Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Module 13: Graph Theory II
Fall 2021

This module corresponds to the following sections from your textbook:

52. Coloring
53. Planar Graphs

1 Colouring

In this section, we will be doing something that we don't usually get to do in a university-level course: colouring. (Hopefully you remembered to bring your crayons!)

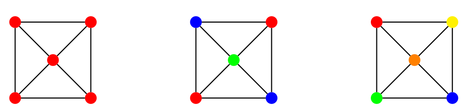
We often label vertices on our graphs with labels like a, b, u, and v. Importantly, each of these symbols had to be unique, since we wanted to distinguish between vertices. A **graph colouring**, on the other hand, is like a method of nonunique labelling of the vertices. Instead of giving each vertex a symbolic label, we assign a "colour" to each vertex that distinguishes it from nearby vertices. These colours need not be unique; we can label more than one vertex with the same colour. In our formal definition, we use numerical values to represent colours.

Definition 1 (Graph colouring)**.** Given a graph $G = (V, E)$ and a set of k colours $\{1, 2, \ldots, k\}$, a k-colouring of G is a function $f : V \to \{1, 2, \ldots, k\}$.

If a k-colouring f has the property that, for all edges $\{u, v\} \in E$, $f(u) \neq f(v)$, then we say that the colouring is **proper**. A proper colouring is one in which no adjacent vertices share the same colour.

If we can construct a proper k-colouring of a graph, then we say that the graph is ***k*-colourable**.

Example 2. Each of the following graphs depicts a graph colouring. However, only the two rightmost graphs depict a proper colouring. Since we can obtain a proper graph colouring using three colours, this graph is 3-colourable.



1.1 Chromatic Numbers

Given a graph G with n vertices, if all we care about is colouring the vertices in some way, then we can use as many as n colours (if we assign each vertex a unique colour) or as few as one colour (if we asslgn all vertices the same colour). Occasionally, however, we might want to see how many colours we require in order to construct a proper colouring of G. This idea leads to the notion of the **chromatic number** of G.

Definition 3 (Chromatic number)**.** Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is equal to the smallest number of colours required to construct a proper colouring of G.

We can attain both the upper bound and the lower bound on the chromatic number if we consider particular classes of graphs.

- Given a complete graph K_n with n vertices and $\frac{n(n-1)}{2}$ edges, we have that $\chi(K_n) = n$. This is because each vertex is adjacent to every other vertex in the graph.
- Given an edgeless graph $\overline{K_n}$ with n vertices and zero edges, we have that $\chi(\overline{K_n}) = 1$. This is because no vertices are adjacent to one another in the graph.

Thus, we know that for any graph G, we have that $1 \leq \chi(G) \leq |V(G)|$.

With a little work, we can obtain a few more results about the chromatic numbers of given graphs. Our first result relates the chromatic number of a graph to the **maximum degree** of any vertex in the graph (i.e., the degree of the vertex that has the most adjacent, denoted by Δ.

Theorem 4. *If a graph G has maximum degree Δ, then $\chi(G) \leq \Delta + 1$.*

Proof. Assume we have a set of $\Delta + 1$ colours. Choose an arbitrary vertex, say v_1, from the graph G and assign to it a colour. Repeat this process of assigning colours to arbitrary vertices of G under the condition that, if two vertices v_i and v_j are adjacent, then both vertices should be assigned different colours.

Since the maximum degree of G is Δ and since we have $\Delta + 1$ colours available to use, we will not encounter the issue of running out of colours to assign to adjacent vertices. Therefore, $\chi(G) \leq \Delta + 1$. ☐

A stronger result, proved by the English mathematician R. Leonard Brooks, reduces the bound given in Theorem 4 to $\chi(G) \leq \Delta$, unless $G = K_n$ or $G = C_{2n+1}$. However, Brooks' result is harder to prove.

Our next result relates the chromatic number of a graph to the chromatic numbers of its subgraphs.

Theorem 5. *If a graph G contains a subgraph H, then $\chi(H) \leq \chi(G)$.*

Proof. Given a proper colouring of the graph G, we can copy those colours to the corresponding vertices of H. Thus, a proper colouring of H requires at most $\chi(G)$ colours. ☐

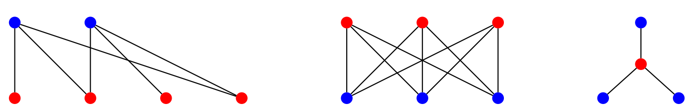
1.2 Bipartite Graphs

Next, recall the notion of a bipartite graph from last week's discussion on special graph classes. A bipartite graph is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that each edge joins a vertex in V_1 to a vertex in V_2. The following result characterizes the set of bipartite graphs in terms of their chromatic number.

Theorem 6. *A graph G is a bipartite graph if and only if $\chi(G) \leq 2$.*

Proof. ($\Rightarrow$): Given a bipartite graph G, colour each of the vertices in V_1 one colour (say, red), and colour each of the vertices in V_2 another colour (say, blue). No vertices in V_1 are adjacent to vertices in the same set, and the same is true for vertices in V_2. Thus, $\chi(G) = 2$.

($\Leftarrow$): Given a graph G with $\chi(G) = 2$, partition each of the vertices of the first colour into one set V_1, and partition each of the vertices of the second colour into another set V_2. Since no two adjacent vertices of G share the same colour, vertices in V_1 are only adjacent to vertices in V_2 and vice versa, making the graph bipartite. ☐



Below, we see two classes of graphs that are bipartite.

Example 7. What is the chromatic number of a cycle graph C_n? If n is even, we can use two colours and alternate the colour of each vertex around the cycle; this is a proper colouring. However, this strategy will not work if n is odd, because the nth vertex will have the same colour as the first vertex. So, we need a third colour for the last vertex. Thus, C_n is two-colourable if n is even and three-colourable if n is odd.

Thus, from the above example, we can conclude that cycle graphs with an even number of vertices are bipartite.

Now we prove that trees are bipartite using Proof Template 25 from the textbook (p. 353), which is a proof by induction technique for proving theorems about trees:

1. **Claim:** A statement about trees.
2. **Basis case:** The theorem is true for all trees on $n = 1$ vertices.
3. **Induction hypothesis:** Suppose the theorem is true for all trees on $n = k$ vertices.
4. **Inductive Step:**
   - Let T be a tree on $n = k + 1$ vertices. Let v be a leaf of T.
   - Let $T' = T - v$. Note that T' is a tree with k vertices, so the claim is true for T' as per the induction hypothesis.
   - **Prove** that since the claim is true for T', it must also be true for T.

Now, we proceed to our proof.

Proposition 8. *Trees are bipartite.*

Proof. We use Proof Template 25:

Basis case: A tree with only one vertex is bipartite, since it is one-colourable and thus also two-colourable (and three-colourable, and four-colourable, and so on—we can keep adding colours, even though we don't actually need more than one for a proper colouring).

Induction hypothesis: Every tree with k vertices is bipartite.

Inductive step:

- Let T be a tree on $n = k + 1$ vertices. Let v be a leaf of T.
- Let $T' = T - v$. Note that T' is a tree with k vertices, so T' is bipartite as per the induction hypothesis.
- To prove that T is also bipartite: Properly colour T' using the two colours black and white. Now, consider the neighbour of the vertex v that we deleted. Since trees are connected and acyclic, there is exactly one such neighbour, which we label w. Whatever colour w has, we can assign the other colour to v and add it back to the tree. This gives us a proper two-colouring of T and thus we have proved that every tree is bipartite.

□

We conclude our discussion by reflecting on the following theorem.

Theorem 9. *A graph is bipartite if and only if it does not contain an odd cycle.*

Proof. ($\Rightarrow$) Let G be a bipartite graph. So, we have $\chi(G) \leq 2$. Suppose, for the sake of contradiction, that G contains an odd cycle C as a subgraph.

Since C is a cycle that contains an odd number of vertices, we know from Example 7 that $\chi(C) = 3$.

By Theorem 5, we know that $\chi(C) \leq \chi(G)$. So, we have

$$3 = \chi(C) \leq \chi(G) \leq 2,$$

which is a contradiction. Therefore, G does not contain an odd cycle.

($\Leftarrow$) We omit this part of the proof (refer to p. 364 of the textbook) but explain the implications below. $\square$

Suppose you inspect a graph and wish to show that it is bipartite. You may do this by finding a proper two-colouring of the graph.

Now, suppose that you inspect a very large and complicated graph and wish to show that it is not bipartite. The above theorem gives us an easy way to do so:

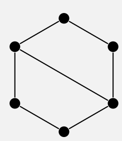
1. Arbitrarily pick a vertex in the graph to start colouring, and colour it white.
2. Colour all of its neighbours black.
3. Colour the neighbours of the black vertices white.
4. Continue the above process. If there are no more adjacent vertices but there still remains uncoloured vertices, this means that the graph is not connected. If so, restart the above process by selecting a vertex from another connected component.

Continue this process, until you either:

1. End up with two adjacent vertices having the same colour, and backtrace your steps to identify an odd cycle.
2. End up with a proper two-colouring of the graph, if there are no more vertices to colour.

Exercise

Let us find in two different ways that the graph below is bipartite. First, find a proper two-colouring by following the above steps. Second, find all the cycles and check that none of them contain an odd number of vertices.



We see that determining whether or not a graph is two-colourable is quite straightforward. Interestingly, the same cannot be said about determining whether or not a graph is three-colourable. In fact, the latter belongs to a class of problems for which there is no known efficient algorithm for solving (although there are some approximate methods that may give "good enough" results).

Example 10. One way of determining whether a graph is three-colourable is by checking every possible colouring to see if it is proper. Suppose we have a graph G with 100 vertices. To check all possible colourings of the graph (proper or not proper) means that we would need to check 3^{100} colourings. If we do this on a computer that can check 1 billion colourings per second, we would be finished in $\frac{3^{100}}{10^9}$ seconds, or about 10^{31} years. For comparison, the age of the universe is about 14 billion (i.e., 1.4×10^{10}) years.

2 Planar Graphs

We often informally refer to the process of depicting a graph on paper as "drawing". However, simply saying that we have "drawn" a graph allows some undesirable properties to sneak in that might make our graph confusing to look at. Thus, we want to formalize the notion of drawing in some way to ensure that our depictions of graphs are clear.

Mathematicians have a name for such a formalism: an **embedding** of a graph G on a surface (for our purposes, a two-dimensional plane) is a representation of the vertices and edges of G that meet the following criteria:

- each vertex $u \in V$ is assigned a point on the surface, and no two vertices share the same point;
- each edge $e \in E$ is assigned a curve on the surface, and no two edges share the same curve;
- the endpoints of some curve e are exactly the two incident vertices u and v; and
- no vertex other than the two incident vertices u and v lie on the edge e.

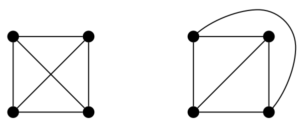
Essentially, an embedding ensures that we are able to see all of the vertices and edges of a graph and that no vertices or edges overlap one another.

Observe that edges overlapping (that is, one edge covering another edge entirely) is different from edges crossing (that is, one edge intersecting another edge at some point on the surface, which includes an edge intersecting itself). Many graphs have edges that cross, and we have seen such graphs throughout these notes. But a special subset of graphs have edges that do not cross, and that subset is the topic of this section.

Definition 11 (Planar graph)**.** A graph $G = (V, E)$ is planar if G has an embedding in the plane that is crossing-free.

Often, if we are given a graph with crossing edges, we can redraw the edges in such a way that no edges cross one another. If we can redraw a graph G into a crossing-free graph G', then we say that G' is a **planar representation** of G.

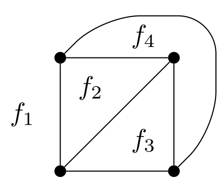
Example 12. The embedding of the graph K_4 on the left is not planar. However, the embedding of the same graph K_4 on the right is planar. Therefore, K_4 is a planar graph.



In the same way that borders on Earth divide land into countries, the edges of a planar graph divide a surface into **faces**. Faces need not be confined to the interior of a graph; there always exists one exterior, unbounded face that surrounds the graph.

Just like vertices, faces have degrees. The degree of a face is equal to the number of edges making up the boundary of that face. Intuitively, the degree of a face can be thought of as the number of lines you would need to draw, without lifting your pencil, to trace out the face on paper and end up at the point where you started.

Example 13. Recall the planar embedding of K_4 from Example 12. This planar graph consists of four faces, each labelled in the following figure. Observe that one face, f_1, is an exterior face. Further observe that each face of the graph has degree 3, since each face is bounded by three edges.



Long ago, Euler discovered a relationship between the numbers of vertices, edges, and faces of certain planar graphs. Summing these numbers in a particular way produces a constant value, known as the **characteristic**. Although Euler's result is easy to remember, it was given the rather uninspired name of "Euler's formula", which isn't particularly helpful once you realize exactly how many formulas Euler discovered throughout his life. Nevertheless, Euler's formula for connected planar graphs is as follows.

Theorem 14 (Euler's formula)**.** *Given a connected planar simple graph $G = (V, E)$ with v vertices, e edges, and f faces,*

$$v - e + f = 2.$$

Proof. Let G be a connected planar simple graph. We will prove the claim using the principle of mathematical induction on the number of edges of G. Let $P(e)$ be the statement "given a graph G with v vertices and e edges, $v - e + f = 2$".

When $e = 0$, we know that G is an edgeless graph. Since G is connected, it must consist of only one vertex, and as a result, G contains one exterior face. This gives $v - e + f = 1 - 0 + 1 = 2$. Therefore, $P(0)$ is true.

Assume that $P(e')$ is true for some $e' \in \mathbb{N}$.

We now show that $P(e'+1)$ is true; that is, we add one edge to the graph G and check whether the statement holds. The new edge can be added in one of two ways:

- If the edge is incident to one existing vertex, then we must add a new vertex to G. This increases both the number of vertices and the number of edges by one, which gives $(v + 1) - (e' + 1) + f = v + 1 - e' - 1 + f = v - e' + f = 2$.
- If the edge connects two existing vertices to one another, then we divide some face within G into two faces. This increases both the number of edges and the number of faces by one, which gives $v - (e' + 1) + (f + 1) = v - e' - 1 + f + 1 = v - e' + f = 2$.

In either case, $P(e' + 1)$ is true. By the principle of mathematical induction, $P(e)$ is true for all $e \in \mathbb{N}$. □

Euler's formula also has the following corollary.

Corollary 15. *Let G be a connected planar graph with $v \geq 3$ vertices and $e \geq 2$ edges. Then*

$$e \leq 3v - 6.$$

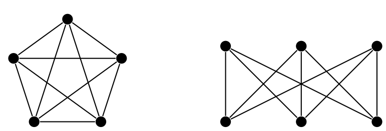
Furthermore, if G does not contain K_3 as a subgraph, then

$$e \leq 2v - 4.$$

Note that the fact that G is connected is crucial to the correctness of Theorem 14. Graphs that are not connected have a slightly different formula.

So, why do we care about Euler's formula? Using this result, we can derive a number of conditions that must be met in order for a graph to be planar.

If a graph has no planar representation, then we say that it is **nonplanar**. Proving the nonplanarity of a graph often reduces to showing that the graph does not meet the conditions set out by Euler's formula (or by some corollary of Euler's formula). As illustrative examples of such proofs, we will focus on proving the nonplanarity of two graphs with which we are already familiar: K_5 and $K_{3,3}$.



Theorem 16. *The complete graph K_5 is nonplanar.*

Proof. Suppose for the sake of contradiction that K_5 is planar. By the first inequality in Corollary 15, we would have

$$10 = e \leq 3v - 6 = 3 \times 5 - 6 = 9,$$

which is a contradiction. Therefore, K_5 is non-planar. □

Theorem 17. *The complete bipartite graph $K_{3,3}$ is nonplanar.*

Proof. Suppose for the same of contradiction that $K_{3,3}$ is planar. Since it does not contain K_3 as a subgraph, by the second inequality in Corollary 15 we would have

$$9 = e \leq 2v - 4 = 2 \times 6 - 4 = 8,$$

which is a contradiction. Therefore, $K_{3,3}$ is non-planar. □

At this point, you might wonder why we chose to prove that these two specific graphs were nonplanar. There was a hidden motive for our choices of proofs: we actually require these results to prove the final result of this section. First, however, we require just one more bit of terminology.

It is possible to define an operation on a graph that, given an edge e incident to vertices u and v, divides the edge into two edges e_1 and e_2, where e_1 is incident to u and a new vertex w and e_2 is incident to w and v. This operation changes nothing about the graph apart from the number of vertices and edges and, in particular, it does not affect the planarity of the graph. We call this operation a **subdivision**.

Now, we are ready to see a fascinating theorem due to the Polish mathematician Kazimierz Kuratowski. This theorem gives us a pair of conditions that identify when certain graphs are nonplanar. Put simply, if we can obtain a graph G by taking subdivisions of either of the nonplanar graphs K_5 or $K_{3,3}$, then G itself is nonplanar.

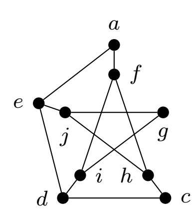
Theorem 18 (Kuratowski's theorem)**.** *A graph G is nonplanar if and only if G contains a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph.*

Proof. ($\Rightarrow$): Omitted.

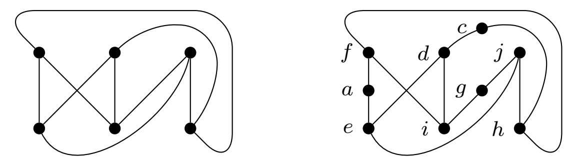
($\Leftarrow$): If G contains a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph, then G is nonplanar by Theorem 16 or Theorem 17, respectively. □

Kuratowski's theorem is biconditional, so we must prove two statements in order to prove the overall theorem. Although one direction of the proof ($\Leftarrow$) followed immediately from other results we proved, the other direction ($\Rightarrow$) is a long and detailed proof. We omit it here, but many textbooks dedicated to graph theory contain the full proof.

Example 19. We can show that the Petersen graph (which we saw last week) is nonplanar by using Kuratowski's theorem. First, delete one vertex from the Petersen graph to obtain the following subgraph, then label the vertices.



Next, let us draw the complete bipartite graph $K_{3,3}$ on the left—note that we have lengthened a few of the edges of the graph for illustrative reasons, but it is still $K_{3,3}$. We take a subdivision of this graph to obtain the graph at right, and again label the vertices.



Finally, observe that the graph on the right is the same as the subgraph of the Petersen graph that we drew above (to verify, you may check that any two vertices that are adjacent in one graph are also adjacent in the other). In other words, the Petersen graph contains a subdivision of $K_{3,3}$ as a subgraph. By Kuratowski's theorem, the Petersen graph is nonplanar.

2.1 Colouring Planar Graphs

Finally, let's investigate chromatic numbers of planar graphs. We do so in the context of maps: a paper map is a plane, land areas are vertices of a graph, and adjacent land areas are connected by non-crossing edges.

In the early days of graph theory, a South African mathematician named Francis Guthrie made a conjecture while trying to colour a map of England. Guthrie asserted that one needs only four colours to colour in any map, no matter how many regions the map depicts. In 1879, an English mathematician named Alfred Kempe claimed to have a proof of Guthrie's conjecture, and for this he received many awards and accolades. However, 11 years later, another English mathematician named Percy Heawood found a flaw in Kempe's proof. Heawood gave his own proof that five colours are in fact sufficient.

From here, work on map colouring lay dormant for many years. Then, in 1976, Kenneth Appel and Wolfgang Haken proved that, indeed, four colours are sufficient to colour any map. The proof of Appel and Haken was a landmark result, since it was one of the first major mathematical theorems to be proved with the aid of a computer; the complete proof consisted of 1 936 different cases. Since then, the proof has been shortened considerably—to a mere 633 cases—but it is still too lengthy to reproduce here.

Theorem 20 (Four colour theorem)**.** *If a graph G is planar, then $\chi(G) \leq 4$.*

Proof. Omitted. $\square$

The textbook provides an an easy-to-follow proof for $\chi(G) \leq 6$, and a more involved proof for $\chi(G) \leq 5$.