Question 1: Of 100 students in a university department, 45 are enrolled in English, 30 in History, 20 in Geography, 10 in at least two of three courses and just 1 student is enrolled in all three courses.

a) How many students take at least one of these courses?
b) How many students take none of these courses?
c) How many students take exactly one course?

Solution.

Let E be the set of students enrolled in English, H be the set of students enrolled in History, and G be the set of students enrolled in Geography. We are given $|E| = 45$, $|H| = 30$, $|G| = 20$, $|(E \cap H) \cup (E \cap G) \cup (H \cap G)| = 10$ and $|E \cap H \cap G| = 1$. (1 point)

(a) By the Principle of Inclusion-Exclusion,

$10 = |(E \cap H) \cup (E \cap G) \cup (H \cap G)| = |(E \cap H)| + |(E \cap G)| + |(H \cap G)| - |(E \cap H) \cap (E \cap G)| - |(E \cap H) \cap (H \cap G)| - |(E \cap G) \cap (H \cap G)| + |(E \cap H) \cap (E \cap G) \cap (H \cap G)|$.

Since each of the last four terms here is $|E \cap H \cap G|$, we obtain

$$10 = |E \cap H| + |E \cap G| + |H \cap G| - 2|E \cap H \cap G|$$

and, therefore,

$$|E \cap H| + |E \cap G| + |H \cap G| = 10 + 2 \cdot 1 = 12.$$

$$\begin{aligned} |(E \cup H \cup G)| &= |(E| + |H| + |G| - |E \cap H| - |E \cap G| - |H \cap G| + |E \cap H \cap G| \\ &= 45 + 30 + 20 - 12 + 1 = 95 - 12 + 1 = 84. \end{aligned}$$

Thus, **84** students are enrolled at least one of these courses. (3 points)

(b) It follows that $100 - 84 = \mathbf{16}$ students are enrolled in none of these courses. (1 point)

(c) There are $84 - 10 = \mathbf{74}$ students enrolled in exactly one course. (1 point)

Question 2: Let the functions $f\ : N\ \ \rightarrow N$ and $g\ : Z\ \ \rightarrow N$ be defined as follows:

$$f(x) = 3x + 2 \text{ and } g(x) = x^2 + 1.$$

Specify the functions

(a) f^{-1}
(b) g^{-1}
(c) $f \circ g$
(d) $g \circ f$

if they exist, and give a valid argument if one/some of them do not exist.

Answer

a) Since $rng\ f \in 3n + 2,\ \ n \in N$ and $rng\ f \neq N$, f is not onto (0.5 point)
   So, f^{-1}does not exist (0.5 point)
b) It can be seen that g is not one-to-one. Counter example: $1 \neq -1$ but $g(1) = g(-1) = 2$ (0.5 point)
   So, g^{-1} does not exist (0.5 point)
c) $f \circ g: Z \rightarrow N$ such that $f \circ g = f\big(g(x)\big) = f(x^2 + 1) = 3(x^2 + 1) + 2 = 3x^2 + 5$ (1 point)
d) $g \circ f$ does not exist because $\mathbb{N} \neq \mathbb{Z}$ (1 point)

Question 3: Let $A\ = \{3,5,7,9\}$, $B\ = \{2,3,5,6,7\}$, and $C\ = \{2,4,6,8\}$ be all subjects of the universe $U\ = \{2,3,4,5,6,7,8,9\}$. Find

(a) the union of A and B;
(b) the intersection of B and C;
(c) $A \oplus B$;
(d) the complement C^c of the set C;
(e) the complement of U.

Solution

(a) $A \cup B = \{3, 5, 7, 9, 2, 6\}$;
(b) $B \cap C = \{2,6\}$;
(c) $A \oplus B = (A \cup B) \setminus (A \cap B) = \{3,5,7,9,2,6\} \setminus \{3,5,7\} = \{9,2,6\}$;
(d) $C^c = \{3,5,7,9\}$;
(e) $U^C = \emptyset$.

Question 4: Check whether $((p \rightarrow q) \rightarrow (r \rightarrow s))$ and $((p \rightarrow r) \rightarrow (q \rightarrow s))$ are logically equivalent.

Answer

Two propositions are logically equivalent if these propositions have same truth or falsity values regardless of the truth or falsity of their propositional variables. To check whether $((p \rightarrow q) \rightarrow (r \rightarrow s))$ and $((p \rightarrow r) \rightarrow (q \rightarrow s))$ are equivalent we compare yellow and blue columns in the following truth table. Since the statement is violated in the highlighted rows, the given propositions are not logically equivalent.

p	q	r	s	$p \rightarrow q$	$r \rightarrow s$	$p \rightarrow r$	$q \rightarrow s$	$((p \rightarrow q) \rightarrow (r \rightarrow s))$	$((p \rightarrow r) \rightarrow (q \rightarrow s))$
0	0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1	1	1
0	0	1	0	1	0	1	1	0	1
0	0	1	1	1	1	1	1	1	1
0	1	0	0	1	1	1	0	1	0
0	1	0	1	1	1	1	1	1	1
0	1	1	0	1	0	1	0	0	0
0	1	1	1	1	1	1	1	1	1
1	0	0	0	0	1	0	1	1	1
1	0	0	1	0	1	0	1	1	1
1	0	1	0	0	0	1	1	1	1
1	0	1	1	0	1	1	1	1	1
1	1	0	0	1	1	0	0	1	1
1	1	0	1	1	1	0	1	1	1
1	1	1	0	1	0	1	0	0	0
1	1	1	1	1	1	1	1	1	1

Question 5: Let $f : R \rightarrow R$ be defined by $f(x) = 2x^2 - 1$.

(a) Find domain, target (or codomain), and range of f.
(b) Is f one-to-one? Justify your answer.
(c) Is f onto? Justify your answer.

Solution:

(a) $dom\, f = R$; $target = R$; $rng\ f = [-1,\infty)$.

(b) f is not one-to-one function, because for example $f(1) = f(-1) = 1$, but $1 \neq -1$.

(c) f is not onto, because $rng\ f \neq target\ f$, where $rng\ f = [-1,\infty)$ and $target = R$.

Question 6: Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(n,m) = \left\lfloor \frac{n}{m} \right\rfloor + 1$, where $\mathbb{N} = \{1,2,3, \dots\}$ is the set of natural numbers.

a) Is f one-to-one? Justify your answer.
b) Is f onto? Justify your answer.

Answer

a) Here is a counter example proving that f is not one-to-one function. By substituting (n,m) by $(4,2)$ and then by $(2,1)$ in $f(n,m) = \left\lfloor \frac{n}{m} \right\rfloor + 1$ we can easily see that $f(4,2) = \left\lfloor \frac{4}{2} \right\rfloor + 1 = 3$ and $f(2,1) = \left\lfloor \frac{2}{1} \right\rfloor + 1 = 3$, meaning that $f(4,2) = f(2,1)$. This counter example proves the fact that f is not one-to-one function. *(3.5 points)*

b) To prove that it is onto we need to find for each natural number n arguments of f where n is taken. Particularly: $f(1,2) = \left\lfloor \frac{1}{2} \right\rfloor + 1 = 0 + 1 = 1$. Further $f(k,1) = \left\lfloor \frac{k}{1} \right\rfloor + 1 = k + 1$ for $k \in \mathbb{N}$ where $k + 1$ takes each natural number $n \geq 2$. These two parts together prove that f is onto, i.e. each natural number is taken by f. *(3.5 points)*

Question 7: Show that $\left(\left(\left((p \vee q) \wedge (p \to r)\right) \wedge (q \to r)\right) \to r\right)$ is tautology (or theorem).

Answer

pqr	$(p \vee q)$	$(p \to r)$	$((p \vee q) \wedge (p \to r))$	$(q \to r)$	$(((p \vee q) \wedge (p \to r)) \wedge (q \to r))$	*
FFF	F	T	F	T	F	T
FFT	F	T	F	T	F	T
FTF	T	T	T	F	F	T
FTT	T	T	T	T	T	T
TFF	T	F	F	T	F	T
TFT	T	T	T	T	T	T
TTF	T	F	F	F	F	T
TTT	T	T	T	T	T	T

According to the last column of the truth table this proposition is true-valued regardless of truth or falsity of its arguments and therefore it is a tautology.

Question 8:

Assume the following predicates: $B(x)$ is "x is a baby", $C(x)$ is "x can manage crocodiles", $D(x)$ is "x is despised" and $L(x)$ is "x is logical".

(a) Assume the domain consists of people. Express each of the following statements using quantifiers, logical connectives and the predicates $B(x)$, $C(x)$, $D(x)$ and $L(x)$.

   i. Babies are illogical
   ii. Nobody is despised who can manage crocodiles
   iii. Illogical people are despised
   iv. Babies cannot manage crocodiles

(b) Prove that iv follows from i, ii and iii.

Answer:

(a) The following are expressed as follows:

   i. Babies are illogical $\forall x(B(x) \rightarrow \neg L(x))$
   ii. Nobody is despised who can manage crocodiles $\neg\exists x(D(x) \wedge C(x))$
   iii. Illogical people are despised $\forall x(\neg L(x) \rightarrow D(x))$
   iv. Babies cannot manage crocodiles $\forall x(B(x) \rightarrow \neg C(x))$

(b) To show iv follows from i, ii and iii notice that ii is equivalent to $\forall x(D(x) \rightarrow \neg C(x))$ using duality of quantifiers, $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$; De Morgans law $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$; and $\neg P \vee Q$ is equivalent to $P \rightarrow Q$. Using transitivity i and iii implies $\forall x(B(x) \rightarrow D(x))$ which with the reformulation of ii implies iv.

□

Question 9:

(a) Assume m and n are both integers. Prove by contraposition, if mn is even then m is even or n is even.

(b) Prove by contradiction that the sum of an irrational number and a rational number is irrational.

(c) Prove that there is not a rational number r such that $r^3 + r + 1 = 0$.

Answer:

(a) Assume m and n are both integers. Prove by contraposition, if mn is even then m is even or n is even.

Solution:

We have to prove

$$mn \text{ even} \rightarrow (m \text{ even} \vee n \text{ even})$$

The contrapositive is

$$\neg(m \text{ even} \vee n \text{ even}) \rightarrow \neg(mn \text{ even})$$

which can be transformed using DeMorgan's law and even $\equiv \neg$ odd

$$(m \text{ odd} \wedge n \text{ odd}) \rightarrow mn \text{ odd}$$

We assume m is odd and by the definition of odd there exists a $k \in \mathbb{Z}$ with $m = 2k + 1$. Similar there exists a $l \in \mathbb{Z}$ with $n = 2l + 1$. Therefore we get

$$\begin{aligned} mn &= (2k + 1) \cdot (2l + 1) \\ &= 4lk + 2k + 2l + 1 \\ &= 2(2lk + k + l) + 1 \\ &= 2l' + 1 \end{aligned}$$

where $l' = 2lk + k + l \in \mathbb{Z}$. By definition mn is therefore odd.

□

(b) Prove by contradiction that the sum of an irrational number and a rational number is irrational.

Solution:

Assume that the sum of an irrational number i and a rational number $\frac{a}{b}$ is rational. Then, let

c and d be integers such that $i + \frac{a}{b} = \frac{c}{d}$. Therefore $i = \frac{c}{d} - \frac{a}{b} = \frac{bc - da}{db}$. Given that a, b, c and d are integers, $bc - da$ and db are also integers, this shows that i is rational and therefore contradicts our initial assumption. Therefore, the sum of a rational and an irrational number must be irrational. □

(c) Prove that there is not a rational number r such that $r^3 + r + 1 = 0$.

Solution:

We prove it by contradiction. Assume that $r = \frac{a}{b}$ is a solution where a, b are in lowest terms, so have no common factors other than 1. So, $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0$; therefore, $a^3 + b^2 a + b^3 = 0$. If a and b are both odd then LHS is a sum of odd numbers; if one is odd and the other even then LHS is odd. That just leaves that both are even which contradicts that $\frac{a}{b}$ is in lowest terms. (We are using here various properties that you may wish to prove such as if n is odd (even) then n^2 and n^3 are odd (even).) □