Times Series and Forecasting (IV)

Chapter 4. Models for Stationary Time Series

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Recall

- In the last chapter, we used regression to "detrend" time series data with the hope of removing non-stationary patterns and producing residuals that resembled a stationary process.
- We have also learned that differencing can be an effective technique to convert a non-stationary process into one which is stationary.
- In this chapter, we consider linear time series models for stationary processes.

General linear processes

- $\{e_t\}$ is a zero mean white noise process with $\text{var}(e_t) = \sigma_e^2$.
- A general linear process is defined by

$$Y_t = e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + \cdots .$$

  At time t, Y_t is a weighted linear combination of WN terms at the current and past times.

- In general, $\text{E}(Y_t) = 0$ and

$$\gamma_k = \text{Cov}(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{i=0}^{\infty} \Psi_i \Psi_{i+k},$$

  (assuming stationarity) for $k \geq 0$, without loss to set $\Psi_0 = 1$.

- The processes that we examine in this chapter are special cases of this general linear process.

Conditions for general linear processes

- For mathematical reasons (to ensure stationarity), we will assume that the Ψ's are square summable, that is,

$$\sum_{i=1}^{\infty} \Psi_i^2 < \infty.$$

- A nonzero mean μ could be added to the right-hand side of the process above; this would not affect the stationarity properties of Y_t. Therefore, without loss of generality, we will henceforth assume that the process $\{Y_t\}$ has zero mean.

4.2. Moving average process of order q, denoted by MA(q)

- The process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

  is called a moving average process of order q, denoted by MA(q).

- Note that this is a special case of the general linear process with $\Psi_0 = 1$, $\Psi_1 = -\theta_1$, $\Psi_2 = -\theta_2$, $\cdots$, $\Psi_q = -\theta_q$ and $\Psi_{q^*} = 0$ for all $q^* > q$.

4.2.1. MA(1) process

- The moving average process of order 1, MA(1) process, is

$$Y_t = e_t - \theta e_{t-1}.$$

- For an MA(1) process, $\mathrm{E}(Y_t) = 0$ and the autocovariance is given by

$$\gamma_k = \begin{cases} \sigma_e^2(1+\theta^2), & k = 0 \\ -\theta\sigma_e^2, & k = 1 \\ 0, & k > 1. \end{cases}$$

- The autocorrelation function is given by

$$\rho_k = \mathrm{corr}(Y_t, Y_{t-k}) = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1, & k = 0 \\ -\theta/(1+\theta^2), & k = 1 \\ 0, & k > 1. \end{cases}$$

Remarks for the MA(1) process

- The MA(1) process has zero correlation beyond lag k=1!
- This fact is important to keep in mind when we entertain models for data using empirical evidence (e.g., estimates of autocorrelations, etc.).
- There are the following facts that
  - the largest ρ_1 can be $1/2$, which occurs when $\theta = -1$
  - the smallest ρ_1 can be $-1/2$, which occurs when $\theta = 1$
  - $\rho_1 = 0$ when $\theta = 0$. Note that when $\theta = 0$, the MA(1) process reduces to a white noise process.

MA(1) with four different value of θ

- MA(1) processes $Y_t = e_t - \theta e_{t-1}$ with $\theta = -0.9, -0.2, 0.2, 0.9,$ where $e_t \sim$ iid$N(0,1)$,
  - $\theta = -0.9 \Longrightarrow \rho_1 = 0.497.$
  - $\theta = 0.9 \Longrightarrow \rho_1 = -0.497.$
  - $\theta = -0.2 \Longrightarrow \rho_1 = 0.192.$
  - $\theta = 0.2 \Longrightarrow \rho_1 = -0.192.$
- θ and ρ_1 are linked through the equation

$$\rho_1 = \frac{-\theta}{1+\theta^2}.$$

Four simulated MA(1) series

- Use sas to simulate four time series for MA(1) with $\theta = -0.9, -0.2, 0.2, 0.9$ and $n = 100$.
- The $\theta = \pm 0.9$ series simulations $(\rho_1 \mp 0.497)$ to have somewhat notable lag 1 (positive/negative) autocorrelation.
- The $\theta = \pm 0.2$ series simulations $(\rho_1 \mp 0.192)$ should have fairly weak lag 1 autocorrelation.
- All processes should have negligible ACF beyond lag 1 (in theory, it is 0).

Figure 4.1. Four simulated MA(1) series

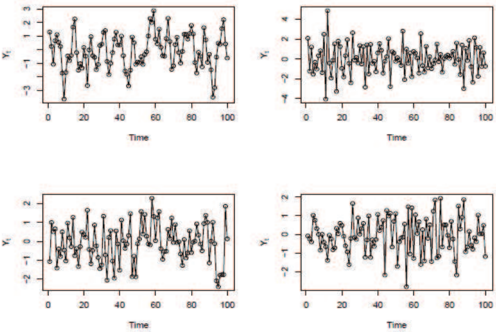


Figure 4.1: *MA(1) process simulations with* $e_t \sim$ *iid* $\mathcal{N}(0,1)$ *and* $n = 100$. *Upper left:* $\theta = -0.9$ ($\rho_1 = 0.497$). *Upper right:* $\theta = 0.9$ ($\rho_1 = -0.497$). *Lower left:* $\theta = -0.2$ ($\rho_1 = 0.192$). *Lower right:* $\theta = 0.2$ ($\rho_1 = -0.192$).

Observations for Figure 4.1

In Figure 4.1, we display four time series simulations from MA(1).

Note the differences between the series

- on the left (which exhibit positive lag 1 autocorrelation) and
- the series on the right (which exhibit negative lag 1 autocorrelation).

Figure 4.2. Sample ACFs of four simulated series

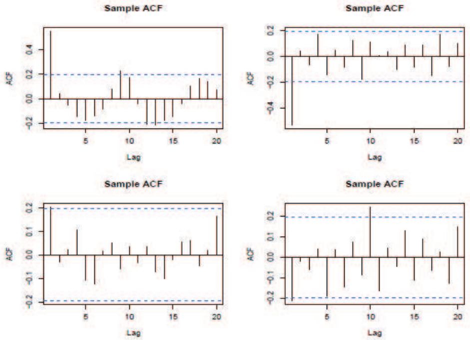


Figure 4.2: *Sample autocorrelation functions for the MA(1) process simulations in Figure 4.1. Upper left:* $\theta = -0.9$ $(\rho_1 = 0.497)$. *Upper right:* $\theta = 0.9$ $(\rho_1 = -0.497)$. *Lower left:* $\theta = -0.2$ $(\rho_1 = 0.192)$. *Lower right:* $\theta = 0.2$ $(\rho_1 = -0.192)$.

Observations of sample ACFs of four simulated series

- In Figure 4.2, we display the sample ACF.
- Note that in the top plots, which correspond to values of $\theta = \mp 0.9$ ($\rho_1 = \pm 0.497$), there is a notably significant lag 1 sample autocorrelation coefficient. The error bounds at ± 0.2 correspond to the error bounds for a white noise process.
- In the bottom plots, we see much milder evidence of significant lag 1 autocorrelations.

Figure 4.3. Lag 1 scatterplots

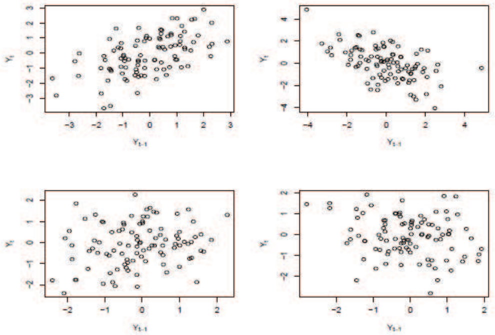


Figure 4.3: *Lag 1 scatterplots, i.e., plots of Y_t versus Y_{t-1}, for the MA(1) process simulations in Figure 4.1. Upper left: $\theta = -0.9$ ($\rho_1 = 0.497$). Upper right: $\theta = 0.9$ ($\rho_1 = -0.497$). Lower left: $\theta = -0.2$ ($\rho_1 = 0.192$). Lower right: $\theta = 0.2$ ($\rho_1 = -0.192$).*

Observations of Lag 1 scatterplots in Figure 4.3

- The last observation is confirmed in the lag 1 scatterplots; i.e., the scatterplots of Y_t versus Y_{t-1}, displayed in Figure 4.3.
- The top plots display a notable linear trend whereas the bottom plots do not.

Figure 4.4. Lag 2 scatterplots

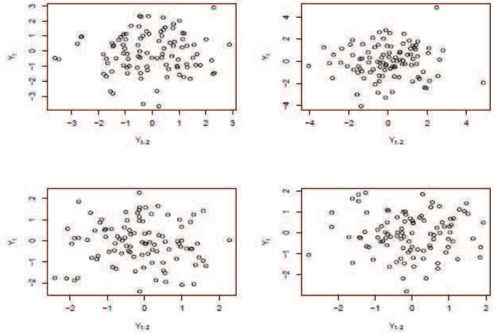


Figure 4.4: *Lag 2 scatterplots, i.e., plots of Y_t versus Y_{t-2}, for the MA(1) process simulations in Figure 4.1. Upper left:* $\theta = -0.9$ ($\rho_1 = 0.497$). *Upper right:* $\theta = 0.9$ ($\rho_1 = -0.497$). *Lower left:* $\theta = -0.2$ ($\rho_1 = 0.192$). *Lower right:* $\theta = 0.2$ ($\rho_1 = -0.192$).

Observations of lag 2 scatterplots in Figure 4.4

In Figure 4.4, we display the lag 2 scatterplots; i.e., the scatterplots of Y_t versus Y_{t-2}. As we would expect from the MA(1) model calculations shown earlier, where

- there is zero autocorrelation beyond lag 1, each plot exhibits random scatter.

4.2.2. MA(2) process

- $\{e_t\}$ is a zero mean white noise process with var$(e_t) = \sigma_e^2$.
- A moving average process of order 2, denoted by MA(2), is defined as

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}.$$

- For this process, E$(Y_t) = 0$ and autocovariance function is

$$\gamma_k = \begin{cases} \sigma_e^2(1+\theta_1^2+\theta_2^2), & k=0 \\ (-\theta_1+\theta_1\theta_2)\sigma_e^2, & k=1 \\ -\theta_2\sigma_e^2, & k=2 \\ 0, & k>2. \end{cases}$$

The MA(2) process (Cont.)

- The autocorrelation function $\rho_k = \text{corr}(Y_t, Y_{t-k})$ is given by

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1, & k = 0 \\ \frac{-\theta_1 + \theta_1\theta_2}{1+\theta_1^2+\theta_2^2}, & k = 1 \\ \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}, & k = 2 \\ 0, & k > 2. \end{cases}$$

An example for the MA(2) process

We use SAS to simulate an MA(2) realization of length $n = 100$ with $e_t \sim \text{iid}N(0,1)$, $\theta_1 = 0.9$, and $\theta_2 = -0.7$ so that

$$Y_t = e_t - 0.9e_{t-1} + 0.7e_{t-2}.$$

For these values of θ_1 and θ_2, we have

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{-0.9 + (0.9)(-0.7)}{1 + (0.9)^2 + (-0.7)^2} \approx -0.665$$

and

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} = \frac{0.7}{1 + (0.9)^2 + (-0.7)^2} \approx 0.304.$$

Figure 4.5. MA(2) simulation with N(0,1) white noise

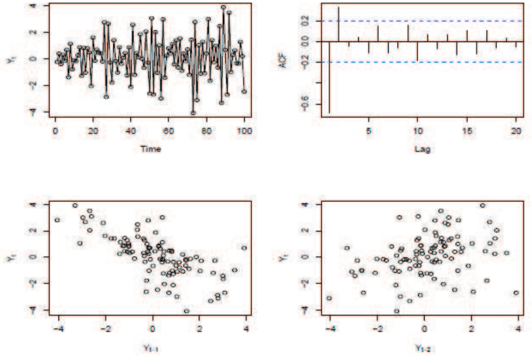


Figure 4.5: *MA(2) simulation of length* $n = 100$ *with* $e_t \sim$ *iid* $\mathcal{N}(0, 1)$, $\theta_1 = 0.9$, *and* $\theta_2 = -0.7$.

Observations of Figure 4.5

Figure 4.5 displays the simulated realization, the sample ACF, and the scatterplots of Y_t versus Y_{t-1} and Y_t versus Y_{t-2}.

- Note how the sample ACF declares the first two sample autocorrelations r_k significant (compared to white noise cutoffs).
- Not how the lagged scatterplots display negative (positive) autocorrelation at lag 1 (2).

4.2.3. MA(q) process

- The autocorrelation function for the general MA(q) process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

is given by

$$\rho_k = \begin{cases} 1, & k = 0 \\ \frac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}, & k = 1, 2, \cdots, q-1 \\ \frac{-\theta_q}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}, & k = q \\ 0, & k > q. \end{cases}$$

- The most important thing to note is that the autocorrelation function ρ_k is nonzero for $k = 1, 2, \cdots, q$ and that $\rho_k = 0$ for all $k > q$.

4.3. AR(p) process

- The process

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

  is called an autoregressive process of order p, denoted by AR(p), where $e_t \sim WN(0, \sigma_e^2)$.
- In this model, Y_t is a weighted linear combination of $Y_{t-1}, \cdots, Y_{t-p}$ plus a "shock" or "innovation" term e_t at time t.
- This model (stationarity) is a special case of the general linear process defined at the beginning of this chapter. It will be demonstrated later.

Convention of AR(p) process

- We continue to assume that $\mathsf{E}(Y_t) = 0$. A nonzero mean could be added to the model by replacing Y_t with $Y_t - \mu$ (for all t), and it would not affect stationarity.

- We assume that e_t, the innovation at time t, is independent of all previous process values $Y_{t-1}, Y_{t-2}, \cdots,$.

4.3.1. AR(1) process

- An AR(1) process is

$$Y_t = \phi Y_{t-1} + e_t.$$

- If $\phi = 1$, this process reduces to a random walk. If $\phi = 0$, this process reduces to white noise.
- The autocovariance function γ_k is

$$\gamma_k = \phi \gamma_{k-1} = \phi^k \left(\frac{\sigma_e^2}{1 - \phi^2} \right),$$

where $-1 < \phi < 1$ because $\gamma_0 > 0$.

- The ACF for the AR(1) process is given by

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{\phi^k \sigma_e^2 / (1 - \phi^2)}{\sigma_e^2 / (1 - \phi^2)} = \phi^k,$$

for $k = 1, 2, \cdots,$.

Remarks for AR(1) process

Importance: In an AR(1) process, because $|\phi| < 1$, the ACF will decay exponentially as k increases.

- If ϕ is close to ± 1, then the decay will be more slowly.
- If ϕ is not close to ± 1, then the decay will take place rapidly.
- If $\phi > 0$, then all of the autocorrelations will be positive.
- If $\phi < 0$, then the autocorrelations will alternate from negative ($k = 1$), to positive ($k = 2$), to negative ($k = 3$), to positive ($k = 4$), and so on.

It is important to remember these theoretical patterns so that, when we see sample ACFs (from real data), we can make good decisions about potential model selection.

Figure 4.6. Four theoretical ACF for AR(1)

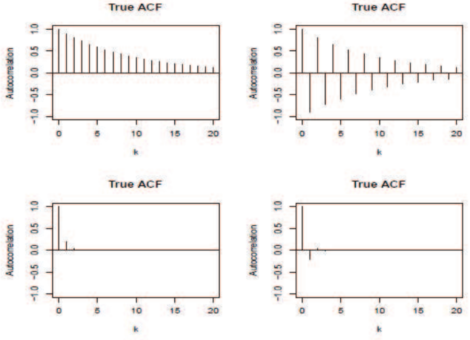


Figure 4.6: *Theoretical autocorrelation functions for AR*(1) *processes. Upper left:* $\phi = 0.9$. *Upper right:* $\phi = -0.9$. *Lower left:* $\phi = 0.2$. *Lower right:* $\phi = -0.2$. *Note that* $\rho_0 = 1$ *is included in all plots.*

Example 4.3

Example

We use SAS to simulate four different AR(1) processes

$$Y_t = \phi Y_{t-1} + e_t,$$

with $e_t \sim$ *iid*$N(0, 1)$ *and* $n = 100$. *We choose*

- $\phi = 0.9$ *(large* ρ_1*, ACF should decay slowly, all* ρ_k *positive)*
- $\phi = -0.9$ *(large* ρ_1*, ACF should decay slowly,* ρ_k *alternating)*
- $\phi = 0.2$ *(small* ρ_1*, ACF should decay quickly, all* ρ_k *positive)*
- $\phi = -0.2$ *(small* ρ_1*, ACF should decay quickly,* ρ_k *alternating).*

These choices of ϕ *are consistent with those in Figure 4.6, which depicts the true AR(1) ACF s.*

Figure 4.7. Four AR(1) process simulation with $\epsilon_t \sim N(0, 1)$

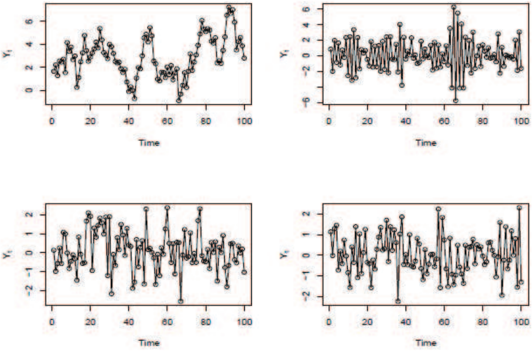


Figure 4.7: *AR*(1) *process simulations with* $e_t \sim$ *iid* $\mathcal{N}(0, 1)$ *and* $n = 100$. *Upper left:* $\phi = 0.9$. *Upper right:* $\phi = -0.9$. *Lower left:* $\phi = 0.2$. *Lower right:* $\phi = -0.2$.

Observation of Figure 4.7

In Figure 4.7, we display the time series simulations.

- The differences between the series on the left (which exhibit positive lag 1 autocorrelation) and
- the series on the right (which exhibit negative lag 1 autocorrelation).

Figure 4.8. Sample ACF for four AR(1) series

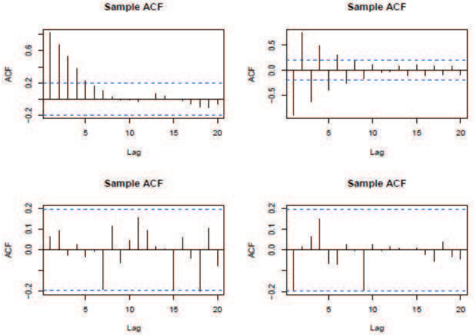


Figure 4.8: *Sample autocorrelation functions for the AR(1) process simulations with $e_t \sim$ iid $\mathcal{N}(0,1)$ and $n = 100$. Upper left: $\phi = 0.9$. Upper right: $\phi = -0.9$. Lower left: $\phi = 0.2$. Lower right: $\phi = -0.2$.*

Observation of Figure 4.8

In Figure 4.8, we display the sample ACF s.

- Compare these sample ACFs to the theoretical ACFs in Figure 4.6.
- Keep this in mind when comparing the sample ACFs to the true ACFs.

Figure 4.9

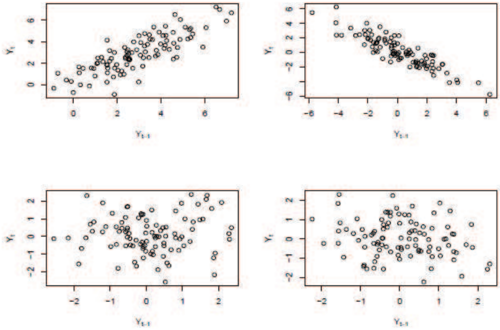


Figure 4.9: *Lag* 1 *scatterplots i.e., plots of* Y_t *versus* Y_{t-1}*, for the* $AR(1)$ *process simulations with* $e_t \sim iid\ \mathcal{N}(0,1)$ *and* $n = 100$*. Upper left:* $\phi = 0.9$*. Upper right:* $\phi = -0.9$*. Lower left:* $\phi = 0.2$*. Lower right:* $\phi = -0.2$*.*

Observation of Figure 4.9

- The lag 1 scatterplots; i.e., the scatterplots of Y_t versus Y_{t-1}, displayed in Figure 4.9 show notable linear trends in the top plots (the left is positive since $\rho_1 = 0.9$; the right is negative since $\rho_1 = -0.9$).
- The bottom plots exhibit weak correlation.

Figure 4.10

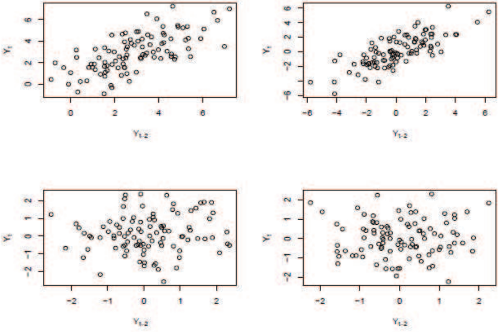


Figure 4.10: *Lag 2 scatterplots i.e., plots of Y_t versus Y_{t-2}, for the AR(1) process simulations with $e_t \sim$ iid $\mathcal{N}(0,1)$ and $n = 100$. Upper left: $\phi = 0.9$. Upper right: $\phi = -0.9$. Lower left: $\phi = 0.2$. Lower right: $\phi = -0.2$.*

Observation of Figure 4.10

- In Figure 4.10, we display the lag 2 scatterplots; i.e., the scatterplots of Y_t versus Y_{t-2}.
- Both top plots now show a positive linear trend because $\rho_2 = 0.81$ for both the $\phi = \pm 0.9$ processes.

General linear process representation of AR(1)

- The AR(1) process is

$$Y_t = \phi Y_{t-1} + e_t.$$

- AR(1) can be written in the form of a general linear process

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots ,$$

which is a special case of the general linear process with $\Psi_j = \phi^j$, for $j = 0, 1, 2, \cdots ,$.

Condition of stationarity of AR(1)

Theorem

Suppose $e_t \sim WN(0, \sigma_e^2)$*. If* e_t *is independent of* Y_{t-k}*, for* $k = 1, 2, \cdots$ *, then the AR(1) process*

$$Y_t = \phi Y_{t-1} + e_t$$

is stationary if and only if $|\phi| < 1$*, that is,* $-1 < \phi < 1$*.*

The AR(1) process is not stationary if and only if $|\phi| \geq 1$.

4.3.2. AR(2) process

- The AR(2) process is

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t.$$

- Assume that $\mathsf{E}(Y_t) = 0$; a nonzero mean μ could be added to model by replacing Y_t with $Y_t - \mu$ for all t.
- Assume that e_t is independent of Y_{t-k}, for all $k = 1, 2, \cdots$,.

Backsift operator

- Define the operator B to satisfy

$$BY_t = Y_{t-1},$$

that is, B "backs up" the current value Y_t one time unit to Y_{t-1}. For this reason, we call B the backshift operator.

- In general, $B^k Y_t = Y_{t-k}$ for $k \geq 0$.
- The AR(2) model can be rewritten as

$$Y_t = \phi_1 BY_t + \phi_2 B^2 Y_t + e_t.$$

Or

$$Y_t - \phi_1 BY_t - \phi_2 B^2 Y_t = e_t \Longleftrightarrow (1 - \phi_1 B - \phi_2 B^2) Y_t = e_t.$$

AR(2) characteristic equation

- Define the AR(2) characteristic polynomial as

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2.$$

- The corresponding AR(2) characteristic equation is

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 = 0.$$

- Discovering the stationarity conditions for the AR(2) model can be accomplished by examining this equation and the solutions to it; i.e., the roots of $\phi(x)$.

Roots of AR(2) characteristic equation

Applying the quadratic formula to the AR(2) characteristic equation, we see that the roots of $\phi(x)$ are

$$x = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}.$$

- The roots are both real if $\phi_1^2 + 4\phi_2 > 0$.
- The roots are both complex if $\phi_1^2 + 4\phi_2 < 0$.
- There is a single real root with multiplicity 2 if $\phi_1^2 + 4\phi_2 = 0$.

Stationarity conditions for AR(2)

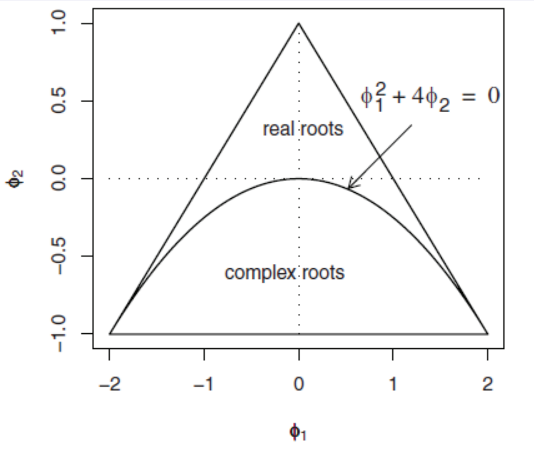
- Important: The AR(2) process is stationary when the roots of $\phi(x)$ both exceed 1 in absolute value (or in modulus if the roots are complex).
- The AR(2) process is stationary if and only if

$$\phi_1 + \phi_2 < 1 \qquad \phi_2 - \phi_1 < 1 \qquad |\phi_2| < 1$$

(see Appendix B, pp 84, CC).

- A sketch of this stationarity region (in the $\phi_1 - \phi_2$ plane) appears in Figure 4.11.

Figure 4.11. Stationary region for the AR(2) model



Yule-Walker equations

- For AR(2),

$$Y_t Y_{t-k} = \phi_1 Y_{t-1} Y_{t-k} + \phi_2 Y_{t-2} Y_{t-k} + e_t Y_{t-k}.$$

- Assuming stationarity and taking expectations gives Yule-Walker equations

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2},$$

and

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}.$$

ACF for AR(2)

- When $k = 1$ and $k = 2$, Yule-walker equations becomes

$$\rho_1 = \phi_1 + \phi_2 \rho_1, \;\; \rho_2 = \phi_1 \rho_1 + \phi_2 \;\; \text{with } \rho_0 = 1.$$

- Solving this system for ρ_1 and ρ_2, we get (verify!)

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} \qquad \text{and} \qquad \rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2}.$$

- For known values of ϕ_1 and ϕ_2, finding ρ_k, for $k \geq 3$, can be done using the recursive relationship above, using ρ_1 and ρ_2 above as "starting values".

General formulae of ACF for AR(2)

- Use $1/G_1$ and $1/G_2$ to denote the roots of the AR(2) characteristic polynomial $\phi(x)$.
- G_1 and G_2 can be written as

$$G_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$
$$G_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}.$$

- The AR(2) process is stationary if and only if these roots are both larger than 1 (in absolute value or modulus); i.e., G_1 and G_2 are both smaller than 1 (in absolute value or modulus).

General formulae of ACF for AR(2) (Cont.)

- If $G_1 \neq G_2$, then

$$\rho_k = \frac{(1-G_2^2)G_1^{k+1} - (1-G_1^2)G_2^{k+1}}{(G_1 - G_2)(1 + G_1 G_2)}.$$

- If $1/G_1$ and $1/G_2$ are complex (i.e., when $\phi_1^2 + 4\phi_2 < 0$), then we can express ρ_k as

$$\rho_k = R^k \frac{\sin(\Theta K + \Phi)}{\sin(\Phi)},$$

for $k \geq 0$, where $R = \sqrt{-\phi_2}$, $\Theta = \arccos(\phi_1/2\sqrt{-\phi_2})$, and $\Phi = \arctan[(1-\phi_2)/(1+\phi_2)]$.

- If $G_1 = G_2$, which occurs when $\phi_1^2 + 4\phi_2 = 0$, then, for $k \geq 0$,

$$\rho_k = \left[1 + k\left(\frac{1+\phi_2}{1-\phi_2}\right)\right]\left(\frac{\phi_1}{2}\right)^k.$$

Remarks to ACF of AR(2)

- In all instances, the autocorrelations ρ_k (in magnitude) decay exponentially as k increases.
- When the roots are complex, the values of ρ_k display a sinusoidal pattern that dampens out as k increases.

Example 4.4 of AR(2)

Example

Use SAS to simulate four different AR(2) processes

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t,$$

with $e_t \sim \text{iid} N(0,1)$ and $n = 100$. Choose

- $(\phi_1, \phi_2) = (0.5, -0.5)$. $\phi(x) = 1 - 0.5x + 0.5x^2$. C roots.
- $(\phi_1, \phi_2) = (1.1, -0.3)$. $\phi(x) = 1 - 1.1x + 0.3x^2$. 2-real roots.
- $(\phi_1, \phi_2) = (-0.5, 0.25)$. $\phi(x) = 1 + 0.5x - 0.25x^2$. 2-r roots.
- $(\phi_1, \phi_2) = (1, -0.5)$. $\phi(x) = 1 - x + 0.5x^2$. C roots.

Figure 4.12. True ACF of four different AR(2) processes

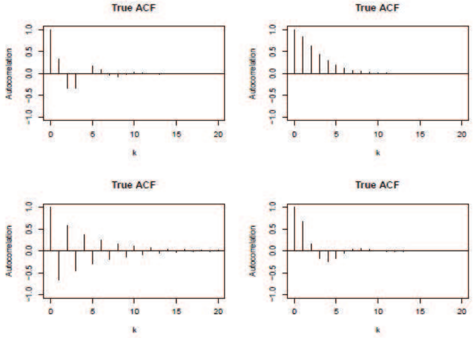


Figure 4.12: *True autocorrelation functions for AR(2) processes. Upper left:* $(\phi_1, \phi_2) = (0.5, -0.5)$. *Upper right:* $(\phi_1, \phi_2) = (1.1, -0.3)$. *Lower left:* $(\phi_1, \phi_2) = (-0.5, 0.25)$. *Lower right:* $(\phi_1, \phi_2) = (1, -0.5)$. *Note that* $\rho_0 = 1$ *is included in all plots.*

Figure 4.13. Time plot of four AR(2) simulated series

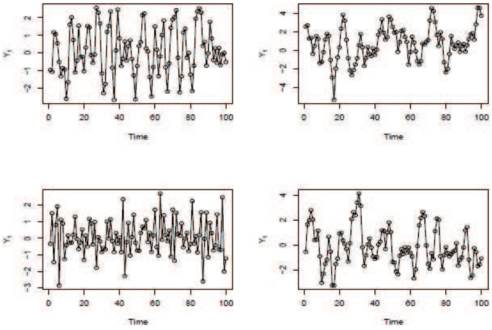


Figure 4.13: *AR(2) process simulations with* $e_t \sim$ *iid* $\mathcal{N}(0,1)$ *and* $n = 100$. *Upper left:* $(\phi_1, \phi_2) = (0.5, -0.5)$. *Upper right:* $(\phi_1, \phi_2) = (1.1, -0.3)$. *Lower left:* $(\phi_1, \phi_2) = (-0.5, 0.25)$. *Lower right:* $(\phi_1, \phi_2) = (1, -0.5)$.

Figure 4.14. Sample ACFs for AR(2) simulated series

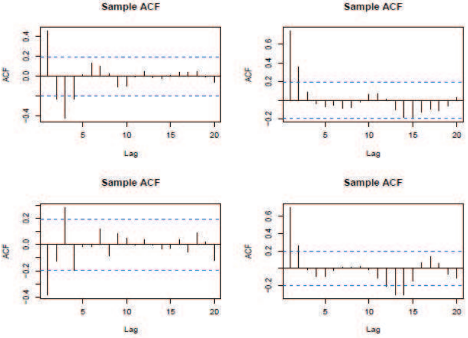


Figure 4.14: *Sample ACFs for the AR(2) simulations in Figure 4.13 with* $e_t \sim$ *iid* $\mathcal{N}(0,1)$ *and* $n = 100$. *Upper left:* $(\phi_1, \phi_2) = (0.5, -0.5)$. *Upper right:* $(\phi_1, \phi_2) = (1.1, -0.3)$. *Lower left:* $(\phi_1, \phi_2) = (-0.5, 0.25)$. *Lower right:* $(\phi_1, \phi_2) = (1, -0.5)$.

General linear process representation for AR(2)

- Variance: For the AR(2) process $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$, the variance of Y_t is

$$\gamma_0 = \text{var}(Y_t) = \left(\frac{1-\phi_2}{1+\phi_2}\right) \frac{\sigma_e^2}{(1-\phi_2)^2 - \phi_1^2}.$$

- Like the AR(1), the AR(2) model can be expressed as a general linear process

$$Y_t = e_t + \Psi_1 e_{t-1} + \Psi_2 e_{t-2} + \cdots .$$

- Coefficients of the linear process are

$$\Psi_j = \begin{cases} \frac{G_1^{j+1} - G_2^{j+1}}{G_1 - G_2}, & \text{if } G_1 \neq G_2 \\ R^j \frac{\sin[(j+1)\Theta]}{\sin(\Theta)}, & \text{if } G_1 \text{ and } G_2 \text{ are complex} \\ (1+j)\left(\frac{\phi_1}{2}\right)^j, & \text{if } G_1 = G_2, \end{cases}$$

for $j = 1, 2, \cdots,$, where $1/G_1$ and $1/G_2$ are the solutions of the AR(2) characteristic equation.

4.3.3. AR(p) process

- The general AR process of order p, denoted AR(p), is

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t.$$

- In backshift operator notation, we write the model as

$$\left(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p\right) Y_t = e_t.$$

- The AR(p) characteristic equation is

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0.$$

Sufficient and necessary conditions for stationarity

- If the roots of $\phi(x)$ are denoted by $1/G_i$, for $i = 1, 2, \cdots, p$, the AR(p) process is stationary if and only if

$$|1/G_i| > 1 \text{ or } |G_i| < 1, \text{ for } i = 1, 2, \cdots, p.$$

- Necessary (but not sufficient) conditions for stationarity of the AR(p) process are that

$$\phi_1 + \phi_2 + \cdots + \phi_p < 1 \quad \text{and} \quad |\phi_p| < 1.$$

Yule-Walker equations for AR(p) process

- Assuming stationarity and zero means, consider the AR(p) process equation and get

$$Y_t Y_{t-k} = \phi_1 Y_{t-1} Y_{t-k} + \cdots + \phi_p Y_{t-p} Y_{t-k} + e_t Y_{t-k}.$$

- Taking expectation for the above equation gets

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_p \gamma_{k-p}.$$

- Yule-Walker equations for AR(p) process are

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p} \quad \text{for } k = 1, \cdots, p.$$

with $\rho_0 = 1$ and $\rho_k = \rho_{-k}$.

Comments for Yule-Walker equations for AR(p) process

- So, for known values of $\phi_1, \phi_2, \cdots, \phi_p$, we can compute the first lag p autocorrelations $\rho_1, \rho_2, \cdots, \rho_p$.
- Values of ρ_k, for $k > p$, can be obtained by using the recursive relation above.
- It is possible to express closed-form formulae for ρ_k, in general, but it is not terribly important to do so.
- The ACF tails off as k gets larger.
- It does so as a mixture of exponential decays and/or damped sine waves, depending on if some of the roots are complex.

Variance for AR(p) process

For the AR(p) model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t,$$

the variance of Y_t is

$$\gamma_0 = \mathsf{var}(Y_t) = \frac{\sigma_e^2}{1 - \phi_1\rho_1 - \phi_2\rho_2 - \cdots - \phi_p\rho_p}.$$

4.4. Invertible processes

- In general, a process $\{Y_t\}$ is said to be invertible if the Y_t can written as a AR(p) or AR(∞) process.
- Obviously, any AR(p) process is invertible.
- Moving average process may not be invertible. So, invertibility is a property to MA(q) processes.
- Invertibility is an important technical property. For prediction purposes, it is important to restrict our attention to the physically sensible class of invertible models.

MA(1) converted into an AR(∞) under some conditions

- Consider the MA(1) model

$$Y_t = e_t - \theta e_{t-1} \text{ or } e_t = Y_t + \theta e_{t-1}.$$

- Repeating substitution reveals that

$$e_t = Y_t + \theta Y_{t-1} + \theta^2 e_{t-2} + \theta^3 e_{t-3} + \cdots ,$$

namely, a "AR(∞)" process,

$$Y_t = -\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots + e_t.$$

- The representation requires $|\theta| < 1$, yielding $\sum_{j=1}^{\infty} \theta^j < \infty$.
- The MA(1) process is invertible if and only if $|\theta| < 1$.
- The "invertibility condition" of $|\theta| < 1$ for MA(1) does correspond to the "stationarity condition" of $|\phi| < 1$ for the AR(1) model.

Importance of Invertibility

- A model must be invertible for us to be able to identify the model parameters associated with it.
- For example, for an MA(1) model, it can be show that both of the following processes have the same ACF :

$$Y_t = e_t - \theta e_{t-1} \text{ or } Y_t = e_t - \frac{1}{\theta} e_{t-1}.$$

- If we knew what their common ACF was, we would not be able to tell if the MA(1) model parameter was θ or $1/\theta$.
- Imposing the condition that $|\theta| < 1$ can help us to ensure invertibility (identifiability).
- Under this condition, the second MA(1) model, rewritten

$$Y_t = -\left(\frac{1}{\theta}\right) Y_{t-1} - \left(\frac{1}{\theta}\right)^2 Y_{t-2} - \left(\frac{1}{\theta}\right)^3 Y_{t-3} - \cdots + e_t,$$

is not meaningful because the series $\sum_{j=1}^{\infty} \left(\frac{1}{\theta}\right)^j$ diverges.

Backshift operator for MA(1) processes

- Rewrite the MA(1) model using backshift notation as

$$Y_t = (1 - \theta B)e_t.$$

- The function $\theta(x) = 1 - \theta x$ is called the MA(1) characteristic polynomial and

$$\theta(x) = 1 - \theta x = 0$$

  is called the MA(1) characteristic equation.
- The root to this equation is

$$x = \frac{1}{\theta}.$$

- For this process to be invertible, we require $|x| > 1$, implying that that $|\theta| < 1$.

Invertibility of MA(q) process

- The MA(q) process

$$\begin{aligned} Y_t =& e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q} \\ =& \left(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right) e_t \end{aligned}$$

is invertible if and only if the roots of the MA(q) characteristic polynomial

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$$

all exceed 1 in absolute value (or modulus).

Summary

- For the AR(p) process to be stationary, we need the roots of the AR characteristic polynomial

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

  to all exceed 1 in absolute value (or modulus).

- For the MA(q) process to be invertible, we need the roots of the MA characteristic polynomial

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$$

  to all exceed 1 in absolute value (or modulus).

Summary (continued)

- All MA processes are stationary. But not all of them is invertible. There exists a problem involving identifiability of the MA process.
- All AR processes are invertible not matter whether it is stationary.
- Any invertible MA(q) process corresponds to an AR(∞).
- Any stationary AR(p) process corresponds to a MA(∞) .

4.5. Autoregressive moving average (ARMA) processes

- The process

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}$$

is called a (mixed) autoregressive moving average process of orders p and q, written ARMA(p, q).

  - The value p is the order the autoregressive part of the model.
  - The value q is the order the moving average part of the model.
  - MA(q) process is the same as the process ARMA(0,q).
  - AR(p) processes is the same as the process ARMA(p,0).

Remark for ARMA processes

- The importance of the ARMA processes lies in the fact that a stationary time series may often be adequately modeled by an ARMA model involving fewer parameters than a pure MA or AR process by itself.
- This is an example of what is often called the Principle of Parsimony. This says that we want to find a model with as few parameters as possible, but which gives an adequate representation of the data.

Backshift expression for ARMA processes

- The ARMA(p, q) process is expressed as as

$$(1 - \phi_1 B - \cdots - \phi_p B^p)\, Y_t = (1 - \theta_1 B - \cdots - \theta_q B^q)\, e_t.$$

- More succinctly, ARMA(p, q) process is expressed as

$$\phi(B) Y_t = \theta(B) e_t,$$

where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$
$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q.$$

Conditions for stationary and invertible ARMA processes

- For the ARMA(p, q) process to be stationary, we need the roots of the AR characteristic polynomial

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$$

  to all exceed 1 in absolute value (or modulus).

- For the ARMA(p, q) process to be invertible, we need the roots of the MA characteristic polynomial

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q$$

  to all exceed 1 in absolute value (or modulus).

Example 4.5 for an ARMA process

Example

Write each of the models

(i) $Y_t = 0.3Y_{t-1} + e_t$

(ii) $Y_t = e_t - 1.3e_{t-1} + 0.4e_{t-2}$

(iii) $Y_t = 0.5Y_{t-1} + e_t - 0.3e_{t-1} + 1.2e_{t-2}$

using backshift notation and determine whether the model is stationary and/or invertible.

Solution for Example 4.5 (i)

Solution for (i). The model in (i) is an AR(1) model with $\phi = 0.3$.

- In backshift notation, this model is $(1 - 0.3B)Y_t = e_t$.
- The characteristic polynomial is

$$\phi(x) = 1 - 0.3x \Longrightarrow x = 10/3 > 1.$$

- This process is stationary.
- The process is also invertible since it is an AR process.

Solution for Example 4.5 (ii)

Solution for (ii). The model in (ii) is an MA(2) model with $\theta_1 = 1.3$ and $\theta_2 = -0.4$.

- In backshift notation, this model is

$$Y_t = (1 - 1.3B + 0.4B^2)e_t.$$

- The characteristic polynomial is

$$\theta(x) = 1 - 1.3x + 0.4x^2 \Longrightarrow x = 2 \text{ and } x = 1.25.$$

- This process is invertible.
- The process is also stationary since it is an MA process.

Solution for Example 4.5 (iii)

Solution for (iii). The model is ARMA(1,2) with $\phi_1 = 0.5$, $\theta_1 = 0.3$ and $\theta_2 = -1.2$.

- In backshift notation, this model is

$$(1 - 0.5B)Y_t = (1 - 0.3B + 1.2B^2)e_t.$$

- The AR characteristic polynomial is

$$\phi(x) = 1 - 0.5x \Longrightarrow x = 2.$$

- This process is stationary because this root is greater than 1.
- The MA characteristic polynomial is

$$\theta(x) = 1 - 0.3x + 1.2x^2 \Longrightarrow x \approx 0.125 \pm 0.904i.$$

- $|x| \approx \sqrt{(0.125)^2 + (0.904)^2} \approx 0.913 < 1$ means that this process is not invertible.

ACF of the ARMA(p, q) process

- The ARMA(p, q) process is

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}.$$

- Autocovariance function satisfies

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \cdots + \phi_p \gamma_{k-p}.$$

- Thus, ACF is given by

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} + \cdots + \phi_p \rho_{k-p}.$$

- The Yule Walker equations are

$$\begin{aligned} \rho_1 &= \phi_1 + \phi_2 \rho_1 + \phi_3 \rho_2 + \cdots + \phi_p \rho_{p-1} \\ &\cdots \\ \rho_p &= \phi_1 \rho_{p-1} + \phi_2 \rho_{p-2} + \phi_3 \rho_{p-3} + \cdots + \phi_p. \end{aligned}$$

- A similar system can be derived which involves $\theta_1, \theta_2, \cdots, \theta_q$.

ACF for the ARMA(p, q)

- The ACF for the ARMA(p, q) process tails off after lag q in a manner similar to the AR(p) process.
- However, unlike the AR(p) process, the first q autocorrelations depend on both $\theta_1, \theta_2, \cdots, \theta_q$ and $\phi_1, \phi_2, \cdots, \phi_p$.

The ARMA(1,1) process

- The process
$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$
is called an ARMA(1,1) process. This is a special case of the ARMA(p, q) process with $p = q = 1$.
- In backshift notation, the process can be written as
$$(1 - \phi B)Y_t = (1 - \theta B)e_t$$
yielding $\phi(x) = 1 - \phi x$ and $\theta(x) = 1 - \theta x$ as the AR and MA characteristic polynomials, respectively.
- As usual, the conditions for stationarity and invertibility are that the roots of both polynomials exceed 1 in absolute value.

ACFs of ARMA(1,1) process

- The calculations on pp 78-79 (CC) show that

$$\gamma_0 = \left(\frac{1 - 2\phi\theta}{1 - \phi^2} \right) \sigma_e^2,$$

$\gamma_1 = \phi\gamma_0 - \theta\sigma_e^2$, and $\gamma_k = \phi\gamma_{k-1}$, for $k \geq 2$.

- The ACF is shown to satisfy

$$\rho_k = \left[\frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\theta\phi + \theta^2} \right] \phi^{k-1}.$$

- ρ_1 is equal to a quantity that depends on ϕ and θ. This is different than the AR(1) process where ρ_1 depends on ϕ only.
- However, as k gets larger, the autocorrelation ρ_k decays in a manner similar to the AR(1) process.

Example 4.6 for four ARMA(1,1) processes

Example

Use SAS to simulate four different ARMA(1,1) processes

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1},$$

with $e_t \sim \text{iid} N(0,1)$ and $n = 100$. We choose

- $\phi = 0.5$ and $\theta = -0.5$
- $\phi = 0.5$ and $\theta = 0.5$
- $\phi = -0.5$ and $\theta = -0.5$
- $\phi = -0.5$ and $\theta = 0.5$.

Figure 4.5. ARMA(1,1) simulation with N(0,1) WN

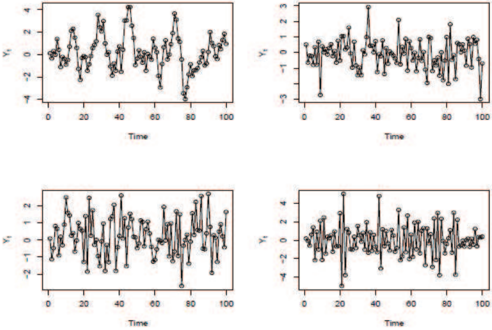


Figure 4.15: *ARMA(1,1) simulations with* $e_t \sim$ *iid* $\mathcal{N}(0,1)$ *and* $n = 100$. *Upper left:* $\phi = 0.5$ *and* $\theta = -0.5$. *Upper right:* $\phi = 0.5$ *and* $\theta = 0.5$. *Lower left:* $\phi = -0.5$ *and* $\theta = -0.5$. *Lower right:* $\phi = -0.5$ *and* $\theta = 0.5$.

Figure 4.16. ARMA(1,1) sample ACFs with N(0,1) WN

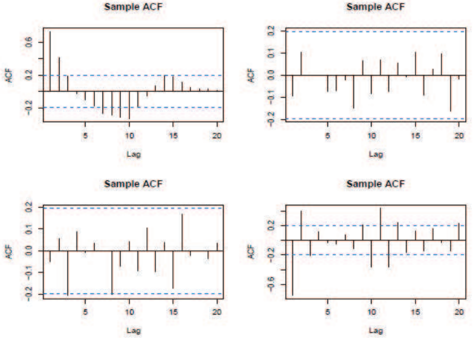


Figure 4.16: *ARMA(1,1) sample autocorrelation functions with* $e_t \sim$ *iid* $\mathcal{N}(0,1)$ *and* $n = 100$. *Upper left:* $\phi = 0.5$ *and* $\theta = -0.5$. *Upper right:* $\phi = 0.5$ *and* $\theta = 0.5$. *Lower left:* $\phi = -0.5$ *and* $\theta = -0.5$. *Lower right:* $\phi = -0.5$ *and* $\theta = 0.5$.

Comments for ARMA(1,1) process

- The AR and MA characteristic polynomial roots are 2 (in absolute value) for all processes. All processes are stationary and invertible.
- This small example shows that there are a wide range of patterns possible for time series and their ACFs in the ARMA(1,1) family, shown in Figure 4.15 and Figure 4.16, respectively.
- The ARMA(1,1) process can be written in the general linear process form defined at the beginning of the chapter, see pp 78-79 (CC).

chapter 4 is over

Thank you for your attention!