Real Closed Field

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Lemma 1. *Fix a prime p, and let M/K be a separable Galois extension of degree $p^k \cdot a$, where $p \nmid a$. Then, for $0 \le j \le k$, there are intermediate fields $K \le L_0 \le \cdots \le L_k \le M$, with $[L_j : K] = p^j \cdot a$.*

Proof. Since M/K is Galois, $|\operatorname{Gal}(M/K)| = p^k \cdot a$. A version of Sylow's first theorem says that each subgroup of order p^j with $0 \le j < k$ is contained in a subgroup of order p^{j+1}. By induction, $\operatorname{Gal}(M/K)$ has a chain of subgroups $H_k \le \cdots \le H_0 \le \operatorname{Gal}(M/K)$ with $|H_j| = p^{k-j}$. By the Galois correspondence, $L_j = M^{H_j}$ are the desired subfields. ☐

Lemma 2. *Let K be a field with $\operatorname{char} K \neq 2$. Then there is a bijection between the quadratic extensions of K (up to K-isomorphism) and the set*

$$\left(\frac{K^*}{(K^*)^2}\right) \setminus \{1 \cdot (K^*)^2\}$$

given by the map $x(K^*)^2 \to K(\sqrt{x})$.

Proof. Consider the map $\Phi : x \to K(\sqrt{x})$ from K^*. We will show it fully respects the relation $x(K^*)^2 = y(K^*)^2$; then Φ descends to a injective map out of the quotient $K^*/(K^*)^2$. In particular, if $x \notin (K^*)^2$, then $\Phi(x) = K(\sqrt{x})$ is not K-isomorphic to K, and is therefore a quadratic extension of K.

Indeed, if $x(K^*)^2 = y(K^*)^2$, then $x = a^2 y$ for some $a \in K$, and so $K(\sqrt{x}) \cong_K K(\sqrt{y})$ via $\sqrt{x} \to a\sqrt{y}$. Conversely, if $\phi : K(\sqrt{x}) \to K(\sqrt{y})$ is a K-isomorphism, then $\phi(\sqrt{x}) = a + b\sqrt{y}$ for some $a, b \in K$, and so $x = a^2 + yb^2 + 2ab\sqrt{y}$. Comparing coefficients in the K-basis $\{1, \sqrt{y}\}$, either $a = 0$ or $b = 0$. Therefore, either $x = a^2 y$ and so $x(K^*)^2 = y(K^*)^2$, or $x = a^2$, in which case $K(\sqrt{y}) \cong_K K(\sqrt{x}) \cong_K K$; that is, $x, y \in (K^*)^2$.

It remains to show all quadratic extensions of K are K-isomorphic to some $L \in \operatorname{im} \Phi$. Fix a quadratic extension L/K, and let $\{1, \alpha\}$ be a K-basis for L; then $\alpha^2 = a\alpha + b$ for some $a, b \in K$. Let $\beta = 2\alpha - a$. Since $\operatorname{char} K \neq 2$, $\alpha = (\beta + a)/2$, and so $L = K + \beta K = K(\beta)$. Now, we compute $\beta^2 = a^2 + 4b$. Therefore $L \cong_K \Phi(a^2 + 4b)$ via $\beta \to \sqrt{a^2 + 4b}$. ☐

Note that we will only use that this map is well-defined and surjective, and not that it is injective (which was the most annoying part to show).

Definition 3. A real closed field is an ordered field in which every positive element has a square root and every odd-degree polynomial has a root.

Let R be a real closed field. Note that, since R is ordered, $\operatorname{char} R = 0$. In particular, its algebraic extensions are separable.

In what follows, all algebraic extensions are given up to isomorphism, as is conventional. Observe that, since -1 is not a square in R, $R(i)/R$ is a quadratic extension. We show that this is the **only** nontrivial algebraic extension of R.

Lemma 4. *Nontrivial algebraic extensions of R have even degree.*

Proof. Let K/R be an odd-degree algebraic extension of R. By the primitive element theorem, $K = R(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over K. Then f is irreducible, but $\deg f = [K : R]$ is odd, so f has a root in R. Therefore, $[K : R] = \deg f = 1$; that is, $K = R$. ☐

Lemma 5. *The field $R(i)$ is the unique quadratic extension of R.*

Proof. Fix $x \in R^*$. Then either $x > 0$ and $x = 1 \cdot (\sqrt{x})^2$, or $x < 0$ and $x = -1 \cdot (\sqrt{-x})^2$. Further, since $-1 \notin (R^*)^2$, $-1 \cdot (R^*)^2 \neq 1 \cdot (R^*)^2$. Therefore $R^*/(R^*)^2 = \{1 \cdot (R^*)^2, -1 \cdot (R^*)^2\}$, and we are done by Lemma 2. $\square$

Lemma 6. *There is no quadratic extension of $R(i)$.*

Proof. By Lemma 2, it suffices to show that every element of $R(i)$ is a square. Indeed, take $x = a + bi \in R(i)$ with $a, b \in R$. If $b = 0$, then either $a \geq 0$ and so x is a square in R, or $a \leq 0$ and so $a = (i\sqrt{-a})^2$ is a square in R. Now let $b \neq 0$. Then we compute $x = (c + di)^2$, where

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \text{ and } d = \frac{b}{2c}.$$

To see that c and d are well-defined elements of R, observe that $a^2 + b^2 > a^2 \geq 0$ (as $b \neq 0$), and so $a + \sqrt{a^2 + b^2} > 0$. Therefore the square roots above lie in R and $c \neq 0$. $\square$

Theorem 7. *The only algebraic extensions of R are R itself and $R(i)$.*

Proof. By separability, every algebraic extension of R is contained in a finite Galois extension. Since $R(i)/R$ has no intermediate fields, it suffices to show the result for finite Galois extensions.

Let K/R be a nontrivial Galois extension of degree $2^k \cdot a$, where $k \geq 0$ and $a \geq 1$ is odd. By Lemma 1 with $p = 2$, there is an intermediate extension of degree a. By Lemma 4, $a = 1$ (and $k > 0$). If $k > 1$, then applying Lemma 1 again yields intermediate extensions $K/L/M/R$ with $[L : M] = [M : R] = 2$. By Lemma 5, $M \cong R(i)$, contradicting Lemma 6.$_{\#}$ Therefore $k = 1$ and (by Lemma 5) $K \cong R(i)$. $\square$

Corollary 8. $\bar{R} = R(i)$.