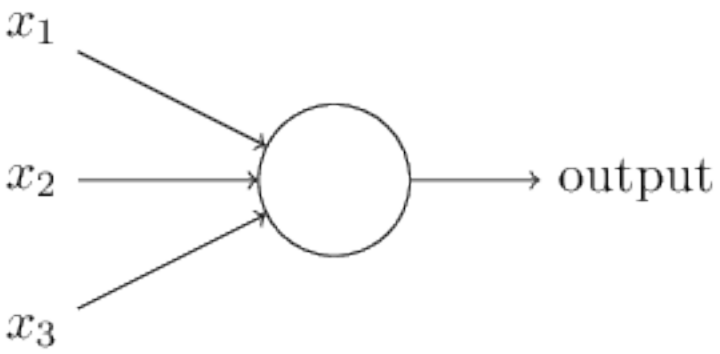
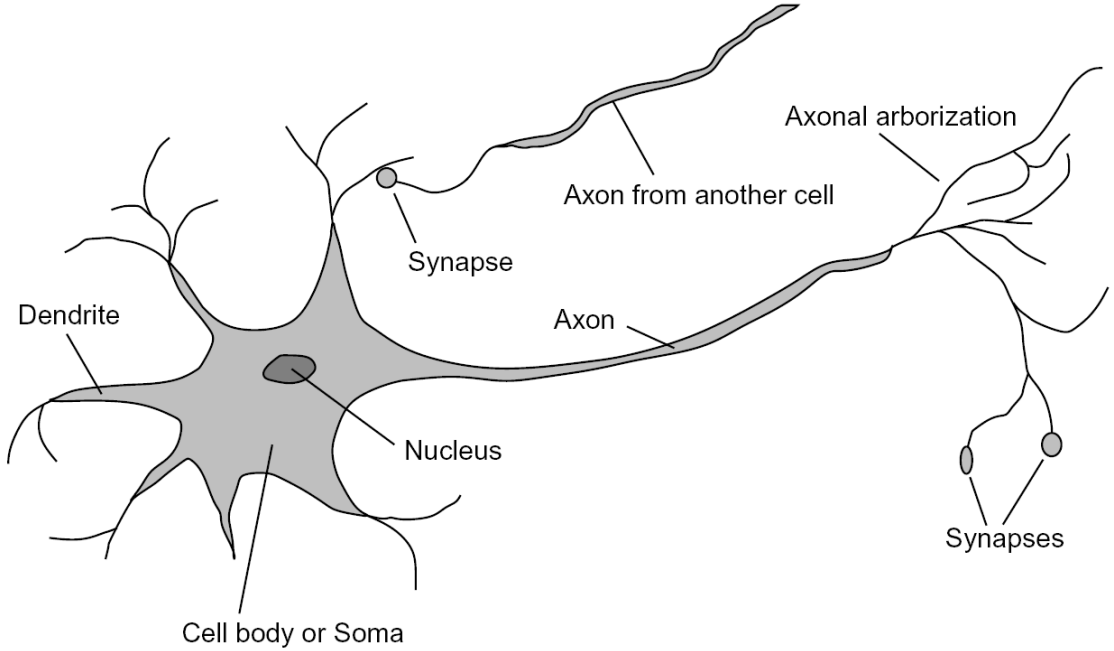
Neural Networks

This lecture is based on Michael Nielsen's *Neural Networks and Deep Learning,* a free online book (with code!).
Please read chapters 1-3.

Review: Perceptron

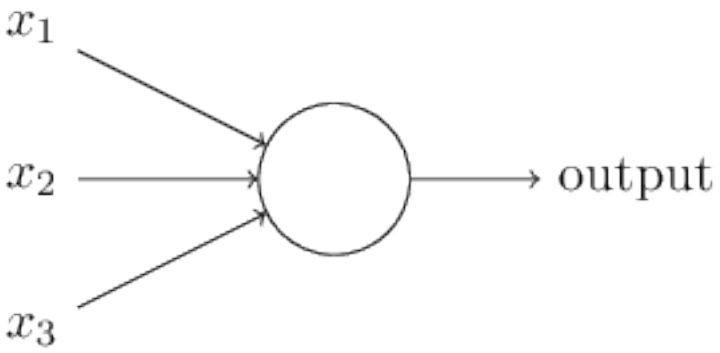
- Perceptrons were developed in the 1950s and 1960s loosely inspired by the neuron.





Review: Perceptron

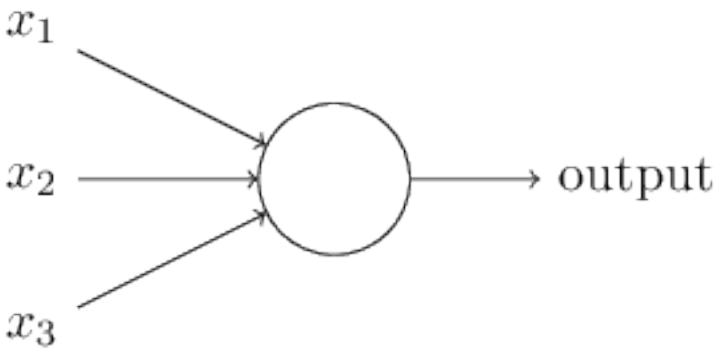
- Perceptron has inputs, $x_1, x_2, .. x_N$, and weights $w_1, w_2, .. w_N$
- The perceptron outputs 0 or 1, based on the weighted sum is less than or greater than a threshold value

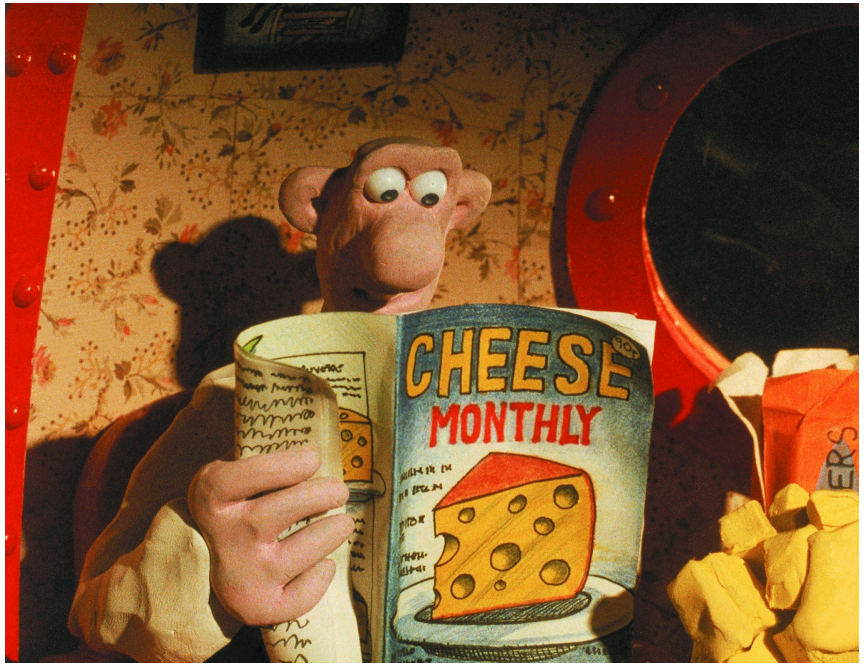


$$\text{output} = \begin{cases} 0 & \text{if } \sum_j w_j x_j \leq \text{threshold} \\ 1 & \text{if } \sum_j w_j x_j > \text{threshold} \end{cases}$$

Perceptrons for decision making

- We can think about the perceptron as a device that makes decisions by weighing up evidence.
- Example: Suppose there's a cheese festival in your town. You like cheese.



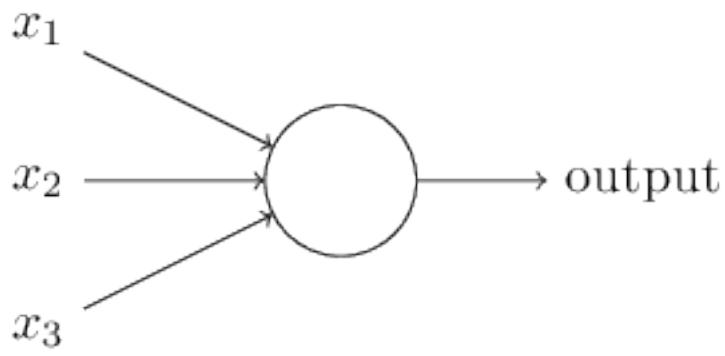


Perceptrons for decision making

- You might use 3 factors to decide whether to go.

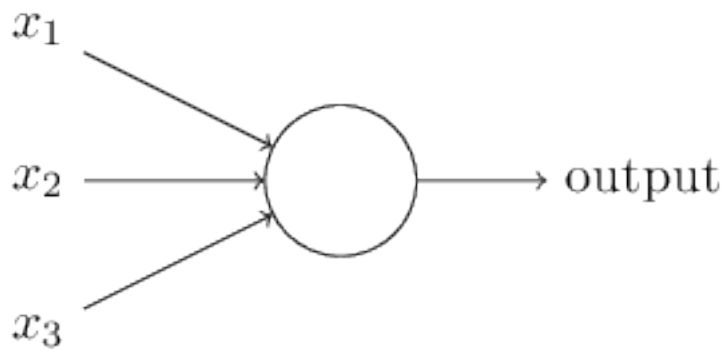
1. Is the weather good?
2. Can your loyal companion come with you?
3. Is the festival near public transit?

- These can be the binary input values to a perceptron



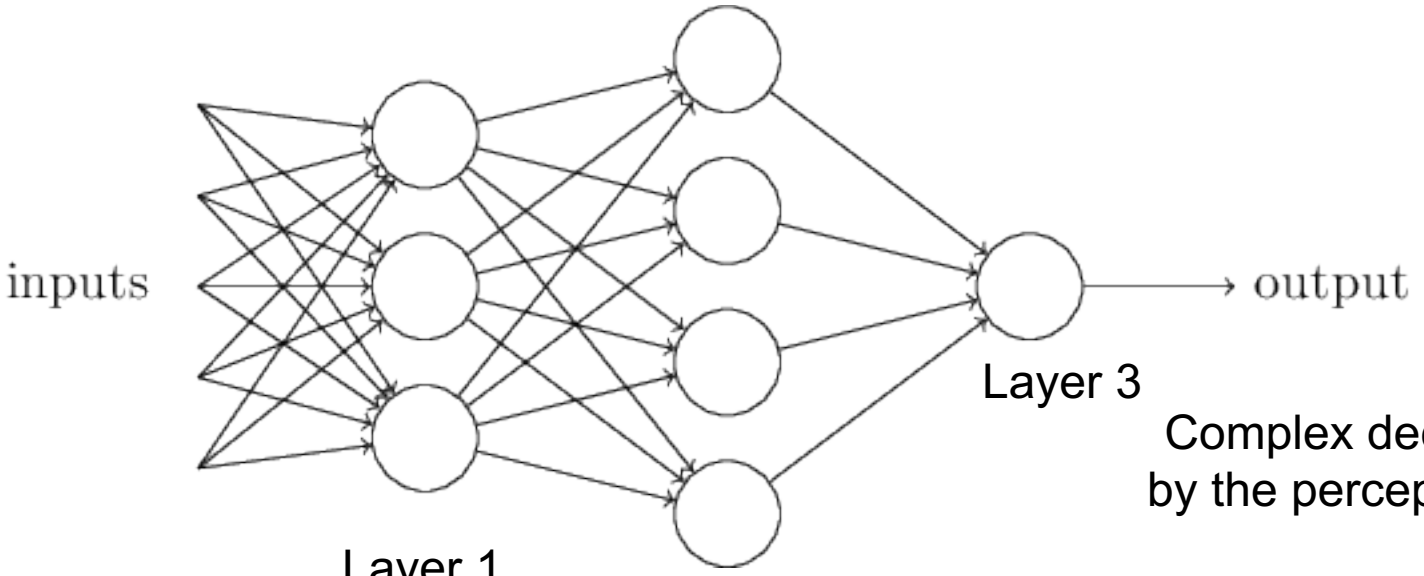
Perceptrons for decision making

- By varying weights and the threshold we get different models of decision making
- Example 1: $w_1 = 6$ $w_2 = 2$ $w_3 = 2$, threshold = 5
- Example 2: $w_1 = 6$ $w_2 = 2$ $w_3 = 2$, threshold = 3



Perceptrons for decision making

- A complex network of perceptrons could make quite subtle decisions:



1st layer makes 3 simple decisions by weighing the input evidence

2nd layer makes a decision by weighing up the results from the 1st layer of decision-making

Complex decisions can be made by the perceptron in the third layer

Weights, bias and dot products

- Two notational changes simplify the way that perceptrons are described.
- The first change is to replace the weighted sum as a dot product

$$w \cdot x \equiv \sum_j w_j x_j$$

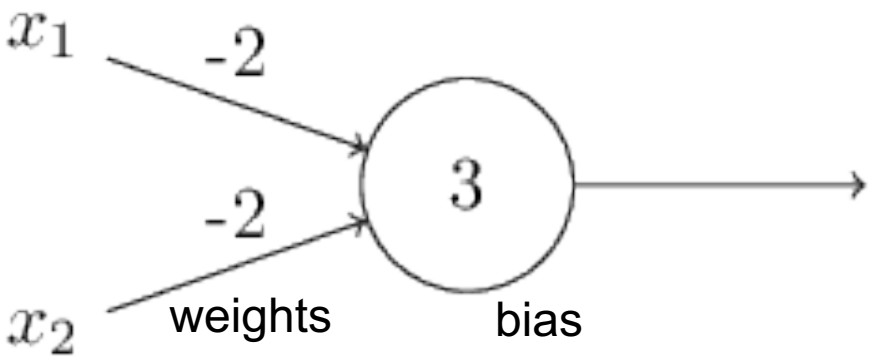
- The second change is to move the threshold to the other side of the inequality, and to replace it by a *bias*, b≡−threshold

$$\text{output} = \begin{cases} 0 & \text{if } \sum_j w_j x_j \leq \text{threshold} \\ 1 & \text{if } \sum_j w_j x_j > \text{threshold} \end{cases}$$

$$\text{output} = \begin{cases} 0 & \text{if } w \cdot x + b \leq 0 \\ 1 & \text{if } w \cdot x + b > 0 \end{cases}$$

Decision making OR logical functions

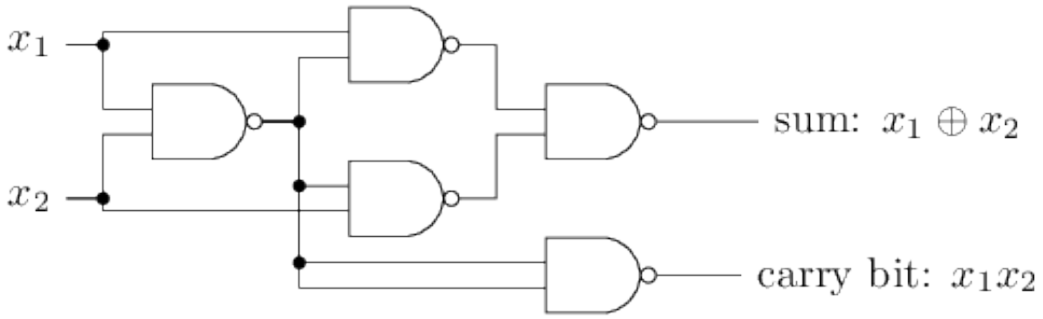
- Perceptrons can be used is to compute logical functions like AND, OR and NAND
- Example:

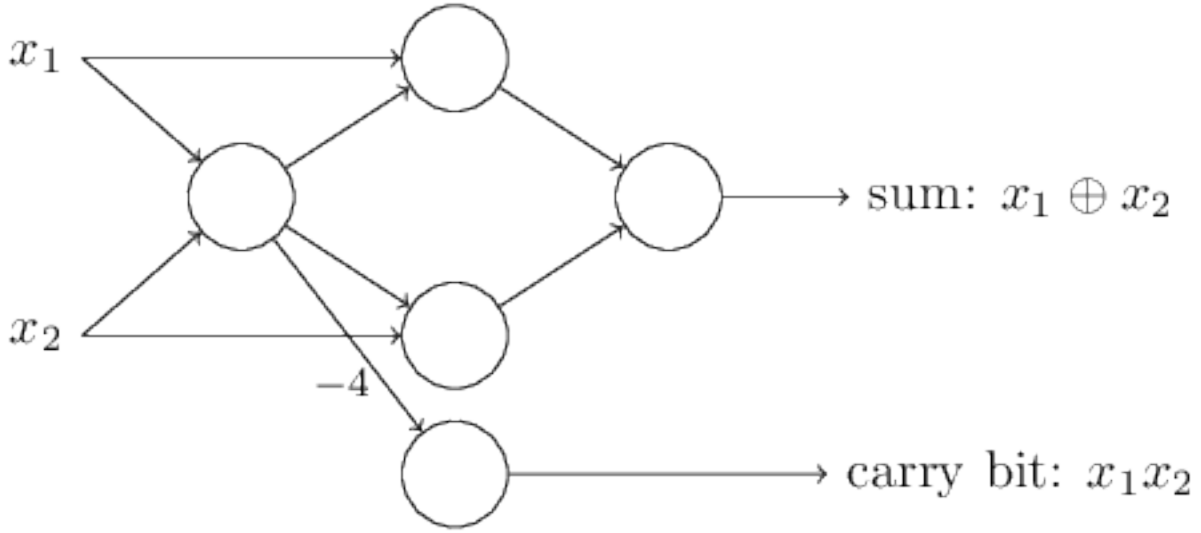


Input	Weighted sum	Output
00	-2*0 + -2*0 +3 = 3	1
10 or 01	-2*1 + -2*0 +3 = 1	1
11	-2*1 + -2*1 +3 = -1	0

Logical functions

- Networks of perceptrons to compute *any* logical function
- We can build any computation up out of NAND gates.
- For example, a circuit which adds two bits x_1 and x_2





All unlabeled weights are -2, all biases =3.

Power of Perceptrons

- Networks of Perceptrons are universal for computation, like NAND gates
- Perceptrons can be as powerful as any other computing device!

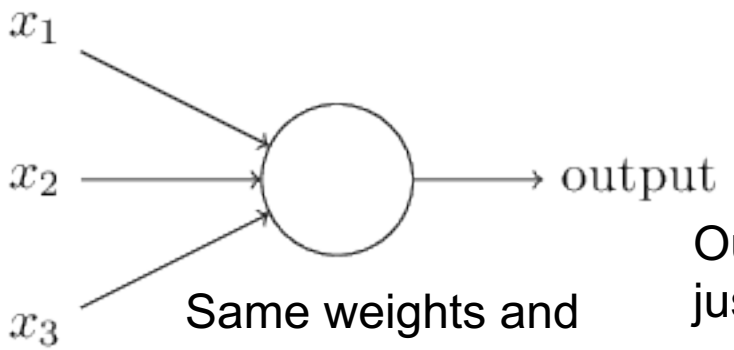
- We can devise *learning algorithms* to automatically tune the weights and biases of a network of artificial neurons
- Instead of laying out a circuit of NAND and other gates, neural networks can simply learn to solve problems

Sigmoid neurons

- Problem: a small change in the weights or bias of any single perceptron in the network can causes the output to completely flip from 0 to 1.
- Solution: sigmoid neuron

Perceptron

$$\text{output} = \begin{cases} 0 & \text{if } w \cdot x + b \leq 0 \\ 1 & \text{if } w \cdot x + b > 0 \end{cases}$$



Same weights and bias as perceptron

Output is no longer just 1 or 0.

Sigmoid neuron

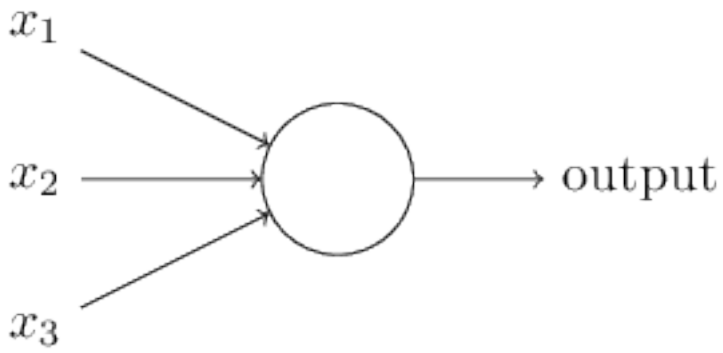
output = $\sigma(w \cdot x + b)$

Sigmoid function:

$$\sigma(z) \equiv \frac{1}{1 + e^{-z}}$$

Inputs: any real-valued number

Perceptron



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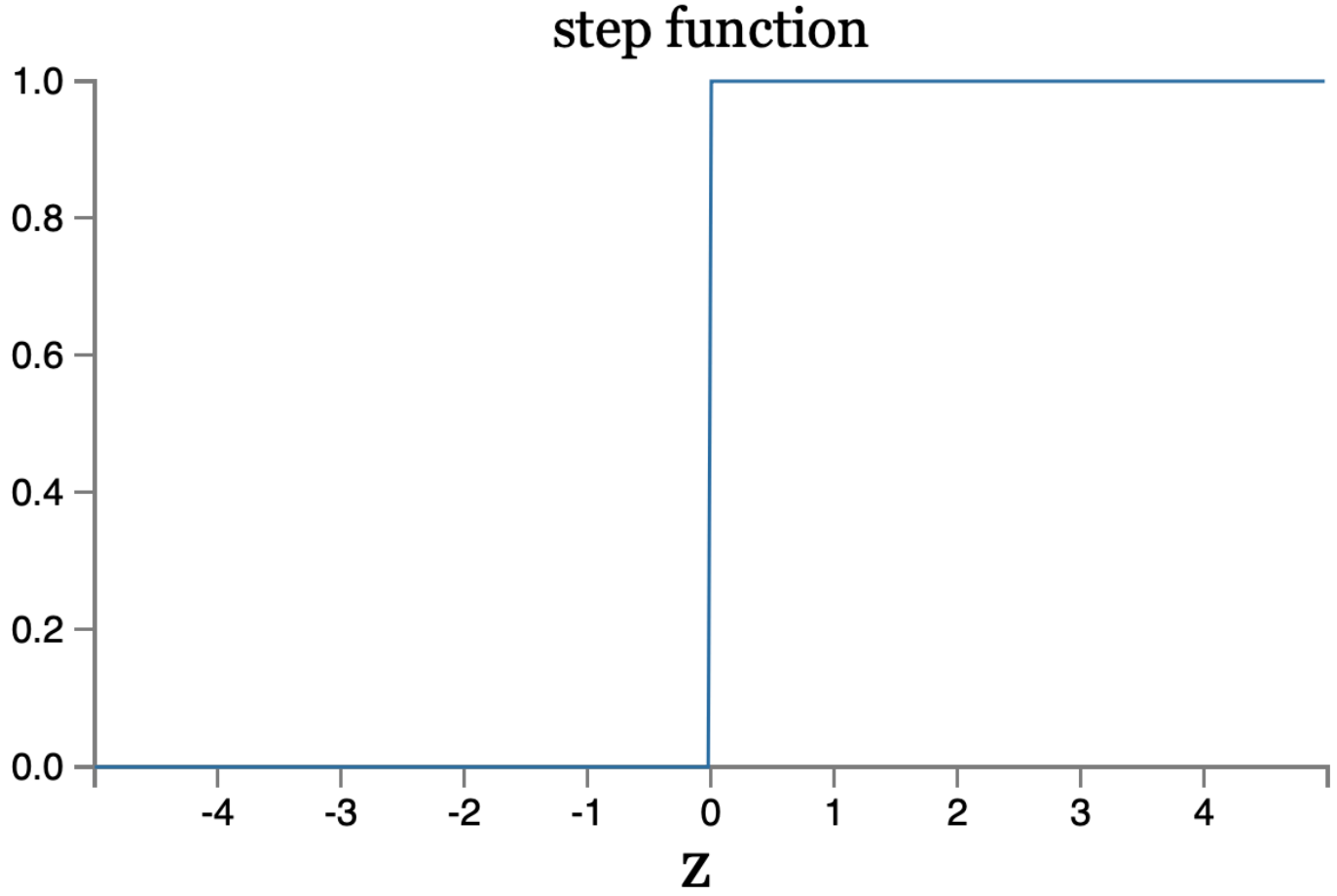
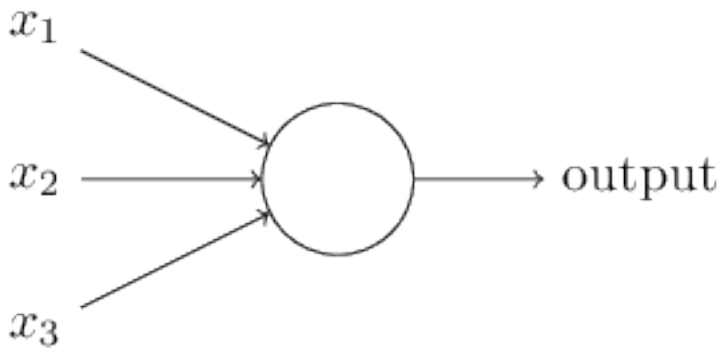
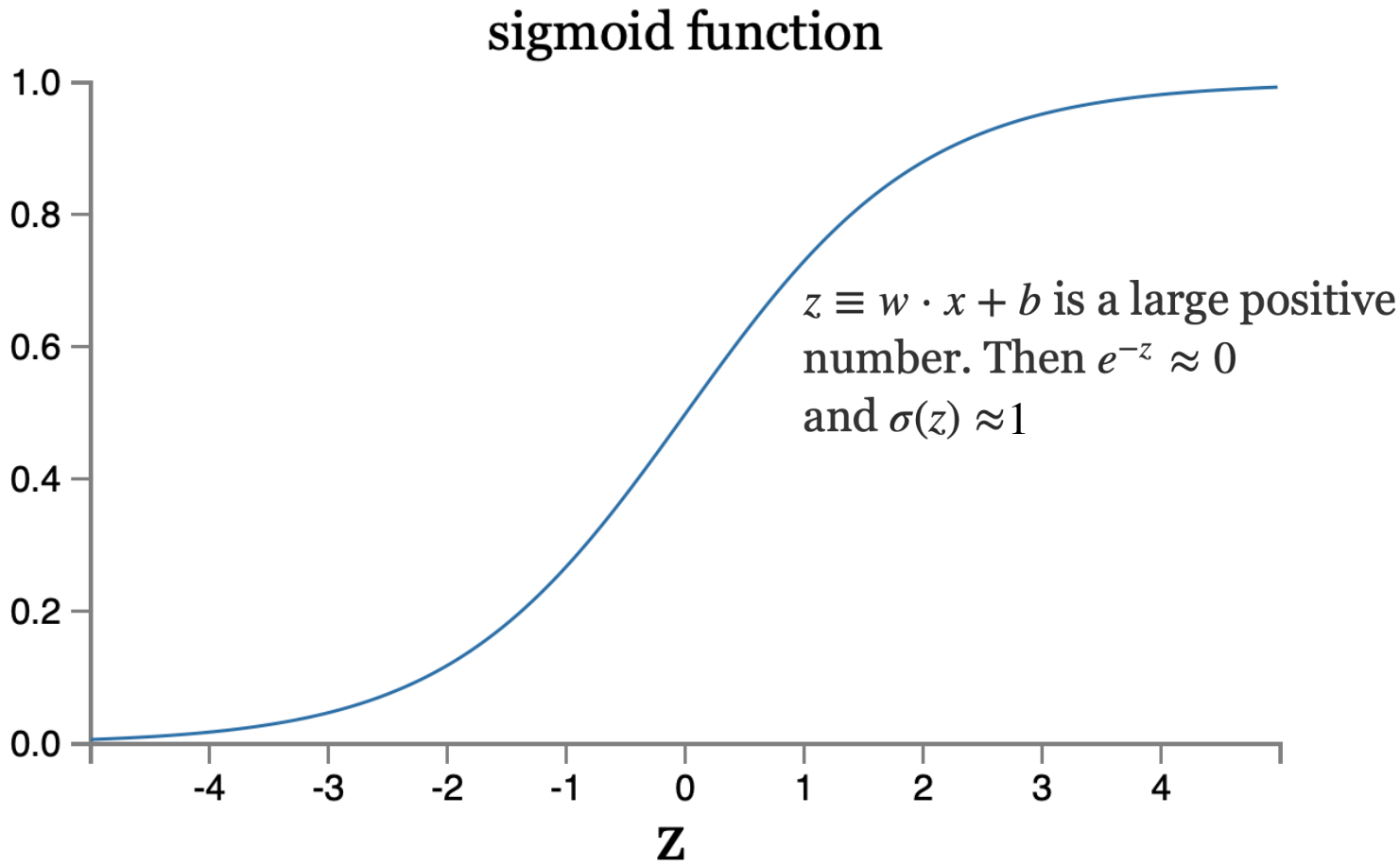


Sigmoid neurons



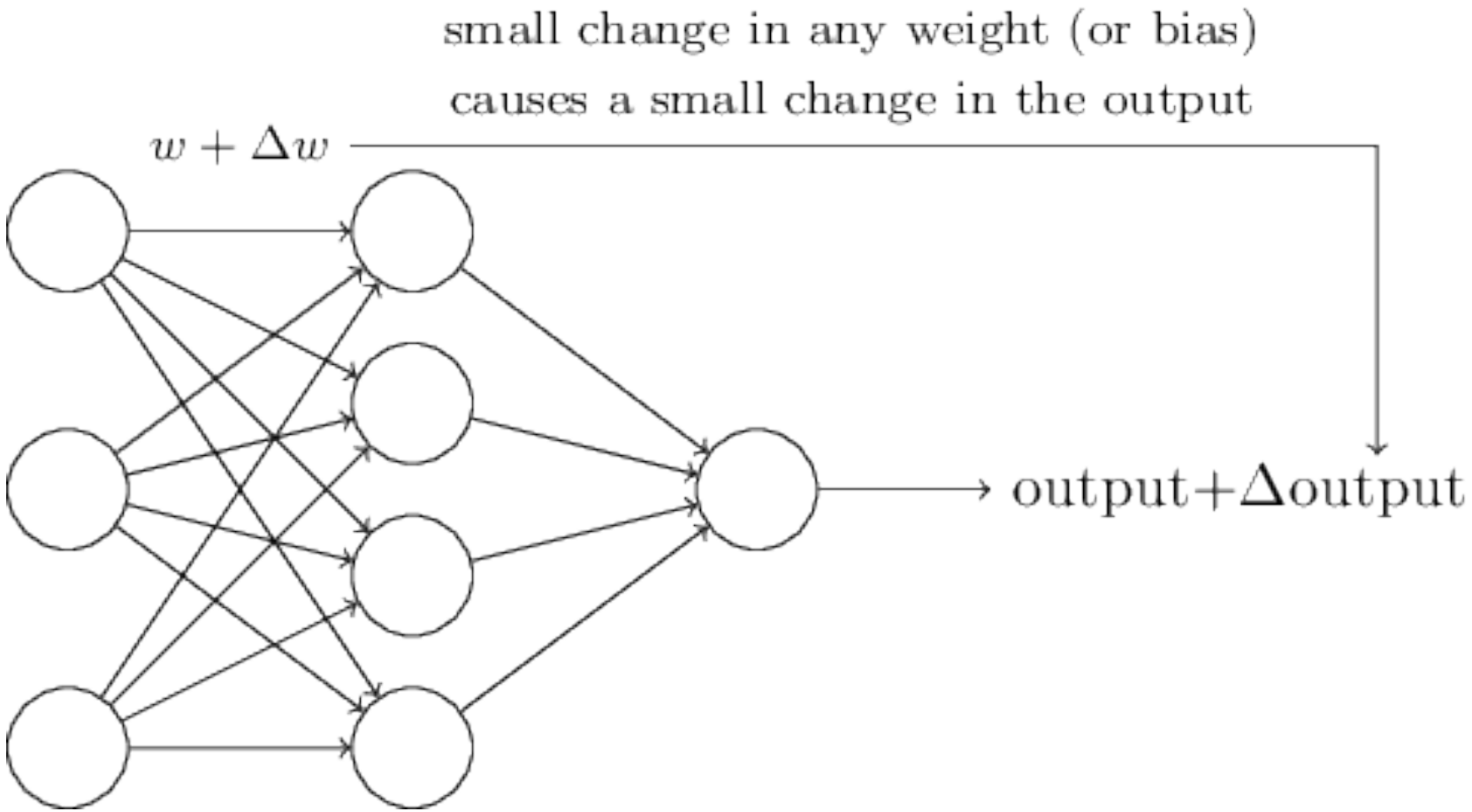
$$z \equiv w \cdot x + b$$

$$\sigma(z) \equiv \frac{1}{1 + e^{-z}}$$



$z = w \cdot x + b$ is very negative.
Then $e^{-z} \to \infty$, and $\sigma(z) \approx 0$

Sigmoid neurons are better for training



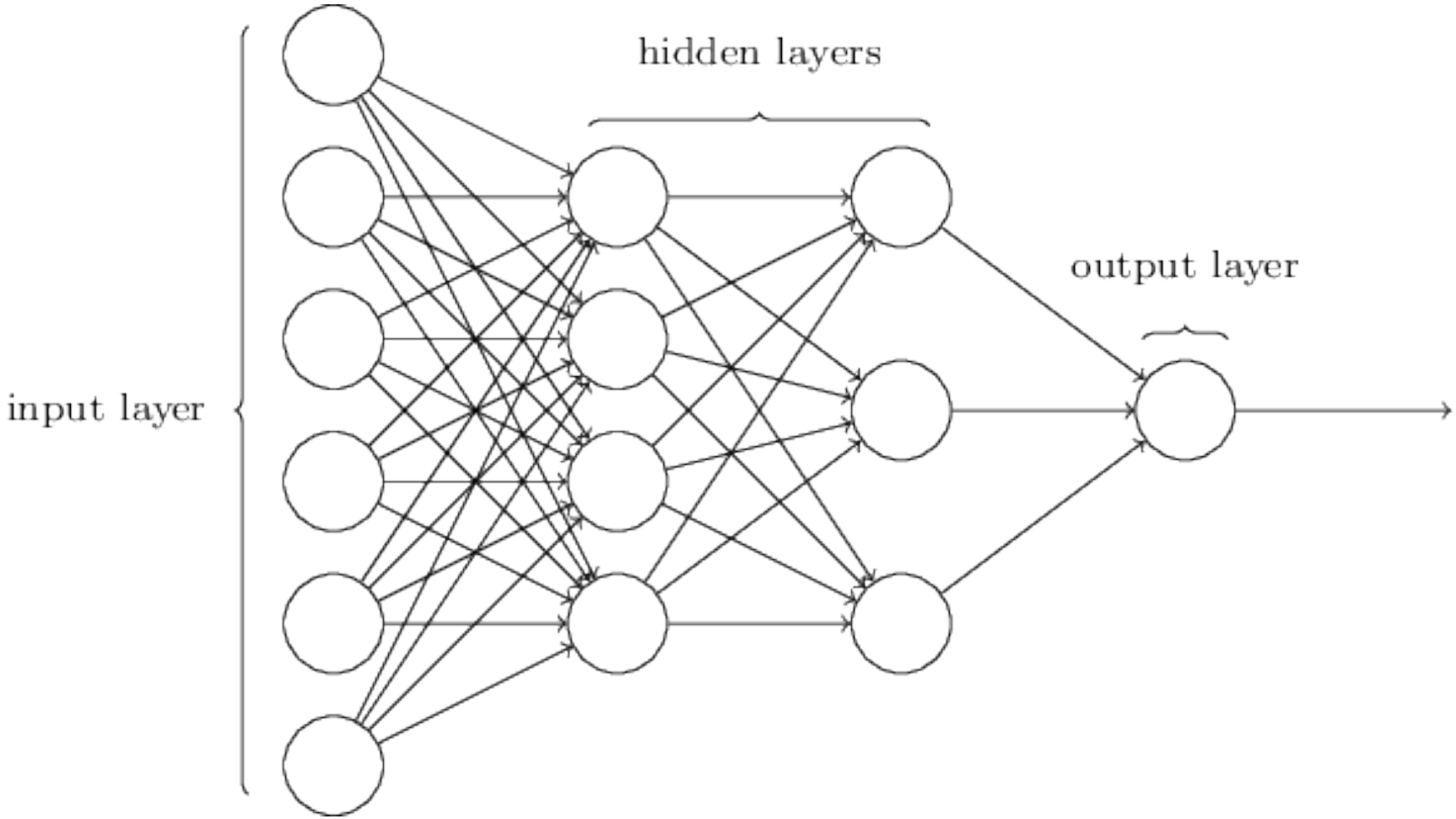
Smoothness is crucial

- Smoothness of σ means that small changes in the weights w_j and in the bias b will produce a small change the output from the neuron

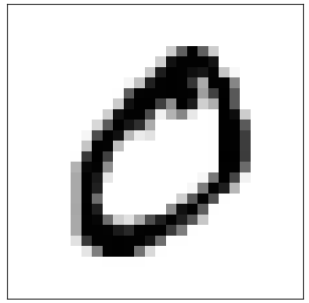
$$\Delta \text{output} \approx \sum_j \frac{\partial \, \text{output}}{\partial w_j} \Delta w_j + \frac{\partial \, \text{output}}{\partial b} \Delta b$$

- Δoutput is a *linear function* of the changes Δwj and Δb
- This makes it easy to choose small changes in the weights and biases to achieve any desired small change in the output

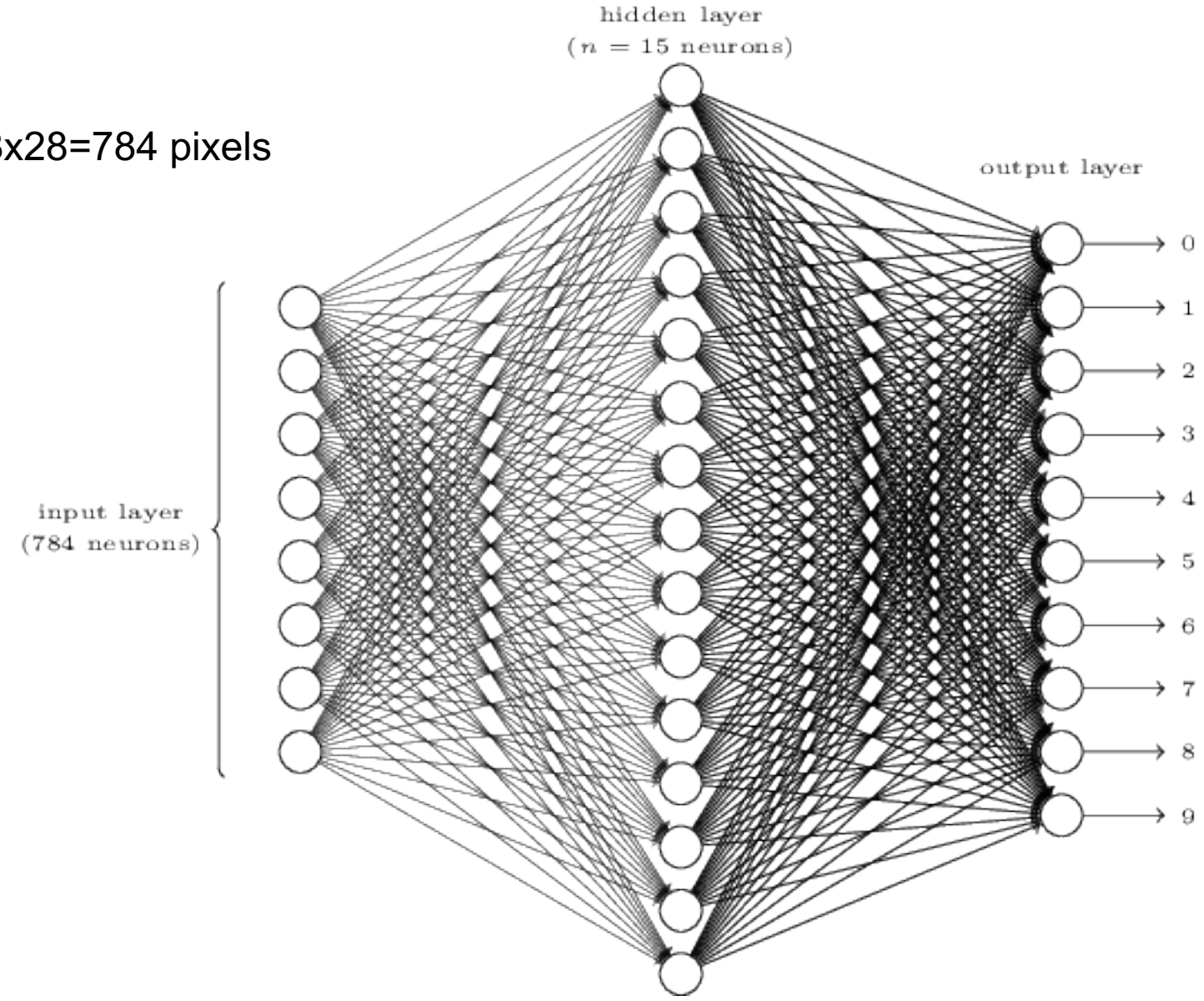
Neural Net Architecture



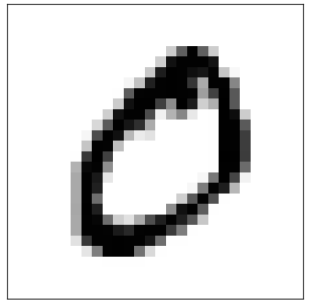
Neural Net Architecture



28x28=784 pixels



Neural Net Architecture



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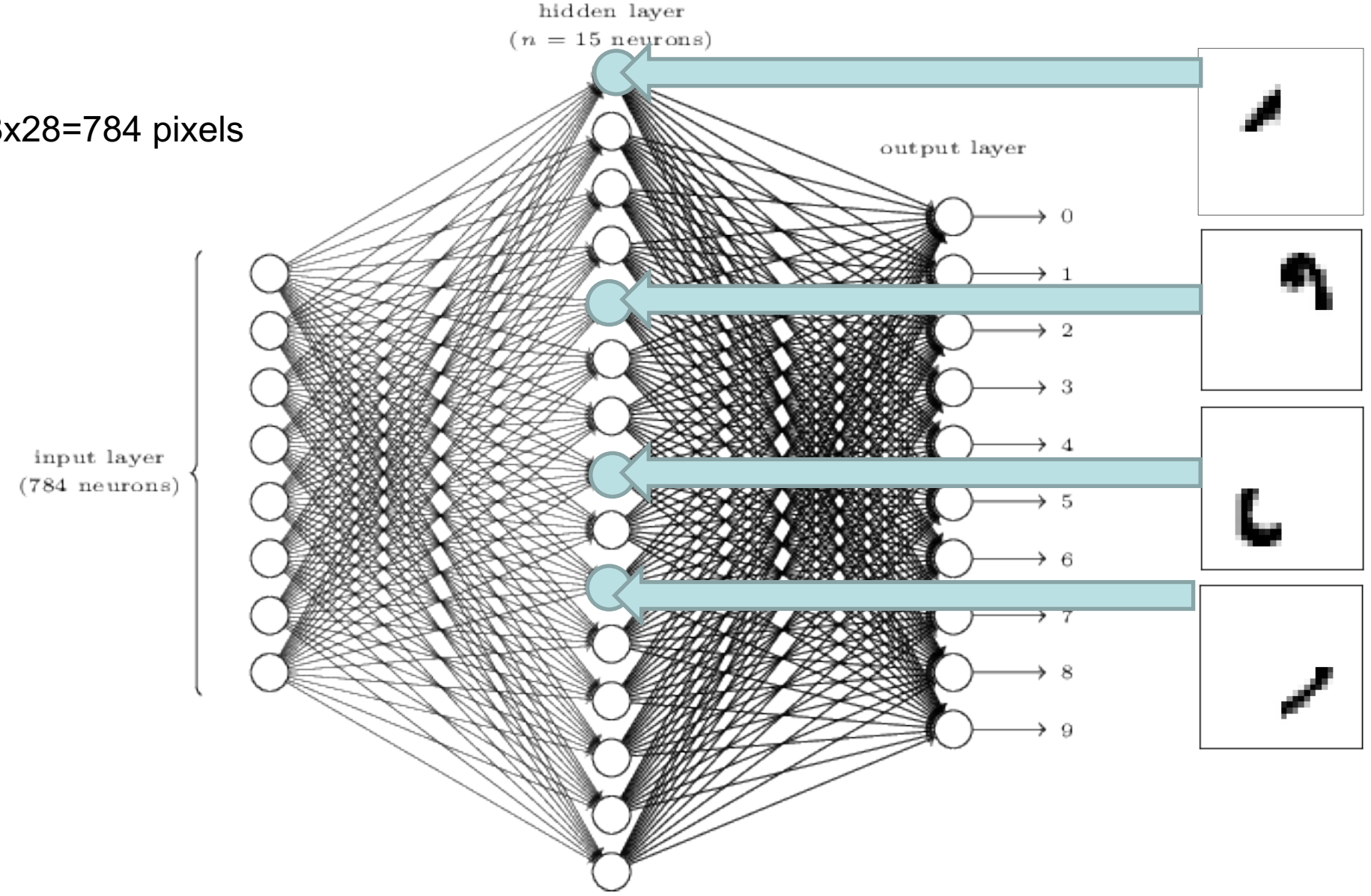
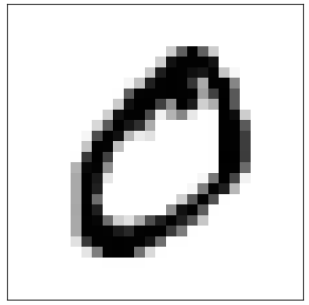


Neural Net Architecture



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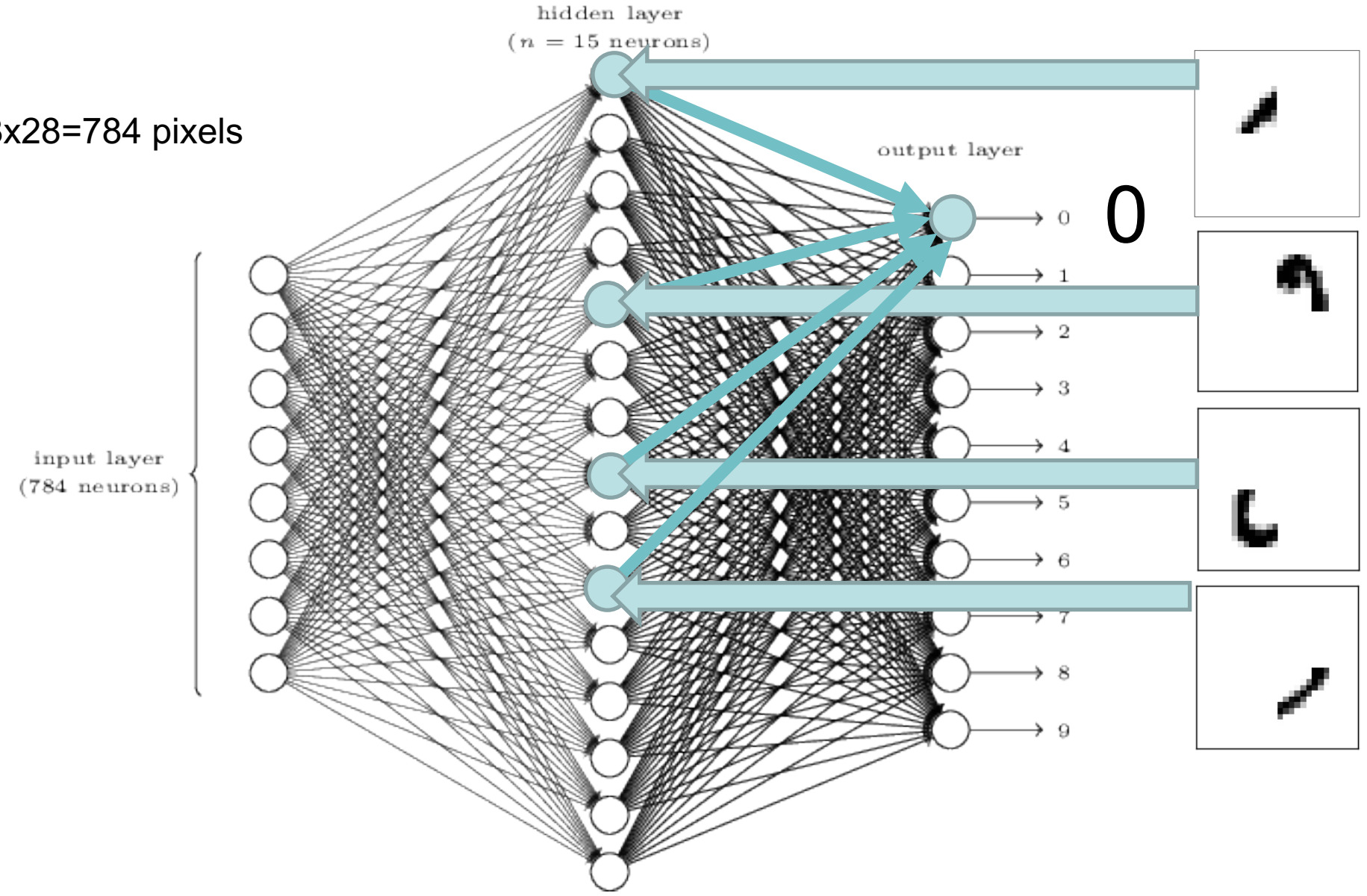

Cost Function

- AKA Loss function or Objective function
- Here's a common one called "Mean Squared Error"

$$C(w, b) \equiv \frac{1}{2n} \sum_x \|y(x) - a\|^2$$

Cost

weights, biases

n = number of training examples

y(x) = correct answer

a = vector of system's outputs

Cost Function

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- Here's a common one called "Mean Squared Error"

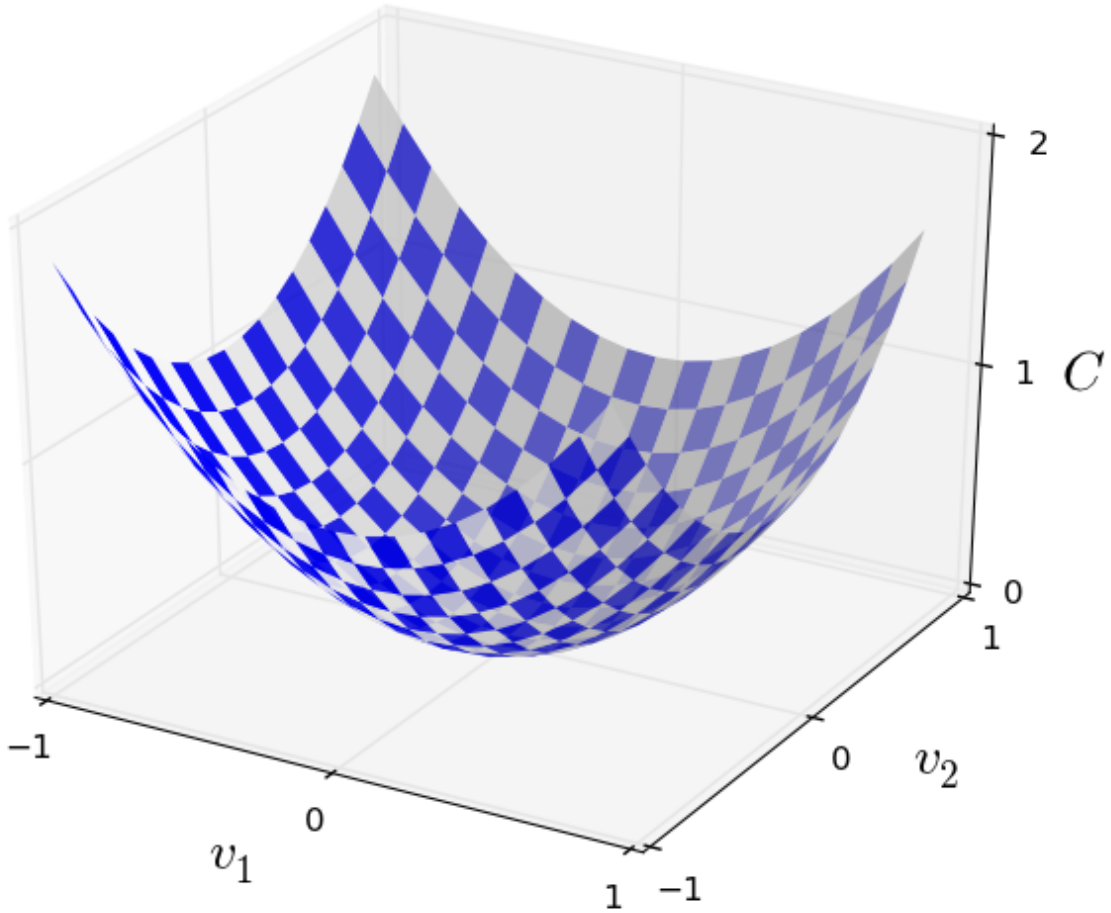
$$C(w,b) \equiv \frac{1}{2n} \sum_x \|y(x) - a\|^2$$

Cost | weights, biases | n = number of training examples | y(x) = correct answer | a = vector of system's outputs

$$\|\boldsymbol{x}\|_2 := \sqrt{x_1^2 + \cdots + x_n^2}$$

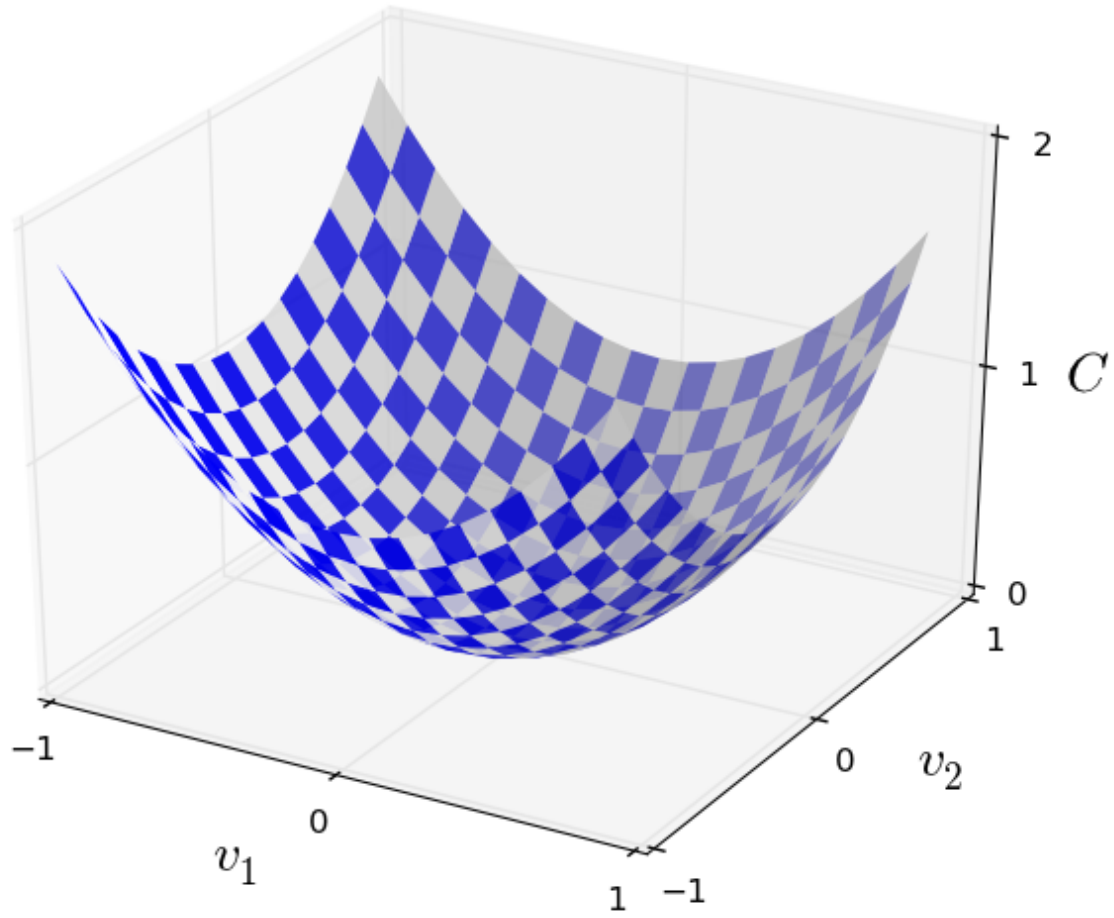
	y(x)	a
0	0	0.01
1	0	0.001
2	0	
3	0	
4	0	
5	0	
6	1	
7	0	
8	0	
9	0	

Minimize the loss function



Gradient Descent

Minimize the loss function



$$\Delta C \approx \frac{\partial C}{\partial v_1}\Delta v_1 + \frac{\partial C}{\partial v_2}\Delta v_2$$

(we want to minimize this)

vector of changes in v

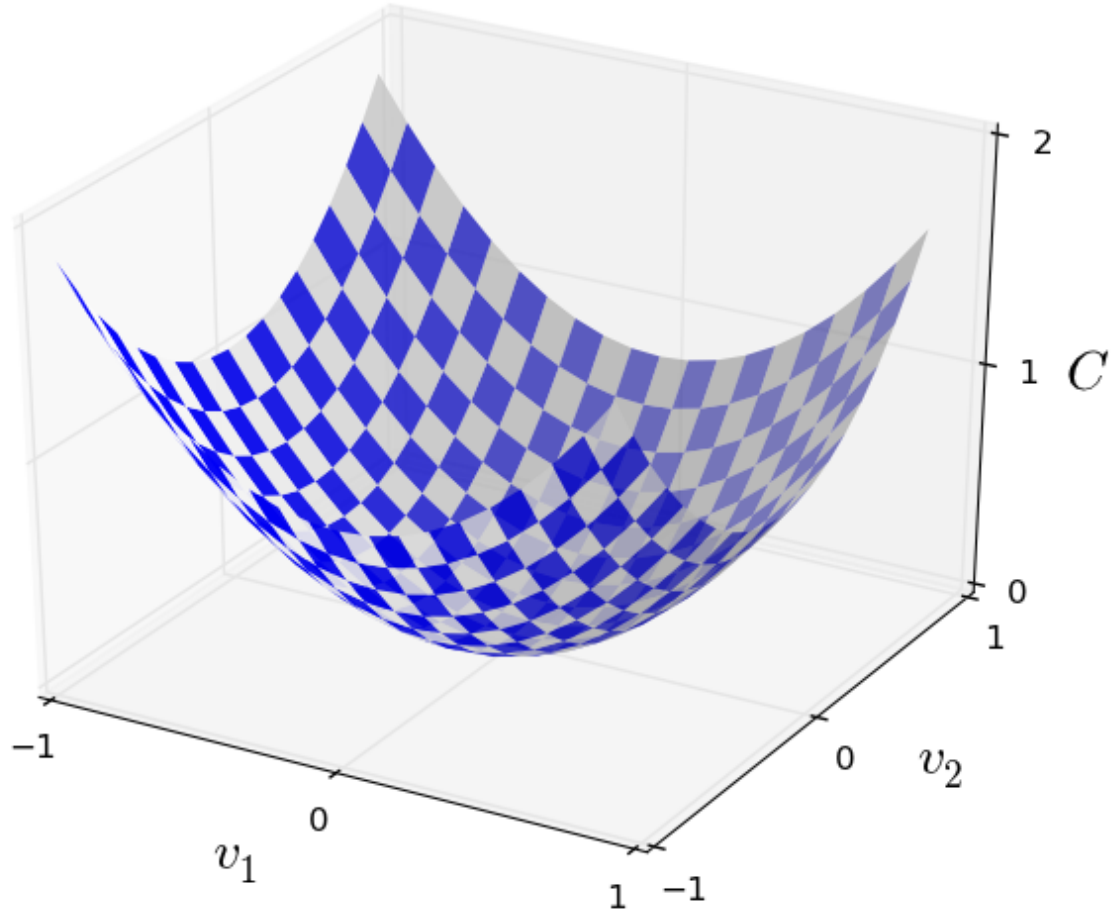
$$\Delta v \equiv (\Delta v_1, \Delta v_2)^T$$

gradient vector

$$\nabla C \equiv \left(\frac{\partial C}{\partial v_1}, \frac{\partial C}{\partial v_2}\right)^T$$

$$\Delta C \approx \nabla C \cdot \Delta v.$$

Minimize the loss function



$$\Delta v = -\eta \nabla C$$

η is a small, positive parameter known as the *learning rate*

$$\Delta C \approx \nabla C \cdot \Delta v.$$

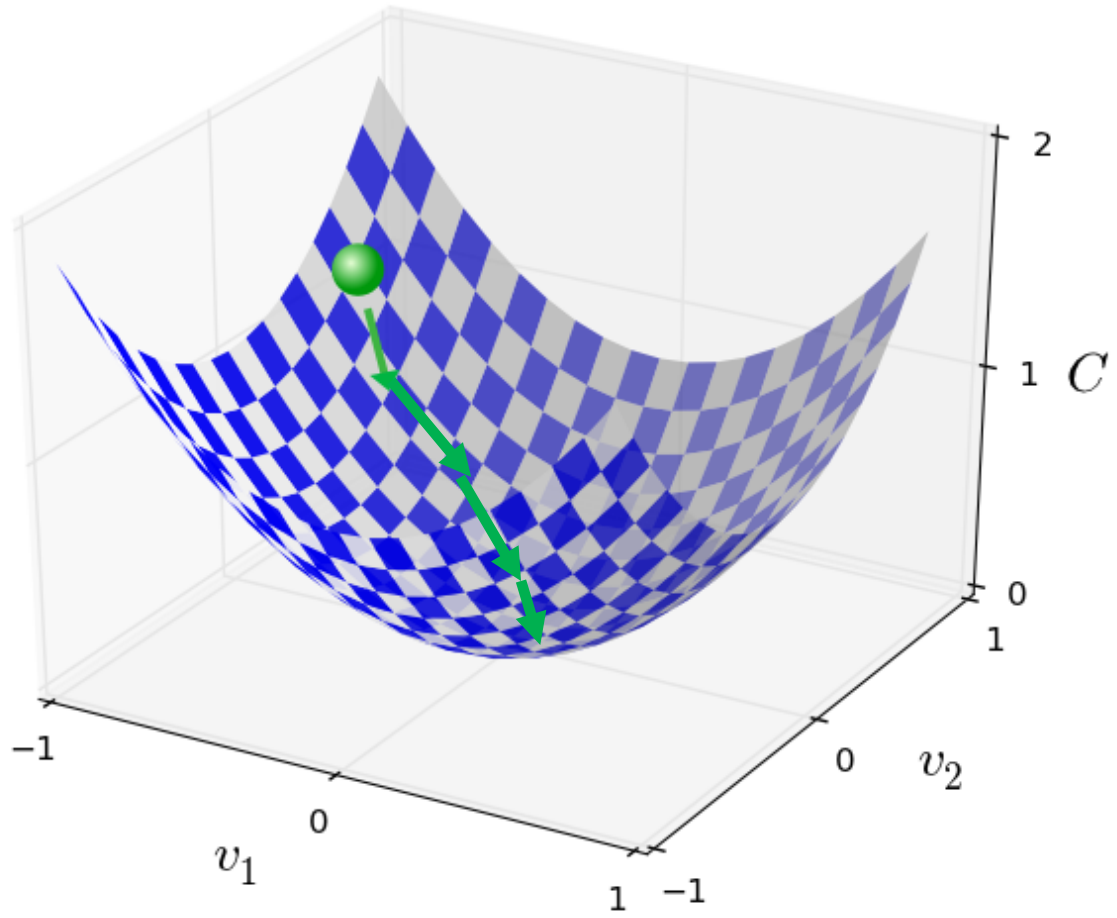
$$\Delta C \approx -\eta \nabla C \cdot \nabla C = -\eta \|\nabla C\|^2$$

$$\text{Because } \|\nabla C\|^2 \geq 0, \ \Delta C \leq 0$$

C will always decrease, if we change v according to $\Delta v = -\eta \nabla C$

$$v \to v' = v - \eta \nabla C.$$

Minimize the loss function



$$v \rightarrow v' = v - \eta \nabla C.$$

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$$v \rightarrow v' = v - \eta \nabla C.$$

$$v \rightarrow v' = v - \eta \nabla C.$$

Gradient descent applied to neural networks

- Use gradient descent to find the weights w_k and biases b_l which minimize

$$C(w,b) \equiv \frac{1}{2n} \sum_x \|y(x) - a\|^2$$

$$w_k \to w_k' = w_k - \eta \frac{\partial C}{\partial w_k}$$

$$b_l \to b_l' = b_l - \eta \frac{\partial C}{\partial b_l}.$$

Gradient descent applied to neural networks

- Problem: our cost function has the form

$$C = \frac{1}{n} \sum_x C_x$$

- It's an average over costs for individual training examples

$$C_x \equiv \frac{\|y(x)-a\|^2}{2}$$

- To compute the gradient ∇C we need to compute the gradients ∇C_x for each training input, x, and then average them

$$\nabla C = \frac{1}{n} \sum_x \nabla C_x$$

- With lots of training data, this can take a long time.

Stochastic gradient descent

- Stochastic gradient descent can be used to speed up learning
- Estimate the gradient ∇C by computing ∇C_x for a small sample of randomly chosen training inputs, called “mini-batches”
- For example, for a training set of size n=60,000 for MNIST, and we choose a mini-batch size of m=10, then we get a 6k speedup
- By averaging over this small sample, we can get a a good estimate of the true gradient ∇C, which speeds up learning

Backpropagation

Learning representations by back-propagating errors

David E. Rumelhart*, Geoffrey E. Hinton† & Ronald J. Williams*

* Institute for Cognitive Science, C-015, University of California, San Diego, La Jolla, California 92093, USA
† Department of Computer Science, Carnegie-Mellon University, Pittsburgh, Philadelphia 15213, USA

We describe a new learning procedure, back-propagation, for networks of neurone-like units. The procedure repeatedly adjusts the weights of the connections in the network so as to minimize a measure of the difference between the actual output vector of the net and the desired output vector. As a result of the weight adjustments, internal 'hidden' units which are not part of the input or output come to represent important features of the task domain, and the regularities in the task are captured by the interactions of these units. The ability to create useful new features distinguishes back-propagation from earlier, simpler methods such as the perceptron-convergence procedure[1].

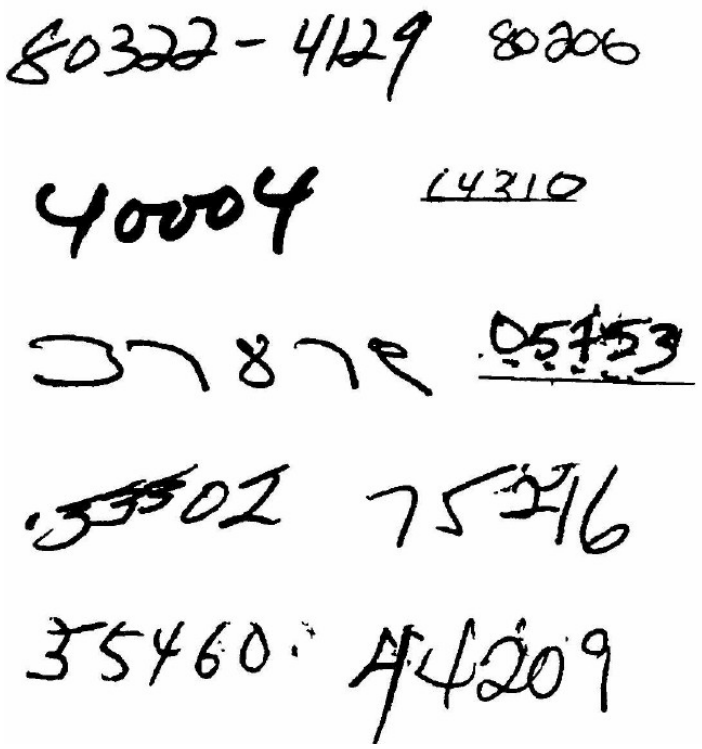
There have been many attempts to design self-organizing neural networks. The aim is to find a powerful synaptic modification rule that will allow an arbitrarily connected neural network to develop an internal structure that is appropriate for a particular task domain. The task is specified by giving the more difficult when we introduce hidden units whose actual or desired states are not specified by the task. (In perceptrons, there are 'feature analysers' between the input and output that are not true hidden units because their input connections are fixed by hand, so their states are completely determined by the input vector: they do not learn representations.) The learning procedure must decide under what circumstances the hidden units should be active in order to help achieve the desired input–output behaviour. This amounts to deciding what these units should represent. We demonstrate that a general purpose and relatively simple procedure is powerful enough to construct appropriate internal representations.

The simplest form of the learning procedure is for layered networks which have a layer of input units at the bottom; any number of intermediate layers; and a layer of output units at the top. Connections within a layer or from higher to lower layers are forbidden, but connections can skip intermediate layers. An input vector is presented to the network by setting the states of the input units. Then the states of the units in each layer are determined by applying equations (1) and (2) to the connections coming from lower layers. All units within a layer have their states set in parallel, but different layers have their states set sequentially, starting at the bottom and working upwards until the states of the output units are determined.

The total input, x_j, to unit j is a linear function of the outputs, y_i, of the units that are connected to j and of the weights, w_{ji}, on these connections

$$x_j = \sum_i y_i w_{ji} \qquad (1)$$

Digit classification

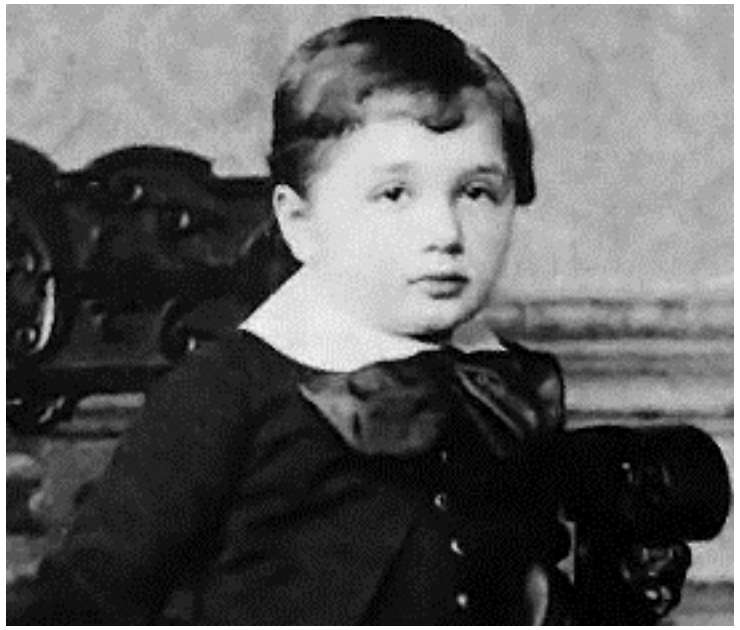


Zip codes

Backpropagation applied to handwritten zip code recognition,
Lecun et al., 1989

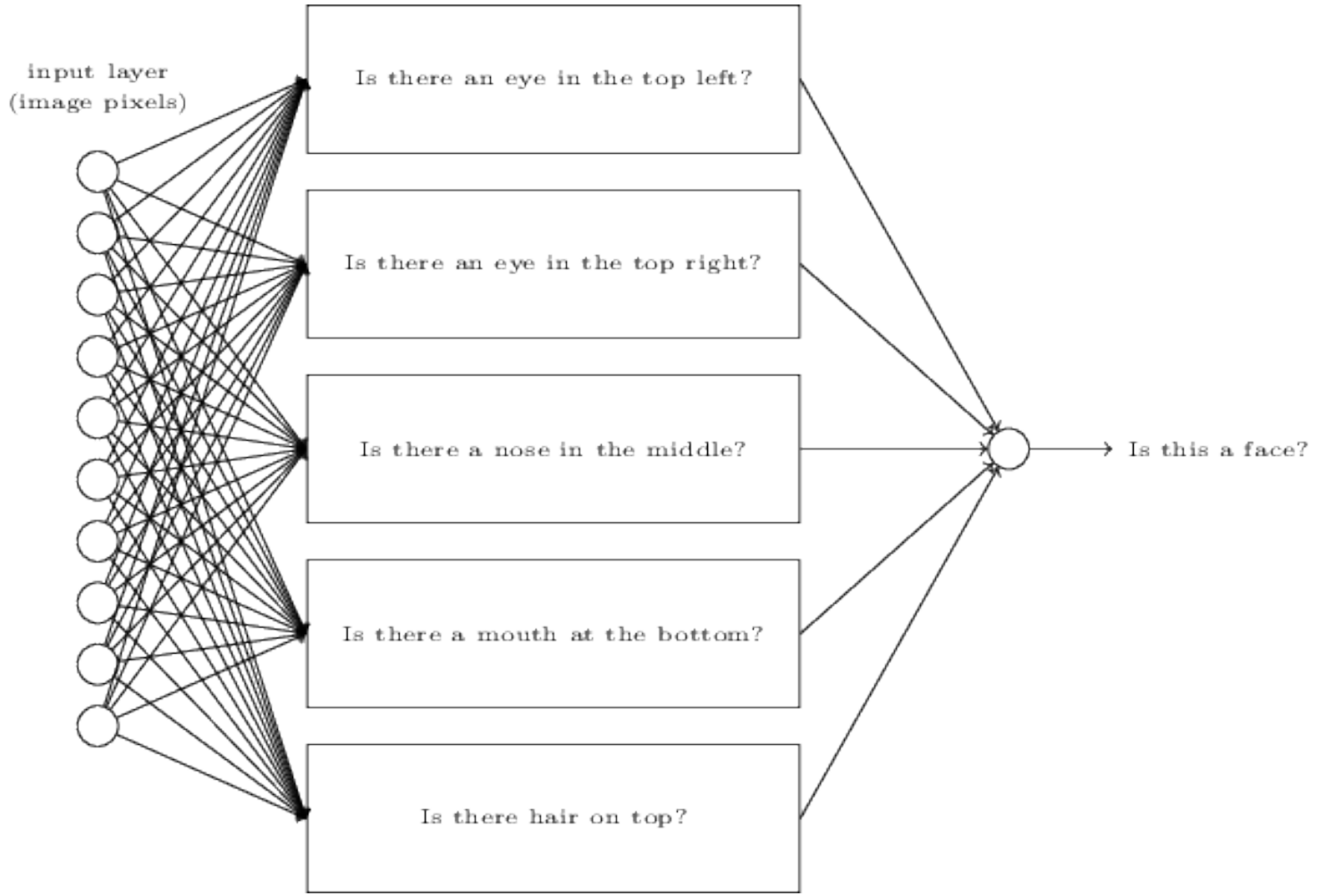
Toward Deep Learning

- Which of the following contain a face?



- Can we approach the problem the same way we do for digits?

Toward Deep Learning



Toward Deep Learning