

Real Closed Field

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We synthesise material from [?], [?], [?] and [?] as convenient.

Note that the meaning of the notation $F(\sqrt{a})$ depends on whether a is a square in F . We take care to write it only in the non-trivial case. The same applies to $F(i)$.

0.1 Ordered Fields

We begin with a purely algebraic characterisation of an ordered field. This relies on the theory of *ring orderings*, which can be found in [?].

Definition 1. A field is real if -1 is not a sum of squares.

Theorem 2. *A field can be ordered if and only if it is real.*

We can also characterise algebraically whether a field F has a unique ordering. This allows us to talk about ‘the ordering’ on F without ambiguity.

Lemma 3. *Let F be a real field. There is a unique ordering on F if and only if, for each $a \in F$, either a or $-a$ is a sum of squares.*

Corollary 4. *Let F be an ordered field. Then F has a unique field ordering if and only if every non-negative element is a sum of squares.*

Proof.

□

Corollary 5. *There is a unique field ordering on \mathbb{Q} .*

Proof. We use Corollary ???. Let $x \in \mathbb{Q}$ be non-negative. Then $x = p/q$ for some integers $p \geq 0$ and $q > 0$, so

$$x = \underbrace{\frac{1}{q^2} + \cdots + \frac{1}{q^2}}_{pq \text{ times}}.$$

□

There is a corresponding algebraic characterisation of an ordered field extension.

Lemma 6. *Let F be an ordered field, and let K/F be a field extension. Then there is an ordering on K making the extension K/F ordered if and only if $\sum_i x_i \alpha_i^2 \neq -1$ for all choices of $\alpha_i \in K$ and $x_i \in F_{\geq 0}$.*

Lemma 7. *Let F be an ordered field, and let K/F be a field extension. Suppose that there is an F -linear functional $\pi : K \rightarrow F$ such that, for all $x \in K$, $\pi(x^2) \geq 0$. Then there is a field ordering on K making K/F ordered.*

Proof. Consider the sum $\sum_i x_i \alpha_i^2$ for some $\alpha_i \in K$ and $x_i \in F_{\geq 0}$. By F -linearity, we compute

$$\pi \left(\sum_i x_i \alpha_i^2 \right) = \sum_i x_i \pi(\alpha_i^2) \geq 0.$$

Since $\pi(-1) = -1 < 0$, we are done by Lemma ???.

□

Corollary 8. *Let F be an ordered field, and suppose $a \in F$ is non-negative (and not a square). Then there is a field ordering on $F(\sqrt{a})$ making $F(\sqrt{a})/F$ ordered.*

Proof. Let $\pi : F(\sqrt{a}) \rightarrow F$ be the projection induced by the F -basis $\{1, \sqrt{a}\}$. For $x, y \in F$, we have $\pi((x + y\sqrt{a})^2) = x^2 + ay^2 \geq 0$. We are done by Lemma ??.

Lemma 9. *Let F be an ordered field, and let K/F be an odd-degree extension. Then there is a field ordering on K making K/F ordered.*

Proof. Since $\text{char } R = 0$, we can apply the primitive element theorem. Let $K = F(\alpha)$ for some $\alpha \in K$, and let f be the minimal polynomial of α over K . Then $\deg f = [K : F]$ is odd. By Lemma ??, we need to show that the congruence

$$\sum_i a_i g_i^2 \equiv -1 \pmod{f} \quad (\star)$$

fails to hold for any non-negative $a_i \in F$ and polynomials $g_i \in F[X]$ each of degree at most $\deg f - 1$. Proceed by induction on $\deg f$; if $\deg f = 1$, then (\star) reduces to an equality of a non-negative element of F with a negative one. Otherwise, suppose for a contradiction that (\star) holds. Without loss of generality, we may assume, for all i , we have $a_i \neq 0$ and that $\deg g_i < \deg f$.

Rearranging (\star) , we have $\sum_i a_i g_i^2 + 1 = hf$ for some $h \in F[X]$. Let $d = \max_i \deg g_i$; note that $d < \deg f$ by construction. Since each a_i is positive, the $2d$ th coefficient on the left-hand side must be positive. Therefore

$$\deg h + \deg f = \deg \left(\sum_i a_i g_i^2 + 1 \right) = 2d.$$

Then $\deg h$ is odd, so h has an odd-degree irreducible factor \tilde{h} . We have

$$\deg \tilde{h} \leq \deg h = 2d - \deg f < \deg f,$$

but $\sum_i a_i g_i^2 \equiv -1 \pmod{\tilde{h}}$. We are done by induction. $\#$ \square

There is an easier way to construct ordered field extensions if we don't care about them being algebraic.

Lemma 10. *Let F be an ordered field, and let $a \in F$. Then there is a unique ordering on the function field $F(X)/F$ making $F(X)/F$ ordered such that $X > a$ but $b > X$ for $b > a$, and a unique one such that $X < a$ but $b < X$ for $b < a$.*

Intuitively, X is infinitesimally close to a . When $a = 0$, we often write $R()$ for $R(X)$ with the first type of ordering.

0.2 Real Closed Fields

Definition 11. A real closed field is an ordered field in which every non-negative element is a square and every odd-degree polynomial has a root.

Lemma 12. *The ordering on a real closed field is unique.*

Proof. Follows directly from Corollary ??.

\square

Fix a real closed field R . In what follows, all algebraic extensions are given up to R -isomorphism, as is conventional. Observe that, since -1 is not a square in R , $R(i)/R$ is a quadratic extension. We show that this is the **only** nontrivial algebraic extension of R .

Lemma 13. *There is no nontrivial odd-degree finite extension of R .*

Proof. Let K/R be an odd-degree extension of R . By the primitive element theorem, $K = R(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over K . Then f is irreducible, but $\deg f = [K : R]$ is odd, so f has a root in R . Therefore, $[K : R] = \deg f = 1$; that is, $K \cong R$. \square

Lemma 14. *The field $R(i)$ is the unique quadratic extension of R .*

Proof. Let K/R be a quadratic extension. Since $\text{char}(R) \neq 2$, we have $K \cong R(\sqrt{a})$ for some $a \in R$. Since nonnegative elements of R are squares, a is negative and so $-a$ is a square in R . Rescaling by $\sqrt{-a}$, we get $R(\sqrt{a}) \cong R(i)$. \square

Lemma 15. *There is no quadratic extension of $R(i)$.*

Proof. Since $\text{char } R(i) \neq 2$, it suffices to show that every element of $R(i)$ is a square. Indeed, take $x = a + bi \in R(i)$ with $a, b \in R$. If $b = 0$, then either $a \geq 0$ and so x is a square in R , or $a \leq 0$ and so $a = (i\sqrt{-a})^2$ is a square in R . Now let $b \neq 0$. Then we compute $x = (c + di)^2$, where

$$c = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} \text{ and } d = \frac{b}{2c}.$$

To see that c and d are well-defined elements of R , observe that $a^2 + b^2 > a^2 \geq 0$ (as $b \neq 0$), and so $a + \sqrt{a^2 + b^2} > 0$. Therefore the square roots above lie in R and $c \neq 0$. \square

Theorem 16. *The only finite extensions of R are R itself and $R(i)$.*

Proof. By separability, every finite extension of R is contained in a finite Galois extension. Since $R(i)/R$ has no intermediate fields, it suffices to show the result for finite Galois extensions.

Let K/R be a nontrivial Galois extension of degree $2^k \cdot a$, where $k \geq 0$ and $a \geq 1$ is odd. Applying the Galois correspondence to a Sylow 2-subgroup of (K/R) yields an intermediate extension of degree a ; by Lemma ??, we have $a = 1$ (and $k > 0$). If $k > 1$, iterating the last construction yields intermediate extensions $K/L/M/R$ with $[L : M] = [M : R] = 2$. By Lemma ??, $M \cong R(i)$, contradicting Lemma ??. $_{\#}$ Therefore $k = 1$ and (by Lemma ??) $K \cong R(i)$. \square

Corollary 17. *The only algebraic extensions of R are R itself and $R(i)$.*

Proof. An infinite algebraic extension contains finite subextensions of arbitrarily large degree. \square

Corollary 18. $\bar{R} = R(i)$.

Proof. \square

The converse to Theorem ?? is much easier.

Lemma 19. *Suppose R is an ordered field whose only nontrivial finite extension is $R(i)$. Then R is real closed.*

Proof. Let $a \in R$ be non-negative. If a is not a square, then $R(\sqrt{a}) \cong R(i)$. Suppose i maps to $x + y\sqrt{a}$ for some $x, y \in R$; then $-1 = x^2 + ay^2 + 2xy\sqrt{a}$. Comparing coefficients, $-1 = x^2 + ay^2 \geq 0$. $_{\#}$ Therefore a is a square in R .

Now, fix a nonlinear odd-degree polynomial $f \in R[X]$. Then $R[X]/(f)$ cannot be a field since R has no nontrivial odd-degree extensions, and so f must be reducible. We are done by induction on the degree. \square

As before, let R be a real closed field. Theorem ?? is a powerful tool for deriving more of its properties.

Lemma 20. *R is maximal with respect to algebraic extensions by ordered fields.*

Proof. Since -1 is a square in $R(i)$, the field $R(i)$ is not formally real and therefore cannot be ordered (by Theorem ??). We are done by Corollary ??.

In particular, R is maximal with respect to ordered algebraic extensions.

The next property is a little less obvious.

Lemma 21. *The monic irreducible polynomials over $R[X]$ have form $X - c$ for some $c \in R$ or $(X - a)^2 + b^2$ for some $a, b \in R$ with $b \neq 0$.*

Proof. Let $f \in R[X]$ be monic and irreducible. The field $R_f = R[X]/(f)$ is a finite extension of R , so it is classified by Theorem ???. If $R_f \cong R$, then $\deg f = 1$, so $f = X - c$ for some $c \in R$. If $R_f \cong R(i)$, let the isomorphism be φ , and suppose $\varphi(X + (f)) = a + bi$ ($a, b \in R$). Note that $b \neq 0$ since φ^{-1} is constant on R . Rearranging, we see that $\varphi((X - a)^2 + b^2 + (f)) = 0$; that is, $(X - a)^2 + b^2 \in (f)$. Since this polynomial is monic and has the same degree as f , it must in fact be equal to f .

Conversely, linear polynomials over a domain are irreducible by degree, and reducible quadratics have a root. A root of $f = (X - a)^2 + b^2$ with $a, b \in R$ is an element $r \in R$ satisfying $(r - a)^2 = -b^2$. Since squares are non-negative, if $b \neq 0$ then f must be irreducible. \square

Lemma 22. *Polynomials over R satisfy the intermediate value property.*

Proof. We will prove that, for all $f \in R[X]$ and all $a, b \in R$ with $a \leq b$, if $f(a) \leq 0 \leq f(b)$, then there is some $c \in [a, b]$ such that $f(c) = 0$.

Fix $a, b \in R$ with $a \leq b$, and proceed by induction on $\deg f$. If $\deg f = 0$, then f is constant and the result is clear. Otherwise, take a monic irreducible factor g of f ; then g is classified by Lemma ??.

Observe that, if $g(a)$ and $g(b)$ are both positive, then f/g satisfies the inductive hypothesis, and so has a root in $[a, b]$. If $g = (X - a)^2 + b^2$ with $a, b \in R$ and $b \neq 0$, then g is everywhere positive. If $g = X - c$ with $c \in R$, then either $c \in [a, b]$ and $g(c) = 0$, or $c \notin [a, b]$ and $f(a)$ and $f(b)$ have the same sign. In this last case, either f/g or $f/(-g)$ satisfies the inductive hypothesis. In all cases, f has a root in $[a, b]$. \square

In fact, the converses to Lemmas ?? and ?? also hold!

Theorem 23. *Let R be an ordered field whose polynomials satisfy the intermediate value property. Then R is real closed.*

Proof. Let $a \in R$ be non-negative, and consider the polynomial $f = X^2 - a$. Then $f(0) = -a \leq 0$, but $f(a + 1) = a^2 + a + 1 > 0$. By the intermediate value property, f has a root in R , and so a is a square in R .

Let f be an odd-degree polynomial over R . Write $f = a_n X^n + \dots + a_0$. We will show f has a root in R . Replacing f by $-f$ if necessary, we may assume $a_n > 0$. For $x > 1$, we compute

$$f(x) \geq x^{n-1}(a_n x - n \max_i |a_i|).$$

Therefore, when $x > \max\{1, n \max_i |a_i|/a_n\}$, $f(x) > 0$. A similar calculation shows that $f(x) < 0$ for sufficiently large negative values of x . We are done by the intermediate value property. \square

Theorem 24. *Let R be an ordered field maximal with respect to ordered algebraic extensions. Then R is real closed.*

Proof. Let $a \in R$ be non-negative, and suppose a is not a square. By Corollary ??, there is an ordering making the nontrivial extension $R(\sqrt{a})/R$ ordered, contradicting maximality. Therefore a is a square in R .

By induction on degree, it suffices to show that irreducible odd-degree polynomials over R are all linear. Let $f \in R[X]$ be such a polynomial, and consider the odd-degree field extension $R_f = R[X]/(f)$. By Lemma ??, there is an ordering making R_f/R an ordered extension; by maximality, $R_f \cong R$, and therefore $\deg f = [R_f : R] = 1$. \square

In particular, an ordered field maximal with respect to algebraic extensions by ordered fields is real closed.

Theorem ?? gives us a way to “construct” real closed fields.

Lemma 25. *An algebraically closed field of characteristic 0 has an index-2 real closed subfield.*

Proof. Let C be an algebraically closed field of characteristic 0. Observe that the prime subfield can be ordered. Further, given an ordered subfield F with $\bar{F} \neq C$, we can use Lemma ?? to adjoin an element transcendental over F , obtaining a strictly bigger ordered subfield.

Apply Zorn’s lemma to obtain a maximal ordered subfield $R \subseteq C$; then $\bar{R} = C$. By Theorem ??, R must be real closed. By Corollary ??, $C \cong R(i)$, and so $[C : R] = 2$. \square

In summary, we have proved the following characterisation of real closed fields.

Theorem 26. *Let R be an ordered field. TFAE:*

1. R is real closed.
2. $\bar{R} = R(i)$ (and -1 is not a square in R).
3. Polynomials over R satisfy the intermediate value property.
4. R is maximal with respect to ordered algebraic extensions.
5. R is maximal with respect to algebraic extensions by ordered fields.

Proof. \square

0.3 Real Closures

Definition 27. Let F be an ordered field. A real closure of F is a real closed ordered algebraic extension of F .

Lemma 28. *Let F be an ordered field. Then F has a real closure.*

Proof. Apply Zorn’s lemma to ordered algebraic extensions of F . We are done by Theorem ??.

Just like with the algebraic closure, it makes sense to talk of the real closure of an ordered field. Proving this uniqueness result requires a method of root-counting in real fields known as Sturm’s theorem.

Theorem 29 (Corollary to Sturm’s Theorem). *Let F be an ordered field, and let f be a polynomial over F . Then f has the same number of roots in any real closure of F .*

Proof. TODO : decide on the generality of the statement of Sturm's Theorem \square

Lemma 30. *Let F be an ordered field with a real closure R , and let K/F be a finite ordered extension. Then there is an F -homomorphism $K \rightarrow R$.*

Proof. By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over F . Since F has a root in K , it has a root in a real closure of K (one exists by Lemma ??). By Theorem ??, f has a root β in R . Therefore define $\varphi : K \rightarrow R$ with $\varphi(\alpha) = \beta$. \square

Lemma 31. *Let F be an ordered field with a real closure R , and let K/F be a finite ordered extension. Then there is a unique order-preserving F -homomorphism $K \rightarrow R$.*

Proof. Fix a real closure R' of K (one exists by Lemma ??).

By the primitive element theorem, $K = F(\alpha)$ for some $\alpha \in K$. Let f be the minimal polynomial of α over F , and let $\alpha_1 < \dots < \alpha_m$ be the roots of f in R' , with $\alpha = \alpha_k$. By Theorem ??, f also has m roots in R ; let them be $\beta_1 < \dots < \beta_m$. Since non-negative elements of R' are squares, and $x_1, \dots, x_{m-1} \in R'$ such that $\alpha_{j+1} - \alpha_j = x_j^2$, and let $L = K(\alpha_1, \dots, \alpha_m, r, x_1, \dots, x_{m-1}) \leq R'$. Now, suppose we have a K -homomorphism $\psi : L \rightarrow R$. Each $\psi(\alpha_j)$ is equal to a different β_i . Then $\psi(\alpha_{j+1}) - \psi(\alpha_j) = \psi(x_j)^2 \geq 0$, so $\psi(\alpha_1) < \dots < \psi(\alpha_m)$, and so $\psi(\alpha_j) = \beta_j$ for all j .

By Lemma ??, there is in fact an F -homomorphism $\varphi : L \rightarrow R$. We will show that φ is order-preserving. Indeed, fix a non-negative element $x \in L$. As before, find $r \in R'$ such that $x = r^2$, and let $M = L(r) \leq R'$. Apply Lemma ?? again to obtain an L -homomorphism $\psi : M \rightarrow R$; then $\varphi(x) = \psi(x) = \psi(r)^2 \geq 0$. Therefore φ maps non-negative elements to non-negative elements, and so is order-preserving. Then $\varphi|_K$ is the map we want. Note that $\varphi(\alpha) = \beta_k$.

To see uniqueness, let $\tilde{\varphi} : K \rightarrow R$ be an order-preserving F -homomorphism; by existence, $\tilde{\varphi}$ extends to a an order-preserving K -homomorphism $\tilde{\psi} : L \rightarrow R$. Then $\tilde{\varphi}(\alpha) = \tilde{\psi}(\alpha_k) = \beta_k = \varphi(\alpha)$, and so $\tilde{\varphi} = \varphi$. \square

Taking $K = F$ above, we see that the order-embedding of a field into its real closure is unique.

Theorem 32. *Let F be an ordered field. Then the real closure of F is unique up to unique F -isomorphism.*

Proof. Let R_1 and R_2 be real closures of F . Applying Zorn's lemma to the set of ordered extensions intermediate between R_1 and F having a unique order-preserving F -embedding into R_2 , and using Lemma ??, we obtain an intermediate extension $R_1/K/F$ with no nontrivial finite ordered extensions and a unique order-preserving F -embedding $\varphi : K \rightarrow R_2$. If the ordered algebraic extension $R_2/\varphi(K)$ were nontrivial, then it would contain a nontrivial ordered finite extension, so φ must be surjective (and so an F -isomorphism). In particular, $K \subseteq R_1$ is real closed; by maximality (??), in fact $K = R_1$ and so φ is an F -isomorphism between R_1 and R_2 . \square

Corollary 33. *A real closed field has no nontrivial field automorphisms.*

Proof. Let R be a real closed field. By Theorem ??, R has no nontrivial order-preserving automorphisms. Since the ordering on R is unique (by Lemma ??), every automorphism of R must be order-preserving. \square

This uniqueness result is stronger than the one in the algebraically closed case: an algebraically closed field has many nontrivial automorphisms.

Uniqueness of algebraic closures allows us to classify ordered algebraic extensions.

Lemma 34. Let F be an ordered field with real closure R , and let K/F be algebraic. Then field orderings on K making K/F ordered correspond to F -homomorphisms $K \rightarrow R$ via the order obtained by restriction from R .

Proof. Fix an ordering on K extending that on F , and let K have real closure R_K (exists by Lemma ??). Then R_K/F is algebraic, so R_K is a real closure of F . By Theorem ??, there is an F -isomorphism to $R_K \cong R$, and this induces an F -homomorphism $K \rightarrow R$. Restricting the order on R to K via this map recovers the original order on K by construction.

Moreover, the inverse to order restriction constructed above is unique. Indeed, an inverse $\varphi : K \rightarrow R$ is order-preserving by definition, so it is an order-embedding from K into its real closure. By Theorem ??, such a map is unique. \square

0.4 The Artin-Schreier Theorem

We didn't actually need to assume the ordering on R in Lemma ??.

Lemma 35. Let F be a field in which -1 is not a square, and suppose every element of $F(i)$ is a square. Then sums of squares in F are squares in F .

Proof. Given $a, b \in F$, find $c, d \in F$ such that $a + bi = (c + di)^2$ in $F(i)$. Then $a = c^2 - d^2$ and $b = 2cd$, so $a^2 + b^2 = (c^2 + d^2)^2$ is a square in F . By induction, every sum of squares in F is a square in F . \square

Lemma 36. Suppose R is a field whose only nontrivial algebraic extension is $R(i)$. Then there is a unique field ordering on R , and moreover R with this ordering is real closed.

Proof. We are very nearly done by Theorem ??, Lemma ??, and Corollary ??; it remains to show that R is real.

We have $\bar{R} \cong R(i)$. Since $R(i)/R$ is nontrivial, -1 is not a square in R . Since every element of $R(i)$ is a square, Lemma ?? says that -1 is not a sum of squares in R . \square

In fact, we can go further. The following is a weak form of the Artin-Schreier theorem. Removing the condition on characteristic is possible, but requires some more involved algebra.

Theorem 37. Let R be a field with $K \neq 2$, and suppose $[\bar{R} : R] = 2$. Then there is a unique field ordering on R , and moreover R with this ordering is real closed.

Proof. By Lemma ??, it suffices to show that $\bar{R} \cong R(i)$.

Since $\text{char } \bar{R} \neq 2$, we have $\bar{R} \cong R(\sqrt{a})$ for some non-square $a \in R$. Since $R(\sqrt{a})$ is algebraically closed, \sqrt{a} is a square in $R(\sqrt{a})$; find $x, y \in R$ such that $\sqrt{a} = (x + y\sqrt{a})^2$. Expanding and comparing coefficients, $x^2 + y^2a = 0$ and $2xy = 1$. Rearranging, $a = -(4x^4) = -1 \cdot (2x^2)^2$. Rescaling by $2x^2$, $R(i) \cong R(\sqrt{a}) = \bar{R}$. \square

We can weaken the hypotheses even further. Here is the full Artin-Schreier theorem.

Theorem 38 (Artin-Schreier Theorem). Let R be a field, and suppose \bar{R} is a finite extension of R . Then there is a unique field ordering on R , and moreover R with this ordering is real closed.

Proof. TODO: this needs a lot more preliminaries eg Artin-Schreier theory, Kummer theory \square

Corollary 39. An algebraically closed field of nonzero characteristic has no finite-index subfields.

Proof. Ordered fields have characteristic 0. \square

Theorem 40. *The finite-index subfields of \mathbb{C} are isomorphic copies of ${}_{\text{alg}} = \cap$ indexed by $({}_{\text{alg}}/(i))$.*

Proof. Since ${}_{\text{alg}}(i) = \bar{i}$, the field ${}_{\text{alg}}$ is a finite-index subfield.

By Theorem ??, any finite-index subfield has a unique ordering making it real closed. Let R be a real closed subfield. Then the ordering on R restricts to the ordering on \mathbb{C} (unique by Corollary ??). Since $R/$ is algebraic, R is a real closure of \mathbb{C} . By Theorem ??, there is a unique \mathbb{C} -isomorphism $\psi : {}_{\text{alg}} \cong R$. Since ${}_{\text{alg}}(i) = R(i) = \bar{i}$, ψ extends uniquely to $\varphi \in ({}_{\text{alg}}/(i))$ by mapping $i \rightarrow i$. The subfield R is then recovered from $\varphi \in ({}_{\text{alg}}/(i))$ by taking $\varphi^{-1}({}_{\text{alg}})$. \square