Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Module 1: Lists and Sets
Fall 2021

This module corresponds to the following sections from your textbook:

8. Lists
10. Sets I: Introduction, Subsets
12. Sets II: Operations
13. Combinatorial Proof

1 Lists

A **list** is an ordered sequence of objects that are related in some way by a common property or rule. The list $(2, 0, 3)$ is not the same as the list $(3, 0, 2)$. Each object in the list is called an **element**.

Theorem 1 (Multiplication Principle)**.** *Consider two-element lists for which there are n choices for the first element and m choices for the second element. Then the number of such lists is nm.*

Example 2. A person's initials consist of the initial letters of their first and last names. For example, the initials of "John Smith" would be "JS". In the English language, how many possible initials are there?

We can think of this problem as the number of ways to construct a two-element list from the 26 letters of the English language. There are 26 choices for the initial letter of the first name (i.e., the first element of the list) and 26 choices for the initial letter of the last name (i.e., the second element of the list). By the Multiplication Principle, there are $26 \times 26 = 676$ possible ways to form initials in the English language.

Example 3. A student club has ten members. They wish to elect one student as a president and another student as a vice-president. In how many ways can these posts be filled?

We can think of this problem as the number of ways to construct a two-element list from ten possible objects. There are ten choices for president (i.e., the first element of the list), which leaves nine choices for vice-president (i.e., the second element of the list). By the Multiplication Principle, there are $10 \times 9 = 90$ possible ways that the positions can be filled.

Example 4. Best Buy has the following laptop models available, classified by manufacturer:

- HP: 5 models
- Lenovo: 4 models
- Acer: 7 models
- Dell: 5 models
- Apple: 3 models

If the customer needs to buy only one laptop, they have 24 choices in total. If the customer needs to purchase one model from each manufacturer (i.e., 5 laptops, since there are 5 manufacturers available), we have a list counting problem where each laptop in the list needs to be from a specific manufacturer; in other words, each of the five elements in the list is chosen from a different pool of elements. So, the customer has $5 \times 4 \times 7 \times 5 \times 3 = 2100$ choices.

While the list in this example consists of five elements (instead of two elements, as in the previous examples), the Multiplication Principle still applies (p. 35 of your textbook provides the relevant proof).

The following formulas for counting lists, both with and without repetition, follow from the Multiplication Principle.

Proposition 5. *For a list of length k composed of elements that are chosen from a pool of n possible objects, the number of all possible lists that can be constructed is*

$$= \begin{cases} n^k & \text{if repetitions are permitted} \\ (n)_k & \text{if repetitions are forbidden.} \end{cases}$$

The notation $(n)_k$ is called **falling factorial**, defined for $0 \leq k \leq n$ as

$$(n)_k = n(n-1)(n-2)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}.$$

If $n = k$, the formula simplifies to $(n)_k = n!$. If $k > n$, $(n)_k$ is defined as 0.

Lists without repetitions are sometimes called permutations, and an alternative notation for $(n)_k$ is $P(n, k)$. Permutations also have another meaning, which we will revisit when we learn about functions.

Example 6. A certain instructor is creating an exam for a discrete mathematics class. Suppose there is a question bank containing 50 different questions. If the exam consists of five questions, the instructor can create a total of $(50)_5 = P(50, 5) = 50 \times 49 \times 48 \times 47 \times 46 = 254\,251\,200$ five-question exams. Note that while some of the possible exams may contain the same questions, each exam will have a unique ordering of questions.

Example 7. The Examinations Office is scheduling three CISC exams and three MATH exams, all of which are different. Three exams can be held per day, and the office wants to schedule all CISC exams on one day and all MATH exams on the other day. Under such constraints, the office has $(3!)(3!)(2!)$ ways to schedule all exams: 3! ways of scheduling three CISC exams, 3! ways of scheduling three MATH exams, and 2! ways to arrange the "CISC exam" day and the "MATH exam" day.

Exercises

1. Consider a two-element list with elements drawn from a pool of n possible objects. What is the total number of possible lists that can be constructed if (i) repetition is allowed and (ii) if repetition is not allowed?
2. How many strings of length 10 can be written using the English alphabet?
3. A 1500m run has 20 competitors. In how many ways can we select the gold, silver, and bronze winners?

2 Sets

A **set** is an unordered sequence of objects that are related in some way by a common property or rule; in other words, a set is an unordered list. Two of the most common sets we will see in this course are the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and the set of integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. We are also interested in the set of binary digits $\mathbb{B} = \{\texttt{0}, \texttt{1}\}$.

2.1 Definitions

A variety of notations exist for illustrating connections between elements and sets, or between sets and other sets. The **inclusion** of an element in a set is denoted by $\in$, and the **exclusion** of an element from a set is denoted by $\notin$.

For two sets A and B, if all elements of A are also elements of B, then A is a **subset** of B and we write $A \subseteq B$. Similarly, if all elements of A are also elements of B and B contains at least one element that A does not contain, then A is a **proper subset** of B and we write $A \subset B$. Not all mathematicians use the symbol $\subset$ to indicate a proper subset. Thus, to avoid potential confusion, the textbook only uses the symbol $\subseteq$ to denote a subset and explicitly states it if A is a proper subset of B.

We say that two sets A and B are **disjoint** if no element of A is an element of B and vice versa.

If we write the subset symbol backwards, we get $A \supseteq B$, which means that A is the **superset** of B. Similarly, $A \supset B$ means that A is the **proper superset** of B.

We can define **equality** of sets by using the subset and superset notations. For two sets A and B, if $A \subseteq B$ and $A \supseteq B$, then $A = B$.

There are two special sets. The **empty set**, denoted by $\emptyset$, contains no elements. We also use $\{\}$ to denote the empty set. Note that $\{\emptyset\}$ is not an empty set.

The **cardinality** (or **size**) of a set A, denoted by $|A|$, is the number of elements in A. If $|A| = m$ where $m \in \mathbb{N}$, then we say that A is a **finite** set. Otherwise, we say that A is an **infinite** set.

The **power set** of a set A, denoted by $\mathcal{P}(A)$, is the set consisting of all subsets of A.

Example 8. Let $A = \{1, 2, 3\}$. The power set of A is

$$\mathcal{P}(A) = 2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},$$

which has 8 elements.

Theorem 9. *Let A be a finite set. The number of subsets of A is $2^{|A|}$.*

Proof. Let A be a finite set, consisting of the elements $a_1, a_2, \dots, a_n$. We can represent each subset of A with a yes-no list, where:

1. The first yes/no in the list indicates whether a_1 is a member of the subset.
2. The second yes/no in the list indicates whether a_2 is a member of the subset.

   ...

n. The n^{th} yes/no in the list indicates whether a_n is a member of the subset.

Each possible subset of A corresponds to exactly one yes-no list, and vice-versa (i.e., each possible yes-no list of up to length $|A|$ corresponds to exactly one possible subset of A).

Therefore, the number of subsets of A is the same as the number of length-n yes-no lists. By the Multiplication Principle, the number of such lists is 2^n, so the number of subsets of A is 2^n, so we have

$$|2^A| = 2^{|A|}.$$

$\square$

2.2 Operations

We apply operations to existing sets to obtain new sets. Assume that our sets are called A and B.

The **union** of two sets, denoted $A \cup B$, consists of all elements that are contained in A or in B (or in both) and is written symbolically as follows:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The **intersection** of two sets, denoted $A \cap B$, consists of all elements that are contained in A and in B and is written symbolically as follows:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

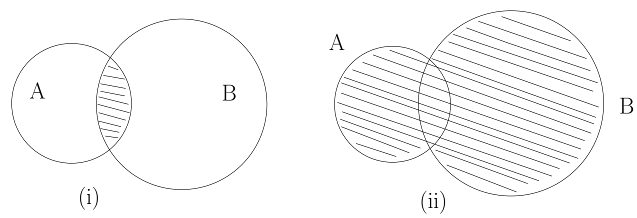

Figure 1: Venn diagrams for (i) $A \cap B$ and (ii) $A \cup B$.

Proposition 10 (Basic Inclusion-Exclusion)**.** *If A and B are finite sets,*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We can further explain Basic Inclusion-Exclusion as follows: If there is a collection of items such that n items fall into Category A, m items fall into Category B, and l items fall into both Category A and Category B, it must be the case that there are $n + m - l$ items in total across Categories A and B. You may convince yourself of this by examining Figure 1.

Corollary 11 (Addition Principle)**.** *It follows from Basic Inclusion-Exclusion that if A and B are disjoint,*

$$|A \cup B| = |A| + |B|.$$

We can further explain the Addition Principle as follows: If there is a collection of items such that n items fall into Category A, and m items fall into Category B, but none of the items fall into both Category A and Category B, it must be the case that there are $n + m$ items in total across Categories A and B.

Example 12. A website requires passwords to be 6 to 8 characters long. Each character is an upper or a lower case letter or a digit, and the password has at least one digit. The total number of allowable passwords are calculated as follows.

Let P be the total number of allowable passwords and let P_i be the total number of passwords of length i. By the Addition Principle, we have

$$P = P_6 + P_7 + P_8.$$

The total number of characters that include upper and lower case letters, but do not include digits, is $26 + 26 = 52$. The total number of characters that include upper and lower case letters and digits is $52 + 10 = 62$. The total number of 6-character passwords is thus

$$P_6 = 62^6 - 52^6,$$

since 62^6 includes all possible 6-character passwords without any restrictions, and subtracting 52^6 removes the disallowed passwords (i.e., those that do not contain at least one digit).

Similarly, $P_7 = 62^7 - 52^7$ and $P_8 = 62^8 - 52^8$.

So, $P = 62^6 - 52^6 + 62^7 - 52^7 + 62^8 - 52^8 = 167,410,949,583,040$ (i.e., about 167 trillion).

Exercise

Count the number of bit strings of length 8 that satisfies one or both of the following conditions:
i. begins with 1
ii. ends with 00

The **difference** of two sets, denoted $A - B$, consists of all elements that are in A but that are not in B and is written symbolically as follows:

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Another common notation for set difference is $A \setminus B$.

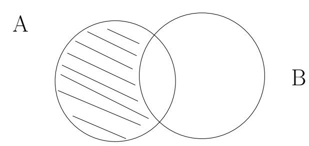

Figure 2: Venn diagram for $A - B$.

The **symmetric difference** of A and B, denoted $A\Delta B$, is the set of all elements in A but not in B or in B but not in A. That is,

$$A\Delta B = (A - B) \cup (B - A).$$

The **complement** of a set A, denoted by $\bar{A}$, contains all elements that are not in A. Other notations include A^{C} (which we use here for readability reasons) or A'.

Example 13. Let U (for "universe") be the set $\mathbb{Z}$ of all integers. Let E be the set of even integers, and let O be the set of odd integers.

Then $E \cup O = U$, $E \cap O = \emptyset$, $E^{\mathsf{C}} = U - E = O$, $O^{\mathsf{C}} = U - O = E$, and $E \setminus O = E$.

Note that if we instead consider U as the set of all real numbers $\mathbb{R}$, the expressions above for E^{C} and O^{C} would no longer be true. Instead, we would write $E^{\mathsf{C}} = U - E = \mathbb{R} - E$ and likewise $O^{\mathsf{C}} = \mathbb{R} - O$. We would also have $E \cup O \neq U$.

Exercise

Draw Venn diagrams for (i) $B - A$, (ii) $A\Delta B$, (iii) $B\Delta A$, and (iv) A^{C}.

You can check your work against the Venn diagrams on p. 60 (after Proposition 12.11) of your textbook.

The **Cartesian product** of two sets A and B is denoted $A \times B$, and is the set of all two-element lists formed by taking each possible element from A together with each possible element from B. Symbolically,

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Exercise

What is the cardinality of $A \times B$?

Some useful properties of sets are listed below.

Proposition 14. *Let A, B, and C be sets. Then each of the following statements are true:*

1. *Idempotent laws:* $A \cup A = A$ *and* $A \cap A = A$.
2. *Identity laws:* $A \cup \emptyset = \emptyset \cup A = A$
3. *Involution law:* $(A^{\mathsf{C}})^{\mathsf{C}} = A$.
4. *Commutative laws:* $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.
5. *Associative laws:* $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$.
6. *Distributive laws:* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
7. *Absorption laws:* $A \cup (A \cap B) = A$ *and* $A \cap (A \cup B) = A$.
8. *De Morgan's laws:* $(A \cup B)^{\mathsf{C}} = A^{\mathsf{C}} \cap B^{\mathsf{C}}$ *and* $(A \cap B)^{\mathsf{C}} = A^{\mathsf{C}} \cup B^{\mathsf{C}}$

Exercise

Study the proof for the associative law for union on p. 57 of your textbook. Then, using a similar approach, prove the distributive law for union.

3 Combinatorial Proofs

Here, we demonstrate how equations can be proven combinatorially by showing that the two sides of an equation are valid answers to the same counting problem. This is summarized by Proof Template 9 on p. 58 of your textbook:

Combinatorial proof. To prove an equation of the form LHS = RHS,
1. Pose a question of the form "In how many ways. . . ?"
2. Argue why LHS is a correct answer to the question.
3. Argue why RHS is also a correct answer to the question.
4. Therefore LHS = RHS.

The following demonstrates the combinatorial proof.

Proposition 15. *Let n be a positive integer. Then,*

$$2^0 + 2^1 + \cdots + 2^{n-1} = 2^n - 1.$$

Proof. Consider the following counting problem: Let n be a positive integer, and let $N = \{1, 2, \ldots, n\}$. How many nonempty subsets does N have?

By Theorem 9, the total number of subsets of the finite set N is 2^n. The total number of non-empty subsets of N is therefore $2^n - 1$, which is equal to the right-hand side of the equation that we are trying to prove.

Now, consider the number of subsets of N whose largest element is j, where $1 \leq j \leq n$, which we call N_j. For example, $N_3 = 4$ is the number of all the subsets of N where the largest element is 3 (refer to the table on p. 67 for further clarity, if needed). Since N_j counts subsets where there can be no more than j elements, but where 1 element must be j (i.e., one of the elements is fixed), we have $N_j = 2^{j-1}$. Adding up N_1 through N_n thus gives us

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}.$$

By the Addition Principle, this is equal to the total number of non-empty subsets of N, and is equal to the left-hand side of the equation that we are trying to prove.

Since the left-hand side and the right-hand side of the equation give the answer to the same counting problem, we have proven that the equation is true. □

It can be challenging to come up with an appropriate counting problem to use in a combinatorial proof. The example above can also be proved by induction, which we will study later. The following exercise asks you to use a list-counting problem, instead of a set-counting problem as was done with the example above.

Exercise

Give a combinatorial proof that $n^2 = n(n-1) + n$.

Hint: Use the counting problem "How many length-two lists can we make from n elements, with repetition allowed?".