Financial Time Series and Their Characteristics

Yanping YI

August 2, 2012

Table of contents

Introduction

- Financial time series analysis is concerned with the theory and practice of asset valuation over time.
- Financial time series analysis is highly related to other time series analysis, but with some added uncertainty.
- Financial time series must deal with the ever-changing business & economic environment and the fact that **volatility** is not directly observed.

Objective of the Course

- Provide some basic knowledge of financial time series data such as skewness, heavy tails, and measure of dependence between asset returns
- Introduce some statistical tools & econometric models useful for analyzing these series
- Gain experience in analyzing financial time series

Objective of the Course

- Simple linear time series models: AR, MA, ARMA
- Unit root nonstationarity
- Volatility modeling: ARCH, GARCH
- Methods for assessing market risk, credit risk, and expected loss. The methods discussed include Value at Risk, expected shortfall and tail dependence
- Analysis of high-dimensional asset returns, including cointegration and ECM

Course Requirements

- Textbook : Ruey S. Tsay (2010): Analysis of Financial Time Series, Third Edition (Wiley Series in Probability and Statistics)
- Dataset is free online http://faculty.chicagobooth.edu/ruey.tsay/teaching/fts3/
- One midterm and one final exam
- Two lab sessions
- 3 or 4 problem sets

Examples of Financial Time Series

- Daily log returns of Apple stock: 2000-2009
- US monthly interest rates (3m & 6m Treasury bills) Relations between the two series? Term structure of interest rates
- Exchange rate between US Dollar vs Euro
- Transformations to achieve stationarity
- ...

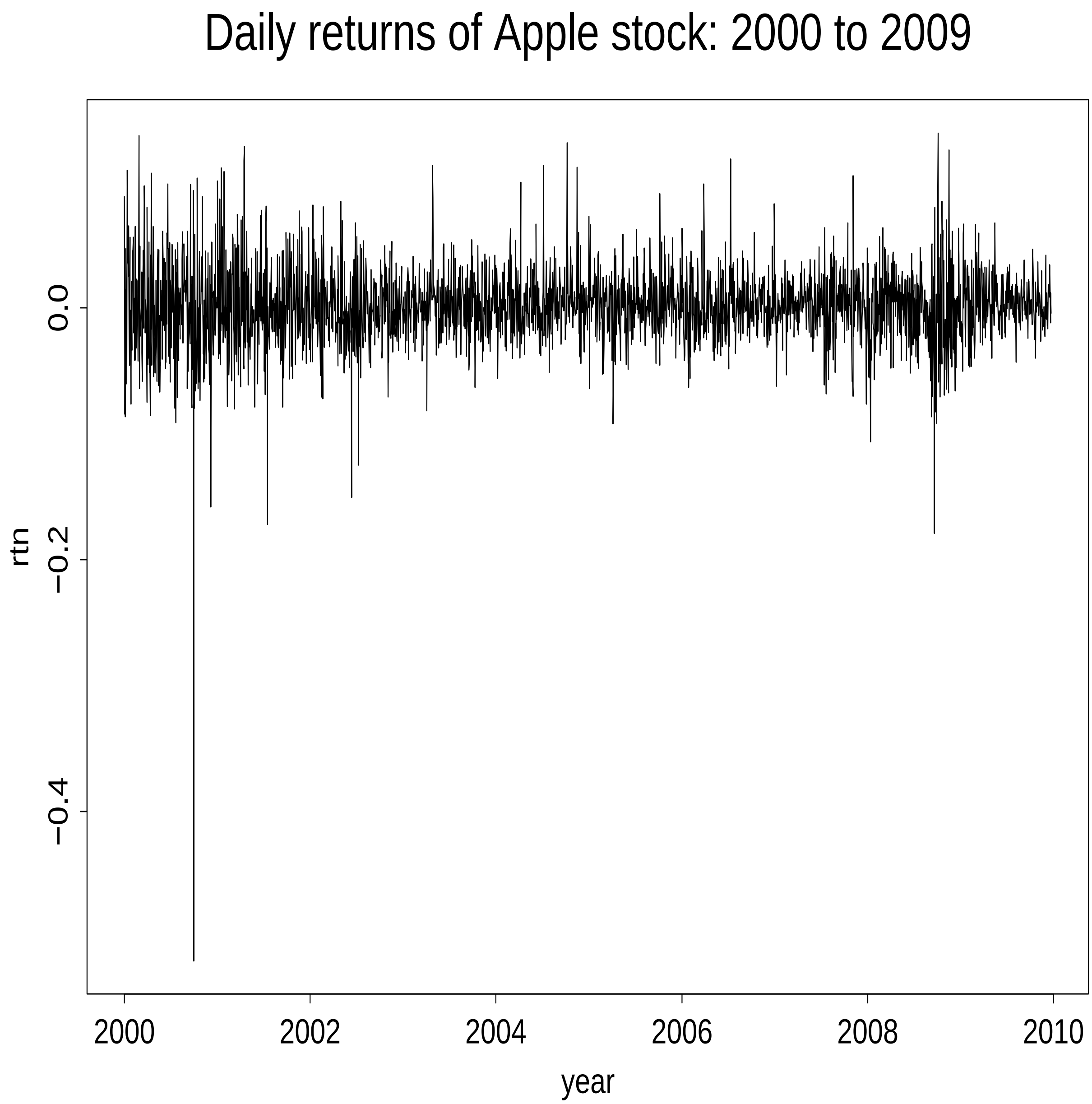

Figure 1: Daily log returns of Apple stock from 2000 to 2009

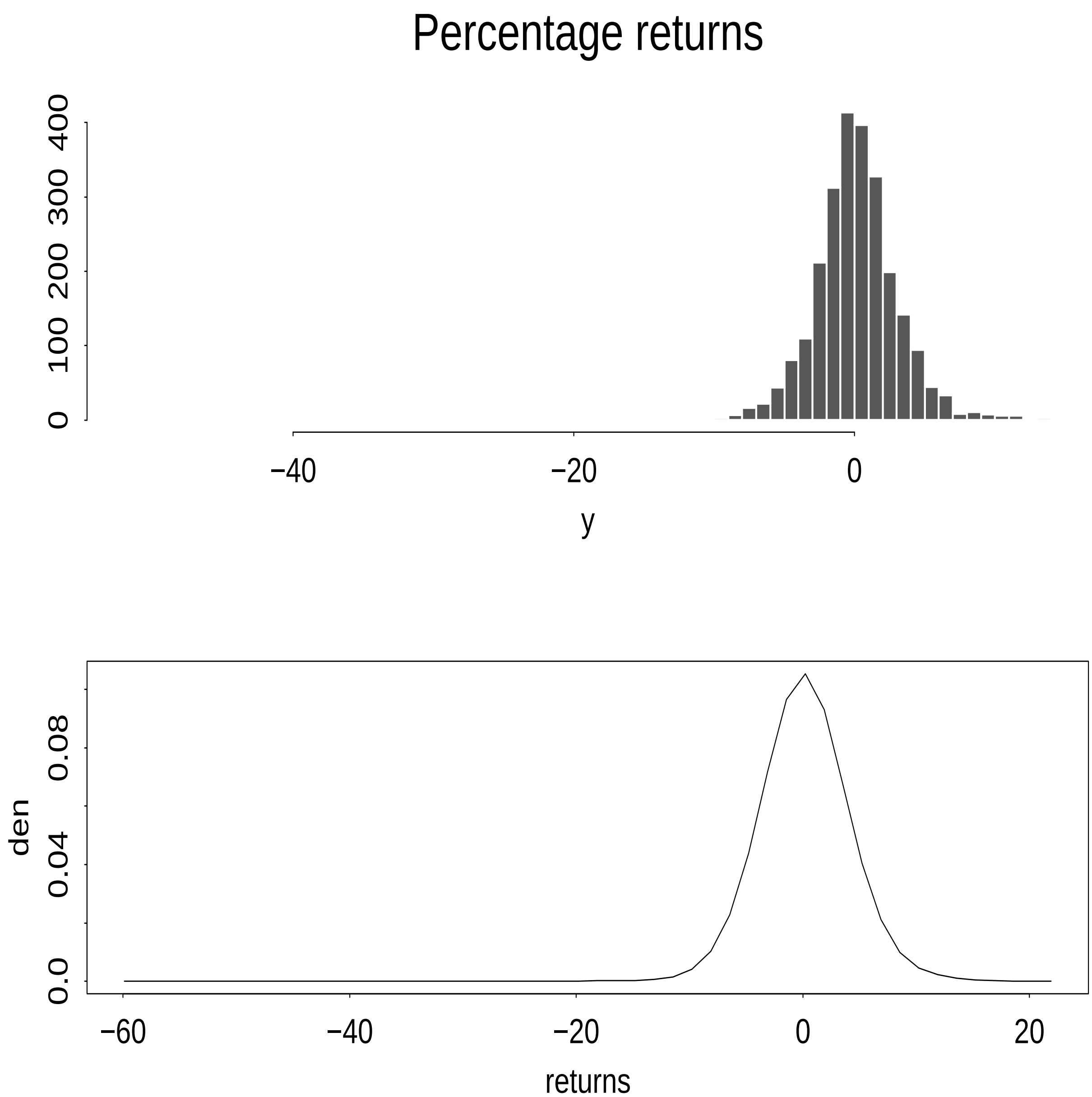

Figure 2: Density of daily Apple stock returns

Figure 5: Daily Exchange Rate: Dollars per Euro

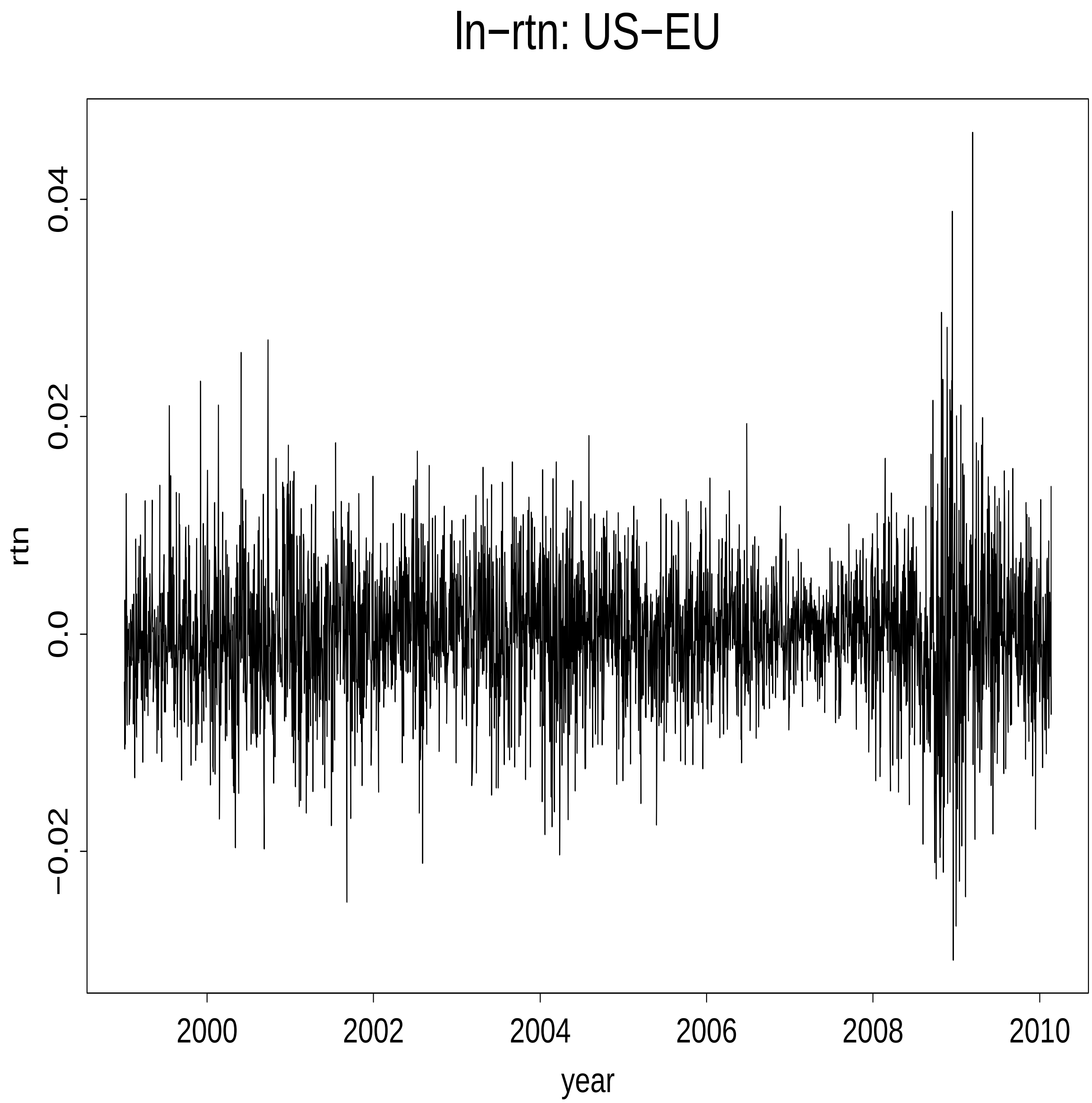

Figure 6: Daily log returns of FX (Dollar vs Euro)

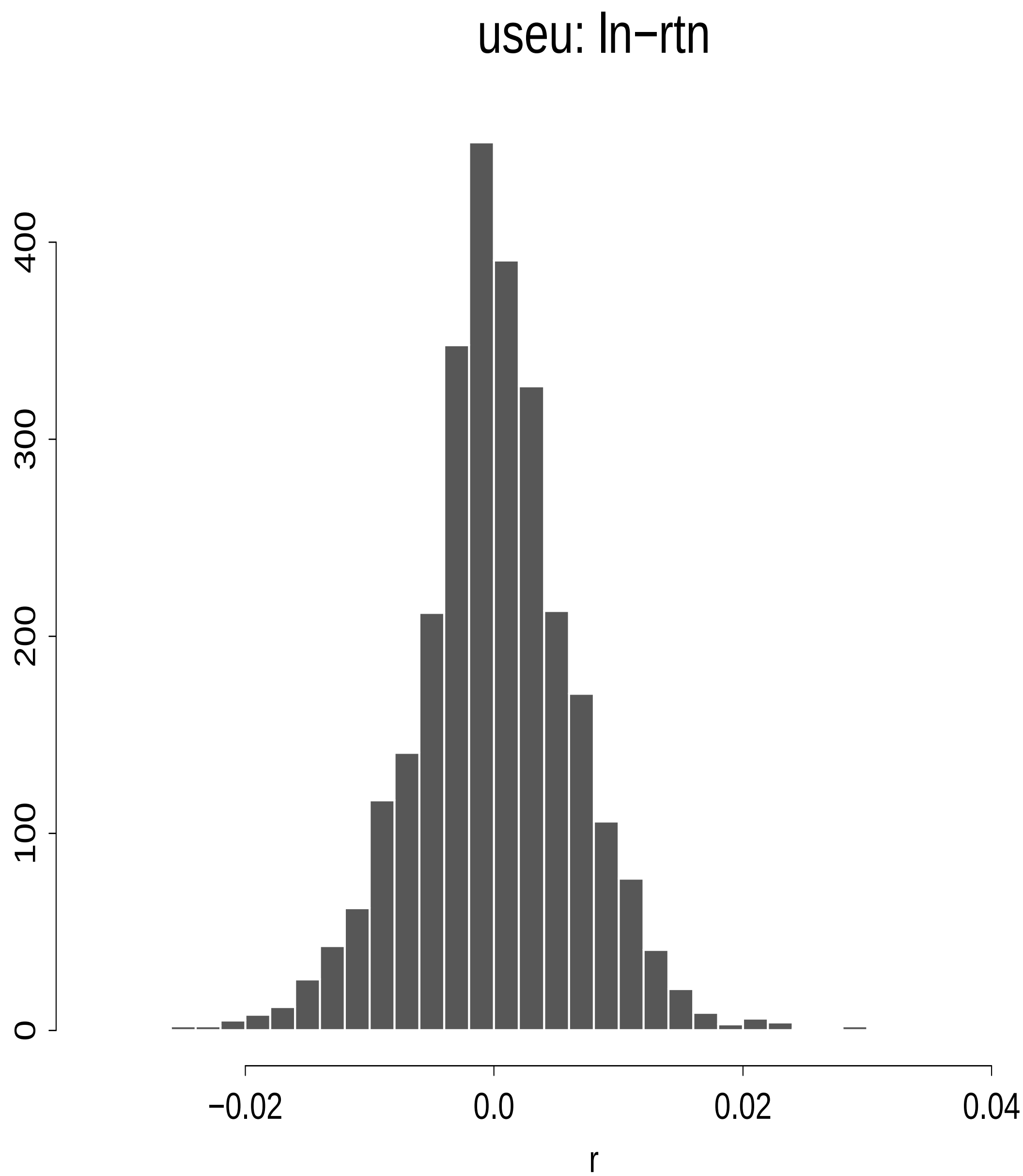

Figure 7: Histogram of daily log returns of FX (Dollar vs Euro)

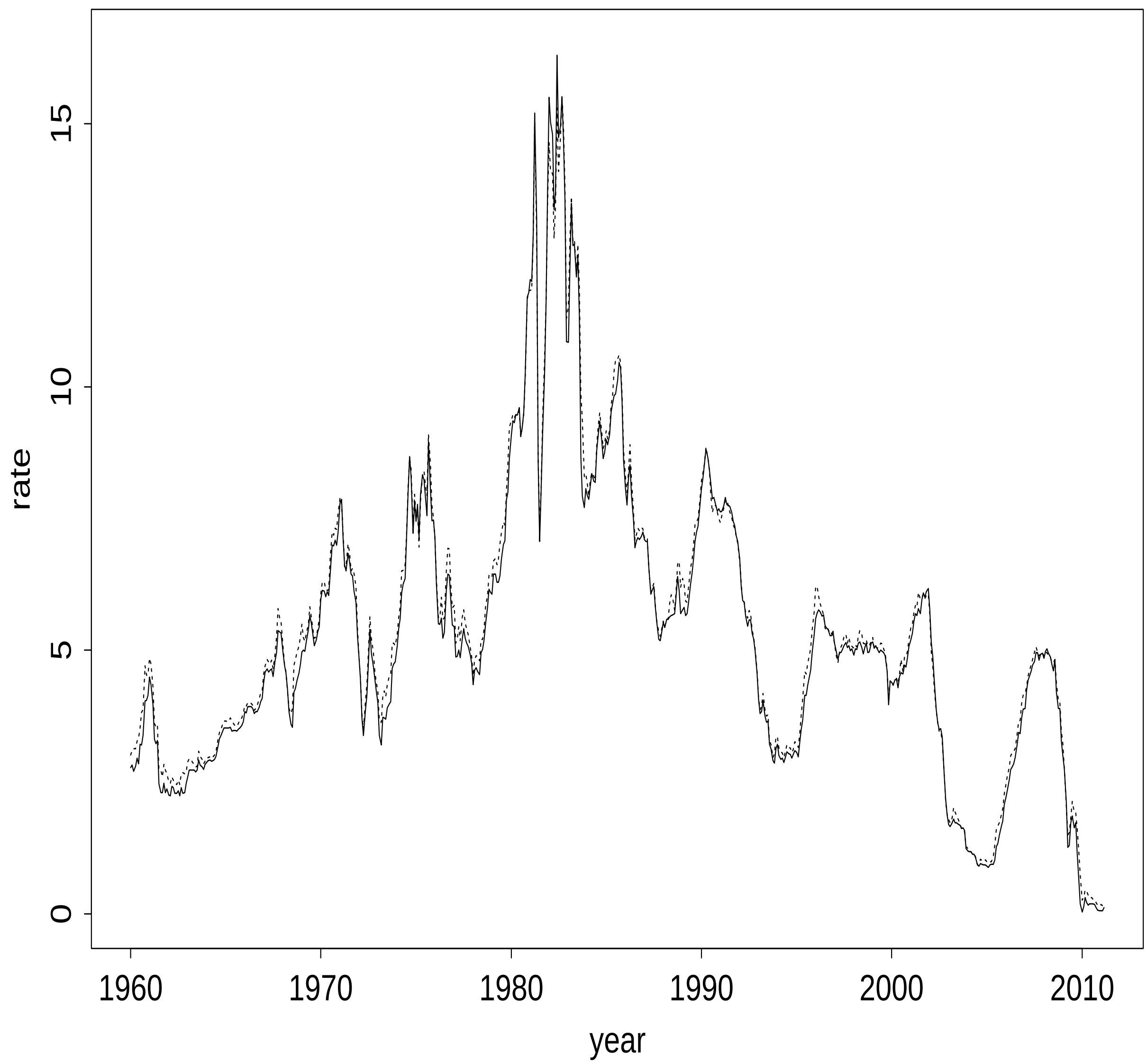

Figure 8: Monthly US interest rates: 3m & 6m TB

Asset Returns

- Most financial studies involve returns, instead of prices of assets, for two main reasons
  1. for average investors, return of an asset is a complete and scale-free summary of the investment opportunity
  2. return series are easier to handle than price series because the former have more attractive statistical properties (for example, stationarity)
- There are, however, several definitions of an asset return. Let P_t be the price of an asset at time t, and assume no dividends.

Discrete Returns

Simple Net Return

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \%\Delta P_t$$

Gross Return

$$1 + R_t = \frac{P_t}{P_{t-1}}$$

2−Period Return

$$R_t(2) = \frac{P_t - P_{t-2}}{P_{t-2}} = \frac{P_t}{P_{t-2}} - 1$$

$$\begin{aligned} R_t(2) &= \frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}} - 1 \\ &= (1 + R_t)(1 + R_{t-1}) - 1. \end{aligned}$$

$k-$ Period Return

$$\begin{aligned} 1 + R_t(k) &= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1}) \\ &= \prod_{j=0}^{k-1} (1 + R_{t-j}). \end{aligned}$$

Example 1

Suppose the daily closing prices of a stock are

Day	1	2	3	4	5
Price	37.84	38.49	37.12	37.60	36.30

1. What is the simple return from day 1 to day 2?
2. What is the simple return from day 1 to day 5?
3. Verify that $1 + R_5(4) = (1 + R_2)(1 + R_3) \cdots (1 + R5)$.

Annualized Returns

$R_t(k) = k-$year return. Define $R_A =$ effective annual rate

$$
\begin{aligned}
(1+R_A)^k &= 1 + R_t(k) \\
R_A &= \left(\prod_{j=0}^{k-1} (1 + R_{t-j}) \right)^{1/k} - 1 \\
&= \text{geometric average}
\end{aligned}
$$

Adjusting for Dividends (Total Returns)

$$R_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}}$$

Adjusting for Inflation (Real Returns)

$$1 + R_t^{\text{Real}} = \frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}$$

Portfolio Return

$$\begin{aligned} P_{p,t} &= \sum_{i=1}^{n} w_i P_{i,t}, \sum_{i=1}^{n} w_i = 1 \\ R_{p,t} &= \sum_{i=1}^{n} w_i R_{i,t} \end{aligned}$$

Excess Returns

$$\begin{aligned} Z_t &= R_t - R_{ft} \\ R_{ft} &= \text{T-bill rate or LIBOR rate} \end{aligned}$$

Example 2

An investor holds stocks of IBM, Microsoft and Citi-Group. Assume that her capital allocation is 30%, 30% and 40%. Use the monthly simple returns in Table 1.2. (textbook) What is the mean simple return of her stock portfolio?

Continuously Compounded Returns

$$
\begin{aligned}
r_t &= \ln(1+R_t) = \ln\left(\frac{P_t}{P_{t-1}}\right) \\
&= \ln(P_t) - \ln(P_{t-1}) \\
&= p_t - p_{t-1} \\
e^{r_t} &= 1 + R_t = \frac{P_t}{P_{t-1}} \\
&\Longrightarrow P_t = P_{t-1}e^{r_t}
\end{aligned}
$$

Note:

$$R_t = e^{r_t} - 1$$

2−period return

$$
\begin{aligned}
r_t(2) &= \ln(1 + R_t(2)) = \ln\left(\frac{P_t}{P_{t-2}}\right) = p_t - p_{t-2} \\
&= \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{P_{t-1}}{P_{t-2}}\right) \\
&= \ln\left(\frac{P_t}{P_{t-1}}\right) + \ln\left(\frac{P_{t-1}}{P_{t-2}}\right) \\
&= r_t + r_{t-1}.
\end{aligned}
$$

$k-$period return

$$
\begin{aligned}
r_t(k) &= \ln(1 + R_t(k)) = \ln\left(\frac{P_t}{P_{t-k}}\right) = p_t - p_{t-k} \\
&= \sum_{j=0}^{k-1} r_{t-j}
\end{aligned}
$$

Example 3

Use the daily prices in Example 1.

1. What is the log return from day 1 to day 2?
2. What is the log return from day 1 to day 5?
3. It is easy to verify $r_5(4) = r_2 + \cdots + r_5$.

Annualized Returns

$r_t(k) = \sum_{j=0}^{k-1} r_{t-j} = k-$year cc return. The average annual cc return, r_A, is

$$\begin{aligned} r_A &= \frac{1}{k}\sum_{j=0}^{k-1} r_{t-j} \\ &= \text{arithmetic average} \end{aligned}$$

Adjusting for Dividends (Total Returns)

$$
\begin{aligned}
r_t &= \ln(1+R_t) = \ln\left(\frac{P_t + D_t}{P_{t-1}}\right) \\
&= \ln(P_t + D_t) - \ln(P_{t-1})
\end{aligned}
$$

Adjusting for Inflation (Real Returns)

$$
\begin{aligned}
r_t^{\text{Real}} &= \ln(1+R_t^{\text{Real}}) = \ln\left(\frac{P_t}{P_{t-1}} \cdot \frac{CPI_{t-1}}{CPI_t}\right) \\
&= r_t - \pi_t
\end{aligned}
$$

Portfolio Return

$$
\begin{aligned}
r_{t,p} &= \ln(1 + R_{t,p}) \\
&= \ln\left(1 + \sum_{i=1}^{n} w_i R_{i,t}\right) \\
&\neq \sum_{i=1}^{n} w_i r_{i,t}
\end{aligned}
$$

But

$$
r_{t,p} \approx \sum_{i=1}^{n} w_i r_{i,t} \text{ if } R_{i,t} \text{ is small}
$$

Excess Returns

$$\begin{aligned} Z_t &= R_t - R_{ft} \\ z_t &= \ln(Z_t) = \ln(R_t - R_{ft}) \neq r_t - r_{ft} \end{aligned}$$

But if Z_t is small then

$$z_t \approx r_t - r_{ft}$$

Distributional Properties of Returns

Distributional Properties of Returns

Let $\tilde{r}_t$ be a random variable denoting the cc return on an asset at time t. Let $\{r_t, \ldots, r_T\}$ denote a sample of size T where r_t is a realization of the random variable $\tilde{r}_t$. We want to characterize

- Unconditional distributions of individual returns, $\tilde{r}_t$

- Unconditional distributions of returns ordered in time, $\{\tilde{r}_1, \ldots, \tilde{r}_T\}$

- Conditional distribution of $\tilde{r}_t$ given $\tilde{r}_{t-1} = r_{t-1}, \tilde{r}_{t-2} = r_{t-2}, \ldots$

Unconditional Distributions

$$
\begin{aligned}
f_t(r_t) &= \text{pdf s.t. } \int f_t(r_t)\, dr_t = 1 \\
F_t(x) &= \Pr(\tilde{r}_t < x) = \text{ CDF}
\end{aligned}
$$

Note: distribution may depend on t

Quantiles

$$
\begin{aligned}
F_t(q_\alpha) &= \alpha,\ 0 \leq \alpha \leq 1 \\
&\Longrightarrow q_\alpha = F_t^{-1}(\alpha)
\end{aligned}
$$

Joint Distribution

$$
\begin{aligned}
f_{1:T}(r_1, \ldots, r_T) &= \text{joint pdf} \\
F_{1:T}(x_1, \ldots, x_T) &= \Pr(\tilde{r}_1 \leq x_1, \ldots, \tilde{r}_T \leq x_T) \\
&= \text{joint CDF}
\end{aligned}
$$

Marginal Distribution

$$
f_1(r_1) = \int \cdots \int f_{1:T}(r_1, \ldots, r_T) dr_2 \cdots dr_T
$$

Stochastic Processes

A *stochastic process* $\{\tilde{r}_t\}_{t=1}^{\infty}$ is a sequence of random variables indexed by time t :

$$\{\ldots, \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t, \tilde{r}_{t+1}, \ldots\}$$

A realization of a stochastic process is the sequence of observed data $\{r_t\}_{t=1}^{\infty}$:

$$\{\ldots, \tilde{r}_1 = r_1, \tilde{r}_2 = r_2, \ldots, \tilde{r}_t = r_t, \tilde{r}_{t+1} = r_{t+1}, \ldots\}$$

Defn: A stochastic process $\tilde{\{r_t\}}_{t=1}^{\infty}$ is *strictly stationary* if, for any given finite integer s and for any set of subscripts $t_1, t_2, \ldots, t_s$ the joint distribution of

$$(\tilde{r}_t, \tilde{r}_{t_1}, \tilde{r}_{t_2}, \ldots, \tilde{r}_{t_s})$$

depends only on $t_1 - t, t_2 - t, \ldots, t_s - t$ but not on t.

Remarks

- For stationary returns, we drop the time subscripts on f and F

- For simplicity, we assume stationary returns for what follows

Conditional Distributions

$$f(r_2|r_1) = f(r_2|\tilde{r}_1 = r_1) = \frac{f(r_1, r_2)}{f(r_1)}, \;\; f(r_1) > 0$$

Useful factorization

$$\begin{aligned} f(r_1, \ldots, r_T) &= f(r_1) f(r_2|r_1) f(r_3|r_2, r_1) \\ &\quad \cdots f(r_T|r_{T-1}, \ldots, r_1) \end{aligned}$$

Stylized Statistical Properties of Asset Returns

- Absence of autocorrelations: (linear) autocorrelations of asset returns are often insignificant, except for very small intraday time scales for which microstructure effects come into play.
- Heavy tails: the (unconditional) distribution of returns displays heavy tails
- Gain/loss asymmetry: one observes large drawdowns in stock prices and stock index values but not equally large upward movements.

Stylized Statistical Properties of Asset Returns

- Aggregational Gaussianity: as one increases the time scale over which returns are calculated, their distribution looks more and more like a normal distribution. In particular, the shape of the distribution is not the same at different time scales.
- Volatility clustering: different measures of volatility display a positive autocorrelation over several days, which quantifies the fact that high-volatility events tend to cluster in time.
- Volume/volatility correlation: trading volume is correlated with all measures of volatility.

Stylized Statistical Properties of Asset Returns

- Conditional heavy tails: even after correcting returns for volatility clustering (e.g. via GARCH-type models), the residual time series still exhibit heavy tails. However, the tails are less heavy than in the unconditional distribution of returns.
- Slow decay of autocorrelation in absolute returns: the autocorrelation function of absolute returns decays slowly as a function of the time lag (a sign of long-range dependence)
- Leverage effect: most measures of volatility of an asset are negatively correlated with the returns of that asset.
- Volatility co-movements: evidence of common factors to explain volatility in multiple series

Shape Characteristics

Let $\tilde{r}$ be a random variable with pdf f

$$
\begin{aligned}
\mu &= E[r] : \text{ center} \\
\sigma^2 &= \text{var}(r) = E[(r-\mu)^2] : \text{ spread} \\
\text{skew}(r) &= E\left[\frac{(r-\mu)^3}{\sigma^3}\right] : \text{ symmetry} \\
\text{kurt}(r) &= E\left[\frac{(r-\mu)^4}{\sigma^4}\right] : \text{ tail thickness}
\end{aligned}
$$

Note: The k^{th} moment and central moment of $\tilde{r}$ is

$$
\begin{aligned}
m_k' &= E[\tilde{r}^k] \\
m_k &= E[(\tilde{r}-\mu)^k]
\end{aligned}
$$

Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty \leq x \leq \infty$$

$$\begin{aligned}
E[X] &= \mu \\
\text{var}(X) &= \sigma^2 \\
\text{skew}(X) &= 0 \\
\text{kurt}(X) &= 3 \\
m_k &= 0 \text{ for } k \text{ odd}
\end{aligned}$$

Sample moments

Let $\{r_t, \ldots, r_T\}$ denote a random sample of size T where r_t is a realization of the random variable $\tilde{r}$.

$$
\begin{aligned}
\tilde{\mu} &= \frac{1}{T}\sum_{t=1}^{T} r_t, \ \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{\mu})^2 = \hat{m}_2 \\
\widehat{\text{skew}} &= \frac{\hat{m}_3}{\hat{\sigma}^3}, \ \widehat{\text{kurt}} = \frac{\hat{m}_4}{\hat{\sigma}^{\cancel{3}\,4}} \\
\hat{m}_k &= \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{\mu})^k,
\end{aligned}
$$

Note: we divide by $T-1$ to get unbiased estimates. Check software to see how moments are computed.

Testing for Normality

- QQ-plot: plot standardized empirical quantiles vs. theoretical quantiles from specified distribution.

- Jarque-Bera (JB) test for normality

$$\begin{aligned} \mathsf{JB} &= \frac{T}{6}\left(\widehat{\mathsf{skew}}^2 + \frac{(\widehat{\mathsf{kurt}}-3)^2}{4}\right) \\ &\overset{A}{\sim} \chi^2(2) \end{aligned}$$

Note: if $r \sim N(\mu, \sigma^2)$ then

$$\sqrt{T}\widehat{\mathsf{skew}} \sim N(0,6), \;\; \sqrt{T}(\widehat{\mathsf{kurt}} - 3) \sim N(0,24)$$

- Shapiro-Wilks (SW) test for normality: correlation coefficient between values used in QQ-plot

- Kolmogorov-Smirnov (KS) test compares the empirical CDF of returns with the CDF of the normal distribution (or any other assumed distribution)

  - Sort returns: $r_{(1)} \leq \cdots \leq r_{(T)}$ and compute empirical CDF $\hat{F}_r(r_{(t)}) = t/T$

  - Evaluate normal CDF: $\Phi\left(\frac{r_{(t)}-\hat{\mu}}{\hat{\sigma}}\right)$

  - Compute KS statistic: $KS = \sup_t \left|\Phi\left(\frac{r_{(t)}-\hat{\mu}}{\hat{\sigma}}\right) - t/T\right|$

KS converges to 0 almost surely under the null

Student's-t distribution

Let $Z \sim N(0,1)$, $W \sim \chi^2(v)$ such that Z and W are independent. Then

$$X = \frac{Z}{\sqrt{W/v}} \sim t_v$$

where t_v denotes a (~~standardized~~) Student's t distribution with v degrees of freedom. Note:

$$\begin{aligned} E[X] &= 0,\ \text{var}(X) = \frac{v}{v-2},\ v > 2 \\ \text{skew} &= 0,\ \text{kurt} - 3 = \frac{6}{\nu - 4},\ v > 4 \end{aligned}$$

Existence of moments depends on degrees of freedom (df) parameter ν. Cauchy = Student's-t with 1 df. Only density exists.

If $X \sim t_v$ then

$$Y = \mu + \frac{\sigma X}{\sqrt{v/(v-2)}}$$

has moments

$$E[Y] = \mu, \ \text{var}(Y) = \sigma^2$$

Density function

$$
\begin{aligned}
f(x;v) &= \left[\frac{\Gamma\{(v+1)/2}{(\pi v)^{1/2}\Gamma(v/2)}\right]\frac{1}{\{1+(x^2/v)\}^{(v+1)/2}} \\
\Gamma(t) &= \int_0^\infty x^{t-1}\,\mathrm{exp}(-x)dx = \text{ gamma function}
\end{aligned}
$$

The d.f. parameter v can be estimated by MLE.

Note: A simple method of moments estimator for v is based on kurtosis:

$$
\text{kurt} - 3 = \frac{6}{\nu - 4} \Rightarrow \nu = 6/(\text{kurt} - 3) + \ 4
$$

Defn: The stochastic process $\{\tilde{r}_t\}$ is *covariance stationary* if

$$\begin{aligned} E[\tilde{r}_t] &= \mu \text{ for all } t \\ cov(\tilde{r}_t, \tilde{r}_{t-j}) &= E[(\tilde{r}_t - \mu)(\tilde{r}_{t-j} - \mu)] = \gamma_j \text{ for all } t \text{ and any } j \end{aligned}$$

The parameter γ_j is called the j^{th} order or lag j *autocovariance* of $\{\tilde{r}_t\}$

The *autocorrelations* of $\{\tilde{r}_t\}$ are defined by

$$\rho_j = \frac{cov(\tilde{r}_t, \tilde{r}_{t-j})}{\sqrt{var(\tilde{r}_t)var(\tilde{r}_{t-j})}} = \frac{\gamma_j}{\gamma_0}$$

and a plot of ρ_j against j is called the *autocorrelation function* (ACF)

The lag j *sample autocovariance* and lag j *sample autocorrelation* are defined as

$$\begin{aligned}
\hat{\gamma}_j &= \frac{1}{T}\sum_{t=j+1}^{T}(r_t - \bar{r})(r_{t-j} - \bar{r}) \\
\hat{\rho}_j &= \frac{\hat{\gamma}_j}{\hat{\gamma}_0}
\end{aligned}$$

where $\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$ is the sample mean.

The sample ACF (SACF) is a plot of $\hat{\rho}_j$ against j.

Example: White noise (GWN) processes

Perhaps the most simple stationary time series is the *independent Gaussian white noise* process $\{\tilde{r}_t\} \sim iid\ N(0, \sigma^2) \equiv GWN(0, \sigma^2)$. This process has $\mu = \gamma_j = \rho_j = 0$ $(j \neq 0)$.

Two slightly more general processes are the independent *white noise* (IWN) process, $\{\tilde{r}_t\} \sim IWN(0, \sigma^2)$, and the *white noise* (WN) process, $\{\tilde{r}_t\} \sim WN(0, \sigma^2)$.

Both processes have mean zero and variance σ^2, but the IWN process has independent increments, whereas the WN process has uncorrelated increments.

The SACF is typically shown with 95% confidence limits about zero. These limits are based on the result that if $\{\tilde{r}_t\} \sim iid\ (0, \sigma^2)$ then

$$\hat{\rho}_j \overset{A}{\sim} N\left(0, \frac{1}{T}\right),\ j > 0.$$

The notation $\hat{\rho}_j \overset{A}{\sim} N\left(0, \frac{1}{T}\right)$ means that the distribution of $\hat{\rho}_j$ is approximated by normal distribution with mean 0 and variance $\frac{1}{T}$ and is based on the central limit theorem result $\sqrt{T}\hat{\rho}_j \xrightarrow{d} N(0, 1)$. The 95% limits about zero are then $\pm\frac{1.96}{\sqrt{T}}$.

Testing for White Noise

Consider testing the null hypothesis

$$H_0 : \{\tilde{r}_t\} \sim WN(0, \sigma^2)$$

Under the null, all of the autocorrelations ρ_j for $j > 0$ are zero. To test this null, Box and Pierce (1970) suggested the *Q-statistic*

$$Q(k) = T \sum_{j=1}^{k} \hat{\rho}_j^2$$

Under the null, $Q(k)$ is asymptotically distributed $\chi^2(k)$. In a finite sample, the Q-statistic may not be well approximated by the $\chi^2(k)$. Ljung and Box (1978) suggested the *modified Q-statistic*

$$MQ(k) = T(T+2) \sum_{j=1}^{k} \frac{\hat{\rho}_j^2}{T-j}$$