CISC-102

PRACTICE PROBLESM WITH SOLUTIONS

Problems

(1) Let F_i be the ith Fibonacci number, and let n be any positive integer.

   (a) Write $F_1 + F_3 + F_5 + \cdots + F_{2n-1}$ as a summation
   (b) Show that $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

(2) Let F_i be the ith Fibonacci number, and let n be any positive integer. Write $F_2 + F_4 + F_6 + \cdots + F_{2n}$ and as a summation and show that

$$F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$$

(3) Let F_i be the ith Fibonacci number, and let n be any positive integer. Prove that F_{3n} is even.

(4) Find a closed form for the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$, with $a_0 = -1$ and $a_1 = 1$. Prove that your closed form is correct.

(5) (a) Find a recurrence relation that defines the sequence $1, 1, 1, 3, 5, 9, 17, 31, \ldots$
   (b) Now find a different sequence that satisfies the recurrence relation you found in (a)

SOLUTIONS

(1)

$$\begin{aligned} F_1 + F_3 + F_5 + \cdots + F_{2n-1} &= F_1 + (F_1 + F_2) + (F_3 + F_4) + \cdots + (F_{2n-3} + F_{2n-2}) \\ &= 1 + \sum_{i=1}^{2n-2} F_i \end{aligned}$$

(2)

$$1 + \sum_{i=1}^{2n-2} F_i = F_{2n}$$

(3)

$$\begin{aligned} F_2 + F_4 + F_6 + \cdots + F_{2n} &= F_2 + (F_2 + F_3) + (F_4 + F_5) + \cdots + (F_{2n-2} + F_{2n-1}) \\ &= F_1 + (F_2 + F_3) + (F_4 + F_5) + \cdots + (F_{2n-2} + F_{2n-1}) \\ &\text{because } F_2 = F_1 \\ &= \sum_{i=1}^{2n-1} F_i \\ &= F_{2n+1} - 1 \quad \text{......Try to prove this...} \end{aligned}$$

(4) Proof by Induction:
We will prove a stronger result: F_{3n-2} and F_{3n-1} are both odd, and F_{3n} is even
Base case: Let $n = 1, 3n = 3$.

$$\begin{aligned} F_{3-2} &= F_1 = 1 \\ F_{3-1} &= F_2 = 1 \\ F_3 &= 2 \end{aligned}$$

Thus the claim is true for $n = 1$.
Inductive Hypothesis: Assume the claim is true for $n = k$, where k is some

integer ≥ 1.
Inductive step: Let $n = k + 1$
Consider F_{3n-2}

$$\begin{aligned} F_{3n-2} &= F_{3(k+1)-2} \\ &= F_{3k+1} \\ &= F_{3k-1} + F_{3k} \end{aligned}$$

By our inductive assumption, F_{3k-1} is odd and F_{3k} is even, therefore their sum is odd, thus F_{3n-2} is odd.
Therefore,

$$\begin{aligned} F_{3n-1} &= F_{3n-3} + F_{3n-2} \\ &= F_{3(k+1)-3} + F_{3n-2} \\ &= F_{3k} + F_{3n-2} \end{aligned}$$

Since F_{3k} is even and F_{3n-2} is odd, we see F_{3n-1} is odd. Therefore, F_{3n} is the sum of two odd numbers, so it is even. Thus the claim is true for $n = k + 1$.
QED

(5) First we look at the first few values in the sequence:

$$\begin{aligned} &a_0 = -1 \\ &a_1 = 1 \\ &a_2 = 2a_1 - a_0 = 2 + 1 = 3 \\ &a_3 = 2a_2 - a_1 = 6 - 1 = 5 \\ &a_4 = 2a_3 - a_2 = 10 - 3 = 7 \\ &a_5 = 2a_4 - a_3 = 14 - 5 = 9 \end{aligned}$$

It looks like the closed form is simply $a_n = 2n - 1$.
Proof by induction: We have already established 6 base cases.
Inductive Hypothesis: Assume $a_n = 2n - 1$ for all non-negative $n \leq k$, where k is some positive integer.

Consider a_{k+1}

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} \\ &= 2(2k-1) - (2(k-1)-1)) \\ &= 4k - 2 - (2k-3) \\ &= 2k+1 \\ &= 2(k+1) - 1 \end{aligned}$$

Therefore the claim is true for $n = k+1$ QED

(6) (a) A brief examination reveals that

$$\begin{aligned} 3 &= 1+1+1 \\ 5 &= 1+1+3 \\ 9 &= 1+3+5 \\ &\text{etc.} \end{aligned}$$

Giving $a_n = a_{n-3} + a_{n-2} + a_{n-1}$ with $a_1 = a_2 = a_3 = 1$

(b) We can define a different sequence satisfying the same recurrence relation by changing the base case values. For example a_1, $a_2 = 1$, and $a_3 = 2$, Then the sequence generated by the recurrence relation is

$$1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$$