## 01OAIQD – Dynamic Design of Machines

**Academic year 2018-2019**

## Discrete linear systems

# TUTORIAL 2

| Emission date: November 22, 2019 | | GROUP: |
|---|---|---|

### EXERCISE 1: Quarter car model

Figure 1 shows the so-called quarter car model, one of the simplest models used to study the dynamic behavior of motor vehicle suspensions. The upper mass $m_5$ simulates the part of the mass of the car body (the sprung mass) that can be considered supported by a given wheel, while the lower mass $m_2$ simulates the wheel and all the parts that can be considered as rigidly connected with the unsprung mass. The two masses are connected by a spring-damper system simulating the suspension ($k_2$,$c_2$) and the silent block (elastometer, $k_{3.1}$, $k_{3.2}$,$c_3$). The unsprung mass is connected to the ground with a second spring simulating the radial stiffness of the tire. The point at which the tire contacts the ground is assumed to move in a vertical direction with a given law $y_1$(t), and it simulates the motion on uneven ground.

For the given quarter car model

- determine the equations of dynamic equilibrium using the Lagrangian approach,
- arrange the equations in matrix form,
- find the natural frequencies and the mode shapes of the system,
- Consider as input a harmonic excitation with amplitude y1 of 3 mm @ 5 Hz. Compute the power dissipated in the elastomeric member.

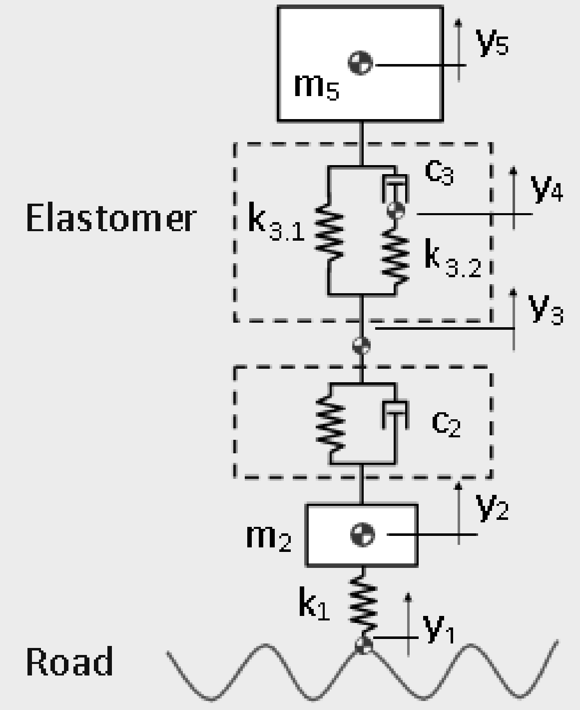


Figure 1

| | |
|---|---|
| sprung mass | $m5$ = 400 kg; |
| unsprung mass | $m2$ = 30 kg; |
| spring stiffness | $k2$ = 24 kN/m |
| spring damping | $c2$ = 1200 Ns/m; |
| elastomer stiffness | $k3.1$ = 150 kN/m; |
| elastomer stiffness | $k3.2$ = 90 kN/m; |
| elastomer damping | $c3$ = 800 Ns/m; |
| tire stiffness | $k1$ = 190 kN/m; |

**SOLUTION**

**1 -**

To compute the equation of motion using the Lagrangian approach it is necessary to determine:

- the kinetic energy $T = \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}m_5\dot{y}_5^2$
- the potential energy $U = \frac{1}{2}k_1(y_2 - y_1)^2 + \frac{1}{2}k_2(y_3 - y_2)^2 + \frac{1}{2}k_{3.1}(y_5 - y_3)^2 + \frac{1}{2}k_{23.2}(y_4 - y_3)^2$
- $\mathcal{F} = \frac{1}{2}C_2(\dot{y}_3 - \dot{y}_2)^2 + \frac{1}{2}C_3(\dot{y}_5 - \dot{y}_4)^2$

Thus, the Lagrange equation related to the 4 DoFs are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = \frac{\partial(\delta L)}{\partial(\delta q_i)}$$

**1)** $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_2}\right) = m_2\ddot{y}_2^2$

$\frac{\partial U}{\partial y_2} = k_1(y_2 - y_1) - k_2(y_3 - y_2)$

$\frac{\partial \mathcal{F}}{\partial \dot{y}_2} = -C_2(\dot{y}_3 - \dot{y}_2)$

$\frac{\partial(\delta L)}{\partial(\delta y_2)} = 0$

The first equation will be :

$$m_2\ddot{y}_2^2 + C_2\dot{y}_2 - C_2\dot{y}_3 + y_2(k_1 + k_2) - k_2y_3 = k_1y_1$$

**2)** $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_3}\right) = 0$

$\frac{\partial U}{\partial y_3} = k_2(y_3 - y_2) - k_{3.1}(y_5 - y_3) - k_{3.2}(y_4 - y_3)$

$\frac{\partial \mathcal{F}}{\partial \dot{y}_3} = C_2(\dot{y}_3 - \dot{y}_2)$

$\frac{\partial(\delta L)}{\partial(\delta y_3)} = 0$

The second equation will be:

$$-C_2\dot{y}_2 + C_2\dot{y}_3 - k_2 y_2 + y_3(k_{3.1} + k_{3.2} + k_2) - k_{3.2}y_4 - k_{3.1}y_5 = 0$$

**3)** $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_4}\right) = 0$

$\frac{\partial U}{\partial y_4} = k_{3.2}(y_4 - y_3)$

$\frac{\partial \mathcal{F}}{\partial \dot{y}_4} = -C_3(\dot{y}_5 - \dot{y}_4)$

$\frac{\partial(\delta L)}{\partial(\delta y_4)} = 0$

The third equation will be :

$$C_3\dot{y}_4 - C_3\dot{y}_5 - k_{3.2}y_3 + k_{3.2}y_4 = 0$$

**4)** $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}_5}\right) = m_5\ddot{y}_5^2$

$\frac{\partial U}{\partial y_5} = k_{3.1}(y_5 - y_3)$

$\frac{\partial \mathcal{F}}{\partial \dot{y}_5} = C_3(\dot{y}_5 - \dot{y}_4)$

$\frac{\partial(\delta L)}{\partial(\delta y_5)} = 0$

The forth equation will be :

$$m_5\ddot{y}_5^2 - C_3\dot{y}_4 + C_3\dot{y}_5 - k_{3.1}y_3 + k_{3.1}y_5 = 0$$

**2 -**

In matrix form:

$$\begin{bmatrix} m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_5 \end{bmatrix}\begin{Bmatrix} \ddot{y}_2 \\ \ddot{y}_3 \\ \ddot{y}_4 \\ \ddot{y}_5 \end{Bmatrix} + \begin{bmatrix} C_2 & -C_2 & 0 & 0 \\ -C_2 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & -C_3 \\ 0 & 0 & -C_3 & C_3 \end{bmatrix}\begin{Bmatrix} \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \\ \dot{y}_5 \end{Bmatrix} +$$

$$+\begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_{3.1} + k_{3.2} & -k_{3.2} & -k_{3.1} \\ 0 & -k_{3.2} & k_{3.2} & 0 \\ 0 & -k_{3.1} & 0 & k_{3.1} \end{bmatrix}\begin{Bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix} = \begin{Bmatrix} k_1 y_1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

**3 -**

For computing the natural frequency

$\{y_i\} = \{y_0\}e^{i\omega t}$ $\{\dot{y}_i\} = i\omega\{y_0\}e^{i\omega t}$ $\{\ddot{y}_i\} = -\omega^2\{y_0\}e^{i\omega t}$

$\left([k] - [m]\omega_{ni}^2\right)\{y_0\} = \{0\}$

det $\left([k] - [m]\omega_{ni}^2\right)\{y_0\} = \{0\}$

$\omega_{n1} = 0,$ $\omega_{n2} = 0.2159,$ $\omega_{n3} = 2.5611,$ $\omega_{n4} = inf$

The mode shapes:

First mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{Bmatrix}$$

Second mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} -0.989 \\ 0.8757 \\ 0.8757 \\ -1 \end{Bmatrix}$$

Third mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.1315 \\ 0.1315 \\ -0.0074 \end{Bmatrix}$$

Fourth mode:

$$\begin{Bmatrix} y_{02} \\ y_{03} \\ y_{04} \\ y_{05} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{Bmatrix}$$

**4 –**

Starting directly from the equation of motion in matrix form:

$$M\ddot{Y} + C\dot{Y} + KY = F$$

Harmonic excitation

$$F = F_0 e^{j\omega t}$$

Where $F_0$ is something like $F_0 = [0\ \cdots k_1 y_1\ \cdots 0]^T$

Hence

$$Y = Y_0 e^{j\omega t}$$
$$\dot{Y} = j\omega Y_0 e^{j\omega t}$$
$$\ddot{Y} = -\omega^2 Y_0 e^{j\omega t}$$
$$\Rightarrow (-\omega^2 M + j\omega C + K)Y_0 = F_0$$

Let's define $(-\omega^2 M + j\omega C + K) = K_{dyn}(\omega)$

So,

$$Y_0 = inv\left(K_{dyn}(\omega)\right) F_0$$

The response to the excitation at $\omega_0 = 2\pi 5$ rad/s is

$$Y_0 = inv\left(K_{dyn}(\omega_0)\right) F_0$$

$Y_0$ is the amplitude of the response. It is something like $Y_0 = [y_{02}\ y_{03}\ y_{04}\ y_{05}]^T$, where the $y_{0i}$ component could be complex quantities.

The power dissipated in the elastomeric member is due to the lumped damping $c_3$ since it is the dissipative member (the spring does not dissipate energy). The power is force times velocity. Hence, the power dissipated by the elastomeric member is

$$P = F_3 \dot{y_4} = (c_3 \dot{y_4})\dot{y_4} = c_3 \dot{y}_4^2$$

Where $y_4 = Re\left(y_{04} e^{j\omega_0 t}\right)$ (note that in the frequency domain we have complex quantities but in the time domain we are interested in the real part only). Consequently, we have

$$\dot{y_4} = Re\left(j\omega_0 y_{04} e^{j\omega_0 t}\right) = Re\left(\omega_0 y_{04} e^{j\omega_0 t + \frac{\pi}{2}}\right)$$

$y_{04}$ could be a complex quantity. Hence, it can be written as

$$y_{04} = |y_{04}| e^{j\phi}$$

By substituting, we have

$$\dot{y_4} = Re\left(\omega_0 |y_{04}| e^{j\left(\omega_0 t + \frac{\pi}{2} + \phi\right)}\right)$$

Where, physically speaking, $\phi$ is the phase delay between the harmonic input $y_1$ and the output displacement $y_4$.

By picking the real part only, it follows

$$\dot{y_4} = |\dot{y_4}| \cos\left(\omega_0 t + \frac{\pi}{2} + \phi\right) = \omega_0 |y_{04}| \sin(\omega_0 t + \phi)$$

Therefore, the instant power is

$$P = c_3\ \omega_0^2 |y_{04}|^2 (\sin(\omega_0 t + \phi))^2$$

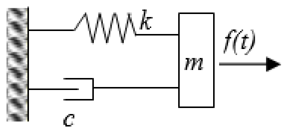
The peak power dissipated by the elastomeric member is

$$P_{peak} = c_3 \omega_0^2 |y_{04}|^2$$

And the RMS value is

$$P_{RMS} = P_{peak}/\sqrt{2}$$

## EXERCISE 2

| | |
|---|---|
| Determine the unit impulse response of the mass-damper-spring system in figure. Plot the response and<br>• evaluate the first peaks of the response;<br>• calculate the damping ratio of the system<br><br>k = 100 N/m, m=2 kg, c = 2 N/m s. |  |

**SOLUTION**

Determine the unit impulse response of the system

Since the impulse determines a variation of the momentum,

$$m(v_2 - v_1) = \int_v f\,dt$$

And if $v_1 = 0$, the system is at rest in the origin

$$m\,v_2 = f \rightarrow v_2 = \frac{f}{m}$$

Then, the initial conditions are $\begin{cases} \dot{x}(0) = f/m \\ x(0) = 0 \end{cases}$

Since impulse is not a proper external force acting on the system, I need to study the associated homogeneous differential equation:

$$m\,\ddot{x} + c\dot{x} + kx = 0$$

Whose solution is:

$$x(t) = e^{-\xi\omega_n t}\,[x_{01}\cos(\omega_p t) + x_{02}\sin(\omega_p t)]$$

To calculate $x_{01}$ and $x_{02}$, the initial conditions are considered

$$\begin{cases} x(0) = x_{01} = 0 \\ \dot{x}(0) = -\xi\,\omega_n\,e^{-\xi\omega_n t}\,[x_{01}] + e^{-\xi\omega_n t}[-x_{01}\,\omega_p\sin(\omega_p t) + \omega_p\,x_{02}\sin(\omega_p t)] = -\xi\omega_n x_{01} + \omega_p x_{02} = \frac{f}{m} \end{cases}$$

So, $\begin{cases} x_{01} = 0 \\ x_{02} = \frac{f}{m\omega_p} \end{cases}$

Then, $x(t) = e^{-\xi\omega_n t}\,[\frac{f}{m\omega_p}\sin(\omega_p t)]$.

The first peak response is when $\sin(\omega_p t)$ is maximum, so:

$$sin(\omega_p t) = 1\text{ , then }\omega_p t = \frac{\pi}{2} \rightarrow t = \omega_p^{-1}\frac{\pi}{2}$$

And substituting in $x(t)$:

$x(t) = e^{-\frac{\xi\omega_n}{\omega_p}\frac{\pi}{2}}\,[\frac{f}{m\omega_p}\sin\left(\frac{\pi}{2}\right)]$ where $\omega_n = 5\sqrt{2}\ rad/s$

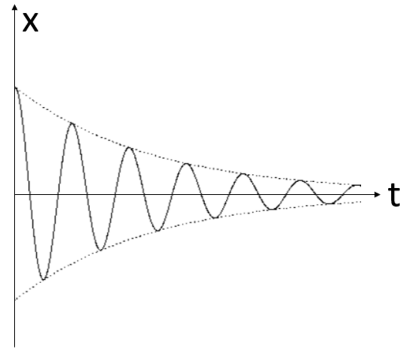
$\omega_p = \omega_n\sqrt{1 - \xi^2} = 6.816$

$\zeta = \frac{c}{2\sqrt{km}} = \sqrt{2}/20$

Then, $x_{max} = 0.065\ m$

2)

The damping ratio of the system is given by the ratio between the amplitudes of two subsequent peaks:

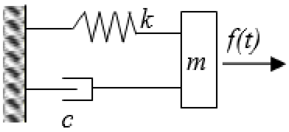



$$\delta = \ln\frac{x_i}{x_{i+1}} = ln\frac{e^{-\xi\omega_n t_i}}{e^{-\xi\omega_n t_{i+1}}} = \ln e^{-\xi\omega_{n(}t_i - t_{i+1)}} = \ln e^{\xi\omega_n\frac{2\,\pi}{\omega_n\sqrt{1-\,\xi^2}}} = \frac{2\,\pi\xi}{\sqrt{1-\,\xi^2}}$$

## EXERCISE 3

Calculate the unit step response $s(t)$ of the mass-damper-spring system of exercise 1 integrating the impulse response. Plot $s(t)$ versus t.

- Determine the response for the undamped system (c=0) and,
- for the damped system as given in the picture.
- Which is the response at infinite?




**SOLUTION**

Calculate the unit step response s(t) of the system

The step response is the integral of the impulse response

$$x(t)=\int_{-\infty}^{t} e^{-\xi\omega_n t}\left[\frac{f}{m\omega_p}\sin\left(\omega_p t\right)\right]dt=\frac{f}{m\omega_p}\int_{-\infty}^{t}\underbrace{e^{-\xi\omega_n t}}_{b\prime}\ \underbrace{\sin\left(\omega_p t\right)}_{a}\ dt$$

$$\int a\,b'=ab-\int a'\,b$$

$$x(t)=\frac{f}{m\omega_p}\left[-\sin(\omega_p t)\right]\frac{e^{-\xi\omega_n t}}{\xi\omega_n}+\frac{\omega_p}{\xi\omega_n}\int_{-\infty}^{t}\cos(\omega_p t)\ e^{-\xi\omega_n t}dt=$$

$$=\frac{f}{m\omega_p}\left[-\sin(\omega_p t)\ \frac{e^{-\xi\omega_n t}}{\xi\omega_n}+\frac{\omega_p}{\xi\omega_n}\left[-\frac{e^{-\xi\omega_n t}}{\xi\omega_n}\cos(\omega_p t)+\int_{-\infty}^{t}-\omega_p\frac{\sin\left(\omega_p t\right)}{\xi\omega_n}\ e^{-\xi\omega_n t}dt\ \right]\right]=$$

$$=\frac{f}{m\omega_p}\left[\left[\int_{-\infty}^{t} e^{-\xi\omega_n t}\sin(\omega_p t)\,dt\ +\frac{\omega_p^2}{\xi^2\ \omega_n^2}\int_{-\infty}^{t}\sin(\omega_p t)\,e^{-\xi\omega_n t}\ dt\ \right]=\right.$$

$$=\frac{f}{m\omega_p}\left[\frac{e^{-\xi\omega_n t}}{\xi\ \omega_n}\ \sin(\omega_p t)-\frac{\omega_p}{\xi^2\omega_n^2}\ \ e^{-\xi\omega_n t}\ \cos(\omega_p t)\right]=$$

$$=\frac{f}{m\omega_p}\ \left(-\frac{e^{-\xi\omega_n t}}{\xi\ \omega_n}\right)\left[\ \sin(\omega_p t)+\frac{\omega_p}{\xi\ \omega_n}\ \cos(\omega_p t)\right]=$$

$$\int_{0}^{t} e^{-\xi\omega_n t}\ \sin(\omega_p t)\,dt=-\left(\frac{\xi^2\omega_n^2}{\xi^2\omega_n^2+\omega_p}\right)\frac{f}{m\omega_p}\ \frac{e^{\xi\omega_n t}}{\xi^2\omega_n^2}\left(\ \xi\omega_n\sin(\omega_p t)+\omega_p\cos\left(\omega_p t\right)\right)=$$

$$-\left(\frac{1}{\xi^2\omega_n^2+\omega_p}\right)\frac{f}{m\omega_p}\quad e^{\xi\omega_n t}\left(\ \xi\omega_n\sin(\omega_p t)+\omega_p\cos(\omega_p t)\right)|_0^t=$$

$$=-\left(\frac{1}{\omega_n\left(\xi^2\omega_n-\sqrt{1-\xi^2}\right)}\right)\frac{f}{m}\quad e^{\xi\omega_n t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_p t)+\cos(\omega_p t)\right)$$

Then,

$$x(t)=\frac{f}{k}\left(1-\ \ e^{-\xi\omega_n t}\left(\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_p t)+\cos(\omega_p t)\right)\right)$$

Determine the response for c=0

$$x(t)=\int_{0}^{t}\frac{f}{m\omega_p}\sin\left(\omega_p t\right)\ dt=-\ \frac{f}{m\omega_p}\frac{\omega_p\cos(\omega_p t)}{\omega_p^2}+\frac{f}{m\omega_p^2}=\frac{f}{m\omega_p^2}\left(1-\cos(\omega_p t)\right)=$$

$$=\frac{f}{k}\left(1-\cos(\omega_n t)\right)$$

$$\omega_p=\omega_n\qquad if\ \ c=0$$

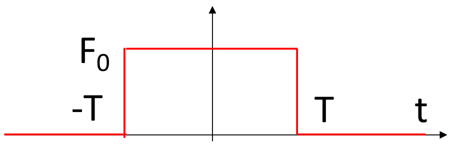
Which is the response at infinite?

$X_{steadyState} = f/k$

## EXERCISE 4

Use the concept of unit step function and calculate the response of system in exercise 2 to the rectangular pulse shown in figure. Plot the first 8 peaks around –T and T.

$F_0 = 50$ N, $T = 5$ s




**SOLUTION**

The initial conditions of the Step input are:

$$x(0) = 0\,, \quad \dot{x}(0) = 0 \;, \omega_n = 7.0711\ \left({}^{rad}/_{s}\right)$$

$$f_0 = 1$$

The response of the system to the excitation can be computed by adding the solution obtained for free oscillations to the steady-state response to the constant force $f_0$.

$$x(t) = e^{-\sigma t}\left[X_{01}\cos(\omega_p t) + X_{02}\sin(\omega_p t)\right] + \frac{f_0}{k}$$

Imposing the initial conditions, it is possible to compute $X_{01}, X_{02}$ from the previous equation:

$$X_{01} = -\frac{f_0}{k}$$

$$X_{02} = \frac{f_0}{k} \cdot \frac{\xi}{\sqrt{1-\xi^2}}$$

Thus,

$$x(t) = \frac{f_0}{k}\, g(t)$$

$$g(t) = 1 - e^{-\xi\omega_n t}\left[\cos\left(\left(\omega_n\sqrt{1-\xi^2}\right)t\right) + \frac{\xi}{\sqrt{1-\xi^2}}\sin\left(\left(\omega_n\sqrt{1-\xi^2}\right)t\right)\right]$$

If C=0 then $\xi = 0\,, X_{01} = -0.01\,, X_{02} = 0$

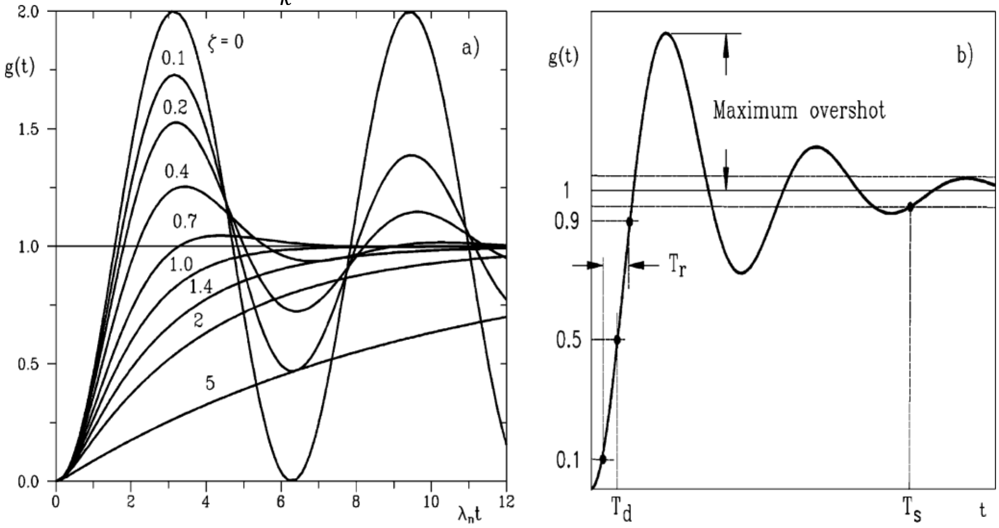
$$g(t) = 1 - [\cos(\omega_n t)]$$

If C=2 $\left({}^{N}/_{m.s}\right)$, $\xi = 0.0707, \omega_p = 7.0534\ \left({}^{rad}/_{s}\right)$

$$g(t) = 1 - e^{-\xi\omega_n t}\left[\cos(\omega_p t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_p t)\right]$$

$$\begin{cases} x(t) = 0 \quad , & t \le 0 \\ x(t) = \dfrac{f_0}{k} g(t), & t > 0 \end{cases}$$

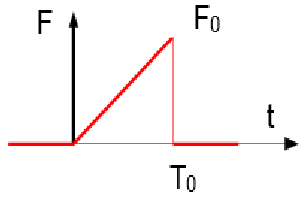
The infinite response converges to $\frac{f_0}{k} = 1$ (mm)



## EXERCISE 5

Derive an expression for the response of the system in exercise 1 to a ramp force $F(t) = F_0 \square t$ in terms of the convolution integral. Consider both undamped and damped systems.

$F_0 = 10$ N/s, $T_0 = 5$ ms

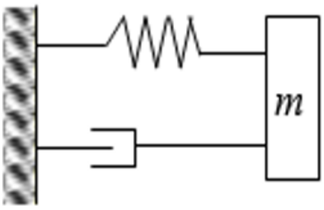



**SOLUTION (complete solution in the dedicated pdf)**

The force is given by:

$$F(t) = \begin{cases} \frac{F_0}{t} & 0 \leq t \leq 5s \\ 0 & t > 5\,s \end{cases}$$

- Damped system: $x(t) = A(t)\sin(\omega_p t) + B(t)\cos(\omega_p t)$




$$\omega_n = \sqrt{\frac{K}{m}} = 5\sqrt{2}\ rad/s$$

$$\omega_p = 6.816\, rad/s$$

$$\xi = \frac{\sqrt{2}}{20}$$

$$A(t) = \frac{1}{m\omega_p}\, e^{\theta t} \int_0^t F(\tau) e^{\theta\tau} \cos(\omega_p \tau)\, d\tau = \frac{1}{m\omega_p}\, e^{\theta t} \frac{F_0}{\Delta t} \int_0^t \tau\, e^{\theta\tau} \cos(\omega_p \tau)\, d\tau$$

$$B(t) = \frac{1}{m\omega_p}\, e^{\theta t} \frac{F_0}{\Delta t} \int_0^t \tau\, e^{\theta\tau} \sin(\omega_p \tau) d\tau$$

**Undamped system**

By introducing the impulse response of an undamped system, the particular integral of the equation of motion can be expressed as

$$x(t) = A(t)\sin(\omega_n t) - B(t)\cos(\omega_n t)$$

Starting from: $A(t) = \frac{1}{m\omega_n} F_0 \int_0^t \tau \cos(\omega_n \tau)\, d\tau$

Integrating by parts,

$$\int a\, b' = ab - \int a'\, b \quad \to a = t, b' = cos(\omega_n t):$$

$$\int_0^t \tau \cos(\omega_n \tau)\, d\tau = \frac{\tau}{\omega_n}\sin(\omega_n \tau) - \frac{1}{\omega_n}\int_0^t \sin(\omega_n \tau)\, d\tau = t\frac{1}{\omega_n}\sin(\omega_n t) + \frac{1}{\omega_n^2}\cos(\omega_n t)|_0^t =$$

$$\frac{1}{\omega_n^2}(\omega_n \sin(\omega_n t)\, t + \cos(\omega_n t) - 1)$$

So, $A(t) = \frac{1}{\sqrt{2}\,50}(5\sqrt{2}\, \sin(5\sqrt{2}\, t)\, t + \cos(5\sqrt{2}\, t) - 1)$

The term B(t) is computed:

$$B(t) = \frac{1}{m\omega_p} F_0 \int_0^t \tau \ \sin(\omega_n \tau)\, d\tau = \int_0^t \tau \ \sin(\omega_n \tau)\, d\tau \ = -\frac{\tau}{\omega_n} \cos(\omega_n t)|_0^t + \frac{1}{\omega_n} \int_0^t \cos(\omega_n \tau)\, d\tau$$

$$= - \ t \frac{1}{\omega_n} \sin(\omega_n t) + \frac{1}{\omega_n} + \frac{1}{\omega_n^2} \sin(\omega_n t) = \frac{1}{\omega_n^2} (1 + \sin(\omega_n t) - \omega_n t \cos(\omega_n t))$$

$$B(t) = \frac{1}{\sqrt{2}\, 50} (1 + \ \sin(5\sqrt{2}\, t) - 5\sqrt{2} t \cos(5\sqrt{2}\, t))$$

So,

$$x(t) = \frac{1}{\sqrt{2}\, 50} (5\sqrt{2} \ \sin^2(5\sqrt{2}\, t)\, t + \sin(5\sqrt{2}\, t) \cos(5\sqrt{2}\, t) - \sin(5\sqrt{2}\, t))$$

$$- \frac{1}{\sqrt{2}\, 50} (\cos(5\sqrt{2}\, t) + \cos(5\sqrt{2}\, t) \ \sin(5\sqrt{2}\, t) - 5\sqrt{2} t \cos^2 \ (5\sqrt{2}\, t))$$

$$= \frac{1}{\sqrt{2}\, 50} \ (5\sqrt{2}\, t - \sin(5\sqrt{2}))$$

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Directly applying the Duhamel integral:

$$x(t) = \int_0^t F_0 (t - \tau) \frac{1}{\omega_n m} \sin(\omega_n \tau)\, d\tau = \frac{1}{\omega_n m} \ F_0 t \int_0^t \sin(\omega_n \tau)\, d\tau - \ \frac{1}{\omega_n m} \ F_0 \int_0^t \tau \sin(\omega_n \tau)\, d\tau =$$

$$= \frac{1}{\omega_n m} \ F_0 t \ \left(-\frac{1}{\omega_n} \cos(\omega_n t) + \frac{1}{\omega_n}\right) - \frac{1}{\omega_n m} \ F_0 \ \int_0^t \tau \sin(\omega_n \tau)\, d\tau \qquad (1)$$

Solving the integration by parts of the term $\int_0^t \tau \sin(\omega_n \tau)\, d\tau$

$$\int a\, b' = ab - \int a\,' \, b \quad \rightarrow a = \tau \qquad b' = \sin(\omega_n \tau))$$

$$\rightarrow -\frac{t}{\omega_n} (\cos(\omega_n t)) - \frac{1}{\omega_n} \int_0^t \cos(\omega_n \tau)\, d\tau = \ -\frac{t}{\omega_n} (\cos(\omega_n t)) - \frac{1}{\omega_n^2} \sin(\omega_n t) \qquad (2)$$

Thus, substituting (2) in equation (1)

$$x(t) = \ \frac{1}{\omega_n m} \ F_0 \left[ t \left( \frac{1}{\omega_n} - \frac{1}{\omega_n} \cos(\omega_n t) \right) + \frac{t}{\omega_n} \ \cos(\omega_n t) - \frac{1}{\omega_n^2} \sin(\omega_n t) \right]$$

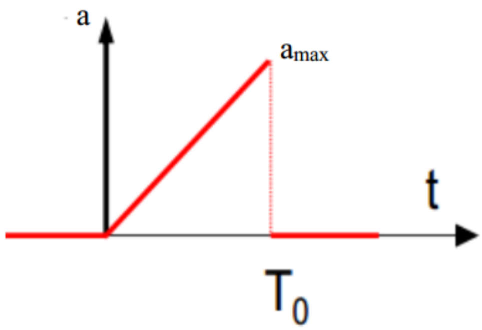
$$= \ \frac{1}{\omega_n m} \ F_0 \ \left[ \frac{t}{\omega_n} - \frac{1}{\omega_n^2} \sin(\omega_n t) \right] = \ \frac{1}{\omega_n^3 m} \ F_0 \ [\omega_n t - \sin(\omega_n t)]$$

## EXERCISE 6

A single degree of freedom dynamic system (k = 2000 N/m, m = 0.3 kg, neglect the damping) is shocked by an acceleration having peak 4000 m/s$^2$ and duration 0.5 ms. Determine the equation of motion by using both the impulse response method and the Duhamel integral (assume a linear acceleration profile). Plot the displacements versus time and compare the results (at least in term of maximum amplitude).

Repeat the calculation when the same shock lasts for 20 ms. Explain the difference with the previous case.

**SOLUTION**




1) Equation of motion using the impulse response method

The initial conditions are $x(0) = 0$ , $\quad \dot{x}(0) = \frac{f_0}{m}$

$x(t) = e^{-\xi\omega_n t}\frac{1}{\omega_p}\cdot\frac{f_0}{m}\sin(\omega_p t) \quad , C = 0 \; so \; \omega_p = \omega_n = \boxed{25.82 \; ({}^{rad}/{}_{s})}$

$f_0 = \frac{1}{2}\, a_m * m * \Delta t$

$\Delta t = 0.5 \; (ms) \; so \;\; f_0 = 0.3 \; (N.s)$

$x(t) = 0.387\sin(\omega_n t) \;\; and \;\; x_{max}(t) = 0.387(m)$

$\Delta t = 20 \; (ms) \; so \;\; f_0 = 12 \; (N.s)$

$x(t) = 0.0387\sin(\omega_n t) \;\; and \;\; x_{max}(t) = 0.0387(m)$

2) Equation of motion using the Duhamel integral

The particular integral of the equation of motion can be expressed as

$x(t) = A_1(t)\sin(\omega_p t) - B_1(t)\cos(\omega_p t)$

where

$A_1(t) = \frac{1}{m\omega_p e^{-\xi\omega_n t}}\int_0^{T_0} F(\tau)e^{\xi\omega_n\tau}\cos(\omega_p\tau)\,d\tau$

$B_1(t) = \frac{1}{m\omega_p e^{-\xi\omega_n t}}\int_0^{T_0} F(\tau)e^{\xi\omega_n\tau}\sin(\omega_p\tau)\,d\tau$

$F(\tau) = \frac{f_0}{T_0}\tau$

When t>$T_0$ so $A_1(t) = A_1(T_0) \; and \; B_1(t) = B_1(T_0)$

Consider that there is no damping and so:

$$A(t) = \frac{1}{m\omega_p}\, F_0 \int_0^t \tau\,\cos(\omega_n\tau)\,d\tau$$

Integrating by parts,

$$\int a\,b' = ab - \int a'\,b \quad \rightarrow a = t, b' = cos(\omega_n t):$$

$$\int_0^t \tau\,\cos(\omega_n\tau)\,d\tau = \frac{\tau}{\omega_n}\sin(\omega_n\tau) - \frac{1}{\omega_n}\int_0^t \sin(\omega_n\tau)\,d\tau = t\frac{1}{\omega_n}\sin(\omega_n t) + \frac{1}{\omega_n^2}\cos(\omega_n t)|_0^t =$$

$$A = \frac{1}{\omega_n^2}(\omega_n \sin(\omega_n t)\, t + \cos(\omega_n t) - 1)$$

The term B(t) is computed as follows:

$$B(t) = \frac{1}{m\omega_p}\, F_0 \int_0^t \tau\, \sin(\omega_n \tau)\, d\tau = \int_0^t \tau\, \sin(\omega_n \tau)\, d\tau\, = -\frac{\tau}{\omega_n}\cos(\omega_n t)|_0^t + \frac{1}{\omega_n}\int_0^t \cos(\omega_n \tau)\, d\tau$$

$$= -\; t\frac{1}{\omega_n}\sin(\omega_n t) + \frac{1}{\omega_n} + \frac{1}{\omega_n^2}\sin(\omega_n t) = \frac{1}{\omega_n^2}(1 + \sin(\omega_n t) - \omega_n t \cos(\omega_n t))$$

**MULTIPLE CHOICE PROBLEMS: select the right answer. Explain in few sentences (eventually with the support of graphs, schemes and formulas) the reason of the correct answer.**

1) The hysteretic damping model is applicable to
   [A] linear models subject to any type of excitation
   [B] only linear models subject to harmonic excitation
   [C] linear and non-linear models subject to harmonic or multi harmonic excitation

**SOLUTION [C]**

2) The Duhamel integral can be applied only
   [A] For linear systems,
   [B] with shock inputs
   [C] if the system is undamped.

**SOLUTION [A]**

3) At resonance
   [A] elastic forces balance exactly inertia forces;
   [B] damping forces balance exactly inertia forces;
   [C] elastic forces balance exactly damping forces.

**SOLUTION [A]**

4) The dynamic compliance of a system with viscous damping
   [A] is expressed by a complex number;
   [B] tends to zero when the forcing function tends to zero;
   [C] is always expressed by a real number.

**SOLUTION [A]**

5) The response of an undamped linear system at its resonant frequency
   [A] is infinitely large;
   [B] grows linearly in time to infinity;
   [C] grows exponentially in time to infinity.

**SOLUTION [B]**