Times Series and Forecasting (X)

Chapter 10. Seasonal Models

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10.1. Introduction: seasonal models

- In this chapter, we introduce extensions of the ARIMA family which account for seasonal behavior.
- With seasonal data, the dependence in the past tends to occur most strongly at multiples of some underlying seasonal lag, denote by s.

Seasonal models

- Consider the following examples:
  - With monthly data, there is a strong yearly component occurring at lags that are multiples of $s = 12$.
  - With quarterly data, there are patterns which repeat every $s = 4$ observation times.

Seasonal models

- Many physical, biological, and economic processes tend to elicit seasonal patterns over time.
- We aim to study new time series models which can account explicitly for these types of patterns.
- This new class of models is called the class of stochastic seasonal ARIMA models.

Example 10.1. U.S. milk production

Example

The data in Figure 10.1 (left) represent monthly U.S. milk production (measured in millions of pounds) from January, 1994 to December, 2005.

Figure 10.1. U.S. milk production

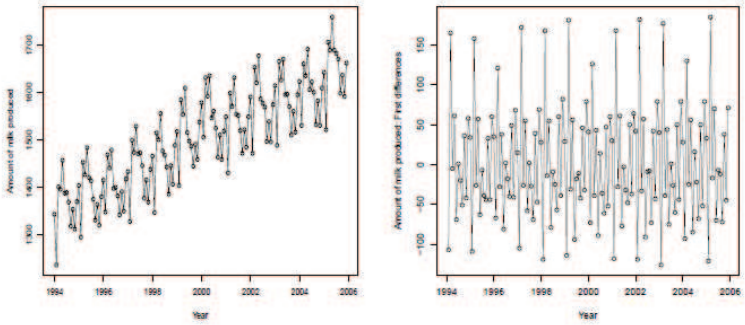


Figure 10.1: *Left: US milk production data. Monthly production, measured in billions of pounds, from January, 1994 to December, 2005. Right: First differences* $\nabla Y_t = Y_t - Y_{t-1}$.

Two trends in U.S. milk production data

- From the plot of the original series, we see that there are two general types of trend: an upward linear trend (over the years) and a seasonal trend (within years).
- We know that the upward linear trend could be "removed" by taking first differences, that is, by computing $\nabla Y_t = Y_t - Y_{t-1}$. Figure 10.1 (right) displays the series of first differences.
- From this plot, it is clear that the upward linear trend over time has been removed (i.e., the first differences look stationary in the mean level).

Seasonal trends in U.S. milk production data

- However, the first difference process ∇Y_t still displays a seasonal pattern within years.
- How can we "remove" this type of trend?
- This is the topic that is investigated in this Chapter.

10.2. Purely seasonal (stationary) ARMA models

- A seasonal moving average model of order q with seasonal period s, denoted by MA$(q)_s$, is

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \cdots - \Theta_q e_{t-qs}.$$

- Let us review a moving average model of order q, denoted by MA(q),

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}.$$

Seasonal MA$(1)_{12}$

- A seasonal moving average model of order $q = 1$ with seasonal lag $s = 12$; i.e., MA$(1)_{12}$, is given by

$$Y_t = e_t - \Theta e_{t-12}.$$

- For this process, note that $\mathrm{E}(Y_t) = 0$ and

$$\gamma_k = \begin{cases} \sigma_e^2(1+\Theta^2), & k = 0 \\ -\Theta\sigma_e^2, & k = 12 \text{ or } k = s \\ 0, & \text{otherwise.} \end{cases}$$

- Because $\mathrm{E}(Y_t) = 0$ and γ_k are both free of t, the MA$(1)_{12}$ process is stationary.

ACF of seasonal MA$(1)_{12}$

- The ACF for the MA$(1)_{12}$ process is given by

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1, & k = 0 \\ -\Theta/(1+\Theta^2), & k = 12 \\ 0, & \text{otherwise.} \end{cases}$$

- This ACF is almost identical to the ACF for the usual (i.e., nonseasonal) MA(1) process.
- The difference is that the only nonzero autocorrelation occurs at the first seasonal lag $k = 12$, as opposed to $k = 1$ in the nonseasonal MA(1).

A seasonal MA$(1)_{12}$ is a special nonseasonal MA(12)

- A seasonal MA$(1)_{12}$ process is the same as a nonseasonal MA(12) process with

$$\theta_1 = \theta_2 = \cdots = \theta_{11} = 0,$$

and $\theta_{12} = \Theta$.

- Because of this equivalence, we can use our already-established methods to fit and diagnose seasonal models (as we shall soon see).

Example 10.2. Simulated $\text{MA}(1)_{12}$

Example

We use SAS to simulate four different $\text{MA}(1)_{12}$ processes

$$Y_t = e_t - \Theta e_{t-12},$$

with $n = 200$, where $e_t \sim \text{iid} N(0, 1)$, using the parameters

- $\Theta = -0.9$ (upper left),
- $\Theta = 0.9$ (upper right),
- $\Theta = -0.3$ (lower left), and
- $\Theta = 0.3$ (lower right).

Figure 10.2. Time plot of the simulated $MA(1)_{12}$

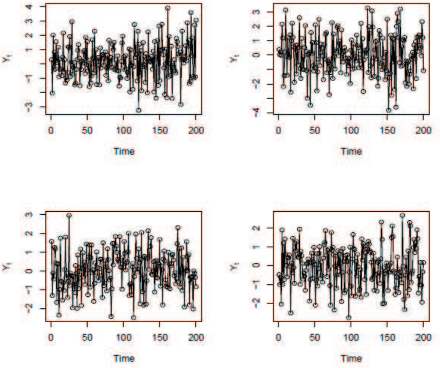


Figure 10.2: *Four* $MA(1)_{12}$ *simulations with* $n = 200$ *and* $\sigma_e^2 = 1$. *Upper left:* $\Theta = -0.9$. *Upper right:* $\Theta = 0.9$. *Lower left:* $\Theta = -0.3$. *Lower right:* $\Theta = 0.3$.

Example 10.2. Simulated MA$(1)_{12}$

- The simulations are in depicted in Figure 10.2 and the sample ACFs are in Figure 10.3.
- The ACFs illustrate clearly the main feature of the MA$(1)_{12}$ process, namely, observations 12 units apart in time are correlated, whereas all other observations are not.

Figure 10.3. ACFs of the simulated MA$(1)_{12}$

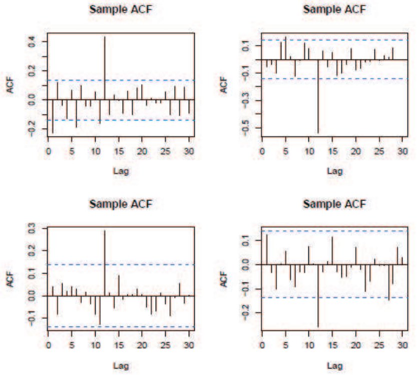


Figure 10.3: *Sample ACFs for MA$(1)_{12}$ simulations in Figure 10.2. Upper left:* $\Theta = -0.9$. *Upper right:* $\Theta = 0.9$. *Lower left:* $\Theta = -0.3$. *Lower right:* $\Theta = 0.3$.

Seasonal MA$(2)_s$

- A seasonal moving average model of order $q = 2$ with seasonal lag $s = 12$ is given by

$$Y_t = e_t - \Theta_1 e_{t-12} - \Theta_2 e_{t-24}.$$

- For this process, it is easy to show that $\mathsf{E}(Y_t) = 0$ and that

$$\gamma_k = \begin{cases} \sigma_e^2(1 + \Theta_1^2 + \Theta_2^2), & k = 0 \\ (-\Theta_1 + \Theta_1\Theta_2)\sigma_e^2, & k = 12 \\ -\Theta_2\sigma_e^2, & k = 24 \\ 0, & \text{otherwise.} \end{cases}$$

Seasonal MA$(2)_s$

- So the MA$(2)_{12}$ process is stationary, and

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1, & k = 0 \\ \frac{-\Theta_1 + \Theta_1\Theta_2}{1+\Theta_1^2+\Theta_2^2}, & k = 12 \\ \frac{-\Theta_2}{1+\Theta_1^2+\Theta_2^2}, & k = 24 \\ 0, & \text{otherwise.} \end{cases}$$

ACF of seasonal $\text{MA}(2)_{12}$

- The ACF for an $\text{MA}(2)_{12}$ process is almost identical to the ACF for the usual (i.e., nonseasonal) MA(2) process.
- The only difference is that nonzero autocovariances occur at the first two seasonal lags $k = 12$ and $k = 24$, as opposed to $k = 1$ and $k = 2$ in the nonseasonal MA(2).
- A seasonal $\text{MA}(2)_{12}$ process is the same as a nonseasonal MA(24) process with

$$\theta_1 = \theta_2 = \cdots = \theta_{11} = \theta_{13} = \theta_{14} = \cdots = \theta_{23} = 0,$$

$\theta_{12} = \Theta_1$, and $\theta_{24} = \Theta_2$.

Backshift expression of the seasonal MA(q)$_s$

- A seasonal MA$(q)_s$ process

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \cdots - \Theta_q e_{t-qs}$$

can be expressed as

$$Y_t = \Theta_q(B^s)e_t,$$

where $\Theta_q(B^s) = 1 - \Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_q B^{qs}$ is the seasonal MA characteristic operator.

Properties of the seasonal MA(q)$_s$

- As with nonseasonal models, a seasonal MA$(q)_s$ process is invertible if and only if all of the roots of $\Theta_q(B^s)$ are greater than one in absolute value (or modulus).
- All seasonal MA$(q)_s$ processes are stationary.

Seasonal $AR(p)_s$

- A seasonal autoregressive model of order P with seasonal period s, denoted by AR$(P)_s$, is

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \cdots + \Phi_P Y_{t-Ps} + e_t.$$

- Let us review Aa autoregressive model of order p, denoted by AR(p),

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

Seasonal AR$(1)_{12}$

- A seasonal autoregressive model of order $P = 1$ with seasonal lag $s = 12$; i.e., AR$(1)_{12}$, is given by

$$Y_t = \Phi Y_{t-12} + e_t.$$

- Similar to a nonseasonal AR(1) process, the seasonal AR$(1)_{12}$ process is stationary if and only if $-1 < \Phi < 1$.

Seasonal AR$(1)_{12}$

- We have $\mathsf{E}(Y_t) = 0$ and

$$\mathsf{var}(Y_t) = \gamma_0 = \frac{\sigma_e^2}{1-\Phi^2}$$

and

$$\rho_k = \begin{cases} \Phi^{k/12}, & k = 0, 12, 24, 36, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

- A seasonal AR$(1)_{12}$ process is the same as a nonseasonal AR(12) process with $\phi_1 = \phi_2 = \cdots = \phi_{11} = 0$ and $\phi_{12} = \Phi$.

Example 10.3. Simulated $\mathrm{AR}(1)_{12}$

Example

Use SAS to simulate four different $\mathrm{AR}(1)_{12}$ processes

$$Y_t = \Phi Y_{t-12} + e_t,$$

with $n = 200$, where $e_t \sim \text{iid} N(0, 1)$, using

- $\Phi = 0.9$ (upper left),
- $\Phi = -0.9$ (upper right),
- $\Phi = 0.3$ (lower left), and
- $\Phi = -0.3$ (lower right).

Figure 10.4. Time plot of four simulated $AR(1)_{12}$

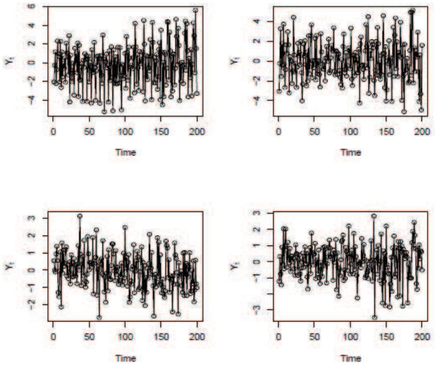


Figure 10.4: *Four* $AR(1)_{12}$ *simulations with* $n = 200$ *and* $\sigma_e^2 = 1$. *Upper left:* $\Phi = -0.9$. *Upper right:* $\Phi = 0.9$. *Lower left:* $\Phi = -0.3$. *Lower right:* $\Phi = 0.3$.

Figure 10.5. ACFs of four simulated $AR(1)_{12}$

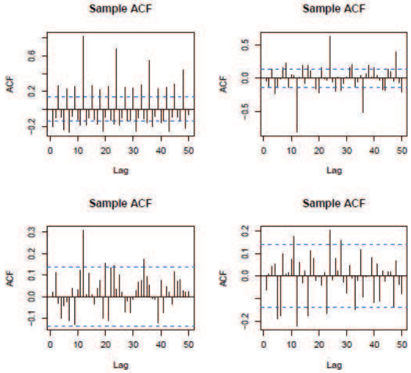


Figure 10.5: *Sample ACFs for* $AR(1)_{12}$ *simulations in Figure 10.4. Upper left:* $\Phi = 0.9$. *Upper right:* $\Phi = -0.9$. *Lower left:* $\Phi = 0.3$. *Lower right:* $\Phi = -0.3$.

Figure 10.6. PACFs of four simulated $AR(1)_{12}$

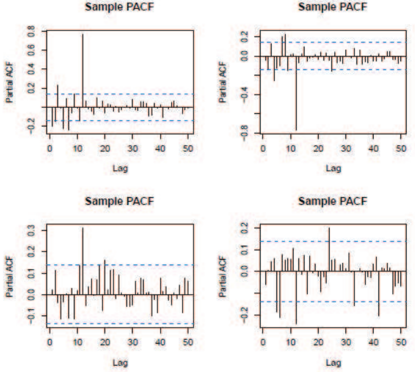


Figure 10.6: *Sample PACFs for* $AR(1)_{12}$ *simulations in Figure 10.2. Upper left:* $\Phi = 0.9$. *Upper right:* $\Phi = -0.9$. *Lower left:* $\Phi = 0.3$. *Lower right:* $\Phi = -0.3$.

Example 10.2 (continued)

- The simulations are in depicted in Figure 10.4 and the sample ACFs are in Figure 10.5.
- When $\Phi = 0.9$, the ACFs at seasonal lags $k = 12, 24, 36, \cdots$, exhibits a slow decay.
- When $\Phi = -0.9$, the ACF alternates from negative to positive at the seasonal lags.
- When $\Phi = \pm 0.3$, the pattern is bit more muddled, but it is safe to say that the ACFs display patterns unlike those we saw in nonseasonal sample ACFs. The sample PACFs are depicted in Figure 10.6.
- When $\Phi = \pm 0.9$ and when $\Phi = 0.3$, we see a pronounced spike at the first seasonal lag, $k = s = 12$, and little elsewhere. When $\Phi = -0.3$, the picture is less clear.

Seasonal $\text{AR}(2)_{12}$

- A seasonal autoregressive model of order $P = 2$ with seasonal lag $s = 12$; i.e., $\text{AR}(2)_{12}$, is given by

$$Y_t = \Phi_1 Y_{t-12} + \Phi_2 Y_{t-24} + e_t.$$

- The seasonal $\text{AR}(2)_{12}$ behaves like the nonseasonal AR(2) at the seasonal lags.
- The ACF ρ_k displays exponential decay or damped sinusoidal patterns across the seasonal lags $k = 12, 24, 36, \cdots$.
- A seasonal $\text{AR}(2)_{12}$ process is the same as a nonseasonal AR(24) process with

$$\phi_1 = \phi_2 = \cdots = \phi_{11} = \phi_{13} = \phi_{14} = \cdots = \phi_{23} = 0,$$

$\phi_{12} = \Phi_1$, and $\phi_{24} = \Phi_2$.

Backshift expression of A seasonal AR$(P)_s$

- A seasonal AR$(P)_s$ process

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \cdots + \Phi_P Y_{t-Ps} + e_t$$

can be expressed as

$$\Phi_p(B^s) Y_t = e_t,$$

where $\Phi_p(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_p B^{ps}$ is the seasonal AR characteristic operator.

Properties of a seasonal AR$(P)_s$

- A seasonal AR$(P)_s$ process is stationary if and only if all of the roots of $\Phi_p(B^s)$ are greater than one in absolute value (or modulus).
- All seasonal AR$(P)_s$ processes are invertible.

A seasonal ARMA(P,Q)$_s$

- A seasonal autoregressive moving average model of orders P and Q with seasonal period s, denoted by ARMA$(P,Q)_s$, is

$$Y_t = \Phi_1 Y_{t-s} + \cdots + \Phi_P Y_{t-Ps} + e_t - \Theta_1 e_{t-s} - \cdots - \Theta_Q e_{t-Qs}.$$

- This is the seasonal analogue of the unseasonal ARMA(P,Q) process with nonzero autocorrelations only at lags $k = s$, $2s, 3s, \cdots$. Using backshift notation, this model can be expressed as

$$\Phi_p(B^s) Y_t = \Theta_q(B^s) e_t,$$

where the seasonal AR and MA characteristic operators are

$$\Phi_p(B^s) = 1 - \Phi_1 B^s - \cdots - \Phi_p B^{ps}$$
$$\Theta_q(B^s) = 1 - \Theta_1 B^s - \cdots - \Theta_q B^{qs}.$$

Conditions for stationarity and invertible ARMA(p,q)$_s$

- Analogous to the unseasonal ARMA(p, q) process,
  - the ARMA$(P, Q)_s$ is stationary when the roots of $\Phi_P(B^s)$ all exceed 1 in absolute value (modulus) and
  - the ARMA$(P, Q)_s$ is invertible when the roots of $\Theta_Q(B^s)$ all exceed 1 in absolute value (modulus).

ARMA(p,q)$_s$ is a special ARMA(Ps, Qs)

- Mathematically, a seasonal ARMA$(P, Q)_s$ process is the same as a nonseasonal ARMA(Ps, Qs) process with

$$\phi_s = \Phi_1, \cdots, \phi_{ps} = \Phi_p, \theta_s = \Theta_1, \cdots, \theta_{qs} = \Theta_q,$$

and all other ϕ and θ parameters equal to 0.

Comparison

The following table succinctly summarizes the behavior of the (theoretical) ACF and PACF for ARMA(p, q)s processes.

	AR$(p)_s$	MA$(q)_s$	ARMA$(p, q)_s$
ACF	Tails off at lags ks $k = 1, 2, \cdots,$	Cuts off after lag Qs	Tails off at lags ks $k = 1, 2, \cdots, s$
PACF	Cuts off after lag Ps	Tails off at lags ks $k = 1, 2, \cdots,$	Tails off at lags ks $k = 1, 2, \cdots,$

Summary:

In this section,

- the stationary ARMA(p, q) models
- are extended to incorporate
- the same type of ARMA(p, q) behavior at seasonal lags $k = s$, $2s$, $3s$, $\cdots$, the so-called the seasonal ARMA$(p, q)_s$ models.

Summary (continued):

- This "extension" is not that much of an extension, because the seasonal ARMA$(p, q)_s$ model is essentially an ARMA(p, q) model applied at the seasonal lags $k = s, 2s, 3s, \cdots,$.
- That is, this model, which incorporates autocorrelation only at the seasonal lags and nowhere else, may be somewhat limited in application.

10.3. Multiplicative seasonal ARMA models

- Combining seasonal ARMA$(p, q)_s$ with nonseasonal ARMA(p, q) models can create a larger class of models applicable for use with seasonal stationary processes.
- This new class of models is called the multiplicative seasonal ARMA class.

Combination of a MA(1) and a MA$(1)_{12}$

- Consider the two models

$$Y_t = e_t - \theta e_{t-1} \Longleftrightarrow Y_t = (1 - \theta B)e_t$$

and

$$Y_t = e_t - \Theta e_{t-12} \Longleftrightarrow Y_t = (1 - \Theta B^{12})e_t,$$

a nonseasonal MA(1) and a seasonal MA$(1)_{12}$, respectively.

  - The defining characteristic of the nonseasonal MA(1) is that the only nonzero autocorrelation occurs at lag $k = 1$.
  - The defining characteristic of the seasonal MA$(1)_{12}$ is that the only nonzero autocorrelation occurs at lag $k = 12$.

A multiplicative seasonal MA(1)×MA(1)$_{12}$ process

- Consider combining the two models to form

$$Y_t = (1 - \theta B)\left(1 - \Theta B^{12}\right) e_t,$$

yielding

$$Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}.$$

- It is called a multiplicative seasonal MA(1)×MA(1)$_{12}$ process.
- The term "multiplicative" arises because the MA characteristic polynomial $(1 - \theta B)\left(1 - \Theta B^{12}\right)$ is the product of $(1 - \theta B)$ and $\left(1 - \Theta B^{12}\right)$.

ACF of the multiplicative seasonal MA(1)×MA(1)$_{12}$ process

- It is easy to see that $\mathsf{E}(Y_t) = 0$ and it can be shown that

$$\rho_1 = -\frac{\theta}{1+\theta^2}, \quad \rho_{11} = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}$$
$$\rho_{12} = -\frac{\Theta}{1+\Theta^2}, \quad \rho_{13} = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}.$$

Remarks for the MA(1)×MA(1)$_{12}$ process

- This process has nonzero autocorrelation at lags $k = 1$ and $k = 12$, from the nonseasonal and seasonal MA models, individually, and additional nonzero autocorrelation at lags $k = 11$ and $k = 13$ which arises from the multiplicative effect of the two models.

- A multiplicative seasonal MA(1)×MA(1)$_{12}$ process

$$Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13},$$

is essentially an MA(13) process with $\theta_1 = \theta$, $\theta_2 = \theta_3 = \cdots = \theta_{11} = 0$, $\theta_{12} = \Theta$, and $\theta_{13} = -\theta\Theta$.

Combination of a MA(1) and a AR$(1)_{12}$

- Consider the two models

$$Y_t = e_t - \theta e_{t-1} \Longleftrightarrow Y_t = (1 - \theta B)e_t$$

and

$$Y_t = \Theta Y_{t-12} + e_t \Longleftrightarrow (1 - \Phi B^{12})Y_t = e_t,$$

a nonseasonal MA(1) and a seasonal AR$(1)_{12}$, respectively.

  - The defining characteristic of the nonseasonal MA(1) is that the only nonzero autocorrelation occurs at lag $k = 1$.
  - The defining characteristic of the seasonal AR$(1)_{12}$ is that the autocorrelation decays across seasonal lags $k = 12, 24, 36, \cdots ,$.

A multiplicative seasonal MA(1)×AR(1)$_{12}$ process

- Consider combining the two models to form

$$\left(1 - \Phi B^{12}\right) Y_t = \left(1 - \theta B\right) e_t,$$

yielding

$$Y_t = \Phi Y_{t-12} + e_t - \theta e_{t-1}.$$

- It is called a multiplicative seasonal MA(1)×AR(1)$_{12}$ process.

A multiplicative seasonal MA(1)×AR$(1)_{12}$ process

- Combining a nonseasonal MA(1) with a seasonal AR$(1)_{12}$ creates a new process which possesses AR-type autocorrelation at seasonal lags $k = 12, 24, 36, \cdots$, and additional MA-type autocorrelation at lag $k = 1$ and at lags one unit in time apart from the seasonal lags.

- A multiplicative seasonal MA(1)×AR$(1)_{12}$ process

$$Y_t = \Phi Y_{t-12} + e_t - \theta e_{t-1},$$

  is essentially an ARMA(12,1) process with $\theta = \theta$, $\phi_i = 0$ for $i = 1, \cdots, 11$, and $\phi_{12} = \Phi$.

- See sas programming example10-multiplicative.sas for a multiplicative seasonal MA(1)×AR$(1)_{12}$ process with parameter $(\Phi = -0.7, \theta = 0.8)$.

Figure 10.7. ACF and PACF of a MA(1)$\times$AR(1)$_{12}$ process

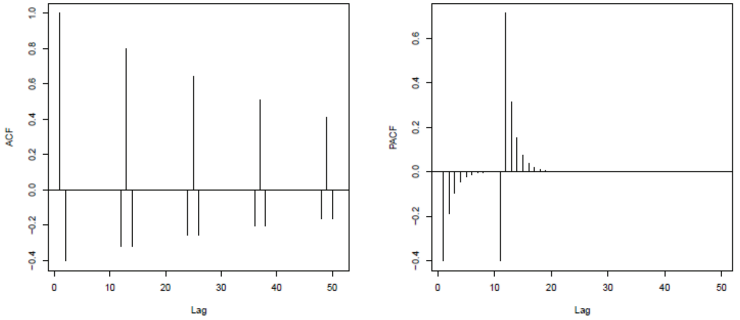


Figure 10.7: *Left: Theoretical ACF for* $MA(1) \times AR(1)_{12}$. *Note that* $\rho_0 = 1$ *is included. Right: Theoretical PACF for* $MA(1) \times AR(1)_{12}$.

A multiplicative seasonal AR(1)×MA(1)$_{12}$ process

- Consider combining the two models to form

$$(1 - \Phi B)\, Y_t = \left(1 - \theta B^{12}\right) e_t,$$

yielding

$$Y_t = \Phi Y_{t-1} + e_t - \theta e_{t-12}.$$

- It is called a multiplicative seasonal AR(1)×MA(1)$_{12}$ process.

A multiplicative seasonal AR(1)×MA(1)$_{12}$ process

- Combining a nonseasonal AR(1) with a seasonal MA(1)$_{12}$ creates a new process which has MA-type autocorrelation at seasonal lags $k = 12, 24, 36, \cdots$, and additional AR-type autocorrelation at lag $k = 1$ and at lags one unit in time apart from the seasonal lags.

- A multiplicative seasonal AR(1)×MA(1)$_{12}$ process

$$Y_t = \Phi Y_{t-1} + e_t - \theta e_{t-12},$$

  is essentially an ARMA(1,12) process with $\Phi = \phi$, $\theta_i = 0$ for $i = 1, \cdots, 11$, and $\theta_{12} = \Theta$.

- See sas programming example10-multiplicative-1.sas for a multiplicative seasonal AR(1)×MA(1)$_{12}$ process with parameter $(\phi = 0.4, \Theta = 0.8)$.

A multiplicative seasonal ARMA$(p,q)\times$ARMA$(P,Q)_s$

- In general, we can combine a nonseasonal ARMA(p,q) process

$$\phi(B)Y_t = \theta(B)e_t$$

with a seasonal ARMA$(P,Q)_s$ process

$$\Phi_P(B^s)Y_t = \Theta_Q(B^s)e_t$$

and write

$$\phi(B)\Phi_P(B^s)Y_t = \theta(B)\Theta_Q(B^s)e_t,$$

called a multiplicative seasonal ARMA$(p,q)\times$ARMA$(P,Q)_s$ model with seasonal period s.

About ARMA$(p, q)\times$ARMA$(P, Q)_s$ model

- The multiplicative seasonal ARMA$(p, q)\times$ARMA$(P, Q)_s$ model

$$\phi(B)\Phi_P(B^s)Y_t = \theta(B)\Theta_Q(B^s)e_t.$$

  is a large ARMA(p, q) model with
  - AR characteristic operator $\phi^*(B) = \phi(B)\Phi_P(B^s)$ and
  - MA characteristic operator $\theta^*(B) = \theta(B)\Theta_Q(B^s)$.

- Therefore, stationarity and invertibility conditions are in terms of the roots of $\phi^*(B)$ and $\theta^*(B)$, respectively.

Differencing in Proc ARIMA procedure

- To take the 1st difference of the series A, use the statement

  identify var=Y(1), i.e. $Y(1) = Y_t - Y_{t-1} = \nabla Y_t.$

- If X is a monthly series, the statement

  identify var=Y(12), i.e. $Y(12) = Y_t - Y_{t-12} = \nabla_{12} Y_t.$

- To take 2nd difference of the series A, use the statement

  identify var=Y(1,1), i.e. $Y(1,1) = \nabla Y_t - \nabla Y_{t-1} = \nabla^2 Y_t.$

- How about identify var=Y(1,12)? The statement denotes

  $$Y(1,12) = \nabla Y_t - \nabla Y_{t-12} = \nabla_{12} \nabla Y_t.$$

Estimate in Proc ARIMA procedure

- The statement estimate p=4 denotes to estimate

$$(1-\phi_1 B-\phi_2 B^2-\phi_3 B^3-\phi_4 B^4)(Y_t-\mu)=e_t.$$

- The statement estimate p=(1 4) denotes to estimate

$$(1-\phi_1 B-\phi_4 B^4)(Y_t-\mu)=e_t.$$

- The statement estimate p=(12) denotes to estimate

$$(1-\Phi_1 B^{12})(Y_t-\mu)=e_t. \quad \Phi_1=\phi_{12}$$

- The statement estimate p=(1 12) denotes to estimate

$$(1-\phi_1 B-\phi_{12} B^{12})(Y_t-\mu)=e_t.$$

Estimate in Proc ARIMA procedure

- The statement estimate $p = (1\ 12\ 13)$ denotes to estimate

$$(1 - \phi_1 B - \phi_{12} B^{12} - \phi_{13} B^{13})(Y_t - \mu) = e_t.$$

- The statement estimate $p = (1)(12)$ denotes to estimate

$$(1 - \phi_1 B)(1 - \phi_{12} B^{12})(Y_t - \mu) = e_t.$$

- The statement estimate $p = (1)(6)\ q = (1\ 3)(12)$ denotes to estimate

$$(1 - \phi_{11} B)(1 - \Phi_1 B^6) Y_t = (1 - \theta_{11} B - \theta_{13} B^3)(1 - \Theta_1 B^{12}) e_t.$$

where $\Phi_1 = \phi_{21}$ and $\Theta_1 = \theta_{21}$ in sas results.

Estimate in Proc ARIMA procedure

- The statement estimate $p = (1)(12\ 24)$ denotes to estimate

$$(1 - \phi_1 B)(1 - \Phi_1 B^{12} - \Phi_2 B^{24})(Y_t - \mu) = e_t.$$

where $\Phi_1 = \phi_{21}$ and $\Phi_2 = \phi_{22}$ in sas results.

Example 10.4. The number of public transit boardings (mostly for bus and light rail) in Denver

Example

Figure 10.8 displays a time series of the number of public transit boardings (mostly for bus and light rail) in Denver, Colorado from $8/2000$ to $3/2006$.

- The data have been log-transformed.
- In Figure 10.9, we display the sample ACF and PACF.

Figure 10.8. Denver boarding data from 8/2000 to 3/2006

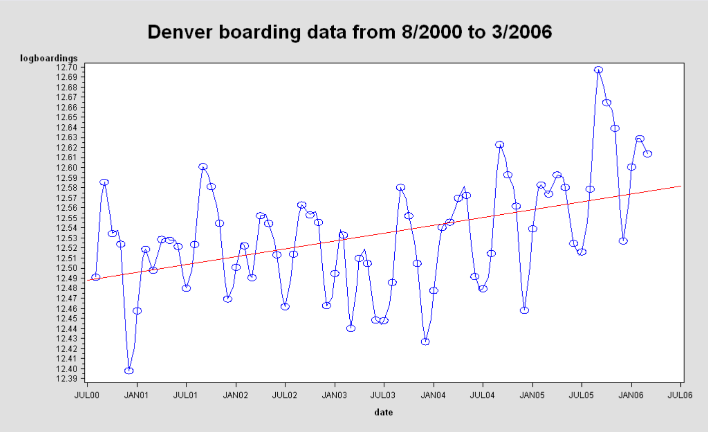


Figure 10.9. Sample ACF and sample PACF for Denver boarding data

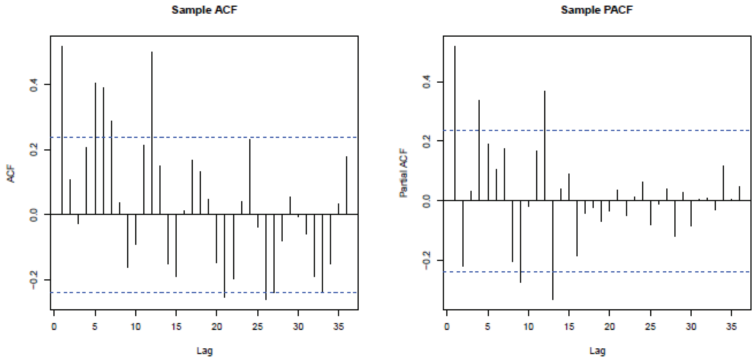


Figure 10.9: *Denver boardings data. Left: Sample ACF. Right: Sample PACF.*

Investigation of Example 10.4

First step: Model selection:

- The sample ACF shows a highly significant sample autocorrelation at lag $k = 12$ and a decay afterward at seasonal lags $k = 24$ and $k = 36$. This suggests a seasonal AR(1)$_{12}$ component.
- The sample PACF shows that there is a highly significant sample PACF at lag $k = 12$, and none at higher seasonal lags. This suggests a seasonal AR(1)$_{12}$ component.

The first step

- So, AR(2)$_{12}$ is a model for consideration.
- Then use a multiplicative seasonal model to fit the data. Try
  - ARMA(i,0)×ARMA(1,0)$_{12}$, $i = 0, 1, 2, \cdots$
  - ARMA(0,j)×ARMA(1,0)$_{12}$ $j = 1, 2, \cdots,$

for further investigation.

The second step: the best model and estimates of parameters

- After model fitting, residual analysis and diagnostics, $AR(1) \times AR(1)_{12}$ model is the best one for Denver boarding data.
- The estimates of parameters are as follows

	μ	ϕ_{11}	ϕ_{21}
estimate	12.54707	0.76837	0.87831
se	0.051	0.069	0.048

with σ^2 estimated as 0.000654 and AIC $= -283.973$.

- The fitted process for Denver boarding data is expressed as

$$(1 - 0.76837B)(1 - 0.87831B^{12})(Y_t - 12.54707) = e_t.$$

The second step: the normality and independence tests for the best model

- The fitted process for Denver boarding data can also be expressed as

$$Y_t = 0.35366 + 0.76837Y_{t-1} + 0.87831Y_{t-12} - 0.67487Y_{t-13} + e_t.$$

- The normality assumptions of the standardized residuals is rejected because Shapiro-wilk test $W = 0.938774$ with $p = 0.0023$.
- The independence assumptions of the standardized residuals looks fine because run test statistic is 37 with expected run 34.2647 and $p = 0.4943$.

The third step: forecasting

- The 12 forecast values for the next 12 months are listed below:

Time	Forecast	Std Error	L95	U95
Apr06	12.6205	0.0256	12.5704	12.6707
May06	12.6019	0.0323	12.5387	12.6651
Jun06	12.5471	0.0356	12.4772	12.6169
Jul06	12.5352	0.0375	12.4617	12.6086
Aug06	12.5865	0.0385	12.5110	12.6619
Sep06	12.6881	0.0391	12.6115	12.7648
Oct06	12.6573	0.0395	12.5800	12.7347
Nov06	12.6333	0.0397	12.5556	12.7111
Dec06	12.5336	0.0398	12.4556	12.6116
Jan07	12.5973	0.0399	12.5191	12.6754
Feb07	12.6213	0.0399	12.5431	12.6995
Mar07	12.6076	0.0399	12.5293	12.6859

Figure 10.10. Forecasting for log Denver boarding data

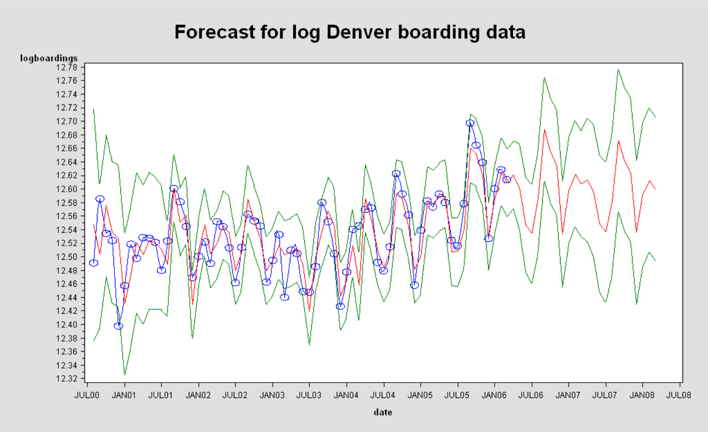
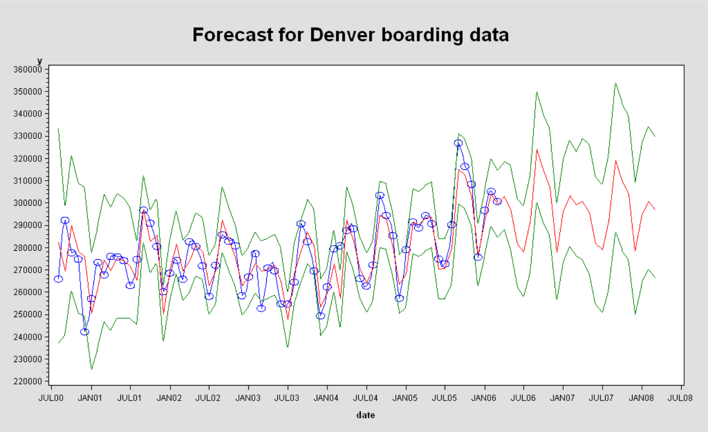


Figure 10.10-1. Forecasting for Denver boarding data



Overfitting

- For an $AR(1) \times ARMA(1,0)_{12}$, there are 4 overfitted models. Here are the models and the results from overfitting the Denver boarding data:

$$\text{ARMA}(2,0) \times \text{ARMA}(1,0)_{12} \Longrightarrow \hat{\phi}_2 \quad \text{not significant}$$

$$\text{ARMA}(1,1) \times \text{ARMA}(1,0)_{12} \Longrightarrow \hat{\theta} \quad \text{not significant}$$

$$\text{ARMA}(1,0) \times \text{ARMA}(1,1)_{12} \Longrightarrow \text{fails to pass WN test}$$

$$\text{ARMA}(1,0) \times \text{ARMA}(2,0)_{12} \Longrightarrow \hat{\Phi}_2 \quad \text{not significant}$$

10.4. Nonstationary seasonal ARIMA (SARIMA) models

- The multiplicative seasonal ARMA$(p, q) \times$ ARMA$(P, Q)_s$ model

$$\phi(B)\Phi_P(B^s)Y_t = \theta(B)\Theta_Q(B^s)e_t.$$

is a very flexible family of time series models for stationary seasonal processes.

Review

- The next step is to extend this class of models to handle two types of nonstationarity:
  - Nonseasonal nonstationary over time (e.g., increasing linear trends, etc.)
  - Seasonal nonstationarity, that is, additional changes in the seasonal mean level, even after possibly adjusting for nonseasonal stationarity over time.

Review

- For a stochastic process $\{Y_t\}$, the first differences are

$$\nabla Y_t = Y_t - Y_{t-1} = (1 - B)Y_t.$$

- This definition can be extended to handle any number of differences; the dth differences are

$$\nabla^d Y_t = (1 - B)^d Y_t.$$

- We know that taking $d = 1$ or (usually at most) $d = 2$ can coerce a nonseasonal nonstationary process into stationarity. However, how do we address seasonal nonstationarity?

The seasonal nonstationarity

- A stochastic process is defined by

$$Y_t = S_t + e_t,$$

where $\{S_t\}$ is a zero mean random walk with period $s = 12$.

- For this process, taking nonseasonal differences will not have an effect on the seasonal nonstationarity (Why?).
- We therefore need to define a new differencing operator that can remove nonstationarity across seasonal lags.

Seasonal difference operator

- The seasonal difference operator ∇_s is defined by

$$\nabla_s Y_t = Y_t - Y_{t-s} = (1 - B^s) Y_t,$$

for a seasonal period s.

- With $s = 12$ and monthly data, the first seasonal differences are

$$\nabla_{12} Y_t = Y_t - Y_{t-12} = (1 - B^{12}) Y_t,$$

that is, the first differences of the January observations, the February observations, and so on.

- Note that for our stochastic process defined above

$$\nabla_{12} Y_t = \nabla_{12} S_t + \nabla_{12} e_t = u_t + e_t - e_{t-12},$$

which has the same ACF as a stationary seasonal MA$(1)_{12}$.

- In other words, taking seasonal differences has coerced this process into stationarity.

Example 10.1 US milk production

Example 10.1 (continued). Consider again the monthly U.S. milk production (measured in millions of pounds) from January, 1994 to December, 2005.

- Figure 10.11 displays the time plot of the data (upper left),
- the first difference process ∇Y_t (upper right),
- the first seasonal difference process $\nabla_{12} Y_t$ (lower left), and
- the combined difference process $\nabla \nabla_{12} Y_t$ (lower right).

Figure 10.11. US milk production data

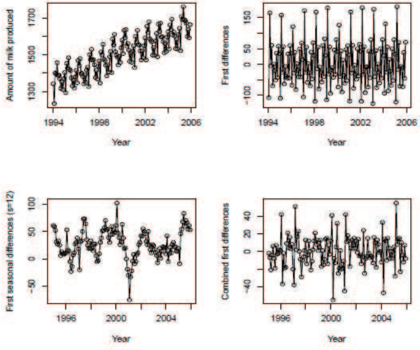


Figure 10.11: *Upper left: US milk production data from 1/94-12/05. Upper right: First differences* ∇Y_t. *Lower left: First seasonal differences* $\nabla_{12} Y_t$. *Lower right: Combined first differences and first seasonal differences* $\nabla\nabla_{12} Y_t$.

Example 10.1 US milk production

Note that

$$\begin{aligned}\nabla\nabla_{12}Y_t &= (1-B)(1-B^{12})Y_t \\ &= (1-B-B^{12}+B^{13})Y_t\end{aligned}$$

- The milk production data looks to display two types of trend: nonstationarity over time and a within-year seasonal pattern.
- Taking first differences ∇Y_t appears to have removed the nonstationarity over time, but ∇Y_t still displays a large amount of seasonality.
- Taking fist seasonal differences $\nabla_{12}Y_t$ appears to have removed the seasonality, but $\nabla_{12}Y_t$ displays a nonstationary pattern over time.
- The combined first differences $\nabla\nabla_{12}Y_t$ look stationarity in every way.

Nonstationarity: nonseasonal and seasonal

- From this example, it should be clear that we can now extend the multiplicative seasonal ARMA$(p, q) \times$ ARMA$(P, Q)_s$ model

$$\phi(B)\Phi_P(B^s)Y_t = \theta(B)\Theta_Q(B^s)e_t.$$

  which is stationary, to incorporate the two types of nonstationarity: nonseasonal and seasonal.

- This leads to the definition of our largest class of ARIMA models.

ARIMA$(p, d, q) \times$ ARIMA$(P, D, Q)_s$

- The multiplicative seasonal autoregressive integrated moving average (SARIMA) model with seasonal period s, denoted ARIMA$(p, d, q) \times$ ARIMA$(P, D, Q)_s$, is

$$\phi(B)\Phi_P(B^s)\nabla^d\nabla_s^D Y_t = \theta(B)\Theta_Q(B^s)e_t,$$

where the nonseasonal AR and MA characteristic operators are

$$\phi(B) = \left(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right)$$
$$\theta(B) = \left(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right),$$

the seasonal AR and MA characteristic operators are

$$\Phi_P(B^s) = \left(1 - \Phi_1 B^s - \Phi_2 B^{2s} - \cdots - \Phi_p B^{Ps}\right)$$
$$\Theta_Q(B^s) = \left(1 - \Theta_1 B^s - \Theta_2 B^{2s} - \cdots - \Theta_Q B^{Qs}\right).$$

and

$$\nabla^d\nabla_s^D Y_t = (1-B)^d(1-B^s)^D Y_t.$$

Figure 10.12. Time plot for combining difference of US milk data

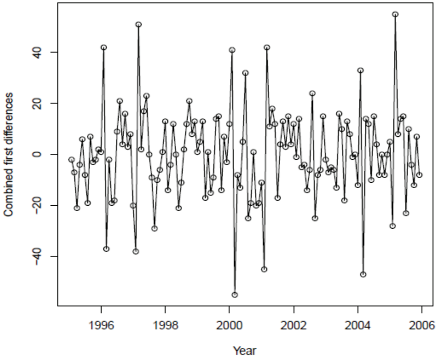


Figure 10.12: *US milk production data. Combined difference process $\nabla\nabla_{12}Y_t$ from Figure 10.11. Here, $d = 1$, $D = 1$, and $s = 12$.*

- In this model, d and D denote the number of nonseasonal and seasonal differences, respectively, to attain stationarity.
- Usually $D = 1$ will achieve seasonal stationarity.
- By definition,

$$Y_t \sim \text{ARIMA}(p, d, q) \times \text{ARIMA}(P, D, Q)_s$$

if and only if

$$\nabla^d \nabla_s^D Y_t \sim \text{ARMA}(p, q) \times \text{ARMA}(P, Q)_s.$$

- This SARIMA class is a very flexible class of models.
- Authors of CC remark that "many series can be adequately fit by these models, usually with a small number of parameters, say, three or four."

Figure 10.13. Sample ACF of US milk data

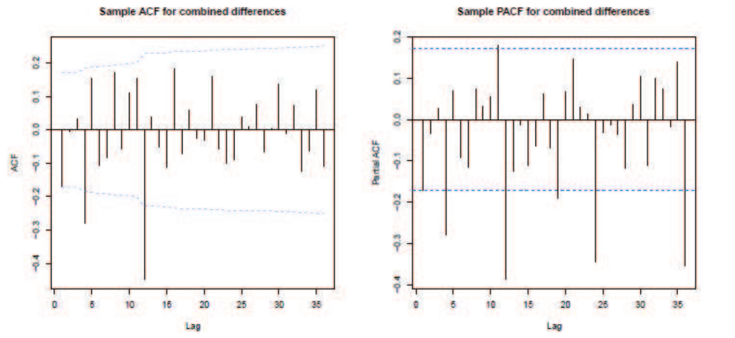


Figure 10.13: *US milk production data. Left: Sample ACF for* $\nabla\nabla_{12}Y_t$*. Right: Sample PACF for* $\nabla\nabla_{12}Y_t$*.*

Identification step for US milk production data

- From the time series plot of US milk production data, there are two trends: seasonal trend and deterministic trend.
- From the time series plot of the first difference of milk production, there is seasonal trend.
- From the time series plot of the seasonal difference of milk production, it seems to be stationary.
- From the time series plot of the combined difference of milk production, it is stationary.

Identification and estimate steps for US milk production data

- If data of milk(12) is viewed as a stationary time series (?), we proceed the model fitting step and tentatively adopt an ARMA(1,0)$\times$ ARMA$(3,0)_{12}$ for $\nabla_{12}Y_t$ or an ARMA(1,0)$\times$ ARMA$(3,1,0)_{12}$ for Y_t.
- By estimate step, the fitting process with ml method is given by

$$(1-0.88453B)(1+0.90656B^{12}+0.80401B^{24}+0.58338B^{36})$$
$$\times\,(\nabla_{12}Y_t-26.35283)=e_t$$

  with

  - $\hat{\sigma}^2=119.2881$, and
  - AIC $=1034.264$ and BIC $=1048.678$.

- They fail to pass WN test at early stages.

Identification step for US milk production data

- If data of milk(1,12) is viewed as a stationary time series (it is for sure), let us look ACF and PACF plots.
- The sample ACF for $\nabla\nabla_{12}Y_t$ reveals a nearly significant lag $k = 1$ sample autocorrelation and a pronounced significant sample autocorrelation at lag $k = 12$.
- The sample PACF for $\nabla\nabla_{12}Y_t$ shows a slow decay across the seasonal lags $k = 12, 24$ and 36.
- We tentatively adopt an ARMA$(3, 0)_{12}$ model for $\nabla\nabla_{12}Y_t$; i.e., an ARIMA$(0, 1, 0) \times$ ARIMA$(3, 1, 0)_{12}$ model for Y_t.

Estimate step for US milk production data

- By estimate step, the fitting process with CLS method is given by
$$(1 + 0.88735B^{12} + 0.79654B^{24} + 0.60539B^{36})$$
$$(1 - B)(1 - B^{12})Y_t = e_t$$
with
  - $\hat{\sigma}^2 = 150.9084$, and
  - AIC $= 1031.911$ and BIC $= 1040.537$.
- This fitted model is Okay for the data set.
- They have too many parameters to give a more acceptable explanation.

Overfitting for the above fitted models

- Trying overfitting the ARIMA$(0,1,0) \times$ ARIMA$(3,1,0)_{12}$ with the four models

$$\text{ARIMA}(1,1,0) \times \text{ARIMA}(3,1,0)_{12} \Longrightarrow \hat{\phi} \quad \text{not} \quad \text{significant}$$
$$\text{ARIMA}(0,1,1) \times \text{ARIMA}(3,1,0)_{12} \Longrightarrow \hat{\theta} \quad \text{not} \quad \text{significant}$$
$$\text{ARIMA}(0,1,0) \times \text{ARIMA}(3,1,1)_{12} \Longrightarrow \widehat{\Theta} \quad \text{not} \quad \text{significant}$$
$$\text{ARIMA}(0,1,0) \times \text{ARIMA}(4,1,0)_{12} \Longrightarrow \widehat{\Phi}_4 \quad \text{not} \quad \text{significant}$$

- The model ARIMA$(0,1,0) \times$ ARIMA$(3,1,0)_{12}$ provided one of the smallest white noise variance estimates and the smallest AIC.
- This analysis suggests that the ARIMA$(0,1,0) \times$ ARIMA$(3,1,0)_{12}$ model is worthy of consideration, but also that it is not immune to criticism to many parameters.

How about the consine trend for the data

- Consider two variables

$$x_1 = cos(2 * 3.14159 * t/12) \quad x_2 = sin(2 * 3.14159 * t/12).$$

- Use autoregresive procedure to fit the data.
- This way can not get a good fitted model as above fitted models.

How about autoreg procedure for the data

- Since $\nabla\nabla_{12}\text{milk}_t$ is a stationary time series, we consider use autoregresive procedure to fit the data.

- By estimate step, the fitting process with ml is given by

$$(1-B)(1-B^{12})Y_t = X_t$$
$$X_t = 0.0937X_{t-4} + 0.885X_{t-12} + 0.7915X_{t-24} + 0.5795X_{t-36} + e_t$$

  with
  - $\hat{\sigma}^2 = 123.12749$, a smaller variance, and
  - AIC $= 1028.82866$ and BIC $= 1040.32945$.
- This fitted model is preferred.
- They also have too many parameters to give a more acceptable explanation.

10.5. Forecasting with seasonal models

- We will adopt the same convention to forecast Y_{t+l} by choosing the function of the observed data $h(Y_1, Y_2, \cdots, Y_t)$ that minimizes the mean squared error of the prediction error, that is, $\mathsf{MSEP} = \mathsf{E}\left\{[Y_{t+l} - h(Y_1, Y_2, \cdots, Y_t)]^2\right\}$.

MMSE forecast

- The MMSE forecast for Y_{t+l} is given by

$$\widehat{Y}_t(l) = \mathsf{E}(Y_{t+l}|Y_1, Y_2, \cdots, Y_t).$$

- Formulae for seasonal MMSE forecasts appear in Section 10.5 (CC) for special cases. The derivations mirror those in the nonseasonal case (Chapter 9).

Prediction limits

- If the $\{e_t\}$ is normal zero mean white noise, then

$$Z = \frac{Y_{t+l} - \widehat{Y}_t(l)}{\sqrt{\mathsf{var}[e_t(l)]}} = \frac{Y_{t+l} - \widehat{Y}_t(l)}{\mathsf{se}[e_t(l)]} \sim N(0,1),$$

  where $e_t(l) = Y_{t+l} - \widehat{Y}_t(l)$ is the l-steps-ahead forecast error.
- This distributional result implies that

$$\left(\widehat{Y}_t(l) - z_{\alpha/2}\mathsf{se}[e_t(l)], \widehat{Y}_t(l) + z_{\alpha/2}\mathsf{se}[e_t(l)]\right)$$

  is a $100(1-\alpha)$ percent prediction interval for Y_{t+l}.
- As with nonseasonal models, normality is needed for the prediction intervals to be applicable.

US milk production data (continued)

- For the milk production data, we use R to compute forecasts and prediction limits for $l = 1, 2, \cdots, 24$ (two years ahead). These are given below:

Date	Forecast	Std Error	$(l95$	$u95)$
$JAN06$	1702.21	11.1459	1680.36	1724.05
$FEB06$	1583.90	15.7626	1553.00	1614.79
$MAR06$	1759.75	19.3052	1721.91	1797.58
$APR06$	1727.43	22.2917	1683.74	1771.12
$MAY06$	1782.47	24.9229	1733.62	1831.31
$JUN06$	1697.10	27.3017	1643.59	1750.61
$\ldots$	$\ldots$	$\ldots$	$\ldots$	
$OCT07$	1675.22	54.3813	1568.63	1781.80
$NOV07$	1629.16	55.7134	1519.96	1738.36
$DEC07$	1707.09	57.0144	1595.34	1818.83

Figure 10.15. Forecasting for US milk production

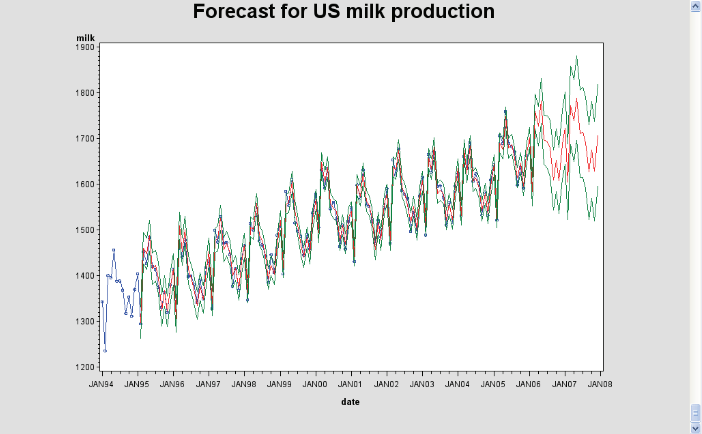


Figure 10.15. Forecasting for US milk production

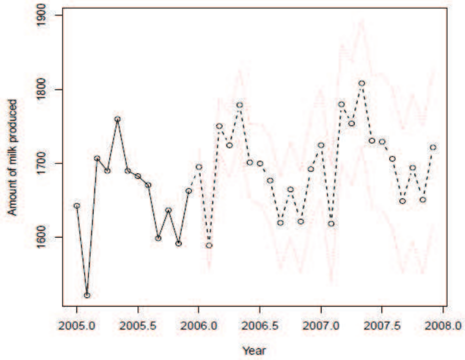


Figure 10.15: *US milk production data. MMSE forecasts for* $l = 1, 2, ..., 24$ *from the* $ARIMA(0,1,1) \times ARIMA(0,1,1)_{12}$ *fit.*

Remarks for the fitted model

- Figure 10.15 display the forecasts, along with estimated standard errors.
- We display the last 12 values of the observed process (from January 2005) and then project out over the next 24 time periods.
- Clearly, the pattern in the forecasts mirror that of the original series.
- Note also the estimated standard errors increase as l increases.

Example 10.5(Example 3.6). Monthly US beer sales data 1980-1990

Example

The data in Figure 3.9 are monthly US beer sales (in millions of barrels) in United States during the period from January of 1980 to December of 1990. The data are monthly, which suggests that the seasonal lag $s = 12$ may be of interest.

Figure 10.16. Monthly US beer sales data 1980-1990

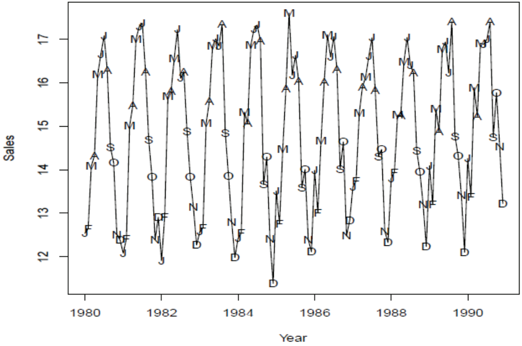


Figure 3.9: *Monthly US beer sales from 1980-1990. The data are measured in millions of barrels.*

Comments from the time plot of monthly US beer sales data

- There looks to be a monthly seasonal component as well.
- There looks to be a constant variance problem (no variation in the series). A Box-Cox analysis is not needed.

Model selection

We will use two procedures to make model selection.

- Box-Jenkins modeling approach:
  - Model identification
  - Model fitting
  - Model diagnostics
- Two procedures:
  - Proc Arima procedure.
  - Proc Autoreg procedure.

The fitting model via proc arima procedure

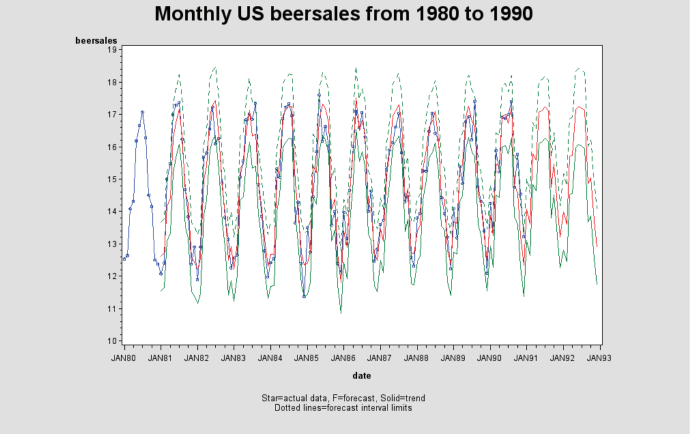
- Using Proc Arima procedure obtains the following fitted model

$$Y_t = 0.07464 + Y_{t-12} + e_t - 0.28392 e_{t-12} - 0.42066 e_{t-24}$$

  with $AIC = 179.1277$ and $\hat{\sigma}^2 = 0.237973$.

- It is an an ARIMA(0,1,2)$_{12}$ model.
- Criticism comes from failing to pass the WN test, too more WN variance.
- The forecast values for the future 24 months of 1991-1992 are listed in Figure 10.16.

Figure 10.16. Forecasting for US beer sales during 1991-1992 by an ARIMA$(0,1,2)_{12}$



The fitting model via proc autoreg procedure

- Using deterministic cosine trend and Proc autreg procedure obtains the following fitted model

$$
\begin{aligned}
Y_t =& 14.4567 + 0.0000201t^2 - 1.7789cos(2\pi t/12) \\
& - 1.3523sin(2\pi t/12) + X_t \\
X_t =& e_t - 0.1712X_{t-6} + 0.1931X_{t-9} + 0.4796X_{t-12} \\
& - 0.3023X_{t-13} - 0.2358X_{t-16} + 0.2231X_{t-25}
\end{aligned}
$$

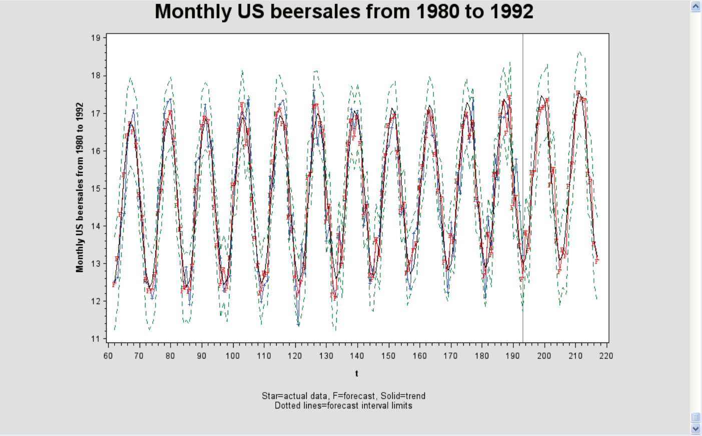
  with $AIC = 171.0914$ and $\hat{\sigma}^2 = 0.1882$. This fitted model is preferred.

- It is a combination model with three parts: quadratic item, cosine and sine items, and an autoregressive process.

- Criticism comes from too items in the autoregressive process.

- The forecast values for the future 24 months of 1991-1992 are listed in Figure 10.17.

Figure 10.17. Forecasting for US beer sales during 1991-1992 by a combination model



Residual analysis

- Run test for independence of errors.
- Shapiro-Wilk test for normality.

10.6. Additional topics

- In this course, we have covered the first 10 chapters of Cryer and Chan (2008).
- This material provides you with a powerful arsenal of techniques to analyze many time series data sets that are seen in practice.
- These chapters also lay the foundation for further study in time series analysis.

Further topics: intervention analysis

- Chapter 11. This chapter provides an introduction to intervention analysis, which deals with incorporating external events in modeling of time series data (e.g., a change in production methods, natural disasters, terrorist attacks, etc.). Techniques for incorporating outliers and external covariate information are presented and analyzing multiple time series is also discussed.

Further topics: GARCH models

- Chapter 12. This chapter deals explicitly with modeling financial time series data (e.g., stock prices, portfolio returns, etc.), mainly with the commonly used ARCH and GARCH models. The salient aspect of these models is that they incorporate additional heteroscedasticity that are common in financial data.

Further topics: spectral analysis

- Chapter 13. This chapter deals with frequency domain methods (spectral analysis) for periodic data which arise in physics, biomedicine, engineering, etc. The periodogram and spectral density are introduced. These methods use linear combinations of sine and cosine functions to model underlying (possibly multiple) frequencies.
- Chapter 14. This chapter is an extension of Chapter 13 which studies the sampling characteristics of the spectral density estimator.

Further topics: nonlinear models

- Chapter 15. This chapter discusses nonlinear models for time series data (a relatively new area in time series research). This class of models assumes that the underlying trend (over time) is nonlinear, which can be a result of nonnormality. These techniques could be helpful in analyzing the data set, where it was clear that nonnormality was present.

Thank You for Your Attention !

Have a nice day !