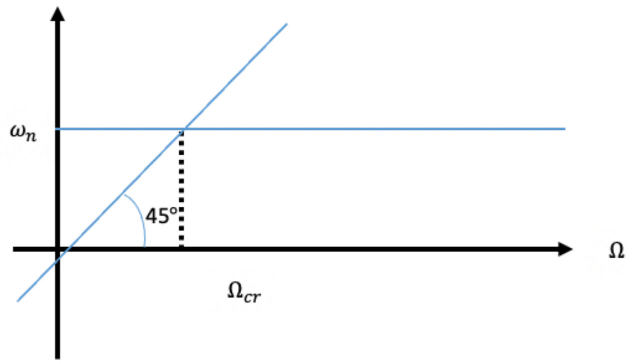


Figure 6. Exercise 3 - Campbell diagram of a Jeffcott rotor.

The cross section of these 2 lines is the Ω_{cr} which means $\Omega_{rotation} = \Omega_{whirl}$ so $\Omega_{cr} = \sqrt{\frac{K}{M}}$ (just in terms of the value.)

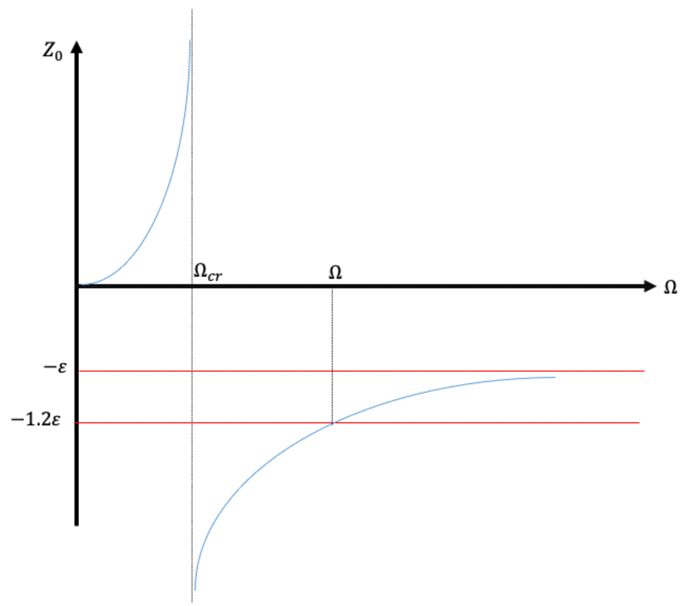


Figure 7. Exercise 3. Amplitude of the orbit as function of the spin speed.ù

Now by having the eccentricity, since the effect of eccentricity is shown as an external force the equation will be:

$$[M]\{\ddot{Z}\} + [K]\{Z\} = m\varepsilon\Omega^2 e^{i\Omega t}$$

$$\{Z\} = z_0 e^{i\Omega t} \quad ,\{\dot{z}\} = i\Omega z_0 e^{i\Omega t}, \quad \{\ddot{z}\} = -\Omega^2 z_0 e^{i\Omega t}$$

$$([K] - [M]\Omega^2) z_0 e^{i\Omega t} = m\varepsilon\Omega^2 e^{i\Omega t}$$

So the radius of orbit, z_0, can be defined as :

$$z_0 = \frac{m\varepsilon\Omega^2}{K - m\Omega^2} = -1.2\varepsilon \qquad ,$$

$$\Omega = 10000\, rpm * \frac{2\pi}{60} = 1047 (rad/s)$$

By subdividing to m

$$-1.2 = \frac{\Omega^2}{\Omega_{cr}^2 - \Omega^2} \qquad , -1.2\Omega_{cr}^2 + 1.2\Omega^2 = \Omega^2 \,, 1.2\Omega_{cr}^2 = 0.2\Omega^2$$

We are in the super critical region $z_0 = -1.2\varepsilon$ so we expected $\Omega > \Omega_{cr}$

$$\Omega_{cr} = \sqrt{\frac{0.2}{1.2}\Omega^2} = 427(rad/s)$$

2) Calculate the diameter of the shaft d

$$\Omega_{cr} = \sqrt{\frac{K}{M}} \quad , \quad K = \frac{48EI}{L^3} \quad , \quad I = \frac{\pi d^4}{64}$$

$$\Omega_{cr} = \sqrt{\frac{48E\pi d^4}{64L^3 m}}$$

Therefore,

$$d = \left(\frac{4\Omega_{cr}^2 l^3 m}{3\pi E}\right)^{\frac{1}{4}} = 29\ [mm]$$

3) Calculate the maximum bending stress on the shaft and the reactions on the supports at rated speed $\boldsymbol{\nu}$.

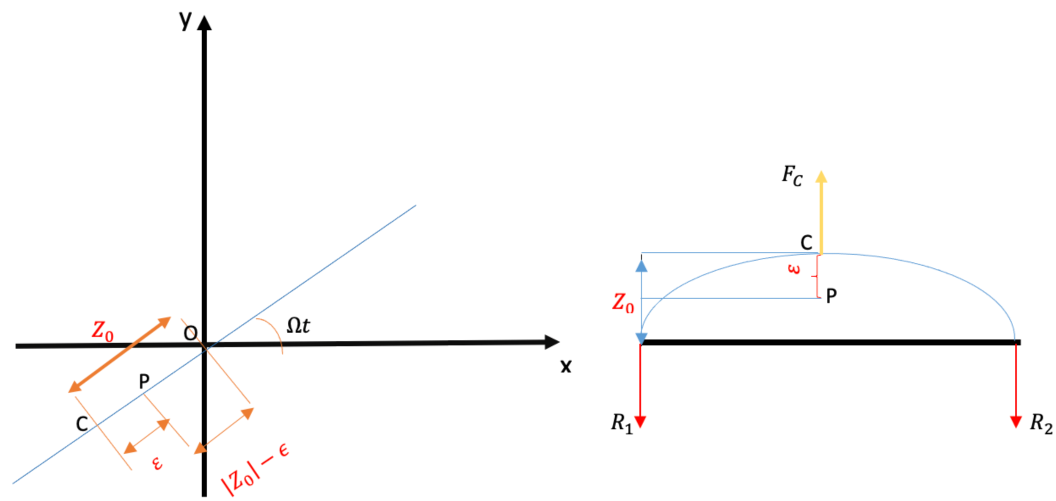


Figure 8. Exercise 3 - Computaation of the reaction forces on the supports.

Now the rotor instead of whirling around point C (center of geometry) is whirling around P (center of mass) so the reaction force at the supports are :

$r =$ the distance between the center of rotation (P) and the axis of two support (point O)

$$R_1 + R_2 = F_C = mr\Omega^2 = m(|Z_0| - \varepsilon)\Omega^2 = 0.2\varepsilon m\Omega^2 = 329[N]$$

Alternatively, we can write the reaction forces are equal to:

$$R_1 + R_2 = KZ_0$$

$$R_1 = R_2 = \frac{F}{2}$$

where $K = 48\frac{EI}{{L_i}^3}$ is the stiffness of the shaft by considering shaft as two parallel spring where the F is applied at their connection point.

So, in this equation $L_i = \frac{L}{2}$, the bending moment of the shaft is

$$M_b = R\frac{L}{2}$$

So the maximum bending stress can be calculated

$$\sigma_b = \frac{M_b}{I} * \frac{d}{2} = \frac{R_1 {}^{L}/_{2}}{\frac{\pi d^4}{64}} * \frac{d}{2} = \frac{16R_1 L}{\pi d^3} = 17[MPa]$$

whenever $\Omega_{spining} = \Omega_{whirling}$, no fatigue is on the rotor and everything is on the stator.

Exercise 4

A rotor with mass m can be modeled as a Jeffcott rotor having a deformable shaft on stiff supports with distance L between the supports.

The shaft has a full circular shaft with stiffness $k = 48EI/L^3 (I = \pi d^4/64)$.

The mass is fixed to the rotor with eccentricity ϵ, and that the rotating damping cr is null.

It is requested to calculate:

a) The critical speed ν_{cr}

b) The nonrotating damping cn necessary to keep the radius of the orbit z_0 below 1.25 ϵ.

c) The radius of the orbit z_0 when crossing the critical speed assuming the value of cn that has been calculated in the previous question.

Data:

E = 2x10^{11} N/m^2,
L = 0.5 m,
d = 0.029 m,
m = 15 kg,
ϵ = 10 μm,
ν = 8000 turns/min.

Results: ν_{cr}= 427 rad/s, c_n = 3.61*10^3 Ns/m, z_0 = 17.5 μm.

Solution

1) Calculate the critical speed ν_{cr}

$$\Omega_{cr} = \sqrt{\frac{K}{m}} = \sqrt{\frac{48EI}{mL^3}} = \sqrt{177.759 * 10^3} = 421.61(rad/s)$$

$$I = \frac{\pi d^4}{64} = 34.7184 * 10^{-9}[m^4]$$

$$K = \frac{48EI}{L^3} = 2.66 * 10^6 \ [{}^N/_m]$$

2) Calculate the nonrotating damping cn necessary to keep the radius of the orbit $\boldsymbol{z_0}$ below 1.25E

$$[M]\{\ddot{Z}\} + [C]\{\dot{Z}\} + [K]\{Z\} = m\varepsilon\Omega^2 e^{i\Omega t}$$

$$\{Z\} = z_0 e^{i\Omega t} \quad ,\{\dot{z}\} = i\Omega z_0 e^{i\Omega t}, \quad \{\ddot{z}\} = -\Omega^2 z_0 e^{i\Omega t}$$

No matter the system has viscous rotating damping or not, in the equation of motion, the non-rotating viscous damping on the stator only is present:

$$(-\Omega^2 m + C_n i\Omega + K) z_0 e^{i\Omega t} = m\varepsilon\Omega^2 e^{i\Omega t}$$

$$Z_0 = \frac{m\varepsilon\Omega^2}{-\Omega^2 m + C_n i\Omega + K}$$

The amplitude of z0 is expressed by a complex number

$$Z_0 = \frac{m\varepsilon\Omega^2(K - \Omega^2 m)}{(K - \Omega^2 m)^2 + (C_n\Omega)^2} - i\,\frac{m\varepsilon\Omega^2(C_n\Omega)}{(K - \Omega^2 m)^2 + (C_n\Omega)^2}$$

The absolute value of z_0 is calculated as

$$|Z_0| = \frac{m\varepsilon\Omega^2}{\sqrt{(k - m\Omega^2)^2 + (C_n\Omega)^2}}$$

$$(Z_0)^2((k - m\Omega^2)^2 + (C_n\Omega)^2) = (m\varepsilon\Omega^2)^2$$

$$(C_n\Omega)^2 = (\frac{m\varepsilon\Omega^2}{Z_0})^2 - (k - m\Omega^2)^2$$

$$C_n = \frac{\sqrt{\left(\frac{m\varepsilon\Omega^2}{Z_0}\right)^2 - (k - m\Omega^2)^2}}{\Omega}$$

$$\Omega = 8000(rpm) = 8000 * \frac{2\pi}{60} = 837.756\left(\frac{rad}{s}\right)$$

$$C_n = 3.58 * 10^3[{}^{Ns}/_{m}]$$

3) Calculate the radius of the orbit $\boldsymbol{Z_0}$ when crossing the critical speed assuming the value of $\boldsymbol{C_n}$ computing in the previous step.

$$\omega = \left(\frac{-C_n}{2m}\right)i \pm \sqrt{\frac{-{C_n}^2 + 4Km}{4m^2}}$$

$$\Omega_{cr} = \omega_{real} = 403.55\ (rad/s)$$

$$(-\Omega^2 m + C_n i\Omega + K)z_0 e^{i\Omega t} = m\varepsilon\Omega^2 e^{i\Omega t}$$

$$|Z_0| = \frac{m\varepsilon\Omega^2}{\sqrt{(k - m\Omega^2)^2 + (C_n\Omega)^2}}$$

$$|Z_0| = \frac{\varepsilon\Omega^2}{\sqrt{({\Omega_{cr}}^2 - \Omega^2)^2 + \left(\frac{C_n\Omega}{m}\right)^2}} = 16.6[\mu m]$$

Exercise 5

A rotor with mass m can be modeled as a Jeffcott rotor having a deformable shaft with bending stiffness k. The bearings have a known viscous damping coefficient cn; the relative total damping $\zeta = (c_n + c_r) / c_{cr}$ is evaluated with tests. It is requested to calculate the threshold of instability of the rotor.

Data:
k = 50 kN/m,
m = 30 kg,
ζ= 0.05,
c_n = 100 Ns/m.

Result: ν_{th} = 226 rad/s.

Solution

The relative total damping is equal to

$$\zeta = \frac{C_n + C_r}{C_{cr}} = \frac{C_n + C_r}{2\sqrt{Km}}$$

Where K is the total stiffness and m is the mass of the system

$$0.05 = \frac{C_n + C_r}{2\sqrt{30 * 50 * 10^3}} \quad \rightarrow \quad C_r \approx 22\ [\frac{Ns}{m}]$$

When viscous rotational damping on the rotor and viscous non-rotational damper on the stator are present, the threshold of instability for the rotor is:

$$\Omega < \left(1 + \frac{C_n}{C_r}\right)\sqrt{\frac{K}{m}}\ , \rightarrow \ \Omega < 226\left(\frac{rad}{s}\right).$$

Exercise 6

A Jeffcott rotor is constituted by a mass m, having eccentricity e, fixed in the middle of a deformable shaft with length l and diameter d suspended on elastic supports having stiffness k_c. Assuming structural rotating damping, having loss factor η_r, and viscous nonrotating damping c_n, it is requested to calculate:

a) The critical speed of the rotor,
b) The viscous nonrotating damping c_n needed to guarantee stability,
c) The amplitude and phase of the response to static unbalance at rated speed Ω.

Solution

1) Compute the critical speed of the rotor

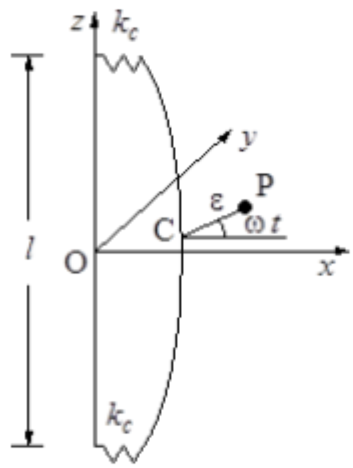


Figure 9. Exercise 6.

The equivalent stiffness is given by the series connection of Ks (Ks= 2*Kc, a parallel connection of two spring) and Kr (stiffness of the rotating part). Thus, the stiffness of the shaft is:

$$\frac{1}{K_{eq}} = \frac{1}{K_s} + \frac{1}{K_r}$$

$$K_s = 2 * 5 * 10^5 = 10^6 \left[\frac{N}{m}\right]$$

$$I = \frac{\pi d^4}{64} = 2.484 * 10^{-9} [m^4]$$

$$K_r = \frac{48EI}{L^3} = 3.726 * 10^5 \left[\frac{N}{m}\right]$$

$$K_{eq} = 2.715 * 10^5 \left[\frac{N}{m}\right]$$

$$\Omega_{cr} = \sqrt{\frac{K_{eq}}{m}} = 184 \; (rad/s)$$

2) Compute the viscous nonrotating damping $\boldsymbol{c_n}$ needed to guarantee stability

In the case of having viscous non-rotating damping on the stator and structural damping on the rotor, for being safe the nonrotating viscous damping coefficient must be

$$\mathrm{C}_n > \frac{K_r \eta_r}{\Omega_{cr}}$$

$$\mathrm{C}_n > \frac{3.726 * 10^5 * 0.002}{184} \rightarrow \quad \mathrm{C}_n = 4.05 \left[\frac{Ns}{m}\right]$$

3) Compute the amplitude and phase of the response to static unbalance at rated speed $\boldsymbol{\Omega}$

Now again by solving the equation of motion by considering the effect of eccentricity:

$$\left(-\Omega^2 m + C_n i\Omega + K_{eq}\right) z_0 e^{i\Omega t} = m\varepsilon\Omega^2 e^{i\Omega t}$$

$$Z_0 = \frac{m\varepsilon\Omega^2}{(\frac{K_{eq}}{m} - \mathrm{m}\Omega^2) + (C_n\Omega)i} = \frac{\varepsilon\Omega^2}{(\Omega_{cr}^2 - \Omega^2) + (\frac{C_n\Omega}{m})i}$$

Dividing the real and imaginary part

$$Z_0 = \frac{\varepsilon\Omega^2(\Omega_{cr}^2 - \Omega^2)}{(\Omega_{cr}^2 - \Omega^2)^2 + \left(\frac{C_n\Omega}{m}\right)^2} - i\, \frac{\varepsilon\Omega^3(\frac{C_n}{m})}{(\Omega_{cr}^2 - \Omega^2)^2 + \left(\frac{C_n\Omega}{m}\right)^2}$$

Therefore,

$$|Z_0| = \frac{\varepsilon\Omega^2}{\sqrt{(\Omega_{cr}^2 - \Omega^2)^2 + \left(\frac{C_n\Omega}{m}\right)^2}} = 1.18 * 10^{-5}[m]$$

$$|\varphi_0| = \tan^{-1}\left(\frac{Z_{0\,imag}}{Z_{0\,real}}\right) = \tan^{-1}\left(\frac{-\frac{C_n\Omega}{m}}{\Omega_{cr}^2 - \Omega^2}\right) \approx -0.4^{\circ}$$

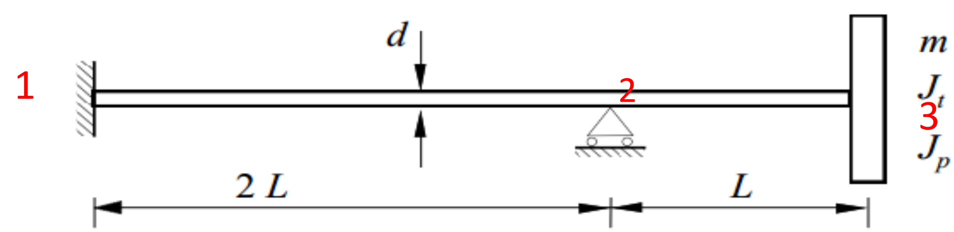
Exercise 7

Obtain the bending critical speeds and the corresponding mode shapes for the system shown in the figure. The system is represented by two beam elements using the lumped parameter approach and neglecting the distribution of the mass along the beam elements.

Data:
E = 2×1011 N/m^2,
L = 0.5 m,
d = 0.015 m,
m = 3 kg,
J_t = 0.04 kgm^2,
J_p = 0.08 kgm^2.

Results: $\Omega_{cr} = 41.5\frac{rad}{s}$

Solution



$$[M]\{\ddot{q}\} - i\Omega[G]\{\dot{q}\} + [K]\{q\} = \{0\}$$

$$\{\ddot{q}\} = q_0 e^{i\omega t} \quad ,\{\dot{q}\} = i\omega q_0 e^{i\omega t}, \quad \{\ddot{q}\} = -\omega^2 q_0 e^{i\omega t}$$

$$(-\omega^2[M] + [G]\Omega\omega + [K])q_0 = \{0\}$$

At critical speed $\Omega = \omega$ means the $\Omega_{rotation} = \Omega_{whirl}$

$$(-\Omega^2([M] - [G]) + [K])q_0 = \{0\}$$

Here again $J_p > J_t$ so there is a coupling in two different plane and for having the symmetric gyroscopic matrix we use the complex coordinate:

$$z = x + iy$$

$$\phi = \phi_y - i\phi_x$$

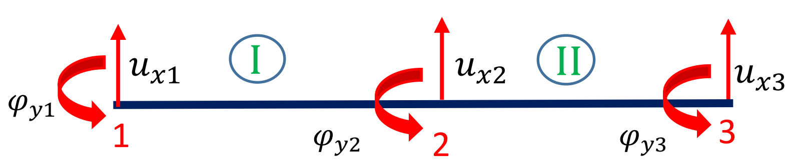


Figure 10. Exercise 7. Set of degrees of freedom.

Element I

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \phi_{y1} \\ u_{x2} \\ \phi_{y2} \end{Bmatrix}$$

Element II

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x2} \\ \phi_{y2} \\ u_{x3} \\ \phi_{y3} \end{Bmatrix}$$

by applying the boundary conditions, node 1 is totally locked $u_{x1}, \varphi_{y1} = 0$ and at node 2 we don't have any movement in the x-direction $u_{x2} = 0$ that we can the overall [K] is

$$\begin{bmatrix} 12 & 6l_1 & -12 & 6l_1 & 0 & 0 \\ 6l_1 & 4l^2{}_1 & 6l_1 & 2l^2{}_1 & 0 & 0 \\ -12 & -6l_1 & 12+12 & -6l_1-6l_2 & -12l_1 & 6l_1 \\ 6l & 2l^2{}_1 & -6l_1-6l_2 & 4l^2{}_1+4l^2{}_2 & -6l_2 & 2l^2{}_2 \\ 0 & 0 & -12 & -6l_2 & 12 & -6l_2 \\ 0 & 0 & 6l_2 & 2l^2{}_2 & -6l_2 & 4l^2{}_2 \end{bmatrix}\begin{Bmatrix} Z_1 \\ \phi_1 \\ Z_2 \\ \phi_2 \\ Z_3 \\ \phi_3 \end{Bmatrix}$$

therefore, a 3x3 matrix is obtained:

$$\begin{bmatrix} \boxed{A+4l_2^2} & -6l_2 & 2l_2^2 \\ -6l_2 & 12 & -6l_2 \\ 2l_2^2 & -6l_2 & 4l_2^2 \end{bmatrix}\begin{Bmatrix} \phi_2 \\ Z_3 \\ \phi_3 \end{Bmatrix} = \begin{bmatrix} 6l^2 & -6l & 2l^2 \\ -6l & 12 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$l_2 = l$, A=$\frac{EI}{l_1^3}(4l_1^2)$

$$\frac{EI}{l_1^3}(4l_1^2) + \frac{EI}{l_2^3}(4l_2^2) = \frac{EI}{l_2^3}(6l^2)$$

[M]=$\begin{bmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_t \end{bmatrix}\begin{Bmatrix} \ddot{\varphi}_2 \\ \ddot{Z}_3 \\ \ddot{\varphi}_3 \end{Bmatrix}$, [G]=$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J_p \end{bmatrix}$

As at the lamped mass is at the last node so this node can be defined as a master DoF

$$\{q_m\} = \begin{Bmatrix} Z_3 \\ \varphi_3 \end{Bmatrix}$$

$$\{q_s\} = \{\varphi_2\}$$

The definition of $[M]_{cond}$, $[G]_{cond}$ and $[K]_{cond}$ is then needed.

$$[M]_{cond} = \begin{bmatrix} m & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{Z}_3 \\ \ddot{\varphi}_3 \\ \ddot{\varphi}_2 \end{Bmatrix},$$

$$[G]_{cond} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & J_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Z_3 \\ \varphi_3 \\ \varphi_2 \end{Bmatrix},$$

$$[K]_{cond} = [K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}]$$

$$[K]_{cond} = \begin{array}{c} \begin{array}{cc} \quad K_{11} \quad\quad\quad & K_{12} \end{array} \\ \begin{bmatrix} 6l^2 & -6l & 2l^2 \\ -6l & 12 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix} \\ \begin{array}{cc} \quad K_{21} \quad\quad\quad & K_{22} \end{array} \end{array}$$

$$[K]_{cond} = \frac{EI}{l^3}\left(\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} - \begin{bmatrix} -6l \\ 2l^2 \end{bmatrix} [-6l \quad 2l^2] \frac{1}{6l^2}\right)$$

$$[K]_{cond} = \begin{bmatrix} \frac{6EI}{l^3} & \frac{-4EI}{l^2} \\ \frac{-4EI}{l^2} & \frac{10EI}{3l} \end{bmatrix}$$

The equation to be solved is:

$$(-\Omega^2([M]_{cond} - [G]_{cond}) + [K]_{cond})q_0 = \{0\}$$

$$det(-\Omega^2([M]_{cond} - [G]_{cond}) + [K]_{cond}) = \{0\}$$

$$det \begin{bmatrix} -m\Omega^2 + \frac{6EI}{l^3} & \frac{-4EI}{l^2} \\ \frac{-4EI}{l^2} & \frac{10EI}{3l} - (J_t - J_p)\Omega^2 \end{bmatrix} = \{0\}$$

Therefore,

$$\Omega^4 m(J_p - J_t) - \Omega^2\left(\frac{EI}{l}\right)\left((J_p - J_t)\frac{6}{l^2} + \frac{10}{3}m\right) + \frac{60E^2I^2}{l^4} - \frac{16E^2I^2}{l^4}$$

$$0.12\Omega^4 + 8985.76\Omega^2 - 1.5808576 = 0$$

$$\Omega^2 = 1720\left(\frac{rad}{s}\right) \;\; so\ \Omega_{cr} = \pm 41.47\left(\frac{rad}{s}\right)$$

The negative result does not have any physical meaning because the Ω_{cr} is in the first quarter and as we expect $J_p > J_t$, we have stiffening effect and just have only one critical speed.

Now for computing the mode shape:

$$\left(-\Omega^2\begin{bmatrix}0 & 0 & 0\\ 0 & m & 0\\ 0 & 0 & J_t - J_p\end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix}6l^2 & -6l & 2l^2\\ -6l & 12 & -6l\\ 2l^2 & -6l & 4l^2\end{bmatrix}\right)\begin{Bmatrix}\phi_{02}\\ Z_{03}\\ \phi_{03}\end{Bmatrix} = \{0\}$$

By setting $\phi_{02} = 1$,

$$\begin{Bmatrix}\phi_{02}\\ Z_{03}\\ \phi_{03}\end{Bmatrix} = \begin{Bmatrix}1\\ 0.822\\ 1.933\end{Bmatrix}$$

Exercise 8

A rotor having stiff supports, deformable shaft whose inertia can be neglected, has inertia properties m, J_t, J_p. The flywheel is attached with an eccentricity ϵ and angular error χ (a = 0). Calculate the critical speed and the amplitude of the response due to unbalance at spin speed equal to Ω.

Data: E = 2×1011 N/m^2, a = 0.08 m, b = 0.32 m, d = 0.022 m, m = 6 kg, J_t = 1.4 kgm2, J_p = 2.6 kgm^2, ϵ=20 mm, χ = 0.01 rad, Ω = 2000 rad/s.

Solution: Ω_{cr} = 1491 rad/s, $z_{0,\epsilon}$ = -5.1·10^{-5} m, $z_{0,\chi}$ = 1.0·10^{-3} m, $\phi_{0,\epsilon}$ = -1.0·10^{-5}rad, $\phi_{0,\chi}$ = -1.0·10^{-2} rad.

Solution

$$[M]\{\ddot{q}\} - i\Omega[G]\{\dot{q}\} + [K]\{q\} = \{0\}$$

$$\{\ddot{q}\} = q_0 e^{i\omega t} \quad ,\{\dot{q}\} = i\omega q_0 e^{i\omega t}, \quad \{\ddot{q}\} = -\omega^2 q_0 e^{i\omega t}$$

$$(-\omega^2[M] + [G]\Omega\omega + [K])q_0 = \{0\}$$

At critical speed, $\Omega = \omega$

$$(-\Omega^2([M] - [G]) + [K])q_0 = \{0\}$$

Complex set of coordinates is considered

$$z = x + iy$$

$$\phi = \phi_y - i\phi_x$$

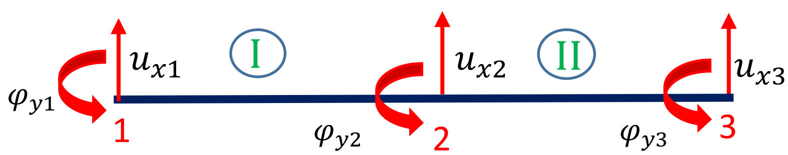


Figure 11. Exercise 8.

Element **I**

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ \phi_{y1} \\ u_{x2} \\ \phi_{y2} \end{Bmatrix}$$

Element **II**

$$[K] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & 6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x2} \\ \phi_{y2} \\ u_{x3} \\ \phi_{y3} \end{Bmatrix}$$

Applying the boundary conditions,

$$\begin{bmatrix} 12 & 6l_1 & -12 & 6l_1 & 0 & 0 \\ 6l_1 & 4l^2{}_1 & 6l_1 & 2l^2{}_1 & 0 & 0 \\ -12 & -6l_1 & 12+12 & -6l_1-6l_2 & -12l_1 & 6l_1 \\ 6l & 2l^2{}_1 & -6l_1-6l_2 & 4l^2{}_1+4l^2{}_2 & -6l_2 & 2l^2{}_2 \\ 0 & 0 & -12 & -6l_2 & 12 & -6l_2 \\ 0 & 0 & 6l_2 & 2l^2{}_2 & -6l_2 & 4l^2{}_2 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ \varphi_{y1} \\ u_{x2} \\ \varphi_{y2} \\ u_{x3} \\ \varphi_{y3} \end{Bmatrix}$$

$I = \frac{\pi d^4}{64} = 1.149 * 10^{-8}\ [m^4]$

$\frac{EI}{a^3} = 4488281.25$

$\frac{EI}{b^3} = 70129.395$

The stiffness matrices are

$$[K]_{\mathrm{I}} = \frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 \\ -6l & 12 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{Bmatrix} \varphi_{y1} \\ u_{x2} \\ \varphi_{y2} \end{Bmatrix} = \begin{bmatrix} 114900 & -2154375 & 57450 \\ -2154375 & 53859375 & -2154375 \\ 57450 & -2154375 & 114900 \end{bmatrix}\begin{Bmatrix} \varphi_{y1} \\ u_{x2} \\ \varphi_{y2} \end{Bmatrix}$$

$$[K]_{\mathrm{II}} = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & 6l \\ 6l & 4l^2 & 2l^2 \\ 2l^2 & 2l^2 & 4l^2 \end{bmatrix}\begin{Bmatrix} u_{x2} \\ \varphi_{y2} \\ \varphi_{y3} \end{Bmatrix} = \begin{bmatrix} 841552.74 & 13468.4384 & 13468.4384 \\ 13468.4384 & 28725 & -2154375 \\ 14362.5 & 14362.5 & 28725 \end{bmatrix}\begin{Bmatrix} u_{x2} \\ \varphi_{y2} \\ \varphi_{y3} \end{Bmatrix}$$

$$[M]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & J_t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{u}_{x1} \\ \ddot{\varphi}_{y1} \\ \ddot{u}_{x2} \\ \ddot{\varphi}_{y2} \\ \ddot{u}_{x3} \\ \ddot{\varphi}_{y3} \end{Bmatrix},$$

$$[G]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{u}_{x1} \\ \dot{\varphi}_{y1} \\ \dot{u}_{x2} \\ \dot{\varphi}_{y2} \\ \dot{u}_{x3} \\ \dot{\varphi}_{y3} \end{Bmatrix}$$

The associated homogenous equation is

$$[M]\{\ddot{q}\} - i\Omega[G]\{\dot{q}\} + [K]\{q\} = \{0\}$$

Therefore,

$$\{\ddot{q}\} = q_0 e^{i\omega t} \quad ,\{\dot{q}\} = i\omega q_0 e^{i\omega t},\ \ \{\ddot{q}\} = -\omega^2 q_0 e^{i\omega t}$$

$$(-\omega^2[M] + [G]\Omega\omega + [K])q_0 = \{0\}$$

At critical speed $\Omega = \omega$

$$(-\Omega^2([M]-[G])+[K])q_0 = \{0\}$$

$$z = x + iy$$

$$\phi = \phi_y - i\phi_x$$

$$\{q\}_m = \begin{Bmatrix} Z_2 \\ \phi_2 \end{Bmatrix}$$

$$\{q\}_s = \begin{Bmatrix} Z_1 \\ Z_3 \end{Bmatrix}$$

$$[M]\text{-}[G]=\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & J_t - J_p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} q_{0m} \\ q_{0s} \end{Bmatrix}$$

$$[K]=\begin{array}{c} K_{11} \\ \\ \\ K_{21} \end{array}\begin{bmatrix} 54700927.74 & -2019726.562 & -2154375 & 13648.4348 \\ -2019726.562 & 143625 & 57450 & 14362.5 \\ -2154375 & 57450 & 114900 & 0 \\ 13468.4348 & 14362.5 & 0 & 28725 \end{bmatrix}\begin{array}{c} K_{12} \\ \\ K_{22} \\ \\ \end{array}$$

$$[K]_{cond} = [K_{11}] - [K_{12}][K_{22}]^{-1}[K_{21}]$$

$$[K]_{cond} = \begin{bmatrix} -3.4988 & -0.1010 \\ -0.09449 & 0.00108 \end{bmatrix} * 10^7$$

Solving the equation

$$7.2\Omega^4 + 42633600\Omega^2 - 4.737194 * 10^{12} = 0$$

$$\Omega_{cr} = 1491\ (rad/s)$$

$$[k - \Omega^2(M-G)]\begin{Bmatrix} Z_0 \\ \varphi_0 \end{Bmatrix} e^{i\Omega t} = \begin{Bmatrix} m\varepsilon e^{i\alpha} \\ \chi(J_t - J_p) \end{Bmatrix} \Omega^2 e^{i\Omega t}$$

$$\Delta = -m(J_p - J_t)\Omega^2 + [K_{11}(J_p - J_t) + mK_{22}]\Omega^2 + K_{11}K_{12} - K_{12}^2$$

$$z_{0\varepsilon} = \frac{m\varepsilon\Omega^2}{\Delta}\left((J_p - J_t)\Omega^2 + K_{22}\right), \varphi_{0\varepsilon} = \frac{m\varepsilon\Omega^2}{\Delta}(K_{12})$$

$$z_{0\chi} = \frac{\chi\Omega^2}{\Delta}\left((J_p - J_t)K_{12}\right), \varphi_{0\chi} = \frac{\chi\Omega^2}{\Delta}(J_p - J_t)(m\Omega^2 - K_{11})$$