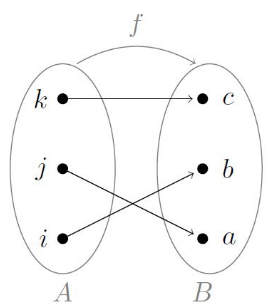
CISC-102

PRACTICE PROBLESM WITH SOLUTIONS

Problems

(1) Decide for each of the following expressions: Is it a function? If so,
(i) what is its domain, codomain, and image? (ii) is it injective? (why or why not)
(iii) is it surjective? (why or why not) (iv) is it invertible? (why or why not)

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \sqrt{(}x)$
(b) $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ defined by $(a, b) \mapsto a/b$
(c) f is shown below



(2) If possible, choose a domain and codomain for the following expressions to make them into functions satisfying

(a) f is an injective but not surjective function;
(b) f is a surjective but not injective function;
(c) f is an invertible function;
(d) f is not a function;

where f is given by:

(a) $f(x) = |x|$

(b) $f(x) = 6$

(c) $f(x, y) = \sqrt{xy}$.

(3) ..

(a) How many distinct functions $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4\}$ are there, from the set $\{1, 2, 3, 4, 5, 6\}$ to the set $\{1, 2, 3, 4\}$ such that for all $i \in \{1, 2, 3\}$ $f(7 - i) = 5 - f(i)$.

(b) How many distinct functions $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ are there, from the set $\{1, 2, 3, 4, 5, 6, 7\}$ to itself, such that there does not exist any $i \in \{1, 2, 3, 4, 5,6,7\}$ such that $f(i) = i$.

(4) Assume functions $g : A \to B$ and $f : B \to C$. Prove or disprove the following statements.

(a) If $f \circ g$ and g are injective then f is injective.

(b) If $f \circ g$ and f are injective then g is injective.

Solutions:

(1) (a) No, f is not a function.
This is because $g(x) = x^2$ is not an injective function; i.e. since $(-2)^2 = 4$ and $2^2 = 4$, we have $\sqrt{4} = \pm 2$, which is not a single value.

(b) No, f is not a function. In particular, for any $a \in \mathbb{Z}$, $f((a, 0))$ is not defined.

(c) Yes, f is a function.

(i) Domain: $\{i, j, k\}$. Codomain: $\{a, b, c\}$. Range: $\{a, b, c\}$

(ii) Yes injective: for all $x \in \{i, j, k\}$, $f^{-1}(x)$ has exactly one element, so f is a bijection. Thus f is injective.

(iii) Yes surjective: see (ii).

(iv) Yes invertible: f is a bijection.

(2) (a) (i) Injective but not surjective: Domain $\mathbb{R}_{>0}$, Codomain $\mathbb{R}$.

(ii) Surjective but not injective: Domain $\mathbb{R}$, Codomain $\mathbb{R}_{\geq 0}$.

(iii) Invertible function: Domain
$mathbbR_{>0}$, Codomain $\mathbb{R}_{>0}$.

(iv) Not a function: Domain $\mathbb{R}_{>0}$, Codomain $\{0\}$.

(b) (i) Injective but not surjective: Domain $\{1\}$, Codomain $\mathbb{R}$.

(ii) Surjective but not injective: Domain $\{1, 2\}$, Codomain $\{6\}$.

(iii) Invertible function: Domain $\{1\}$, Codomain $\{6\}$.

(iv) Not a function: Domain $\{1, 2\}$, Codomain $\{0\}$.

(c) (i) Injective but not surjective: Domain $\mathbb{Z}_{>0} \times \{1\}$, Codomain $\mathbb{R}$.

(ii) Surjective but not injective: Domain $\mathbb{Z}_{>0} \times \{0\}$, Codomain $\{0\}$.

(iii) Invertible function: Domain $\{0\} \times \{0\}$, Codomain $\{0\}$.

(iv) Not a function: Domain $\{1\} \times \{1\}$, Codomain $\{\pm 1\}$.

(3) (a) Note that since we know that for $i \in \{1, 2, 3\}, f(7 - i) = 5 - f(i)$, we know that the values $f(i)$ of the function, for all $i \in \{1, 2, 3\}$, determine the values of the function on the entire domain $\{1, 2, 3, 4, 5, 6\}$. (Namely, $f(1)$ determines $f(6)$, and $f(2)$ determines $f(5)$, and $f(3)$ determines $f(4)$.) Moreover, for each $i \in \{1, 2, 3\}$, we are free to let $f(i)$ be any value in $i \in \{1, 2, 3, 4\}$, Thus, the number of such functions is the number of distinct functions $g : \{1, 2, 3\} \to \{1, 2, 3, 4\}$. There are thus $4^3 = 64$ distinct such functions.

(b) Each function $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ can be described by a sequence $(f(1), f(2), \cdots, f(7))$ of numbers, each in $\{1, 2, 3, 4, 5, 6, 7\}$ such that the i'th number $f(i)$ is not i. Thus, there are $7-1 = 6$ possible choices for the i'th number, f(i), for all $i \in \{1, 2, 3, 4, 5, 6, 7\}$. By the product rule, there are thus 6^7 such functions.

(4) (a) This statement is not correct; let $A = \{a, b\} = C$ and $B = \{a, b, c\}$ let $g(a) = a$ and $g(b) = b$, and $f(a) = a; f(b) = b$ and $f(c) = a$. Now $f \circ g$ is injective since $(f \circ g)(a) \neq (f \circ g)(b)$; similarly g is injective; however, f is not injective because $f(a) = f(c)$. □

(b) This statement is true. In fact, we prove the slightly stronger: if $f \circ g$ is injective then g is injective. By way of contradiction assume $f \circ g$ is injective and g is not. So, for some $a, a' \in A, a \neq a'$ and $g(a) = g(a')$ so, $f(g(a)) = f(g(a'))$, so $(f \circ g)(a) = (f \circ g)(a')$ which contradicts that $f \circ g$ is injective. □