CISC 102 (Fall 20)
Homework #3: Proofs (20 Points)

Student Name/ID:. . . . .

Solutions are due before 11:59 PM on **Friday Midnight October 10, 2020** .

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?
   If George does not have eight legs, then he is not a spider.
   George is a spider.

   $\therefore$ George has eight legs.
   Solution:
   This is modus tollens. The

   first statement is $p \rightarrow q$, where p is "George does not have eight legs" and q is "George is not a spider." The second statement is $\neg q$. The third is $\neg p$. Modus tollens is valid. We can therefore conclude that the conclusion of the argument (third statement) is true, given that the hypotheses (the first two statements) are true.

2. What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."
   Solution: Universal instantiation is used to conclude that "If Socrates is a man, then Socrates is mortal." Modus ponens is then used to conclude that Socrates is mortal.

3. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."
   Solution:

   Let r be the proposition "It rains," let f be the proposition "It is foggy," let s be the proposition "The sailing race will be held," let l be the proposition "The life saving demonstration will go on," and let t be the proposition "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \rightarrow (s \wedge l)$, $s \rightarrow t$, and $\neg t$. We want to conclude r. We set up the proof in two columns, with reasons, as in Example 6. Note that it is valid to replace subexpressions by other expressions logically equivalent to them.

Step	Reason
1. $\neg t$	Hypothesis
2. $s \to t$	Hypothesis
3. $\neg s$	Modus tollens using (1) and (2)
4. $(\neg r \vee \neg f) \to (s \wedge l)$	Hypothesis
5. $(\neg(s \wedge l)) \to \neg(\neg r \vee \neg f)$	Contrapositive of (4)
6. $(\neg s \vee \neg l) \to (r \wedge f)$	De Morgan's law and double negative
7. $\neg s \vee \neg l$	Addition, using (3)
8. $r \wedge f$	Modus ponens using (6) and (7)
9. r	Simplification using (8)

4. Prove, or find a counterexample: the sum of the squares of two consecutive positive integers is odd.

   Solution:

   Let n and $n+1$ be two consecutive positive integers
   Suppose n is even. Then $n+1$ is odd.
   Since n is even, n^2 is even. Thus we can write $n^2 = 2k$ for some integer k
   Since $n+1$ is odd, $(n+1)^2$ is odd. Thus we can write $(n+1)^2 = 2m+1$ for some m
   Thus $n^2 + (n+1)^2 = 2k + 2m + 1$
   Thus $n^2 + (n+1)^2 = 2(k+m) + 1$
   Thus $n^2 + (n+1)^2$ is odd
   Suppose n is odd. Then $n+1$ is even.
   Since n is odd, n^2 is odd. Thus we can write $n^2 = 2k+1$ for some integer k.
   Since $n+1$ is even, $(n+1)^2$ is even. Thus we can write $(n+1)^2 = 2m$ for some m.
   Thus $n^2 + (n+1)^2 = 2k + 1 + 2m$
   Thus $n^2 + (n+1)^2 = 2(k+m) + 1$
   Thus $n^2 + (n+1)^2$ is odd

5. Prove that if n is any integer then $n^3 + 2n^2 + n + 4$ is even
   Hint: do two cases: one for when n is even, and one for when n is odd.

   Solution:

   Suppose n is even, so $n = 2k$ for some integer k
   Then n^2 is even, so $n^2 = 2m$ for some integer m
   So $n^3 = 2k * 2m = 2 * (2 * k * m)$ so n^3 is even, so $n^3 = 2p$ for some integer p
   $4 = 2 * 2$
   So $n^3 + 2^2 + n + 4 = 2p + 2 + 2 * 2m + 2k + 2 * 2$
   $= 2 * (p + 2m + k + 2)$
   which is even

   Suppose n is odd, so $n = 2k+1$ for some integer k
   Then n^2 is odd, so $n^2 = 2m+1$ for some integer m
   So $n^3 = (2k+1) * (2m+1) = 2 * (2km + k + m) + 1$ so n^3 is odd, so $n^3 = 2p+1$ for some integer p
   $4 = 2 * 2$

$$
\begin{aligned}
n^3 + 2n^2 + n + 4 = 2p + 1 + 2 * (2m + 1) + 2k + 1 &= 2 * 2 \\
\text{So } &= 2 * (p + 2m + 1 + k + 2) + 1 + 1 \\
&= 2 * (p + 2m + 1 + k + 2) + 2 \\
&= 2 * (p + 2m + 1 + k + 2 + 1)
\end{aligned}
$$

which is even.

6. Prove by contradiction the following. For all rational number x and irrational number y, the sum of x and y is irrational.

   Solution:

   Assume that this statement is not true. From this assumption we have to get a contradiction. Since this statement is not true we must have a rational number x and an irrational number y such that $x+y$ is rational, say $x+y = \frac{p}{q}$ where p and q are integers. Since x is rational we have $x = \frac{a}{b}$ for some integers a and b. Then $y = \frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{bq}$ a contradiction, since y is irrational, it cannot be the fraction of two integers.

7. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?

   Solution:

   We need to prove the proposition "If 1 is a positive integer, then $1^2 \geq 1$." The conclusion is the true statement $1 \geq 1$. Therefore the conditional statement is true. This is an example of a trivial proof, since we merely showed that the conclusion was true.

8. Prove, by contradiction, that at least three of any 25 days chosen must fall in the same month of the year.

   Solution:

   By contradiction, if there were at most two days falling in the same month, then we could have at most $2 \times 12 = 24$ days, since there are 12 months. Since we have chosen 25 days, at least three of them must fall in the same month.

9. Use prroof by contraposition to show that these statements about the integer x are equivalent: (i) $3x + 2$ is even, (ii) $x + 5$ is odd, (iii) x^2 is even.

   Solution:

   Proof by contraposition: assume that x is not even; thus, x is odd and we can write $x = 2k + 1$ for some integer k. Then $3x + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$, which is odd (i.e., not even), because it has been written in the form $2t + 1$, where $t = 3k + 2$. Similarly, $x + 5 = 2k + 1 + 5 = 2(k + 3)$, so $x + 5$ is even (i.e., not odd). For $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ which is also odd.

10. Showthat the propositions $p_1, p_2, p_3,$ and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4, p_2 \leftrightarrow p_3,$ and $p_1 \leftrightarrow p_3$.

    Solution:

    The only conditional statements not shown directly are $p1 \leftrightarrow p_2, p_2 \leftrightarrow p_4$ and $p_3 \leftrightarrow p_4$. But these each follow with one or more intermediate steps: $p_1 \leftrightarrow p_2$, since $p_1 \leftrightarrow p_3$ and

$p_3 \leftrightarrow p_2$; $p_2 \leftrightarrow p_4$, since $p_2 \leftrightarrow p_1$ (just established) and $p_1 \leftrightarrow p_4$; and $p_3 \leftrightarrow p_2$, since $p_3 \leftrightarrow p_1$ and $p_1 \leftrightarrow p_4$.