Queen's University
School of Computing

CISC 203: Discrete Mathematics for Computing II
Module 7: Approximating Functions
Fall 2021

This module corresponds to the following sections from your textbook:

29. Assorted Notation

1 Big Oh

Definition 1 (Big Oh)**.** Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ and $g : \mathbb{N} \longrightarrow \mathbb{R}$. We say $f(n)$ is $\mathcal{O}(g(n))$ if there is a positive number C such that, with at most finitely many exceptions,

$$|f(n)| \leq C|g(n)|.$$

Example 2. Let $f(n) = n^2 + 5n + 6$. So, if we take $C = 3$ and $g(n) = n^2$, we have that $f(n)$ is $\mathcal{O}(n^2)$, since $|f(n)| \leq 3n^2$ for $n \geq 4$. In other words, $|f(n)|$ is no greater than $C|g(n)|$, except for a finite number of exceptions (i.e., when $0 \leq n \leq 3$). Note that we could have also found a different C that works for $n \geq 1$, e.g., by the technique as follows:

$$\begin{aligned} f(n) &= n^2 + 5n + 6 \\ &\leq n^2 + 5n^2 + 6n^2 = 12n^2 \end{aligned}$$

So, we have $|f(n)| \leq 12n^2$ for $n \geq 1$ and our conclusion is unchanged: $f(n)$ is $\mathcal{O}(n^2)$.

For the next example, we will need to use the Triangle Inequality.

Proposition 3 (Triangle inequality)**.** *Let* $a, b \in \mathbb{R}$. *Then,* $|a + b| \leq |a| + |b|$.

Proof. Try to prove this yourself by considering all possible cases such as when a and b are both positive, when one is positive and the other is negative, or when both are negative. Then you may refer to the proof for Proposition 29.4 on p. 205 in the textbook. □

Example 4. Let $f(n) = 2n^5 - 8n^3$. So for $n \geq 0$, we have

$$\begin{aligned} |f(n)| &= |2n^5 - 8n^3| = |2n^5 + (-8n^3)| \\ &\leq |2n^5| + |-8n^3| = 2n^5 + 8n^3 \\ &\leq 2n^5 + 8n^5 = 10n^5. \end{aligned}$$

So, $f(n)$ is $\mathcal{O}(n^5)$, since for $C = 10$ and $g(n) = n^5$ we have $|f(n)| \leq C|g(n)|$.

Example 5. Let $f(n) = 2^n - 10n^2 + 15$. So for $n \geq 5$, we have

$$\begin{aligned} |f(n)| &\leq 2^n + 10n^2 + 15 \\ &\leq 2^n + 10 \cdot 2^n + 2^n = 13 \cdot 2^n. \end{aligned}$$

Since $n^2 < 2^n$ when $n \geq 5$, as we proved already by the well-ordering principle in Module 4 (see also Proposition 21.11, p. 132). So, $f(n)$ is $\mathcal{O}(2^n)$, since for $C = 13$ and $g(n) = 2^n$ we have $|f(n)| \leq C|g(n)|$.

Exercise

Let $f(n) = \binom{n}{2}$ and suppose that $f(n)$ is $\mathcal{O}(g(n))$. Find $g(n)$, and check your answer against Example 29.2 on p. 205 of your textbook.

Note that some math books define $g(n)$ as a non-negative function (i.e., $g : \mathbb{N} \longrightarrow \mathbb{R}_+ \cup \{0\}$). In that case, the inequality $|f(n)| \leq C|g(n)|$ in Definition 1 becomes $|f(n)| \leq Cg(n)$. In computer science books, you may find that $f(n)$ is also defined as a non-negative function, since Big Oh notation is typically used to measure computational complexity of algorithms (time or space requirements would never be negative).

2 Ω and Θ

Definition 6 (Ω)**.** Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ and $g : \mathbb{N} \longrightarrow \mathbb{R}$. We say $f(n)$ is $\Omega(g(n))$ if there is a positive number C such that, with at most finitely many exceptions,

$$|f(n)| \geq C|g(n)|.$$

Note that $f(n)$ is $\mathcal{O}(g(n))$ if and only if $g(n)$ is $\Omega(f(n))$. This should be intuitive, since Big Oh notation establishes the upper bound on the growth of a function, and Ω notation establishes a lower bound on the growth of a function.

Example 7. Let $f(n) = n^2 - n$. We have, for $n \geq 2$,

$$\begin{aligned} |f(n)| &= |n^2 - n| = n^2 - n \\ &\geq n^2 - \frac{1}{2}n^2 = \frac{1}{2}n^2 \end{aligned}$$

So, $f(n)$ is $\Omega(n^2)$ since for $C = \frac{1}{2}$ and $g(n) = n^2$ we have $|f(n)| \geq C|g(n)|$ for $n \geq 2$. Note that (although it would be sloppy) it is also correct to say that $f(n)$ is $\Omega(n)$ or that $f(n)$ is $\mathcal{O}(n^3)$. However, it is **not** correct to say that $f(n)$ is $\Omega(n^3)$ or that $f(n)$ is $\mathcal{O}(n)$.

Definition 8 (Θ)**.** Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ and $g : \mathbb{N} \longrightarrow \mathbb{R}$. We say $f(n)$ is $\Theta(g(n))$ if there are positive numbers A and B such that, with at most finitely many exceptions,

$$A|g(n)| \leq |f(n)| \leq B|g(n)|.$$

Exercise

We already showed above that $f(n) = n^2 - n$ is $\Omega(n^2)$. Show that it is also $\mathcal{O}(n^2)$ and therefore is $\Theta(n^2)$.

3 Little Oh

We remarked in Example 10 that referring to $f(n) = n^2 - n$ as $\mathcal{O}(n^3)$ would be correct, but sloppy. We say this because $f(n)$ grows "much" slower than n^3. In this case, it would be more appropriate to say that $f(n)$ is $o(n^3)$. To determine precisely when this is appropriate, we define the Little Oh notation below.

Definition 9 (Little Oh)**.** Let $f : \mathbb{N} \longrightarrow \mathbb{R}$ and $g : \mathbb{N} \longrightarrow \mathbb{R}$. We say that $f(n)$ is $o(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

Example 10. Let $f(n) = n^2 - n$. We have,

$$\lim_{n\to\infty} \frac{n^2 - n}{n^3} = \lim_{n\to\infty} \frac{n^2(1 - \frac{1}{n})}{n^3} = \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{n} = 0.$$

So, that means $f(n)$ is $o(n^3)$.

Exercise

Let $f(n) = \sqrt{n}$. Show that $f(n)$ is $o(n)$. Compare your answer with Example 29.13 on p. 208 of your textbook.