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Dynamic Design of Machines
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Mechanical Engineering
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Tutorial 8 – Torsional dynamics

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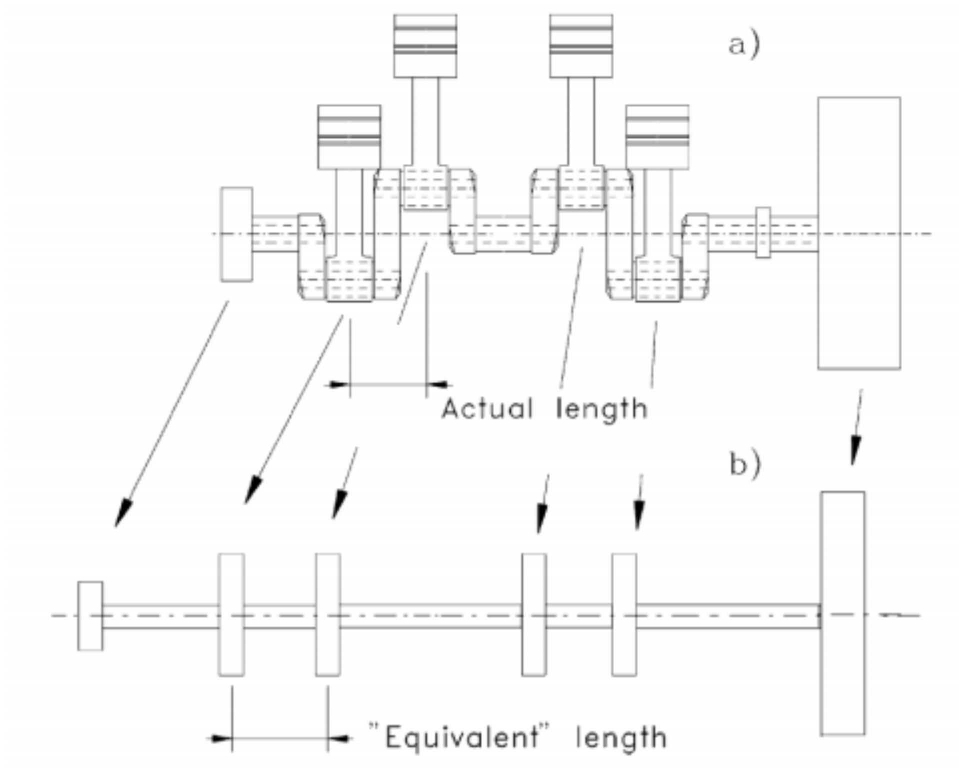
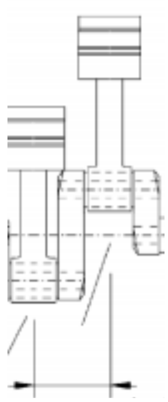
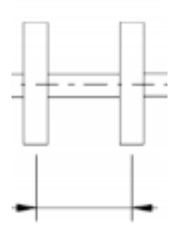
Exercise 1



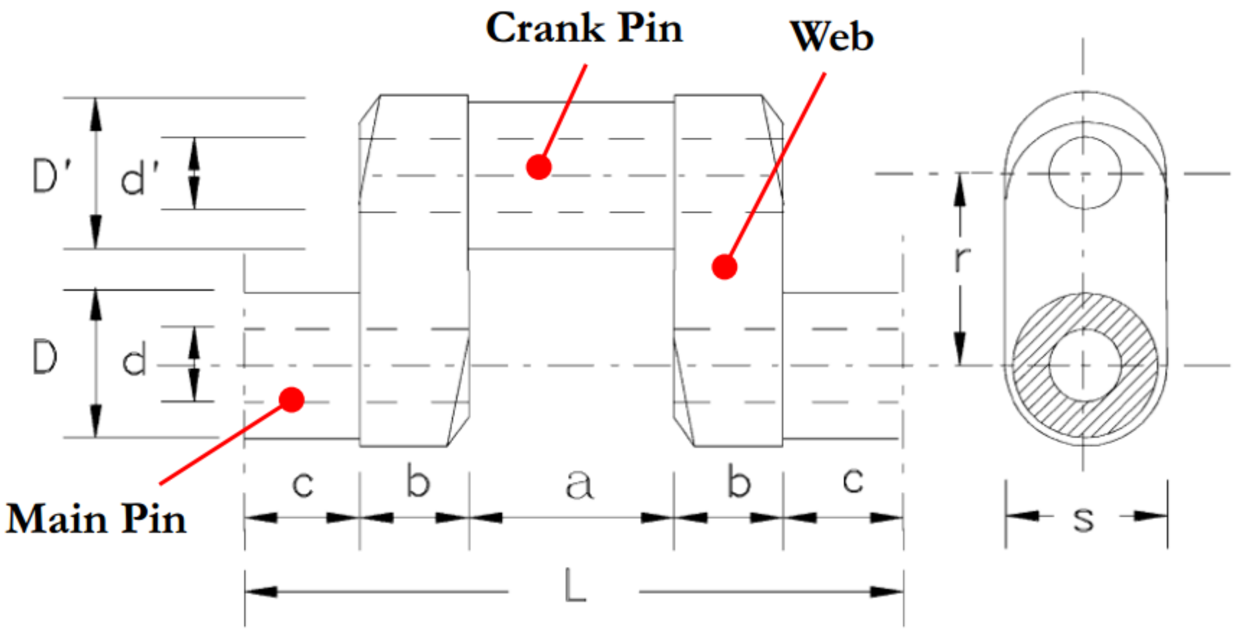
For studying the torsional behavior of the reciprocating machine, the system is modeled as a single shaft with torsional stiffness only.



Actual length



Equivalent length



The equivalent length is computed according to Carter's model :

$$L_{eq} = 2c + 0.8b + \frac{3}{4} * \frac{D^4 - d^4}{D'^4 - d'^4} + \frac{3}{2} * \frac{D^4 - d^4}{bs^3} r = 149.54[mm] \approx 0.150\ [m]$$

The equivalent torsion stiffness of the crankshaft is :

$$K_{eq,C} = \frac{GI_{p,mp}}{l_{eq}} = 4.040 * 10^4 [Nm/rad]$$

The second moment of area of the Main Pin is:

$$I_{p,MP} = \pi \frac{D^4 - d^4}{32} = 7.854 * 10^{-8}\ [m^4]$$

The equivalent moment of inertia of the Crank J_C

$$J_C = 2J_{MP} + 2J_W + J_{CP}$$

$$J_{MP} = \pi \frac{D^4 - d^4}{32} . c . \rho = 1.2252 * 10^{-5} [Kgm^2]$$

$$J_{CP} = \pi \frac{D'^4 - d'^4}{32} . a . \rho + \pi \frac{D'^2 - d'^2}{4} . a . \rho . r^2 = 3.1571 * 10^{-4} [Kgm^2]$$

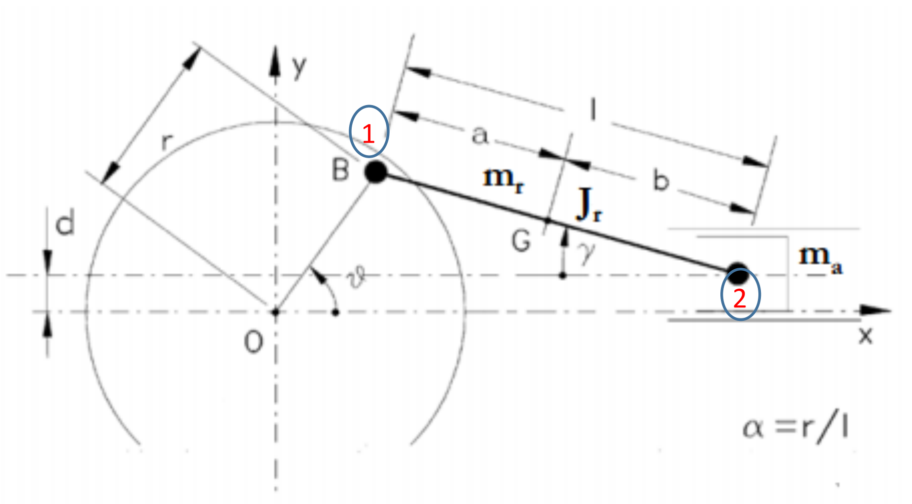
$$J_W = \frac{sr^3}{12} . b . \rho + m_W (r/2)^2 = 1.8588 * 10^{-4}\ [Kgm^2]$$

$$J_C = 2J_{MP} + 2J_W + J_{CP} = 7.1217 * 10^{-4}\ [Kgm^2]$$

The equivalent moment of inertia of the Crank/Rod/Reciprocating masses is:

$$J_{eq} = J_C + 2m_{b1}r^2 + (m_{b2+}m_p)r^2 * f_1(\theta) + J_0 * f_2(\theta)$$

In the equivalent Rod:



1) The same mass as the real Rod:

$$m_b = m_{b1} + m_{b2}$$

2) The same center of mass:

$$x_G m_b = x_1 m_1 + x_2 m_2$$

$$a * m_1 = b * m_2$$

$$m_1 = m_2 * \frac{b}{a} = (m_b - m_1) * \frac{b}{a}$$

$$m_1 \underbrace{\left(1 + \frac{b}{a}\right)} = m_b * \frac{b}{a}$$

$$\frac{a+b}{a} = \frac{l}{a}$$

$$\begin{cases} m_1 = \frac{b}{l} * m_b = 1.0011 \ \ [Kg] \\ m_2 = \frac{a}{l} * m_b = 0.402 \ \ [Kg] \end{cases}$$

3) The same moment of inertia:

$$J_0 = J_b - (m_1 a^2 + m_2 b^2) = -3.878 * 10^{-3} [Kgm^2]$$

$$f_1(\theta) = a_0 + \sum_{i=1}^{7} a_i \cos(i\theta) + \sum_{i=1}^{6} b_i \sin(i\theta)$$

$$f_2(\theta) = c_0 + \sum_{i=1}^{4} c_i \cos(i\theta) + \sum_{i=1}^{3} d_i \sin(i\theta)$$

If the equivalent moment of inertia is constant (it does not depend anymore on θ)

$$f_1(\theta) = a_0 = \frac{8 + 2\alpha(1 + 6\beta^2) + 8\beta^2(1 + \beta^2) + \alpha^4}{16} = 0.514$$

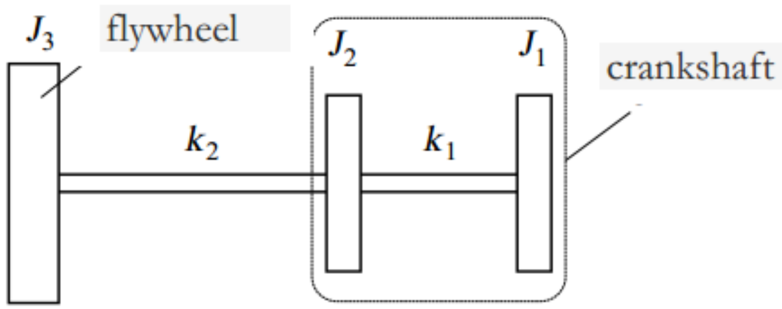
$$f_2(\theta) = c_0 = \alpha^2 * \frac{4 + \alpha^2(1 + 6\beta^2) + 4\beta^2}{8} = 0.054$$

$$\beta = \frac{d}{l} = 0$$

$$\alpha = \frac{r}{l} = 0.325$$

$$J_{eq} = J_C + 2m_{b1} r^2 + \left(m_{b2+} m_p\right) r^2 * f_1(\theta) + J_0 * f_2(\theta) = 4.467 * 10^{-3} [kgm^2]$$

Exercise 2



By adopting Lagrange approach:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)+\frac{\partial U}{\partial q_i}+\frac{\partial \mathcal{F}}{\partial \dot{q}_i}=\frac{\partial(\delta L)}{\partial(\delta q_i)}$$

Where there is not any external force and dissipation function,($\frac{\partial \mathcal{F}}{\partial \dot{q}_i}=0,\quad \frac{\partial(\delta L)}{\partial(\delta q_i)}=0$)

The potential energy in the system is equal to:

$$U=\frac{1}{2}*k_2(\theta_3-\theta_2)^2+\frac{1}{2}*k_1(\theta_2-\theta_1)^2$$

$$\frac{\partial U}{\partial \theta_1}=-k_1(\theta_2-\theta_1)$$

$$\frac{\partial U}{\partial \theta_2}=-k_2(\theta_3-\theta_2)+k_1(\theta_2-\theta_1)$$

$$\frac{\partial U}{\partial \theta_3}=k_2(\theta_3-\theta_2)$$

Kinetic energy is defined as :

$$T=\frac{1}{2}J_1\dot{\theta}_1^2+\frac{1}{2}J_2\dot{\theta}_2^2+\frac{1}{2}J_3\dot{\theta}_3^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)=J_1\ddot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)=J_2\ddot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right)=J_3\ddot{\theta}_3$$

$$\begin{cases} J_1\ddot{\theta}_1 - k_1(\theta_2 - \theta_1) \ = 0 \\ J_2\ddot{\theta}_2 - k_1\theta_1 + (k_1 + k_2)\theta_2 - k_2\theta_3 = 0 \\ J_3\ddot{\theta}_3 + k_2(\theta_3 - \theta_2) = 0 \end{cases}$$

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The equivalent stiffness of the Crank mechanism can be computed by this equation

$$k_1 = \frac{GI_{p,mp}}{l_{eq}} = 3.90 * 10^4[Nm/rad]$$

$$I_{p,MP} = \ \pi\frac{D^4 - d^4}{32} = 7.85 * 10^{-8}[m^4]$$

$$\{\theta\} = \theta_0 e^{i\omega t} \ \ ,\{\dot{\theta}\} = i\omega\theta_0 e^{i\omega t}, \ \ \{\ddot{\theta}\} = -\omega^2\theta_0 e^{i\omega t}$$

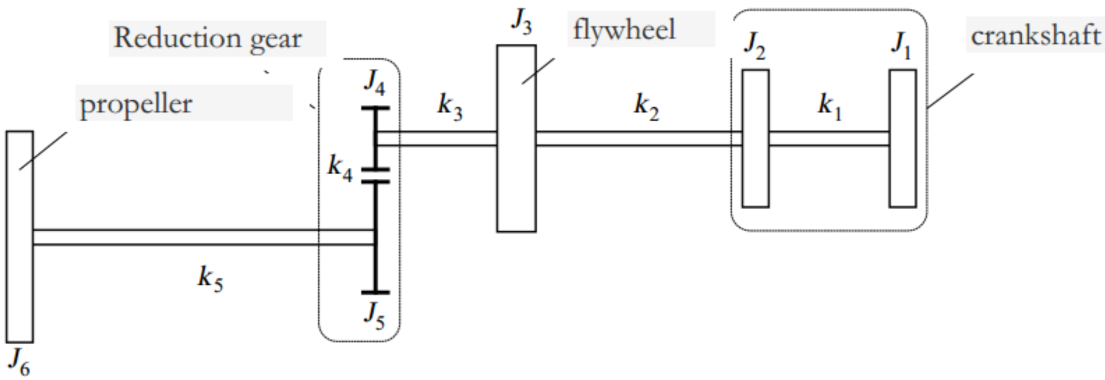
$$\det\left([k] - [M]\omega^2\right) = 0$$

Because of rigid body motion, the system is not properly constrained, the first natural frequency is equal to zero, the two others are:

$$\omega_{n1} = 2.16 * 10^3[rad/s]$$

$$\omega_{n1} = 5.02 * 10^3[rad/s]$$

Exercise 3



The transmission ratio is the ratio of the rotating velocity

$$\tau = \frac{\omega_5}{\omega_4} = \frac{\dot{\theta}_5}{\dot{\theta}_4} = \frac{\theta_5}{\theta_4}$$

k_4 (teeth of the gear and also the contact point stiffness)is negligible

By Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} + \frac{\partial \mathcal{F}}{\partial \dot{q}_i} = \frac{\partial(\delta L)}{\partial(\delta q_i)}$$

where there is not any external force and dissipation function, ($\frac{\partial \mathcal{F}}{\partial \dot{q}_i} = 0,\ \ \frac{\partial(\delta L)}{\partial(\delta q_i)} = 0$)

The potential energy in the system is equal to:

$$U = \frac{1}{2} * \underline{k_5(\theta_6 - \theta_5)}^2 + \frac{1}{2} * k_3(\theta_4 - \theta_3)^2 + \frac{1}{2} * k_2(\theta_3 - \theta_2)^2 + \frac{1}{2} * k_1(\theta_2 - \theta_1)^2$$

Now by considering transmission factor :

$$U = \frac{1}{2} * \underline{k_5(\theta_6 - \tau\theta_4)}^2 + \frac{1}{2} * k_3(\theta_4 - \theta_3)^2 + \frac{1}{2} * k_2(\theta_3 - \theta_2)^2 + \frac{1}{2} * k_1(\theta_2 - \theta_1)^2$$

$$k_5(\theta_6 - \tau\theta_4)^2 = \ k_5\tau^2(\frac{\theta_6}{\tau} - \theta_4)^2 = \ \underline{k_5^*(\theta_6^* - \theta_4)}^2$$

$$U = \frac{1}{2} * k_5^*(\theta_6^* - \theta_4)^2 + \frac{1}{2} * k_3(\theta_4 - \theta_3)^2 + \frac{1}{2} * k_2(\theta_3 - \theta_2)^2 + \frac{1}{2} * k_1(\theta_2 - \theta_1)^2$$

$$\frac{\partial U}{\partial \theta_1} = -\, k_1(\theta_2 - \theta_1)$$

$$\frac{\partial U}{\partial \theta_2} = -k_2(\theta_3 - \theta_2) + \ k_1(\theta_2 - \theta_1) = -k_1\theta_1 + \ \theta_2(k_1 + k_2) - k_2\theta_3$$

$$\frac{\partial U}{\partial \theta_3} = -k_3(\theta_4 - \theta_3) + \ k_2(\theta_3 - \theta_2) = \ -k_2\theta_2 + \theta_3(k_2 + k_3) - \ k_3\theta_4$$

$$\frac{\partial U}{\partial \theta_4} = -k_5^*(\theta_6^* - \theta_4) + k_3(\theta_4 - \theta_3) = -k_3\theta_3 + \theta_4(k_3 + \ k_5^*) - k_5^* \, \theta_6^*$$

$$\frac{\partial U}{\partial \theta_6^*} = k_5^*(\theta_6^* - \theta_4)$$

The kinetic energy is computed as:

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \ \frac{1}{2}J_2\dot{\theta}_2^2 + \ \frac{1}{2}J_3\dot{\theta}_3^2 + \ \frac{1}{2}J_4\dot{\theta}_4^2 + \ \frac{1}{2}J_5\dot{\theta}_5^2 + \ \frac{1}{2}J_6\dot{\theta}_6^2$$

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \ \frac{1}{2}J_2\dot{\theta}_2^2 + \ \frac{1}{2}J_3\dot{\theta}_3^2 + \ \frac{1}{2}J_4\dot{\theta}_4^2 + \ \frac{1}{2}J_5\tau^2\dot{\theta}_4^2 + \ \frac{1}{2}J_6\tau^2\dot{\theta}_6^*$$

$$\frac{1}{2}J_6\tau^2\dot{\theta}_6^* = \frac{1}{2}J_6^*\dot{\theta}_6^* \quad , \quad \frac{1}{2}J_5\tau^2\dot{\theta}_4^2 = \frac{1}{2}J_5^* \, \dot{\theta}_4^2$$

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \ \frac{1}{2}J_2\dot{\theta}_2^2 + \ \frac{1}{2}J_3\dot{\theta}_3^2 + \ \frac{1}{2}(J_4 + J_5^* \,)\dot{\theta}_4^2 + \ \frac{1}{2}J_6^*\dot{\theta}_6^*$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = J_1\ddot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = J_2\ddot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = J_3\ddot{\theta}_3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_4}\right) = (J_4 + J_5^* \,)\ddot{\theta}_4$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_6^*}\right) = J_6^*\ddot{\theta}_6^*$$

$$\begin{cases} J_1\ddot{\theta}_1 \ + k_1\theta_1 - k_1\theta_2 = 0 \\ J_2\ddot{\theta}_2 - k_1\theta_1 + \ \theta_2(k_1 + k_2) - k_2\theta_3 = 0 \\ J_3\ddot{\theta}_3 - k_2\theta_2 + \theta_3(k_2 + k_3) - \ k_3\theta_4 = 0 \\ (J_4 + J_5^* \,)\ddot{\theta}_4 - k_3\theta_3 + \theta_4(k_3 + \ k_5^*) - k_5^* \, \theta_6^* = 0 \\ J_6^*\ddot{\theta}_6^* - \ k_5^*\theta_4 + k_5^*\theta_6^* = 0 \end{cases}$$

$$\begin{bmatrix} J_1 & 0 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & J_3 & 0 & 0 \\ 0 & 0 & 0 & J_4 + J_5^* & 0 \\ 0 & 0 & 0 & 0 & J^*{}_6 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \\ \ddot{\theta}_6^* \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & 0 & -k_3 & k_3 + k_5^* & -k_5^* \\ 0 & 0 & 0 & -k_5^* & -k_5^* \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_6^* \end{Bmatrix} = \{0\}$$

$\{\theta\} = \theta_0 e^{i\omega t}$, $\{\dot{\theta}\} = i\omega\theta_0 e^{i\omega t}$, $\{\ddot{\theta}\} = -\omega^2\theta_0 e^{i\omega t}$

$\det([k] - [M]\omega^2) = 0$

$\omega_{n1} = 2.20 * 10^2 \left[\frac{rad}{s}\right]$, $\omega_{n2} = 2.16 * 10^3 \left[\frac{rad}{s}\right]$, $\omega_{n3} = 5.02 * 10^3 \left[\frac{rad}{s}\right]$,

$\omega_{n4} = 1.74 * 10^4 \left[\frac{rad}{s}\right]$, $\omega_{n5} = 3.72 * 10^5 \left[\frac{rad}{s}\right]$