CISC 102 (Fall 20)
Homework #1 (Solutions): Sets (20 Points)

Student Name/ID:. . . . .

Solutions are due before 11:59 PM on **Thursday 2 September 2020**.

1. (3 pts) List the elements in the following sets:

   (a) $\{n \in \mathbb{Z} | n^2 < 7\}$
   $\{n \in \mathbb{Z} | n^2 < 7\} = \{-2, -1, 0, 1, 2\}$

   (b) $\{x^2 | x \in \mathbb{N}_0 \wedge x < 5\}$
   $\{x^2 | x \in \mathbb{N}_0 \wedge x < 5\} = \{0, 1, 4, 9, 16\}$

   (c) $\{m \in \mathbb{Q} | m^2 = 7\}$.
   $\{m \in \mathbb{Q} | m^2 = 7\} = \varnothing$

2. (3 pts) Let $A = \{1, 2, 3\}$ and $B = \{1, 3\}$

   (a) List the elements of $A \times B \times A$

$$\begin{aligned} A \times B \times A = \{&(1,1,1),(1,3,1),(1,1,2),(1,3,2),(1,1,3),(1,3,3),\\ &(2,1,1),(2,3,1),(2,1,2),(2,3,2),(2,1,3),(2,3,3),\\ &(3,1,1),(3,3,1),(3,1,2),(3,3,2),(3,1,3),(3,3,3)\} \end{aligned}$$

   (b) List the elements of $(A \times B) \cap (B \times A)$

$$A \times B = \{(1,1),(1,3),(2,1),(2,3),(3,1),(3,3)\}$$
$$B \times A = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3)$$

   The intersection is
$$\{(1,1),(1,3),(3,1),(3,3)\}$$

   (c) List the elements of $(A \times A) \setminus (A \times B)$

$$A \times A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$
$$A \times B = \{(1,1),(1,3),(2,1),(2,3),(3,1),(3,3)\}$$
$$(A \times A) \setminus (A \times B) = \{(1,2),(2,2),(3,2)\}$$

3. (3 pts) For the following sets decide whether they are finite or infinite. If the set is finite, write down its size.

   (a) $\{x \in \mathbb{N}_0 | x > 10\}$
   infinite

   (b) $\{x \in \mathbb{N}_0 | x \leq 10\}$.
   finite, size 11 (since $\mathbb{N}_0$ includes zero)

   (c) $\{4, \{4\}, \{4, \{4\}\}, \{\mathbb{N}\}\}$
   infinite, size 4

4. (2 pts) Show that if A, B, C are sets with $A \subseteq B$ and $B \subset C$ then $A \subset C$.
   We first show that $A \subset C$. To see this, take $x \in A$. Since $A \subseteq B$ this implies$x \in B$. But $B \subset C$, so we must have $x \in C$. To see that in $A \subset C$ we need to find $x \in C \setminus A$. Since $B \subset C$ there exists $x \in C$ with $x \notin B$. The contrapositive of $A \subseteq B$ now implies that $x \notin A$. Thus $x \in C \setminus A$ as required.

5. (4 pts) For each of the following statements about sets, decide whether they are true or false. Justify your assertion, either way.

   (a) $\mathbb{Z}$ is finite
   False: suppose that $\mathbb{Z}$ were finite. Then there exists a largest integer x. But if x is an integer, so is $x + 1$ and $x + 1 > x$ – contradiction.

   (b) $\{\{\mathbb{Z}\}\}$ is finite
   True: $\{\{\mathbb{Z}\}\}$ only contains one element, namely $\{\mathbb{Z}\}$.

   (c) $\{x \in \mathbb{Q} | x^2 = 2\}$ is finite.
   True: as we saw in class that set is empty.

   (d) If A is a finite set then $|\mathcal{P}(A)| > |A|$
   True: if $A = \varnothing$ then $|\mathcal{P}(A)| = |\{\varnothing\}| = 1 > 0 = |\varnothing|$. Otherwise,$\mathcal{P}(A)$ contains $\varnothing$ and $\{x\}$ any $x \in A$, which already gives $|A| + 1 > |A|$ elements.

6. (2 pts) Let $A = \{1, 2, 3, 4\}$. Write down $\mathcal{P}(A)$. What is $|\mathcal{P}(A)|$?

$$\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \\ \{1,2,3,4\}\}$$

   $|\mathcal{P}(A)| = 16 = 2^4$

7. (3 pts) In a class of 65 students, 25 speak Spanish, 32 are excellent cooks, and 50 love dogs. Each student is in at least one of these categories.
   There are 18 Spanish speakers who don't cook. There are 21 dog lovers who are excellent cooks. There are 4 cooks who speak Spanish and do not love dogs. Determine the number of students in the class in each of the following categories:

   (a) Speak Spanish and love dogs

   (b) Love dogs and cannot cook

(c) Speak Spanish, are excellent cooks, and love dogs

Define the following sets: S = people who speak Spanish, do not cook, and do not love dogs
SC = people who speak Spanish, cook, and do not love dogs
SD = people who speak Spanish, love dogs, and do not cook
SCD = people who speak Spanish, are excellent cooks, and love dogs
D = people who love dogs, do not speak Spanish, and are not cooks
DC = people who love dogs, are excellent cooks, and do not speak Spanish
C = people who are excellent cooks, do not like dogs, and do not speak Spanish
These sets correspond to the regions of the standard 3-set Venn Diagram for this problem
Then we start with these equations:

$$
\begin{aligned}
S + SC + SD + SCD + D + DC + C &= 65 && (1)\\
S + SC + SD + SCD &= 25 && (2)\\
C + SC + DC + SCD &= 32 && (3)\\
D + SD + DC + SCD &= 50 && (4)\\
\text{and we also know} & && (5)\\
S + SD &= 18 && (6)\\
DC + SCD &= 21 && (7)\\
SC &= 4 && (8)
\end{aligned}
$$

Equation (6) lets us replace S by 18 - SD in equations (1) and (2)

Equation (7) lets us replace SCD by 21 - DC in equations (1), (2), (3) and (4)

Equation (8) lets us replace SC by 4 in equations (1), (2) and (3)

This gives

$$
\begin{aligned}
(18 - SD) + 4 + SD + (21 - DC) + D + DC + C &= 65 && (9)\\
(18 - SD) + 4 + SD + (21 - DC) &= 25 && (10)\\
C + 4 + DC + (21 - DC) &= 32 && (11)\\
D + SD + DC + (21 - DC) &= 50 && (12)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
D + C &= 22 && (13)\\
DC &= 18 && (14)\\
C &= 7 && (15)\\
D + SD &= 29 && (16)
\end{aligned}
$$

Which rapidly yields

$$\begin{aligned} D &= 15 \\ SD &= 14 \\ S &= 4 \\ SDC &= 3 \end{aligned}$$

Now we can answer the specific questions:

(a) Speak Spanish and love dogs = |SD| + |SDC| = 14 + 3 = 17

(b) Love dogs and cannot cook = |SD| + |D| = 14 + 15 = 29

(c) Speak Spanish, are excellent cooks, and love dogs = |SDC| = 3

Another Solution Using Venn Diagram: