Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Module 3: Counting and Relations (Continued)
Fall 2021

This module corresponds to the following sections from your textbook:

17. Binomial Coefficients
18. Counting Multisets
19. Inclusion-Exclusion

1 Binomial Coefficients

As you likely learned in high school,

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y) \\ &= x^2 + 2xy + y^2.\end{aligned}$$

We now consider $(x+y)^3$:

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= (x^2+2xy+y^2)(x+y) \\ &= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

Let's also consider $(x+y)^4$ while we're at it:

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y) \\ &= (x^3+3x^2y+3xy^2+y^3)(x+y) \\ &= x^4 + x^3y + 3x^3y + 3x^2y^2 + 3x^2y^2 + 3xy^3 + xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.\end{aligned}$$

By this point, you might notice that a pattern is developing. Each term in the expansion of the binomial $(x+y)^n$ is of the form ax^by^c, where $b+c=n$ and where a is some coefficient.

How can we calculate the value of the coefficient a for some arbitrary term without writing the entire expansion? Let's begin by determining what this value a is counting. We can write the general binomial $(x+y)^n$ as

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

Each term in the expansion of $(x+y)^n$ will be a product of one term (either the x or y) from inside each pair of parentheses. To illustrate using $n=3$:

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

To illustrate more generally for any n, consider the term that is the product of only the x terms from inside each of the n pairs of parentheses. If we take x a total of n times, we will add the term x^n to our expansion. On the other hand, if we take x a total of $n-4$ times, then we must take y a total of 4 times in order to collect n terms overall. Thus, we will add the term $x^{n-4}y^4$ to our expansion.

To formulate this as a combinatorial problem, we say that we need to choose x a total of b times and y a total of c times, where $b+c=n$. This is equivalent to calculating the number of ways we can choose b occurrences of x from n binomials, which we represent as $\binom{n}{b}$. Or conversely, this is equivalent to choosing c occurrences of y from n binomials, which we represent as $\binom{n}{c}$. Since both of these formulations represent the same counting problem (i.e., counting the same thing in different ways), we have

$$\binom{n}{b} = \binom{n}{c} = \binom{n}{n-b} = \binom{n}{n-c}.$$

We call our new notation the binomial coefficient, since we will use it to express the coefficient of each term in the expansion of $(x+y)^n$. We then write the Binomial Theorem as follows.

Theorem 1 (Binomial theorem)**.** *Let x and y be variables, and let n be a natural number. Then*

$$\begin{aligned}(x+y)^n &= \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.\end{aligned}$$

Immediately from the statement of the binomial theorem, we get a variant of the theorem as a corollary.

Corollary 2. *Let x be a variable and let n be a natural number. Then*

$$(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Proof. Follows from the binomial theorem when $y=1$. □

Using what we have seen so far, we construct the two triangles below, where each line in the triangle represents the coefficients of the expansion of $(x+y)^n$, with

(i) binomial coefficients, obtained from the Binomial Theorem, and

(ii) the corresponding numerical values, obtained by expanding $(x+y)^n$ and collecting like terms.

$$\begin{array}{ccccccccccccc}
&&&&&&\binom{0}{0}&&&&&& \\
&&&&&\binom{1}{0}&&\binom{1}{1}&&&&& \\
&&&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&&& \\
&&&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&&& \\
&&\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}&& \\
&\binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5}& \\
\binom{6}{0}&&\binom{6}{1}&&\binom{6}{2}&&\binom{6}{3}&&\binom{6}{4}&&\binom{6}{5}&&\binom{6}{6}
\end{array}
\quad = \quad
\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&1&&1&&&&& \\
&&&&1&&2&&1&&&& \\
&&&1&&3&&3&&1&&& \\
&&1&&4&&6&&4&&1&& \\
&1&&5&&10&&10&&5&&1& \\
1&&6&&15&&20&&15&&6&&1
\end{array}$$

This is called Pascal's Triangle. Pascal's triangle reveals many relationships between binomial coefficients. One of the most important properties we can observe, called Pascal's Identity, is that each term $\binom{n}{k}$ is the sum of the two terms $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ that are written directly above it (blank or nonexistent entries are taken to be zero). We prove Pascal's Identity below.

Theorem 3 (Pascal's Identity)**.** *Let n and k be integers with $1 \leq k \leq n$. Then, we have*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. We prove this combinatorially, by showing that the left-hand side and right-hand side are solutions to the same counting problem. Let S be a set containing n elements.

The left-hand side of the equation gives us the number of ways to choose k elements from S.

The right-hand side of the equation represents another approach to count the number of ways to choose k elements from S, as follows. Let us "hide" one of the elements x and call it the "missing" element, and then consider these two scenarios:

- Suppose we first retrieve the "missing" element. Now, we have $\binom{n-1}{k-1}$ ways to complete the subset of k elements that include the "missing" element.
- Suppose we give up on retrieving the "missing" element. Now we have $\binom{n-1}{k}$ ways to pick a subset of k elements that don't include the "missing" element.

By the Addition Principle, the above two mutually-exclusive scenarios together give us all the possible ways to choose k elements from S. $\square$

Using the Binomial Theorem and Pascal's Identity, we can expand certain expressions or calculate certain terms within an expression much more efficiently than multiplying out all the terms. Multiplication is an intensive operation for a computer, and many multiplications at once can slow down a computation considerably.

We present a few other interesting identities here that can be observed from Pascal's Triangle.

Proposition 4. *The following identities hold:*

1. $\binom{r}{k} = \binom{r}{r-k}$ *for all* $0 \leq k \leq r$ *(row symmetry);*
2. $\sum_{k=0}^{r} \binom{r}{k} = 2^r$ *for all* $r \geq 0$ *(row sum);*
3. $\sum_{k=0}^{r} \binom{r}{k}^2 = \binom{2r}{r}$ *for all* $r \geq 0$ *(row sum of squares);*
4. $\sum_{k=0}^{r} \binom{k}{c} = \binom{r+1}{c+1}$ *for all* $r, c \geq 0$ *(column sum).*

For convenience, we also derive a closed-form expression that can be used for calculating binomial coefficients.

Theorem 5 (Formula for $\binom{n}{k}$)**.** *Let n, k be integers with $0 \leq k \leq n$. Then,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Let us first illustrate with $k = 3$. We know from Module 1 that the number of 3-element repetition-free lists that can be composed from an n-element set is $(n)_3 = n(n-1)(n-2)$. Below we write out all possible 3-element lists, but each row contains the 3-element lists containing the same elements but in different order.

$$\begin{array}{rrrrrr}
(1,2,3) & (1,3,2) & (2,1,3) & (2,3,1) & (3,1,2) & (3,2,1) \\
(1,2,4) & (1,4,2) & (2,1,4) & (2,4,1) & (4,1,2) & (4,2,1) \\
\dots & \dots & \dots & \dots & \dots & \dots \\
(n-2,n-1,n) & \dots & \dots & \dots & \dots & (n,n-1,n-2)
\end{array}$$

We are interested in finding the number of possible 3-element sets (i.e., unordered lists). To do that, we divide the total number of lists, $n(n-1)(n-2)$, by the number of possible orderings of each 3-element list, which is $3! = 6$. The result is then

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}.$$

Exercise

Complete the example above for $n = 4$ and $k = 3$ to help convince yourself of the formula.

Proof. We now mimic our analysis of the above example for any $\binom{n}{k}$. We know that the number of k-element repetition-free lists that can be composed from an n-element set is

$$(n)_k = n(n-1)(n-2)\dots(n-(k-1)) = \frac{n!}{(n-k)!}.$$

We are interested in finding the number of possible k-element sets (i.e., unordered lists). To do that, we divide the total number of lists by the number of possible orderings of each k-element list, which is $k!$. So, the result is

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

□

Exercise

Write the binomial coefficient that gives the number of bit strings of length 10 that contain exactly four 1s.

Note that the textbook defines an equivalence relation "has-the-same-elements-as", for which all lists that are different orderings of the same elements form an equivalence class. Then, the total number of k-element lists is $(n)_k$, and each equivalence class contains $k!$ lists, and the number of equivalence classes is $\binom{n}{k}$.

You can observe from your illustration for $n = 4$ and $k = 3$ in the previous exercise that each row of your table constitutes an equivalence class, and the union of all the equivalence classes gives the set of all possible 3-element lists from a pool of 4 elements.

2 Multisets

If we modify the definition of a set to allow multiple copies of each element, we get what is known as a **multiset**. For instance, $\{1, 2\}$ is a set, but $\langle 1, 2, 2\rangle$ is a multiset. Notice, as denoted in your textbook, that angular brackets are used with multisets in place of the curly braces.

Note that with multisets, just as with sets, order does not matter; the multisets $\langle 1, 2, 2\rangle$ and $\langle 2, 1, 2\rangle$ are the same multiset. However, $\langle 1, 2, 2\rangle$ and $\langle 1, 2, 2, 2\rangle$ are *distinct* multisets.

We call repeated elements within a multiset **indistinguishable**. In order to count the number of permutations of a set (i.e., the number of ordered lists that can be composed from the set) with indistinguishable objects, we:

1. Count the number of permutations of the set as usual (i.e., counting as if all objects in the list are distinct).
2. Compensate for the overcounting of identical permutations, by dividing the total number of permutations by the number of ways we can permute just the indistinguishable elements.

Example 6. An often used example of a word with indistinguishable letters is MISSISSIPPI. In this word, we have four I's, four S's, and two P's, all of which are indistinguishable. We can "distinguish" the letters for illustrative purposes by subscripting them:

$$\mathrm{M}\ \mathrm{I}_1\ \mathrm{S}_1\ \mathrm{S}_2\ \mathrm{I}_2\ \mathrm{S}_3\ \mathrm{S}_4\ \mathrm{I}_3\ \mathrm{P}_1\ \mathrm{P}_2\ \mathrm{I}_4.$$

If we permute two or more indistinguishable letters, then we should not get two or more different permutations. For instance, MISSISSIP$_1$P$_2$I and MISSISSIP$_2$P$_1$I should be considered the same permutation.

We have a total of 11! ways to permute the word MISSISSIPPI. We then divide that by 4!, 4!, and 2! to compensate for the indistinguishable letters I, S, and P, respectively, which gives us a total of $\frac{11!}{(4!)(4!)(2!)}$ permutations.

Note that we already solved a similar example in the previous module using equivalence classes.

We now formulate this counting technique as follows.

Theorem 7. *The number of n-element lists that can be composed from a set with n elements, where n_1 objects are indistinguishable from each other, n_2 objects are indistinguishable from each other, ..., and finally, n_r objects are indistinguishable from each other, and $n_1 + n_2 + \cdots + n_r = n$, is:*

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

Proof. Our goal is to construct all possible n element lists from the n element multiset:

1. Suppose we first place the n_1 indistinguishable objects into the n "slots" in the list. We can do this in $\binom{n}{n_1}$ ways. Now, we have $n - n_1$ empty slots remaining.
2. We now place the n_2 indistinguishable objects into the remaining $n - n_1$ slots. We can do this in $\binom{n-n_1}{n_2}$ ways. Now, we have $n - n_1 - n_2$ empty slots remaining.

...

r. We now place the n_r indistinguishable objects into the remaining $n - n_1 - n_2 - \cdots - n_{r-1}$ remaining slots. We can do this in $\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$ ways. Now we have filled up all the slots.

Then by the multiplication principle, the total number of ways in which we can compose the list is

$$\begin{aligned}
&\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} \\
&\qquad = \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \cdots \times \frac{(n-n_1-\cdots-n_{r-1})!}{n_r!0!} \\
&\qquad = \frac{n!}{n_1!n_2!\ldots n_r!}.
\end{aligned}$$

□

Definition 8 (Multinomial Coefficient)**.** The notation

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}$$

is called a **multinomial coefficient**.

Example 9. The Queen's hockey team has six games scheduled for the season. The number of ways they can end the season with two wins, three losses, and one tie, is

$$\binom{6}{2, 3, 1} = \frac{6!}{2! \cdot 3! \cdot 1!} = 60.$$

2.1 Combinations with Repetition

We now consider the following counting problem: Let $n, k \in \mathbb{N}$. How many k-element multisets can we form by choosing elements from an n-element set? In other words, how many ways can we choose k elements with repetition allowed?

Definition 10. $\left(\!\binom{n}{k}\!\right)$ denotes the number of multisets with cardinality k, composed from an n-element set.

Example 11. Let n, k be positive integers. So, we have the following

$$\left(\!\!\binom{n}{1}\!\!\right) = n \qquad \left(\!\!\binom{1}{k}\!\!\right) = 1 \qquad \left(\!\!\binom{2}{2}\!\!\right) = 3 \qquad \left(\!\!\binom{3}{3}\!\!\right) = 10$$

> **Exercise**
>
> Convince yourself that each of the examples above are correct by listing out the possible multisets, assuming that they are being drawn from a set $\{0, 1, 2, \ldots, n\}$. Afterwards, compare your results with Examples 18.2 through 18.5 on p. 102 of your textbook.

Proposition 12. *Let n, k be positive integers. Then,*

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right).$$

> **Exercise**
>
> Complete the proof for Proposition 12 above, which is similar to that of Pascal's Identity (Theorem 3). Compare with the proof for Proposition 18.6 on p. 103 of your textbook. Consult also the table on the same page for a variant of Pascal's Triangle for $\left(\!\binom{n}{k}\!\right)$.

We can express $\left(\!\binom{n}{k}\!\right)$ in terms of a binomial coefficient, as follows.

Theorem 13. *Let n, k be positive integers. Then,*

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

Before proving the above theorem, we introduce the so-called **stars and bars** method with an example.

Example 14. Suppose a student is enrolling in courses for the upcoming academic year. They plan to enrol in ten courses, selected from the set of five departments {CISC, MATH, PHYS, BIOL, CHEM}. We wish to count the number of ways that the student can divide their course load between the given set of departments. In other words, our counting problem is $\left(\!\binom{5}{10}\!\right)$.

If the student registers in five CISC courses, three MATH courses, one PHYS course, no BIOL courses, and one CHEM course, we can encode their course selection using stars and bars as follows:

$$\bigstar\bigstar\bigstar\bigstar\bigstar \mid \bigstar\bigstar\bigstar \mid \bigstar \mid \ \mid \bigstar$$

We see in the encoding above that the ten courses are represented by ten $\bigstar$ symbols, and the courses from each of the five departments are separated from each other using the four $\mid$ symbols. Each arrangement of ten stars and four bars corresponds to exactly one selection of ten courses from the five departments, and vice-versa (i.e., each selection of ten courses from the five departments corresponds to exactly one arrangement of ten stars and four bars).

We see now that we have reformulated the multiset-counting problem $\left(\!\binom{5}{10}\!\right)$ into a list-counting problem, where we wish to determine the number of ways that we can re-arrange ten stars and four bars. This is equivalent to selecting which of the four out of the 14 positions will be used to place a $|$. Equivalently, which of the ten out of the 14 positions to place a $\bigstar$. Thus,

$$\left(\!\binom{5}{10}\!\right) = \binom{14}{4} = \binom{14}{10} = 1001.$$

Proof. Now, let us use the stars and bars method to prove the general case $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$.

If $n = 0$ and $k = 0$, we have $\left(\!\binom{0}{0}\!\right) = 1$ (the empty multiset) and $\binom{-1}{0} = 1$ (by definition). If only one of n or k is zero, it is easy to see that the identity holds. For other positive values of n and k, following from the example above, we can represent each of the possible $\left(\!\binom{n}{k}\!\right)$ multisets as a list of k stars and $n-1$ bars. Thus, our problem is equivalent to choosing which of the k positions out of the $n+k-1$ positions will be used to place a $\bigstar$. So, we have

$$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}.$$

$\square$

Exercise

Suppose Tim Horton's has five varieties of donuts. In how many ways can we select a dozen donuts?

3 Inclusion-Exclusion

We start with a simple example.

Example 15. Consider the sets $A = \{1, 2, 4, 8\}$ and $B = \{2, 4, 6, 8\}$. How many elements are in the union of the two sets, $A \cup B$?

We know from Module 1 that $|A \cup B| = |A| + |B| - |A \cap B|$. We have $|A| = 4$, $|B| = 4$, and $|A \cap B| = |\{2, 4, 8\}| = 3$. So, $|A \cup B| = 4 + 4 - 3 = 5$. A quick check reveals that, indeed, $A \cup B = \{1, 2, 4, 6, 8\}$.

Exercise

Draw a Venn diagram for the above example and place each element in the appropriate region.

We generalize the formula to three sets as follows:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Exercise

Let $A = \{1, 2, 3, 5, 8, 13\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$. Find $|A \cup B \cup C|$ using the formula above. Draw a Venn diagram and place each element in the appropriate region.

The intuitive explanation for obtaining the formula for $|A \cup B \cup C|$ above is as follows:

1. Add the cardinality of the three sets, i.e., $|A|$, $|B|$, and $|C|$. But the result is greater than $|A \cup B \cup C|$ if any pairwise intersections are non-empty.
2. Subtract the cardinality of the pairwise intersections, i.e., $|A \cap B|$, $|A \cap C|$, and $|B \cap C|$. But now the result is less than $|A \cup B \cup C|$ if the triplewise intersection is non-empty.
3. Add the triplewise intersection of the sets, i.e., $|A \cap B \cap C|$.

This technique generalizes to find the union of n sets as follows:

1. Add the cardinality of all n sets.
2. Subtract the cardinality of all pairwise intersections of the sets.
3. Add the cardinality of all triplewise intersections of the sets.
4. Subtract the cardinality of all quadruple-wise intersections of the sets.
   ...

n. Add (if n is odd) or subtract (if n is even) the n-wise intersection of all the sets.

Writing out a formula for the generalized procedure gives us the **Inclusion-Exclusion Principle** as follows.

Theorem 16 (Inclusion-exclusion principle)**.** *Let $A_1, A_2, \ldots, A_n$ be sets, each of finite cardinality. Then*

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n| =& |A_1| + \cdots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ & - \cdots + \cdots \\ & \pm |A_1 \cap A_2 \ldots \cap A_n|.\end{aligned}$$

If we wish to appear more sophisticated, we can rewrite this expression using summation notation as

$$\left|\bigcup_{k=1}^{n} A_k\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\leq a_1<\cdots<a_k\leq n}\left|\bigcap_{j=1}^{k} A_{a_j}\right|\right).$$

Proof. Refer to the proof of Theorem 19.1 on p. 110 of your textbook. □

The part of the expression to the left of the equals sign, $|\bigcup_{k=1}^{n} A_k|$, is a shorthand way to say $|A_1 \cup \cdots \cup A_n|$. To the right of the equals sign, we have two parts to consider. The sum, $\sum_{k=1}^{n}(-1)^{k+1}$, alternates between adding and subtracting the k^{th} term of the expression; we add the term if k is odd and subtract the term if k is even. The term we are adding or subtracting, $\left(\sum_{1\leq a_1<\cdots<a_k\leq n}\left|\bigcap_{j=1}^{k} A_{a_j}\right|\right)$, is a notation-heavy way of writing "add together the cardinalities of all possible intersections of k sets for $1 \leq k \leq n$".

Example 17. The number of length-4 lists whose elements are chosen from the set $\{1, 2, 3\}$ is $3^4 = 81$. We want to count the number of lists that use all the elements in $\{1, 2, 3\}$ at least once—we call these "good" lists.

Let U (for "Universe") be the set of all 81 lists. We define "bad" lists as those lists that are missing at least one element from $\{1, 2, 3\}$. If we count the number of bad lists, we can subtract them from 81 to obtain the number of good lists.

Let B_i be the set of bad lists that are missing the element i, where $1 \leq i \leq 3$. So, B_1 is the set of bad lists that are missing the element 1. This means that the lists in B_1 consist of the elements in $\{2, 3\}$. So, we have $|B_1| = 2^4 = 16$. Similarly, lists in B_2 consist of the elements $\{1, 3\}$, so we have $|B_2| = 2^4 = 16$. It should be clear that $|B_3| = 16$ as well.

We now count the lists in $B_1 \cap B_2$. These lists consist only of the element 3. So, we have $|B_1 \cap B_2| = 1$. Similarly, $|B_1 \cap B_3| = 1$ and $|B_2 \cap B_3| = 1$.

We now count the lists in $B_1 \cap B_2 \cap B_3$. These lists cannot have any elements from $\{1, 2, 3\}$. So, $B_1 \cap B_2 \cap B_3 = \emptyset$, and thus $|B_1 \cap B_2 \cap B_3| = 0$.

By Theorem 16, we have

$$\begin{aligned}
\#\text{ bad lists} = |B_1 \cup B_2 \cup B_3| = &|B_1| + |B_2| + |B_3| \\
&- |B_1 \cap B_2| - |B_1 \cap B_3| - |B_2 \cap B_3| \\
&+ |B_1 \cap B_2 \cap B_3| \\
&= 16 + 16 + 16 - 1 - 1 - 1 + 0 \\
&= 45.
\end{aligned}$$

So, we have

$$\#\text{ good lists} = |U| - \#\text{ bad lists} = 81 - 45 = 36.$$

Exercise

The number of length-5 lists whose elements are chosen from the set $\{1, 2, 3.4\}$ is $4^5 = 1024$. Count the number of lists that use all the elements in $\{1, 2, 3, 4\}$ at least once. Check your answer using the general formula given on the bottom of p. 113 of your textbook.

Definition 18 (Derangement)**.** A list of length n using the elements of $\{1, 2, 3, \ldots, n\}$ is called a derangement if the number j does not occupy position j of the list for any $j = 1, 2, 3, \ldots, n$.

Example 19. The derangements of $\{1, 2, 3\}$ are $(3, 1, 2)$, and $(2, 3, 1)$. Note that $(3, 2, 1)$ and $(2, 1, 3)$ are not derangements.

Optional Exercise

Counting derangements is another application of the inclusion-exclusion principle, covered in Examples 19.4 and 19.5 of your textbook.

Using inclusion-exclusion, find the number of derangements of $\{1, 2, 3, 4\}$. Check your answer with Example 19.5 on p. 114 and the formula at the bottom of p. 115.