INTEGRATION BY PARTS IDENTITIES

for parametrized Feynman integrals

November 7, 2024
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based on 2310.03939 and work in progess with L. Magnea

RTG 2575:
Rethinking
Quantum Field Theory

MOTIVATIONS

PREVIOUSLY IN THIS ROOM

48 hours ago

- How can we use symmetries to reduce the number of Feynman diagrams to compute
- Bootstrapping integrals and correlators (in presence of a Wilson line)

Today

- Techniques for solving large sets of Feynman integrals
- Testing an alternative approach based on a parametric representation of Feynman integrals

WHY FEYNMAN INTEGRALS

- For particle scattering

Graded transcendental functions: an application to four-point amplitudes with one off-shell leg
Thomas Gehrmann, Johannes Henn, Petr Jakubčík, Jungwon Lim, Cesare Carlo Mella et al. (Oct 24, 2024)
e-Print: 2410.19088 [hep-th]

- For gravitational physics

Gravitational scattering and beyond from extreme mass ratio effective field theory
Clifford Cheung (Caltech), Julio Parra-Martinez (IHES, Bures-sur-Yvette), Ira Z. Rothstein (Carnegie Mellon U.), Nabha Shah (Caltech), Jordan Wilson-Gerow (Caltech) (Jun 20, 2024)
Published in: *JHEP* 10 (2024) 005 • e-Print: 2406.14770 [hep-th]

Classifying post-Minkowskian geometries for gravitational waves via loop-by-loop Baikov
Hjalte Frellesvig (Bohr Inst.), Roger Morales (Bohr Inst.), Matthias Wilhelm (Bohr Inst.) (May 27, 2024)
Published in: *JHEP* 08 (2024) 243 • e-Print: 2405.17255 [hep-th]

- For cosmology

Algebraic Approaches to Cosmological Integrals
Claudia Fevola, Guilherme L. Pimentel, Anna-Laura Sattelberger, Tom Westerdijk
Oct 18, 2024

WHY FEYNMAN PARAMETRIZATION

The Monodromy Rings of a Class of Self-Energy Graphs

G. PONZANO, T. REGGE, E. R. SPEER*, and M. J. WESTWATER*
Institute for Advanced Study, Princeton, New Jersey

Received April 18, 1969

Abstract. The monodromy rings of self-energy graphs, with two vertices and an arbitrary number of connecting lines, are determined.

§ 1. Introduction

This paper is the first of a series of publications in which we hope to elucidate in a systematic way the properties of Feynman integrals. The motivation for this work is clear: we hope to develop sufficiently the methods of investigating functions of several complex variables defined by integrals to give a basis for the determination of the analytic structure of the S-matrix itself. This is admittedly not an easy task and one whose outcome we cannot guarantee. An ideal research program should be carried out in three steps:

WHY FEYNMAN PARAMETRIZATION

Generalized Cuts of Feynman Integrals in Parameter Space

Ruth Britto
Institute for Advanced Study, Einstein Drive, Princeton NJ 08540, USA and
School of Mathematics and Hamilton Mathematics Institute, Trinity College, Dublin 2, Ireland

Feynman integral relations from parametric annihilators

Thomas Bitoun[1] · Christian Bogner[2] · René Pascal Klausen[2] · Erik Panzer[3]

Two-loop helicity amplitudes for $gg \to ZZ$ with full top-quark mass effects

Bakul Agarwal,[a] Stephen P. Jones,[b,c] and Andreas von Manteuffel[a]

Feynman Integral Relations from GKZ Hypergeometric Systems

Henrik J. Munch[a,b,*]

***D*-module techniques for solving differential equations in the context of Feynman integrals**

Johannes Henn[1] · Elizabeth Pratt[2] · Anna-Laura Sattelberger[3,4] · Simone Zoia[5,6,7]

FEYNMAN PARAMETERS
and
NOTATION

LOOP INTEGRALS AND FEYNMAN TRICK

Working in d dimensions, with $d = 4 - 2\epsilon$, and allowing for the possibility of raising propagators to integer powers ν_i we associate to each graph G a family of Feynman integrals

$$\boxed{I_G(\nu_i, d) = (\mu^2)^{\nu - ld/2} \int \prod_{r=1}^{l} \frac{d^d k_r}{\mathrm{i}\pi^{d/2}} \prod_{i=1}^{n} \frac{1}{(-q_i^2 + m_i^2)^{\nu_i}}}$$

The integration over loop momenta can be performed in full generality by means of the Feynman parameter technique, using the identity

$$\prod_{i=1}^{n} \frac{1}{\left(-q_i^2 + m_i^2\right)^{\nu_i}} = \frac{\Gamma(\nu)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z\, \delta\left(1 - \sum_{j=1}^{n} z_j\right) \frac{\prod_{j=1}^{n} z_j^{\nu_j - 1}}{\left(\sum_{j=1}^{n} z_j \left(-q_j^2 + m_j^2\right)\right)^{\nu}}$$

SYMANZIK POLYNOMIALS [Weinzierl '22]

$$I_G\left(\nu_i, d\right) = \frac{\Gamma(\nu - ld/2)}{\prod_{j=1}^{n} \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z \, \delta\left(1 - \sum_{j=1}^{n} z_j\right) \left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\nu-(l+1)d/2}}{\mathcal{F}^{\nu - ld/2}}$$

co-tree $\mathcal{T}_G \subset \mathcal{I}_G$

set of internal lines of G such that that the lines in its complement $\overline{\mathcal{T}}_G \subset \mathcal{I}_G$ form a spanning tree

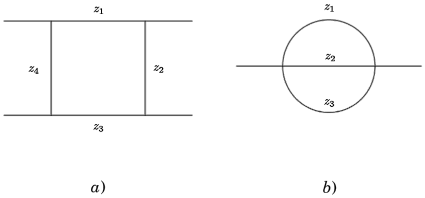
cut $\mathcal{C}_G \subset \mathcal{I}_G$

subset $\mathcal{C}_G \subset \mathcal{I}_G$ with the property that, upon omitting the lines of $\mathcal{C}_G$ from G, the graph becomes a disjoint union of two connected subgraphs.

$$\mathcal{U} = \sum_{\mathcal{T}_G} \prod_{i \in \mathcal{T}_G} z_i$$

$$\mathcal{F} = \sum_{\mathcal{C}_G} \frac{\hat{s}\left(\mathcal{C}_G\right)}{\mu^2} \prod_{i \in \mathcal{C}_G} z_i - \mathcal{U} \sum_{i \in \mathcal{I}_G} \frac{m_i^2}{\mu^2} z_i$$

SYMANZIK POLYNOMIALS



a One-loop equal-mass box

$$\mathcal{U} = z_1 + z_2 + z_3 + z_4$$

$$\mathcal{F} = \frac{s}{\mu^2} z_1 z_3 + \frac{t}{\mu^2} z_2 z_4 - \frac{m^2}{\mu^2} (z_1 + z_2 + z_3 + z_4)^2$$

b Two-loop equal-mass sunrise

$$\mathcal{U} = z_1 z_2 + z_2 z_3 + z_1 z_3$$

$$\mathcal{F} = \frac{p^2}{\mu^2} z_1 z_2 z_3 - \frac{m^2}{\mu^2} (z_1 z_2 + z_2 z_3 + z_1 z_3)(z_1 + z_2 + z_3)$$

CONSISTENCY RELATIONS [DA, Magnea '23]

The definitions of the Symanzik polynomials already provide linear relations among integrals

$$\int_{z_j \geq 0} d^n z\, \delta\left(1 - \sum_{j=1}^{n} z_j\right)\left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right)\frac{\mathcal{U}^{\,\nu-(l+1)d/2}}{\mathcal{F}^{\,\nu-ld/2}} =$$

$$= \int_{z_j \geq 0} d^n z\, \delta\left(1 - \sum_{j=1}^{n} z_j\right)\left(\mathcal{U}\prod_{j=1}^{n} z_j^{\nu_j - 1}\right)\frac{\mathcal{U}^{\,\nu-1-(l+1)d/2}}{\mathcal{F}^{\,\nu-ld/2}}$$

$$= \int_{z_j \geq 0} d^n z\, \delta\left(1 - \sum_{j=1}^{n} z_j\right)\left(\mathcal{F}\prod_{j=1}^{n} z_j^{\nu_j - 1}\right)\frac{\mathcal{U}^{\,\nu-(l+1)d/2}}{\mathcal{F}^{\,1+\nu-ld/2}}$$

For example

$$I(1,1,1,3\epsilon) = p^2 I(2,2,2,3\epsilon) - 3m^2 I(2,1,1,1+3\epsilon)$$

PROJECTIVE FORMS
and
INTEGRATION BY PARTS

PROJECTIVE VOLUME FORMS [Regge '67]

$$\omega_A \;=\; dz_{i_1} \wedge \ldots \wedge dz_{i_a} \,,$$

The volume form ω_A can be ‘integrated’

$$\boxed{\eta_A \;=\; \sum_{i \in A} \epsilon_{i,A-i}\, z_i\, \omega_{A-i}}$$

meaning that

$$d\eta_A \;=\; a\,\omega_A \,.$$

As an example, consider $A = \{1,2,3\}$: the form η_A is then given by

$$\eta_{\{1,2,3\}} \;=\; z_1\, dz_2 \wedge dz_3 - z_2\, dz_1 \wedge dz_3 + z_3\, dz_1 \wedge dz_2 \,,$$

PROJECTIVE FORMS

Affine q-forms

$$\psi_q = \sum_{|A|=q} R_A(z_i)\,\omega_A$$

where R_A is a homogeneous rational function of the variables z_i with degree $-|A| = -q$.

Projective form

Affine form that can be identically re-written as a linear combination of the 'integrated' forms η_A

$$\boxed{\psi_q = \sum_{|B|=q+1} T_B(z_i)\,\eta_B}$$

where the homogeneous functions $T_B(z_i)$ are obtained by suitably combining the functions $R_A(z_i)$ with appropriate factors of z_i arising from the definition of η_A

PROJECTIVE FORMS

Affine q-forms

$$\psi_2\left(\nu_i,\lambda,r\right)=\frac{z_1^{\nu_1}\,z_2^{\nu_2-1}\,z_3^{\nu_3-1}\left(z_1z_2+z_2z_3+z_3z_1\right)^{\lambda}}{\left(r\,z_1z_2z_3-\left(z_1z_2+z_2z_3+z_3z_1\right)\left(z_1+z_2+z_3\right)\right)^{\frac{2\lambda+\nu}{3}}}dz_2\wedge dz_3$$

Projective form

$$\psi_2\left(\nu_i,\lambda,r\right)=\frac{z_1^{\nu_1-1}\,z_2^{\nu_2-1}\,z_3^{\nu_3-1}\left(z_1z_2+z_2z_3+z_3z_1\right)^{\lambda}}{\left(r\,z_1z_2z_3-\left(z_1z_2+z_2z_3+z_3z_1\right)\left(z_1+z_2+z_3\right)\right)^{\frac{2\lambda+\nu}{3}}}\eta_{\{1,2,3\}}$$

FEYNMAN INTEGRALS

$$\int_{z_j \geq 0} d^n z \, \delta\left(1 - \sum_{j=1}^{n} z_j\right) \left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\nu-(l+1)d/2}}{\mathcal{F}^{\nu-ld/2}} =$$

$$\int_{S_{n-1}} \eta_{n-1} \left(\prod_{j=1}^{n} z_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\nu-(l+1)d/2}}{\mathcal{F}^{\nu-ld/2}}$$

Theorem

Given two integration domains, $O, O' \in \mathbb{C}^n$, if their image in $\mathbb{PC}^{n-1}$ is the same simplex S_{n-1}, then $\int_O \alpha_{n-1} = \int_{O'} \alpha_{n-1}$.

CHENG-WU THEOREM [Cheng, Wu ’87]

Theorem

Given two integration domains, $O, O' \in \mathbb{C}^n$, if their image in $\mathbb{PC}^{n-1}$ is the same simplex, then $\int_O \alpha_{n-1} = \int_{O'} \alpha_{n-1}$.

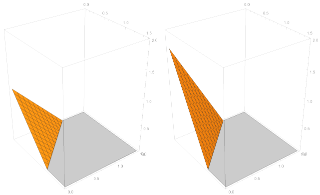
Proof.

This follows immediately from the fact that

i) α_{n-1} is a closed form;

ii) η_{n-1} is null on each surface defined by $z_i = 0$

$\square$



THE CORE THEOREM [Regge '67][DA, Magnea '23]

Theorem

The boundary of a projective form is itself projective.

$$\boxed{d \circ p + p \circ d = 0}$$

$$p \; : \; \sum_{|A|=q} R_A(z_i)\, \omega_A \quad \to \quad \sum_{|A|=q} R_A(z_i)\, \eta_A$$

THE CORE THEOREM - PROOF [DA, Magnea '23]

$$\left(d \circ p + p \circ d\right) \psi_q = \sum_{j \in A,\, i \notin A} \frac{\partial R_A}{\partial z_i}\, z_j \left((-1)^{|A_i|} (-1)^{|(A \cup i)_j|} + (-1)^{|A_j|} (-1)^{|(A-j)_i|} \right) \omega_{A \cup i - j}$$
$$+ \sum_{j \in A,\, i = j} R_A\, \delta_{i,j}\, \omega_A + \sum_{i \in D} \frac{\partial R_A}{\partial z_i}\, z_i\, \omega_A \;=\; 0\,,$$

As an example, consider

$$\psi_2 \;=\; \frac{z_1 + z_3}{(z_1 + z_2)^3}\, dz_1 \wedge dz_2\,,$$

which implies

$$d\psi_2 \;=\; \frac{1}{(z_1 + z_2)^3}\, dz_1 \wedge dz_2 \wedge dz_3 \quad \longrightarrow \quad \left(p \circ d\right)\psi_2 \;=\; \frac{1}{(z_1 + z_2)^3}\, \eta_{\{1,2,3\}}\,.$$

On the other hand

$$\begin{aligned}\left(d \circ p\right)\psi_2 =& d\left(\frac{z_1(z_1 + z_3)}{(z_1 + z_2)^3}\, dz_2 - \frac{z_2(z_1 + z_3)}{(z_1 + z_2)^3}\, dz_1 \right) \\ =& -\frac{z_3}{(z_1 + z_2)^3}\, dz_1 \wedge dz_2 - \frac{z_1}{(z_1 + z_2)^3}\, dz_2 \wedge dz_3 + \frac{z_2}{(z_1 + z_2)^3}\, dz_1 \wedge dz_3 \\ =& -\frac{1}{(z_1 + z_2)^3}\, \eta_{\{1,2,3\}}\,,\end{aligned}$$

PHYSICAL MEANING [DA, Magnea '23]

This theorem implies that if we consider a Feynman integrand (*i.e.* a projective form),

$$\omega_{n-2} \equiv \sum_{i=1}^{n} (-1)^i \, \eta_{\{z\}-z_i} \, \frac{H_i(z)}{(P-1)\,\big(D(z)\big)^{P-1}}$$

then the differentiation of this form is still made of Feynman integrands

$$\boxed{d\omega_{n-2} = \frac{1}{(P-1)\,\big(D(z)\big)^{P-1}} \, \eta_{\{z\}} \sum_{i=1}^{n} \frac{\partial H_i(z)}{\partial z_i} - \frac{\eta_{\{z\}}}{\big(D(z)\big)^{P}} \sum_{i=1}^{n} H_i \, \frac{\partial D(z)}{\partial z_i}}$$

This creates linear relations between Feynman integrals in parameter space.

REMEMBER YOUR BOUNDARIES

$$\boxed{d\omega_{n-2} = \frac{1}{(P-1)\big(D(z)\big)^{P-1}}\,\eta_{\{z\}}\sum_{i=1}^{n}\frac{\partial H_i(z)}{\partial z_i} - \frac{\eta_{\{z\}}}{\big(D(z)\big)^{P}}\sum_{i=1}^{n}H_i\,\frac{\partial D(z)}{\partial z_i}}$$

The integral of the form $d\omega_{n-2}$ is not trivially zero (as boundary terms in the momentum space approach): the boundary of S_{n-1} consists in n copies of S_{n-2}! Stokes theorem leads to integral over S_{n-2} with a parameter set to zero.

Physically, this correspond to shrinking a line in the diagram to a point.

EXAMPLE: ONE-LOOP FIVE-POINT DIAGRAM

[Bern, Dixon, Kosower'93]

We derive the dimensional-shift relation for the one-loop pentagon by considering the equations generated by terms as

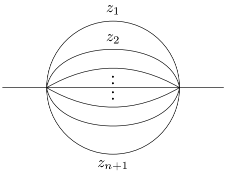
$$\omega_3 = -\eta_{\{z_2,z_3,z_4,z_5\}} \frac{(z_1+z_2+z_3+z_4+z_5)^{2\epsilon}}{(2+\epsilon)\,\mathcal{F}(z)^{2+\epsilon}}$$

generating non-vanishing boundary terms corresponding to box-integrals

$$\begin{aligned} 2(2+\epsilon)\,I(1,1,1,1,1;1+2\epsilon) = \Bigg\{ & \frac{s_{13}s_{24}-s_{13}s_{25}-s_{14}s_{25}+s_{14}s_{35}-s_{24}s_{35}}{s_{13}s_{14}s_{25}} I^{(1)}_{\text{box}} \\ & - \frac{s_{13}s_{24}+s_{13}s_{25}-s_{14}s_{25}+s_{14}s_{35}-s_{24}s_{35}}{s_{13}s_{24}s_{25}} I^{(2)}_{\text{box}} \\ & - \frac{s_{13}s_{24}-s_{13}s_{25}+s_{14}s_{25}-s_{14}s_{35}+s_{24}s_{35}}{s_{13}s_{24}s_{35}} I^{(3)}_{\text{box}} \\ & + \frac{s_{13}s_{24}-s_{13}s_{25}+s_{14}s_{25}-s_{14}s_{35}-s_{24}s_{35}}{s_{14}s_{24}s_{35}} I^{(4)}_{\text{box}} \\ & - \frac{s_{13}s_{24}-s_{13}s_{25}+s_{14}s_{25}+s_{14}s_{35}-s_{24}s_{35})}{s_{14}s_{25}s_{35}} I^{(5)}_{\text{box}} \Bigg\} \end{aligned}$$

MULTILOOP EXAMPLES [Laporta, Remiddi '04]

We tested our method on equal-mass 2-point banana diagrams up to 4-loops with a naive seeding.



- 2-loop: 302 equations $\rightarrow$ < 10 used
- 3-loop: 968 equations $\rightarrow$ < 70 used
- 4-loop: 3749 equations $\rightarrow$ < 150 used

$$\frac{r}{3}\,\frac{d^2}{dr^2}I(1,1,1;0) + \left(\frac{1}{3}+\frac{3}{r-9}+\frac{1}{3(r-1)}\right)\frac{d}{dr}I(1,1,1;0) - \left(\frac{1}{4(r-9)}+\frac{1}{12(r-1)}\right)I(1,1,1;0) = \frac{2}{(r-1)(r-9)}$$

AVAILABLE TOOLS

- Laporta algorithm

  [Laporta '01]
- Computer programs with finite field reduction (Kira, FIRE, LiteRed ...)

  [Maierhöfer, Usovitsch, Uwer, Klappert, Lange, Smirnov, Lee..]
- Non-linear algebra based packages (NeatIBP)

  [Wu, Bohem, Ma, Xu, Zhang '23]
- bypassing the problem: intersection theory

  [Mastrolia, Mizera '19][Frellesvig et al. '19]

NON-LINEAR ALGEBRA ENTERS THE ROOM

A DIRECTION FOR SIMPLER INTEGRALS

Staring at the IBP equation, we notice that it relates integrals of denominator power P with integrals of denominator power $P-1$

$$\boxed{d\omega_{n-2} = \frac{1}{(P-1)\big(D(z)\big)^{P-1}}\,\eta_{\{z\}}\sum_{i=1}^{n}\frac{\partial H_i(z)}{\partial z_i} - \frac{\eta_{\{z\}}}{\big(D(z)\big)^{P}}\sum_{i=1}^{n} H_i\,\frac{\partial D(z)}{\partial z_i}}$$

This means we can reduce more complicated integrals to simpler ones, provided that

$$\sum_{i=1}^{n} H_i\,\frac{\partial D(z)}{\partial z_i} = Q(z)$$

is the numerator of the integral we want to reduce ($\rightarrow$similar to sygyzy idea [Agarwal, Jones, von Manteuffel '21])

POLYNOMIAL IDEALS [Michalek, Sturmfels '20]

Definition:
Consider R the ring of polynomials in the variables $\{ x_1, ..., x_n \}$. An ideal $I \subset R$ in a non-empty subset such that

1. if $f \in R$ and $g \in I$, then $fg \in I$
2. if $f \in I$ and $g \in I$, then $f + g \in I$

The set of polynomials

$$\sum_{i=1}^{n} H_i \frac{\partial D(z)}{\partial z_i}$$

is the ideal generated from the derivative of the second Symanzik polynomials

SUNRISE REVISITED [DA, Magnea *wip*]

Consider the reduction for the 2-loop sunrise integral (~ 300 naive equations, only ~ 10 used). The whole reduction problem amounts to finding the $Q_i(z)$ for

$$\sum_{i=1}^{3} Q_i \frac{\partial \mathcal{F}(z)}{\partial z_i} = z_1^2 z_2 z_3$$

this is not a trivial problem, but can be solved via the software *Singular* giving

```
Q2[1, 1] = -4 / (3*y^2 - 30*y + 27) *x[1] *x[2] + (2*y^2 - 3*y + 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[2]^2 -
   1 / 3*x[1] *x[3] + (2*y^2 - 10*y + 6) / (3*y^3 - 30*y^2 + 27*y) *x[2] *x[3] +
   (y^3 - 8*y^2 + 2*y + 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[3]^2;
Q2[2, 1] = (y^2 + 6*y - 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[1] *x[2] + (-y^2 + 5*y - 2) / (3*y^3 - 33*y^2 + 57*y - 27) *x[2]^2 +
   (-2*y^3 + 19*y^2 - 16*y + 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[1] *x[3] +
   (y^4 - 11*y^3 + 32*y^2 - 21*y + 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[2] *x[3] +
   (-y^3 + 8*y^2 - 2*y - 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[3]^2;
Q2[3, 1] = (-y^2 - 6*y + 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[1] *x[2] +
   (-2*y^2 + 3*y - 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[2]^2 +
   (2*y^3 - 19*y^2 + 16*y - 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[1] *x[3] +
   (2*y^3 - 3*y^2 - 6*y + 3) / (3*y^4 - 33*y^3 + 57*y^2 - 27*y) *x[2] *x[3] + (-5*y + 3) / (3*y^3 - 33*y^2 + 57*y - 27) *x[3]^2;
```

SUNRISE SOLUTION (and more)

The clever choice of the polynomials solves the reduction in one step, immediately giving the correct decomposition

$$B(3,2,2,3\epsilon+1) =$$

$$\frac{\left(y^2(\epsilon+1)+10y\epsilon-9(3\epsilon+1)\right)B(2,1,1,3\epsilon+1)+(y-3)(3\epsilon+1)B(1,1,1,3\epsilon)}{2(y-9)(y-1)y(\epsilon+1)}+$$

+ boundary terms

The reduction for the 3-loop banana integral was solved in 2 *Singular* calls with this method. However, the second call also needed to investigate the structure of the ideal via Gröbner basis determination.

OUTLOOK

TOWARDS STATE OF THE ART

- Feynman parametrization is now automatized in pySecDec
- Graph polynomials best encode the symmetries of the integral
- By using properties of projective forms, we built linear relations among parametric integrals
- Non-linear algebra tools led to an extremely efficient solution for banana integrals

- The integrals analyzed so far belong to interesting families of functions, but their reduction is easy to deal with
- A study of a two-loop four-point integral is under way to pave the way towards state of the art calculations

OTHER DIRECTIONS

Sector Decomposition [Heinrich '08]

- The integrals appearing in the sector decomposition (numerical) integration method are still integrals of similar kind
- Linear relation among sector integrals may offer performance improvement

Cutting Propagators [Britto '23s]

- Being able to *cut* propagators (*i.e.* putting them on-shell) is a crucial improvement in other algorithms
- Recent papers have analyzed the meaning of cutting propagators in parameter space and an extension of our relations to cut integrals may be of use

THANK YOU!