CISC 102 (Fall 20)
Homework #5: Sequences, Recursion & Induction (25 Points)

Student Name/ID:. . . . .

Solutions are due before 11:59 PM on **November 1, 2020** .

1. (2pts)
   Prove by induction that
   $$\sum_{j=1}^{n} 2^j = 2^{n+1} - 2 \ , \forall\, n \geq 1$$

   Basis Step:
   $P(1)$ holds since $\sum_{j=1}^{1} 2^1 = 2^1 = 2^{1+1} - 2 \rightarrow 2 = 2$

   Inductive Step:
   The inductive hypothesis is that $P(K)$ is true as an integer equal to or greater than 1, so our statement is $2^1 + 2^2 + 2^3 + ... + 2^k = 2^{k+1} - 2$. If $P(k)$ is true, then $P(k+1)$ must also be true. To show its true we add 2^{k+1} to both sides.
   $2^1 + 2^2 + 2^3 + ... + 2^k + 2^{k+1} = 2^{k+1} - 2 + 2^{k+1}$
   $2^{k+1} - 2 + 2^{k+1} = 2(2)^{k+1} - 2$
   $= 2^{k+2} - 2$
   This also shows that if hypothesis $P(k)$, then $P(k+1)$ must also be true.

   This shows since the basis and inductive have been completed, so $P(n)$ is true for all nonnegative integers greater to or equal to 1. Shows the formula for the sum of the geometric sequence is indeed correct.

2. (2pts)
   Prove by induction:
   $$\forall n \geq 1, (n^3 - n) \text{ is divisible by } 3$$

   Basis Step:
   P(1) is true since $P(1) = (1)^3 - (1) = 0$, and 0 is divisible by 3.

   Inductive Step:
   We assume that $p(k)$ holds for a positive integer k so $k^3 - k$ is divisible by 3 for an arbitrary value of k, and it follows that $P(k+1)$ is divisible by 3. $(k+1)^3 - (k+1)$. If we expand we get $(k^3 + 3k^2 + 3k + 1) - (k+1)$, (we get rid of the ones but do not

simplify for k). Then we rearrange as $(k^3 - k) + (3k^2 + 3k) = (k^3 - k) + 3(k^2 + k)$. We can now conclude that using our hypothesis that $k^3 - k$ is divisible by 3, and the second term is divisible by 3 since it is 3 times an integer.

Since we have completed both the basis and inductive steps, we proved that $n^3 - n$ is divisible by 3 for any positive integer.

3. (2pts)
Find a closed form for

$$a_1 = 2, a_n = a_{n-1} + n + 6$$

$a_1 = 2$
$a_2 = a_1 + 2 + 6 = 2 + 2 + 6 = 10$
$a_3 = a_2 + 3 + 6 = 10 + 3 + 6 = 19$
$a_4 = a_3 + 4 + 6 = 19 + 4 + 6 = 29$
$a_5 = a_4 + 5 + 6 = 29 + 5 + 6 = 40$
$a_6 = a_5 + 6 + 6 = 40 + 6 + 6 = 52$
Closed formula: $a_n = \frac{1}{2}(n^2 + 13n - 10)$

4. (2pts)
Find a closed form for the recurrence relation

$$a_n = a_{n-1} + 2n, \text{ with } a_1 = 2$$

Prove that your closed form is correct.
$a_1 = 2$
$a_2 = 2 + 2(2) = 6$
$a_3 = 6 + 2(3) = 12$
$a_4 = 12 + 2(4) = 20$
$a_5 = 20 + 2(5) = 30$
$a_6 = 30 + 2(6) = 42$
Closed formula: $a_n = n^2 + n$
Proof:
Case 1: $a_1 = 1^2 + 1 = 2$, Case 2: $a_3 = 3^2 + 3 = 12, Case3 : 6^2 + 6 = 42$
$\therefore$This closed formula works for all of the terms in this sequence.

5. (4pts)

   (a) Find a recurrence relation that defines the sequence 1, 1, 1, 1, 2, 3, 5, 9, 15, 26, ... (Hint: each number in the sequence is based on the four numbers just before it in the sequence.)
   $a_n = -(a_{n-4}) + a_{n-3} + a_{n-2} + a_{n-1}$ Where $a_1 = a_2 = a_3 = a_4 = 1$

   (b) Now find a different sequence that satisfies the recurrence relation you found in (a)
   1, 2, 3, 4, 8, 13, 22, 39, 66, 114

6. (2pts)
   Consider the following recurrence relation:

$$a_n = a_{n-1} + 2n, \text{ with } a_1 = 3$$

   Prove by induction that

$$a_n = n^2 + n + 1 \ \forall \ n \geq 1$$

   Basis Step:
   a_1 is true since $a_1 = (1)^2 + (1) + 1 = 3$

   Inductive Step:
   We assume that the value of a_k, where k is a positive integer equal to or greater to 1, $a_k = k^2 + k + 1$. If a_k is true, which is a solution to $a_k = a_{k-1} + 2k$ with $a_1 = 3$. If this is true then $k + 1$ must also be true.

   $a_{k+1} = a_k + 2(k+1) = a_k + 2k + 2$ (First equation)
   $a_{k+1} = (k+1)^2 + (k+1) + 1 = k^2 + 2k + 1 + k + 1 + 1 = k^2 + 3k + 3$ (What we want)
   (Sub in a_k into the first equation)
   $= (k^2 + k + 1) + 2k + 2 = k^2 + 3k + 3$
   $\therefore$This proves that a_{k+1} is true.

   Since we have completed both basis and inductive steps we have proven that $a_n = n^2 + n + 1$ is the closed form for the recurrence relation $a_n = a_{n-1} + 2n$ for all values of n greater or equal to 1.

7. (2pts)
   Consider the following recurrence relation:

$$a_n = 2 \cdot a_{n-1} - 3 \text{ with } a_1 = 5$$

   Prove by induction that

$$a_n = 2^n + 3 \ \forall \ n \geq 1$$

   Basis Step:
   a_1 is true since $a_1 = 2^1 + 3 = 5$.

   Inductive Step:
   We assume the value of a_k where k, where $k \geq 1$, $a_k = 2^k + 3$ (first equation), which is a solution to $a_k = 2a_{k-1} - 3$ (Second equation). If these statements of k are true, then $k + 1$ must also be true.

   $a_{k+1} = 2a_k - 3$ (Third equation)
   $a_{k+1} = 2^{k+1} + 3$ (Fourth equation: What we want)
   Sub in first equation into third equation
   $= 2(2^k + 3) - 3$

$= 2^{k+1} + 3$
$\therefore$This proves a_{k+1} is true

Since we have proven both the basis and inductive steps we have proven that $a_n = 2^n + 3$ is the solution to $a_n = 2 \cdot a_{n-1} - 3$ for $n \geq 1$.

8. (4pts)

   (a) Find a closed-form solution for this recurrence relation:

   $$a_n = 2 \cdot a_{n-1} - n + 1 \text{ with } a_1 = 2$$

   $a_1 = 2$
   $a_2 = 2(2) - 2 + 1 = 3$
   $a_3 = 2(3) - 3 + 1 = 4$
   $a_4 = 2(4) - 4 + 1 = 5$
   Closed formula: $a_n = 2n - n + 1 = n + 1$

   (b) Prove that your closed-form solution is correct Case 1: $a_1 = 1 + 1 = 2$, Case 2: $a_2 = 2 + 1 = 3$, Case 3: $a_3 = 3 + 1 = 4$, Etc...

9. (5 pts)
   Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.

   (a) Show that the statements $P(18), P(19), P(20)$, and $P(21)$ are true, completing the basis step of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$.

   $P(18)$ can be formed by two 7-cent stamps and one 4-cent stamp $(2(7) + 4 = 18)$. $P(19)$ can be formed by one 7-cent stamp and three 4-cent stamps $(7+3(4) = 19)$. $P(20)$ can be formed by two five 4-cent stamps $(4(5) = 20)$. $P(21)$ can be formed by three 7-cent stamps $(3(7) = 21)$.

   (b) What is the inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$?

   The inductive hypothesis states that $P(j)$ is true for $18 \leq j \leq k$, where k is an integer with $k \geq 21$.

   (c) What do you need to prove in the inductive step of a proof that P(n) is true for all integers $n \geq 18$?

   You need to prove that under the assumption above that $P(k + 1)$ is true, to get $k + 1$ cents.

(d) Complete the inductive step for $k \geq 21$.

Using the inductive hypothesis we can assume that $P(k-3)$ is true since $k-3 \geq 18$, where we can form $k-3$ stamps using 7-cent and 4-cent stamps. To form $k+1$ stamps, we need only add another 4-cent to the stamps used in $k-3$ cents. From this we have shown the inductive hypothesis is true, so $P(k+1)$ is also true.

(e) Explain why these steps show that $P(n)$ is true for all integers $n \geq 18$.

Since we have completed the basis step and the inductive step of this strong proof, we know by strong induction that $P(n)$ is true for all integers n with $n \geq 12$. We know that every postage stamp of n cents, where n is at least 18, can be formed using 7-cent and 4-cent stamps.