Real Closed Field

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January 3, 2026

We synthesise material from [1], [4], [2] and [3] as convenient.

Note that the meaning of the notation $F(\sqrt{a})$ depends on whether a is a square in F. We take care to write it only in the non-trivial case. The same applies to $F(i)$.

0.1 Ordered Fields

We begin with a purely algebraic characterisation of an ordered field. This relies on the theory of *ring orderings*, which can be found in [5].

Theorem 1. *A field can be ordered if and only if it is real (that is, -1 is not a sum of squares).*

Proof. □

We can also characterise algebraically whether a field F has a unique ordering. This allows us to talk about 'the ordering' on F without ambiguity.

Lemma 2. *Let F be a real field. There is a unique ordering on F if and only if, for each $a \in F$, either a or $-a$ is a sum of squares. In this ordering, the non-negative elements are precisely the squares in F.*

Proof. □

Corollary 3. *Let F be an ordered field. Then F has a unique field ordering if and only if every non-negative element is a sum of squares.*

Proof. □

Corollary 4. *There is a unique field ordering on $\mathbb{Q}$.*

Proof. We use Corollary 3. Let $x \in \mathbb{Q}$ be non-negative. Then $x = p/q$ for some integers $p \geq 0$ and $q > 0$, so

$$x = \underbrace{\frac{1}{q^2} + \cdots + \frac{1}{q^2}}_{pq \text{ times}}.$$

□

There is a corresponding algebraic characterisation of an ordered field extension.

Theorem 5. *Let F be an ordered field, and let K/F be a field extension. Then there is an ordering on K making the extension K/F ordered if and only if $\sum_i x_i \alpha_i^2 \neq -1$ for all choices of $\alpha_i \in K$ and $x_i \in F_{\geq 0}$.*

Proof. □

Lemma 6. *Let F be an ordered field, and let K/F be a field extension. Suppose that there is an F-linear functional $\pi : K \to F$ such that, for all $x \in K$, $\pi(x^2) \geq 0$. Then there is a field ordering on K making K/F ordered.*

Proof. Consider the sum $\sum_i x_i \alpha_i^2$ for some $\alpha_i \in K$ and $x_i \in F_{\geq 0}$. By F-linearity, we compute

$$\pi\left(\sum_i x_i \alpha_i^2\right) = \sum_i x_i \pi(\alpha_i^2) \geq 0.$$

Since $\pi(-1) = -1 < 0$, we are done by Theorem 5. □

Corollary 7. *Let F be a ordered field, and suppose $a \in F$ is non-negative (and not a square). Then there is a field ordering on $F(\sqrt{a})$ making $F(\sqrt{a})/F$ ordered.*

Proof. Let $\pi : F(\sqrt{a}) \to F$ be the projection induced by the F-basis $\{1, \sqrt{a}\}$. For $x, y \in F$, we have $\pi((x + y\sqrt{a})^2) = x^2 + ay^2 \geq 0$. We are done by Lemma 6. $\square$

Lemma 8. *Let F be an ordered field, and let K/F be an odd-degree extension. Then there is a field ordering on K making K/F ordered.*

Proof. Since $\operatorname{char} R = 0$, we can apply the primitive element theorem. Let $K = F(\alpha)$ for some $\alpha \in K$, and let f be the minimal polynomial of α over K. Then $\deg f = [K : F]$ is odd. By Theorem 5, we need to show that the congruence

$$\sum_i a_i g_i^2 \equiv -1 \bmod f \tag{$\star$}$$

fails to hold for any non-negative $a_i \in F$ and polynomials $g_i \in F[X]$ each of degree at most $\deg f - 1$. Proceed by induction on $\deg f$; if $\deg f = 1$, then ($\star$) reduces to an equality of a non-negative element of F with a negative one. Otherwise, suppose for a contradiction that ($\star$) holds. Without loss of generality, we may assume, for all i, we have $a_i \neq 0$ and that $\deg g_i < \deg f$.

Rearranging ($\star$), we have $\sum_i a_i g_i^2 + 1 = hf$ for some $h \in F[X]$. Let $d = \max_i \deg g_i$; note that $d < \deg f$ by construction. Since each a_i is positive, the $2d$th coefficient on the left-hand side must be positive. Therefore

$$\deg h + \deg f = \deg \left(\sum_i a_i g_i^2 + 1 \right) = 2d.$$

Then $\deg h$ is odd, so h has an odd-degree irreducible factor $\tilde{h}$. We have

$$\deg \tilde{h} \leq \deg h = 2d - \deg f < \deg f,$$

but $\sum_i a_i g_i^2 \equiv -1 \bmod \tilde{h}$. We are done by induction.$_\#$ $\square$

There is an easier way to construct ordered field extensions if we don't care about them being algebraic.

Lemma 9. *Let F be an ordered field, and let $a \in F$. Then there is a unique ordering on the function field $F(X)$ making $F(X)/F$ ordered such that $X > a$ but $b > X$ for $b > a$, and a unique one such that $X < a$ but $b < X$ for $b < a$.*

Intuitively, X is infinitesimally close to a. When $a = 0$, we often write $R(\varepsilon)$ for $R(X)$ with the first type of ordering.

0.2 Real Closed Fields

Definition 10. A real closed field F is a real field such that

1. for all x in F, either x or $-x$ is a square in F, and
2. every odd-degree polynomial over F has a root in F.

Lemma 11. *An ordered field is real closed if*

1. *every non-negative element is a square, and*
2. *every odd-degree polynomial over F has a root in F.*

Proof. Squares in an ordered field are non-negative. $\square$

Fix a real closed field R.

Lemma 12. *Every sum of squares in R is a square in R.*

Proof. If the negative of a sum of squares is a square, then, dividing through, -1 is a sum of squares, contradicting the realness of R. $\square$

Lemma 13. *R has a unique ordering making it an ordered field. In this ordering, the non-negative elements are exactly the squares.*

Proof. Follows directly from Lemmas 2 and 12. $\square$

In what follows, all algebraic extensions are given up to R-isomorphism, as is conventional. Observe that, since -1 is not a square in R, $R(i)/R$ is a quadratic extension. We will show this is the **only** nontrivial algebraic extension of R.

Lemma 14. *There is no nontrivial odd-degree finite extension of R.*

Proof. Let K/R be an odd-degree extension of R. By the primitive element theorem, $K = R(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over K. Then f is irreducible, but $\deg f = [K : R]$ is odd, so f has a root in R. Therefore, $[K : R] = \deg f = 1$; that is, $K \cong R$. $\square$

Lemma 15. *The field $R(i)$ is the unique quadratic extension of R.*

Proof. Let K/R be a quadratic extension. Since $\operatorname{char} R \neq 2$, we have $K \cong R(\sqrt{a})$ for some $a \in R$. Since a cannot be a square in R, we know $-a$ must be one. Rescaling by $\sqrt{-a}$, we get $R(\sqrt{a}) \cong R(i)$. $\square$

Lemma 16. *There is no quadratic extension of $R(i)$.*

Proof. Since $\operatorname{char} R(i) \neq 2$, it suffices to show that every element of $R(i)$ is a square. Observe that every $x \in R$ is a square in $R(i)$: if x is not a square in R, then $-x$ is, and so $x = (i\sqrt{-x})^2$. Now, fix $x = a + bi \in R(i)$ with $a, b \in R$. If $b = 0$, then $a = x$ is a square in $R(i)$. Otherwise, we have $x = (c + di)^2$, where

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \text{ and } d = \frac{b}{2c}.$$

To see these are well-defined elements of $R(i)$, note that $\sqrt{a^2 + b^2}$ is in R by Lemma 12 and that $c \neq 0$ because otherwise $-a = \sqrt{a^2 + b^2}$ and so $b = 0$.$_{\#}$ $\square$

Theorem 17. *The only finite extensions of R are R itself and $R(i)$.*

Proof. By separability, every finite extension of R is contained in a finite Galois extension. Since $R(i)/R$ has no intermediate fields, it suffices to show the result for finite Galois extensions.

Let K/R be a nontrivial Galois extension of degree $2^k \cdot a$, where $k \geq 0$ and $a \geq 1$ is odd. Applying the Galois correspondence to a Sylow 2-subgroup of $\operatorname{Gal}(K/R)$ yields an intermediate extension of degree a; by Lemma 14, we have $a = 1$ (and $k > 0$). If $k > 1$, iterating the last construction yields intermediate extensions $K/L/M/R$ with $[L : M] = [M : R] = 2$. By Lemma 15, $M \cong R(i)$, contradicting Lemma 16.$_{\#}$ Therefore $k = 1$ and (by Lemma 15) $K \cong R(i)$. $\square$

Corollary 18. *The only algebraic extensions of R are R itself and $R(i)$.*

Proof. An infinite algebraic extension contains finite subextensions of arbitrarily large degree. □

Corollary 19. $\bar{R} = R(i)$.

Proof. □

The converse to Theorem 17 is much easier.

Lemma 20. *Let F be a field in which -1 is not a square, and suppose every element of $F(i)$ is a square. Then sums of squares in F are squares in F.*

Proof. Given $a, b \in F$, find $c, d \in F$ such that $a + bi = (c + di)^2$ in $F(i)$. Then $a = c^2 - d^2$ and $b = 2cd$, so $a^2 + b^2 = (c^2 + d^2)^2$ is a square in F. By induction, every sum of squares in F is a square in F. □

In particular, F as in the last lemma is real.

Lemma 21. *Suppose R is a field with unique nontrivial finite extension $R(i)$. Then R is real closed.*

Proof. Since $R(i)$ has no quadratic extensions, we can apply Lemma 20: sums of squares in R are squares in R, and R is real.

Take a non-square $a \in R$; then $R(\sqrt{a}) \cong R(i)$. Suppose i maps to $x + y\sqrt{a}$ for some $x, y \in R$. Then $-1 = x^2 + ay^2 + 2xy\sqrt{a}$; comparing coefficients, we get $-1 = x^2 + ay^2$. Since -1 is not a square, y must be nonzero, and so $-a = (x/y)^2 + (1/y)^2$ is a square.

Now, fix a nonlinear odd-degree polynomial $f \in R[X]$. Then $R[X]/(f)$ cannot be a field since R has no nontrivial odd-degree extensions, and so f must be reducible. We are done by induction on the degree. □

Corollary 22. *Suppose R is a field with nontrivial algebraic closure $R(i)$. Then R is real closed.*

Proof. □

As before, let R be a real closed field. Theorem 17 is a powerful tool for deriving more of its properties.

Lemma 23. *R has no nontrivial real algebraic extensions.*

Proof. The field $R(i)$ is not real since -1 is a square in it. We are done by Corollary 18. □

Corollary 24. *R has no nontrivial ordered algebraic extensions (with respect to the unique order).*

Proof. Since ordered fields are real, we are done by Lemma 23. □

The next property is a little less obvious.

Lemma 25. *The monic irreducible polynomials over $R[X]$ have form $X - c$ for some $c \in R$ or $(X - a)^2 + b^2$ for some $a, b \in R$ with $b \neq 0$.*

Proof. Let $f \in R[X]$ be monic and irreducible. The field $R_f = R[X]/(f)$ is a finite extension of R, so it is classified by Theorem 17. If $R_f \cong R$, then $\deg f = 1$, so $f = X - c$ for some $c \in R$. If $R_f \cong R(i)$, let the isomorphism be φ, and suppose $\varphi(X + (f)) = a + bi$ $(a, b \in R)$. Note that $b \neq 0$ since φ^{-1} is constant on R. Rearranging, we see that $\varphi((X - a)^2 + b^2 + (f)) = 0$; that is, $(X - a)^2 + b^2 \in (f)$. Since this polynomial is monic and has the same degree as f, it must in fact be equal to f.

Conversely, linear polynomials over a domain are irreducible by degree, and reducible quadratics have a root. A root of $f = (X - a)^2 + b^2$ with $a, b \in R$ is an element $r \in R$ satisfying $(r - a)^2 = -b^2$. If $b \neq 0$, then dividing through would give a contradiction to the realness of R.Therefore f must be irreducible. □

Lemma 26. *Polynomials over R satisfy the intermediate value property (with respect to the unique order).*

Proof. We will prove that, for all $f \in R[X]$ and all $a, b \in R$ with $a \leq b$, if $f(a) \leq 0 \leq f(b)$, then there is some $c \in [a, b]$ such that $f(c) = 0$.

Fix $a, b \in R$ with $a \leq b$, and proceed by induction on $\deg f$. If $\deg f = 0$, then f is constant and the result is clear. Otherwise, take a monic irreducible factor g of f; then g is classified by Lemma 25.

Observe that, if $g(a)$ and $g(b)$ are both positive, then f/g satisfies the inductive hypothesis, and so has a root in $[a, b]$. If $g = (X - a)^2 + b^2$ with $a, b \in R$ and $b \neq 0$, then g is everywhere positive. If $g = X - c$ with $c \in R$, then either $c \in [a, b]$ and $g(c) = 0$, or $c \notin [a, b]$ and $f(a)$ and $f(b)$ have the same sign. In this last case, either f/g or $f/(-g)$ satsifies the inductive hypothesis. In all cases, f has a root in $[a, b]$. □

In fact, the converses to Lemmas 23 and 26 also hold!

Theorem 27. *Let R be an ordered field whose polynomials satisfy the intermediate value property. Then R is real closed.*

Proof. We use Lemma 11. Let $a \in R$ be non-negative, and consider the polynomial $f = X^2 - a$. Then $f(0) = -a \leq 0$, but $f(a + 1) = a^2 + a + 1 > 0$. By the intermediate value property, f has a root in R, and so a is a square in R.

Let f be an odd-degree polynomial over R. Write $f = a_n X^n + \cdots + a_0$. We will show f has a root in R. Replacing f by $-f$ if necessary, we may assume $a_n > 0$. For $x > 1$, we compute

$$f(x) \geq x^{n-1}(a_n x - n \max_i |a_i|).$$

Therefore, when $x > \max\{1, n \max_i |a_i|/a_n\}$, $f(x) > 0$. A similar calculation shows that $f(x) < 0$ for sufficiently large negative values of x. We are done by the intermediate value property. □

Theorem 28. *Let R be an ordered field with no nontrivial ordered algebraic extensions. Then R is real closed.*

Proof. We use Lemma 11.

Let $a \in R$ be non-negative, and suppose a is not a square. By Corollary 7, there is an ordering making the nontrivial extension $R(\sqrt{a})/R$ ordered.$_\#$ Therefore a is a square in R.

By induction on degree, it suffices to show that irreducible odd-degree polynomials over R are all linear. Let $f \in R[X]$ be such a polynomial, and consider the odd-degree field extension $R_f = R[X]/(f)$. By Lemma 8, there is an ordering making R_f/R an ordered extension. Therefore $\deg f = [R_f : R] = 1$. □

Corollary 29. *Let R be a real field with no nontrivial real algebraic extensions. Then R is real closed.*

Proof. Since R can be ordered and ordered fields are real, we are done by Theorem 28. ☐

Theorem 28 gives us a way to "construct" real closed fields.

Lemma 30. *An algebraically closed field of characteristic 0 has an index-2 real closed subfield.*

Proof. Let C be an algebraically closed field of characteristic 0. Observe that the prime subfield $\mathbb{Q}$ can be ordered. Further, given an ordered subfield F with $\bar{F} \neq C$, we can use Lemma 9 to adjoin an element transcendental over F, obtaining a strictly bigger ordered subfield.

Apply Zorn's lemma to obtain a maximal ordered subfield $R \subseteq C$; then $\bar{R} = C$. By Theorem 28, R must be real closed. By Corollary 19, $C \cong R(i)$, and so $[C : R] = 2$. ☐

In summary, we have the following characterisations of real closed fields.

Theorem 31. *Let R be a field. TFAE:*

1. *R is real closed.*
2. *$\bar{R} = R(i)$ (and $R(i) \neq R$).*
3. *R is real, but has no nontrivial real algebraic extensions.*

Proof. ☐

Theorem 32. *Let R be an ordered field. TFAE:*

1. *R is real closed.*
2. *R is maximal with respect to ordered algebraic extensions.*
3. *Polynomials over R satisfy the intermediate value property.*

Proof. ☐

0.3 Real Closures

Definition 33. Let F be an ordered field. A real closure of F is a real closed ordered algebraic extension of F.

Lemma 34. *Let F be an ordered field. Then F has a real closure.*

Proof. Apply Zorn's lemma to ordered algebraic extensions of F. We are done by Theorem 28. ☐

Just like with the algebraic closure, it makes sense to talk of *the* real closure of an ordered field. Proving this uniqueness result requires a method of root-counting in real fields known as Sturm's theorem.

Theorem 35 (Corollary to Sturm's Theorem)**.** *Let F be an ordered field, and let f be a polynomial over F. Then f has the same number of roots in any real closure of F.*

Proof. TODO : decide on the generality of the statement of Sturm's Theorem ☐

Lemma 36. *Let F be an ordered field with a real closure R, and let K/F be a finite ordered extension. Then there is an F-homomorphism $K \to R$.*

Proof. By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over F. Since F has a root in K, it has a root in a real closure of K (one exists by Lemma 34). By Theorem 35, f has a root β in R. Therefore define $\varphi : K \to R$ with $\varphi(\alpha) = \beta$. □

Lemma 37. *Let F be an ordered field with a real closure R, and let K/F be a finite ordered extension. Then there is a unique order-preserving F-homomorphism $K \to R$.*

Proof. Fix a real closure R' of K (one exists by Lemma 34).

By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over F, and let $\alpha_1 < \cdots < \alpha_m$ be the roots of f in R', with $\alpha = \alpha_k$. By Theorem 35, f also has m roots in R; let them be $\beta_1 < \cdots < \beta_m$. Since non-negative elements of R' are squares, and $x_1, \ldots, x_{m-1} \in R'$ such that $\alpha_{j+1} - \alpha_j = x_j^2$, and let $L = K(\alpha_1, \ldots, \alpha_m, r, x_1, \ldots, x_{m-1}) \leq R'$. Now, suppose we have a K-homomorphism $\psi : L \to R$. Each $\psi(\alpha_j)$ is equal to a different β_i. Then $\psi(\alpha_{j+1}) - \psi(\alpha_j) = \psi(x_j)^2 \geq 0$, so $\psi(\alpha_1) < \cdots < \psi(\alpha_m)$, and so $\psi(\alpha_j) = \beta_j$ for all j.

By Lemma 36, there is in fact an F-homomorphism $\varphi : L \to R$. We will show that φ is order-preserving. Indeed, fix a non-negative element $x \in L$. As before, find $r \in R'$ such that $x = r^2$, and let $M = L(r) \leq R'$. Apply Lemma 36 again to obtain an L-homomorphism $\psi : M \to R$; then $\varphi(x) = \psi(x) = \psi(r)^2 \geq 0$. Therefore φ maps non-negative elements to non-negative elements, and so is order-preserving. Then $\varphi|_K$ is the map we want. Note that $\varphi(\alpha) = \beta_k$.

To see uniqueness, let $\tilde{\varphi} : K \to R$ be an order-preserving F-homomorphism; by existence, $\tilde{\varphi}$ extends to a an order-preserving K-homomorphism $\tilde{\psi} : L \to R$. Then $\tilde{\varphi}(\alpha) = \tilde{\psi}(\alpha_k) = \beta_k = \varphi(\alpha)$, and so $\tilde{\varphi} = \varphi$. □

Taking $K = F$ above, we see that the order-embedding of a field into its real closure is unique.

Theorem 38. *Let F be an ordered field. Then the real closure of F is unique up to unique F-isomorphism.*

Proof. Let R_1 and R_2 be real closures of F. Applying Zorn's lemma to the set of ordered extensions intermediate between R_1 and F having a unique order-preserving F-embedding into R_2, and using Lemma 37, we obtain an intermediate extension $R_1/K/F$ with no nontrivial finite ordered extensions and a unique order-preserving F-embedding $\varphi : K \to R_2$. If the ordered algebraic extension $R_2/\varphi(K)$ were nontrivial, then it would contain a nontrivial ordered finite extension, so φ must be surjective (and so an F-isomorphism). In particular, $K \subseteq R_1$ is real closed; by maximality (Lemma 23), in fact $K = R_1$ and so φ is an F-isomorphism between R_1 and R_2. □

Corollary 39. *A real closed field has no nontrivial field automorphisms.*

Proof. Let R be a real closed field. By Theorem 38, R has no nontrivial order-preserving automorphisms. Since the ordering on R is unique (by Lemma 13), every automorphism of R must be order-preserving. □

This uniqueness result is stronger than the one in the algebraically closed case: an algebraically closed field has many nontrivial automorphisms.

Uniqueness of algebraic closures allows us to classify ordered algebraic extensions.

Lemma 40. *Let F be an ordered field with real closure R, and let K/F be algebraic. Then field orderings on K making K/F ordered correspond to F-homomorphisms $K \to R$ via the order obtained by restriction from R.*

Proof. Fix an ordering on K extending that on F, and let K have real closure R_K (exists by Lemma 34). Then R_K/F is algebraic, so R_K is a real closure of F. By Theorem 38, there is an F-isomorphism to $R_K \cong R$, and this induces an F-homomorphism $K \to R$. Restricting the order on R to K via this map recovers the original order on K by construction.

Moreover, the inverse to order restriction constructed above is unique. Indeed, an inverse $\varphi : K \to R$ is order-preserving by definition, so it is an order-embedding from K into its real closure. By Theorem 38, such a map is unique. $\square$

0.4 The Artin-Schreier Theorem

We can go much further than Lemma 22. The following is a weak form of the Artin-Schreier theorem. Removing the condition on characteristic is possible, but requires some more involved algebra.

Theorem 41. *Let R be a field with $\operatorname{char} K \neq 2$, and suppose $[\bar{R} : R] = 2$. Then R is real closed.*

Proof. By Lemma 22, it suffices to show that $\bar{R} \cong R(i)$.

Since $\operatorname{char} \bar{R} \neq 2$, we have $\bar{R} \cong R(\sqrt{a})$ for some non-square $a \in R$. Since $R(\sqrt{a})$ is algebraically closed, $\sqrt{a}$ is a square in $R(\sqrt{a})$; find $x, y \in R$ such that $\sqrt{a} = (x + y\sqrt{a})^2$. Expanding and comparing coefficients, $x^2 + y^2 a = 0$ and $2xy = 1$. Rearranging, $a = -(4x^4) = -1 \cdot (2x^2)^2$. Rescaling by $2x^2$, $R(i) \cong R(\sqrt{a}) = \bar{R}$. $\square$

We can weaken the hypotheses even further. Here is the full Artin-Schreier theorem.

Theorem 42 (Artin-Schreier Theorem)**.** *Let R be a field, and suppose $\bar{R}$ is a finite extension of R. Then R is real closed.*

Proof. TODO: this needs a lot more preliminaries eg Artin-Schreier theory, Kummer theory $\square$

Corollary 43. *An algebraically closed field of nonzero characteristic has no finite-index subfields.*

Proof. Ordered fields have characteristic 0. $\square$

Theorem 44. *The finite-index subfields of $\bar{\mathbb{Q}}$ are isomorphic copies of $\mathbb{Q}_{alg} = \bar{\mathbb{Q}} \cap \mathbb{R}$ indexed by* $\mathrm{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$.

Proof. Since $\mathbb{Q}_{\mathrm{alg}}(i) = \bar{\mathbb{Q}}$, the field $\mathbb{Q}_{\mathrm{alg}}$ is a finite-index subfield.

By Theorem 42, any finite-index subfield has a unique ordering making it real closed. Let R be a real closed subfield. Then the ordering on R restricts to the ordering on $\mathbb{Q}$ (unique by Corollary 4). Since $R/\mathbb{Q}$ is algebraic, R is a real closure of $\mathbb{Q}$. By Theorem 38, there is a unique $\mathbb{Q}$-isomorphism $\psi : \mathbb{Q}_{\mathrm{alg}} \cong R$. Since $\mathbb{Q}_{\mathrm{alg}}(i) = R(i) = \bar{\mathbb{Q}}$, ψ extends uniquely to $\varphi \in \mathrm{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$ by mapping $i \to i$. The subfield R is then recovered from $\varphi \in \mathrm{Gal}(\mathbb{Q}_{alg}/\mathbb{Q}(i))$ by taking $\varphi^{-1}(\mathbb{Q}_{\mathrm{alg}})$. $\square$

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