Final Exam: Solutions

Problem 1. Use truth tables to verify the absorption laws:
(a) $p \vee (p \wedge q) \equiv p$.

(b) $p \wedge (p \vee q) \equiv p$.

Solution. We have the following truth table:

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
0	0	0	0	0	0
0	1	0	0	1	0
1	0	0	1	1	1
1	1	1	1	1	1

Since the values in column 1 and column 4 coincide, (a) is true. Since the values in column 1 and column 6 coincide, (b) is true.

Problem 2. Determine the truth value of each of the following statements if the universe of discourse for x is the set of all real numbers.

(a) $\exists x(x^2 = 2)$.

(b) $\exists x(x^2 = -1)$.

(c) $\forall x(x^2 + 2 \geq 1)$;

(d) $\forall x(x^2 \neq x)$.

Solution. (a) True, e.g., for $x = \sqrt{2}$.
(b) False, since the square of a real number is nonnegative.
(c) True, since for every real x we have $x^2 + 2 \geq 2 > 1$.
(d) False, e.g., $0^2 = 0$, i.e., for $x = 0$ the inequality fails.

Problem 3. Let $A = \{a, b, c, d, e\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Find

(a) $A \cup B$.

(b) $A \cap B$.

(c) $A - B$.

(d) $B - A$.

Solution. (a) $A \cup B = \{a, b, c, d, e, f, g, h\}$.

(b) $A \cap B = A = \{a, b, c, d, e\}$.

(c) $A - B = \emptyset$.

(d) $B - A = \{f, g, h\}$.

Problem 4. Let n be an integer. How many different integers are there in the set

$$\left\{n, \left\lfloor \frac{2n+1}{2} \right\rfloor, n+\frac{1}{2}, \left\lceil \frac{2n-1}{2} \right\rceil \right\}?$$

Solution. Only one: $\left\lfloor \frac{2n+1}{2} \right\rfloor = \left\lfloor n+\frac{1}{2} \right\rfloor = n$, $n+\frac{1}{2}$ is not an integer, and $\left\lceil \frac{2n-1}{2} \right\rceil = \left\lceil n-\frac{1}{2} \right\rceil = n$.

Problem 5. Let $a_n = 2^n + 5 \cdot 3^n$.

(a) Find a_0, a_1, a_2, a_3, and a_4.

(b) Show that $a_2 = 5a_1 - 6a_0$, $a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.

(c) Show that the sequence $\{a_n\}$ satisfies the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for every $n \geq 2$.

Solution. (a) $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 = 6$, $a_1 = 2^1 + 5 \cdot 3^1 = 17$, $a_2 = 2^2 + 5 \cdot 3^2 = 49$, $a_3 = 2^3 + 5 \cdot 3^3 = 143$, and $a_4 = 2^4 + 5 \cdot 3^4 = 421$.

(b) $5a_1 - 6a_0 = 5 \cdot 17 - 6 \cdot 6 = 85 - 36 = 49 = a_2$.
$5a_2 - 6a_1 = 5 \cdot 49 - 6 \cdot 17 = 245 - 102 = 143 = a_3$.
$5a_3 - 6a_2 = 5 \cdot 143 - 6 \cdot 49 = 715 - 294 = 421 = a_4$.

(c) $5a_{n-1} - 6a_{n-2} = 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) = 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 30 \cdot 3^{n-2} = 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 3 \cdot 2^{n-1} - 10 \cdot 3^{n-1} = 2 \cdot 2^{n-1} + 15 \cdot 3^{n-1} = 2^n + 5 \cdot 3^n = a_n$.

Problem 6. Find the closed form solution of each of the following recurrence relations and initial conditions. Justify your answer.

(a) $a_n = a_{n-1} + 3$, $a_0 = 1$.

(b) $a_n = 2a_{n-1} - 1$, $a_0 = 1$. **Hint:** You may or may not use the formula for the sum of a geometric progression:

$$1 + q + q^2 + \cdots + q^m = \frac{q^{m+1} - 1}{q - 1}, \quad q \neq 1.$$

Solution. (a) $a_0 = 1$, $a_1 = 1 + 3 = 4$, $a_2 = 1 + 3 + 3 = 7$, $a_3 = 1+3+3+3 = 10$. We conjecture that $a_n = 1+3n$ for all n. Let's prove this conjecture is true. We have $a_{n-1} = 1+3(n-1) = 1+3n-3 = a_n - 3$, hence $a_n = a_{n-1} + 3$. We also have $a_0 = 1 + 3 \cdot 0 = 1$. Therefore the sequence $a_n = 1 + 3n$ is the closed form solution of the recurrence relation $a_n = a_{n-1} + 3$ and the initial condition $a_0 = 1$.

(b) Solution 1. We have $a_1 = 2a_0 - 1 = 2 \cdot 1 - 1$, $a_2 = 2a_1 - 1 = 2(2a_0 - 1) - 1 = 2^2 a_0 - 2 - 1 = 2^2 - 2 - 1$. Similarly, $a_3 = 2^3 - 2^2 - 2 - 1$, and by induction one can show that $a_n = 2^n - (2^{n-1} + 2^{n-2} + \cdots + 2 + 1)$. By the formula given in the hint, we obtain that $a_n = 2^n - \frac{2^n - 1}{2-1} = 2^n - (2^n - 1) = 1$. Thus $a_n = 1$ for every n.

Solution 2. We have $a_1 = 2a_0 - 1 = 2 \cdot 1 - 1 = 1$, $a_2 = 2a_1 - 1 = 2 \cdot 1 - 1 = 1$, and so on. We guess that $a_n = 1$ for all integer $n \geq 0$. Since it is true for $n = 0$, the initial condition $a_0 = 1$ is satisfied. Since $1 = 2 \cdot 1 - 1$, the recurrence relation $a_n = 2a_{n-1} - 1$ is also satisfied, so that our guess is true.

Comment. Solution 1 seems to be more complicated in this case, however it works for any initial condition, while solution 2 works only for the given initial condition $a_0 = 1$.

Problem 7. Use mathematical induction to prove the formula for the sum of a geometric progression given in Problem 7(b).

Solution. Base step. $1 = \frac{q^1 - 1}{q-1}$.

Induction step. Suppose we have the statement hold for $m = k$: $1 + q + \cdots + q^k = \frac{q^{k+1} - 1}{q-1}$. Then

$$1+q+\cdots+q^k+q^{k+1} = \frac{q^{k+1} - 1}{q-1} + q^{k+1} = \frac{q^{k+1} - 1 + q^{k+2} - q^{k+1}}{q-1} = \frac{q^{k+2} - 1}{q-1},$$

i.e., the statement holds for $m = k + 1$.

Problem 9. Which of the following relations on the set of all people are equivalence relations, and which are not? Explain your answers.

(a) $\{(a, b)|$ a and b are the same age$\}$.

(b) $\{(a, b)|$ a and b have the same parents$\}$.

(c) $\{(a, b)|$ a and b share a common parent$\}$.

Solution. The relations in (a) and (b) are equivalence relations (check that they are reflexive, symmetric, and t ransitive!). The relation in (c) is not an equivalence relation, because it is not transitive. E.g., a and b have the same mother but different fathers, and b and c have the same father but different mothers. Then a and c cannot share a common parent.