CISC 102 (Fall 20)
Homework #9: Relations (33 Points)

Student Name/ID:. . . . .

1. (6 pts) Let R be the relation on the natural numbers defined by

$$R = \{(x, y) : x, y \in \mathbb{N}, x^2 + y^2 \leq 8\}.$$

   (a) Write out the elements of R as a set of ordered pairs

   (b) Is R an equivalence relation, and explain why or why not?

   (c) Is R a partial order and explain why or why not?

2. (2 pts)
   Let A be a set with n elements, and let B be a set with m elements ($n \geq 0, m \geq 0$)
   How many relations are there from A to B?

3. (4 pts) Let A be a set and let R and S be relations on A.

   (a) Suppose R is anti-symmetric. Prove that $R \cap S$ is also anti-symmetric.

   (b) Suppose R and S are both transitive. Prove that $R \cap S$ is also transitive.

4. (2 pts) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$
   Let R be the relation on A defined by
   $(a, b) \in R$ if and only if $b = a + 3m$ for some non-negative integer m
   For example, $(5, 11) \in R$ because $11 = 5 + 3 \cdot 2$
   Is R a partial ordering on A? Prove your answer.

5. (2 pts) Let R be a relation defined on the set $\{0, 1, 2, 3, \dots\}$ as follows:
   $(a, b) \in R$ if and only if $(a + b)$ is a multiple of 2
   (a) is R reflexive?
   (b) is R symmetric?

6. (6 pts) Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.$$

   (a) Draw the directed graph of R

   (b) Find the equivalence class of every element of A.

   (c) Find the distinct equivalence classes of the relation (Usually several of the classes will contain exactly the same elements, so you must take a careful look at the classes to determine which are the same. You then indicate the distinct equivalence classes by describing them without duplication.)

7. (5 pts)
Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? For those that are not, what properties do they lack?

   (i) $\{0 \sim 0, 1 \sim 1, 2 \sim 2, 3 \sim 3\}$
   (ii) $\{0 \sim 0, 0 \sim 2, 2 \sim 0, 2 \sim 2, 2 \sim 3, 3 \sim 2, 3 \sim 3\}$
   (iii) $\{0 \sim 0, 1 \sim 1, 1 \sim 2, 2 \sim 1, 2 \sim 2, 3 \sim 3\}$
   (iv) $\{0 \sim 0, 1 \sim 1, 1 \sim 3, 2 \sim 2, 2 \sim 3, 3 \sim 1, 3 \sim 2, 3 \sim 3\}$
   (v) $\{0 \sim 0, 0 \sim 1, 0 \sim 2, 1 \sim 0, 1 \sim 1, 1 \sim 2, 2 \sim 0, 2 \sim 2, 3 \sim 3\}$

8. (4 pts)
For the following relations on A determine whether they are reflexive, symmetric, and/or transitive. State whether they are equivalence relations or not, and if they are describe their equivalence classes.

   (a) Let $A = \mathbb{Z}$ and define $\sim$ by $a \sim b$ whenever $a - b$ is odd.
   (b) Let $A = \mathbb{R}$ and define $\sim$ by $a \sim b$ whenever $ab \neq 0$.

1. SOLUTION:

   (a)
   $$R = \{(1,1),(1,2),(2,1),(2,2)\}$$

   (b) R is an equivalence relation if it is reflexive, symmetric, and transitive.
   Clearly, R is not reflexive at it is not applicable to every point in the domain.

   (c) R is a partial order if it is reflexive, antisymmetric, and transitive.
   Clearly, R is not reflexive at it is not applicable to every point in the domain. It is also not antisymmetric.

2. SOLUTION:

   Every relation from A to B maps each element of A to a subset of B. Since B has 2^m subsets and each of the n elements of A can map to any one of them, the number of different relations is $(2^m)^n$, or $2^{(m \cdot n)}$

   OR

   Every relation from A to B is a subset of $A \times B$. $A \times B$ has $n \cdot m$ elements, so it has $2^{(n \cdot m)}$ subsets. Thus there are $2^{(n \cdot m)}$ relations.

3. SOLUTION:

   (a) Let (a,b) be an ordered pair in $R \cap S$.
   $\Rightarrow (a,b) \in R$
   $\Rightarrow a = b \quad \text{or} \quad (b,a) \notin R$
   $\Rightarrow a = b \quad \text{or} \quad (b,a) \notin R \cap S$
   $\therefore R \cap S$ is anti-symmetric

   (b) Let (a,b) and (b,c) be ordered pairs in $R \cap S$
   $\Rightarrow (a,b) \in R$ and $(b,c) \in R$
   and
   $(a,b) \in S$ and $(b,c) \in S$
   $\Rightarrow (a,c) \in R$
   and
   $(a,c) \in S$
   $\Rightarrow (a,c) \in R \cap S$
   $\therefore R \cap S$ is transitive.

4. SOLUTION:

   To be a partial ordering, R must be
   - reflexive
   - anti-symmetric
   - transitive

   Proof that R is reflexive: every integer $a = a + 3 \cdot 0$ so all pairs of the form (a,a)

are in R

Proof that R is anti-symmetric: Suppose R is not anti-symmetric: let a and b be elements of A such that both (a, b) and (b, a) are in R, and $a \neq b$

$\Rightarrow b = a + 3m_1$ and $a = b + 3m_2$ for some non-negative integers m_1 and m_2

Since $a \neq b, m_1 \neq 0$ and $m_2 \neq 0$.

But then we can write

$b = b + 3(m_2 + m_1)$ where $3(m_2 + m_1) > 0$, which is not possible. CONTRADICTION.

Therefore R is anti-symmetric.

Proof that R is transitive: Suppose (a, b) and (b, c) are both in R.

$$\Rightarrow b = a + 3m_1 \text{ and } c + b + 3m_2 \qquad \text{for some non-negative integers } m_1 \text{ and } m_2$$

$$\Rightarrow c = a + 3m_1 + 3m_2 = a + 3(m_1 + m_2)$$

$$\Rightarrow (a, c) \in R$$

Therefore R is transitive.

Since R is reflexive, anti-symmetric and transitive, R is a partial ordering on A.

5. SOLUTION:

(a)

For every integer a,

$a + a = 2a,$ so $(a, a) \in R$

Therefore R is reflexive

(b)

Let (a, b) be an ordered pair in R.

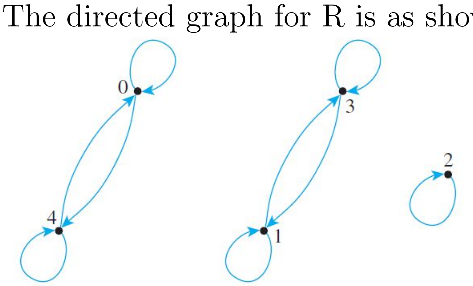
$$\Rightarrow (a + b) = 2m, \text{ for some integer m}$$

$$\Rightarrow (b + a) = 2m$$

$$\Rightarrow (b, a) \in R$$

Therefore R is symmetric

6. (a) The directed graph for R is as shown below



(b)

$$[0] = \{x \in A |\ x\ R\ 0\} = \{0, 4\}$$

$$[1] = \{x \in A |\ x\ R\ 1\} = \{1, 3\}$$

$$[2] = \{x \in A |\ x\ R\ 2\} = \{2\}$$

$$[3] = \{x \in A |\ x\ R\ 3\} = \{1, 3\}$$

$$[4] = \{x \in A |\ x\ R\ 4\} = \{0, 4\}$$

(c) The distinct equivalence classes of the relation are $\{0, 4\}, \{1, 3\}, \{2\}$.

7. Which of these relations on $\{0,1,2,3\}$ are equivalence relations? For those that are not, what properties do they lack?

   (i) This is an equivalence relation.

   (ii) This relation is symmetric, but is not reflexive ($1 \not\sim 1$) and is not transitive ($0 \sim 2$ and $2 \sim 3$, but $0 \not\sim 3$).

   (iii) This is an equivalence relation.

   (iv) This relation is symmetric and reflexive, but is not transitive ($1 \sim 3$ and $3 \sim 2$, but $1 \not\sim 2$).

   (v) This relation is reflexive, but not symmetric ($1 \sim 2$ but $2 \not\sim 1$) and is not transitive ($2 \sim 0$ and $0 \sim 1$, but $2 \not\sim 1$).

8. For the following relations on A determine whether they are reflexive, symmetric, and/or transitive. State whether they are equivalence relations or not, and if they are describe their equivalence classes.

   (a) This is symmetric, since $a-b = 2k+1$ implies $b-a = -(2k+1) = 2(-k-1)+1$. But it is not an equivalence relation since it isn't reflexive or transitive. In particular, for any $a \in \mathbb{Z}$, we have $a - a = 0 = 2 \cdot 0$, which is even. And for any $a, b, c \in \mathbb{Z}$ satisfying $a - b = 2k + 1$ and $b - c = 2\ell + 1$, we have

   $$a - c = (a - b) + (b - c) = 2k + 1 + 2\ell + 2 = 2(k + \ell + 1),$$

   which is even.

   (b) This is symmetric, since $ab \neq 0$ implies $ba = ab \neq 0$. It is also transitive, since $ab \neq 0$ implies $a \neq 0$; and $bc \neq 0$ implies $c \neq 0$; and hence $ac \neq 0$. But this is not an equivalence relation since it's not reflexive: $0 \nsim 0$ since $0 \cdot 0 = 0$.