Value at Risk, Expected Shortfall & Risk Management

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Classification of Financial Risk

- Credit risk
- Market risk
- Operational risk

Market Risk

We start with the market risk, because

- more high-quality data are available
- easier to understand
- the idea applicable to other types of risk

Three Most Common Risk Measures

- Volatility (σ)
- Value-at-Risk (VaR)
- Expected Shortfall (ES)
  (or Expected Tail Loss (ETL), conditional VaR (cVaR))

Value at Risk (VaR)

- Value at risk (VaR) has become the standard measure of market risk in risk management.
- Various methods for calculating VaR and the statistical theories behind these methods.
- Unconditional approach to VaR calculation for a financial position uses the historical returns of the instruments involved to compute VaR.
- Conditional approach uses the historical data and explanatory variables to calculate VaR.

What is Value at Risk (VaR)?

- Value at risk (VaR) is mainly concerned with market risk, but the concept is also applicable to other types of risk.
- A measure of **maximum** loss of a financial position within a certain period of time for a given (small) probability (from the viewpoint of the financial institution)
- Alternatively, VaR can be defined as the **minimal** loss under extraordinary market circumstances (from the viewpoint of a regulatory committee)

What is Value at Risk (VaR)?

- VaR is a point estimate of potential financial loss. It contains a certain degree of uncertainty.
- It also has a tendency to underestimate the actual loss if an extreme event actually occurs.
- Other risk measures such as expected shortfalls are also used.

A Formal Definition: VaR

A probabilistic framework

- Time period given : $\triangle t = l$
- $\triangle V(l)$ is the change in value of the underlying assets of the financial position from time t to $t + l$
- Loss in value : $L(l) = \triangle V(l)$ if holding a short position of the underlying assets; $L(l) = -\triangle V(l)$ if holding a long position of the underlying assets.

A Formal Definition: VaR

- CDF of the loss $L(l)$: $F_l(x)$
- Given (upper tail) probability: p, (say 5%)
- VaR is defined as

$$
\begin{aligned}
p &= Pr\left[L(l) \geq VaR\right] = 1 - Pr\left[L(l) < VaR\right] \\
1 - p &= F_l(VaR)
\end{aligned}
$$

which can be interpreted as

1. the probability that the position holder would encounter a loss greater than or equal to VaR over the time horizon l is p.
2. or, with probability $1 - p$, the potential loss encountered by the holder of the financial position over the time horizon l is less than VaR.

A Formal Definition: VaR

- Quantile: x_q is the $100qth$ quantile of the distribution $F_l(x)$ if

$$q = F_l(x_q), \quad i.e., \quad q = Pr(L(l) \leq x_q)$$

  and $F_l(\cdot)$ is continuous.

  For discrete distribution, we have

$$\begin{aligned} x_q &= \inf\{x | Pr(L(l) \leq x) \geq q\} \\ &= \inf\{x | F_l(x) \geq q\} \end{aligned}$$

- If the CDF of the loss $L(l)$, $F_l(x)$, is known, then VaR is simply the $100(1-p)th$ quantile of $F_l(x)$ (i.e. $VaR = x_{1-p}$).

Factors Affect VaR

1. The probability p.
2. The time horizon l.
3. The CDF $F_l(x)$ (or CDF of loss).
4. The mark-to-market value of the position.

- The CDF is unknown in practice .
- Studies of VaR are essentially concerned with estimation of the CDF and/or its quantile, especially the upper tail behavior of the loss CDF.
- Different methods for estimating the CDF give rise to different approaches to VaR calculation.

In what follows, we shall use log returns in the analysis (simple returns can also be used).
Why use log returns?

- log returns $\approx$ percentage changes.
- $VaR = Value \times VaR\ (of\ \ log\ \ return)$.

Methods Available For Market Risk

1. RiskMetrics
2. Econometric modeling
3. Empirical quantile
4. Traditional extreme value theory (EVT)
5. EVT based on exceedance over a high threshold

Data used in illustrations

Data used in Figure 7.1., Example 7.2., Example 7.3. : Daily log returns of IBM stock

- span: July 3, 1962 to Dec. 31, 1998.
- size: 9190 points
- Position: long on $10 million.
- For a long position, loss occurs at the left (or lower) tail of the returns. This is equivalent to using the right (or upper) tail if **negative** returns are used.

RiskMetrics

- It was developed by J.P. Morgan to VaR calculation.
- r_t given F_{t-1} : $N(0, \sigma_t^2)$
- σ_t^2 follows the special IGARCH(1,1) model

$$\sigma_t^2 = \alpha\sigma_{t-1}^2 + (1-\alpha)r_{t-1}^2 \quad 0 < \alpha < 1$$

- $VaR = 1.65\sigma_t$ if $p = 0.05$.
- k-horizon: $r_t[k]|F_{t-1} \sim N(0, k\sigma_t^2)$, therefore $VaR[k] = \sqrt{k}VaR$ (the square root of time rule).
- Pros: simplicity and transparence.
- Cons: model is not adequate. As security returns tend to have heavy tails (or fat tails), the normality assumption used often results in underestimation of VaR.

- Because RiskMetrics assumes log returns are normally distributed with mean zero, the loss function is **symmetric** and VaR are the same for long and short financial positions.
- This property does **NOT** hold for other methods to VaR calculation.

Model:

$$\begin{aligned} r_t &= a_t, \quad a_t = \sigma_t \epsilon_t, \\ \sigma_t^2 &= 0.9396\sigma_{t-1}^2 + (1-0.9396)a_{t-1}^2 \end{aligned}$$

- Because $r_{9190} = -0.0128$ and $\hat{\sigma}_{9190}^2 = 0.0003472$, $\hat{\sigma}_{9190}^2(1) = 0.9396 \times 0.0003472 + (1-0.9396) \times (-0.0128)^2 = 0.000336$.
- RiskMetrics : VaR are the same for long and short financial positions.

RiskMetrics : Example 7.2.

- For $p = 0.05$, VaR of
$r_{9191} = 1.65 \times \hat{\sigma}_{9190}(1) = 1.65 \times \sqrt{0.000336} = 0.03025.$

$$VaR = \$10,000,000 \times 0.03025 = \$302,500.$$

- For $p = 0.01$, VaR of
$r_{9191} = 2.3262 \times \hat{\sigma}_{9190}(1) = 2.3262 \times \sqrt{0.000336} = 0.04265.$

$$VaR = \$10,000,000 \times 0.04265 = \$426,500.$$

Desirable Properties for Risk Measures

Let $R(L)$ be the risk associated with loss L. From a theoretical point, $R(L)$ must possess the following basic properties:

1. Monotonicity: If $L_1 \leq L_2$ for all possible outcomes, then $R(L_1) \leq R(L_2)$.
2. Sub-additivity: $R(L_1 + L_2) \leq R(L_1) + R(L_2)$ for any two portfolios.
3. Positive homogeneity: $R(hL) = hR(L)$, where $h > 0$.
4. Translation invariance: $R(L + a) = R(L) + a$, where a is a positive real number.

Desirable Properties for Risk Measures

- The sub-additivity is associated with risk diversification. The equality holds when the two portfolios are perfectly positively correlated.
- A risk measure is called coherent if it satisfies the above four properties.

Desirable Properties for Risk Measures

If the loss involved is normally distributed, then VaR is a coherent risk measure. The sub-additivity can be seen because

- $L_1 \sim N(\mu_1, \sigma_1^2)$, $L_2 \sim N(\mu_2, \sigma_2^2)$, the correlation between L_1 and L_2 is ρ.
- $VaR(L_1) = \mu_1 + X_{1-p}\sigma_1$, $VaR(L_2) = \mu_2 + X_{1-p}\sigma_2$.

$$\begin{aligned} VaR(L_1 + L_2) &= \mu_1 + \mu_2 + X_{1-p}\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} \\ &\leq \mu_1 + \mu_2 + X_{1-p}(\sigma_1 + \sigma_2) \\ &= VaR(L_1) + VaR(L_2) \end{aligned}$$

Desirable Properties for Risk Measures

- However, VaR fails to meet the sub-additive property under certain conditions.
- This is the reason that we shall also discuss expected shortfall or conditional VaR (CVaR), which is a **coherent** risk measure.
- Expected shortfall is the expected loss when the VaR is exceeded.
- ES is also called Tail VaR (TVaR) or expected tail loss (ETL).
- In the insurance literature, expected shortfall is called Conditional Tail Expectation or Tail Conditional Expectation (TCE).

Example of Incoherent VaR

Let Z denote a continuous loss random variable with the following CDF values:

$$F_Z(1) = 0.91, \quad F_Z(90) = 0.95, \quad F_Z(100) = 0.96$$

It is clear that $VaR_{.95}(Z) = 90$.

Now, define loss variables X and Y such that $Z = X + Y$, where

$$\begin{aligned} X &= \begin{cases} Z, & \textit{if } Z \leq 100 \\ 0, & \textit{if } Z > 100, \end{cases} \\ Y &= \begin{cases} 0, & \textit{if } Z \leq 100 \\ Z, & \textit{if } Z > 100, \end{cases} \end{aligned}$$

Example of incoherent VaR

The CDF of X satisfies

$$\begin{aligned} F_X(1) &= Pr(X \leq 1) = 0.91 + 0.04 = 0.95 \\ F_X(90) &= 0.95 + 0.04 = 0.99 \\ F_X(100) &= 1 \end{aligned}$$

Therefore, $VaR_{.95}(X) = 1$.

The CDF of Y satisfies $F_Y(0) = 0.96$ so that $VaR_{.95}(Y) = 0$. Consequently,

$$VaR_{.95}(X) + VaR_{.95}(Y) = 1 < VaR_{.95}(Z).$$

When does VaR Violate Subadditivity?

- When the tails of assets are super fat!
- When assets are subject to occasional very large returns
  - Exchange rates in countries that peg currency but are subject to occasional large devaluations
  - Electricity prices subject to occasional large price swings
  - Defaultable bonds when most of the time the bonds deliver a steady positive return but may occasionally default
  - Protection seller portfolios - portfolios that earn a small positive amount with high probability but suffer large losses with small probabilities (carry trades, short options, insurance contracts)

Expected Shortfall

- From the prior discussion, VaR is simply the $100(1-p)th$ quantile of the loss function, where p is the upper tail probability.
- When an extreme loss occurs, i.e. VaR is exceeded, the actual loss can be much higher than VaR.
- To better quantify the loss and to employ a coherent risk measure, we consider the expected loss once the VaR is exceeded.

Expected Shortfall: A Formal Definition

Let $q = 1 - p$. The expected shortfall (ES) is then

$$ES_q = E(L(l)|L(l) > VaR_q)$$

Expected Shortfall : Standard Normal Distribution

For standard normal distribution, the conditional density

$$f(x|L > VaR_q) = \frac{\phi(x)}{p}, \quad x > VaR_q$$

where $\phi(x)$ is the density function of the standard normal distribution.

Since $\phi'(x) = -x\phi(x)$,

$$ES_q = E(L(l)|L(l) > VaR_q) = -\int_{VaR_q}^{\infty} \frac{\phi'(x)}{p} dx = \frac{\phi(VaR_q)}{p}$$

- Consequently, the ES for a normal return $N(0, \sigma_t^2)$ is

$$ES_q = \frac{\phi(VaR_q)}{p} \times \sigma_t$$

where σ_t is the volatility and $\phi(x)$ denotes the density function of $N(0,1)$. Therefore, for a $N(0, \sigma_t^2)$ loss function, we have $ES_{0.99} = 2.6652\sigma_t$.

- In general, for a normal distribution $N(\mu, \sigma_t^2)$, the ES is

$$ES_q = \mu + \frac{\phi(VaR_q)}{p} \times \sigma_t$$

Expected Shortfall

Expected shortfall can also be defined as the average VaR for small tail probabilities, i.e.

$$ES_{1-p} = \frac{1}{p} \int_0^p VaR_{1-u} du$$

Econometric Models to VaR Calculations

- $r_t = \mu_t + \sigma_t \epsilon_t$ given F_{t-1}
- μ_t : a mean equation (Chapter 2)
- σ_t^2 : a volatility model (Chapter 3 or 4)
- Pros: sound theory
- Cons: a bit complicated

Econometric Models to VaR Calculations : Gaussian Distribution

- $\hat{r}_t(1) = E(r_{t+1}|F_t) = \mu_{t+1}$.
- $\hat{\sigma}_t^2(1) = E(\sigma_{t+1}^2|F_t)$.
- If one further assumes that ϵ_t is Gaussian, then the conditional distribution of r_{t+1} given the information available at time t is $N(\hat{r}_t(1), \hat{\sigma}_t^2(1))$.
- For $p = 0.05$, VaR of $r_{t+1} = \hat{r}_t(1) + 1.65\hat{\sigma}_t(1)$.
- For $p = 0.05$, ES of $r_{t+1} = \hat{r}_t(1) + \frac{\phi(VaR_{0.95})}{0.05} \times \hat{\sigma}_t(1)$. where VaR_q is the qth quantile of the standard normal distribution.

- If one assumes that ϵ_t is a standardized Student-t distribution with v degrees of freedom,
  VaR of $r_{t+1} = \hat{r}_t(1) + t_v^*(1-p)\hat{\sigma}_t(1)$, where $t_v^*(1-p)$ is the $(1-p)th$ quantile of a standardized Student-t distribution with v degrees of freedom.
- t_v^* denotes a standardized Student-t distribution with v degrees of freedom, and t_v denotes a Student-t distribution with v degrees of freedom.

$$
\begin{aligned}
t_v^* &= \frac{t_v}{\sqrt{v/(v-2)}} \\
t_v^*(1-p) &= \frac{t_v(1-p)}{\sqrt{v/(v-2)}}
\end{aligned}
$$

where $t_v(1-p)$ is the $(1-p)th$ quantile of a Student-t distribution with v degrees of freedom.

If ϵ_t of the GARCH model is a standardized Student-t distribution with v degrees of freedom, and the upper tail probability is p, VaR of r_{t+1} is

$$\hat{r}_t(1) + \frac{t_v(1-p)\hat{\sigma}_t(1)}{\sqrt{v/(v-2)}}$$

- We use two volatility models to calculate VaR of 1-day horizon at $t = 9190$ for a long position of $10 million.
- Because the position is long, we use $r_t = -r_t^c$, where r_t^c is the usual log return of IBM stock shown in Figure 7.1. In other words, we fit econometric models to $-r_t^c$.

CASE 1. Assume that ϵ_t is standard normal. The fitted model is

$$\begin{aligned} r_t &= -0.00066 - 0.0247 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t, \\ \sigma_t^2 &= 0.00000389 + 0.9073 \sigma_{t-1}^2 + 0.0799 a_{t-1}^2 \end{aligned}$$

From the data, we have $r_{9189} = 0.00201$, $r_{9190} = 0.0128$, and $\sigma_{9190}^2 = 0.00033455$.

- We have $\hat{r}_{9190}(1) = -0.00071$ and $\hat{\sigma}_{9190}^2(1) = 0.0003211$.
- If $p = 0.05$, then
  $VaR\ of\ r_{9191} = -0.00071 + 1.6449 \times \sqrt{0.0003211} = 0.02877$.
  $VaR = \$10,000,000 \times 0.02877 = \$287,700$.
- If $p = 0.01$, then
  $VaR\ of\ r_{9191} = -0.00071 + 2.3262 \times \sqrt{0.0003211} = 0.0409738$.
  $VaR = \$10,000,000 \times 0.0409738 = \$409,738$.

CASE 2. Assume that ϵ_t is a standardized Student-t distribution with 5 degrees of freedom. The fitted model is

$$\begin{aligned} r_t &= -0.0003 - 0.0335 r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t, \\ \sigma_t^2 &= 0.000003 + 0.9350 \sigma_{t-1}^2 + 0.0559 a_{t-1}^2 \end{aligned}$$

$r_{9189} = 0.00201$, $r_{9190} = 0.0128$, and $\sigma_{9190}^2 = 0.000349$.

- We have $\hat{r}_{9190}(1) = -0.000367$ and $\hat{\sigma}_{9190}^2(1) = 0.0003386$.
- If $p = 0.05$, then
  $VaR \text{ of } r_{9191} = -0.000367 + \frac{2.015}{\sqrt{5/3}} \times \sqrt{0.0003386} = 0.028354.$
  $VaR = \$10,000,000 \times 0.028354 = \$283,540.$
- If $p = 0.01$, then $VaR \text{ of } r_{9191} =$
  $-0.000367 + \frac{3.3649}{\sqrt{5/3}} \times \sqrt{0.0003386} = 0.0475943.$
  $VaR = \$10,000,000 \times 0.0475943 = \$475,943.$

Comparing with that of Case 1, we see the heavy-tail effect of using a Student-t distribution with 5 degrees of freedom; it increases the VaR when the tail probability becomes smaller.

Assignment 3

- Tsay : 7.1.(a) (b) (c) ; 7.2. (a) (b) (c);
- Due March 27 in class