RTG 2575: Rethinking Quantum Field Theory

IBPs and differential equations in parameter space

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work done in collaboration with L. Magnea

A projective framework for the construction of Integration by Parts (IBP) identities and differential equations for Feynman integrals in Feynman-parameter space is presented. Adapting and generalising these results to the modern language, simple tools of projective geometry are used to generate sets of IBP identities and differential equations in parameter space, with a technique applicable to any loop order.

1 Notations

Consider the integral

$$I_G(\nu_i, d) = \frac{\Gamma(\nu - ld/2)}{\prod_{j=1}^n \Gamma(\nu_j)} \int_{z_j \geq 0} d^n z \, \delta\left(1 - \sum_{j=1}^n z_j\right) \left(\prod_{j=1}^n z_j^{\nu_j - 1}\right) \frac{\mathcal{U}^{\nu-(l+1)d/2}}{\mathcal{F}^{\nu - ld/2}}$$

where the Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$,

$$\mathcal{U} = \sum_{\mathcal{T}_G} \prod_{i \in \mathcal{T}_G} z_i \,, \qquad \mathcal{F} = \sum_{C_G} \frac{\hat{s}(C_G)}{\mu^2} \prod_{i \in C_G} z_i - \mathcal{U} \sum_{i \in \mathcal{I}_G} \frac{m_i^2}{\mu^2} z_i \,,$$

are defined purely from the graph properties. Denote by $\mathcal{I}_G$ the set of the internal lines of G, each endowed with a Feynman parameter z_i.

- A **co-tree** $\mathcal{T}_G \subset \mathcal{I}_G$ is a set of internal lines of G such that the lines in its complement $\overline{\mathcal{T}}_G \subset \mathcal{I}_G$ form a spanning tree.
- Subsets $C_G \subset \mathcal{I}_G$ with the property that, upon omitting the lines of C_G from G, the graph becomes a disjoint union of two connected subgraphs, define a **cut** of graph G, containing $l+1$ lines;

An invariant mass $\hat{s}(C_G)$ can be associated with each cut, by squaring the sum of the momenta flowing in one of the two subgraphs. The Symanzik polynomial $\mathcal{U}$ is homogeneous of degree l, while the Symanzik polynomial $\mathcal{F}$ is homogeneous of degree $l+1$, so that **the integrand (measure included) is homogeneous of degree 0.**

2 A projective framework

A crucial mathematical property of Feynman integrals is that their integrands are **projective forms** in the space of Feynman parameters, which can be identified with $\mathbb{PC}^{n-1}$. In order to provide an overview of this idea, begin by considering a generic subset A, $|A| = a$, of the set $D = \{1, \dots, N\}$, and define the a-form

$$\omega_A = dz_{i_1} \wedge \dots \wedge dz_{i_a} \,,$$

with $i_1 < \dots < i_a$. One can show that ω_A integrates to the projective $(a-1)$-form

$$\eta_A = \sum_{i \in A} \epsilon_{i, A-i} \, z_i \, \omega_{A-i} \,, \qquad d\eta_A = a\omega_A \,,$$

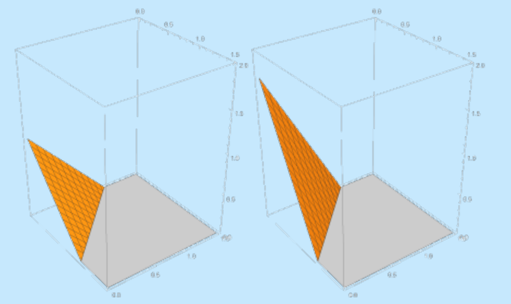
where the signature factor is

$$\epsilon_{k,B} = (-1)^{|B_k|} \,, \qquad B_k = \{i \in B, i < k\} \,.$$

Parametric Feynman integrands can be represented in the general form

$$\alpha_{n-1} = \eta_{n-1} \frac{Q(\{z_i\})}{D^P(\{z_i\})}$$

where $D(\{z_i\})$ and $Q(\{z_i\})$ are polynomials of degrees such that the form (measure included) is homogeneous of degree 0. Two theorems naturally emerge within this projective framework.



Theorem 1. *The boundary of a projective form is itself projective.*

Theorem 2. *Given two integration domains, $O, O' \in \mathbb{C}^n$, if their image in $\mathbb{PC}^{n-1}$ is the same simplex, then $\int_O \alpha_{n-1} = \int_{O'} \alpha_{n-1}$.*

3 Integration by parts in projective space

The correspondence between projective forms and parametric integrals is obtained from the usual choice of chart in projective space identifying a coordinate symplex in $\mathbb{R}^n$ with the choice $\sum_{i=1}^n z_i = 1$.

$$\int_{S_{n-1}} \eta_{n-1} \frac{Q(z)}{D^P(z)} = \int_{z_i \geq 0} dz_1 \dots dz_n \, \delta\left(1 - \sum_{i=1}^n z_i\right) \frac{Q(z)}{D^P(z)} \,.$$

Consider the projective $(n-2)$-forms

$$\omega_{n-2} \equiv \sum_{i=1}^n (-1)^i \, \eta_{\{z\}-z_i} \frac{H_i(z)}{(P-1)\big(D(z)\big)^{P-1}} \,,$$

Differentiating these forms generates (at the integrand level) a set of **integration by parts** identities among parametric integrals.

$$d\omega_{n-2} = \frac{1}{(P-1)\big(D(z)\big)^{P-1}} \, \eta_{\{z\}} \sum_{i=1}^n \frac{\partial H_i(z)}{\partial z_i} - \frac{\eta_{\{z\}}}{\big(D(z)\big)^P} \sum_{i=1}^n H_i \frac{\partial D(z)}{\partial z_i}$$

These identities apply for any number of loops or external legs. At one loop the identity becomes

$$\begin{aligned} d\omega_{n-2} + \sum_{k=1}^{n+1} (s_{kh} + s_{k,n+1}) \, f\big(\{\mathcal{R}-k\}_0, \{k\}_1\big) &= \frac{\nu_h - 1}{\nu - (d+1)/2} f\big(\{h\}_{-1}, \{\mathcal{R}-h\}_0\big) \\ &+ \frac{\nu - d}{\nu - (d+1)/2} f\big(\{n+1\}_{-1}, \{\mathcal{R} - \{n+1\}\}_0\big) \,, \end{aligned}$$

where we can raise or lower the exponents ν_i by adding $\{-1, 0, 1\}$ in the subsets $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ of set $\mathcal{R}$ respectively. Note also that the action of raising and lowering exponents is subject to a constraint, arising from the definition of $\mathcal{U}$. Specifically, the following sum rule holds

$$\sum_{i=1}^n f\big(\{\mathcal{R}-i\}_0, \{i\}_1\big) = f\big(\{\mathcal{R} - \{n+1\}\}_0, \{n+1\}_1\big)$$

4 One-loop example

The equations generated in parameter space effectively include also dimensional-shift identities, and they connect the desired integrals to lower-point integrals through non-vanishing boundary terms. As an example, consider the following identity for five-point integrals:

$$\int_{S_{\{1,2,3,4,5\}}} d\omega_3 + s_{13}\, I(1,1,2,1,1;2\epsilon) + s_{14}\, I(1,1,1,2,1;2\epsilon) = \frac{2\epsilon}{2+\epsilon} I(1,1,1,1,1;-1+2\epsilon)$$

with

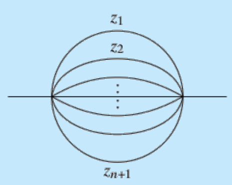
$$d\omega_3 = d\left[-\eta_{\{2,3,4,5\}} \frac{(z_1+z_2+z_3+z_4+z_5)^{2\epsilon}}{(2+\epsilon)\,(s_{13}z_1z_3 + s_{14}z_1z_4 + s_{24}z_2z_4 + s_{25}z_2z_5 + s_{35}z_3z_5)^{2+\epsilon}}\right] .$$

The integration of this form using Stokes theorem produces a **non-vanishing boundary term**, corresponding to the one-loop box integral with one external leg off-shell. Using similar identities, dimensional-shift relations for the one-loop pentagon can easily be reproduced. One finds

$$\begin{aligned} 2(2+\epsilon)\, I(1,1,1,1,1;1+2\epsilon) = & \Bigg\{ \frac{s_{13}s_{24} - s_{13}s_{25} - s_{14}s_{25} + s_{14}s_{35} - s_{24}s_{35}}{s_{13}s_{14}s_{25}} I^{(1)}_{\text{box}}(s_{25}) \\ & - \frac{s_{13}s_{24} + s_{13}s_{25} - s_{14}s_{25} + s_{14}s_{35} - s_{24}s_{35}}{s_{13}s_{24}s_{25}} I^{(2)}_{\text{box}}(s_{13}) \\ & - \frac{s_{13}s_{24} - s_{13}s_{25} + s_{14}s_{25} - s_{14}s_{35} + s_{24}s_{35}}{s_{13}s_{24}s_{35}} I^{(3)}_{\text{box}}(s_{24}) \\ & + \frac{s_{13}s_{24} - s_{13}s_{25} + s_{14}s_{25} - s_{14}s_{35} - s_{24}s_{35}}{s_{14}s_{24}s_{35}} I^{(4)}_{\text{box}}(s_{35}) \\ & - \frac{s_{13}s_{24} - s_{13}s_{25} + s_{14}s_{25} + s_{14}s_{35} - s_{24}s_{35})}{s_{14}s_{25}s_{35}} I^{(5)}_{\text{box}}(s_{14}) \Bigg\} \\ & + 2\epsilon\, I(1,1,1,1,1;-1+2\epsilon) \,. \end{aligned}$$

Since the integral in the last line is finite in $d = 4$, this gives the well-known result stating that the massless pentagon integral is given by a linear combination of box integrals, up to corrections vanishing in four dimensions.

5 Two-loop example



A interesting case beyond one-loop is given by the family of l-loop sunrise diagrams, *i.e.* diagrams contributing to a two-point function, with two vertices connected by $l+1$ propagators. The first Symanzik polynomial for l-loop sunrise integrals is given by

$$\mathcal{U}_l = \sum_{i=1}^{l+1} z_1 \dots \hat{z}_i \dots z_{l+1} \,,$$

where $\hat{z}_i$ denotes the omission of z_i. In the specific case of $l = 2$ and equal internal masses, the Feynman parametric integral is

$$I(\nu_1, \nu_2, \nu_3; \lambda_4) = \int_{S_{\{1,2,3\}}} \frac{\eta_3\, z_1^{\nu_1-1} z_2^{\nu_2-1} z_3^{\nu_3-1} \,(z_1z_2 + z_2z_3 + z_3z_1)^{\lambda_4}}{\Big[r\, z_1z_2z_3 - (z_1+z_2+z_3)(z_1z_2+z_2z_3+z_3z_1)\Big]^{\frac{2\lambda_4+\nu}{3}}} \,.$$

By choosing a suitable numerator in our master identity,

$$H(z) = z_1^{\nu_1-1} z_2^{\nu_2-1} z_3^{\nu_3-1} \big(z_1z_2 + z_2z_3 + z_3z_1\big)^{\lambda_4} \,,$$

one easily derives integration by parts identities. Interestingly, also in this case the non-vanishing boundary term provides an **inhomogeneous contribution to the system**. It arises form the basis integral

$$\int d\omega_1 = \frac{1}{2(1+\epsilon)} \int_{S_{\{1,2\}}} \eta_{\{1,2\}} \frac{(z_1z_2)^\epsilon}{\big[-(z_1+z_2)\big]^{2+2\epsilon}} = \frac{(-1)^{2\epsilon}}{2+2\epsilon} \frac{\Gamma^2(1+\epsilon)}{\Gamma(2+2\epsilon)}$$

corresponding to the massive one-loop tadpole. In two space-time dimensions, the sunrise integral is finite and the linear differential equation system can be analysed for $\epsilon = 0$. The first-order differential equations can be combined into a single second-order equation for the equal-mass sunrise, $(r = \frac{p^2}{m^2})$

$$\begin{aligned} \frac{r}{3} \frac{d^2}{dr^2} I(1,1,1;0) &+ \left(\frac{1}{3} + \frac{3}{r-9} + \frac{1}{3(r-1)}\right) \frac{d}{dr} I(1,1,1;0) \\ &- \left(\frac{1}{4(r-9)} + \frac{1}{12(r-1)}\right) I(1,1,1;0) = \frac{2}{(r-1)(r-9)} \,. \end{aligned}$$

6 Outlook

Comparing the parameter-space method to momentum-space approaches, it's clear that the organisation of calculations differs significantly.

- The differential equations generated by the projective framework are **in general distinct** from the conventional ones.
- Boundary terms, which vanish in the momentum-space approach, play here a **crucial role**, linking complex integrals to simpler ones.
- Parameter-space integrands closely mirror the **graph symmetries**, and circumvent issues related to loop-momentum routing and irreducible numerators, which can complicate momentum-space algorithms.
- The projective framework aligns closely with the **algebraic structures** related to the monodromy ring of Feynman integrals, which may provide direction for future progress.

The present work is largely a feasibility study: for the future, the goal is clearly to extend these techniques to more complex integrals, including higher-loop and multi-scale examples, possibly developing automated tools. Besides the obvious interest in direct physics applications, this will allow for a necessary detailed comparison of parameter-space and momentum-space approaches, including computational aspects.

References

[1] D. Artico and L. Magnea, *"Integration-by-parts identities and differential equations for parametrised Feynman integrals"*, [arXiv:2310.03939 [hep-ph]].

[2] D. Artico and L. Magnea, *"IBPs and differential equations in parameter space"*, [arXiv:2311.02457 [hep-ph]].

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