Times Series and Forecasting (II)

Chapter 2. Fundamental Concepts

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2.1. Discrete stochastic processes and times series

- The sequence of random variables $\{Y_t : t = 0, 1, 2, \cdots,\}$, or $\{Y_t\}$, is called a discrete stochastic process (DSP).
- It is a collection of random variables indexed by time t, i.e.

$$\begin{aligned} Y_0 &= \text{value of the process at time } t = 0 \\ &\cdots \\ Y_n &= \text{value of the process at time } t = n. \end{aligned}$$

- A DSP can be described as "a statistical phenomenon that evolves through time according to a set of probabilistic laws".
- An equally spaced time DSP is a time series process. Any realization of a time series process is a times series.

2.1. DSP and times series (continued)

- A complete probabilistic time series model for $\{Y_t\}$ would specify all of the joint distributions of random vectors $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_n)$, for all $n = 1, 2, \cdots$, or, equivalently, specify the joint probabilities

$$P\left(Y_1 \leq y_1, Y_2 \leq y_2, \cdots, Y_n \leq y_n\right),$$

for all $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ and $n = 1, 2, \cdots$.

- In practice, it is hard to get joint distributions for all n.
- In this course, we specify only the first and second-order moments; namely, $\mathsf{E}(Y_t)$ and $\mathsf{E}(Y_t Y_{t-k})$, for $k, t = 0, 1, 2, \cdots$.

2.2. Means and autocovariance

- For the stochastic process $\{Y_t : t = 0, 1, 2, \cdots,\}$, the mean function is defined as

$$\mu_t = \mathsf{E}(Y_t),$$

for $t = 0, 1, 2, \cdots$.

- The autocovariance function (variances) is defined as

$$\gamma_{t,s} = \mathsf{Cov}(Y_t, Y_s),$$

for $t, s = 0, 1, 2, \cdots$, with

$$\mathsf{Cov}(Y_t, Y_s) = \mathsf{E}(Y_t Y_s) - \mathsf{E}(Y_t)\mathsf{E}(Y_s).$$

Autocorrelation

The autocorrelation function (ACF) is given by

$$\rho_{t,s} = \text{corr}(Y_t, Y_s) = \frac{\text{Cov}(Y_t, Y_s)}{\sqrt{\text{var}(Y_t)\text{var}(Y_s)}} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}}.$$

- $\rho_{s,t}$ is close to $\pm 1 \Longrightarrow$ strong linear dependence between Y_t and Y_s.
- $\rho_{s,t}$ is close to $0 \Longrightarrow$ weak linear dependence between Y_t and Y_s.
- $\rho_{s,t} = 0 \Longrightarrow Y_t$ and Y_s are uncorrelated.

2.3.1. White noise process

A process $\{e_t : t = 0, 1, 2, \cdots, \}$ is called a white noise process if it is a sequence of uncorrelated (independent) and identically distributed random variables with

$$\mathrm{E}(e_t) = \mu_e \text{ and } \mathrm{var}(e_t) = \sigma_e^2.$$

- μ_e and σ_e^2 are constant, free of t. Write $\{e_t\} \sim WN(\mu_e, \sigma_e^2)$.
- Assume that $\{e_t\} \sim WN(0, \sigma_e^2)$ throughout our course.
- If $e_t \sim N(0, \sigma_e^2)$ for all t, or $\{e_t\} \sim$ normal WN$(0, \sigma_e^2)$, $\{e_t\}$ is a sequence of iid (for linear time series models).

Figure 2.1. Four simulated realizations of $e_t \sim N(0, 1)$

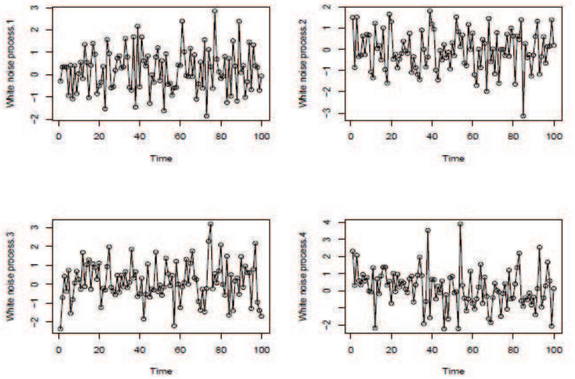


Figure 2.1: *Four white noise processes* $e_t \sim$ *iid* $\mathcal{N}(0, \sigma_e^2)$, *where* $n = 100$ *and* $\sigma_e^2 = 1$.

SAS programming for generating data of white noise N(0,1)

```
Data figure21;
do t=0 to 100;
y1=normal(0);
output;
end;
run;
Proc gplot data= figure21; symbol i= v=circle h=1.5;
title 'Scatter plot of the 1st simulated realization of WN N(0,1)';
plot y1 * t;
run;
symbol i=spline v=circle h=1.5;
title 'The first simulated realization of WN N(0,1)';
plot y1 * t;
run;
```

Mean and autocovariances of white noise process

- For $t \neq s$, the independence of $\{e_t\}$ gives

$$\text{Cov}(e_t, e_s) = 0.$$

- Thus, the autocovariance function of $\{e_t\}$ is

$$\gamma_{t,s} = \begin{cases} \sigma_e^2, & t = s \\ 0, & t \neq s. \end{cases}$$

- Thus, the autocorrelation function of $\{e_t\}$ is

$$\rho_{t,s} = \begin{cases} 1, & t = s \\ 0, & t \neq s. \end{cases}$$

Remarks for white noise processes

- A white noise is the simplest time series model.
- A white noise process, itself, is generally not interesting!
- However, white noise processes play a crucial role in the analysis of time series data! Why? Many important time series models can be constructed from white noise.
- Engineers and physicists use the term "white noise" to describe a random signal of every frequency in the audio or visual spectrum, all of which have an average uniform power level.

Remarks for white noise processes (continued)

- To understand the crucial role in the analysis of time series data, let us look at analysis approaches to time series processes $\{Y_t\}$.
- Time series process $\{Y_t\}$ contain two different types of variation:
  - systematic variation (that we would like to capture and model; e.g., trends, seasonal components, etc.)
  - random variation (that is just inherent background noise in the process).

Remarks for white noise processes (continued)

- Our goal as data analysts is to extract the systematic part of the variation in the data (and incorporate this into our model).

- If we do an adequate job of extracting the systematic part, then the only variation "left over" should be just random noise, which can be modeled by a white noise.

2.3.2. Random walk

- Define

$$
\begin{aligned}
Y_1 &= e_1 \\
Y_2 &= e_1 + e_2 \\
&\vdots \\
Y_n &= e_1 + e_2 + \cdots + e_n.
\end{aligned}
$$

- By this definition, note that we can write, for $t > 1$,

$$Y_t = Y_{t-1} + e_t.$$

- The process $\{Y_t\}$ is called a random walk process.

Mean and autocorrelation of a random walk process

- The mean of Y_t is $\mu_t = \mathsf{E}(Y_t) = \mathsf{E}(e_1 + \cdots + e_t) = 0$.
- The autocovariance of Y_t and Y_s is

$$\gamma_{t,s} = \mathsf{Cov}(e_1 + \cdots + e_t, e_1 + \cdots + e_s) = \min\{s, t\}\sigma_e^2.$$

- The autocorrelation function is

$$\rho_{t,s} = \mathsf{corr}(Y_t, Y_s) = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} = \frac{\min\{s, t\}}{\sqrt{st}}.$$

Mean and autocorrelation of a random walk process

- $\rho_{t,s}$ is closer to 1 when t is close to s. That is, observations that are close together in time are more positively correlated than observations which are far apart.
- For t fixed, the correlation becomes smaller as $d(t,s)$ increases. In fact, for t fixed, it is easy to see that

$$\lim_{s\to\infty}\rho_{t,s} = 0.$$

- Random walk processes often are used to model stock prices, movements of molecules in gases and liquids, wild animal locations, etc.

Figure 2.2. Four simulated realizations of a N(0,1) walk

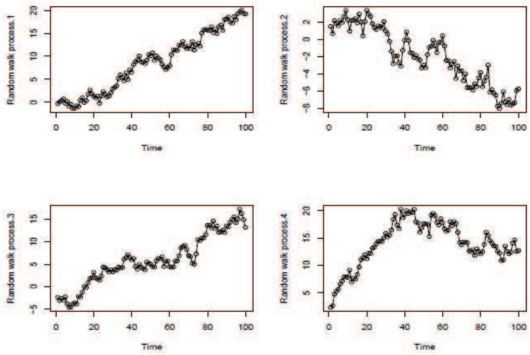


Figure 2.2: *Four random walk processes* $Y_t = Y_{t-1} + e_t$, *where* $e_t \sim iid\ \mathcal{N}(0, \sigma_e^2)$, $n = 100$, *and* $\sigma_e^2 = 1$. *These were constructed from the white noise processes in Figure 2.1.*

SAS programming for random walk with the normal WN(0,1)

```
Data figure22;
y1=0;
do t=0 to 100;
y1=y1+normal(0);
output;
end;
run;
Proc gplot data=figure22;
symbol i= v=circle h=1;
title 'Scatter plot of the 1st simulated realization of a N(0,1) walk';
plot y1 * t;
run; symbol i=spline v=circle h=1.5;
title 'The 1st simulated realization of a N(0,1) walk';
plot y1 * t;
run;
```

2.3.3. Random walk with drift

- Define

$$\begin{aligned} Y_1 &= \theta_0 + e_1 \\ Y_2 &= 2\theta_0 + e_1 + e_2 \\ &\vdots \\ Y_n &= n\theta_0 + e_1 + e_2 + \cdots + e_n. \end{aligned}$$

- By this definition, note that we can write, for $t > 1$,

$$Y_t = \theta_0 + Y_{t-1} + e_t.$$

- The process $\{Y_t\}$ is called a random walk process with drift.
- The constant θ_0 is called the drift parameter. Note that if $\theta_0 = 0$, then this process becomes a random walk.

Mean and autocorrelations

- The mean of Y_t is

$$\mu_t = \mathsf{E}(Y_t) = \mathsf{E}(t\theta_0 + e_1 + \cdots + e_t) = t\theta_0.$$

  Thus, the mean function μ_t changes with time (compare this to the random walk, where the mean function is zero for all t).

- For the random walk process with drift, the autocovariance function is

$$\gamma_{t,s} = \mathsf{Cov}(Y_t, Y_s) = \min\{s,t\}\sigma_e^2.$$

- The autocorrelation function is

$$\rho_{t,s} = \mathsf{corr}(Y_t, Y_s) = \frac{\min\{s,t\}}{\sqrt{ts}}.$$

Figure 2.3. Four realizations of a random walks with drift

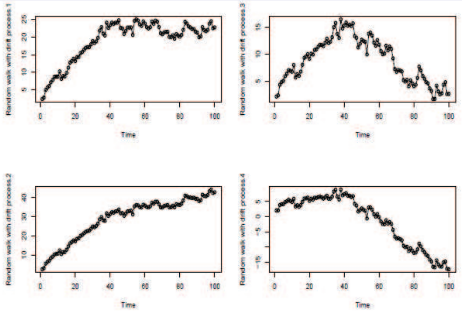


Figure 2.3: *The top two processes are the same. Each is the fourth random walk process in Figure 2.2. The middle processes are the drift versions of the top process with* $\theta_0 = 0.1$ *(left) and* $\theta_0 = -0.1$ *(right). The bottom processes are the drift versions of the top process with* $\theta_0 = 0.3$ *(left) and* $\theta_0 = -0.3$ *(right).*

SAS programming for a random walk with drift

```
Data figure23;
y=0; theta=0.1;
do t=1 to 100;
y=theta+y+normal(0);
output;
end;
run;
Proc gplot data= figure23;
symbol i=spline v=circle h=1.5;
title '1st simulated realization of a N(0,1) random walk with drift 0.1';
plot y * t;
run;
```

2.3.4. A moving average model

- Define
$$Y_t = \frac{1}{3}\left(e_t + e_{t-1} + e_{t-2}\right),$$
that is, Y_t is a running (or moving average of the white noise process (averaged across the most recent 3 time periods).

- On pp 14-15 in textbook CC, consider
$$Y_t = \frac{1}{2}\left(e_t + e_{t-1}\right),$$

Mean and autocovariances

- The mean of Y_t is $\mu_t = \mathsf{E}(Y_t) = \mathsf{E}(e_t + e_{t-1} + e_{t-2})/3 = 0.$
- If $s > t + 2$, then $\gamma_{t,s} = 0$ because Y_t and Y_s are uncorrected.
- And

$$\begin{aligned}
\gamma_{t,t} &= \mathsf{Cov}(Y_t, Y_t) = \mathsf{var}(Y_t) = \frac{1}{3}\sigma_e^2;\\
\gamma_{t,t+1} &= \mathsf{Cov}(Y_t, Y_{t+1}) = \frac{1}{9}\mathsf{Cov}\left(\sum_{i=0}^{2} e_{t-i}, \sum_{j=-1}^{1} e_{t+j}\right) = \frac{2}{9}\sigma_e^2;\\
\gamma_{t,t+2} &= \frac{1}{9}\mathsf{Cov}(Y_t, Y_{t+2}) = \mathsf{Cov}\left(\sum_{i=0}^{2} e_{t-i}, \sum_{j=0}^{2} e_{t+j}\right) = \frac{1}{9}\sigma_e^2.
\end{aligned}$$

Autocovariance and autocorrelation

- Thus, the autocovariance function can be written as

$$\gamma_{t,s} = \begin{cases} \sigma_e^2/3, & |t-s| = 0 \\ 2\sigma_e^2/9, & |t-s| = 1 \\ \sigma_e^2/9, & |t-s| = 2 \\ 0, & |t-s| > 2. \end{cases}$$

- The autocorrelation function for this moving average process is

$$\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} = \begin{cases} 1, & |t-s| = 0 \\ 2/3, & |t-s| = 1 \\ 1/3, & |t-s| = 2 \\ 0, & |t-s| > 2. \end{cases}$$

Figure 2.4. Four realizations of the moving average process

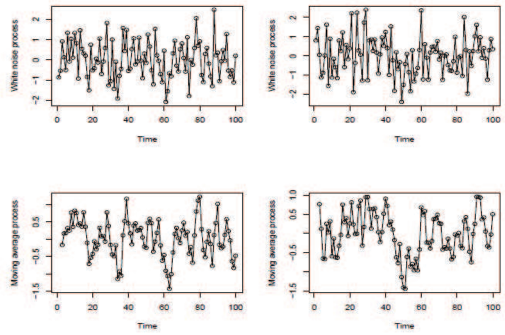


Figure 2.4: *The top plots are two different normal zero-mean white noise processes with* $\sigma_e^2 = 1$*. The bottom plots are the moving average* $Y_t = \frac{1}{3}(e_t + e_{t-1} + e_{t-2})$ *versions of the top plots.*

SAS programming for MA $Y_t = (e_t + e_{t-1} + e_{t-2})/3$

```
Data figure24_1;
do t=1 to 100; y_t = normal(0); output; end; run;
Data figure24_2;
set figure24_1;
if t=1 then delete; y_1 = y_t; t = t - 1; drop y_t; run;
Data figure24_3; set figure24_1;
if t < 3 then delete; y_2 = y_t; t = t - 2; drop y_t; run;
Data figure24;
merge figure24_1 figure24_2 figure24_3;
by t; y = (y_t + y_1 + y_2)/3; t = t + 2; run;
Proc gplot data=figure24; ...; run;
```

2.3.5. An autoregressive model

- Consider the stochastic process defined by

$$Y_t = 0.7Y_{t-1} + e_t.$$

- Y_t is related to the (downweighted) previous value of Y_{t-1} and e_t (a "shock" or "innovation" that occurs at time t).
- This is called an autoregressive model. Autoregression means "regression on itself". Essentially, we can envision "regressing" Y_t on Y_{t-1}.
- The usual calculations for this autoregressive model are postponed to Chapter 4.

Figure 2.5. Four realizations of the autoregressive model

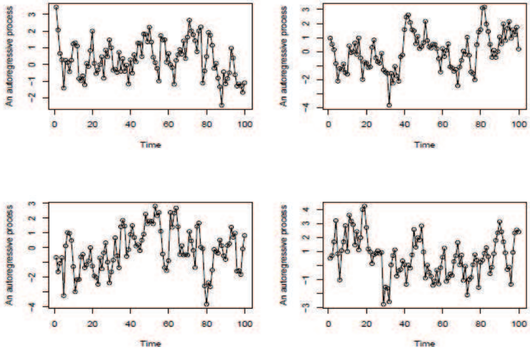


Figure 2.5: *Four different realizations of the autoregressive model* $Y_t = 0.7Y_{t-1} + e_t$ *with* $n = 100$ *and* $\sigma_e^2 = 1$.

SAS programming for the AR $Y_t = 0.7Y_{t-1} + e_t$

```
Data figure25;
y=0; a=0.7; do t=1 to 100;
y=a*y+normal(0); output; end; run;

Proc gplot data=figure25;
symbol i=spline v=circle h=1.5;
plot y * t;
title '1st simulated realization of the AR model Y_t = 0.7Y_{t-1} + e_t';
run;
```

2.3.6. A sinusoidal model

- Many time series processes in practice exhibit seasonal patterns that correspond to different weeks, months, years, etc.
- One way to capture these seasonal patterns is to use models with deterministic parts which are trigonometric in nature.
- Consider the process defined by

$$Y_t = a\mathsf{sin}(2\pi\omega t + \phi) + e_t,$$

  where a is the amplitude, ω is the frequency of oscillation, and $\phi/2\pi\omega$ is the phase shift.
- With $a = 2$, $\omega = 1/50$ (one cycle/50 time points), and $\phi = 0.6\pi$, we have

$$\mathsf{E}(Y_t) = 2\mathsf{sin}(2\pi t/50 + 0.6\pi) \text{ and } \mathsf{var}(Y_t) = \sigma_e^2.$$

Figure 2.6. Sinusoidal model illustration and realizations

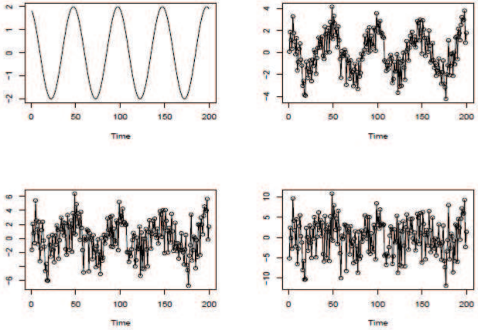


Figure 2.6: *Sinusoidal model illustration. Top left:* $E(Y_t) = 2\sin(2\pi t/50 + 0.6\pi)$. *The other plots are simulated realizations of this process* ($n = 200$) *with* $\sigma_e^2 = 1$ *(top right),* $\sigma_e^2 = 4$ *(bottom left), and* $\sigma_e^2 = 16$ *(bottom right).*

SAS programming for a sinusoidal model

Data figure26_1;
do t=1 to 200;
$y = 2 * sin(2 * 3.14159 * t/50 + 0.6 * 3.14159);$
output; end; run;

Proc gplot data= figure26_1;
symbol i=spline v=circle h=1.5;
plot y * t;
title 'the curve of $2sin(2\pi t/50 + 0.6\pi)$';
run;

2.4. Stationarity

Stationarity is a very important concept in the analysis of time series data.

Broadly speaking, a time series is said to be stationary

- if there is no systematic change in mean (no trend),
- if there is no systematic change in variance, and
- if strictly periodic variations have been removed.

In other words, the properties of one section of the data are much like those of any other section.

Importance of Stationarity

IMPORTANCE: Much of the theory of time series is concerned with stationary time series.

- For this reason, time series analysis often requires one to transform a non-stationary time series into a stationary one as to use this theory.

- For example, it may be of interest to remove the trend and seasonal variation from a set of data and then try to model the variation in the residuals by means of a stationary stochastic process.

2.4.1. Strict stationarity

- The stochastic process $\{Y_t : t = 0, 1, 2, \cdots, n\}$ is said to be strictly stationary if the joint distribution of

$$Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n}$$

is the same as

$$Y_{t_1-k}, Y_{t_2-k}, \cdots, Y_{t_n-k}$$

for all time points $t_1, t_2, \cdots, t_n$ and for all time lags k.

- In other words, shifting the time origin by an amount k has no effect on the joint distributions, which must therefore depend only on the intervals between $t_1, t_2, \cdots, t_n$. This is a very strong condition.

Implication of strict stationarity

- When $n = 1$, this implies Y_t and Y_{t-k} have the same marginal distribution for all t and k.
- For all t and k,

$$\mathsf{E}(Y_t) = \mathsf{E}(Y_{t-k})$$
$$\mathsf{var}(Y_t) = \mathsf{var}(Y_{t-k}).$$

- Therefore, for a strictly stationary process, both $\mu_t = \mathsf{E}(Y_t)$ and $\gamma_{t,t} = \mathsf{var}(Y_t)$ are constant over time.

Implication of strict stationarity

- When $n = 2$, this implies (Y_t, Y_s) and (Y_{t-k}, Y_{s-k}) have the same joint distribution for all t, s, and k.
- For all t, s and k,

$$\mathrm{Cov}(Y_t, Y_s) = \mathrm{Cov}(Y_{t-k}, Y_{s-k}).$$

- For any s, t,

$$\gamma_{t,s} = \mathrm{Cov}(Y_t, Y_s) = \mathrm{Cov}(Y_0, Y_{|t-s|}) = \gamma_{0,|t-s|}.$$

  This means that the covariance between Y_t and Y_s does not depend on the actual values of t and s; it only depends on the time difference $|t - s|$.

New notation

- For a (strictly) stationary process, the covariance $\gamma_{t,s}$ depends only on the time difference $|t-s|$.
- The covariance between Y_t and any observation $k=|t-s|$ time points from it only depends on the lag k.
- Therefore, we write

$$\gamma_k = \mathsf{Cov}(Y_t, Y_{t-k})$$
$$\rho_k = \mathsf{corr}(Y_t, Y_{t-k}).$$

- We use this simpler notation only when we refer to a process which is stationary. Thay are

$$\gamma_0 = \mathsf{Cov}(Y_t, Y_t) = \mathsf{var}(Y_t),$$
$$\rho_k = \mathsf{corr}(Y_t, Y_{t-k}) = \frac{\gamma_k}{\gamma_0}.$$

Necessary conditions of strict stationarity

For a (strictly) stationary process,

1. the mean function $\mu_t = \mathsf{E}(Y_t)$ is constant throughout time; i.e., μ_t is free of t.

2. the covariance between any two observations depends only the time lag between them; i.e., $\gamma_{t,t-k} \equiv \gamma_k$ depends only on k.

2.4.2. Weak stationarity

A weak form of stationarity:

Definition: The stochastic process $\{Y_t : t = 0, 1, 2, \cdots, n\}$ is said to be weakly stationary (or **second-order stationary**) if

1. The mean function $\mu_t = \mathrm{E}(Y_t)$ is constant throughout time; i.e., μ_t is free of t.
2. The covariance between any two observations depends only the time lag between them; i.e., $\gamma_{t,t-k}$ depends only on k.

Remarks for weak stationarity

- Nothing is assumed about the collection of joint distributions of the process. Instead, we only are specifying the characteristics of the first two moments of the process.

- Strict stationarity implies weak stationarity. It is also clear that the converse to statement is not true.

- The additional assumption of multivariate normality (for the Y_t process) is given, then

$$\text{weak stationarity} + \text{multivariate normality} \Longrightarrow \text{strict stationarity.}$$

- Convention: When the term "stationary process" is used in this courese, this is understood to mean that the process is weakly stationary.

A white noise process is stationary

- Suppose that $\{e_t\}$ is a white noise process with $\mathsf{E}(e_t) = \mu_e$ and $\mathsf{var}(e_t) = \sigma_e^2$, both constant (free of t).
- Clearly, the mean process $\mu_t = \mathsf{E}(e_t)$ is constant over time.
- In addition, the autocovariance function $\gamma_k = \mathsf{cov}(Y_t, Y_{t-k})$ is given by

$$\gamma_k = \begin{cases} \sigma_e^2, & k = 0 \\ 0, & k \neq 0, \end{cases}$$

  which is free of time t (i.e., γ_k depends only on k).
- Thus, a white noise process is stationary.

A random walk process is not stationary

- Suppose that $\{Y_t\}$ is a random walk process. That is,

$$Y_t = Y_{t-1} + e_t.$$

- $\mu_t = \mathsf{E}(Y_t) = 0$, for all t, which is constant over time.
- However,

  $$\mathsf{Cov}(Y_t, Y_{t-k}) = \mathsf{Cov}(e_1 + \cdots + e_t, e_1 + \cdots + e_{t-k}) = (t-k)\sigma_e^2,$$

  which clearly depends on time t.
- Thus, a random walk process is not stationary.

A random walk with drift process is not stationary

- Suppose that $\{Y_t\}$ is a random walk with drift process; that is,

$$Y_t = \theta_0 + Y_{t-1} + e_t.$$

- $\mu_t = \mathsf{E}(Y_t) = t\theta_0$, which clearly is not free of time t.
- Additionally, $\mathsf{cov}(Y_t, Y_{t-k}) = (t-k)\sigma_e^2$ remains unchanged.
- Thus, a random walk with drift process is not stationary.

A moving average process is stationary

- Suppose that $\{Y_t\}$ is a moving average process given by

$$Y_t = \frac{1}{3}\left(e_t + e_{t-1} + e_{t-2}\right).$$

- We calculated $\mu_t = \mathsf{E}(Y_t) = 0$ (which is free of t) and $\gamma_k = \mathsf{cov}(Y_t, Y_{t-k})$ to be

$$\gamma_k = \begin{cases} \sigma_e^2/3, & k = 0 \\ 2\sigma_e^2/9, & k = 1 \\ \sigma_e^2/9, & k = 2 \\ 0, & k > 2. \end{cases}$$

- Because $\mathsf{Cov}(Y_t, Y_{t-k})$ is free of time t, this moving average process is stationary.

An autoregressive process is stationary

- Suppose that $\{Y_t\}$ is the autoregressive process

$$Y_t = 0.7Y_{t-1} + e_t.$$

- We avoided the calculation of $\mu_t = \mathsf{E}(Y_t)$ and $\mathsf{Cov}(Y_t, Y_{t-k})$ for this process, so we will not make a definite determination here.

- It turns out that if e_t is independent of $Y_{t-1}, Y_{t-2}, \cdots$, and if $\sigma_e^2 > 0$, then this autoregressive process is stationary (details coming later).

A sinusoidal process is not stationary

- Suppose that $\{Y_t\}$ is the sinusoidal process defined by

$$Y_t = a\,\mathsf{sin}(2\pi\omega t + \phi) + e_t.$$

- Clearly $\mu_t = \mathsf{E}(Y_t) = a\,\mathsf{sin}(2\pi\omega t + \phi)$ is not free of t, so this sinusoidal process is not stationary.

A random cosine wave process is stationary

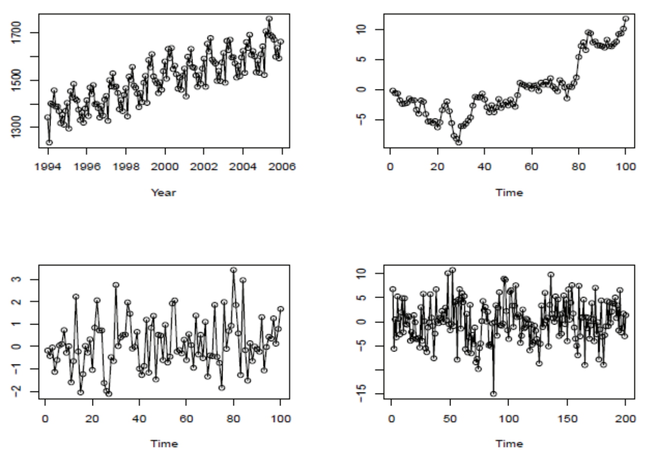
- Consider the random cosine wave process

$$Y_t = \cos\left[2\pi\left(\frac{t}{12} + \Phi\right)\right],$$

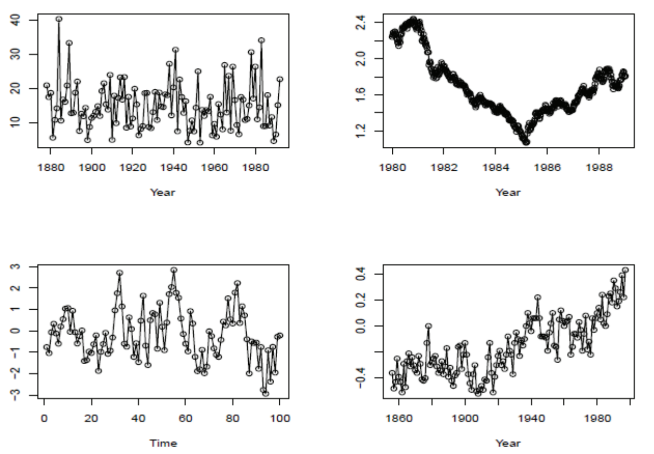
where Φ is a uniform random variable from 0 to 1; i.e., $\Phi \sim U(0,1)$.

- The calculations on pp 18-19 (CC) show that this process is (perhaps unexpectedly) stationary.

Which plots appear stationary?



Which plots appear stationary?



Remarks for stationary processes

Importance for stationary processes:

- In order to start thinking about viable stationary time series models for real data, we need to have a stationary process.
- However, as we have just seen, many data sets exhibit non-stationary behavior.
- A simple, but effective, technique to convert a non-stationary process into a stationary one is to examine data differences.

Transforming a non-stationary process to a stationary one

- Definition: Consider the process $\{Y_t : t = 0, 1, 2, \cdots, n\}$. The (first) difference process of $\{Y_t\}$ is defined by

$$\nabla Y_t = Y_t - Y_{t-1}, \text{ for } t = 1, 2, \cdots, n.$$

- In many realistic examples, a non-stationary process can be transformed into a stationary process by taking (first-order) differences.

- For example, the random walk $Y_t = Y_{t-1} + e_t$ is not stationary. However, it follows immediately that $\nabla Y_t = Y_t - Y_{t-1}$ is stationary. For a visual depiction, see Figure 2.7.

Figure 2.7. non-stationary RW and its stationary difference

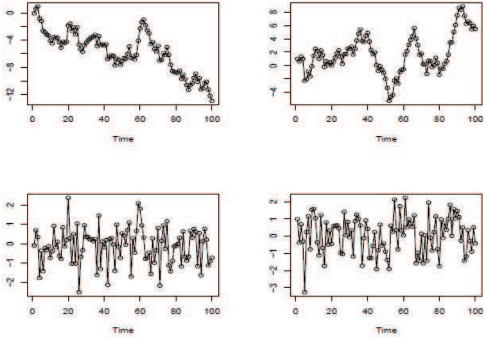


Figure 2.7: *The top two processes are each random walks* $Y_t = Y_{t-1} + e_t$, *where* $e_t \sim$ *iid* $\mathcal{N}(0, 1)$. *The bottom processes are the difference processes, respectively.*

Thank you for your attention !