https://artie2000.github.io/real$_{c}losed_{f}ieldhttps : //github.com/artie2000/real_{c}losed_{f}ieldhttps : //artie20$

Real Closed Field

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We follow [**?**] for the material on field orderings and the algebraic criterion for field orderability. For the material on real closed fields, we synthesise material from [**?**], [**?**], [**?**] and [**?**] as convenient.

Note that the meaning of the notation $F(\sqrt{a})$ depends on whether a is a square in F. We take care to write it only in the non-trivial case. The same applies to $F(i)$.

0.1 Preliminaries

0.2 Ordered Fields

We begin with a purely algebraic characterisation of an ordered field. This relies on the theory of *ring orderings*, which can be found in [**?**].

Definition 1. IsSemireal A field is real if -1 is not a sum of squares.

Theorem 2. *def:semireal* $Field.exists_isStrictOrderedRing_iff_isSemirealAfieldcanbeorderedifandonlyifitis$

We can also characterise algebraically whether a field F has a unique ordering. This allows us to talk about 'the ordering' on F without ambiguity.

Lemma 3. *def:semireal* $IsSemireal.existsUnique_isStrictOrderedRingLetFbearealfield.Thereisauniqueorderi$
F, either a or $-a$ is a sum of squares.

Corollary 4. *def:ordered*$_f$$ieldIsStrictOrderedRing.unique_isStrictOrderedRing_iffLetFbeanorderedfield.Th$
$negativeelementisasumofsquares.$

Proof. lem:unique$_o rd_c ond$ ☐

Corollary 5. *def:ordered*$_f$$ieldRat.unique_isStrictOrderedRingThereisauniquefieldorderingonQ.$

Proof. cor:unique$_o rd_c ond_o rderedLet$x$\in \mathbb{Q}$ be non-negative. Then $x = p/q$ for some integers $p \geq 0$ and $q > 0$, so

$$x = \underbrace{\frac{1}{q^2} + \cdots + \frac{1}{q^2}}_{pq \text{ times}}$$

is a sum of squares. We are done by Corollary **??**. ☐

There is a corresponding algebraic characterisation of an ordered field extension. Note that, since real fields have characteristic 0, their algebraic extensions are separable.

Lemma 6. *$Field.exists_isOrderedAlgebra_iff_neg_one_notMem_span_nonneg_isSquareLetFbeanorderedfield, andle$* *$-1$ for all choices of $\alpha_i \in K$ and $x_i \in F_{\geq 0}$.*

Lemma 7. *$Field.exists_isOrderedAlgebra_of_projectionLetFbeanorderedfield, andletK/Fbeafieldextension.Su$* *$K \to F$ such that, for all $x \in K$, $\pi(x^2) \geq 0$. Then there is a field ordering on K making K/F ordered.*

Proof. lem:ext$_ord_condConsiderthesum\sum_i x_i\alpha_i^2$ for some $\alpha_i \in K$ and $x_i \in F_{\geq 0}$. By F-linearity, we compute

$$\pi(\sum_i x_i\alpha_i^2) = \sum_i x_i\pi(\alpha_i^2) \geq 0.$$

Since $\pi(-1) = -1 < 0$, we are done by Lemma **??**. ☐

Corollary 8. *$adj_sqrt_orderedLetFbeaorderedfield, andsupposea\in$ F is non-negative (and not a square). Then there is a field ordering on $F(\sqrt{a})$ making $F(\sqrt{a})/F$ ordered.*

Proof. lem:ext$_ord_functional_suffLet\pi : F(\sqrt{a}) \to F$ be the projection induced by the F-basis $\{1, \sqrt{a}\}$. For $x, y \in F$, we have $\pi((x + y\sqrt{a})^2) = x^2 + ay^2 \geq 0$. We are done by Lemma **??**. ☐

Lemma 9. *$odd_deg_orderedLetFbeanorderedfield, andletK/Fbeanodd-degreeextension.Thenthereisafieldord$*

Proof. lem:ext$_ord_condBytheprimitiveelementtheorem$,K=F($\alpha$) for some $\alpha \in K$. Let f be the minimal polynomial of α over K; then $\deg f = [K : F]$ is odd. By Lemma **??**, we need to show that the congruence

$$\sum_i a_i g_i^2 \equiv -1 \bmod f \tag{$\star$}$$

fails to hold for any non-negative $a_i \in F$ and polynomials $g_i \in F[X]$ each of degree at most $\deg f - 1$. Proceed by induction on $\deg f$; if $\deg f = 1$, then ($\star$) reduces to an equality of a non-negative element of F with a negative one. Otherwise, suppose for a contradiction that ($\star$) holds. Without loss of generality, we may assume, for all i, we have $a_i \neq 0$ and that $\deg g_i < \deg f$.

Rearranging ($\star$), we have $\sum_i a_i g_i^2 + 1 = hf$ for some $h \in F[X]$. Let $d = \max_i \deg g_i$; note that $d < \deg f$ by construction. Since each a_i is positive, the $2d$th coefficient on the left-hand side must be positive. Therefore

$$\deg h + \deg f = \deg(\sum_i a_i g_i^2 + 1) = 2d.$$

Then $\deg h$ is odd, so h has an odd-degree irreducible factor $\tilde{h}$. We have

$$\deg \tilde{h} \leq \deg h = 2d - \deg f < \deg f,$$

but $\sum_i a_i g_i^2 \equiv -1 \bmod \tilde{h}$. We are done by induction.$_\#$ ☐

There is an easier way to construct ordered field extensions if we don't care about them being algebraic.

Lemma 10. *def:ordered$_e$xtLetFbeanorderedfield, andleta* $\in F$*. Then there is a unique ordering on the function field $F(X)$ making $F(X)/F$ ordered such that $X > a$ but $b > X$ for $b > a$, and a unique one such that $X < a$ but $b < X$ for $b < a$.*

Intuitively, X is infinitesimally close to a. When $a = 0$, we often write $R(\varepsilon)$ for $R(X)$ with the first type of ordering.

0.3 Real Closed Fields

Definition 11. def:ordered$_{f}ieldIsRealClosedArealclosedfieldisanorderedfieldinwhicheverynon-negativeelementisasquareandeveryodd-degreepolynomialhasaroot.$

Lemma 12. *def:RCF IsRealClosed.unique$_i$sStrictOrderedRingTheorderingonarealclosedfieldisunique.*

Proof. cor:unique$_ord_cond_orderedFollowsdirectlyfromCorollary$**??**. □

Fix a real closed field R. In what follows, all algebraic extensions are given up to R-isomorphism, as is conventional. Observe that, since -1 is not a square in R, $R(i)/R$ is a quadratic extension. We show that this is the **only** nontrivial algebraic extension of R.

Lemma 13. *def:RCF IsRealClosed.even$_f$inrank$_e$xtensionNontrivialfiniteextensionsofRhaveevendegree.*

Proof. Let K/R be an odd-degree extension of R. By the primitive element theorem, $K = R(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over K. Then f is irreducible, but $\deg f = [K : R]$ is odd, so f has a root in R. Therefore, $[K : R] = \deg f = 1$; that is, $K \cong R$. □

Lemma 14. *def:RCF IsRealClosed.isAdjoinRootIOfFinrankExtensionEqTwo The field $R(i)$ is the unique quadratic extension of R.*

Proof. Let K/R be a quadratic extension. Since $\mathrm{char}(R) \neq 2$, we have $K \cong R(\sqrt{a})$ for some $a \in R$. Since nonnegative elements of R are squares, a is negative and so $-a$ is a square in R. Rescaling by $\sqrt{-a}^{-1}$, we get $R(\sqrt{a}) \cong R(i)$. □

Lemma 15. *def:RCF IsRealClosed.finrank$_a$djoinRoot$_i$$_e$xtension$_neq_t$woThereisnoquadraticextensionofR(i).*

Proof. Since char$R(i) \neq 2$, it suffices to show that every element of $R(i)$ is a square. Indeed, take $x = a + bi \in R(i)$ with $a, b \in R$. If $b = 0$, then either $a \geq 0$ and so x is a square in R, or $a \leq 0$ and so $a = (i\sqrt{-a})^2$ is a square in R. Now let $b \neq 0$. Then we compute $x = (c + di)^2$, where

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \text{ and } d = \frac{b}{2c}.$$

To see that c and d are well-defined elements of R, observe that $a^2 + b^2 > a^2 \geq 0$ (as $b \neq 0$), and so $a + \sqrt{a^2 + b^2} > 0$. Therefore the square roots above lie in R and $c \neq 0$. □

Theorem 16. *def:RCF IsRealClosed.finite$_e$xtension$_c$lassifyTheonlyfiniteextensionsofRareRitselfandR(i).*

Proof. lem:ext$_d$eg$_2$, $lem : Ri_ext_deg_2, lem : ext_ord_odd_degByseparability, everyfiniteextensionof$R$iscontainedi$

Let K/R be a nontrivial Galois extension of degree $2^k \cdot a$, where $k \geq 0$ and $a \geq 1$ is odd. Applying the Galois correspondence to a Sylow 2-subgroup of Gal(K/R) yields an intermediate extension of degree a; by Lemma **??**, we have $a = 1$ (and $k > 0$). If $k > 1$, iterating the last construction yields intermediate extensions $K/L/M/R$ with $[L : M] = [M : R] = 2$. By Lemma **??**, $M \cong R(i)$, contradicting Lemma **??**.$_\#$ Therefore $k = 1$ and (by Lemma **??**) $K \cong R(i)$. □

Corollary 17. *def:RCF IsRealClosed.algebraic$_e$xtension$_c$lassifyTheonlyalgebraicextensionsofRareRitselfa*

Proof. thm:FTAlg □

Corollary 18. *def:RCF IsRealClosed.isAdjoinRootIOfIsAlgClosure* $\bar{R} = R(i)$.

Proof. cor:FTAlg$_a$*lg* □

The converse to Theorem **??** is much easier.

Lemma 19. *def:RCF IsRealClosed.of$_i$sAdjoinRoot$_{io}$r$_{1e}$xtensionSupposeRisanorderedfieldwhoseonlynontriv*

Proof. Let $a \in R$ be non-negative. If a is not a square, then $R(\sqrt{a}) \cong R(i)$. Suppose i maps to $x + y\sqrt{a}$ for some $x, y \in R$; then $-1 = x^2 + ay^2 + 2xy\sqrt{a}$. Comparing coefficients, $-1 = x^2 + ay^2 \geq 0$.$_\#$ Therefore a is a square in R.

Now, fix a nonlinear odd-degree polynomial $f \in R[X]$. Then $R[X]/(f)$ cannot be a field since R has no nontrivial odd-degree extensions, and so f must be reducible. We are done by induction on the degree. □

As before, let R be a real closed field. Theorem **??** is a powerful tool for deriving more of its properties.

Lemma 20. *def:RCF R is maximal with respect to algebraic extensions by ordered fields.*

Proof. cor:FTAlg$_a lgSince$-$1isasquarein$R(i)$, thefield$R(i)$isnotformallyrealandthereforecannotbeordered.W$

In particular, R is maximal with respect to ordered algebraic extensions.

Lemma 21. *def:RCF The monic irreducible polynomials over $R[X]$ have form $X - c$ for some $c \in R$ or $(X - a)^2 + b^2$ for some $a, b \in R$ with $b \neq 0$.*

Proof. thm:FTAlg Let $f \in R[X]$ be monic and irreducible. The field $R_f = R[X]/(f)$ is a finite extension of R, so it is classified by Theorem **??**. If $R_f \cong R$, then $\deg f = 1$, so $f = X - c$ for some $c \in R$. If $R_f \cong R(i)$, let the isomorphism be φ, and suppose $\varphi(X + (f)) = a + bi$ $(a, b \in R)$. Note that $b \neq 0$ since φ^{-1} is constant on R. Rearranging, we see that $\varphi((X - a)^2 + b^2 + (f)) = 0$; that is, $(X - a)^2 + b^2 \in (f)$. Since this polynomial is monic and has the same degree as f, it must in fact be equal to f.

Conversely, linear polynomials over a domain are irreducible by degree, and reducible quadratics have a root. A root of $f = (X - a)^2 + b^2$ with $a, b \in R$ is an element $r \in R$ satisfying $(r - a)^2 = -b^2$. Since squares are non-negative, if $b \neq 0$ then f must be irreducible. □

The next property is a little less obvious.

Lemma 22. *def:RCF R satisfies the intermediate value property for polynomials.*

Proof. lem:irreds$_c lassWewillprovethat, forall$f$\in$ $R[X]$ and all $a, b \in R$ with $a < b$, if $f(a) \cdot f(b) < 0$, then there is some $c \in (a, b)$ such that $f(c) = 0$.

Fix $a, b \in R$ with $a < b$. First, suppose $f \in R[X]$ is linear. Then $f = m(X - c)$ for some $m, c \in R$ with $m \neq 0$; then $f(c) = 0$. If $m > 0$, then $f(x) < 0$ for $x < c$ and $f(x) > 0$ for $x > c$, and vice versa if $m < 0$. In either case, if $c \notin [a, b]$, then $f(a) \cdot f(b) > 0$. Taking into account the cases $c = a$ and $c = b$, if $f(a) \cdot f(b) < 0$ then $c \in (a, b)$.

Now suppose $f(a) \cdot f(b) < 0$, and proceed by induction on $\deg f$. If $\deg f = 0$, write $f = x \in R$; then $f(x) \cdot f(x) = x^2 \leq 0$, so, since squares are non-negative, $x = 0$ and $f((a + b)/2) = 0$. The above validates the property for $\deg f = 1$. Now, take a monic irreducible factor g of f; then g is classified by Lemma **??**. If $g = (X - a)^2 + b^2$ with $a, b \in R$ and $b \neq 0$, then g is everywhere positive. If $g = X - c$ with $c \in R$, then either $c \in (a, b)$ and $g(c) = 0$, or $c \notin (a, b)$ and $g(a)$ and $g(b)$ have the same sign (they are nonzero since $f(a)$ and $f(b)$ are). In the second case, f has a root in (a, b); in the first and third cases, f/g satsifies the induction hypothesis, so it has a root in (a, b). In all cases, a factor of f has a root in (a, b), and therfore so does f. □

In fact, the converses to Lemmas **??** and **??** also hold!

Theorem 23. *def:RCF IsRealClosed.of$_{I}ntermediateValuePropertyLetRbeanorderedfieldsatisfyingtheinter$*

Proof. Let $a \in R$ be non-negative, and consider the polynomial $f = X^2 - a$. Then $f(0) = -a \leq 0$, but $f(a+1) = a^2 + a + 1 > 0$. By the intermediate value property, f has a root in R, and so a is a square in R.

Let f be an odd-degree polynomial over R. Write $f = a_n X^n + \cdots + a_0$. We will show f has a root in R. Replacing f by $-f$ if necessary, we may assume $a_n > 0$. For $x > 1$, we compute

$$f(x) \geq x^{n-1}(a_n x - n \max_i |a_i|).$$

Therefore, when $x > \max\{1, n \max_i |a_i|/a_n\}$, $f(x) > 0$. A similar calculation shows that $f(x) < 0$ for sufficiently large negative values of x. We are done by the intermediate value property. □

Theorem 24. *def:RCF IsRealClosed.of$_{m}aximalIsOrderedAlgebraLetRbeanorderedfieldmaximalwithrespec$*

Proof. cor:ext$_{o}rd_{t}o_{a}dj_{s}qrt, lem : ext_{o}rd_{o}dd_{d}egLet$a$\in$ R be non-negative, and suppose a is not a square. By Corollary **??**, there is an ordering making the nontrivial extension $R(\sqrt{a})/R$ ordered, contradicting maximality.$_{\#}$ Therefore a is a square in R.

By induction on degree, it suffices to show that irreducible odd-degree polynomials over R are all linear. Let $f \in R[X]$ be such a polynomial, and consider the odd-degree field extension $R_f = R[X]/(f)$. By Lemma **??**, there is an ordering making R_f/R an ordered extension; by maximality, $R_f \cong R$, and therefore $\deg f = [R_f : R] = 1$. □

In particular, an ordered field maximal with respect to algebraic extensions by ordered fields is real closed.

Theorem **??** gives us a way to "construct" real closed fields.

Lemma 25. *def:RCF An algebraically closed field of characteristic 0 has an index-2 real closed subfield.*

Proof. lem:order$_{f}un_{f}ield, thm : ord_{m}ax_{i}mp_{R}CF, cor : RCF_{a}cLet$C$beanalgebraicallyclosedfieldofcharacteris$
C, we can use Lemma **??** to adjoin an element transcendental over F, obtaining a strictly bigger ordered subfield.

Apply Zorn's lemma to obtain a maximal ordered subfield $R \subseteq C$; then $\bar{R} = C$. By Theorem **??**, R must be real closed. By Corollary **??**, $C \cong R(i)$, and so $[C : R] = 2$. □

In summary, we have proved the following characterisation of real closed fields.

Theorem 26. *def:RCF, def:ordered$_{e}xtLetRbeanorderedfield.TFAE$:*

R is real closed.

$\bar{R} = R(i)$ *(and* -1 *is not a square in* R*).*

R *satisfies the intermediate value property for polynomials.*

R *is maximal with respect to ordered algebraic extensions.*

R *is maximal with respect to algebraic extensions by ordered fields.*

Proof. cor:RCF$_a c, lem : FTAlg_c onverse, lem : IVP_p oly, thm : IVP_p oly_i mp_R CF, lem : RCF_m ax, thm : ord_m ax_i mp_R CF$ □

0.4 Real Closures

Definition 27. def:ordered$_e xt, def : RCFIsRealClosureLet$F$beanorderedfield.Arealclosureof$F$isarealclose$

***Lemma* 28.** *def:real$_c$losureLetFbeanorderedfield.ThenFhasarealclosure.*

Proof. thm:ord$_m ax_i mp_R CFApplyZorn'slemmatoorderedalgebraicextensionsof$F$.WearedonebyTheorem$**??**.

Just like with the algebraic closure, it makes sense to talk of *the* real closure of an ordered field. Proving this uniqueness result requires a method of root-counting in real fields known as Sturm's theorem.

***Theorem* 29** (Corollary to Sturm's Theorem)**.** *def:real$_c$losureLetFbeanorderedfield, andletfbeapolynomialove*

Proof. TODO : decide on the generality of the statement of Sturm's Theorem □

***Lemma* 30.** *def:real$_c$losureLetFbeanorderedfieldwitharealclosureR, andletK/Fbeafiniteorderedextension.T*
R.

Proof. lem:real$_c losure_e xists, thm : SturmBytheprimitiveelementtheorem$,K=F$(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over F. Since F has a root in K, it has a root in a real closure of K (one exists by Lemma **??**). By Theorem **??**, f has a root β in R. Therefore define $\varphi : K \to R$ with $\varphi(\alpha) = \beta$. □

***Lemma* 31.** *def:real$_c$losureLetFbeanorderedfieldwitharealclosureR, andletK/Fbeafiniteorderedextension.T preservingF−homomorphismK→ R.*

Proof. lem:real$_c$*losure$_e$xists, thm : Sturm, lem : closure$_e$mb$_e$xt$_u$norderedFixarealclosure*R'*of*K(*oneexistsby*

By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over F, and let $\alpha_1 < \cdots < \alpha_m$ be the roots of f in R', with $\alpha = \alpha_k$. By Theorem **??**, f also has m roots in R; let them be $\beta_1 < \cdots < \beta_m$. Since non-negative elements of R' are squares, and $x_1, \ldots, x_{m-1} \in R'$ such that $\alpha_{j+1} - \alpha_j = x_j^2$, and let $L = K(\alpha_1, \ldots, \alpha_m, r, x_1, \ldots, x_{m-1}) \leq R'$. Now, suppose we have a K-homomorphism $\psi : L \to R$. Each $\psi(\alpha_j)$ is equal to a different β_i. Then $\psi(\alpha_{j+1}) - \psi(\alpha_j) = \psi(x_j)^2 \geq 0$, so $\psi(\alpha_1) < \cdots < \psi(\alpha_m)$, and so $\psi(\alpha_j) = \beta_j$ for all j.

By Lemma **??**, there is in fact an F-homomorphism $\varphi : L \to R$. We will show that φ is order-preserving. Indeed, fix a non-negative element $x \in L$. As before, find $r \in R'$ such that $x = r^2$, and let $M = L(r) \leq R'$. Apply Lemma **??** again to obtain an L-homomorphism $\psi : M \to R$; then $\varphi(x) = \psi(x) = \psi(r)^2 \geq 0$. Therefore φ maps non-negative elements to non-negative elements, and so is order-preserving. Then $\varphi|_K$ is the map we want. Note that $\varphi(\alpha) = \beta_k$.

To see uniqueness, let $\tilde{\varphi} : K \to R$ be an order-preserving F-homomorphism; by existence, $\tilde{\varphi}$ extends to a an order-preserving K-homomorphism $\tilde{\psi} : L \to R$. Then $\tilde{\varphi}(\alpha) = \tilde{\psi}(\alpha_k) = \beta_k = \varphi(\alpha)$, and so $\tilde{\varphi} = \varphi$. □

Taking $K = F$ above, we see that the order-embedding of a field into its real closure is unique.

***Theorem* 32.** *def:real$_c$losureLetFbeanorderedfield.ThentherealclosureofFisuniqueuptouniqueF−isomorph*

Proof. lem:closure$_e$*mb$_e$xt, lem : RCF$_m$axLet*R$_1$ and R_2 be real closures of F. Applying Zorn's lemma to the set of ordered extensions intermediate between R_1 and F having a unique order-preserving F-embedding into R_2, and using Lemma **??**, we obtain an intermediate extension $R_1/K/F$ with no nontrivial finite ordered extensions and a unique order-preserving F-embedding $\varphi : K \to R_2$. If the ordered algebraic extension $R_2/\varphi(K)$ were nontrivial, then it would contain a nontrivial ordered finite extension, so φ must be surjective (and so an F-isomorphism). In particular, $K \subseteq R_1$ is real closed; by maximality (**??**), in fact $K = R_1$ and so φ is an F-isomorphism between R_1 and R_2. □

***Corollary* 33.** *def:RCF A real closed field has no nontrivial field automorphisms.*

Proof. thm:real$_c$*losure*$_u$*nique*, *lem* : *RCF*$_o$*rd*$_u$*niqueLet*R*bearealclosedfield.ByTheorem* **??**,R*hasnonontrivial*
*preservingautomorphisms.Sincetheorderingon*R*isunique*(*byLemma* **??**), *everyautomorphismof*R*mustbeorde*
preserving. □

This uniqueness result is stronger than the one in the algebraically closed case: an algebraically closed field has many nontrivial automorphisms.

Uniqueness of algebraic closures allows us to classify ordered algebraic extensions.

Lemma 34. *def:real*$_c$*losureLetFbeanorderedfieldwithrealclosureR*, *andletK/Fbealgebraic.Thenfieldorderin*
R via the order obtained by restriction from R.

Proof. lem:real$_c$*losure*$_e$*xists*, *thm* : *real*$_c$*losure*$_u$*niqueFixanorderingon*K*extendingthaton*F, *andlet*K*havereal*
(exists by Lemma **??**). Then R_K/F is algebraic, so R_K is a real closure of F. By Theorem **??**, there is an F-isomorphism to $R_K \cong R$, and this induces an F-homomorphism $K \to R$. Restricting the order on R to K via this map recovers the original order on K by construction.

Moreover, the inverse to order restriction constructed above is unique. Indeed, an inverse $\varphi : K \to R$ is order-preserving by definition, so it is an order-embedding from K into its real closure. By Theorem **??**, such a map is unique. □

0.5 The Artin-Schreier Theorem

We didn't actually need to assume the ordering on R in Lemma **??**.

Lemma 35. *isSumSq*$_o$*f*$_i$*sSquareLetFbeafieldinwhich-1isnotasquare*, *andsupposeeveryelementofF(i)isasqu*

Proof. Given $a, b \in F$, find $c, d \in F$ such that $a + bi = (c + di)^2$ in $F(i)$. Then $a = c^2 - d^2$ and $b = 2cd$, so $a^2 + b^2 = (c^2 + d^2)^2$ is a square in F. By induction, every sum of squares in F is a square in F. □

Lemma 36. *def:RCF Suppose R is a field whose only nontrivial algebraic extension is $R(i)$. Then there is a unique field ordering on R, and moreover R with this ordering is real closed.*

Proof. thm:ord$_iff_real, lem : FTAlg_converse, lem : RCF_ord_unique, lem : sumsq_is_sqWeareverynearlydonebyT$

We have $\bar{R} \cong R(i)$. Since $R(i)/R$ is nontrivial, -1 is not a square in R. Since every element of $R(i)$ is a square, Lemma **??** says that -1 is not a sum of squares in R. ☐

In fact, we can go further. The following is a weak form of the Artin-Schreier theorem. Removing the condition on characteristic is possible, but requires some more involved algebra.

***Theorem* 37.** *def:RCF Let R be a field with $\operatorname{char} K \neq 2$, and suppose $[\bar{R} : R] = 2$. Then there is a unique field ordering on R, and moreover R with this ordering is real closed.*

Proof. lem:FTAlg$_converse_strong, lem : deg_2classifyByLemma$**??**$, itsufficestoshowthat \cong$ $R(i)$.

By Lemma **??**, $\bar{R} \cong R(\sqrt{a})$ for some non-square $a \in R$. Since $R(\sqrt{a})$ is algebraically closed, $\sqrt{a}$ is a square in $R(\sqrt{a})$; find $x, y \in R$ such that $\sqrt{a} = (x + y\sqrt{a})^2$. Expanding and comparing coefficients, $x^2 + y^2 a = 0$ and $2xy = 1$. Rearranging, $a = -(4x^4) = -1 \cdot (2x^2)^2$. In particular, -1 is not a square. By Lemma **??**, $R(i) \cong R(\sqrt{a}) = \bar{R}$. ☐

We can weaken the hypotheses even further. Here is the full Artin-Schreier theorem.

***Theorem* 38** (Artin-Schreier Theorem)**.** *def:RCF Let R be a field, and suppose $\bar{R}$ is a finite extension of R. Then there is a unique field ordering on R, and moreover R with this ordering is real closed.*

Proof. thm:Artin-Schreier$_weakTODO : thisneedsalotmorepreliminarieseg Artin- Schreiertheory, Kummertheory$ ☐

***Corollary* 39.** *An algebraically closed field of nonzero characteristic has no finite-index subfields.*

Proof. thm:Artin-Schreier Ordered fields have characteristic 0. ☐

***Theorem* 40.** *The finite-index subfields of $\bar{\mathbb{Q}}$ are isomorphic copies of $\mathbb{Q}_{alg} = \bar{\mathbb{Q}} \cap \mathbb{R}$ indexed by $\operatorname{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$.*

Proof. thm:Artin-Schreier,thm:real$_c losure_u nique, cor : unique_o rd_Q Since \mathrm{Q}_{\mathrm{alg}}(i) = \bar{\mathbb{Q}}$, the field $\mathbb{Q}_{\mathrm{alg}}$ is a finite-index subfield.

By Theorem **??**, any finite-index subfield has a unique ordering making it real closed. Let R be a real closed subfield. Then the ordering on R restricts to the ordering on $\mathbb{Q}$ (unique by Corollary **??**). Since $R/\mathbb{Q}$ is algebraic, R is a real closure of $\mathbb{Q}$. By Theorem **??**, there is a unique $\mathbb{Q}$-isomorphism $\psi : \mathbb{Q}_{\mathrm{alg}} \cong R$. Since $\mathbb{Q}_{\mathrm{alg}}(i) = R(i) = \bar{\mathbb{Q}}$, ψ extends uniquely to $\varphi \in \mathrm{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$ by mapping $i \to i$. The subfield R is then recovered from $\varphi \in \mathrm{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$ by taking $\varphi^{-1}(\mathbb{Q}_{\mathrm{alg}})$. $\square$