https://artie2000.github.io/real$_c$$losed_fieldhttps : //github.com/artie2000/real_closed_fieldhttps : //artie20$

Real Closed Field

Artie2000

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Lemma 1. *Fix a prime p, and let M/K be a separable Galois extension of degree $p^k \cdot a$, where $p \nmid a$. Then, for $0 \leq l \leq k$, there is an intermediate field $M/L/K$ of degree $[L : K] = p^l \cdot a$.*

Proof. Since M/K is Galois, $|\operatorname{Gal}(M/K)| = p^k \cdot a$. By Sylow's theorems at p, $\operatorname{Gal}(M/K)$ has a subgroup H of degree p^{k-l}. By the Galois correspondence, $[M^H : K] = p^l \cdot a$. □

Lemma 2. *The separable quadratic extensions of a field K are classified by the set*

$$\left(\frac{K^*}{(K^*)^2}\right) \setminus \{1 \cdot (K^*)^2\}$$

via the map $x(K^*)^2 \to K(\sqrt{x})$.

Definition 3. A real closed field R is an ordered field in which every positive element has a square root and every odd-degree polynomial has a root.

Let R be a real closed field. Note that, since R is ordered, $\operatorname{char} R = 0$. In particular, its algebraic extensions are separable.

Lemma 4. *Nontrivial algebraic extensions of R have even degree.*

Proof. Let K/R be an algebraic extension of R. By the primitive element theorem, $K = R(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over K; then f is irreducibile but $\deg f = [K : R]$. Therefore, if $[K : R]$ is odd, then $[K : R] = 1$; that is, $K = R$. □

Lemma 5. *Let K/R be a quadratic extension. Then $K = R(i)$.*

Proof. Fix $x \in R^*$. Then either $x > 0$ and $x = 1 \cdot (\sqrt{x})^2$, or $x < 0$ and $x = -1 \cdot (\sqrt{-x})^2$. Therefore $R^*/(R^*)^2 = \{1, -1\}$, and we are done by Lemma **??**. □

Lemma 6. *There is no quadratic extension of $R(i)$.*

Proof. By Lemma **??**, it suffices to show that every element of $R(i)$ is a square. Indeed, take $x = a + bi \in R(i)$ with $a, b \in R$. If $b = 0$, then either $a \geq 0$ and so x is a square in R, or $a \leq 0$ and so $a = (i\sqrt{-a})^2$ is a square in K. Now let $b \neq 0$. Then we compute $x = (c + di)^2$, where

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \text{ and } d = \frac{b}{2c}.$$

These square roots lie in $R(i)$ because squares are non-negative and $\frac{a^2+b^2}{\geq}a^2$ so $a + \sqrt{a^2 + b^2} \geq 0$. □

Theorem 7. *The only algebraic extensions of R are R itself and $R(i)$.*

Proof. By separability, every algebraic extension of R is contained in a Galois extension. Since $R(i)/R$ has no intermediate fields, it suffices to show the result for Galois extensions.

Let K/R be a nontrivial Galois extension of degree $2^k a$, where $k \geq 0$ and $a \geq 1$ is odd. By Lemma **??** with $p = 2$, there is an intermediate extension of degree a. By Lemma **??**, $a = 1$ (and $k \geq 1$). Applying Lemma **??**, there is an intermediate extension $K/L/R$ of degree 2. By Lemma **??**, $L = K(i)$. If $L \neq K(i)$, then one more application of Lemma **??** yields an intermediate quadratic extension $L/M/K(i)$, contradicting Lemma **??**.$_{\#}$ $\square$

Corollary 8. *$\bar{R} = R(i)$.*