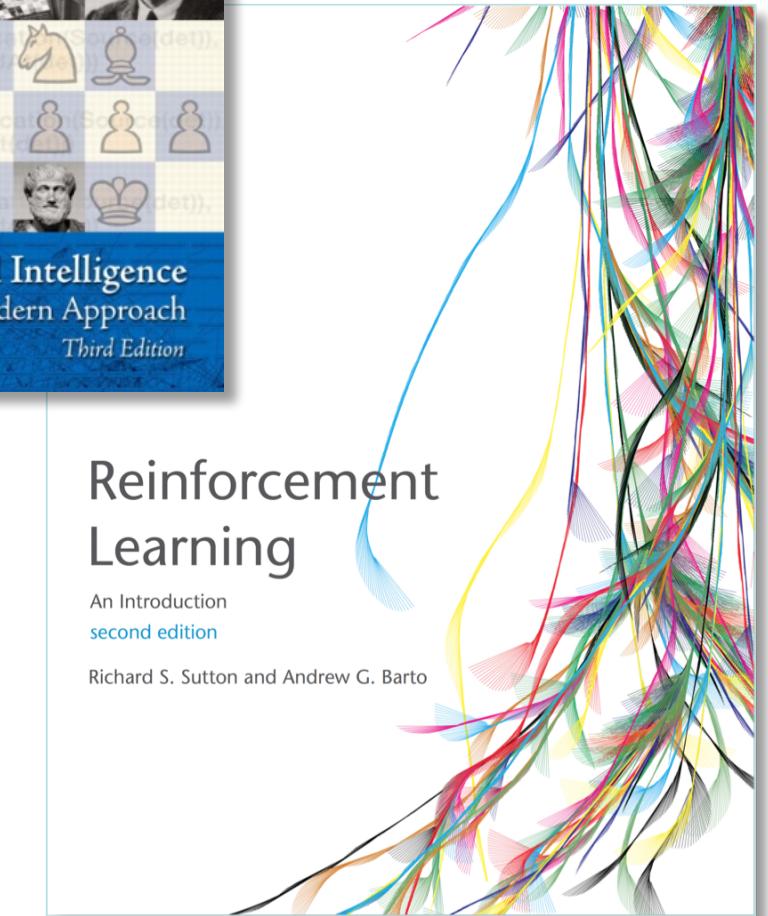
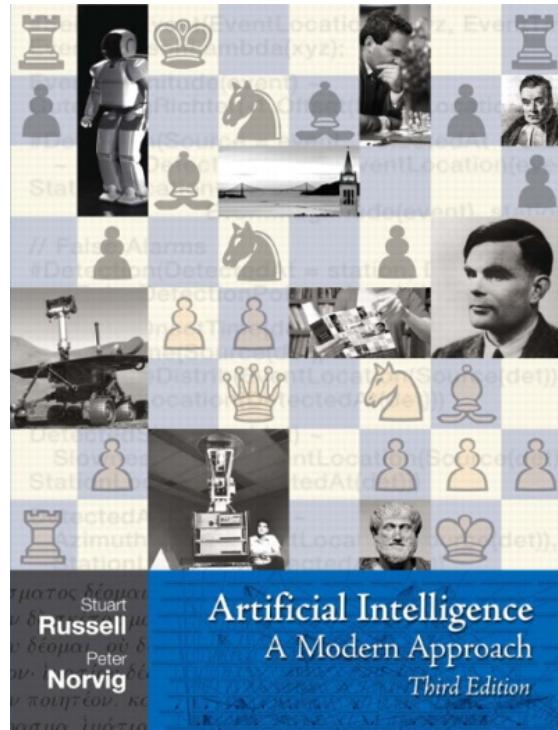


Markov Decision Processes

Sutton and Barto, Chapters 3
and Chapter 6.1, 6.2, 6.5
AIMA Chapter 17.1-17.3

Slides courtesy of Dan Klein and Pieter Abbeel
University of California, Berkeley



Review: Money

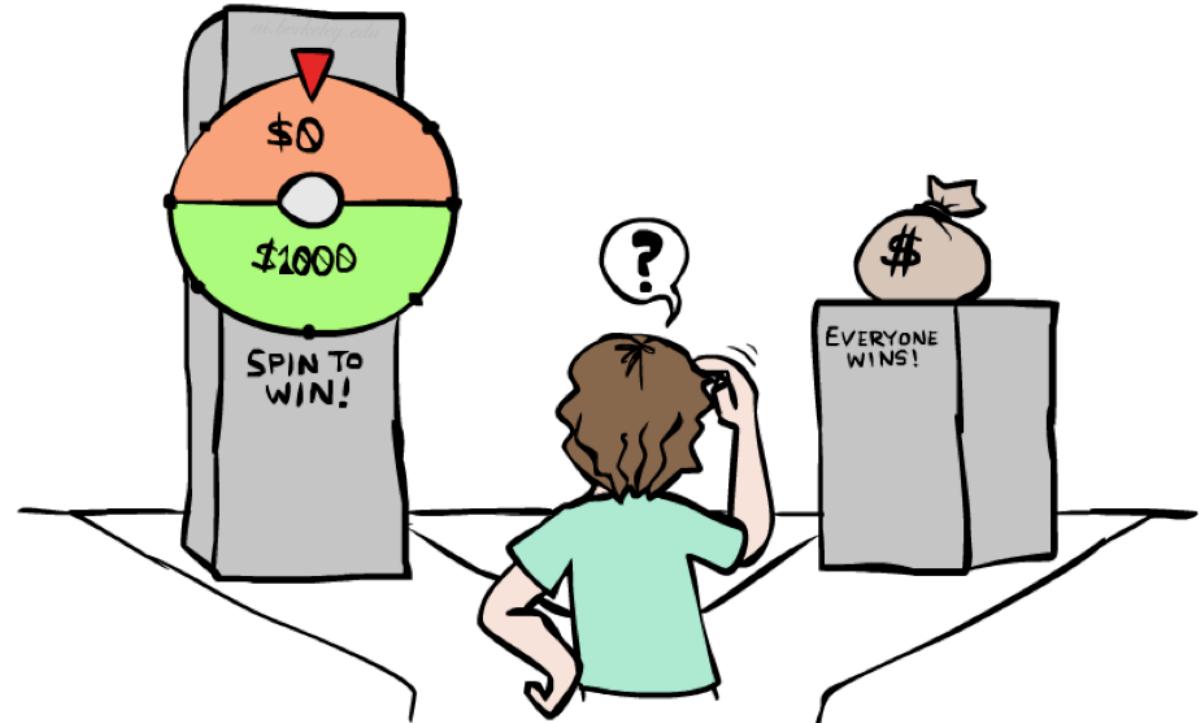
- Money does not behave as a utility function, but we can talk about the utility of having money (or being in debt)
- Given a lottery $L = [p, \$X; (1-p), \$Y]$
 - The **expected monetary value** $EMV(L)$ is $p*X + (1-p)*Y$
 - The **utility of the lottery** is $U(L) = p*U(\$X) + (1-p)*U(\$Y)$
 - Typically, people express their utilities as: $U(L) < U(EMV(L))$
 - In this sense, people are **risk-averse**.

They would rather have a smaller amount of money than the gamble on the chance of having more.



Example: Insurance

- Consider the lottery [0.5, \$1000; 0.5, \$0]
 - What is its **expected monetary value**? (\$500)
 - What is its **certainty equivalent**?
 - Monetary value acceptable in lieu of lottery
 - \$400 for most people
 - Difference of \$100 is the **insurance premium**
 - There's an insurance industry because people will pay to reduce their risk
 - If everyone were risk-neutral, no insurance needed!
 - It's win-win: you'd rather have the \$400 and the insurance company would rather have the lottery (their utility curve is linear and they have many lotteries)



Example: Human Rationality?

- Famous example of Allais (1953)

- A: [0.8, \$4k; 0.2, \$0] ←
- B: [1.0, \$3k; 0.0, \$0]
- C: [0.2, \$4k; 0.8, \$0]
- D: [0.25, \$3k; 0.75, \$0]



- Most people prefer B > A, C > D
- But if the utility of \$0 is 0, that is $U(\$0)=0$, then
 - $B > A \Rightarrow 1.0 U(\$3k) > 0.8 U(\$4k)$
 - $C > D = \text{chance of } C^*4 > \text{chance of } D^*4 \Rightarrow 0.8 U(\$4k) > 1.0 U(\$3k)$

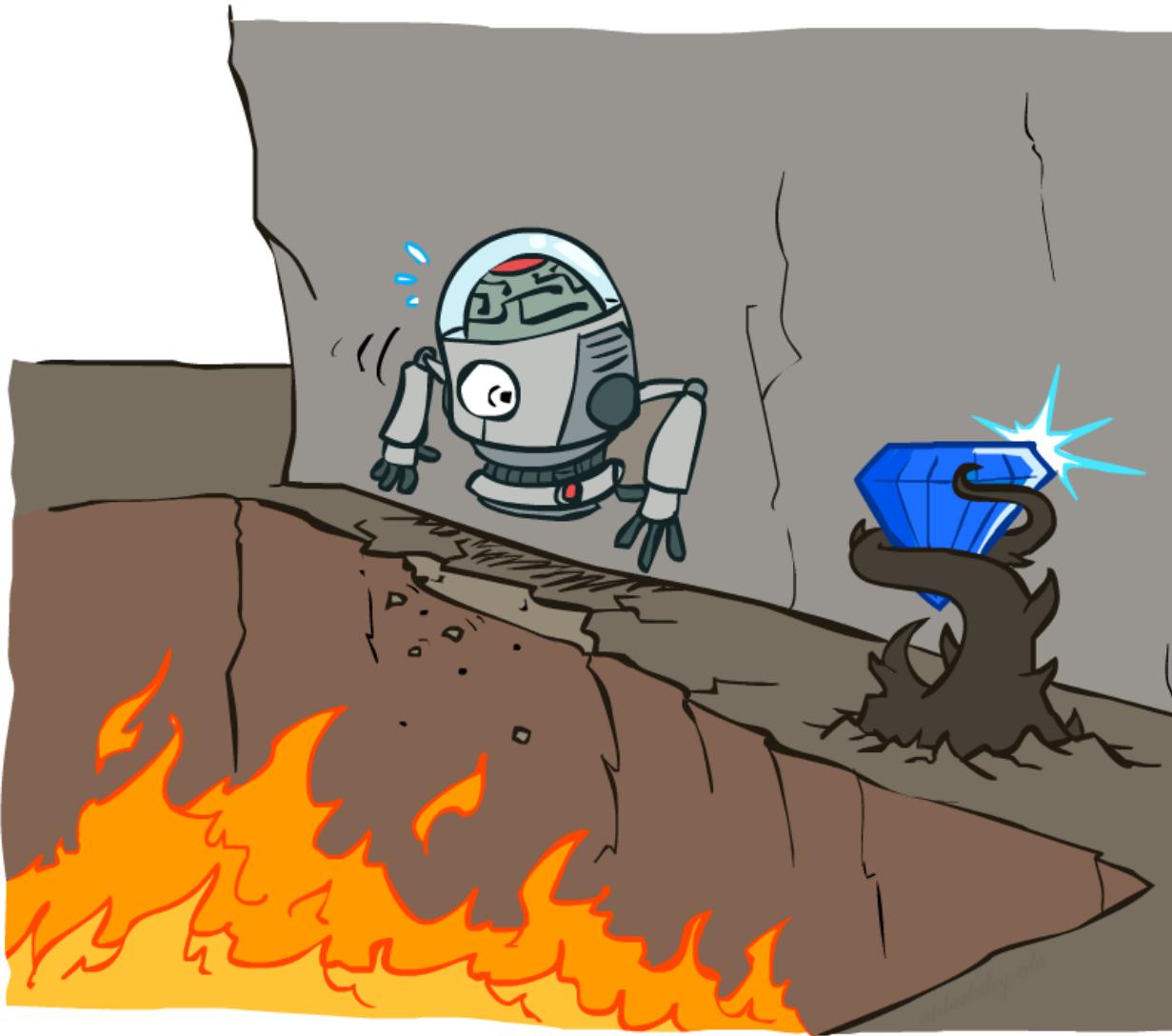
Markov Decision Processes



Slides courtesy of Dan Klein and Pieter Abbeel

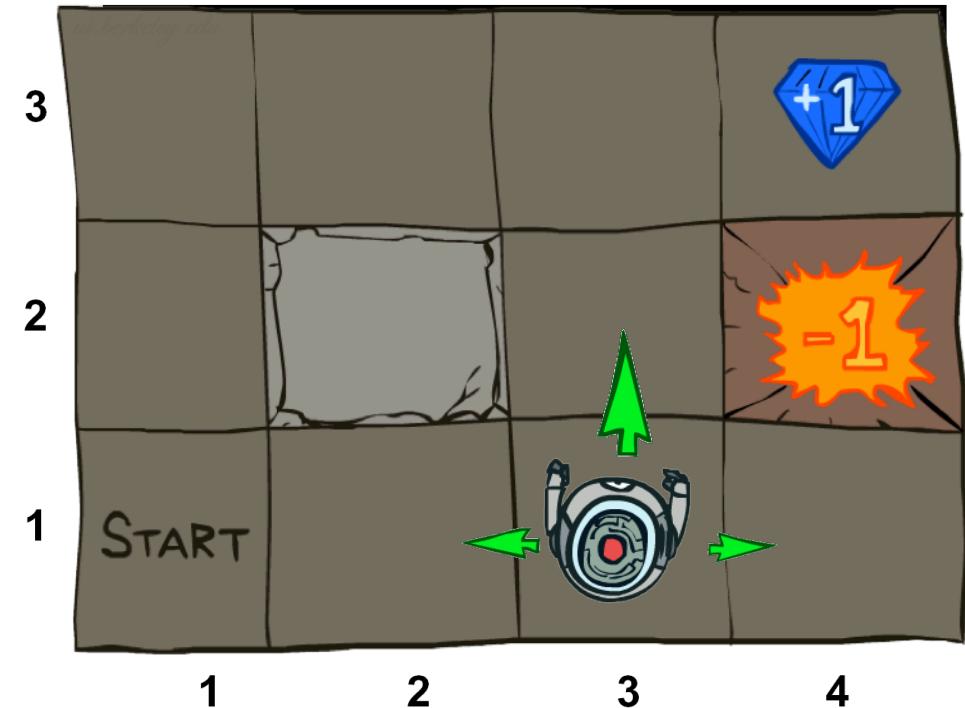
University of California, Berkeley

Non-Deterministic Search



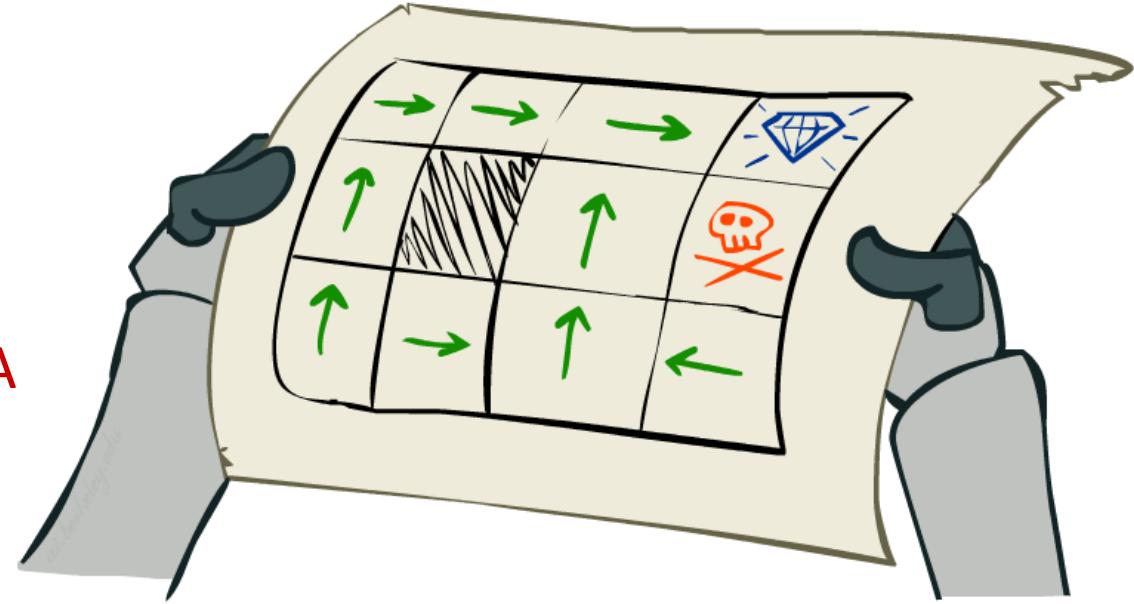
Example: Grid World

- A maze-like problem
 - The agent lives in a grid
 - Walls block the agent's path
- Noisy movement: actions do not always go as planned
 - 80% of the time, the action North takes the agent North (if there is no wall there)
 - 10% of the time, North takes the agent West; 10% East
 - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
 - Small “living” reward each step (can be negative)
 - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards



Policies

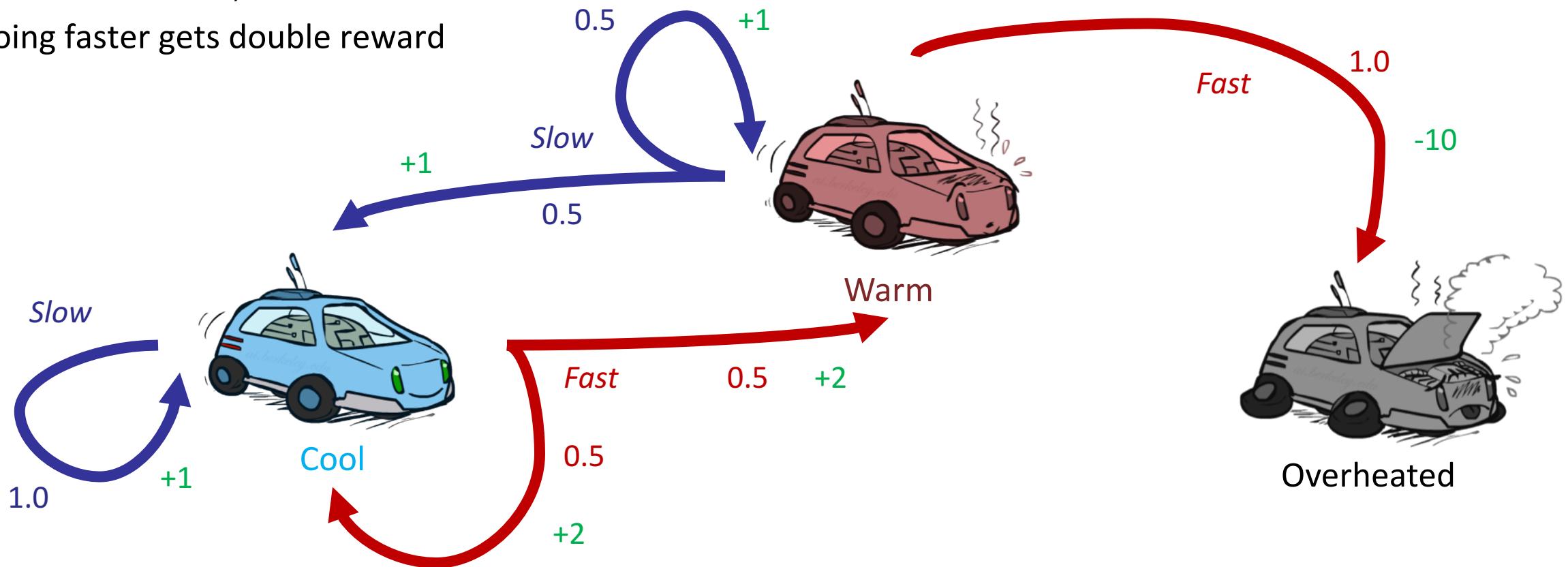
- In deterministic single-agent search problems, we wanted an optimal **plan**, or sequence of actions, from start to a goal
- For MDPs, we want an optimal **policy** $\pi^*: S \rightarrow A$
 - A policy π gives an action for each state
 - An optimal policy is one that maximizes expected utility if followed
 - An explicit policy defines a reflex agent
- Expectimax didn't compute entire policies
 - It computed the action for a single state only



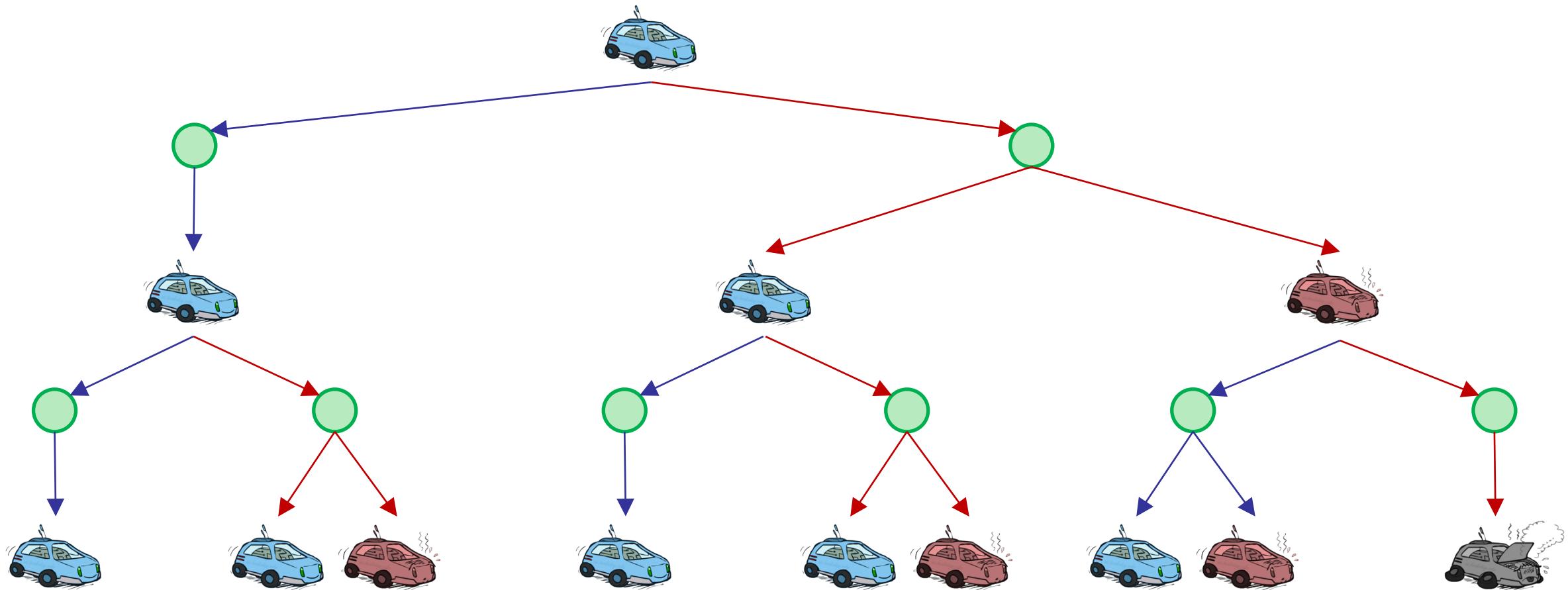
Optimal policy when $R(s, a, s') = -0.03$
for all non-terminals s

Example: Racing

- A robot car wants to travel far, quickly
- Three states: Cool, Warm, Overheated
- Two actions: *Slow*, *Fast*
- Going faster gets double reward

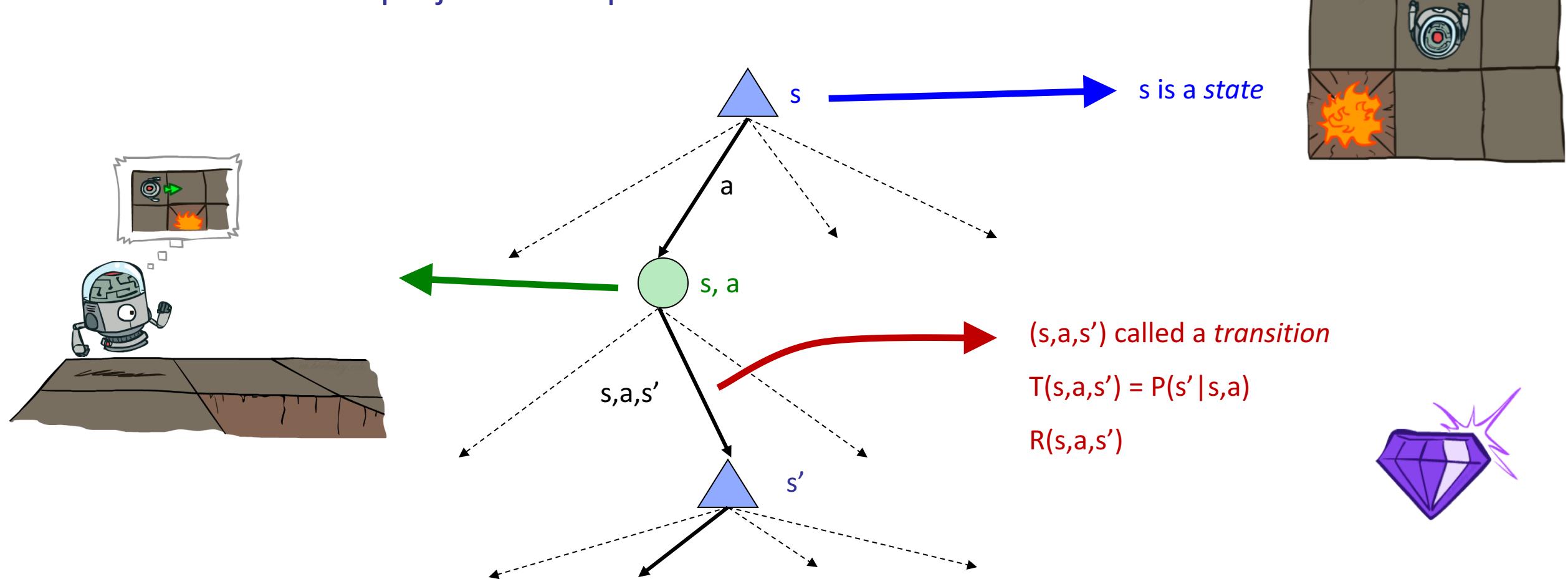


Racing Search Tree

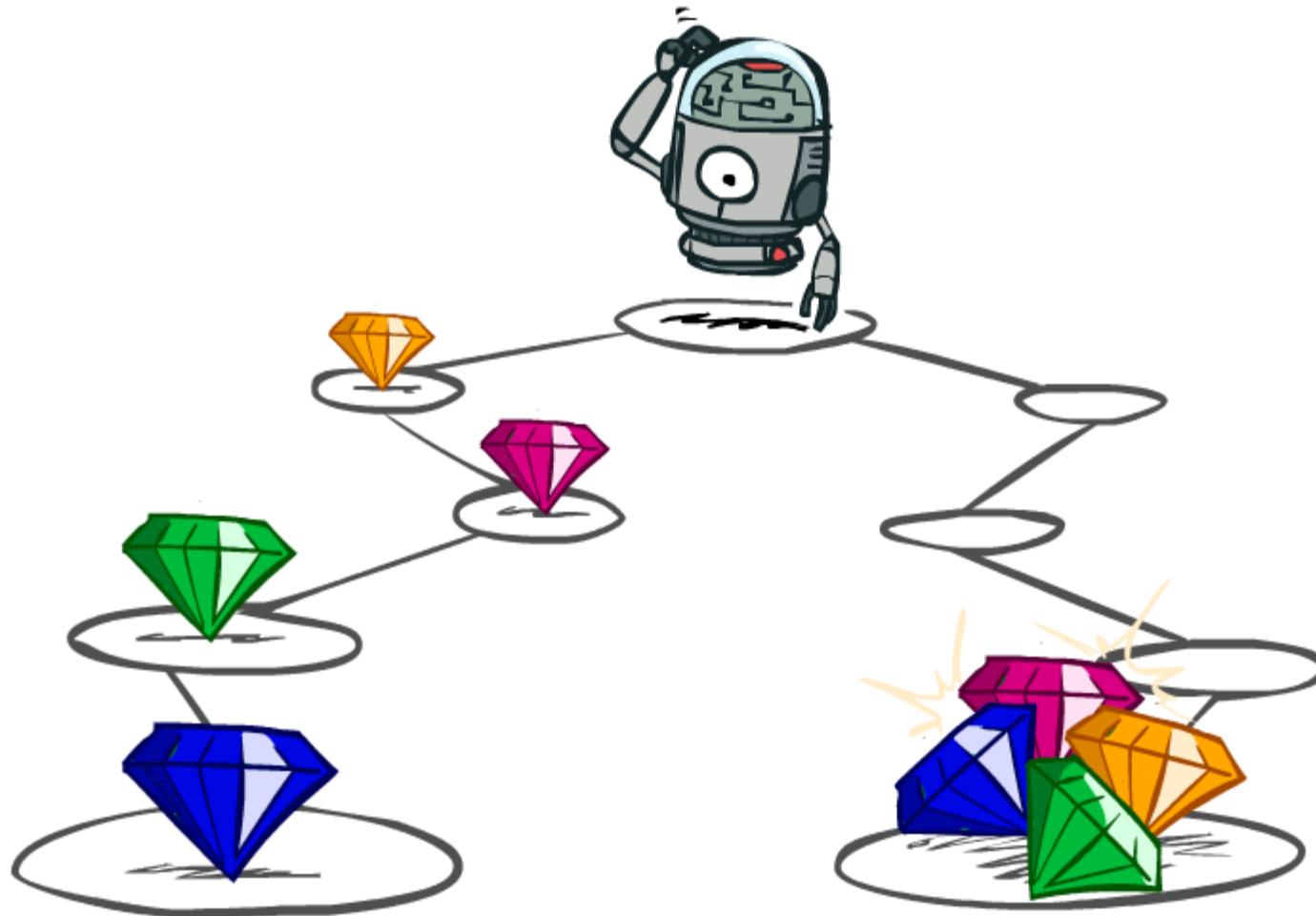


MDP Search Trees

- Each MDP state projects an expectimax-like search tree

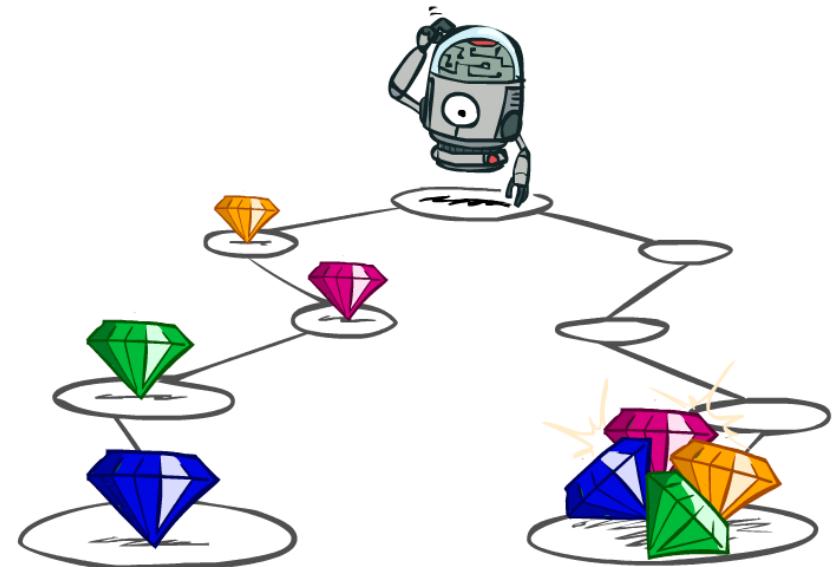


Utilities of Sequences



Utilities of Sequences

- What preferences should an agent have over reward sequences?
- More or less? $[1, 2, 2]$ or $[2, 3, 4]$
- Now or later? $[0, 0, 1]$ or $[1, 0, 0]$



Discounting

- It's reasonable to maximize the sum of rewards
- It's also reasonable to prefer rewards now to rewards later
- One solution: values of rewards decay exponentially



1

Worth Now



γ

Worth Next Step



γ^2

Worth In Two Steps

Discounting

- How to discount?

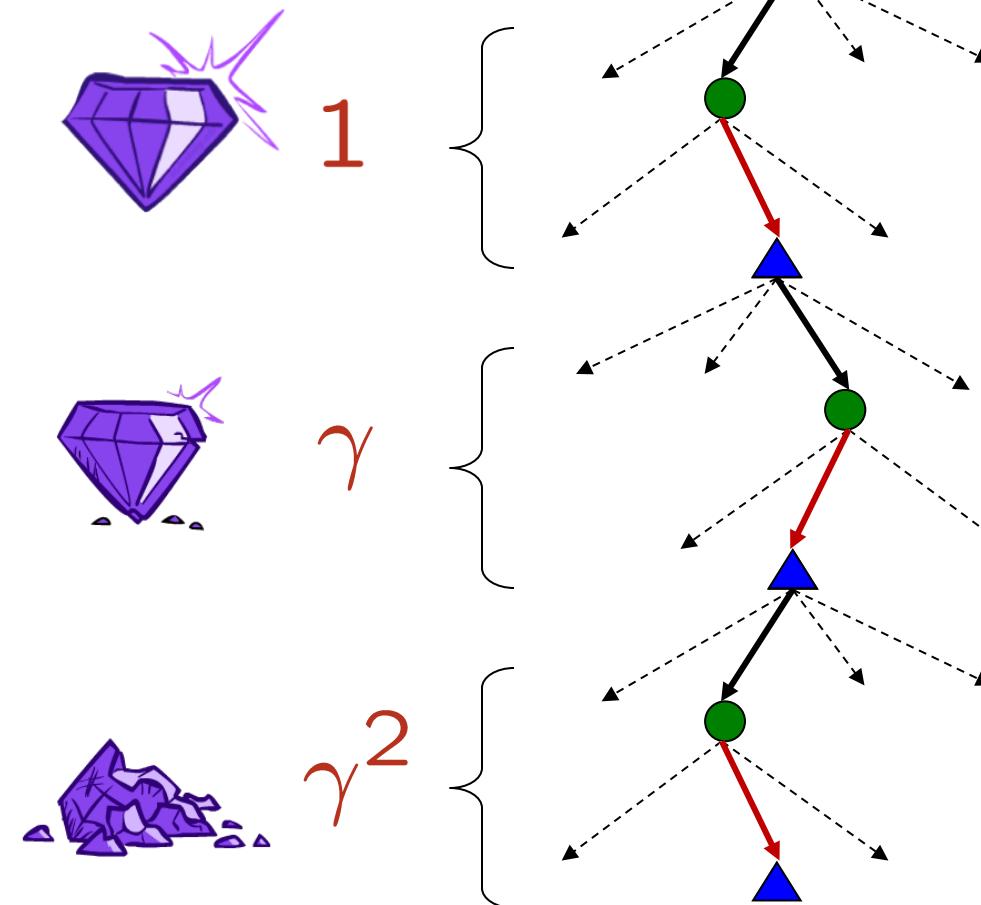
- Each time we descend a level, we multiply in the discount once

- Why discount?

- Sooner rewards probably do have higher utility than later rewards
- Also helps our algorithms converge

- Example: discount of 0.5

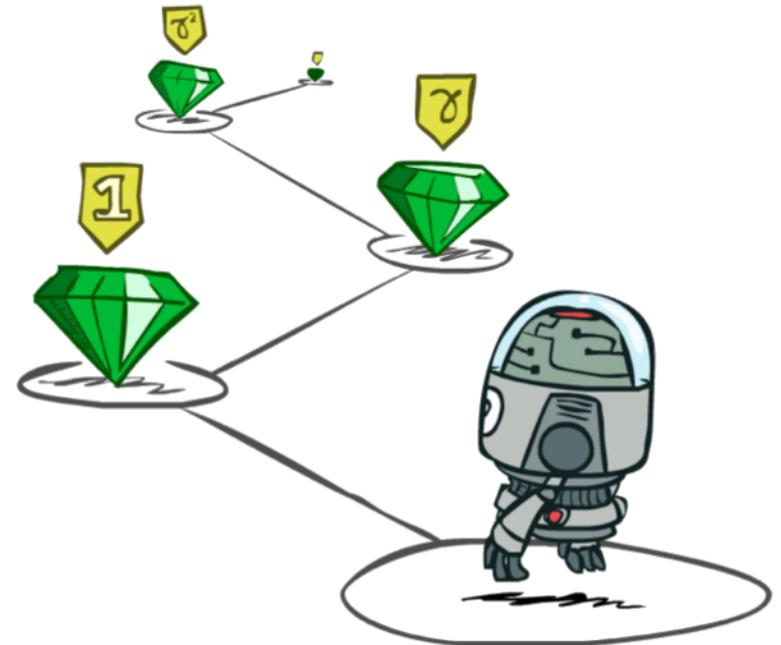
- $U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3$
- $U([1,2,3]) < U([3,2,1])$



Stationary Preferences

- Theorem: if we assume stationary preferences:

$$\begin{aligned}[a_1, a_2, \dots] &\succ [b_1, b_2, \dots] \\ \Updownarrow \\ [r, a_1, a_2, \dots] &\succ [r, b_1, b_2, \dots]\end{aligned}$$



- Then: there are only two ways to define utilities

- Additive utility: $U([r_0, r_1, r_2, \dots]) = r_0 + r_1 + r_2 + \dots$
- Discounted utility: $U([r_0, r_1, r_2, \dots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \dots$

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?

- Solutions:

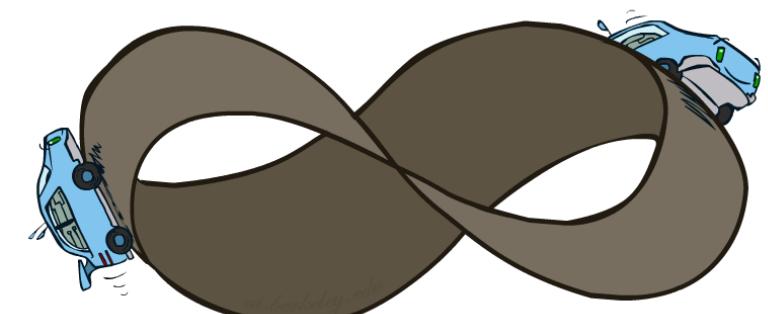
- Finite horizon: (similar to depth-limited search)

- Terminate episodes after a fixed T steps (e.g. life)
 - Gives nonstationary policies (π depends on time left)

- Discounting: use $0 < \gamma < 1$

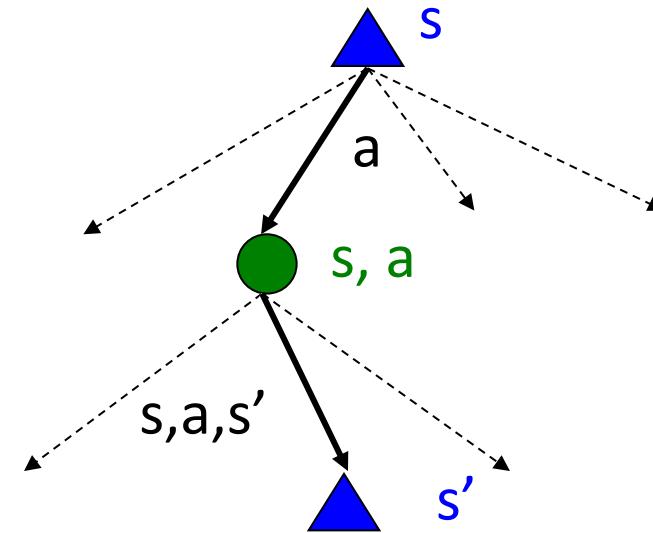
$$U([r_0, \dots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1 - \gamma)$$

- Smaller γ means smaller “horizon” – shorter term focus
 - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like “overheated” for racing)

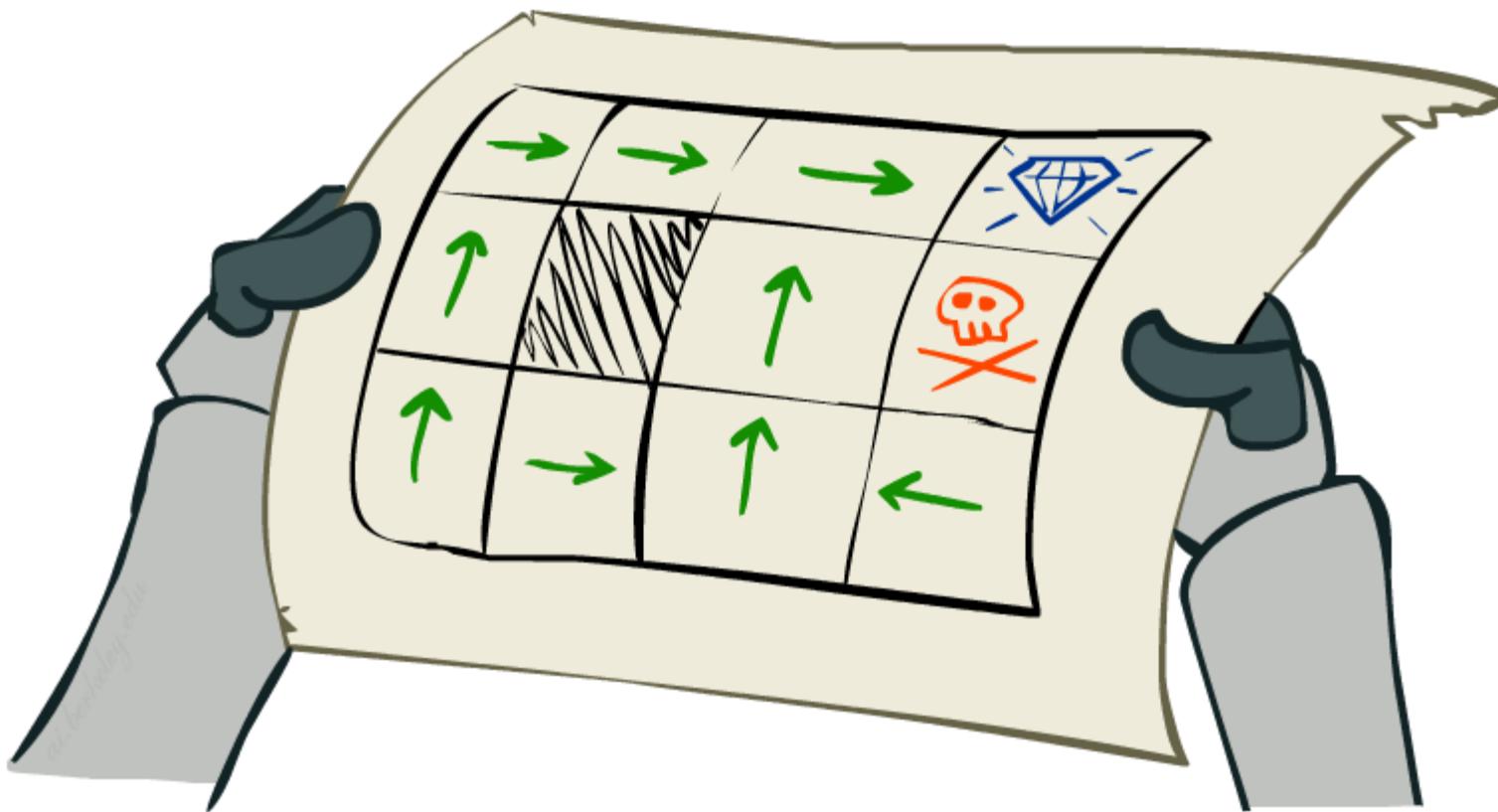


Recap: Defining MDPs

- Markov decision processes:
 - Set of states S
 - Start state s_0
 - Set of actions A
 - Transitions $P(s'|s,a)$ (or $T(s,a,s')$)
 - Rewards $R(s,a,s')$ (and discount γ)
- MDP quantities so far:
 - Policy = Choice of action for each state
 - Utility = sum of (discounted) rewards

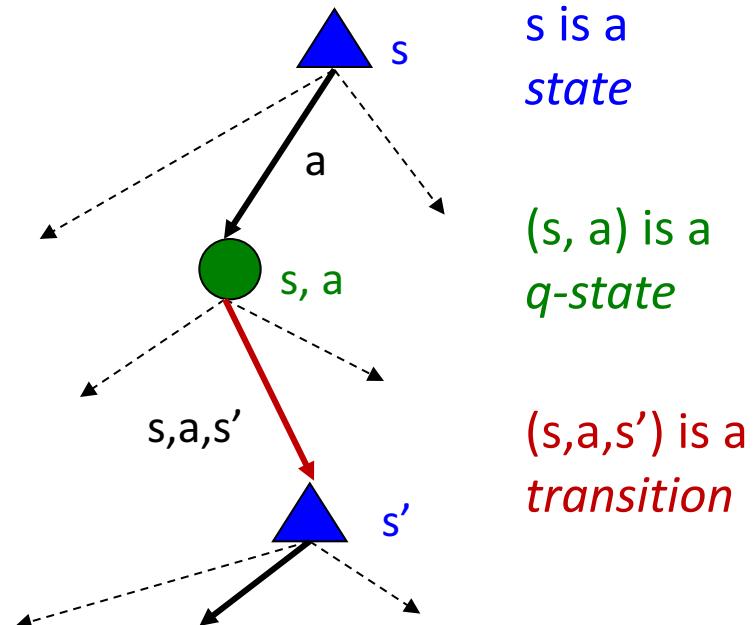


Solving MDPs



Optimal Quantities

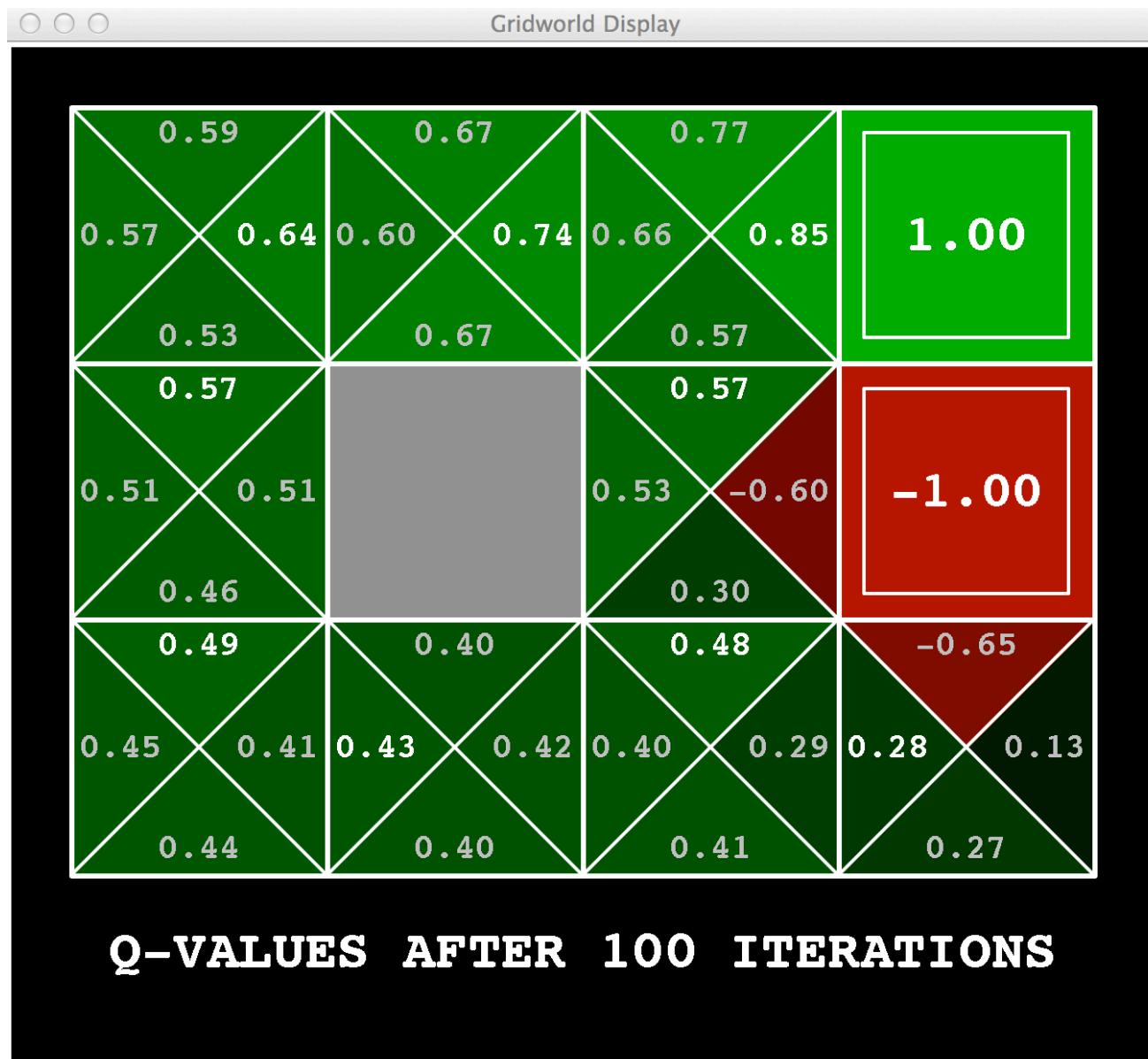
- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s



Gridworld Values V^*



Gridworld: Q*



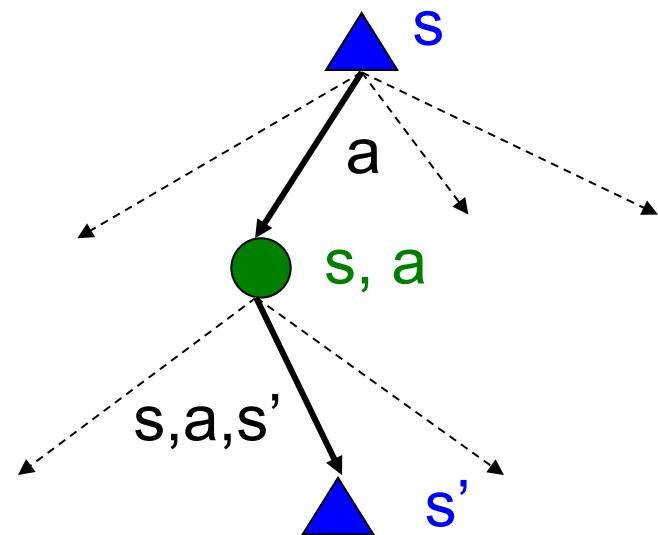
Values of States

- Fundamental operation: compute the (expectimax) value of a state
 - Expected utility under optimal action
 - Average sum of (discounted) rewards
 - This is just what expectimax computed!
- Recursive definition of value:

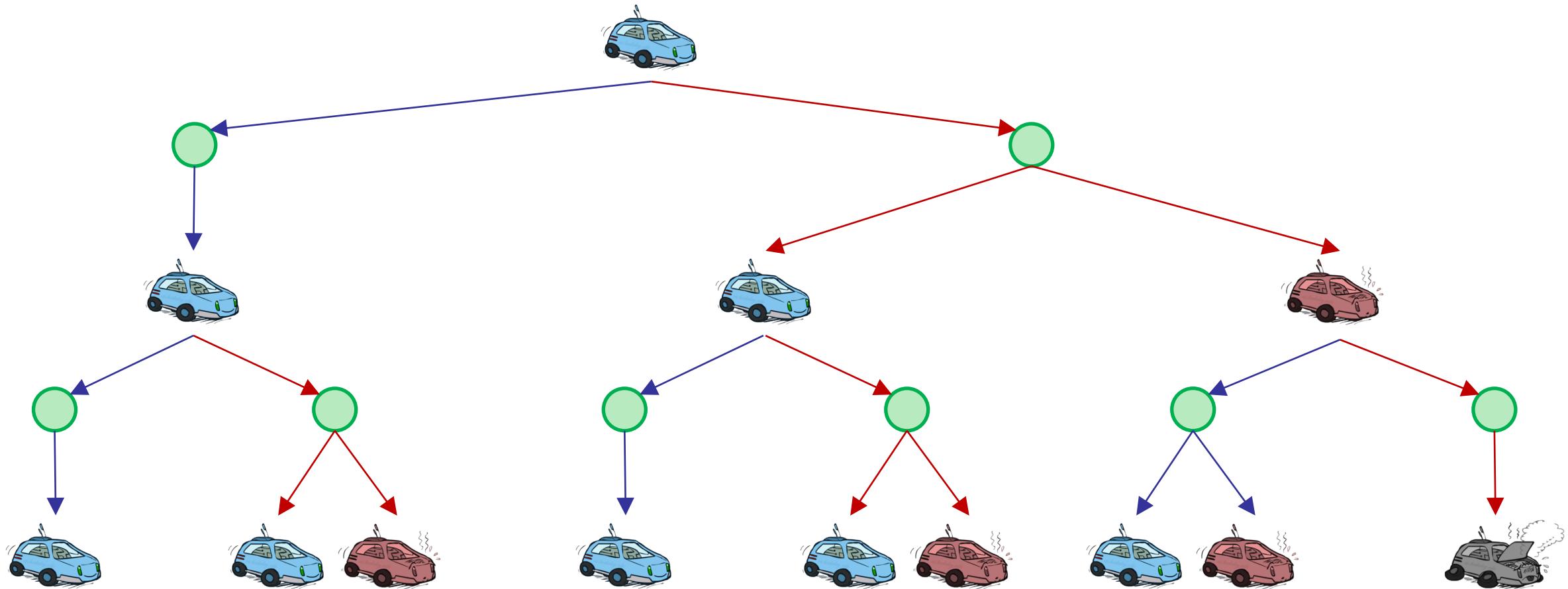
$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

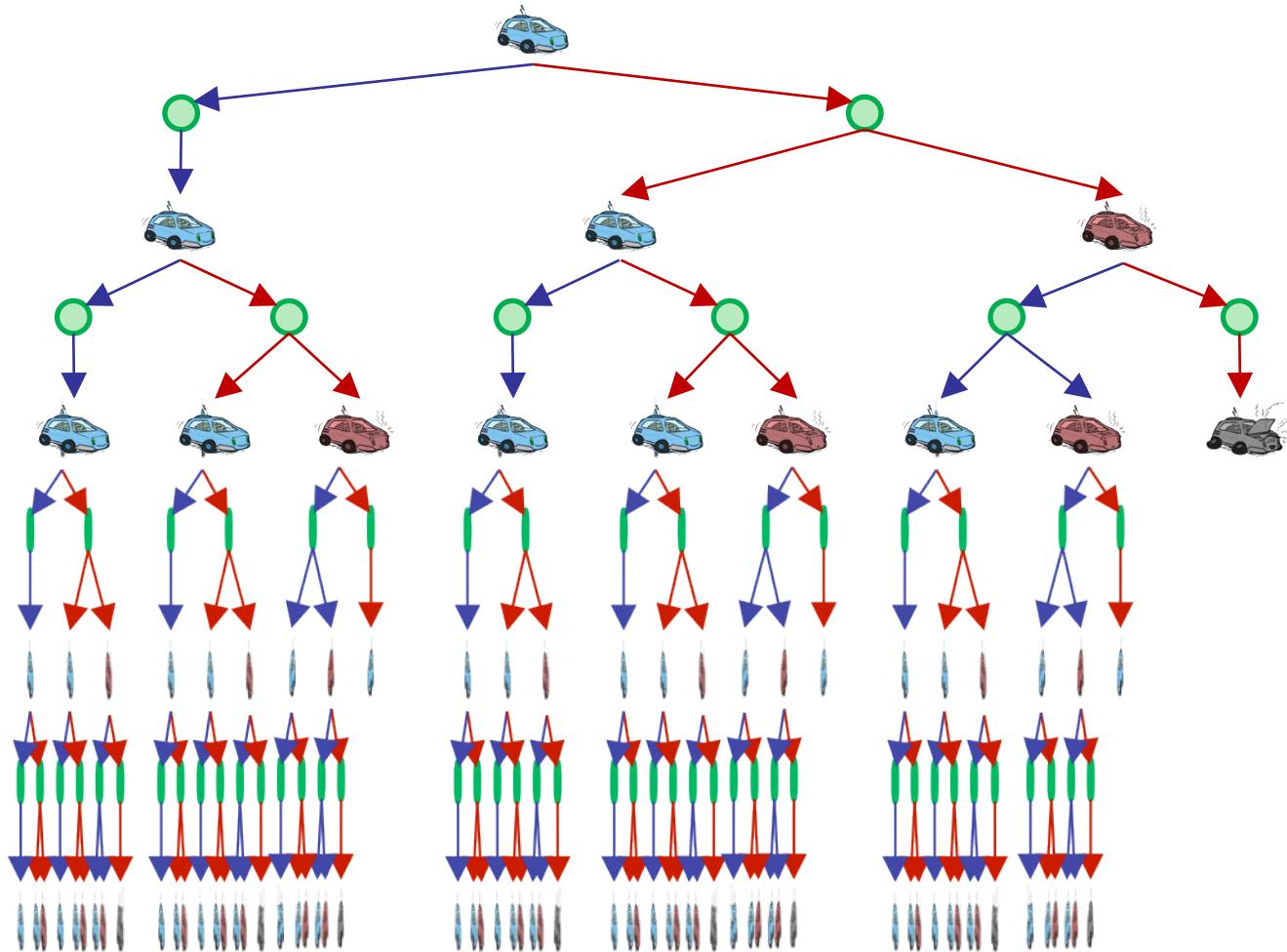
$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



Racing Search Tree

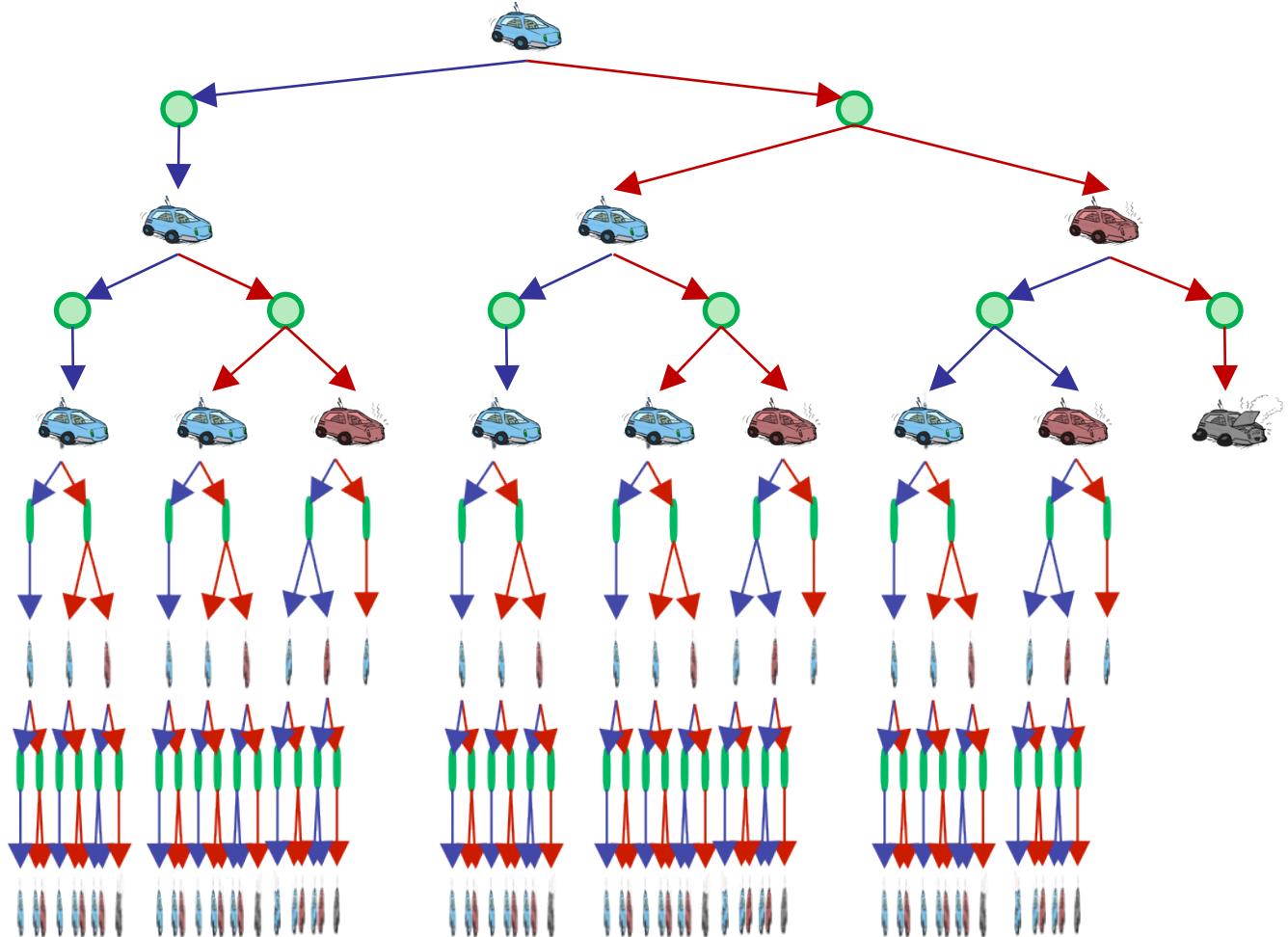


Racing Search Tree



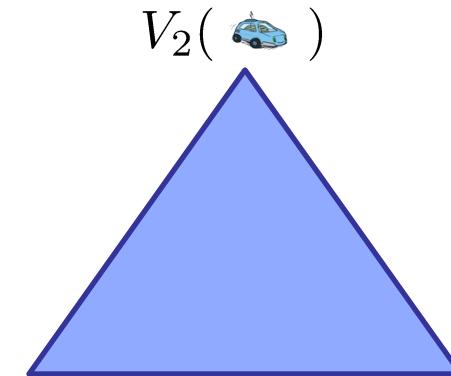
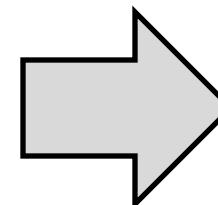
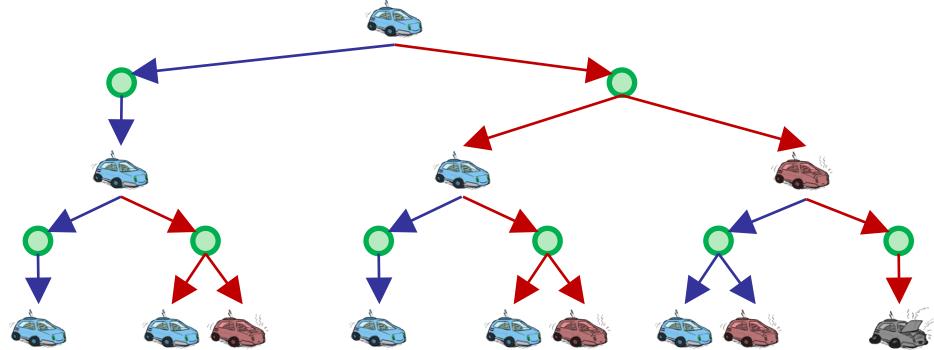
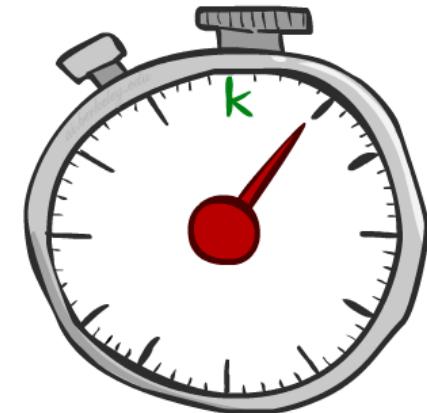
Racing Search Tree

- We're doing way too much work with expectimax!
 - Problem: States are repeated
 - Idea: Only compute needed quantities once
 - Problem: Tree goes on forever
 - Idea: Do a depth-limited computation, but with increasing depths until change is small
 - Note: deep parts of the tree eventually don't matter if $\gamma < 1$

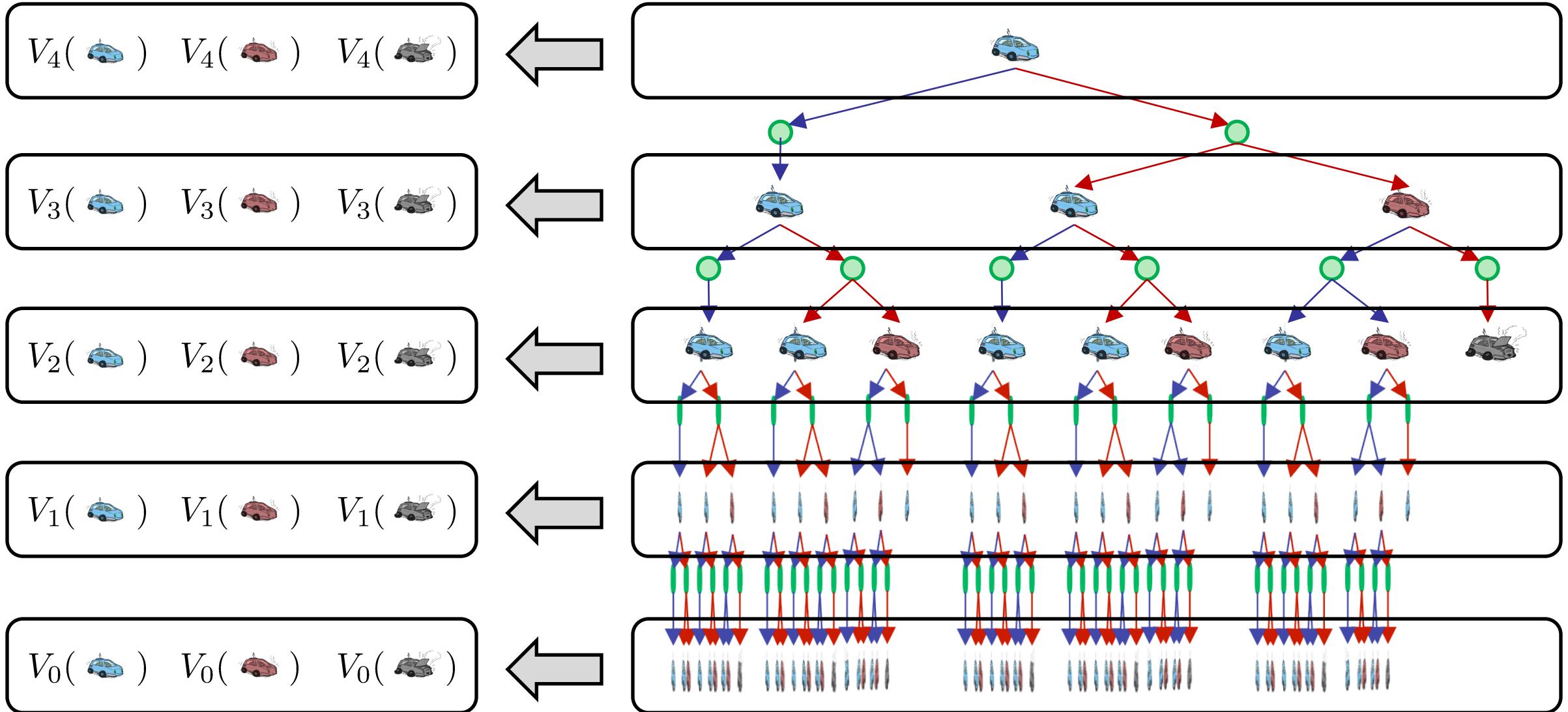


Time-Limited Values

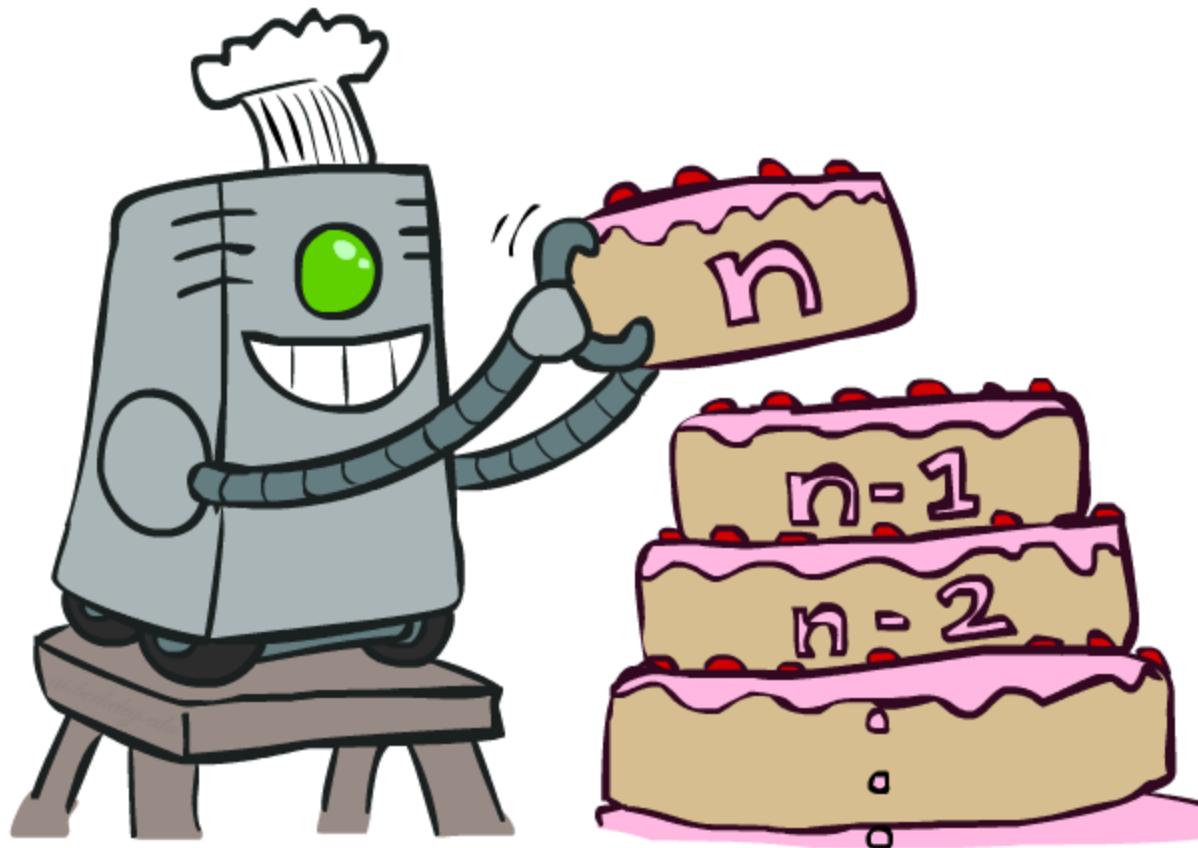
- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
 - Equivalently, it's what a depth- k expectimax would give from s



Computing Time-Limited Values



Value Iteration

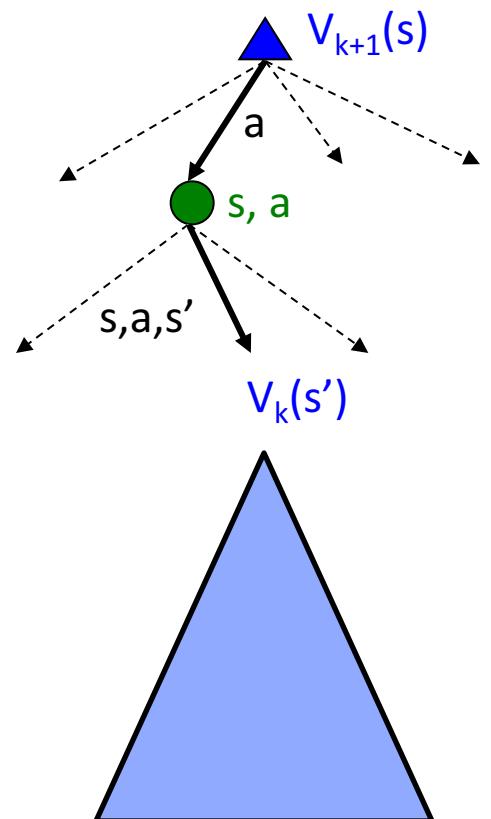


Value Iteration

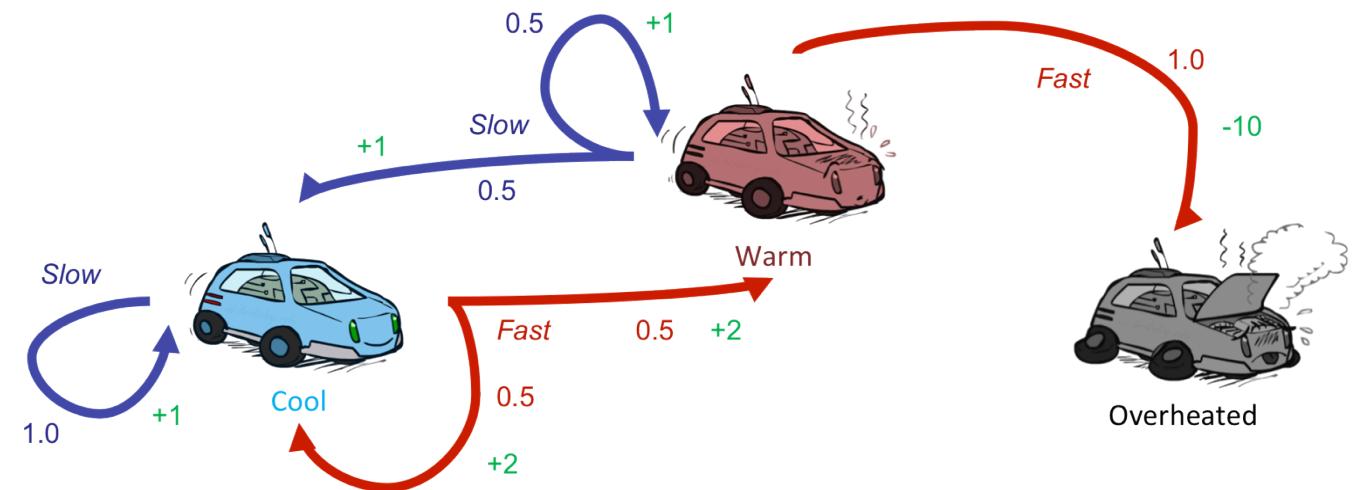
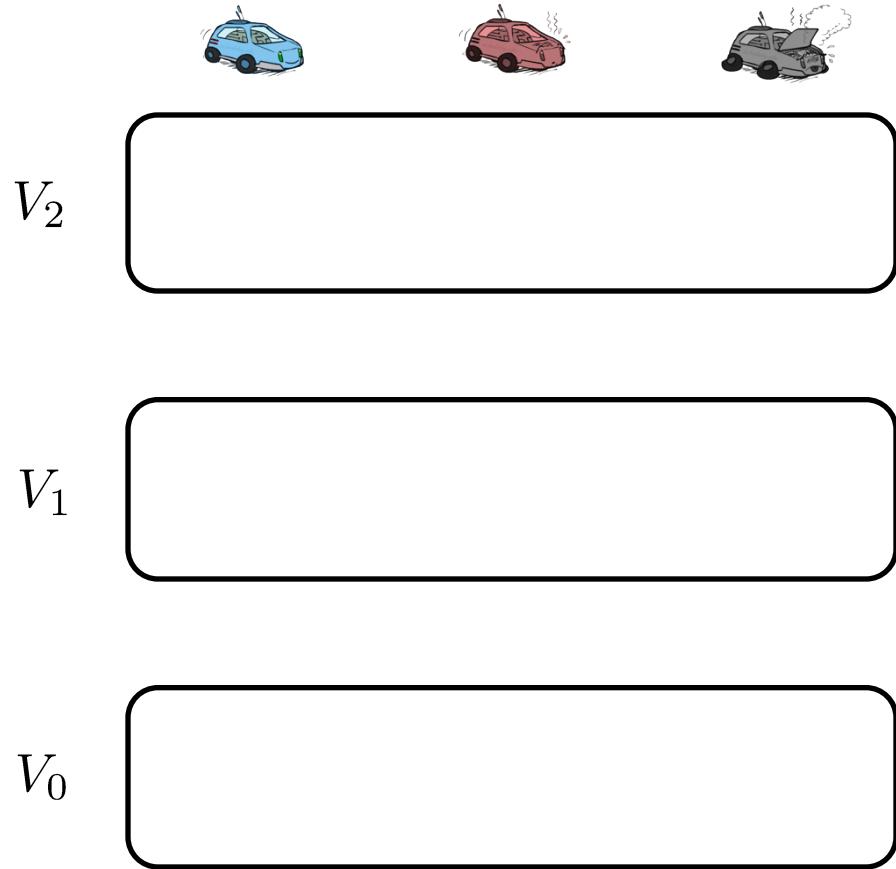
- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- **Theorem: will converge to unique optimal values**
 - Basic idea: approximations get refined towards optimal values
 - Policy may converge long before values do



Example: Value Iteration

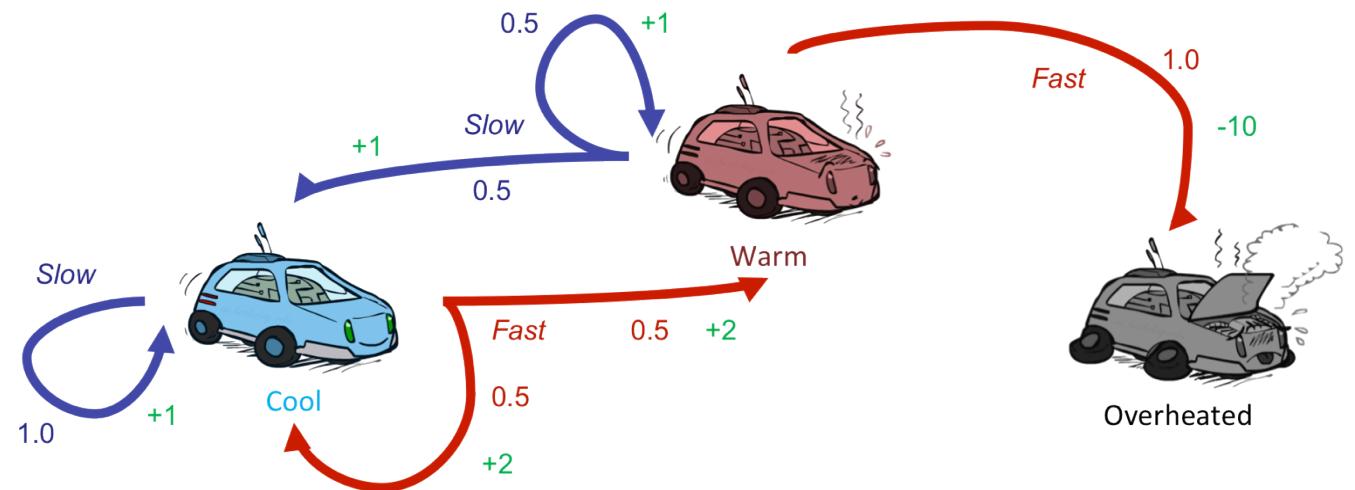


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

Example: Value Iteration

			
V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0

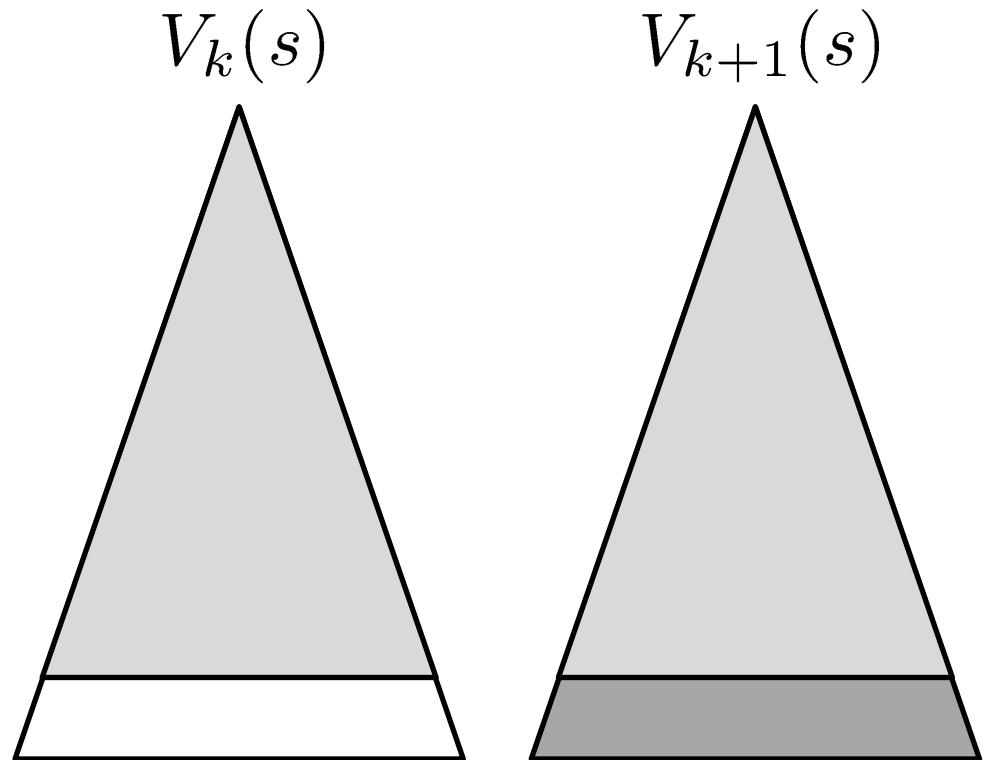


Assume no discount!

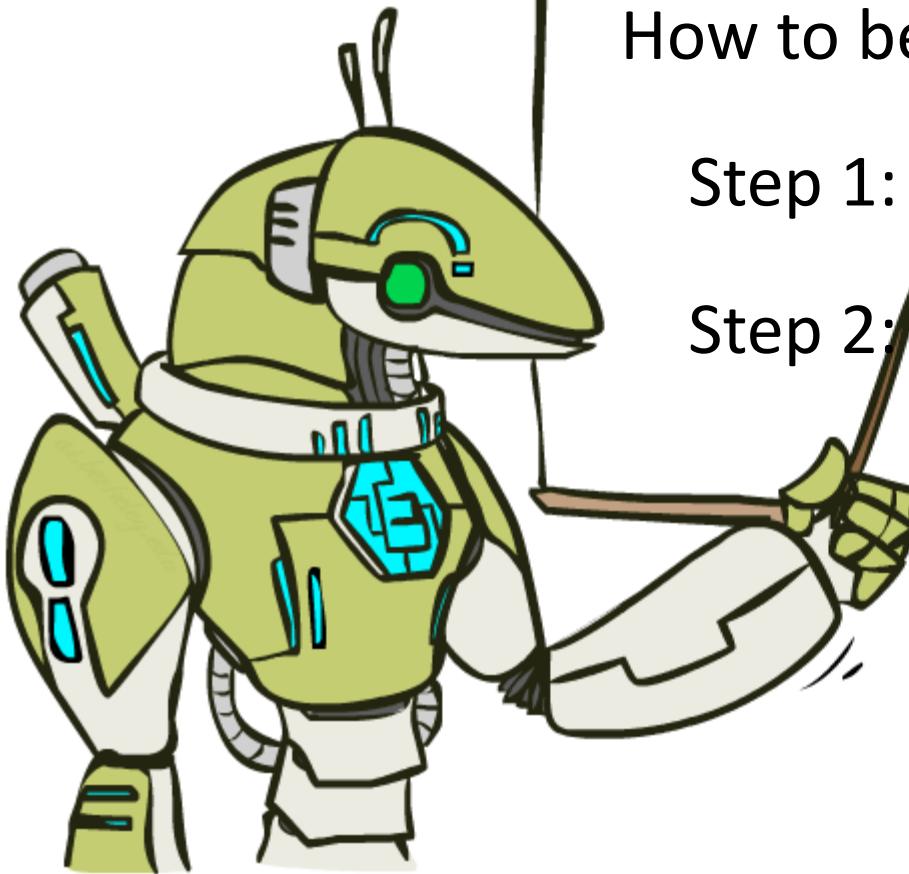
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

Convergence*

- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M , then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
 - Sketch: For any state V_k and V_{k+1} can be viewed as depth $k+1$ expectimax results in nearly identical search trees
 - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - That last layer is at best all R_{MAX}
 - It is at worst R_{MIN}
 - But everything is discounted by γ^k that far out
 - So V_k and V_{k+1} are at most $\gamma^k \max|R|$ different
 - So as k increases, the values converge



The Bellman Equations



How to be optimal:

Step 1: Take correct first action

Step 2: Keep being optimal

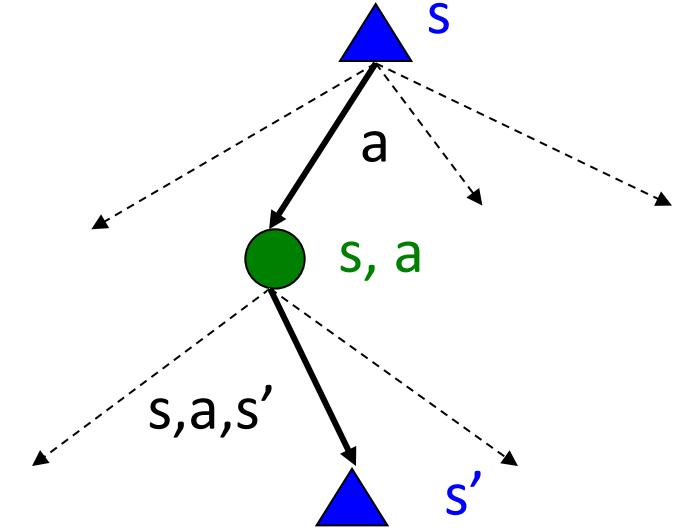
The Bellman Equations

- Definition of “optimal utility” via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values

$$V^*(s) = \max_a Q^*(s, a)$$

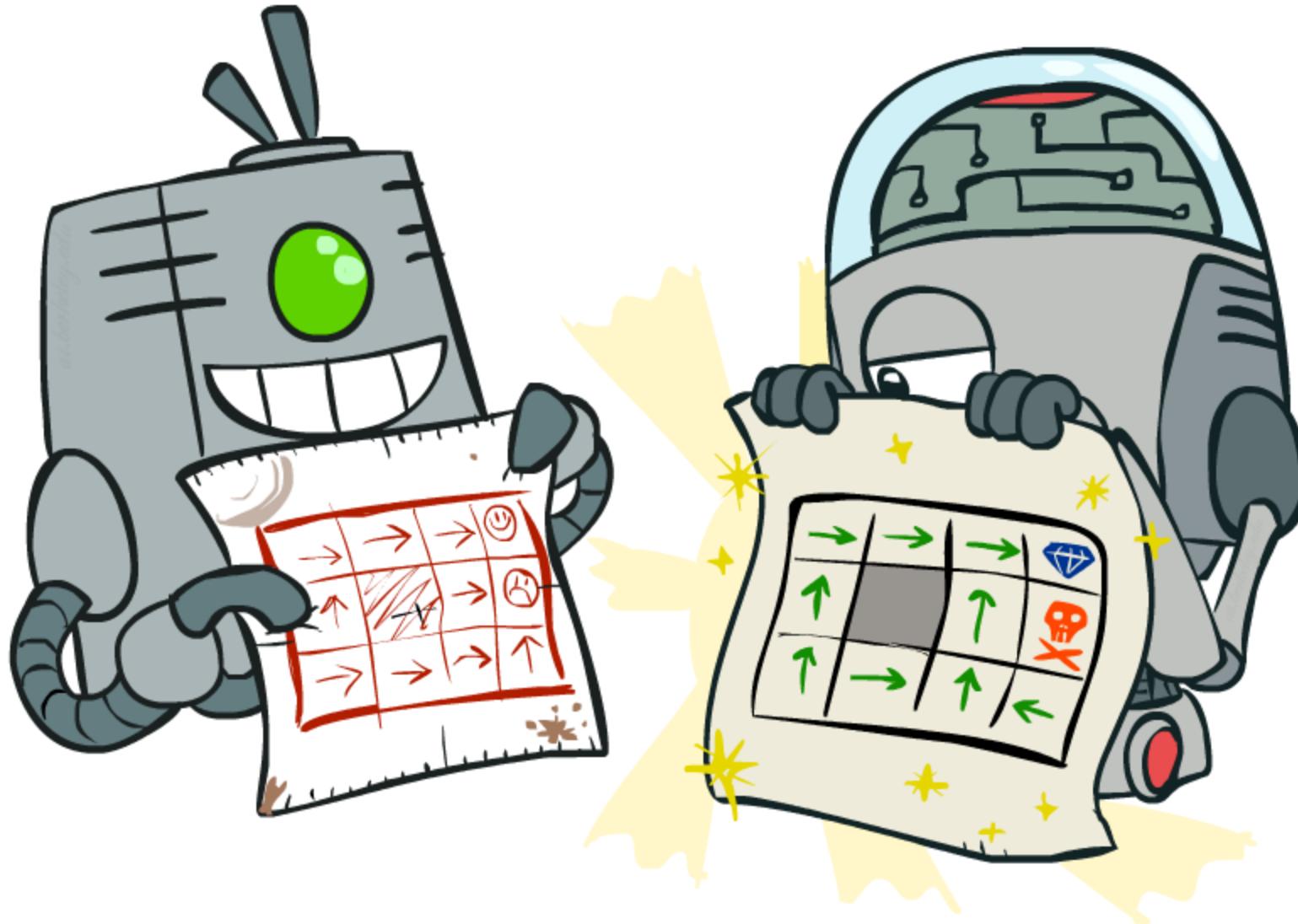
$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

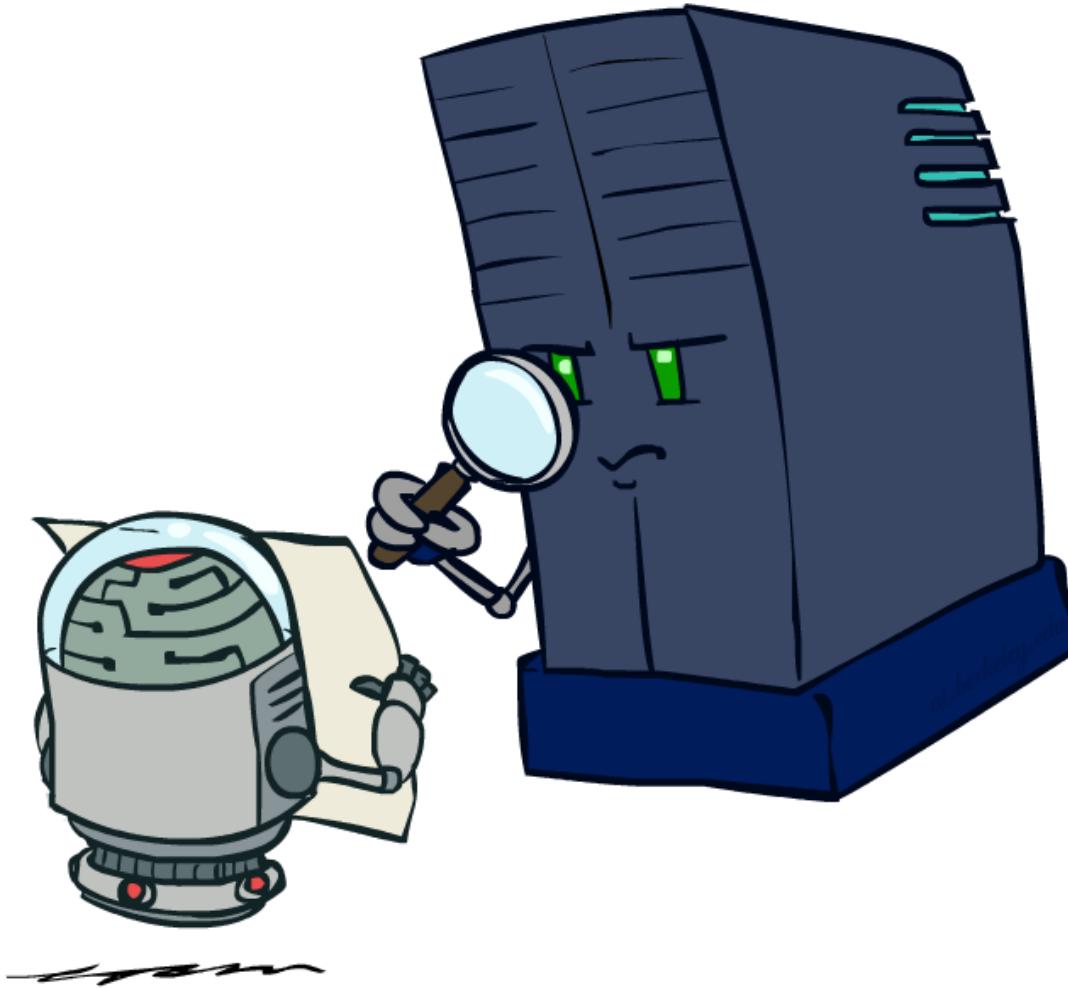


- These are the Bellman equations, and they characterize optimal values in a way we'll use over and over

Policy Methods

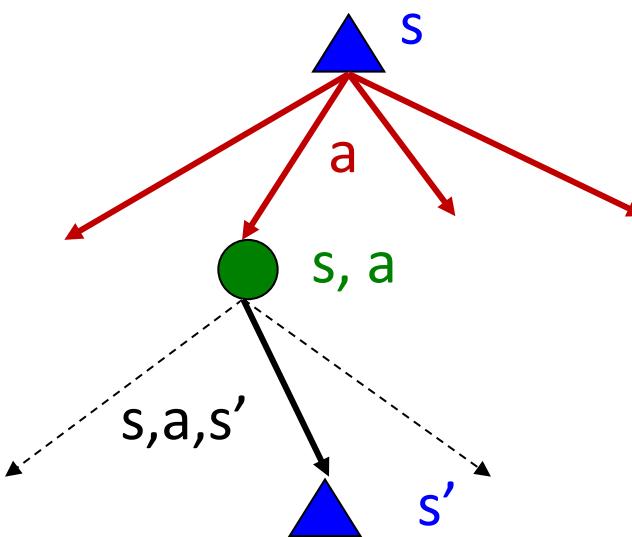


Policy Evaluation

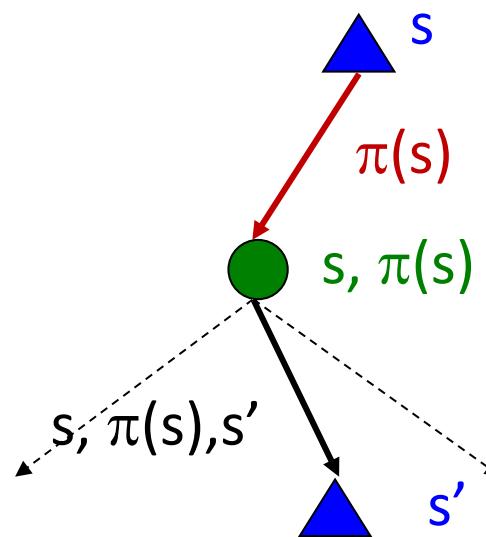


Fixed Policies

Do the optimal action



Do what π says to do

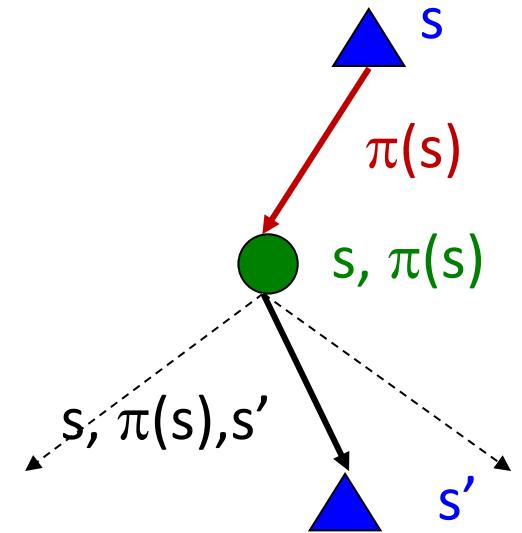


- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
 - ... though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy

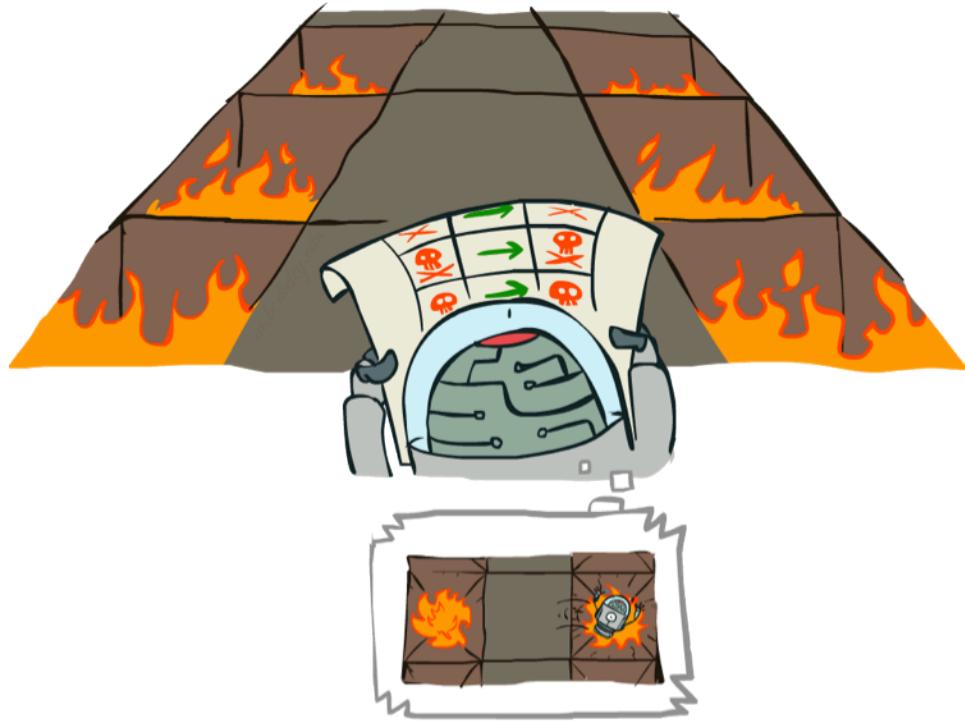
- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s , under a fixed policy π :
 $V^\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^\pi(s')]$$

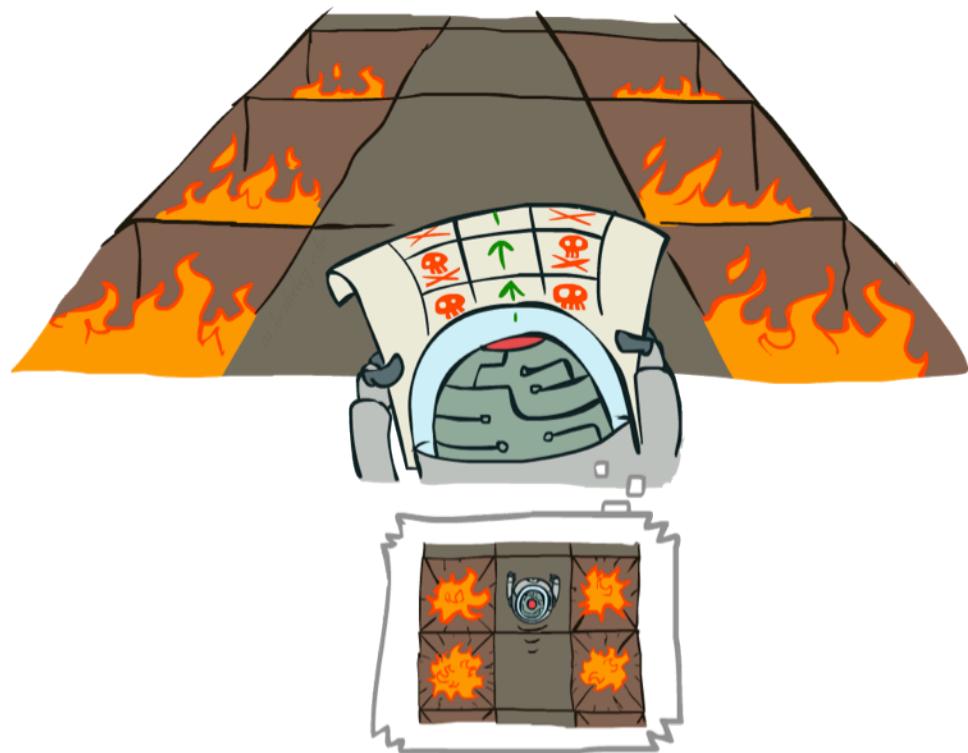


Example: Policy Evaluation

Always Go Right



Always Go Forward



Example: Policy Evaluation

Always Go Right



Always Go Forward

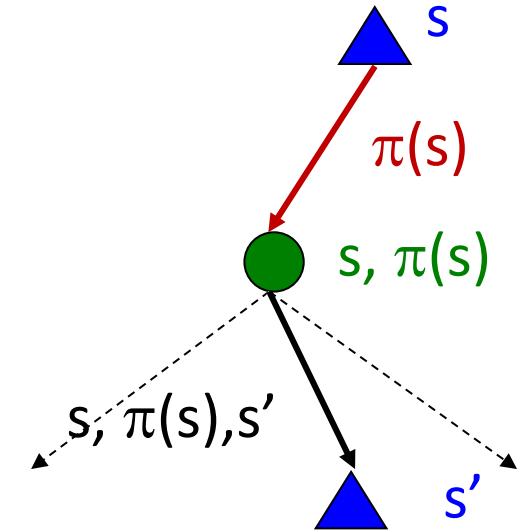


Policy Evaluation

- How do we calculate the V's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

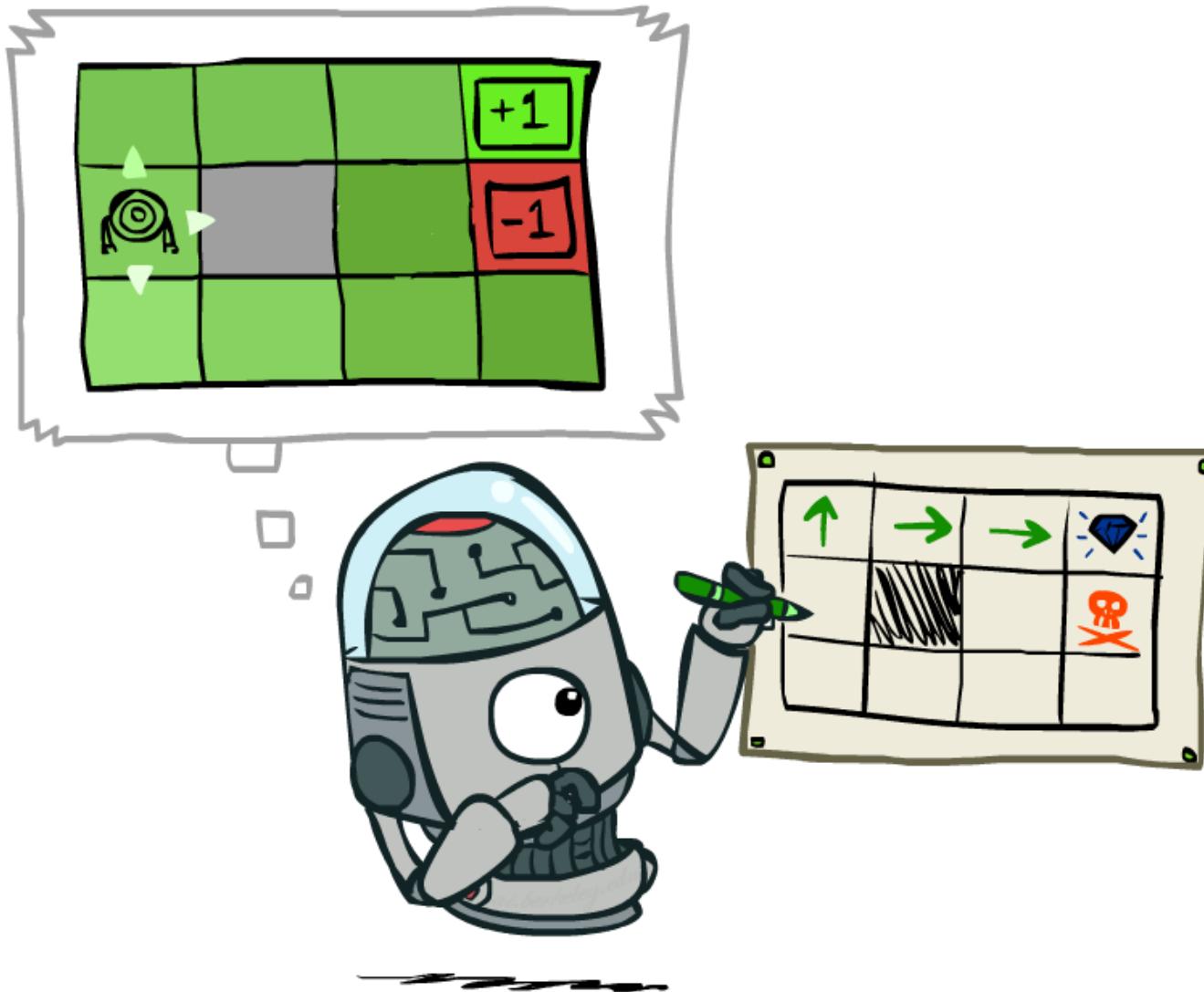
$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$



- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
 - Solve with Matlab (or your favorite linear system solver)

Policy Extraction



Computing Actions from Values

- Let's imagine we have the optimal values $V^*(s)$
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)



$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- This is called **policy extraction**, since it gets the policy implied by the values

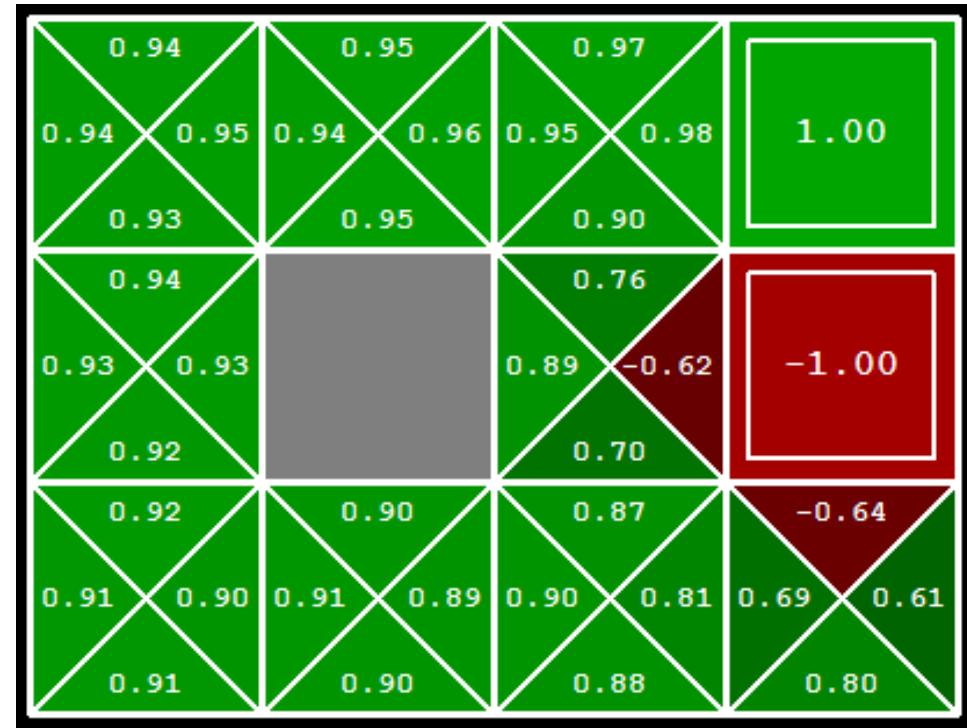
Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:

- How should we act?

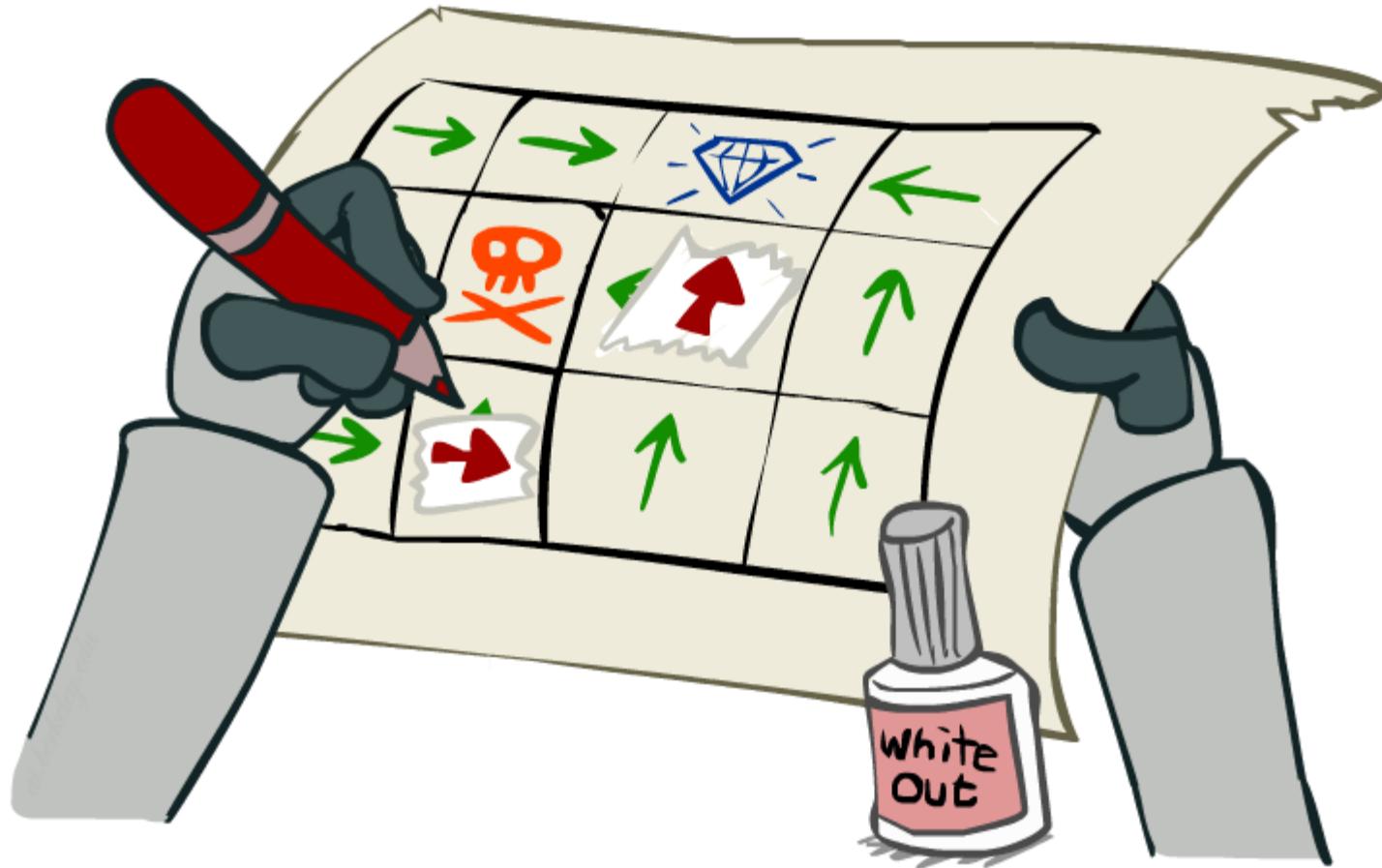
- Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$



- Important lesson: actions are easier to select from q-values than values!

Policy Iteration

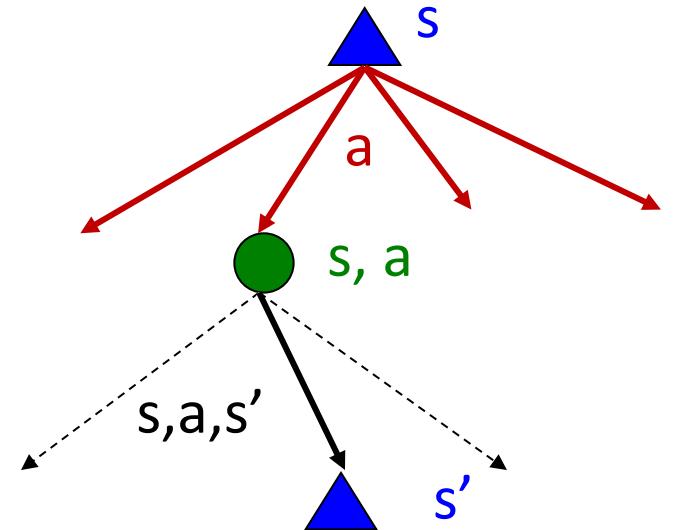


Problems with Value Iteration

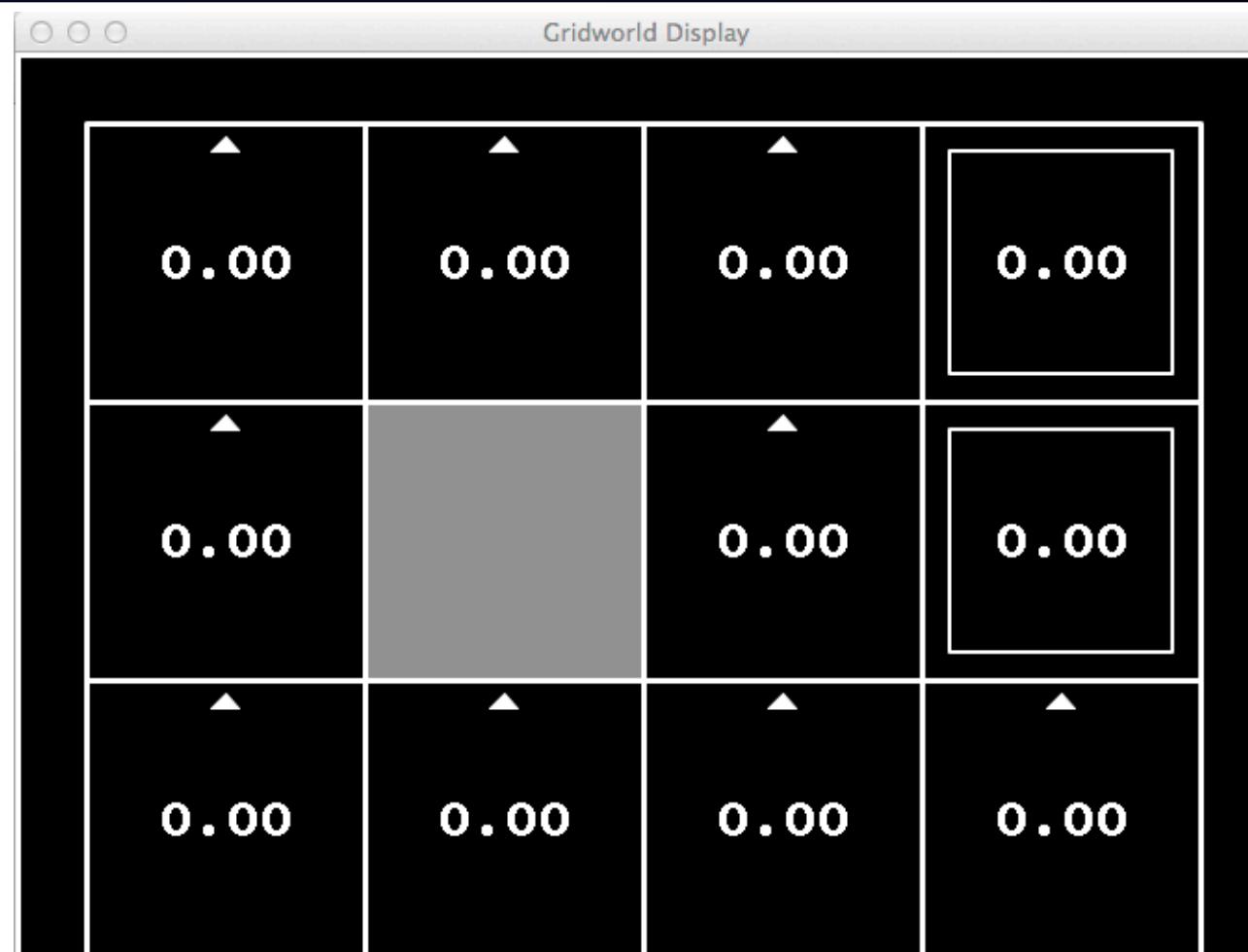
- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values



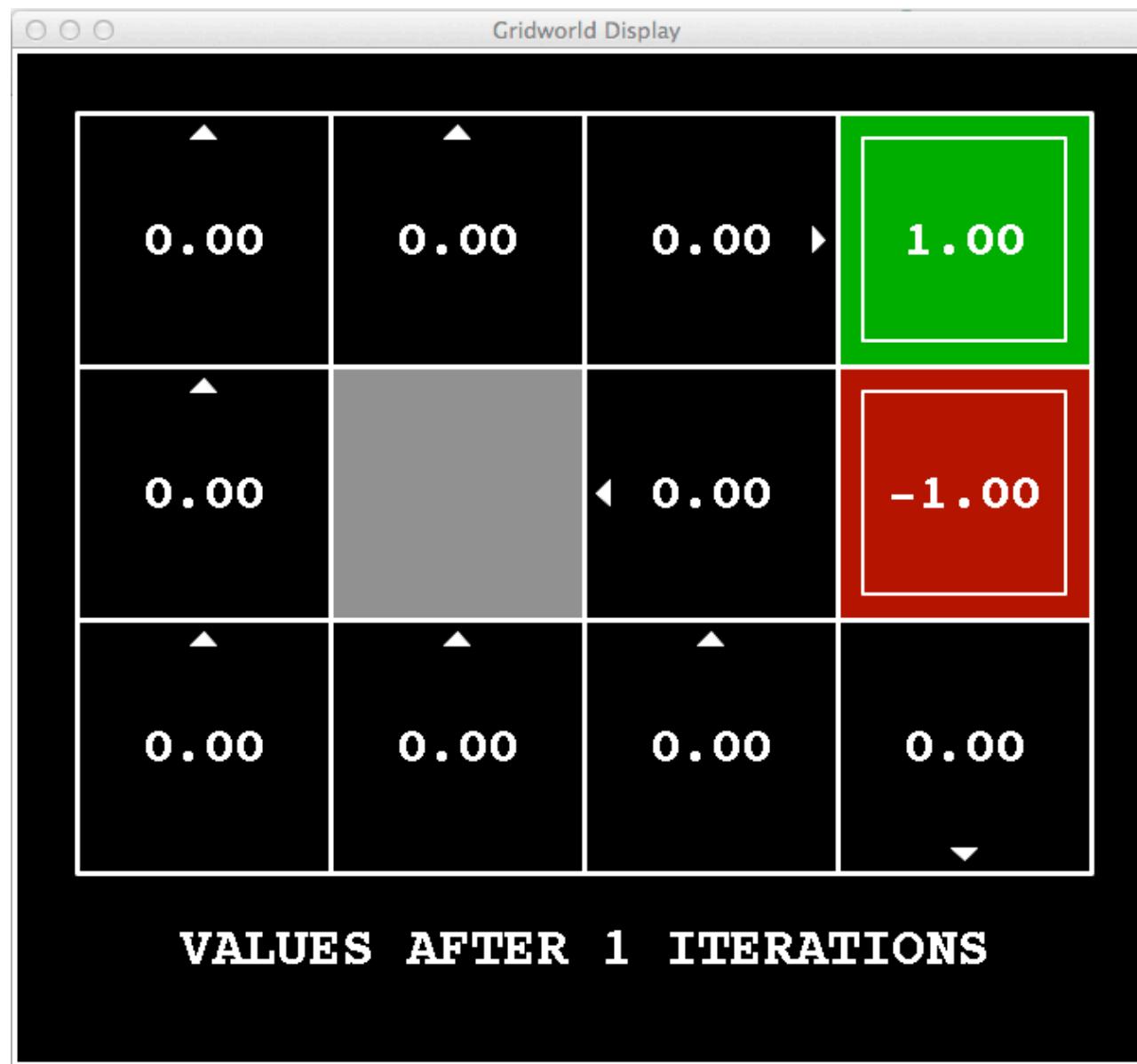
$k=0$



VALUES AFTER 0 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

$k=1$



$k=2$



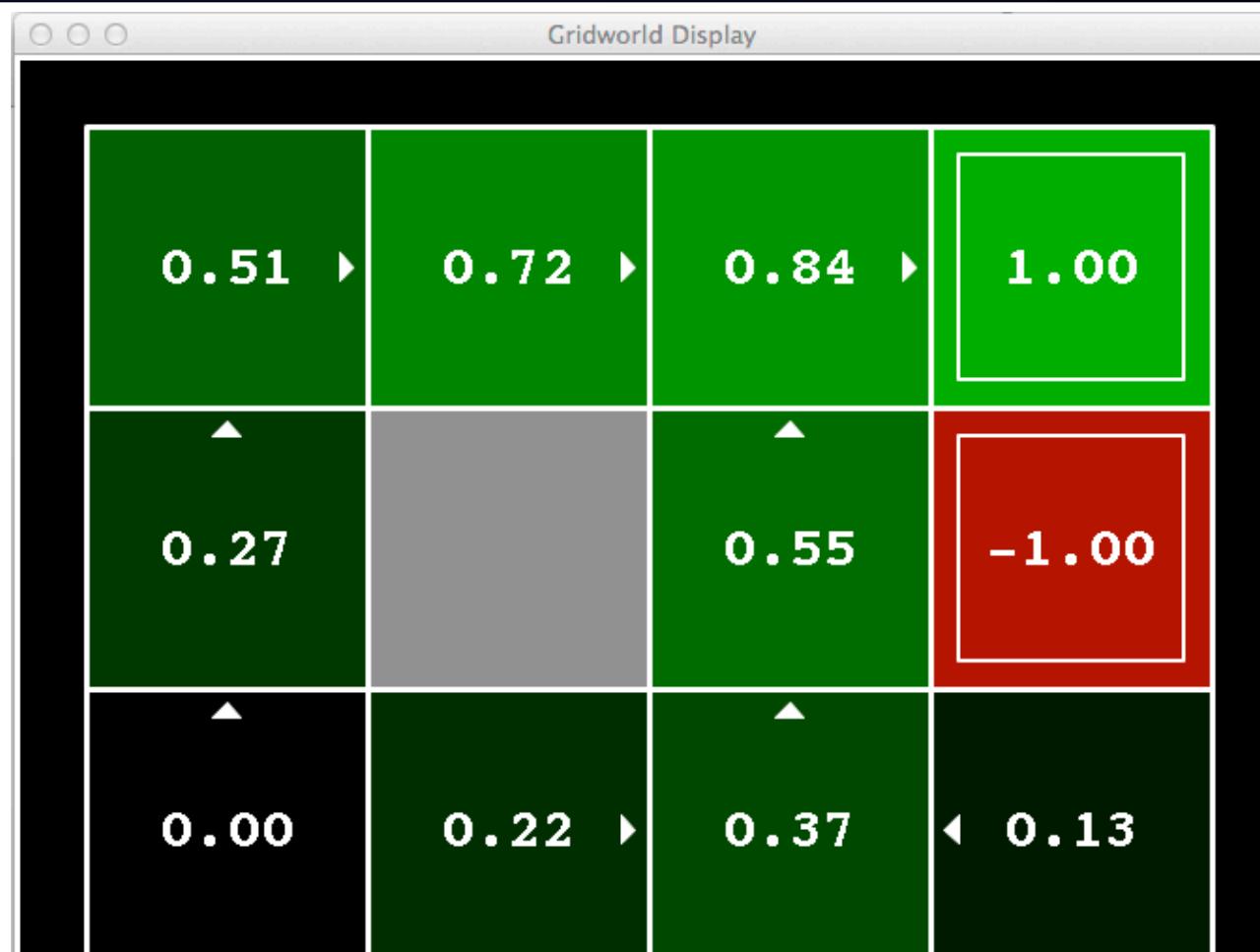
k=3



k=4



k=5



VALUES AFTER 5 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



k=7



k=8



k=9



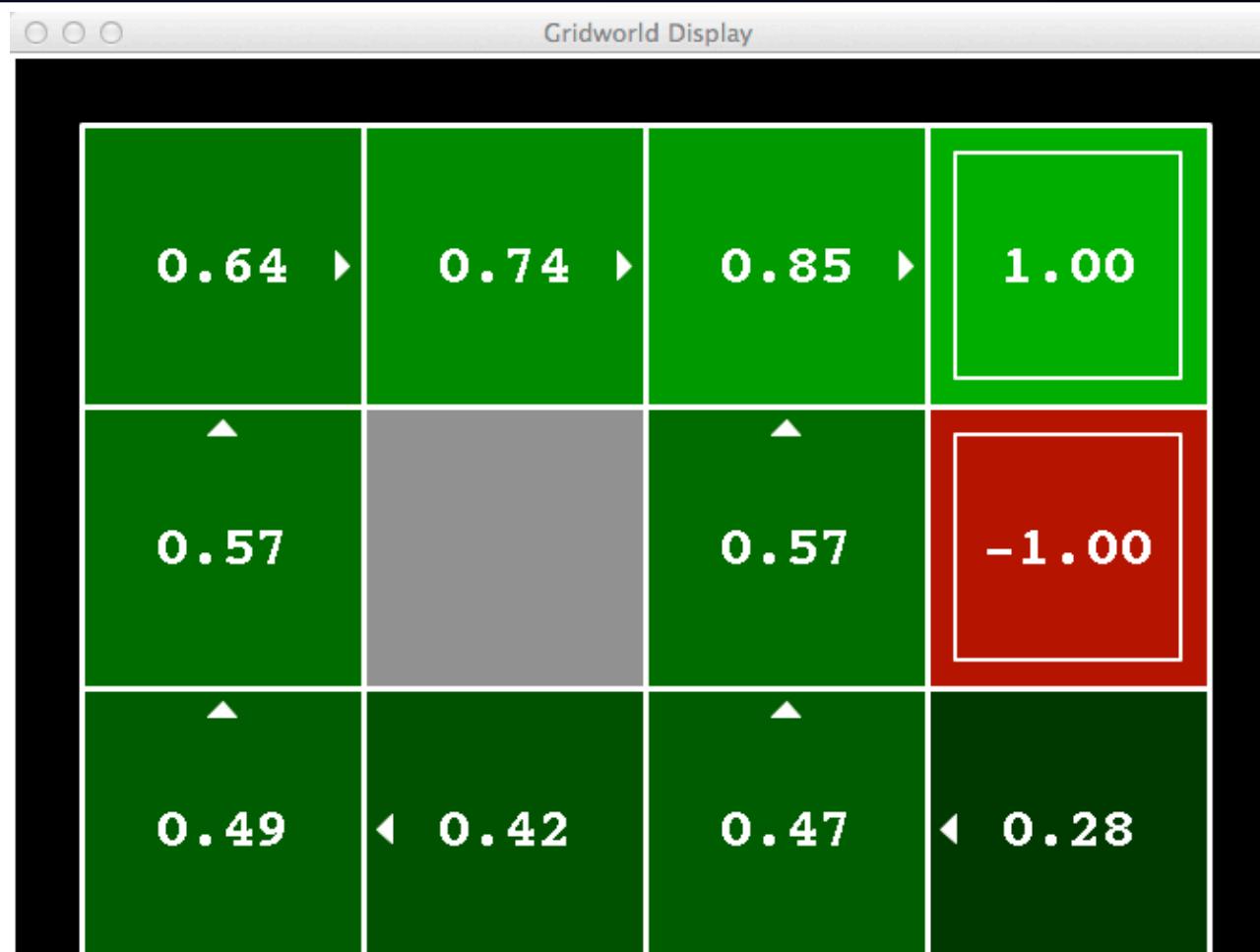
k=10



k=11



k=12



VALUES AFTER 12 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

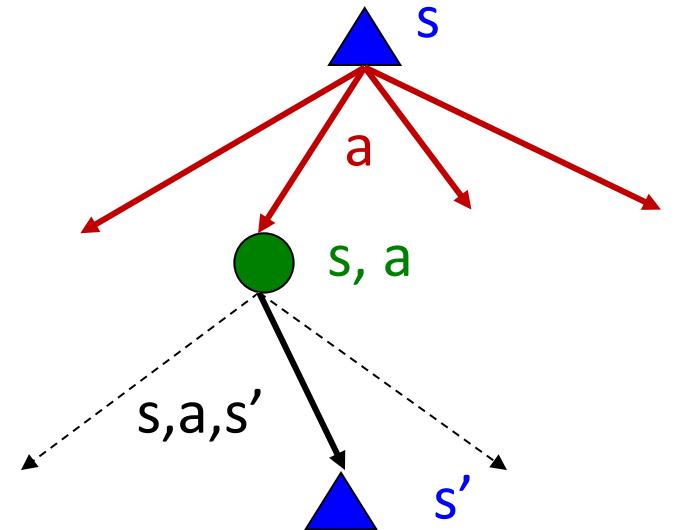


Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values



Policy Iteration

- Alternative approach for optimal values:
 - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges
- This is policy iteration
 - It's still optimal!
 - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') [R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:

$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

- So you want to...
 - Compute optimal values: use value iteration or policy iteration
 - Compute values for a particular policy: use policy evaluation
 - Turn your values into a policy: use policy extraction (one-step lookahead)
- These all look the same!
 - They basically are – they are all variations of Bellman updates
 - They all use one-step lookahead expectimax fragments
 - They differ only in whether we plug in a fixed policy or max over actions

Next Time: Reinforcement Learning!
