PERTURBATIVE BOOTSTRAP

of the Wilson line defect CFT

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RTG 2575:
Rethinking
Quantum Field Theory

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1794
ENS
ÉCOLE NORMALE SUPÉRIEURE

MOTIVATIONS

WHY DEFECTS

"[...] all the available tools in Quantum Field Theory have been applied to this subject [defects]: perturbation theory, lattice, conformal bootstrap, holographic methods, supersymmetric localization, and more"
from the webpage of the workshop *Defects, from condensed matter to quantum gravity*, Pollica 2024

$\mathcal{N}=4$ SUSY set-up

- AdS/CFT correspondence
  [Giombi, Roiban, Tseytlin, '17][Erikson, Semenoff, Zarembo, '00][Kristjansen, Zarembo '24]
- Supersymmetric localization
  [Giombi, Komatsu, '18][Drukker, '20]
- Superconformal Ward identity
  [Liendo, Meneghelli, '16][Liendo, Meneghelli, Mitev, '18][Barrat, Liendo, Peveri, Plefka, '21][Bliard, '24]
- Superconformal bootstrap
  [Ferrero, Meneghelli, '23][Ferrero, Meneghelli, '23][Ferrero, Meneghelli, '23]
- Bootstrability
  [Cavaglià, Gromov, Julius, Preti, '22][Cavaglià, Gromov, Julius, Preti, '22][Cavaglià, Gromov, Julius, Preti, '22][Cavaglià, Gromov, Julius, Preti, '22]

non-SUSY set-up

- Conformal bootstrap
  [Billò, Goncalves,Lauria,Meineri, '17][Poland, Rychkov, Vichi'18][Iliesiu, Koloğlu, Mahajan, Perlmutter, Simmons-Duffin, '18]
- Analytic bootstrap for $O(N)$ model
  [Bianchi, Bonomi, de Sabbata, '22][Bianchi, Bonomi, de Sabbata, Gimenez-Grau, '23]
- Locality
  [Levine, Paulos, '23][Levine, Paulos, '24]
- ϵ-expansion
  [Cuomo, Komargodski, Mezei, '21][Barrat, Liendo, Van Vliet '23][Giombi, Helfenberger, Khanchandani, '23]
- Applications to gravity
  [Bachas, Chen, '24]

WHY $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM: the maximally extended supersymmetric theory in four space-time dimensions with a multiplet representation of spin $\ell \leq 1$.

Although the ultimate goal may be calculating non-SUSY amplitudes, SUSY theories provide an excellent testing ground. Looking at super-Yang-Mills offers a lot of insight into how one can deal with the problems in QCD.

Some selected topics of interest

- Leading transcendental weight contribution identical in QCD and $\mathcal{N} = 4$ SYM
- Presence of exact and all order results (e.g. Γ_{cusp} [Beisert, Eden, Staudacher, '07], 4-gluon amplitude [Anastasiou, Bern, Dixon, Kosower '03])
- Duality between amplitudes and Wilson loops [Alday, Maldacena, '07]
- Position space description of event shapes [Henn, Sokatchev, Yan, Zhiboedov, '19]
- Analytic bootstrap [Bern, Dixon, Dunbar, Kosower, '94][Morales, Spiering, Wilhelm, Yang, Zhang '23]

THE PERTURBATIVE BOOTSTRAP

The perturbative bootstrap framework puts together Feynman diagram computation for weak coupling expansion of correlators and non-perturbative information given by the symmetries of the set-up.

SET-UP AND WORKFLOW

LINE DEFECTS

Defects: extended operators preserving a specific subgroup of the original symmetry.

p-dimensional defects preserve a p-dimensional conformal symmetry too.

$$SO(d+1,1) \longrightarrow SO(p+1,1) \otimes SO(q)$$

A defect extends the set of conformal data to compute

BULK CFT

- $\langle \mathcal{O}_i \mathcal{O}_j \rangle \rightarrow$ anomalous dimension γ_i
- $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle \rightarrow$ 3-point coefficient λ_{ijk}

DEFECT CFT

- $\langle \hat{\mathcal{O}}_i \rangle \rightarrow$ anomalous dimension $\hat{\gamma}_i$, 1-point coefficient a_i
- $\langle \hat{\mathcal{O}}_i \mathcal{O}_j \rangle \rightarrow$ 2-point coefficient b_{ij}
- $\langle \hat{\mathcal{O}}_i \hat{\mathcal{O}}_j \hat{\mathcal{O}}_k \rangle \rightarrow$ 3-point coefficient $\hat{\lambda}_{ijk}$

WILSON LINE DEFECT

Conformal line defects preserve some of the original conformal symmetry

$$SO(5,1) \to SO(2,1) \otimes SO(3)$$

Starting from $N = 4$ SYM, we can also preserve half of the supercharges by defining the Wilson line defect

$$\mathcal{W}_\ell = \frac{1}{N} \mathrm{tr}\, \mathcal{P} \exp \int_{-\infty}^{+\infty} d\tau \left(i\dot{x}^\mu A_\mu + |\dot{x}| \theta^i \phi_i(\tau) \right)$$

where $\theta^2 = 1$ to preserve half of the supercharges.
The expectation value of the Wilson line is protected [Drukker, Gross '01][Semenoff, Zarembo '01].

$$\langle \mathcal{W}_\ell \rangle = 1$$

The resulting supersymmetry break is

$$\boxed{PSU(2,2|4) \to OSP(4^*|4) \supset \text{1d CFT}}$$

SYMMETRY BREAKING [Liendo, Meneghelli '16]

$$PSU(2,2|4) \to OSP(4^*|4)$$

$N = 4$ SYM

- 32 supercharges
- $SO(5,1)$ conformal symmetry
- $SO(6)$ R-simmetry

Wilson line defect

- 16 supercharges
- $SO(2,1)$ conformal symmetry
- $SO(5)$ R-simmetry

$$\left\{\phi^1, ..., \phi^6\right\} \longrightarrow \left\{\phi^1, ..., \phi^5\right\} + \phi^6$$

SCALAR INSERTIONS [Giombi, Roiban, Tseytlin '17]

The operators we consider are scalar insertions on the line

$$\hat{\mathcal{O}}_{\hat{\Delta}}(u,\tau) = \frac{1}{\sqrt{n_{\hat{\Delta}}}} \mathcal{W}_\ell \left[(u \cdot \phi(\tau))^{\hat{\Delta}} \right]$$

where $n_{\hat{\Delta}}$ is a normalization factor and u is the $SO(5)$ vector describing the insertion of the protected scalar fields at the point τ on the line.

We can define multipoint correlation functions for defect operators using the $4d$ theory

$$\left\langle \hat{\mathcal{O}}_{\hat{\Delta}_1} ... \hat{\mathcal{O}}_{\hat{\Delta}_n} \right\rangle_{1d} := \frac{1}{N} \left\langle \operatorname{tr} \mathcal{P}\, \hat{\mathcal{O}}_{\hat{\Delta}_1} ... \hat{\mathcal{O}}_{\hat{\Delta}_n} \exp \int_{-\infty}^{+\infty} d\tau \left(i\dot{x}^\mu A_\mu + |\dot{x}| \phi_6(\tau) \right) \right\rangle_{4d}$$

and bulk-defect-defect form factors in a similar fashon

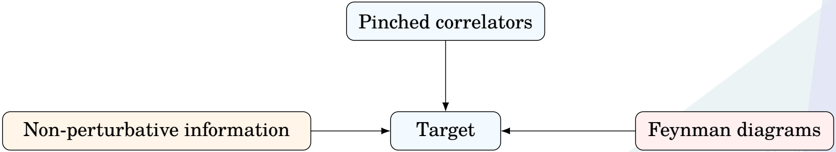
$$\langle \mathcal{O}_{\Delta_1} \hat{\mathcal{O}}_{\hat{\Delta}_2} \hat{\mathcal{O}}_{\hat{\Delta}_3} \rangle = \frac{1}{N} \left\langle \mathcal{O}_{\Delta_1} \operatorname{tr} \mathcal{P} \left[\hat{\mathcal{O}}_{\hat{\Delta}_2} \hat{\mathcal{O}}_{\hat{\Delta}_3} \exp \int_{-\infty}^{+\infty} d\tau \left(i\dot{x}^\mu A_\mu + |\dot{x}| \phi_6(\tau) \right) \right] \right\rangle_{4d} .$$

PERTURBATIVE BOOTSTRAP: WORKFLOW

[DA, Barrat, Peveri ’24][DA, Barrat, Xu ’24]

The computation of correlation functions consists in the following steps:

1. Use of super-conformal symmetry (Ward identities) to constrain the correlator
2. Design of a suitable ansatz (using symbols) for the solution function
3. Input of perturbative information (minimal set of Feynman diagrams) to constrain the ansatz
4. Input of lower-point results (pinching) to fix the last degrees of freedom



BULK-DEFECT-DEFECT CORRELATORS

DEFINITION AND CONVENTIONS

$$\boxed{\langle\, \mathcal{O}_{\Delta_1}(u_1, x_1)\hat{\mathcal{O}}_{\hat{\Delta}_2}(\hat{u}_2, \tau_2)\hat{\mathcal{O}}_{\hat{\Delta}_3}(\hat{u}_3, \tau_3)\,\rangle = \mathcal{K}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}\mathcal{A}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}(\zeta;\chi)\,,}$$

The prefactor $\mathcal{K}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}$ is defined as

$$\mathcal{K}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3} := \frac{(u_1\cdot\hat{u}_2)^{\hat{\Delta}_2}(u_1\cdot\hat{u}_3)^{\hat{\Delta}_3}(u_1\cdot\theta)^{2\Delta_{123}}}{x_{12}^{2\hat{\Delta}_2}x_{13}^{2\hat{\Delta}_3}|\vec{x}_1|^{2\Delta_{123}}}\,,$$

with

$$x_{1j}^2 := \vec{x}_1^2 + \tau_{1j}^2\,, \qquad \Delta_{123} := \Delta_1 - \hat{\Delta}_2 - \hat{\Delta}_3\,.$$

The spacetime and R-symmetry cross-ratios are defined as

$$\chi := \frac{\vec{x}_1^2\tau_{23}^2}{(\vec{x}_1^2+\tau_{12}^2)(\vec{x}_1^2+\tau_{13}^2)}\,, \qquad \zeta := \frac{(u_1\cdot\theta)^2(\hat{u}_2\cdot\hat{u}_3)}{(u_1\cdot\hat{u}_2)(u_1\cdot\hat{u}_3)}\,.$$

The form of the function $\mathcal{A}$ is

$$\boxed{\mathcal{A}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}(\zeta;\chi) = \sum_{j=0}^{n}\left(\frac{\zeta}{\chi}\right)^j F_j(\chi)\,.}$$

NON-PERTURBATIVE CONSTRAINTS

SUPERCONFORMAL SYMMETRY

Given the particularly reduced number of superconformal invariants, the correlators obey a superconformal Ward identity

$$\left((\partial_\chi + \partial_\xi)\mathcal{A}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}\right)|_{\xi\to\chi^2} = 0 \longrightarrow \boxed{\sum_{j=0}^{n} F_j(\chi) = \mathbb{F}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}} \,.$$

BLOCK EXPANSION

The correlator can be expanded in superconformal blocks

$$\mathcal{A}_{\Delta_1\hat{\Delta}_2\hat{\Delta}_3}(\zeta;\chi) = \sum_{\hat{\Delta},\ell,k} b_{\Delta_1\hat{\Delta}}\lambda_{\hat{\Delta}_2\hat{\Delta}_3\hat{\Delta}}\mathcal{G}_{\hat{\Delta},\ell,k}(\chi,\zeta)$$

with non-physical singularities and branch cuts. Such singularities must cancel in correlation functions. [Kabat, Lifschytz, '16][Levine, Paulos, '23]. At strong coupling only even dimensions contribute [Alday, Maldacena '07].

LIMITS

$$\frac{\sqrt{n_{\hat{\Delta}_2} n_{\hat{\Delta}_3}}}{\sqrt{n_{\hat{\Delta}}}} \lim_{3\to 2} \langle \Delta_1\hat{\Delta}_2\hat{\Delta}_3 \rangle = b_{\Delta_1\hat{\Delta}} (12)^{\hat{\Delta}} (1\theta)^{\Delta_1 - \hat{\Delta}} \,, \quad \lim_{\chi\to 0} \langle \Delta_1\hat{\Delta}_2\hat{\Delta}_3 \rangle = \langle \Delta_1 \rangle \langle \hat{\Delta}_2\hat{\Delta}_3 \rangle \,.$$

PERTURBATIVE INFORMATION

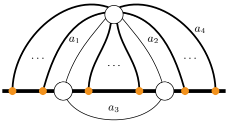
NUMBER OF R-SYMMETRY CHANNELS

The number of R-symmetry channels that are non-zero reduces to 2 at NLO at weak coupling, meaning that the correlation function becomes

$$\mathcal{A}_{\Delta_1 \hat{\Delta}_2 \hat{\Delta}_3}(\zeta;\chi) = F_0(\chi) + \left(\frac{\zeta}{\chi}\right) F_1(\chi)\,.$$

MASTER INTEGRALS

The diagrams appearing in $F_1(\chi)$ at NLO are of the kind [DA, Barrat, Xu '24]



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The integrals to solve are all of the kind

$$\int_{-\infty}^{a} d\tau_1 I_{1\tau_1} = \frac{\pi + 2\arctan\left(\frac{a}{|x_\perp|}\right)}{2|x_\perp|} \qquad \int_{a}^{+\infty} d\tau_1 I_{1\tau_1} = \frac{\pi - 2\arctan\left(\frac{a}{|x_\perp|}\right)}{2|x_\perp|}$$

RESULTS AND TRANSCENDENTALITY

The solution to the WI combined with the knowledge of the master integrals gives the NLO result of bulk-defect-defect correlation functions for operators of arbitrary dimensions.

$$F_2^{(1)}(x) = c_1(\Delta_1, \hat{\Delta}_2, \hat{\Delta}_3) \sum_{\pm} \left(\pi \pm 2\text{arctan}\left(\sqrt{\frac{1-x}{x}}\right)\right)^{a+2},$$

The NLO expression of the correlator does not present any power of $\log(\chi)$ in the series expansion for $\chi \to 0$.

How can we explain this absence?

For $\langle \mathcal{O}_2 \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_1 \rangle$ the superblock expansion predict a total coefficient of $\log(\chi)$ that is the sum of $b^{(0)}_{2\hat{\mathcal{O}}} \lambda^{(0)}_{11\hat{\mathcal{O}}} \gamma_{\hat{\mathcal{O}}}$.
This term of the expansion vanishes, but we lack an explanation that is valid beyond the single example.

RESULTS AT STRONG COUPLING

The strong-coupling regime for $\langle 2\hat{1}\hat{1} \rangle$ is expressed through a perturbative expansion at large N of the form

$$\mathcal{A}_{2\hat{1}\hat{1}}(\zeta;x) = \frac{\sqrt{\lambda}}{N}\left(\mathcal{A}^{(0)}_{2\hat{1}\hat{1}}(\zeta;x) + \frac{1}{\sqrt{\lambda}}\mathcal{A}^{(1)}_{2\hat{1}\hat{1}}(\zeta;x) + \frac{1}{\lambda}\mathcal{A}^{(2)}_{2\hat{1}\hat{1}}(\zeta;x) + \dots\right) + \dots .$$

The Witten diagrams contributing to the first two orders are trivial and give

LO $F_2^{(0)}(x) = -2a_2^{(0)} = -\frac{1}{\sqrt{2}}\,, \quad F_1^{(0)} = 0\,.$

NLO $F_2^{(0)}(x) = \frac{3}{2\sqrt{2}}\,, \quad F_1^{(0)} = \frac{3}{\sqrt{2}}\,.$

At **NNLO** the correlator takes the form [DA, Barrat, Xu '24]

$$\mathcal{A}^{(2)}_{2\hat{1}\hat{1}}(\zeta;x) = r_0(\zeta;x) + \frac{3}{\sqrt{2}}(x-\zeta)\log x\,,$$

where the rational function in front of $\log x$ is determined via superblock expansion and the function $r_0(\zeta;x)$ can be further constraint via Ward identities and locality sum rules.

MULTIPOINT CORRELATORS

DEFINITION AND CONVENTIONS

2-point

$$\boxed{\langle \hat{\Delta}_1 \hat{\Delta}_2 \rangle = \delta_{\hat{\Delta}_1 \hat{\Delta}_2} (12)^{\hat{\Delta}_1}}\,,$$

$$(ij) = \frac{u_i \cdot u_j}{\tau_{ij}^2}\,.$$

3-point

$$\boxed{\langle \Delta_1 \Delta_2 \Delta_3 \rangle = \lambda_{\Delta_1 \Delta_2 \Delta_3} (12)^{2\Delta_{123}} (23)^{2\Delta_{231}} (31)^{2\Delta_{312}}}\,,$$

$$\Delta_{ijk} = \frac{1}{2} (\Delta_i + \Delta_j - \Delta_k)\,.$$

4-point

$$\boxed{\langle \Delta_1 \Delta_2 \Delta_3 \Delta_4 \rangle = \mathcal{K}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \mathcal{A}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} (x; r, s)}\,,$$

$$x := \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}\,, \qquad r := \frac{(u_1 \cdot u_2)(u_3 \cdot u_4)}{(u_1 \cdot u_3)(u_2 \cdot u_4)}\,, \qquad s := \frac{(u_1 \cdot u_4)(u_2 \cdot u_3)}{(u_1 \cdot u_3)(u_2 \cdot u_4)}\,.$$

5-point

$$\boxed{\langle 11112 \rangle = \mathcal{K}_{11112} \mathcal{A}_{11112} (\{x; r, s, t\})}\,,$$

$$\mathcal{A}_{11112} = \sum_{i=1}^{6} R_i F_i (x_1, x_2)\,, \qquad \{R_i\} = \left\{ 1, \frac{r_1}{x_1^2}, \frac{s_1}{\bar{x}_1^2}, \frac{r_2}{x_2^2}, \frac{s_2}{\bar{x}_2^2}, \frac{t_{12}}{x_{12}^2} \right\}\,,$$

WHAT IS KNOWN

WEAK COUPLING

- n-point recursion relation up to NLO [Barrat, Liendo, Peveri, Plefka '23]
- 4-point function at NNLO using integrability methods [Cavaglià, Gromov, Julius, Preti, '22]
- ϕ_6 scaling dimension up to 5 loop [Grabner, Gromov, Julius, '20]

STRONG COUPLING

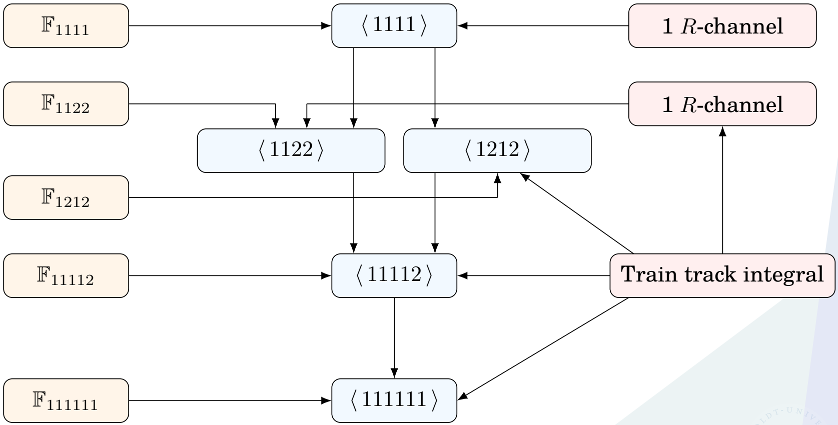
- 4-point function up to $\mathrm{N}^3\mathrm{LO}$ [Ferrero, Meneghelli '22]
- 5-point function up to NLO [Barrat, Bliard, Ferrero, Meneghelli, Peveri, *to appear*]
- ϕ_6 scaling dimension up to 4 loop [Ferrero, Meneghelli '22]

EXACT RESULTS

- Closed form of the topological sector $\mathbb{F}$ [Giombi, Komatsu, '18]
- Precise numerical results for 4-point function [Cavaglià, Gromov, Preti, '23]

PERTURBATIVE BOOTSTRAP: MULTIPOINT

The perturbative bootstrap table for multipoint correlators



NON-PERTURBATIVE CONSTRAINTS - 1

PINCHING

Higher-weight operators formed by bringing together fields from distinct points

$$\left\langle \hat{\mathcal{O}}_{\hat{\Delta}_1}(u_1, \tau_1)...\hat{\mathcal{O}}_{\hat{\Delta}_{n-1}+\hat{\Delta}_n}(u_{n-1}, \tau_{n-1}) \right\rangle_{1d} =$$

$$\lim_{n \to n-1} \left\langle \hat{\mathcal{O}}_{\hat{\Delta}_1}(u_1, \tau_1)...\hat{\mathcal{O}}_{\hat{\Delta}_{n-1}}(u_{n-1}, \tau_{n-1})\hat{\mathcal{O}}_{\hat{\Delta}_n}(u_n, \tau_n) \right\rangle_{1d}$$

CROSSING

Relations arising after identifying the line endpoints at infinity, consisting in the exchange of external points

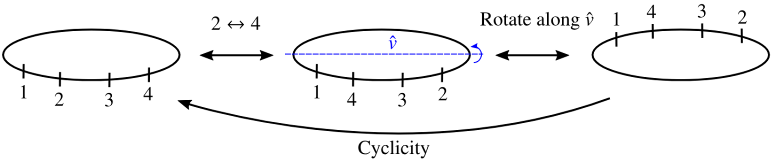


figure adapted from [Liendo, Meneghelli, '18]

NON-PERTURBATIVE CONSTRAINTS - 2

SUPERCONFORMAL WARD IDENTITIES

Set of constraints encoding the effect of superconformal symmetry on correlators [Bliard, '24][Barrat, Meneghelli, Müller, '24]

$$\sum_{i\neq j}^{n_R} \beta_i \left(\frac{1}{2}\partial_{x_i} + \alpha_i \partial_{r_i} - \bar{\alpha}_i \partial_{s_i} + \alpha_{ij}\partial_{t_{ij}} \right) \mathcal{A}_{\Delta_1 \dots \Delta_n} \Big|_{r_i \to \alpha_i x_i, s_i \to \bar{\alpha}_i \bar{x}_i, t_{ij} \to \alpha_{ij} x_{ij}} = 0\,,$$

The easiest form of the solution to the WI proceeds in (5+1) steps [DA, Barrat, Peveri '24]:

1 Introduce a new basis for the correlator:

$$\mathcal{A}_n = \sum_{j=1}^{n_R} \tilde{R}_j G_j(x_1, \dots, x_{n-3})\,.$$

2 Impose the topological limit to be $G_1(x_1, \dots, x_{n-3})$

3 The maximum number of derivatives of G_i do not appear after applying the WI

$$\tilde{R}_j \Big|_{r_i \to \alpha_i x_i, s_i \to \bar{\alpha}_i \bar{x}_i, t_{ij} \to \alpha_{ij} x_{ij}} = 0\,.$$

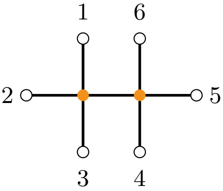
4 The remaining R_i are chosen such to satisfy the WI

5 The functions corresponding to the elements in point 3 are directly related to the simplest channels at weak coupling

6 The functions G_i corresponding to the elements in point 4 can be chosen to satisfy simple crossing relations

ONE INTEGRAL TO RULE THEM ALL

A central part of the perturbative bootstrap is to demonstrate that higher-point correlators at NNLO can be computed by imposing symmetry constraints, provided we know **one** integral.

$B_{123,456} =$ 

$$B_{123,456} = \frac{b_{123,456}(x_1, x_2, x_3)}{8192\pi^{10}\tau_{15}^2\tau_{24}^2\tau_{36}^2} \, ,$$

ONE INTEGRAL TO RULE THEM ALL [Rodrigues, '24]

$$
\begin{aligned}
b_{123,456} = \frac{x_{13}^2}{x_1 x_2 \bar{x}_3 x_{12}} \Big(& -G(1,x_1)G(1,x_2)G(1,x_3) + G(1,x_2)G(x_3,x_1)G(1,x_3) \\
& - G(1,x_1)G(x_3,x_2)G(1,x_3) + G(x_3,x_1)G(x_3,x_2)G(1,x_3) \\
& - 2G(1,0,x_1)G(1,x_3) + 2G(1,x_2,x_1)G(1,x_3) + 2G(x_3,0,x_1)G(1,x_3) \\
& - 2G(x_3,x_2,x_1)G(1,x_3) + G(x_3,x_1)G(0,1,x_2) - G(1,x_1)G(0,x_3,x_2) \\
& + G(x_3,x_1)G(1,0,x_2) + G(0,x_3)(-G(1,x_2)G(x_3,x_1) + G(1,x_1)(G(1,x_2) \\
& + G(x_3,x_2)) + 2G(1,0,x_1) - 2G(1,x_2,x_1)) - G(x_3,x_2)G(1,x_2,x_1) \\
& + G(x_3,x_2)G(1,x_3,x_1) + G(1,x_1)G(1,x_3,x_2) - G(x_3,x_1)G(1,x_3,x_2) \\
& - G(1,x_1)G(x_3,0,x_2) - G(1,x_2)G(x_3,1,x_1) + G(0,x_2)(-2G(0,x_3)G(1,x_1) \\
& + 2G(1,x_3)G(1,x_1) - 2G(1,x_3)G(x_3,x_1) - G(1,x_3,x_1) + G(x_3,1,x_1)) \\
& + G(1,x_1)G(x_3,1,x_2) - G(x_3,x_1)G(x_3,1,x_2) + G(1,x_2)G(x_3,x_2,x_1) \\
& + G(1,0,x_3,x_1) - G(1,x_2,x_3,x_1) + G(1,x_3,0,x_1) - G(1,x_3,x_2,x_1) \\
& - G(x_3,0,1,x_1) - G(x_3,1,0,x_1) + G(x_3,1,x_2,x_1) + G(x_3,x_2,1,x_1) \Big) \,.
\end{aligned}
$$

INTERLUDE: GONCHAROV POLYLOGARITHMS

Iterated integrals of the kind

$$G(a_1, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)\,,$$

with $G(x) = 1$. An example is

$$G(0, \ldots, 0, a, x) = -\mathrm{Li}_n\left(\frac{x}{a}\right).$$

The symbol is an effective way of describing Goncharov polylogarithm and a crucial part of our ansatz building.

$$\mathcal{S}\left(G(a_1, \ldots, a_n, x)\right) = x - a_n \otimes \ldots \otimes x - a_1$$

Symbols obey the useful properties as

$$A \otimes (x \cdot y) \otimes B = A \otimes x \otimes B + A \otimes y \otimes B\,.$$

$$\mathcal{S}\left(G(a_1, \ldots, a_n, x) G(b_1, \ldots, b_m, x)\right) = \mathcal{S}\left(G(a_1, \ldots, a_n, x)\right) ⧢ \mathcal{S}\left(G(b_1, \ldots, b_m, x)\right)\,,$$

FOUR-POINT CORRELATORS

For our examples of four point correlators, there are 3 R-symmetry channels, therefore three G_i to solve for. A solution to the WI is for the correlator to have the form

$$\boxed{\mathcal{A} = \frac{1}{2}\left(\frac{r}{x^2} + \frac{s}{\bar{x}^2}\right)\mathbb{F} + \partial_x(\xi f(x))\,,}$$

where the auxiliary function ξ is defined as

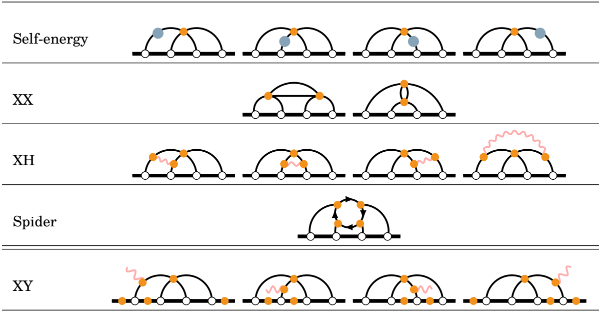
$$\xi = 1 - \frac{r}{x} - \frac{s}{\bar{x}}\,.$$

The algorithm presented for solving the WI in this configuration leads to

1. $G_1(x) = \mathbb{F}$
2. $\tilde{R}_2\Big|_{r\to\alpha x,\, s\to\bar{\alpha}\bar{x}} = 0\,.$
3. $f'(x) = G_2(x) \sim F_1(x)$ corresponding to the simplest perturbative channel

For each (protected) configuration of four scalar operators, the knowlegde of $f(x)$ is sufficient to determine the whole correlator.

$\langle\, 1111 \,\rangle_{NNLO}$: DIAGRAMS AND SOLUTION



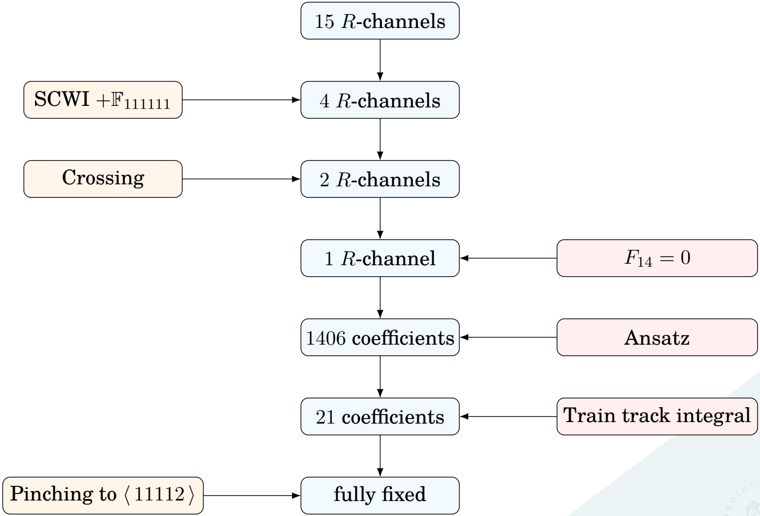
$$f^{(2)}(x) = \frac{1}{64\pi^4}\left(\frac{\pi^4}{15} + 3\zeta_3 G(1,x) + \frac{\pi^2}{3}(G(1,0,x) - G(0,1,x) + G(1,1,x))\right.$$

$$+ 2(G(1,1,0,1,x) - G(0,0,1,0,x) + G(0,0,1,1,x) - G(1,1,0,0,x)$$

$$+ G(0,1,0,0,x) - G(1,0,1,1,x)) + G(1,0,1,0,x) - G(0,1,0,1,x)$$

$$\left. + G(1,0,0,1,x) - G(0,1,1,0,x)\right).$$

$\langle\, 111111 \,\rangle_{NNLO}$: PERTURBATIVE BOOTSTRAP



$\langle 111111 \rangle_{NNLO}$: SOME DETAILS

ANSATZ

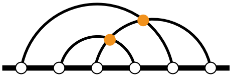
To construct the appropriate Goncharov polylogarithms, we write the Ansatz in symbols

$$f^{(2)}(x_1, x_2, x_3) = c_0 + \sum_{i,j} c_{ij} \xi_i \otimes \xi_j + \sum_{i,j,k,l} c_{ijkl} \xi_i \otimes \xi_j \otimes \xi_k \otimes \xi_j \,, \quad \xi_l = \{x_i, 1 - x_i, x_{ij}\} \,.$$

The ansatz is not quite yet in a usable form, as it is not finite for every choice of coefficients. Requiring finiteness fixes 5237 out of the 6643 free coefficients.

TRAIN TRACK INTEGRAL

On the input side, it is crucial to note that the only non-zero necessary channel is determined by a single diagram:



$$\sim B_{146,345} \,,$$

FORM OF RESULT

To illustrate the results, here are some of the contributing terms [DA, Barrat, Peveri '24]

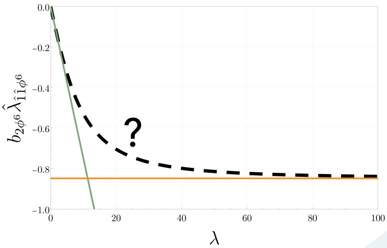
$$f_1^{(2)}(x_1, x_2, x_3) = \frac{1}{64\pi^4} \left(G(x_2, x_3, 1, 0, x_1) - \frac{8}{3} G(x_3, x_3, x_3, x_2, x_1) + \dots \right) ,$$

$$f_2^{(2)}(x_1, x_2, x_3) = 0 \,.$$

CONCLUSIONS AND OUTLOOK

BULK-DEFECT-DEFECT - SUMMARY

- We use superconformal symmetry and pinching limits to reduce the amount of perturbative information needed
- We compute the master integrals necessary to describe all bulk-defect-defect correlators at NLO weak coupling, observing the lack of $\log x$ terms
- We observe the same loss of transcendentality in a non-SUSY set-up, the $O(N)$ model
- We compute $\langle\, 2\hat{1}\hat{1}\,\rangle$ at NNLO strong coupling up to a rational function



BULK-DEFECT-DEFECT - PERSPECTIVES

FURTHER CONSTRAINTS FOR LOCALITY

- We can write an expansion of the correlators in local blocks, serving as a second block-expansion [Levine, Paulos, '23][Levine, Paulos, '24]
- This expansion can be used to determine relations among conformal data and can further constrain the correlator

THE SPACE OF FUNCTIONS

- The NLO analysis did not give access to the space of transcendental functions describing bulk-defect-defect correlators
- There are NNLO correlators with 3 channels, with one of them simpler than the others. Further constraints may be needed to apply the perturbative bootstrap set-up

MULTIPOINT - SUMMARY

- We use superconformal symmetry and pinching limits to reduce the amount of perturbative information needed
- We develop a systematic algorithm to find a suitable solution to the WI for this application
- We benchmark our method by computing $\langle\, 1111 \,\rangle$ at NNLO and $\langle\, 111111 \,\rangle$ at NLO
- We compute new 4-point correlators at NNLO: $\langle\, 1212 \,\rangle$, $\langle\, 1122 \,\rangle$
- Using non-perturbative information and one single integral, we determine $\langle\, 11112 \,\rangle$ and $\langle\, 111111 \,\rangle$ at NNLO

MULTIPOINT - PERSPECTIVES

BEYOND WEAK COUPLING

- Numerical analysis of the correlator $\langle\, 111111 \,\rangle$, building upon numerical bootstrap [Antunes, Harris, Kaviraj, Schomerus, '23].
- Connection with the bootstrability techniques applied so far to four-point functions [Cavaglià, Gromov, Julius, Preti, '22][Cavaglià, Gromov, Julius, Preti, '22][Cavaglià, Gromov, Julius, Preti, '22][Cavaglià, Gromov, Julius, Preti, '22]
- A preliminary study of the superconformal blocks is necessary and under way [Barrat, Bliard, Ferrero, Meneghelli, Peveri, *to appear*]

THE ROLE OF THE TRAIN-TRACK

- Study of the role of train-track integrals at further perturbative orders
- The collapse of elliptic complexity for 6-point train-track may hold for higher points
- Train-track integrals would be the ones appearing in a line-defect of the fishnet theory
- Fascinating integrable non-supersymmetric framework to study exact correlators

MERCI!