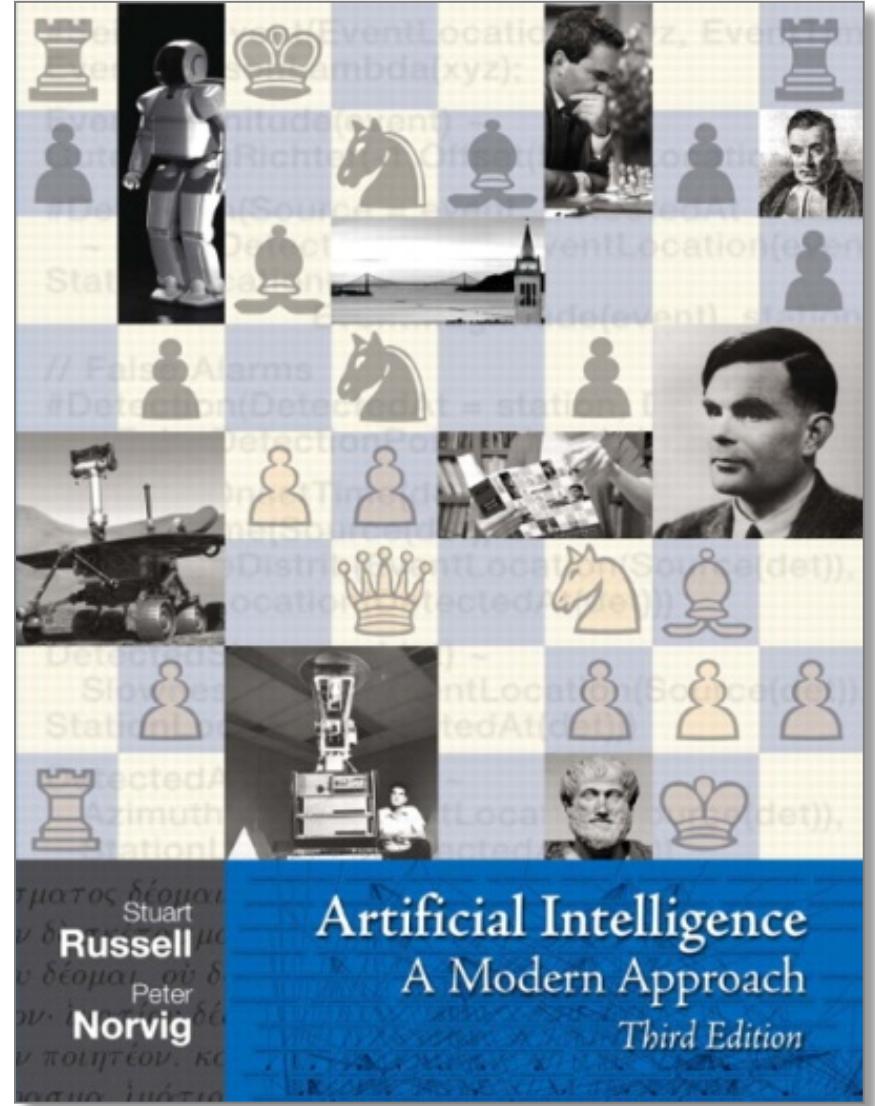


# Bayes' Nets

# Read AIMA

## Chapter 14 “Probabilistic Reasoning” (Sections 14.1, 14.2 and 14.4)



Slides courtesy of Dan Klein and Pieter Abbeel – University of California, Berkeley

# Probabilistic Models

- Models describe how (a portion of) the world works
- **Models are always simplifications**
  - May not account for every variable
  - May not account for all interactions between variables
  - “All models are wrong; but some are useful.”
    - George E. P. Box
- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)



# Review: Independence

- Two variables are *independent* if:

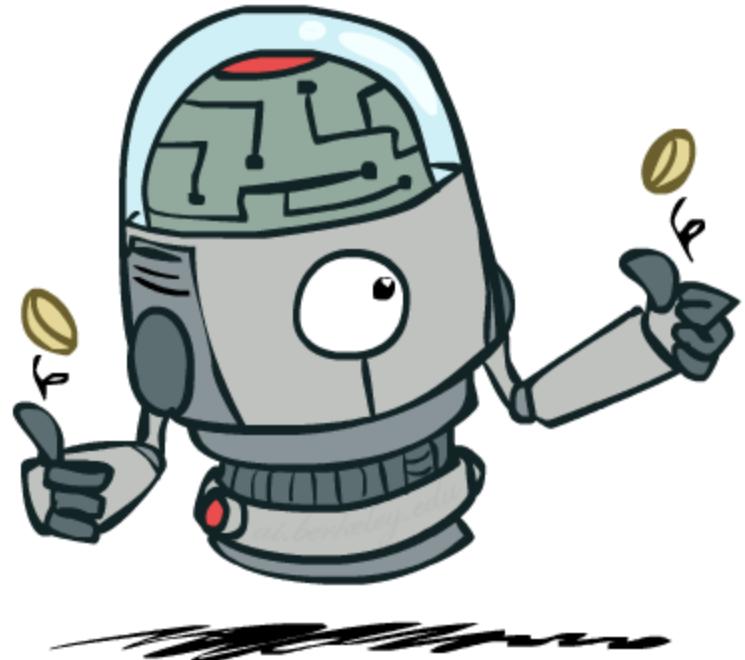
$$\forall x, y : P(x, y) = P(x)P(y)$$

- This says that their joint distribution *factors* into a product two simpler distributions
- Another form:

$$\forall x, y : P(x|y) = P(x)$$

- We write:  $X \perp\!\!\!\perp Y$
- Independence is a simplifying *modeling assumption*

- *Empirical* joint distributions: at best “close” to independent
- What could we assume for {Weather, Traffic, Cavity, Toothache}?



# Review: Independence?

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$P_2(T, W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

# Review: Independence

- N fair, independent coin flips:

$$P(X_1)$$

H	0.5
T	0.5

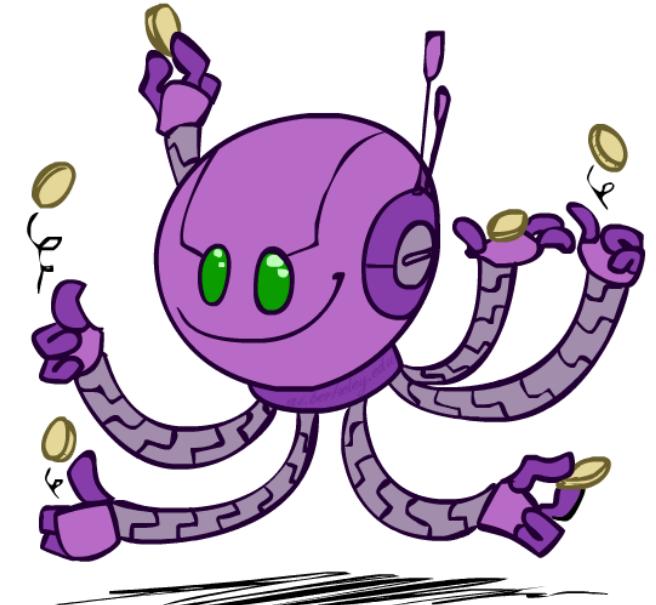
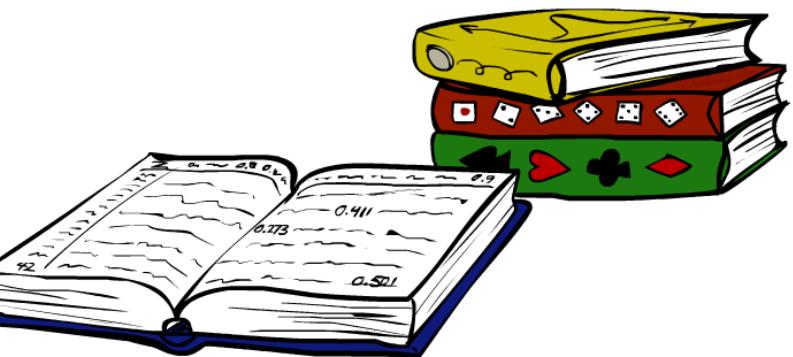
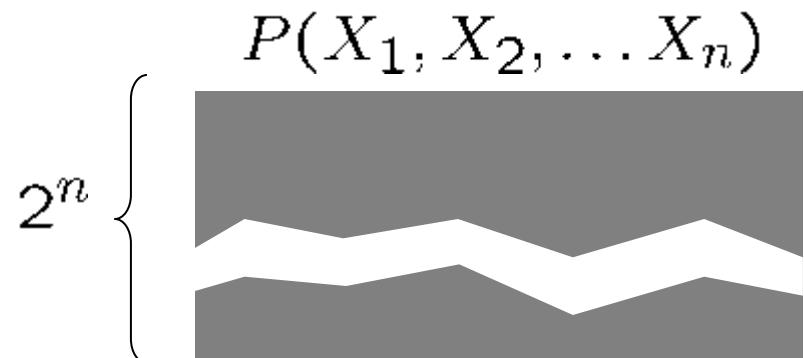
$$P(X_2)$$

H	0.5
T	0.5

...

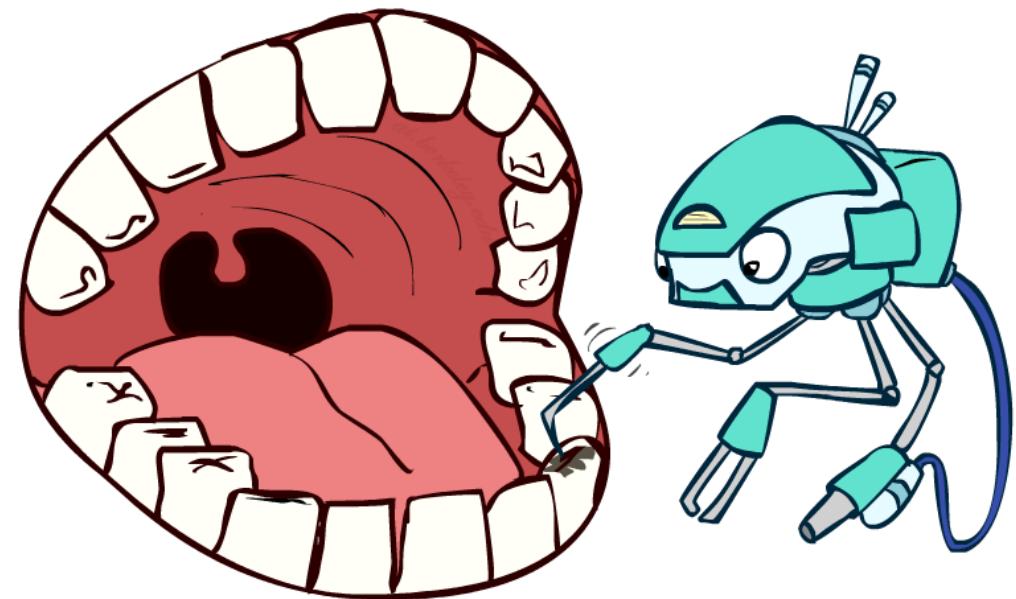
$$P(X_n)$$

H	0.5
T	0.5



# Review: Conditional Independence

- $P(\text{Toothache}, \text{Cavity}, \text{Detect})$
- If I have a cavity, the probability that the probe detects it doesn't depend on whether I have a toothache:
  - $P(+\text{detect} | +\text{toothache}, +\text{cavity}) = P(+\text{detect} | +\text{cavity})$
- The same independence holds if I don't have a cavity:
  - $P(+\text{detect} | +\text{toothache}, -\text{cavity}) = P(+\text{detect} | -\text{cavity})$
- Detect is *conditionally independent* of Toothache given Cavity:
  - $P(\text{Detect} | \text{Toothache}, \text{Cavity}) = P(\text{Detect} | \text{Cavity})$
- Equivalent statements:
  - $P(\text{Toothache} | \text{Detect}, \text{Cavity}) = P(\text{Toothache} | \text{Cavity})$
  - $P(\text{Toothache}, \text{Detect} | \text{Cavity}) = P(\text{Toothache} | \text{Cavity}) P(\text{Detect} | \text{Cavity})$
  - One can be derived from the other using the chain rule



# Review: Conditional Independence

---

- Unconditional (absolute) independence very rare, and it doesn't help us make inferences about other variables.
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- $X$  is conditionally independent of  $Y$  given  $Z$   $X \perp\!\!\!\perp Y | Z$

if and only if:

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

# Conditional Independence

---

- What about this domain:

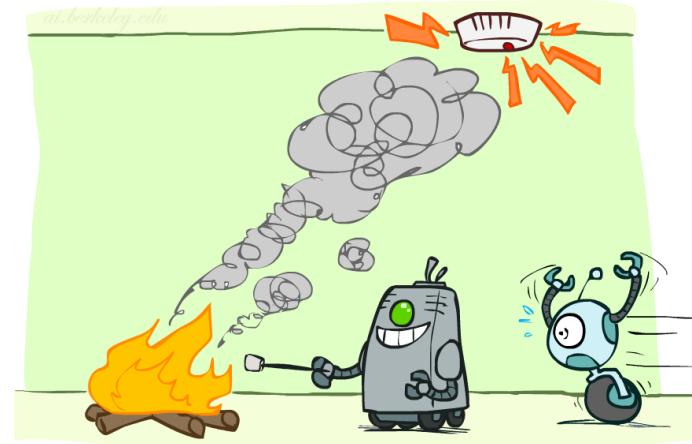
- Traffic
- Umbrella
- Raining



# Conditional Independence

- What about this domain:

- Fire
- Smoke
- Alarm



# Conditional Independence and the Chain Rule

- Chain rule:

$$P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots$$

- Trivial decomposition:

$$P(\text{Traffic, Rain, Umbrella}) =$$

$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain, Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic, Rain, Umbrella}) =$$

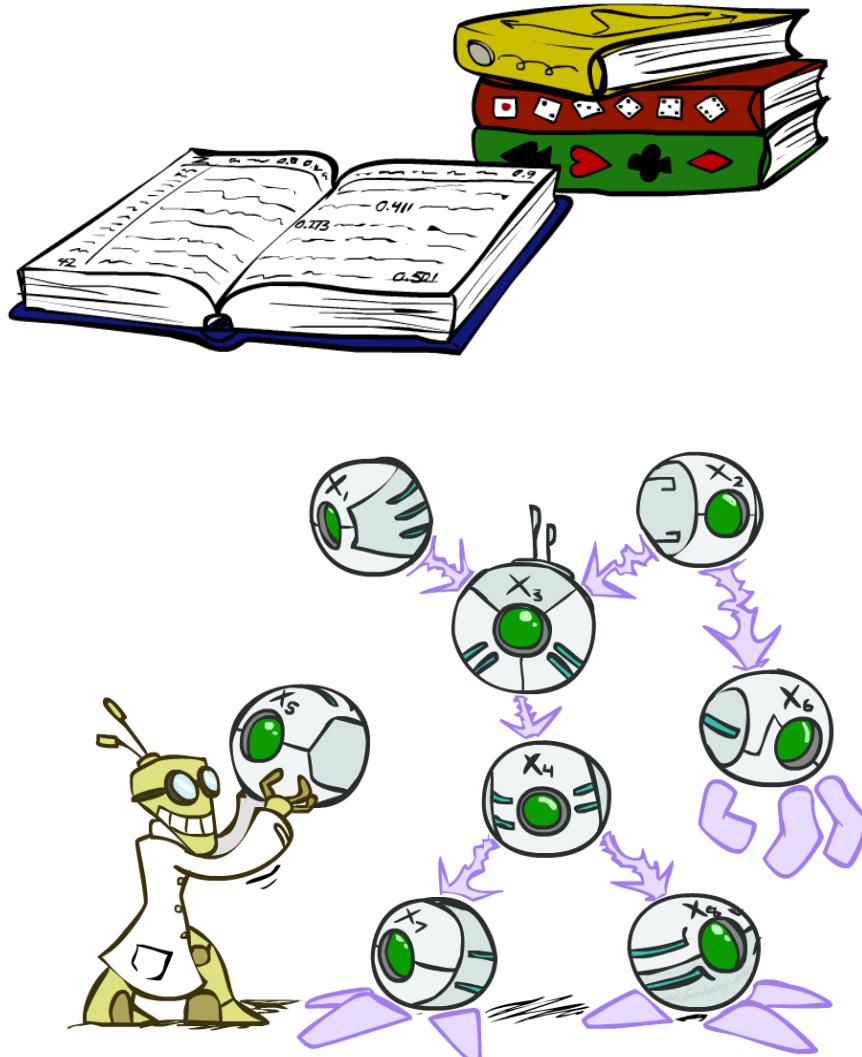
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$



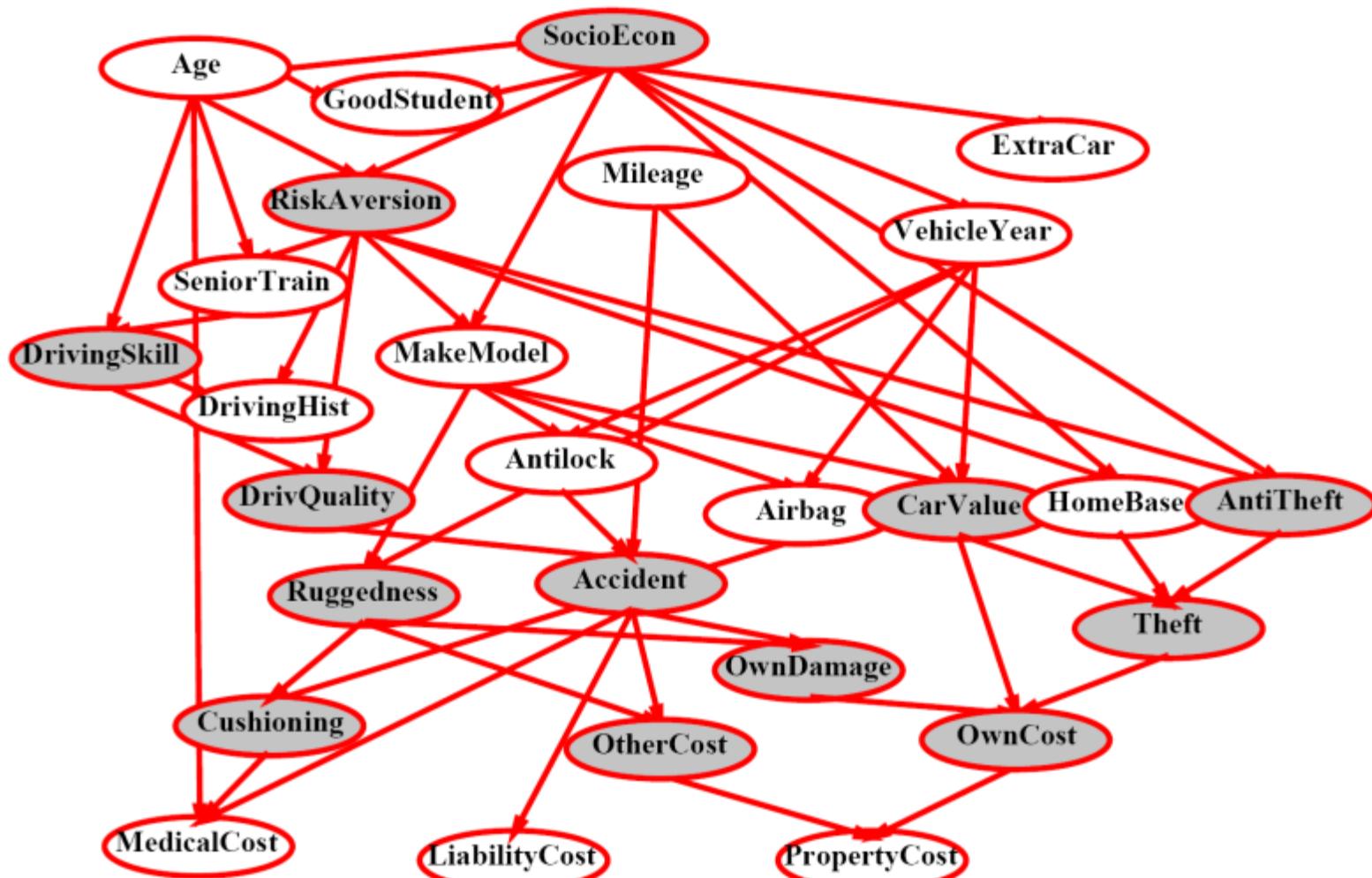
- Bayes' nets / graphical models help us express conditional independence assumptions

# Bayes' Nets: Big Picture

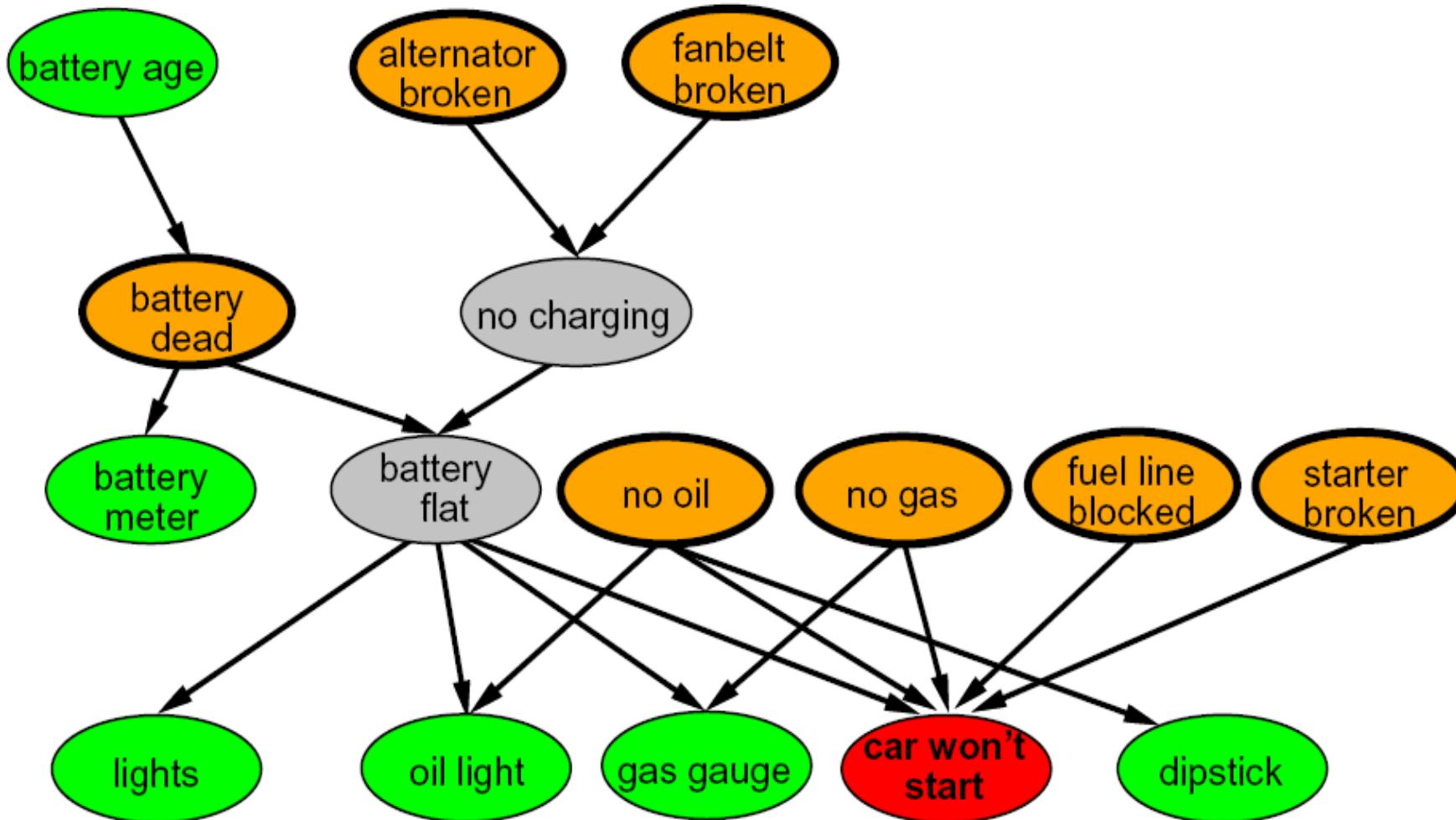
- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions



# Example Bayes' Net: Insurance

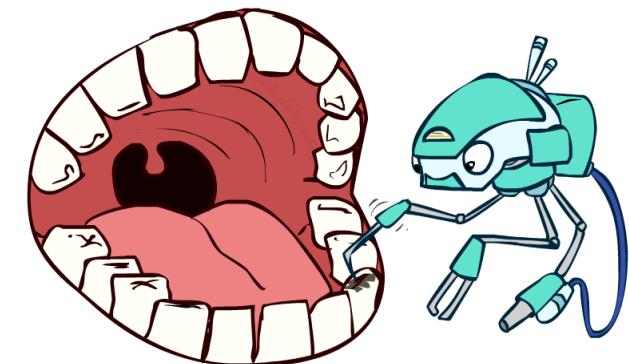
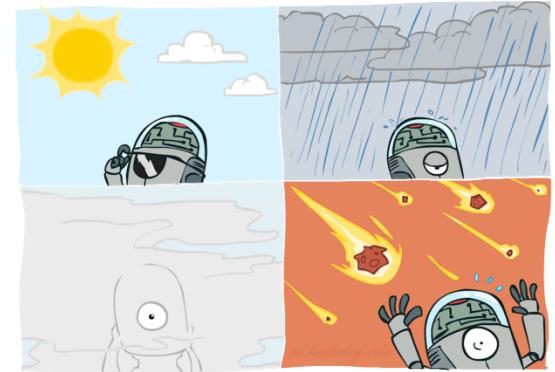
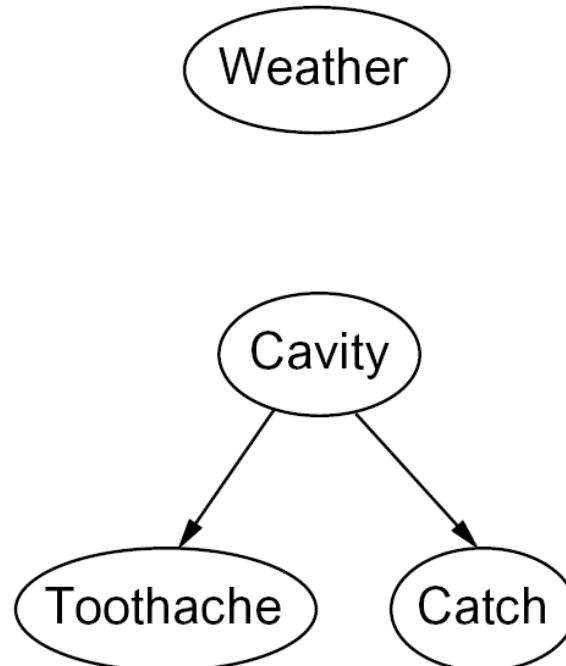


# Example Bayes' Net: Car



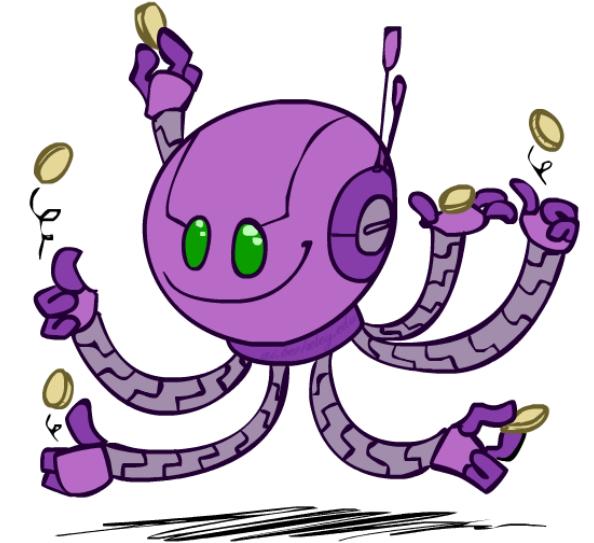
# Graphical Model Notation

- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)
- Arcs: interactions
  - Indicate “direct influence” between variables
  - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don’t!)



# Example: Coin Flips

- N independent coin flips



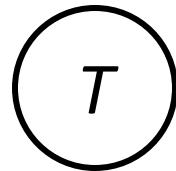
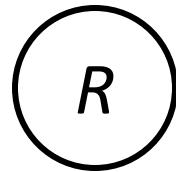
- No interactions between variables: absolute independence

# Example: Traffic

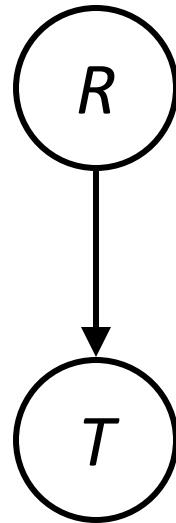
- Variables:
  - $R$ : It rains
  - $T$ : There is traffic



- Model 1: independence



- Model 2: rain causes traffic



- Why is an agent using model 2 better?

# Example: Traffic II

- Let's build a causal graphical model!
- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity

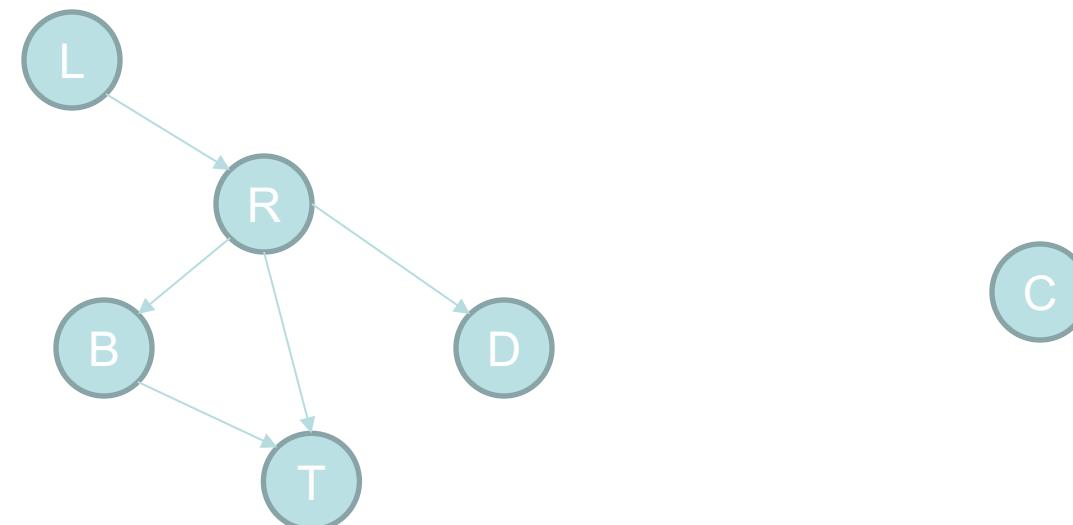


# Example: Traffic II

- Let's build a causal graphical model!

- Variables

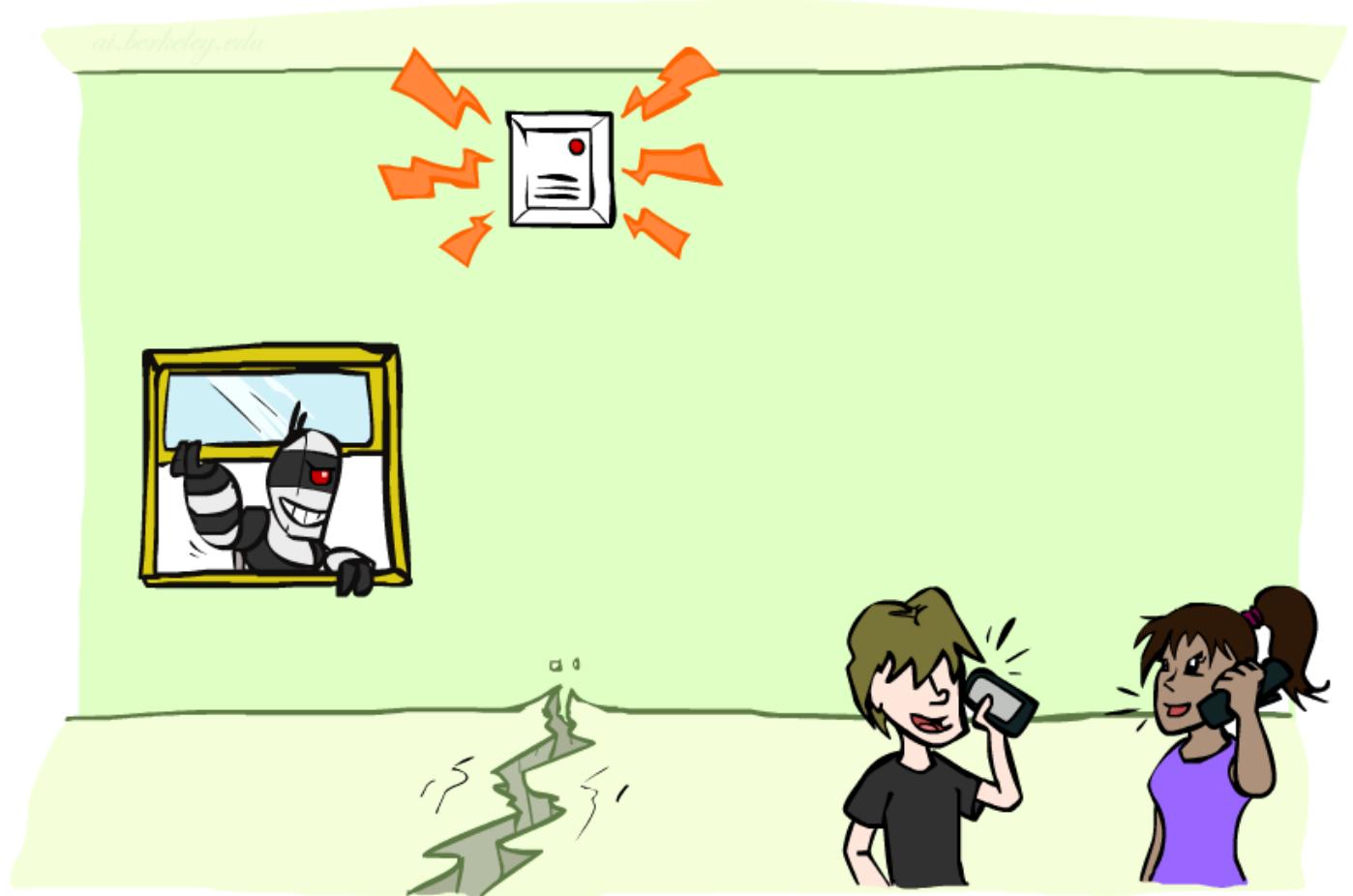
- T: Traffic
- R: It rains
- L: Low pressure
- D: Roof drips
- B: Ballgame
- C: Cavity



# Example: Alarm Network

## ■ Variables

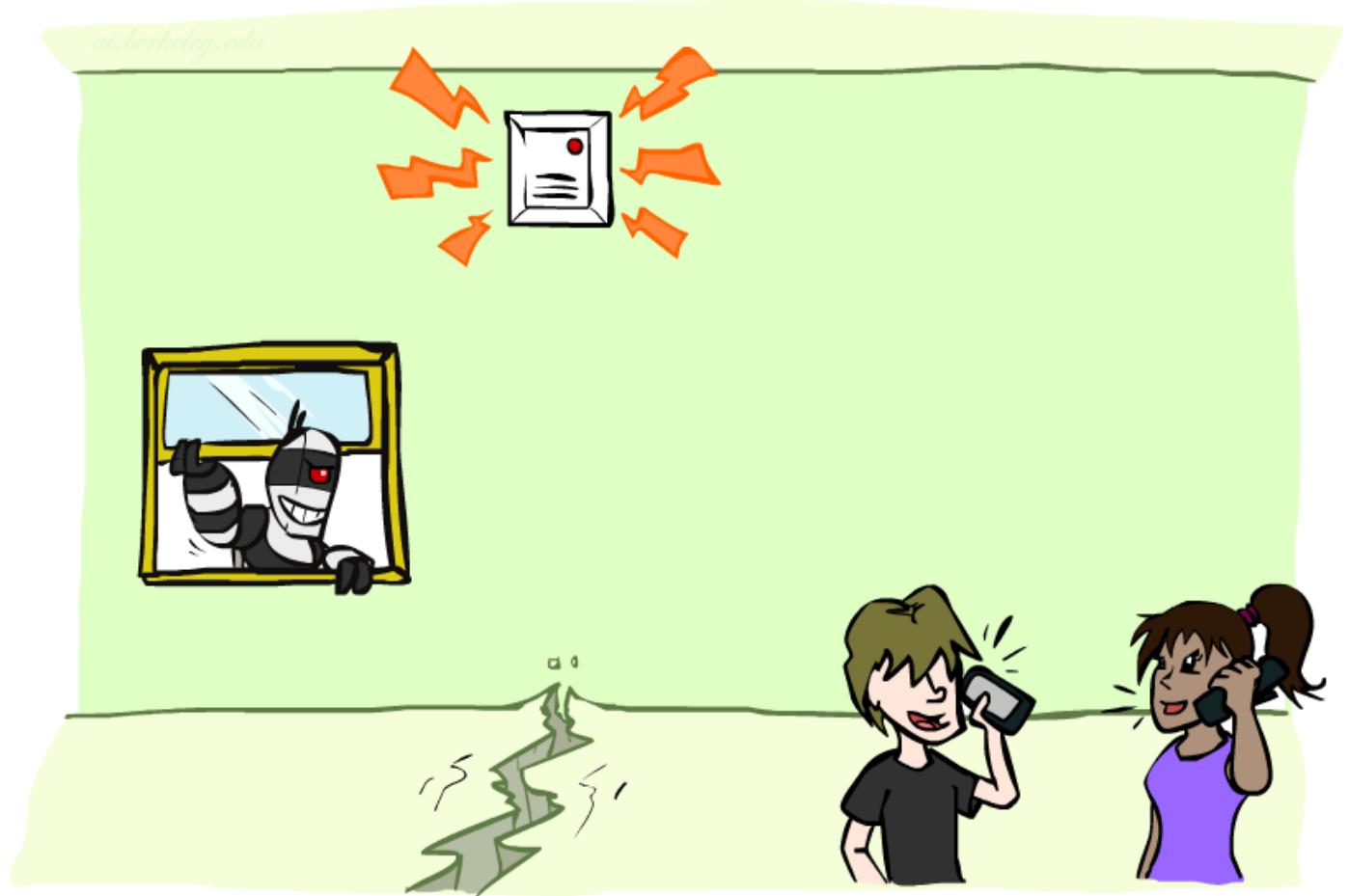
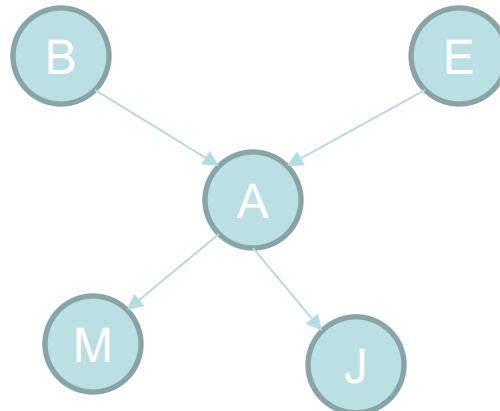
- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



# Example: Alarm Network

## Variables

- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



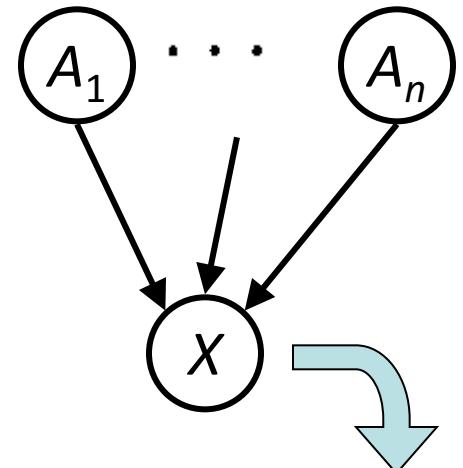


# Bayes' Net Semantics

- A set of nodes, one per variable  $X$
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over  $X$ , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- CPT: conditional probability table
- Description of a noisy “causal” process



$$P(X|A_1 \dots A_n)$$

*A Bayes net = Topology (graph) + Local Conditional Probabilities*

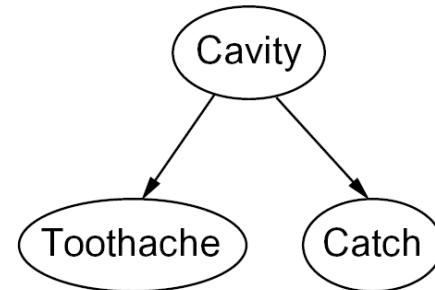
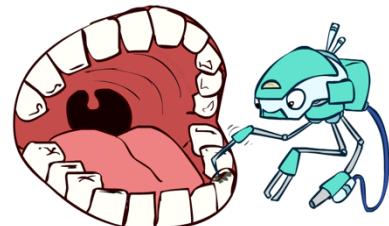
# Probabilities in BNs



- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- Example:



$P(+\text{cavity}, +\text{detect}, -\text{toothache})$

# Probabilities in BNs



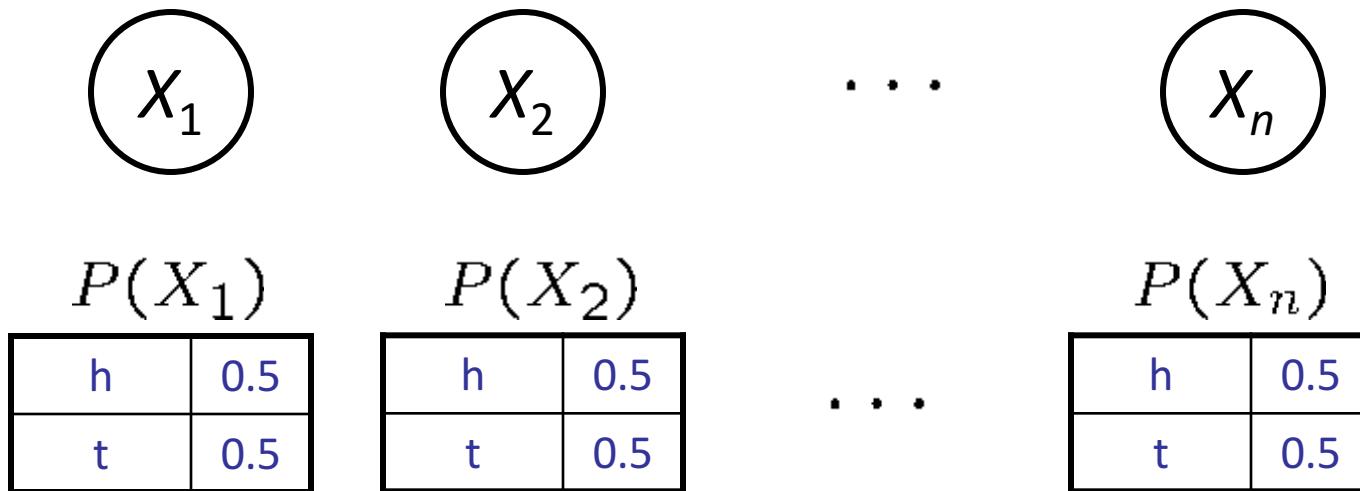
- Why are we guaranteed that setting

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

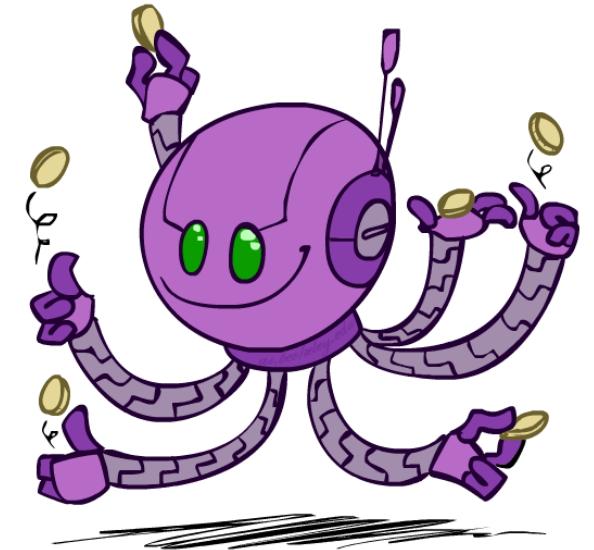
results in a proper joint distribution?

- Chain rule (valid for all distributions):  $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1 \dots x_{i-1})$
- Assume conditional independences:  $P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$   
→ Consequence:  $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$
- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

# Example: Coin Flips

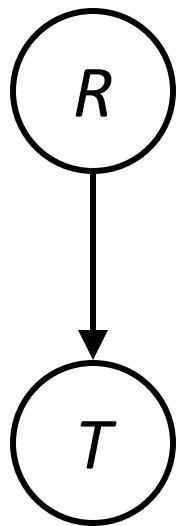


$$P(h, h, t, h) =$$



*Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.*

# Example: Traffic



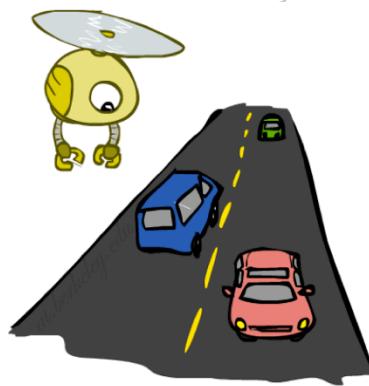
$P(R)$

$+r$	$1/4$
$-r$	$3/4$

$$P(+r, -t) =$$

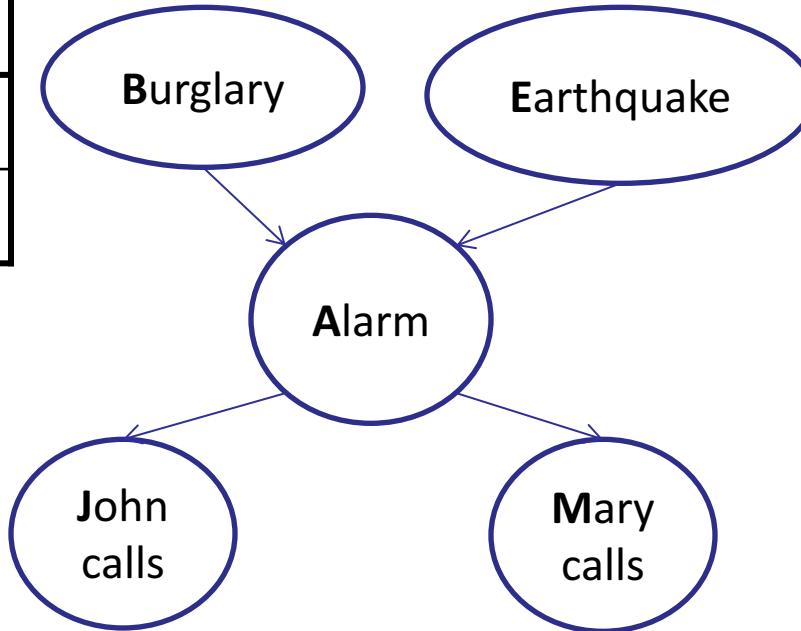
$P(T|R)$

$+r$	$+t$	$3/4$
	$-t$	$1/4$
$-r$	$+t$	$1/2$
	$-t$	$1/2$



# Example: Alarm Network

B	$P(B)$
+b	0.001
-b	0.999



E	$P(E)$
+e	0.002
-e	0.998



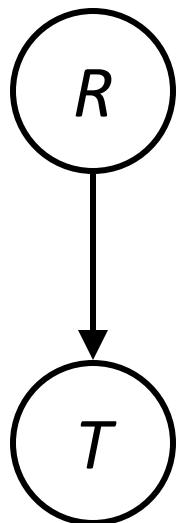
A	J	$P(J A)$
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	$P(M A)$
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	$P(A B,E)$
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

# Example: Traffic

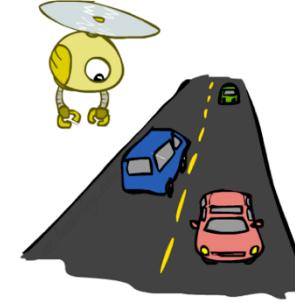
- Causal direction

 $P(R)$ 

+r	1/4
-r	3/4

 $P(T|R)$ 

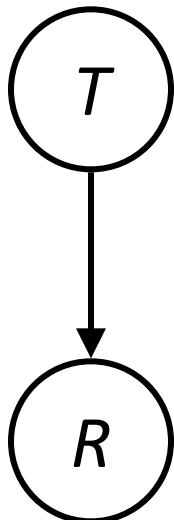
+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

 $P(T, R)$ 

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

# Example: Reverse Traffic

- Reverse causality?



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3
-t	+r	1/7
	-r	6/7



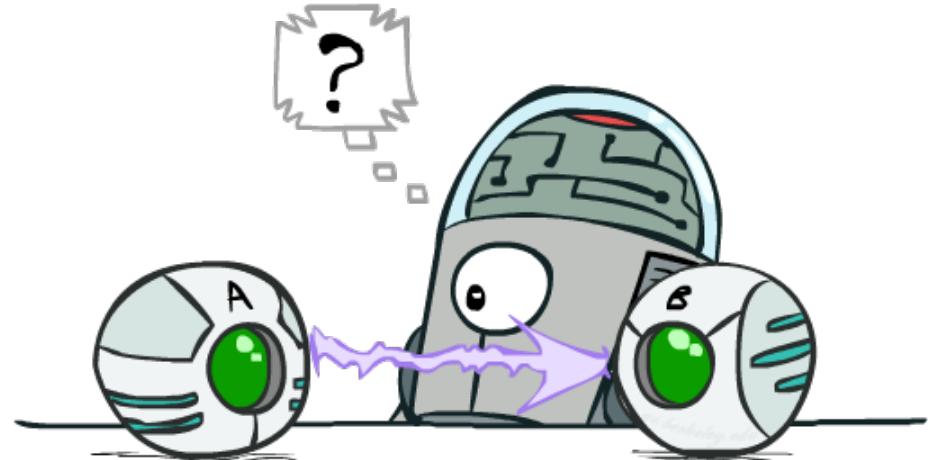
$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

# Causality?

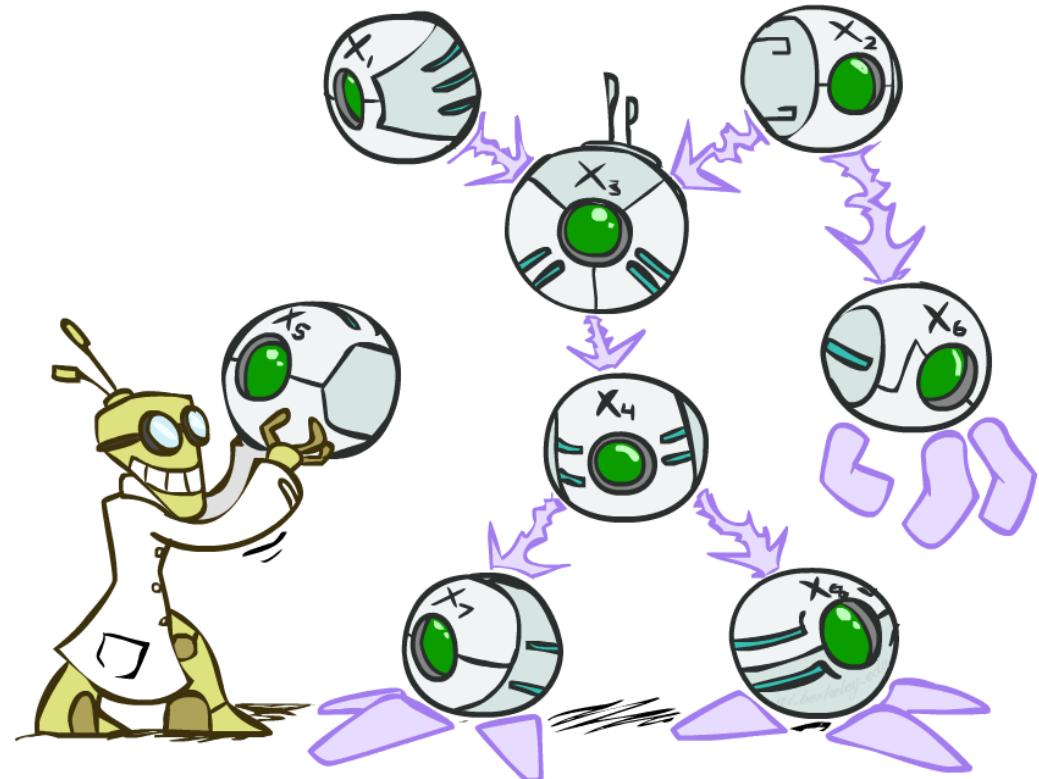
- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts
- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - **Topology really encodes conditional independence**

$$P(x_i|x_1, \dots x_{i-1}) = P(x_i|\text{parents}(X_i))$$



# Bayes' Nets

- So far: how a Bayes' net encodes a joint distribution
- Next: how to answer queries about that distribution
  - Today:
    - First assembled BNs using an intuitive notion of conditional independence as causality
    - Then saw that key property is conditional independence
  - Main goal: answer queries about conditional independence and influence
- After that: how to answer numerical queries (inference)



# Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

$$2^N$$

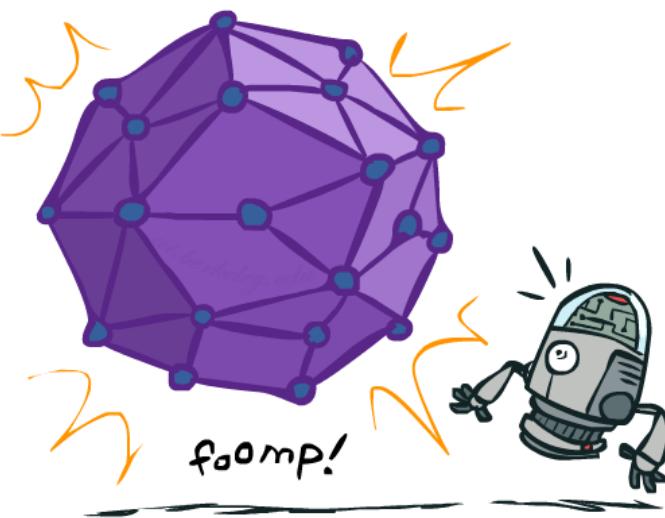
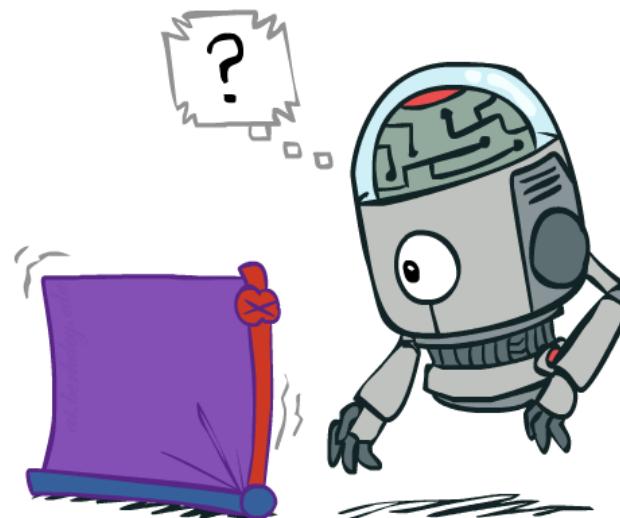
- How big is an N-node net if nodes have up to k parents?

$$O(N * 2^{k+1})$$

- Both give you the power to calculate

$$P(X_1, X_2, \dots, X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



# Conditional Independence

- X and Y are independent if

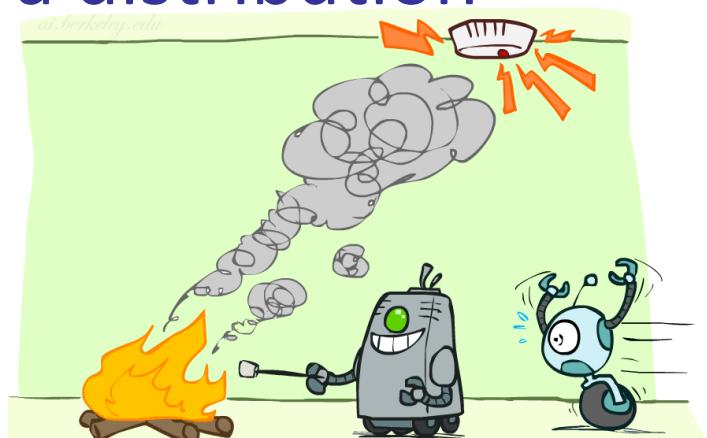
$$\forall x, y \ P(x, y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

- X and Y are conditionally independent given Z

$$\forall x, y, z \ P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example:  $Alarm \perp\!\!\!\perp Fire|Smoke$

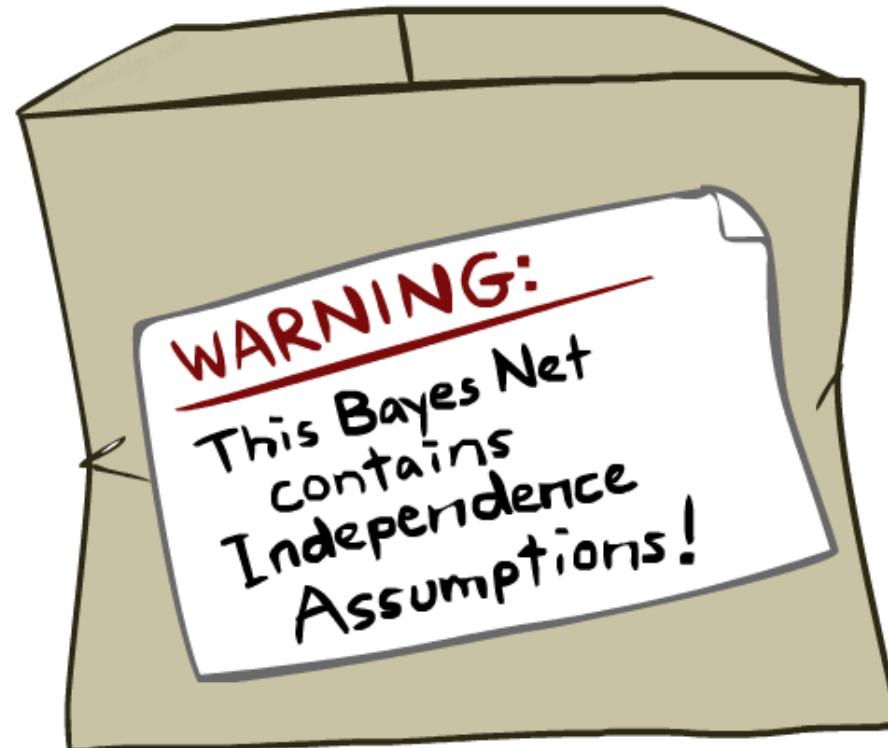


# Bayes Nets: Assumptions

- Assumptions we are required to make to define the Bayes net when given the graph:

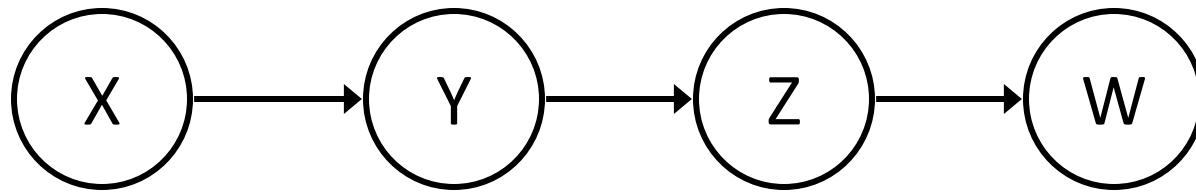
$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- Beyond above “chain rule → Bayes net” conditional independence assumptions
  - Often additional conditional independences
  - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



# Example

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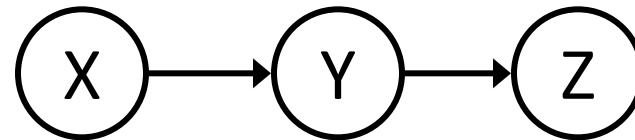


- Conditional independence assumptions directly from simplifications in chain rule:
- Additional implied conditional independence assumptions?

# Independence in a BN

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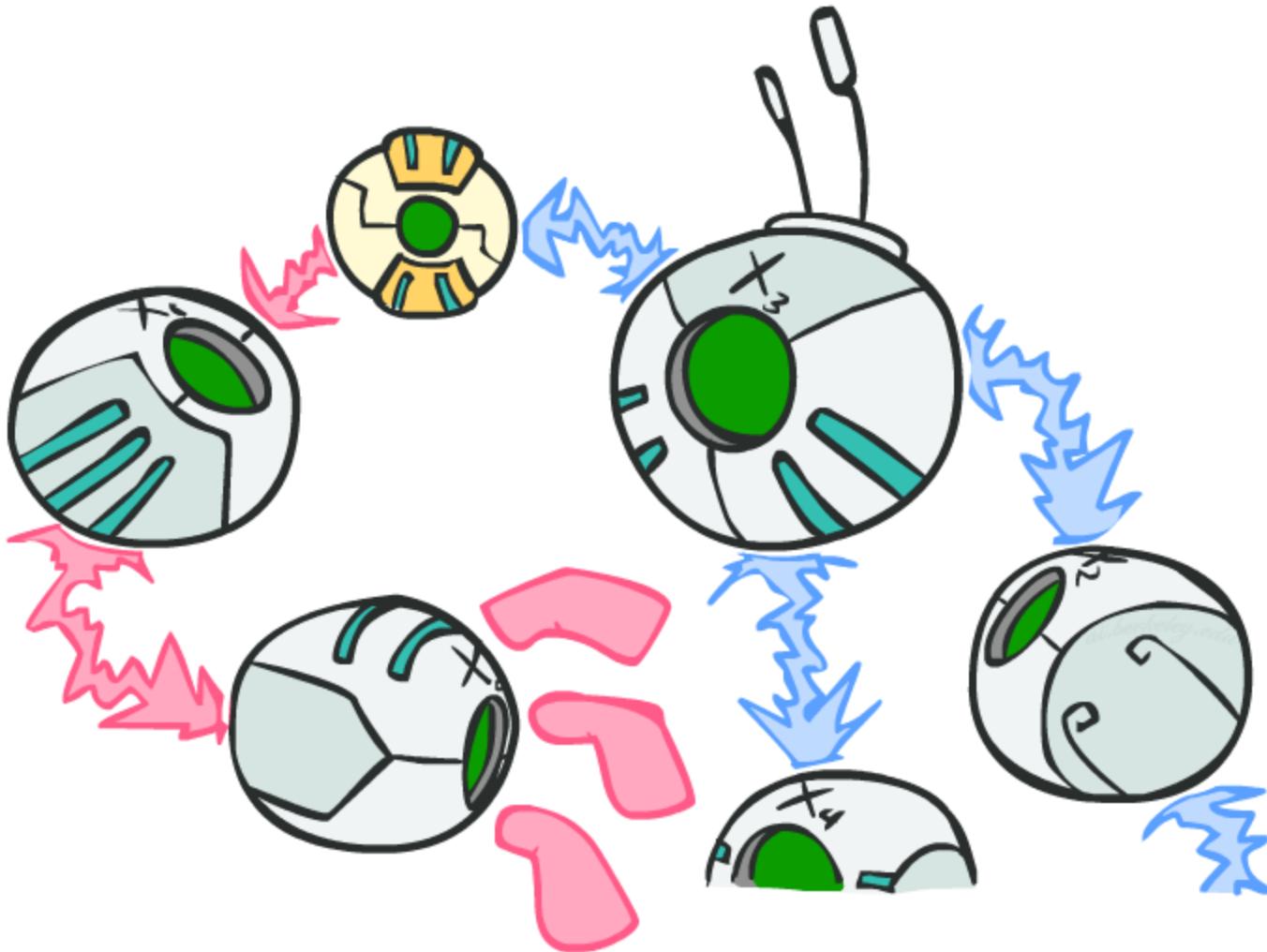
- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can prove using algebra (sometimes tedious)
  - If no, can prove with a counter example
  - Example:



- Question: are X and Z necessarily independent?
  - Answer: no. Example: low pressure causes rain, which causes traffic.
  - X can influence Z, Z can influence X (via Y)
  - Addendum: they *could* be independent: how?

# D-separation: Outline

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# D-separation: Outline

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- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

# Causal Chains

- This configuration is a “causal chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
- Example:
  - Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic
- In numbers:

$$\begin{aligned} P(+y | +x) &= 1, P(-y | -x) = 1, \\ P(+z | +y) &= 1, P(-z | -y) = 1 \end{aligned}$$

# Causal Chains

- This configuration is a “causal chain”
- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

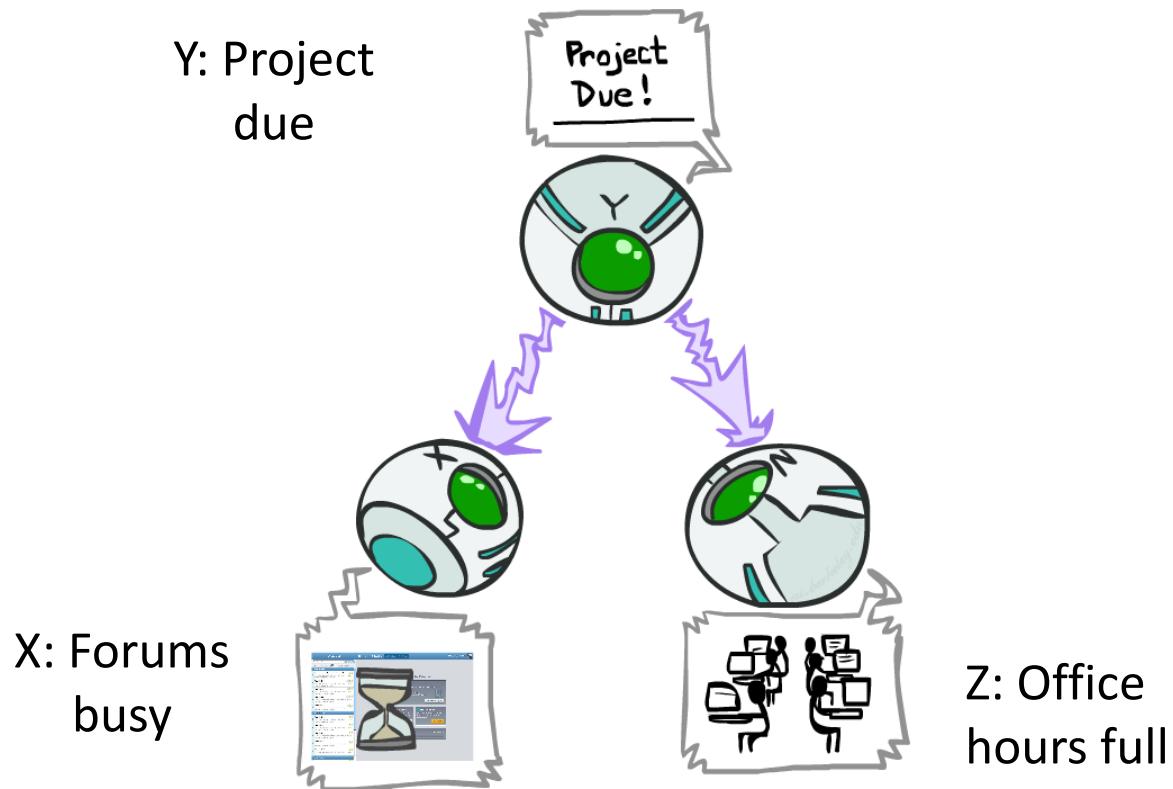
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Evidence along the chain “blocks” the influence

# Common Cause

- This configuration is a “common cause”



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? *No!*

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

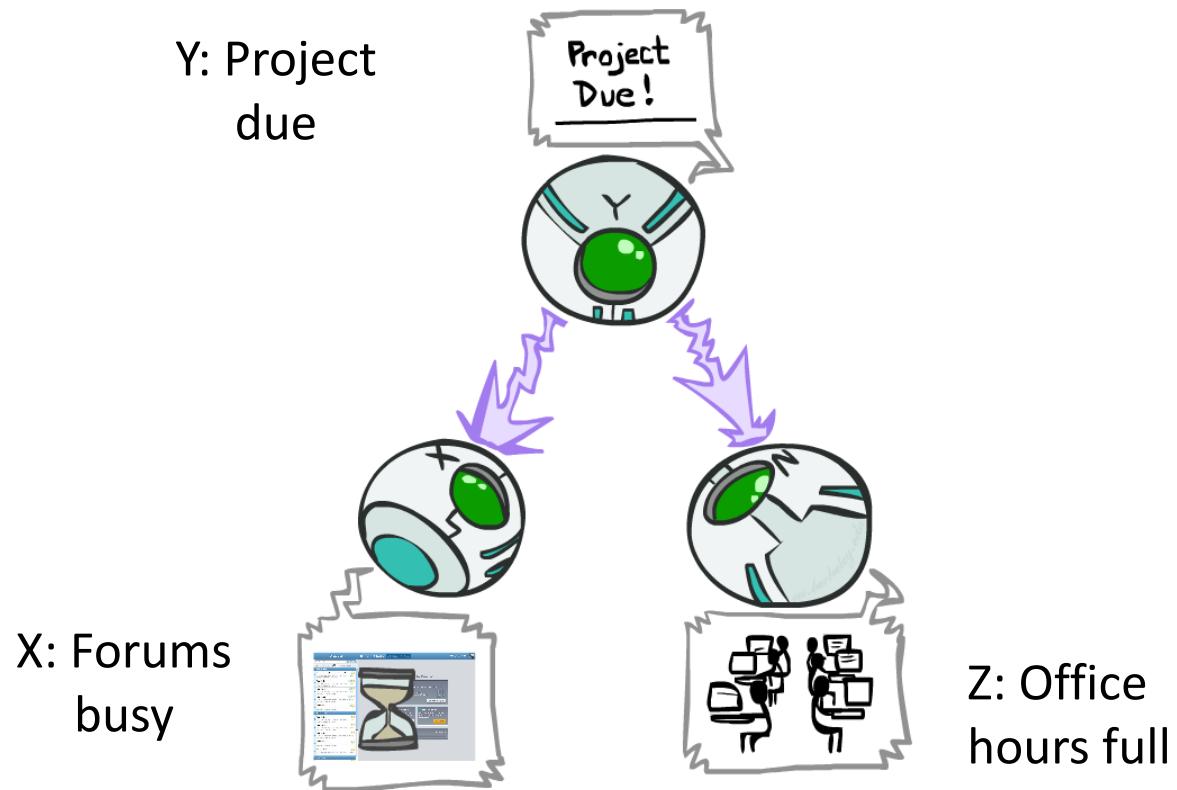
- Project due causes both forums busy and lab full

- In numbers:

$$\begin{aligned}P(+x | +y) &= 1, P(-x | -y) = 1, \\P(+z | +y) &= 1, P(-z | -y) = 1\end{aligned}$$

# Common Cause

- This configuration is a “common cause”
- Guaranteed X and Z independent given Y?



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

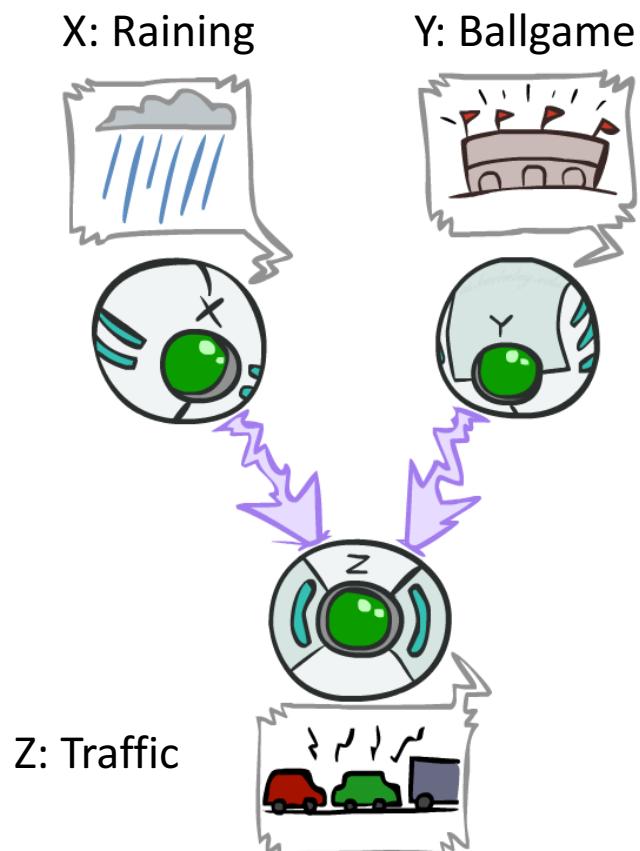
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Observing the cause blocks influence between effects.

# Common Effect

- Last configuration: two causes of one effect (v-structures)



- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
  - Observing an effect **activates** influence between possible causes.

# The General Case

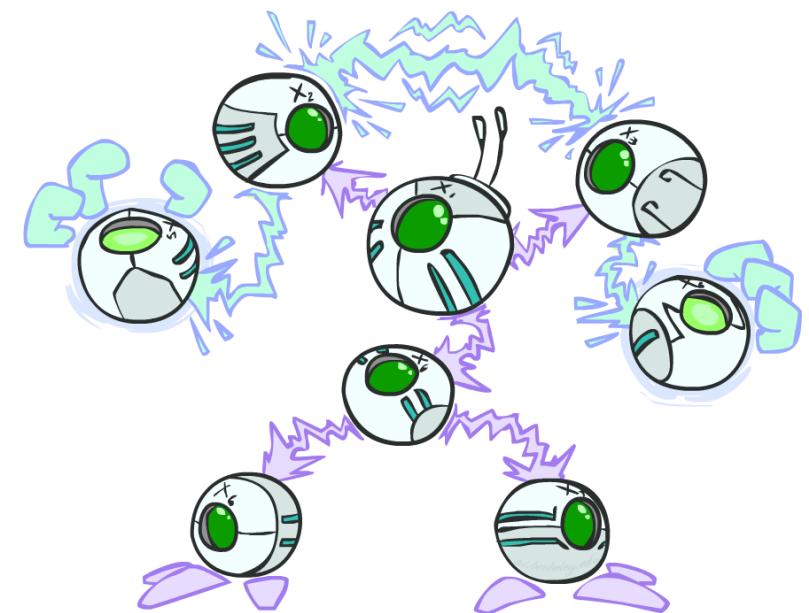
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# The General Case

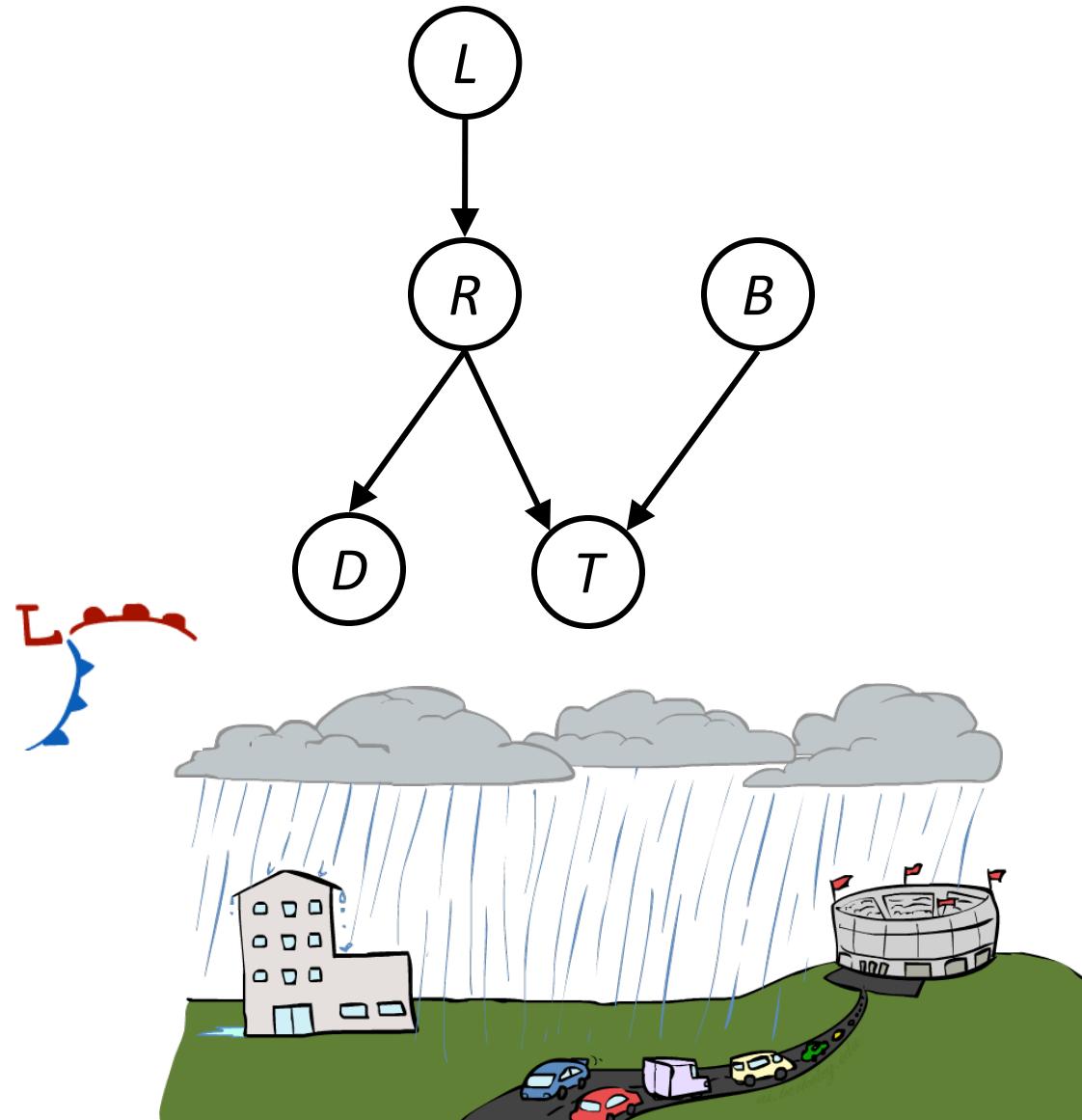
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- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



# Reachability

- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



# Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables  $\{Z\}$ ?

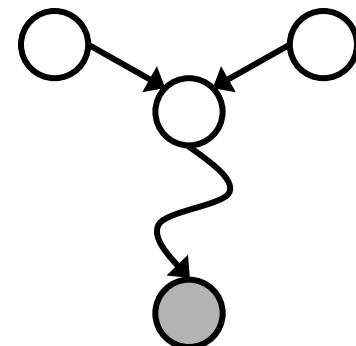
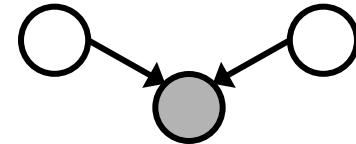
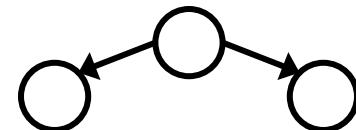
- Yes, if X and Y “d-separated” by Z
- Consider all (undirected) paths from X to Y
- No active paths = independence!

- A path is active if each triple is active:

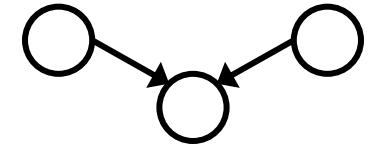
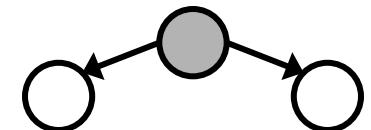
- Causal chain  $A \rightarrow B \rightarrow C$  where B is unobserved (either direction)
- Common cause  $A \leftarrow B \rightarrow C$  where B is unobserved
- Common effect (aka v-structure)  
 $A \rightarrow B \leftarrow C$  where B or one of its descendants is observed

- All it takes to block a path is a single inactive segment

Active Triples



Inactive Triples



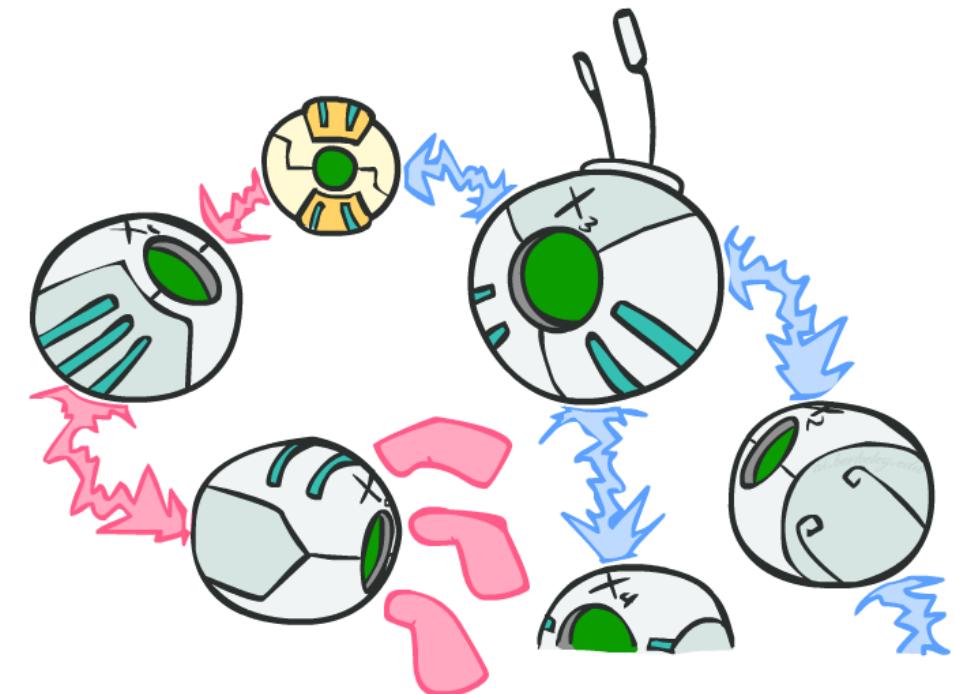
# D-Separation

- Query:  $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$  ?
- Check all (undirected!) paths between  $X_i$  and  $X_j$ 
  - If one or more active, then independence not guaranteed

$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$

- Otherwise (i.e. if all paths are inactive),  
then independence is guaranteed

$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$



# Example

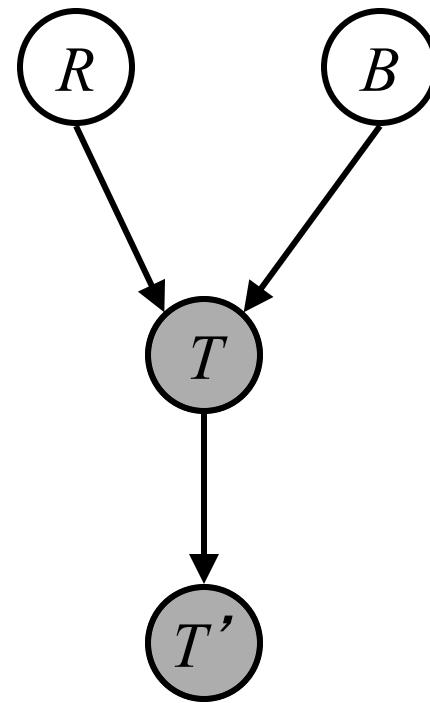
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$R \perp\!\!\!\perp B$

*Yes*

$R \perp\!\!\!\perp B | T$

$R \perp\!\!\!\perp B | T'$



# Example

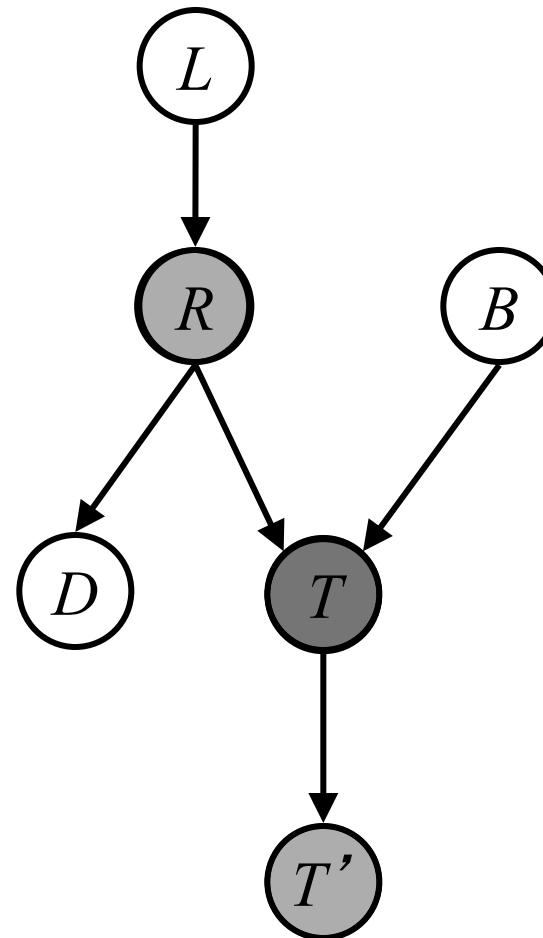
$L \perp\!\!\!\perp T' | T$       Yes

$L \perp\!\!\!\perp B$       Yes

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$       Yes



# Example

- Variables:

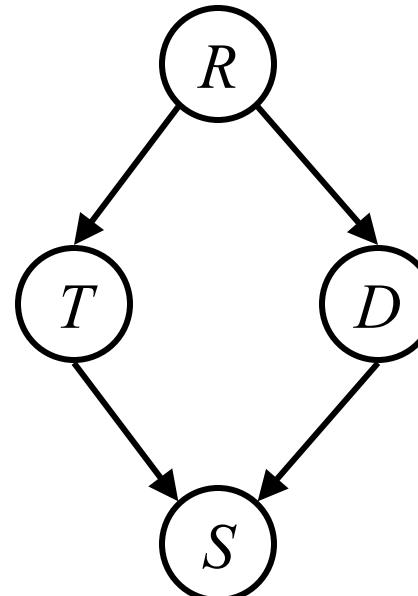
- R: Raining
- T: Traffic
- D: Roof drips
- S: I'm sad

- Questions:

$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R \quad \text{Yes}$$

$$T \perp\!\!\!\perp D | R, S$$

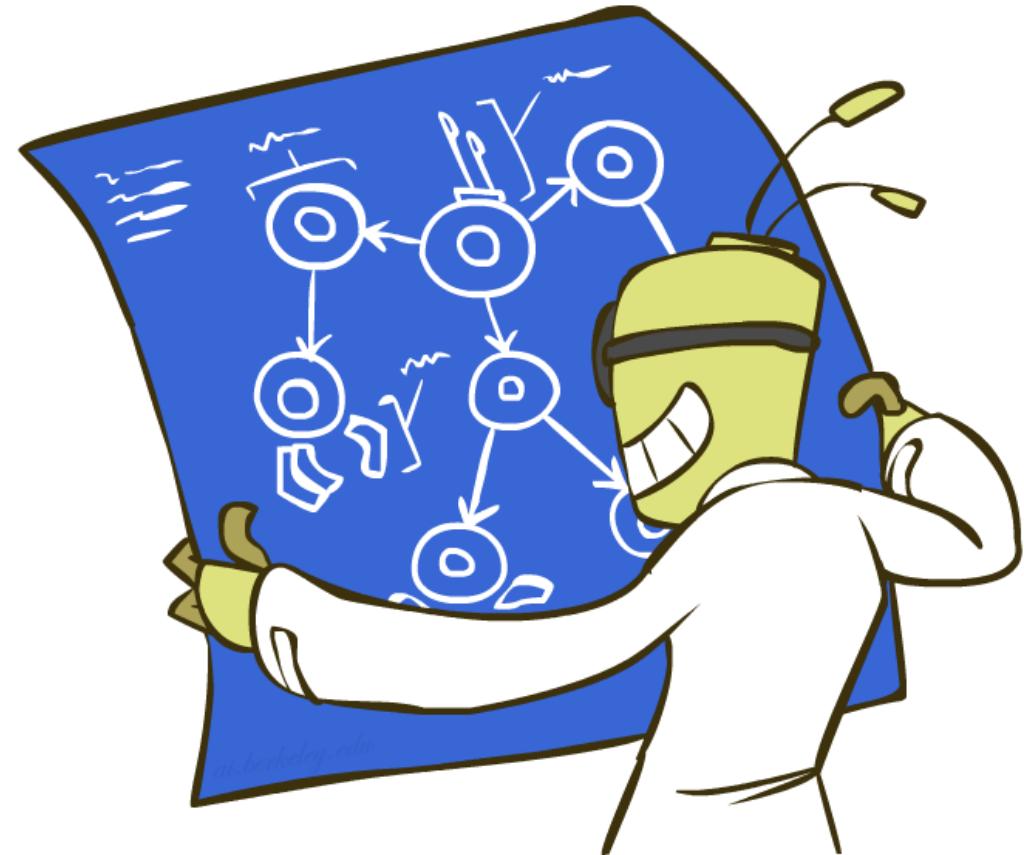


# Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

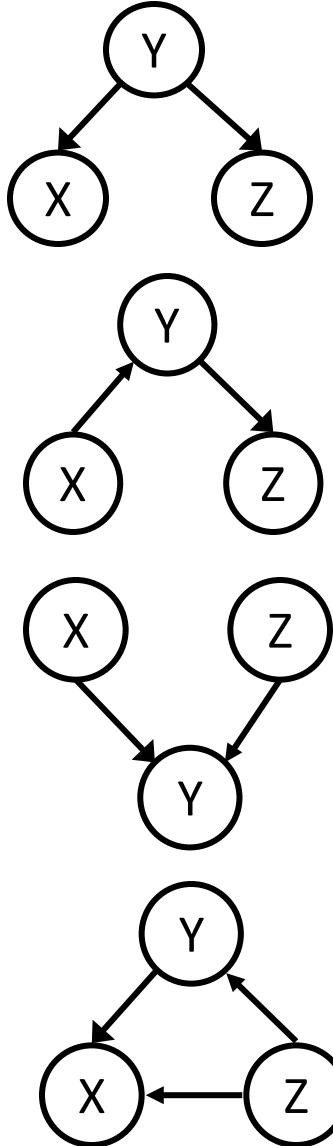
- This list determines the set of probability distributions that can be represented



# Computing All Independences

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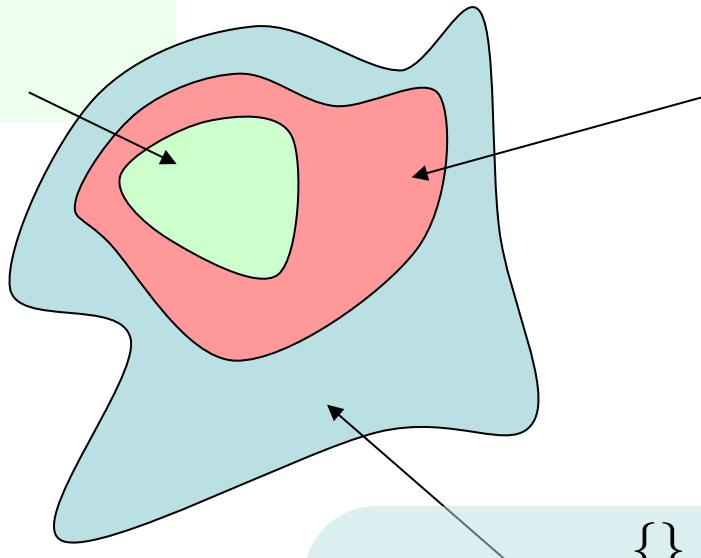
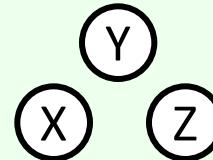
COMPUTE ALL THE INDEPENDENCES!



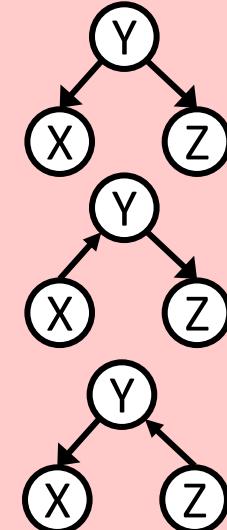
# Topology Limits Distributions

- Given some graph topology  $G$ , only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs
- Full conditioning can encode any distribution

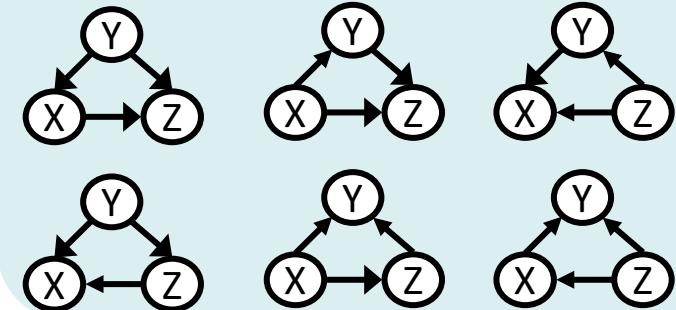
$$\{X \perp\!\!\!\perp Y, X \perp\!\!\!\perp Z, Y \perp\!\!\!\perp Z, \\ X \perp\!\!\!\perp Z \mid Y, X \perp\!\!\!\perp Y \mid Z, Y \perp\!\!\!\perp Z \mid X\}$$



$$\{X \perp\!\!\!\perp Z \mid Y\}$$



$$\{\}$$



# Bayes Nets Representation Summary

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- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' Net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution