Name: ______________________________

**Question 1.** ($20 = 7 + 7 + 6$ points). Consider a standard deck of $52$ cards.

(a) How many different five-card hands are possible?

(b) Of all the possible hands, how many are a full house? (Two cards of one face value, and three of another).

(c) How many of the hands have only one suit?

**Solution.** (a) The number of hands is the number of subsets of five cards, which is

$$\begin{aligned} \binom{52}{5} &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 13 \cdot 51 \cdot 5 \cdot 49 \cdot 16 \\ &= 2,598,960. \end{aligned}$$

(b) The number of ways to have a full house with two kings and three queens is $\binom{4}{2} \cdot \binom{4}{3} = 24$. We can choose one of thirteen face values for the pair and one of the remaining twelve for the triplet. The total number of ways to have a full house is therefore

$$24 \cdot 13 \cdot 12 = 3,744.$$

(c) The number of hands with five hearts is $\binom{13}{5} = 1,287$. There are four different suits, hence the number of hands with five cards all the same suit is $5,148$.

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**Question 2.** (20 points). How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \quad = \quad 29,$$

where each $x_i$ is an integer larger than $-1$?

**Solution.** The number of solutions is the same as the number of ways to write

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 \quad = \quad 29,$$

where each $y_i$ is a non-negative integer. Using five dividers for the twenty-nine copies of 1, we get

$$\begin{aligned} \binom{34}{5} &= \frac{34!}{29!5!} \\ &= \frac{34 \cdot 33 \cdot 32 \cdot 31 \cdot 30}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\ &= 34 \cdot 33 \cdot 8 \cdot 31 \\ &= 278,256 \end{aligned}$$

different sequences, each corresponding to a different way of writing 29.

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**Question 3.** ($20 = 10 + 10$ points). Prove or Disprove that the following are equivalence relations.

(a) Let $x, y \in \mathbb{R}$. We say $x \sim y$ if $x \leq y$.

(b) Let $x, y \in \mathbb{R}$. We say $x \sim y$ if $\lfloor x \rfloor = \lfloor y \rfloor$.

**Solution.** (a) The less than relation is not an equivalence relation because it is not symmetric: $1 \leq 2$ but not $2 \leq 1$.

(b) The floor function is an equivalence relation because it is reflexive, symmetric, and transitive. All three properties are easy to check, since $\lfloor x \rfloor$ is an integer and the set of integers under equality is symmetric, transitive, and reflexive.

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**Question 4.** (20 points). Show that if $p$ is a positive prime number then $\binom{p}{r}$ is an integer multiple of $p$ for every integer $0 < r < p$.

**Solution.** Recall that

$$\binom{p}{r} = p \cdot \frac{(p-1) \cdot \ldots \cdot (p-r+1)}{1 \cdot \ldots \cdot r}.$$

We know that $\binom{p}{r}$ is an integer, but none of the integers $1$ to $r$ in the denominator divides $p$. So they divide the product of the other integers in the numerator, that is,

$$\frac{(p-1) \cdot (p-2) \cdot \ldots \cdot (p-r+1)}{1 \cdot 2 \cdot \ldots \cdot r}$$

is an integer. It follows that $\binom{p}{r}$ is an integer multiple of $p$.

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**Question 5.** (20 points). Find the greatest common divisor of $55$ and $89$ as well as the multiplicative inverse of $55$ modulo $89$.

**Solution.** We run the extended gcd algorithm for $j = 144$ and $k = 233$.

| $j$ | $k$ | $q$ | $r$ | $g$ | $x$ | $y$ |
|---|---|---|---|---|---|---|
| 55 | 89 | 1 | 34 | 1 | 34 | -21 |
| 34 | 55 | 1 | 21 | 1 | -21 | 13 |
| 21 | 34 | 1 | 13 | 1 | 13 | - 8 |
| 13 | 21 | 1 | 8 | 1 | - 8 | 5 |
| 8 | 13 | 1 | 5 | 1 | 5 | - 3 |
| 5 | 8 | 1 | 3 | 1 | - 3 | 2 |
| 3 | 5 | 1 | 2 | 1 | 2 | - 1 |
| 2 | 3 | 1 | 1 | 1 | - 1 | 1 |
| 1 | 2 | 2 | 0 | 1 | 1 | 0 |

We see that $\gcd(55, 89) = 1$. The multiplicative inverse of $55$ modulo $89$ is $x \bmod 89 = 34$.