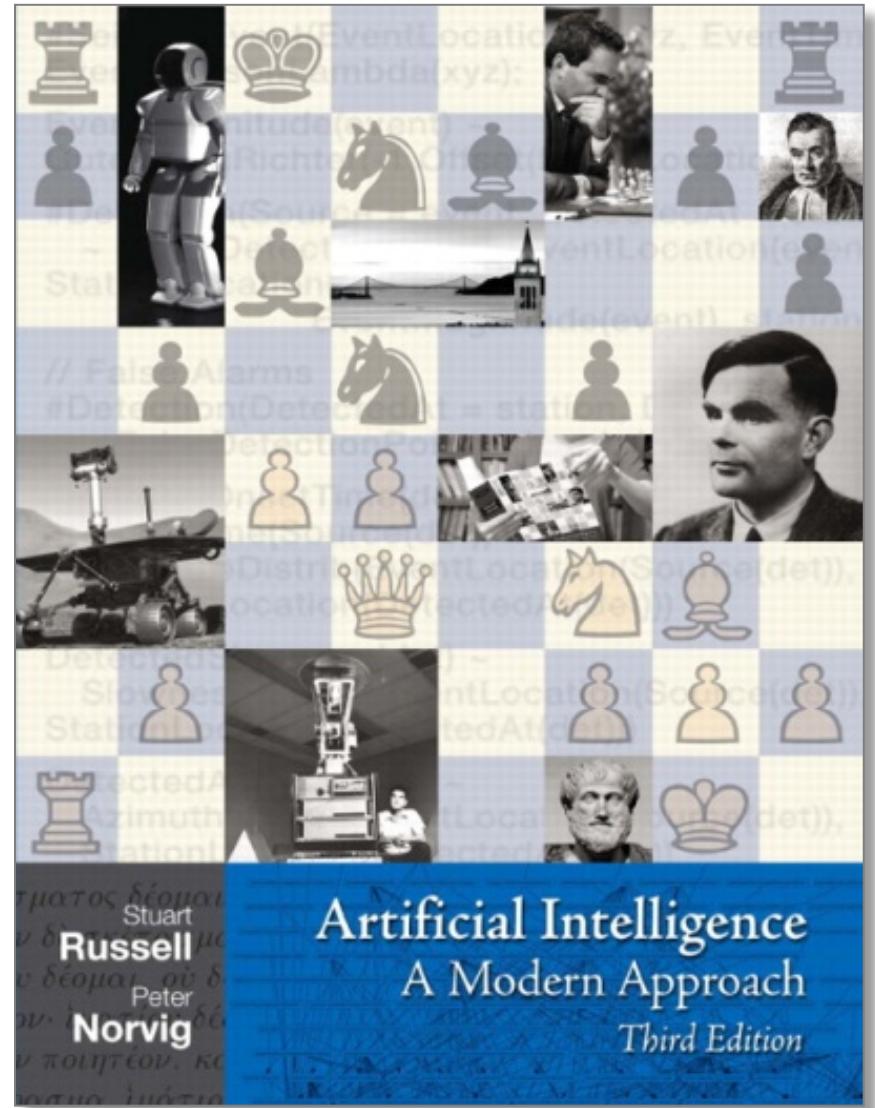


Bayes' Nets

Read AIMA

Chapter 14 "Probabilistic Reasoning" (Sections 14.1, 14.2 and 14.4)



Slides courtesy of Dan Klein and Pieter Abbeel --- University of California, Berkeley

Probabilistic Models

- Models describe how (a portion of) the world works
- **Models are always simplifications**
 - May not account for every variable
 - May not account for all interactions between variables
 - “All models are wrong; but some are useful.”
 - George E. P. Box
- What do we do with probabilistic models?
 - We (or our agents) need to reason about unknown variables, given evidence
 - Example: explanation (diagnostic reasoning)
 - Example: prediction (causal reasoning)



Review: Independence

- Two variables are *independent* if:

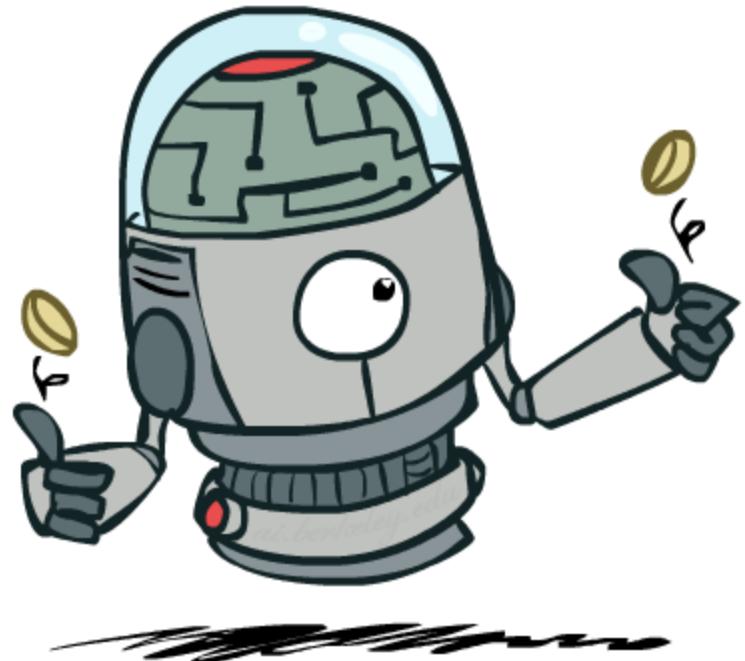
$$\forall x, y : P(x, y) = P(x)P(y)$$

- This says that their joint distribution *factors* into a product two simpler distributions
- Another form:

$$\forall x, y : P(x|y) = P(x)$$

- We write: $X \perp\!\!\!\perp Y$
- Independence is a simplifying *modeling assumption*

- *Empirical* joint distributions: at best “close” to independent
- What could we assume for {Weather, Traffic, Cavity, Toothache}?



Review: Independence?

$P_1(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$P_2(T, W)$

T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

$P(W)$

W	P
sun	0.6
rain	0.4

Review: Independence

- N fair, independent coin flips:

$$P(X_1)$$

H	0.5
T	0.5

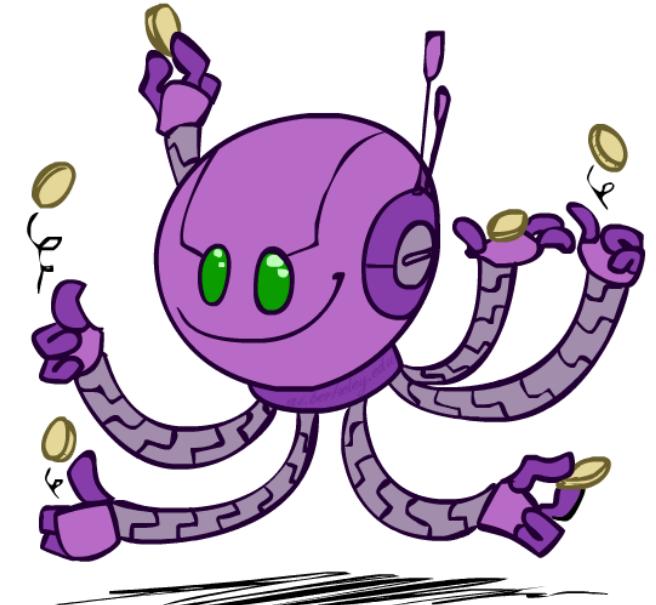
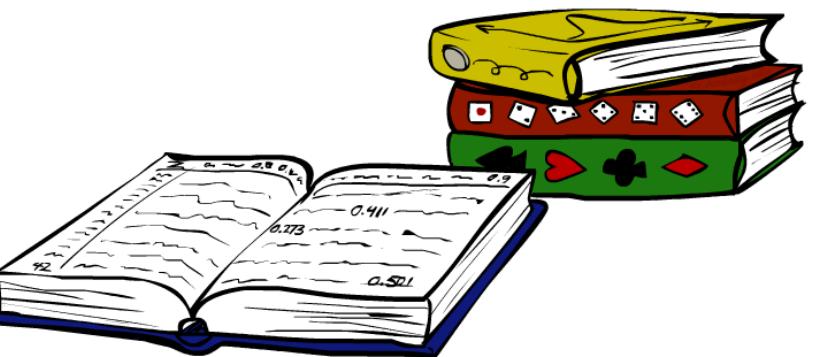
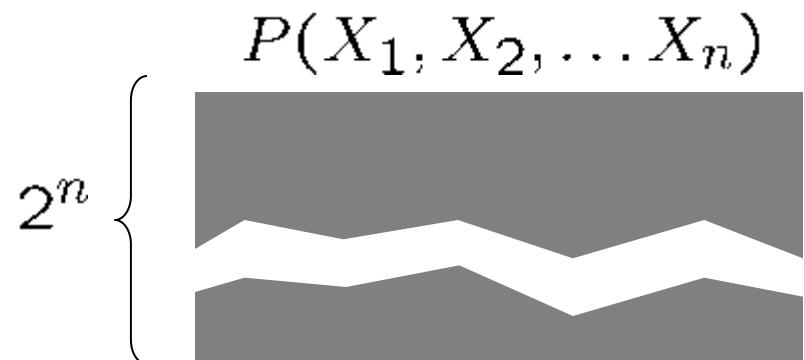
$$P(X_2)$$

H	0.5
T	0.5

...

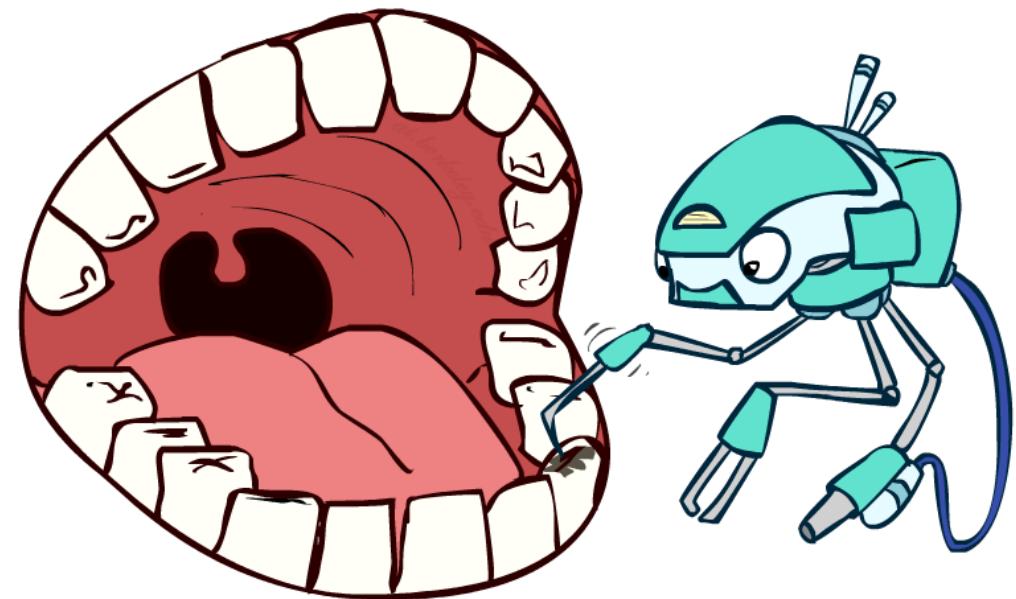
$$P(X_n)$$

H	0.5
T	0.5



Review: Conditional Independence

- $P(\text{Toothache}, \text{Cavity}, \text{Detect})$
- If I have a cavity, the probability that the probe detects it doesn't depend on whether I have a toothache:
 - $P(+\text{detect} | +\text{toothache}, +\text{cavity}) = P(+\text{detect} | +\text{cavity})$
- The same independence holds if I don't have a cavity:
 - $P(+\text{detect} | +\text{toothache}, -\text{cavity}) = P(+\text{detect} | -\text{cavity})$
- Detect is *conditionally independent* of Toothache given Cavity:
 - $P(\text{Detect} | \text{Toothache}, \text{Cavity}) = P(\text{Detect} | \text{Cavity})$
- Equivalent statements:
 - $P(\text{Toothache} | \text{Detect}, \text{Cavity}) = P(\text{Toothache} | \text{Cavity})$
 - $P(\text{Toothache}, \text{Detect} | \text{Cavity}) = P(\text{Toothache} | \text{Cavity}) P(\text{Detect} | \text{Cavity})$
 - One can be derived from the other using the chain rule



Review: Conditional Independence

- Unconditional (absolute) independence very rare, and it doesn't help us make inferences about other variables.
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z $X \perp\!\!\!\perp Y | Z$

if and only if:

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

Conditional Independence

- What about this domain:

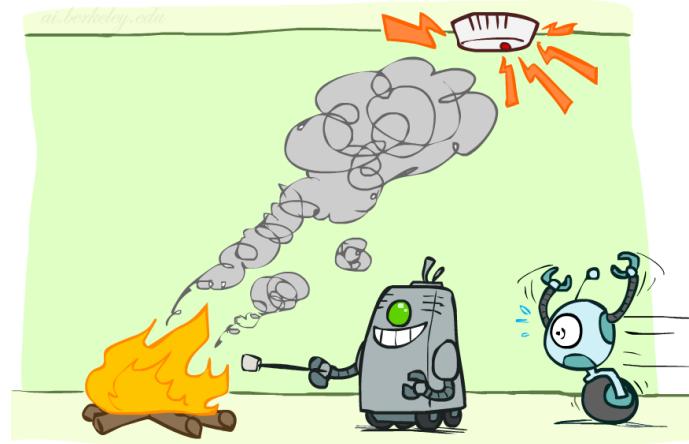
- Traffic
- Umbrella
- Raining



Conditional Independence

- What about this domain:

- Fire
- Smoke
- Alarm



Conditional Independence and the Chain Rule

- Chain rule:

$$P(X_1, X_2, \dots, X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2) \dots$$

- Trivial decomposition:

$$P(\text{Traffic, Rain, Umbrella}) =$$

$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain, Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic, Rain, Umbrella}) =$$

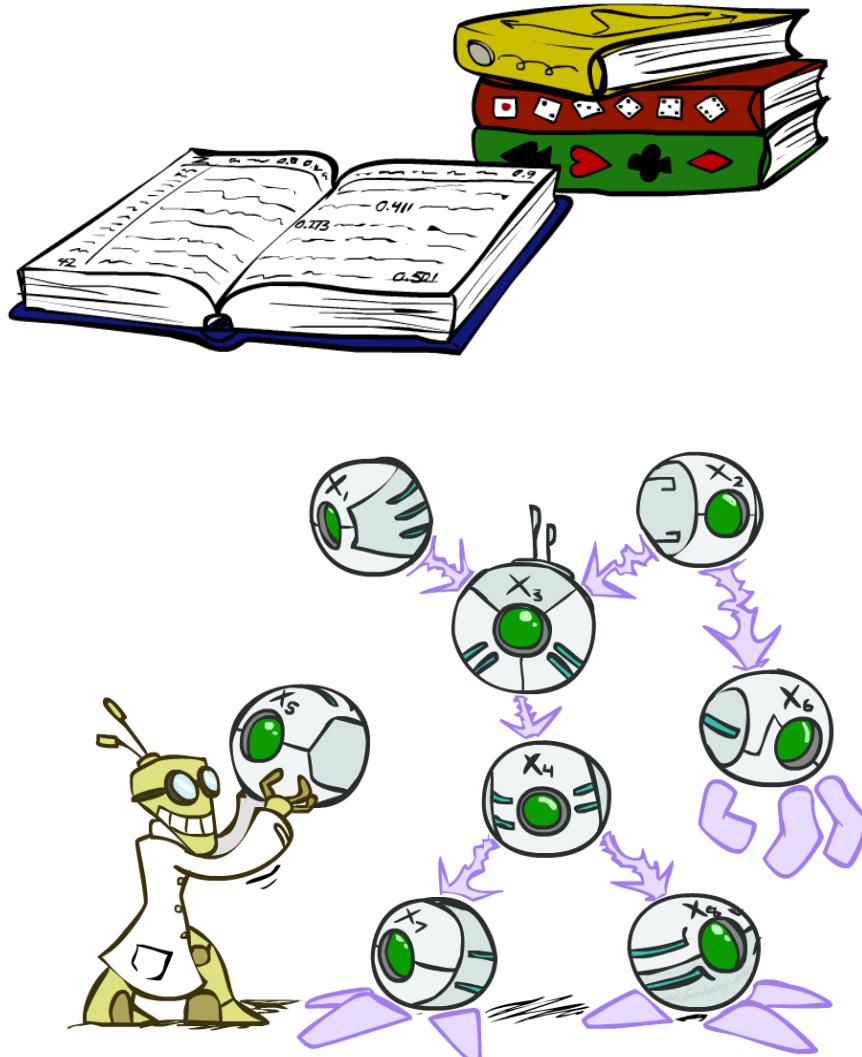
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$



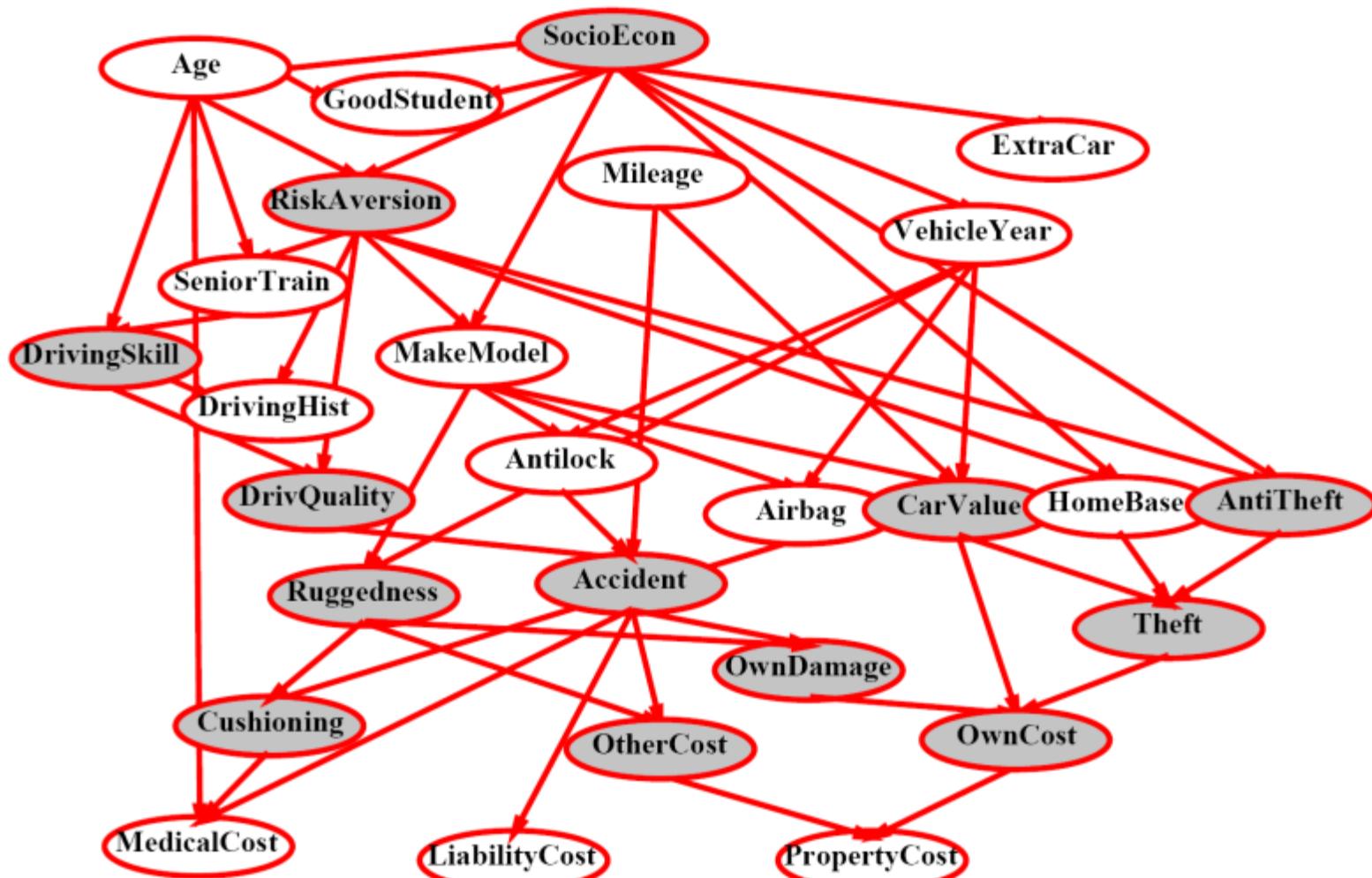
- Bayes' nets / graphical models help us express conditional independence assumptions

Bayes' Nets: Big Picture

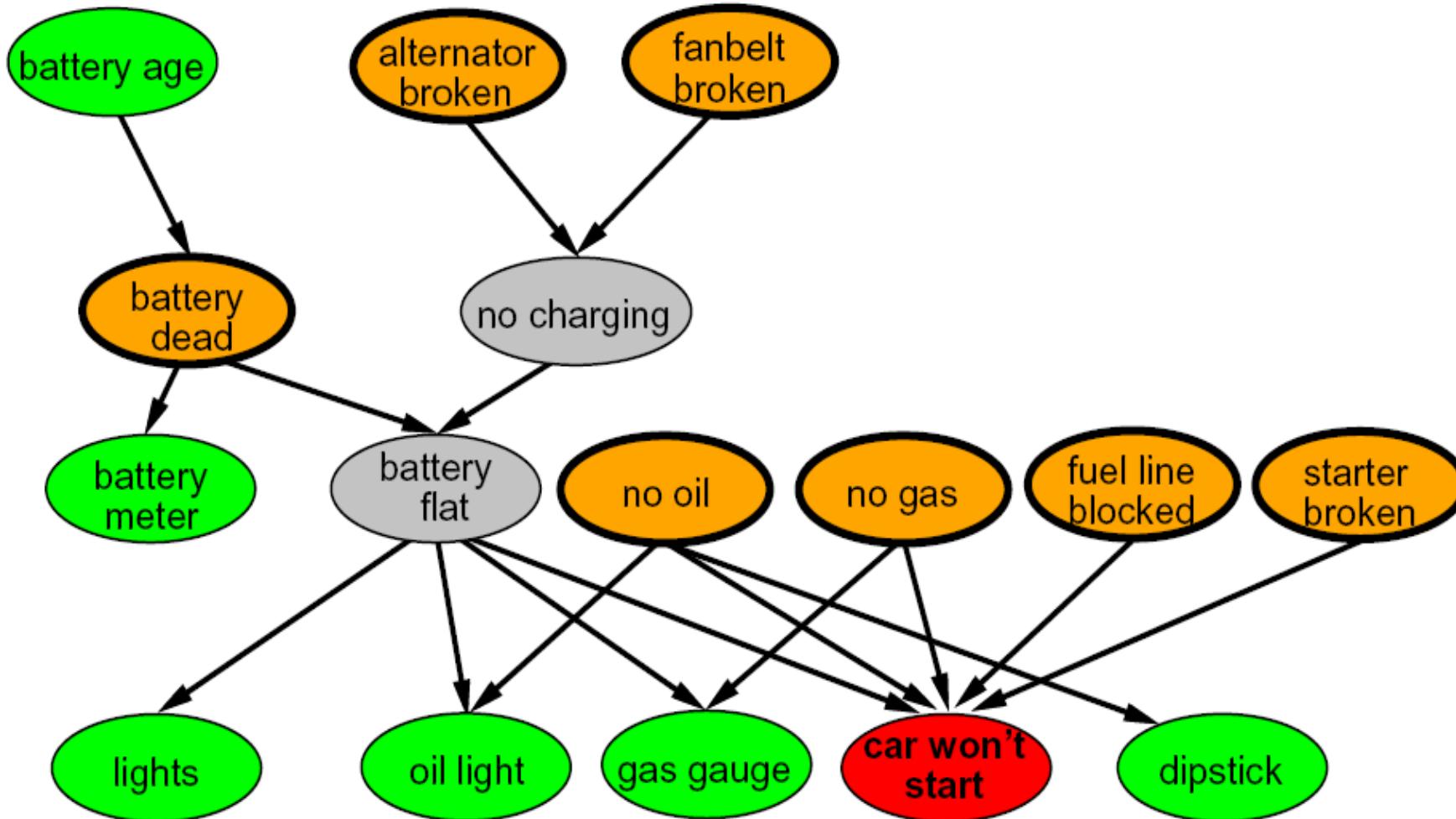
- Two problems with using full joint distribution tables as our probabilistic models:
 - Unless there are only a few variables, the joint is WAY too big to represent explicitly
 - Hard to learn (estimate) anything empirically about more than a few variables at a time
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
 - More properly called graphical models
 - We describe how variables locally interact
 - Local interactions chain together to give global, indirect interactions



Example Bayes' Net: Insurance

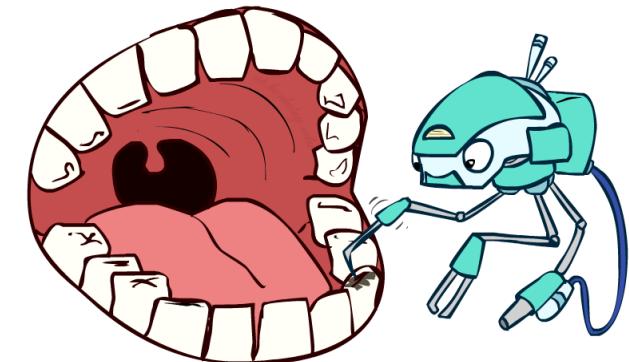
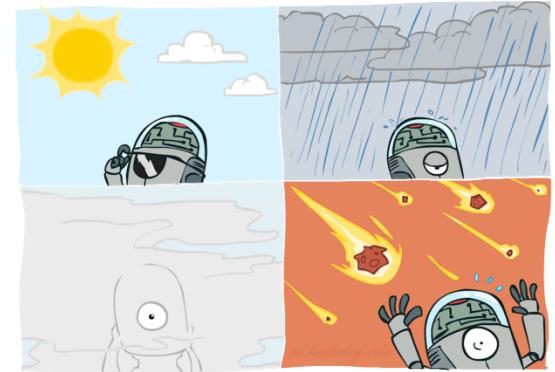
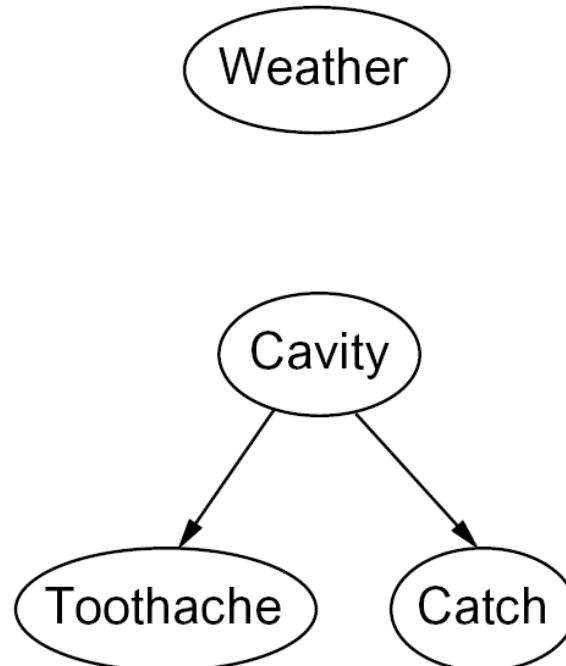


Example Bayes' Net: Car



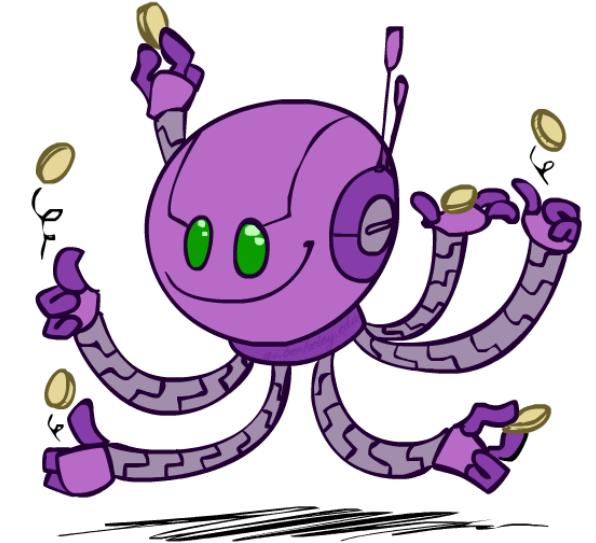
Graphical Model Notation

- Nodes: variables (with domains)
 - Can be assigned (observed) or unassigned (unobserved)
- Arcs: interactions
 - Indicate “direct influence” between variables
 - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don’t!)



Example: Coin Flips

- N independent coin flips



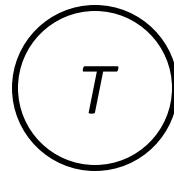
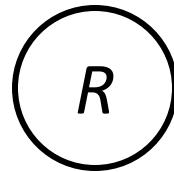
- No interactions between variables: absolute independence

Example: Traffic

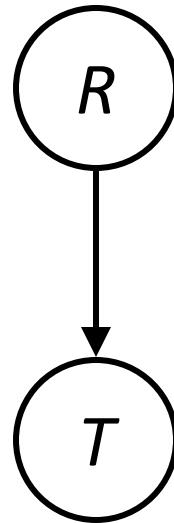
- Variables:
 - R : It rains
 - T : There is traffic



- Model 1: independence



- Model 2: rain causes traffic



- Why is an agent using model 2 better?

Example: Traffic II

- Let's build a causal graphical model!
- Variables
 - T: Traffic
 - R: It rains
 - L: Low pressure
 - D: Roof drips
 - B: Ballgame
 - C: Cavity

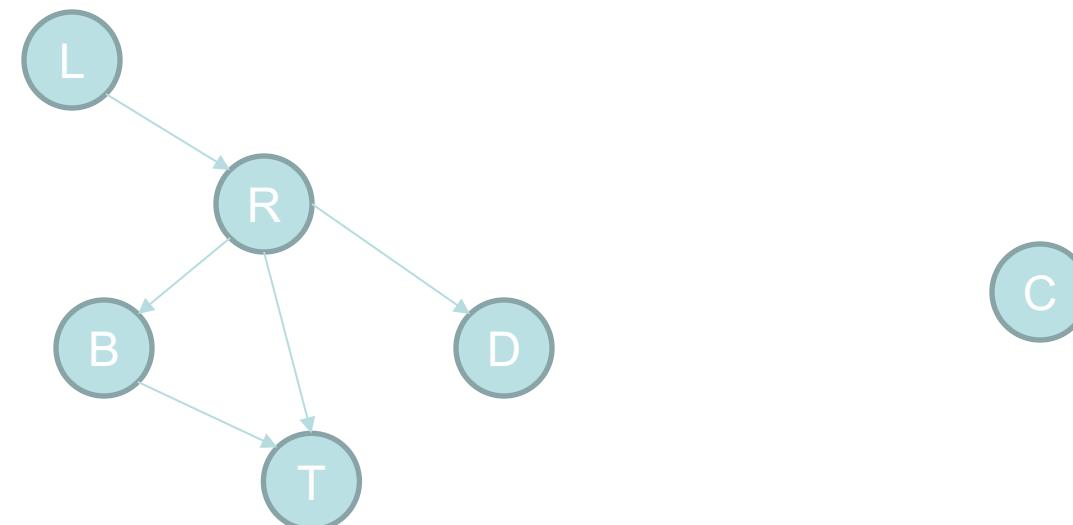


Example: Traffic II

- Let's build a causal graphical model!

- Variables

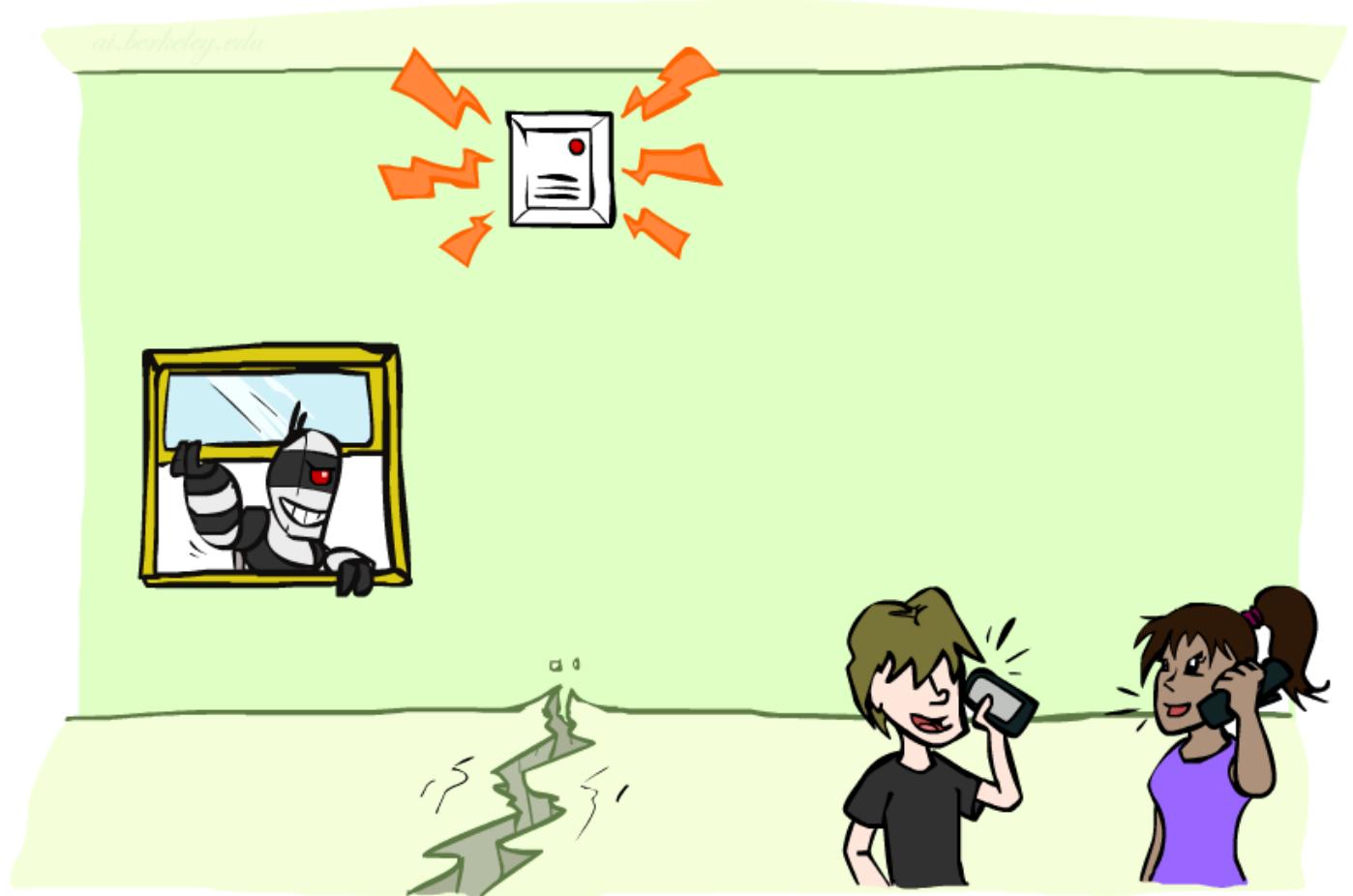
- T: Traffic
- R: It rains
- L: Low pressure
- D: Roof drips
- B: Ballgame
- C: Cavity



Example: Alarm Network

■ Variables

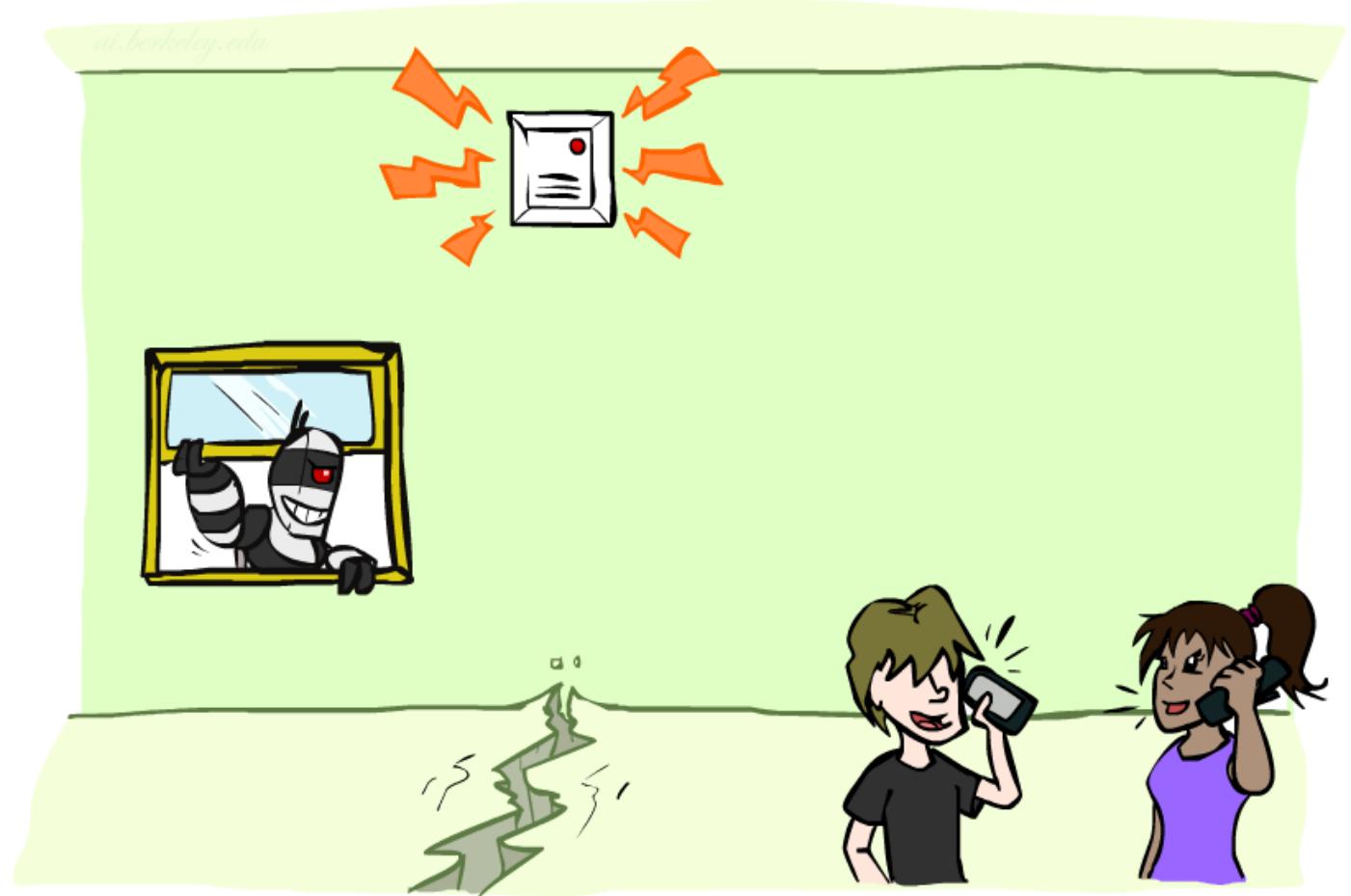
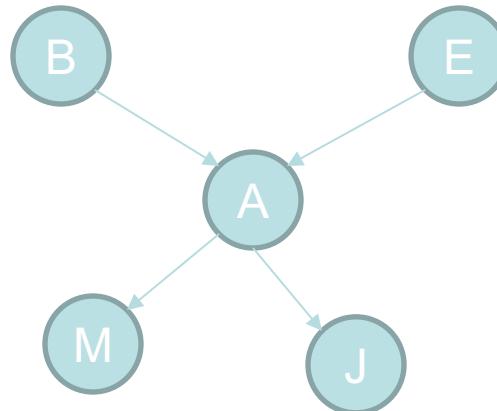
- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



Example: Alarm Network

Variables

- B: Burglary
- A: Alarm goes off
- M: Mary calls
- J: John calls
- E: Earthquake!



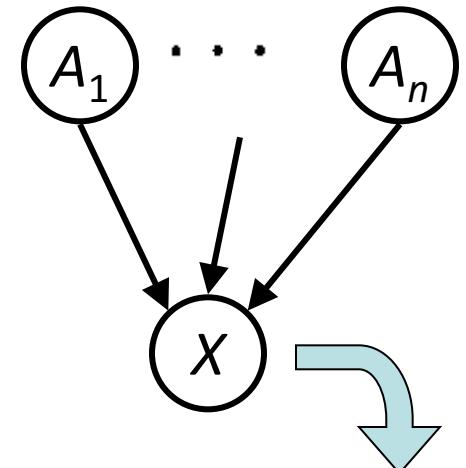


Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
 - A collection of distributions over X , one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

- CPT: conditional probability table
- Description of a noisy “causal” process



$$P(X|A_1 \dots A_n)$$

A Bayes net = Topology (graph) + Local Conditional Probabilities

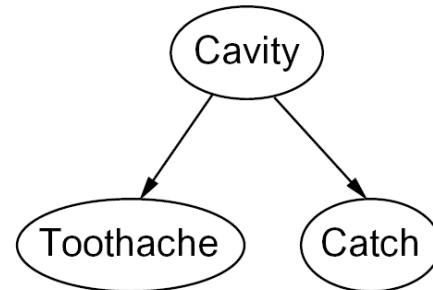
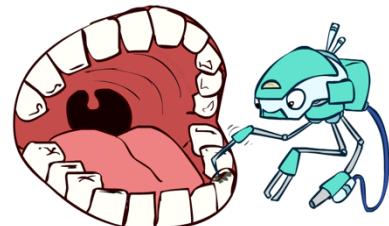
Probabilities in BNs



- Bayes' nets implicitly encode joint distributions
 - As a product of local conditional distributions
 - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

- Example:



$P(+\text{cavity}, +\text{detect}, -\text{toothache})$

Probabilities in BNs



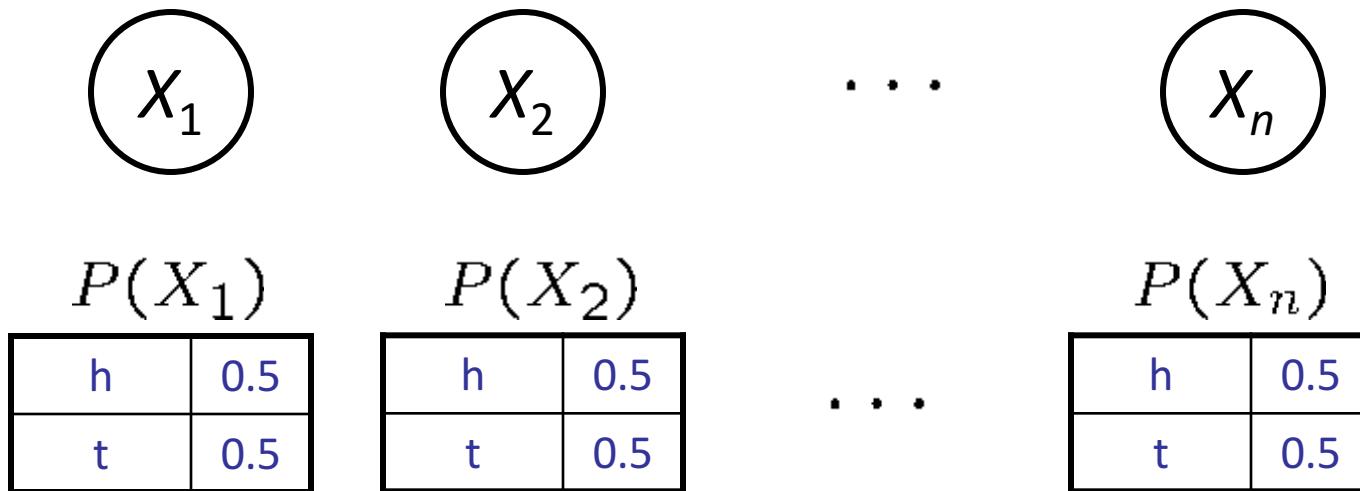
- Why are we guaranteed that setting

$$P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

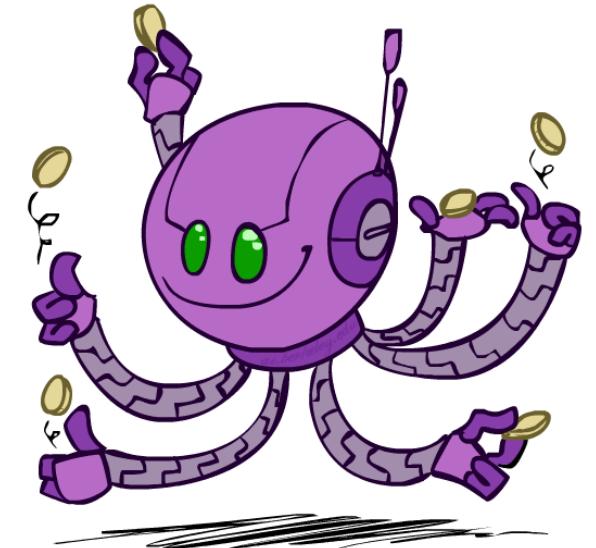
results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | x_1 \dots x_{i-1})$
- Assume conditional independences: $P(x_i | x_1, \dots, x_{i-1}) = P(x_i | \text{parents}(X_i))$
→ Consequence: $P(x_1, x_2, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$
- Not every BN can represent every joint distribution
 - The topology enforces certain conditional independencies

Example: Coin Flips

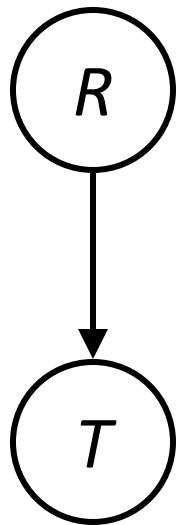


$$P(h, h, t, h) =$$



Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic



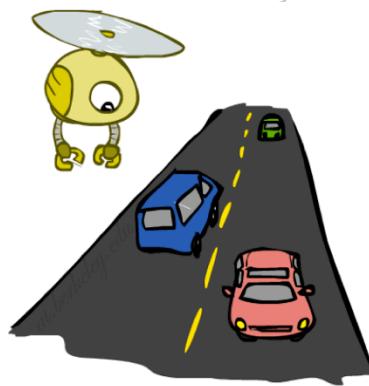
$P(R)$

$+r$	$1/4$
$-r$	$3/4$

$$P(+r, -t) =$$

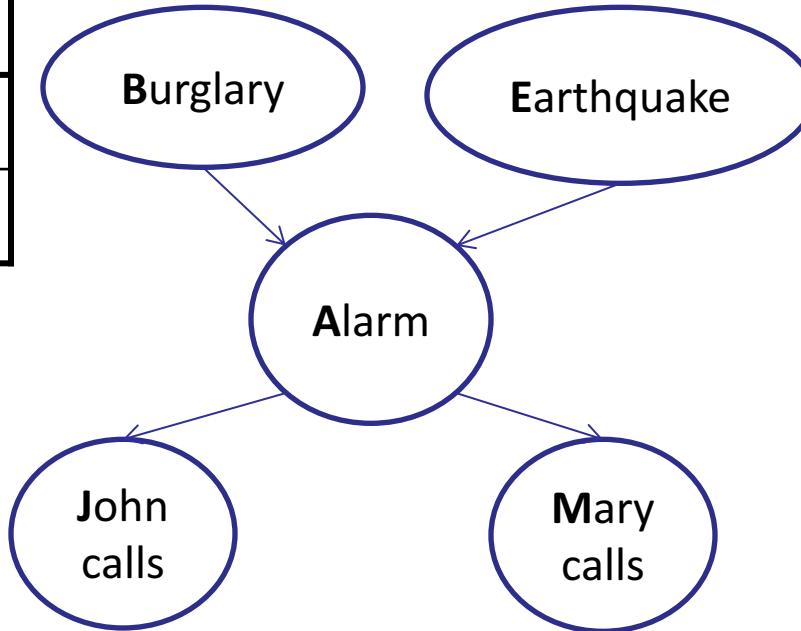
$P(T|R)$

$+r$	$+t$	$3/4$
	$-t$	$1/4$
$-r$	$+t$	$1/2$
	$-t$	$1/2$



Example: Alarm Network

B	$P(B)$
+b	0.001
-b	0.999



E	$P(E)$
+e	0.002
-e	0.998



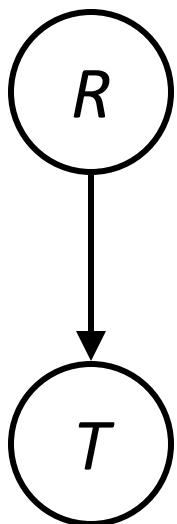
A	J	$P(J A)$
+a	+j	0.9
+a	-j	0.1
-a	+j	0.05
-a	-j	0.95

A	M	$P(M A)$
+a	+m	0.7
+a	-m	0.3
-a	+m	0.01
-a	-m	0.99

B	E	A	$P(A B,E)$
+b	+e	+a	0.95
+b	+e	-a	0.05
+b	-e	+a	0.94
+b	-e	-a	0.06
-b	+e	+a	0.29
-b	+e	-a	0.71
-b	-e	+a	0.001
-b	-e	-a	0.999

Example: Traffic

- Causal direction

 $P(R)$

+r	1/4
-r	3/4

 $P(T|R)$

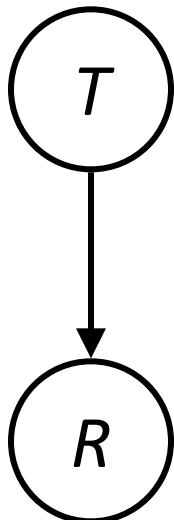
+r	+t	3/4
	-t	1/4
-r	+t	1/2
	-t	1/2

 $P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Example: Reverse Traffic

- Reverse causality?



$P(T)$

+t	9/16
-t	7/16

$P(R|T)$

+t	+r	1/3
	-r	2/3
-t	+r	1/7
	-r	6/7



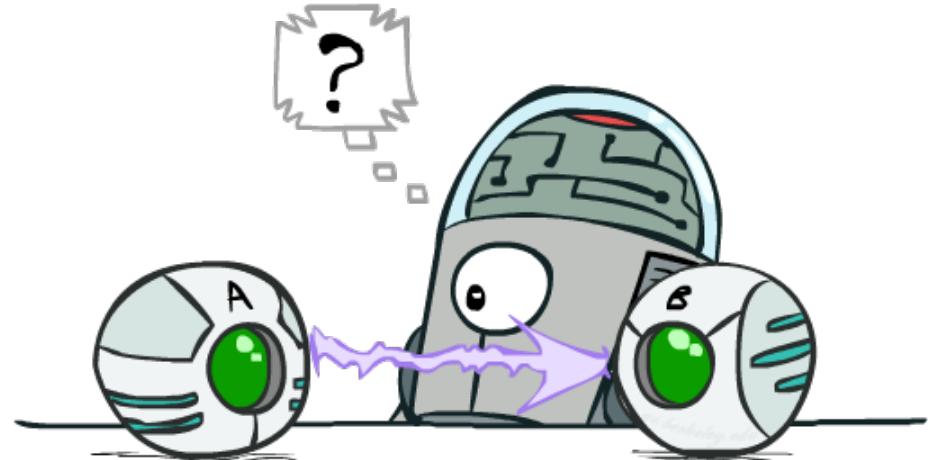
$P(T, R)$

+r	+t	3/16
+r	-t	1/16
-r	+t	6/16
-r	-t	6/16

Causality?

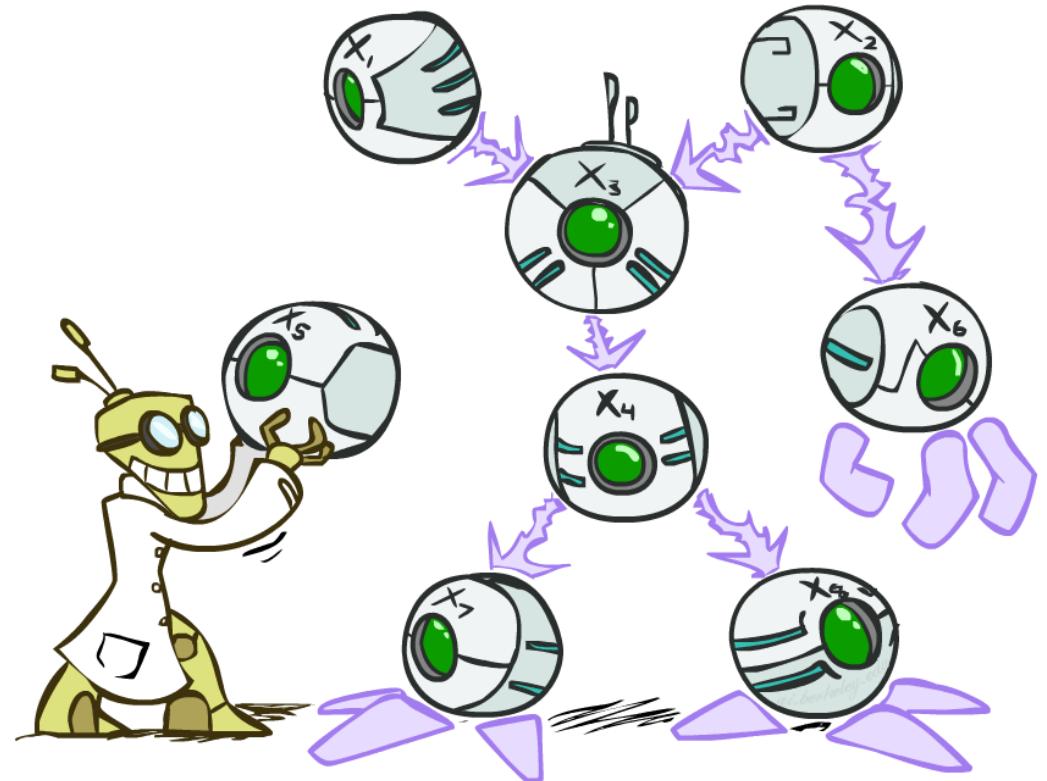
- When Bayes' nets reflect the true causal patterns:
 - Often simpler (nodes have fewer parents)
 - Often easier to think about
 - Often easier to elicit from experts
- BNs need not actually be causal
 - Sometimes no causal net exists over the domain (especially if variables are missing)
 - E.g. consider the variables *Traffic* and *Drips*
 - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
 - Topology may happen to encode causal structure
 - **Topology really encodes conditional independence**

$$P(x_i|x_1, \dots x_{i-1}) = P(x_i|\text{parents}(X_i))$$



Bayes' Nets

- So far: how a Bayes' net encodes a joint distribution
- Next: how to answer queries about that distribution
 - Today:
 - First assembled BNs using an intuitive notion of conditional independence as causality
 - Then saw that key property is conditional independence
 - Main goal: answer queries about conditional independence and influence
- After that: how to answer numerical queries (inference)



Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?

$$2^N$$

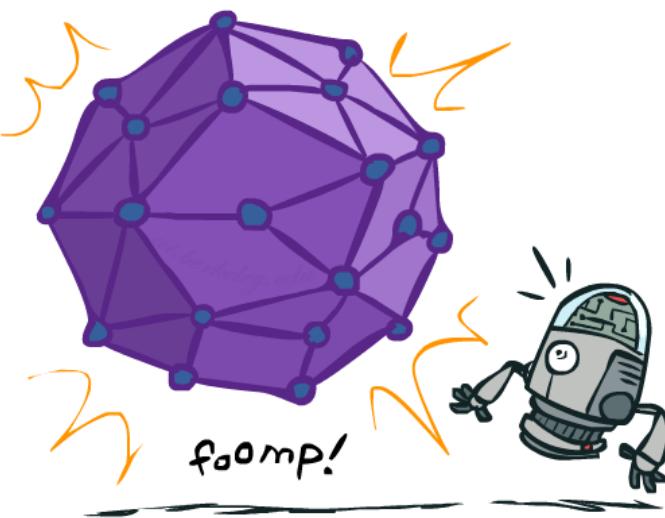
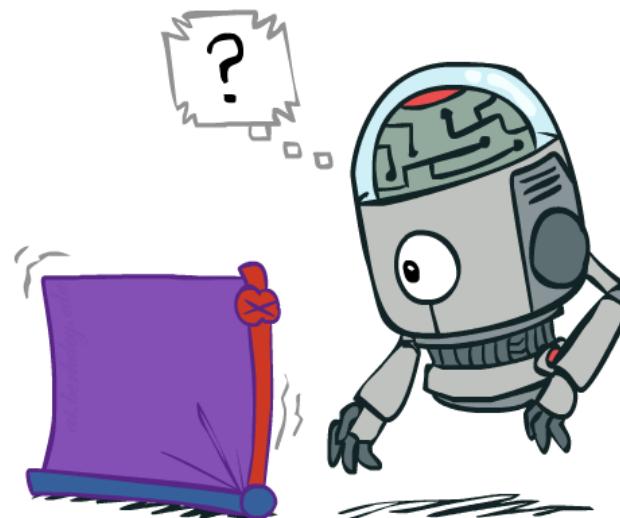
- How big is an N-node net if nodes have up to k parents?

$$O(N * 2^{k+1})$$

- Both give you the power to calculate

$$P(X_1, X_2, \dots, X_n)$$

- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also faster to answer queries (coming)



Conditional Independence

- X and Y are independent if

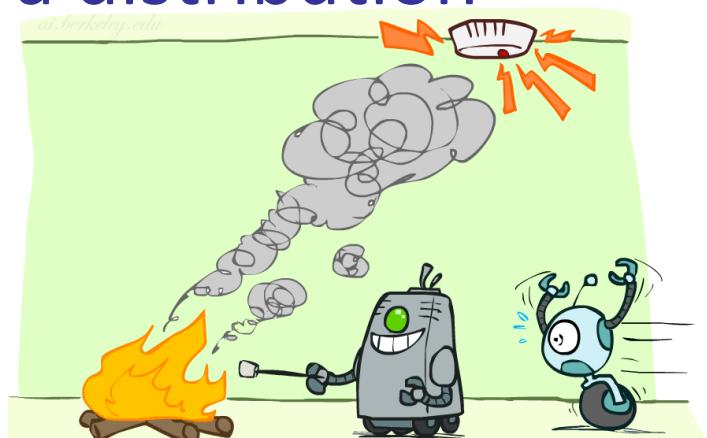
$$\forall x, y \ P(x, y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

- X and Y are conditionally independent given Z

$$\forall x, y, z \ P(x, y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y|Z$$

- (Conditional) independence is a property of a distribution

- Example: $Alarm \perp\!\!\!\perp Fire|Smoke$

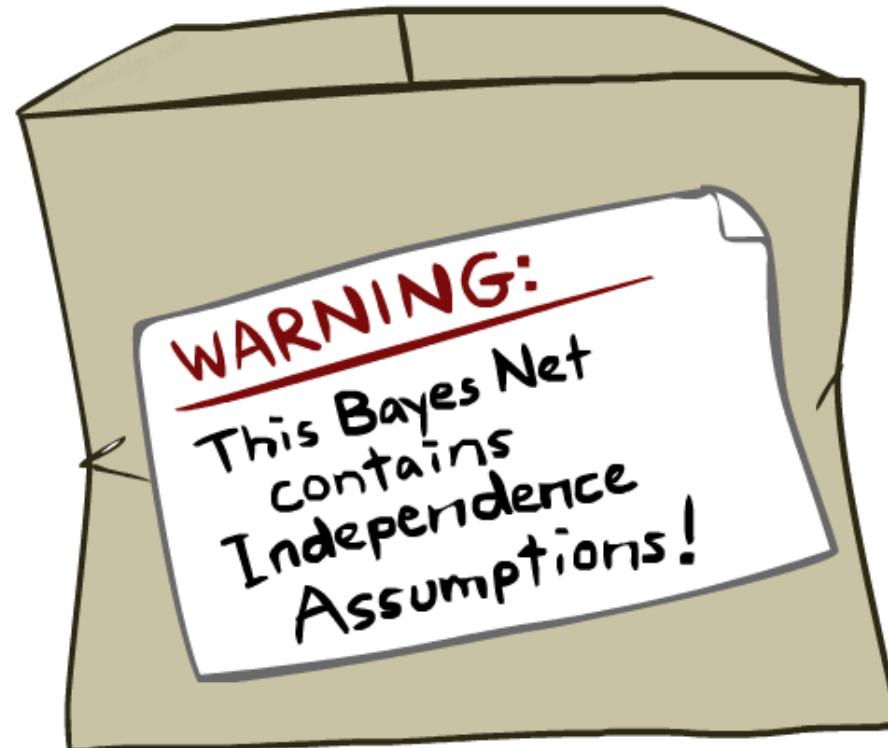


Bayes Nets: Assumptions

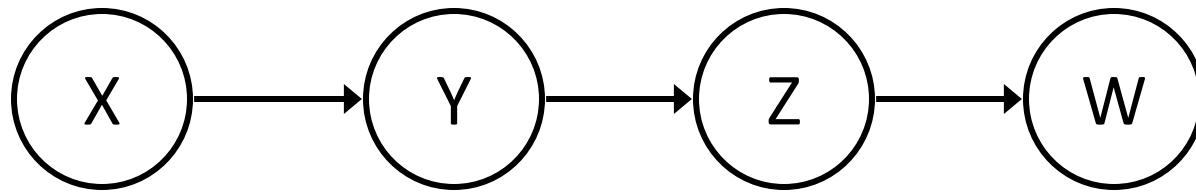
- Assumptions we are required to make to define the Bayes net when given the graph:

$$P(x_i | x_1 \cdots x_{i-1}) = P(x_i | \text{parents}(X_i))$$

- Beyond above “chain rule → Bayes net” conditional independence assumptions
 - Often additional conditional independences
 - They can be read off the graph
- Important for modeling: understand assumptions made when choosing a Bayes net graph



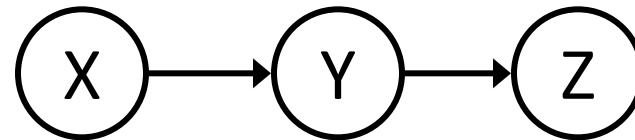
Example



- Conditional independence assumptions directly from simplifications in chain rule:
- Additional implied conditional independence assumptions?

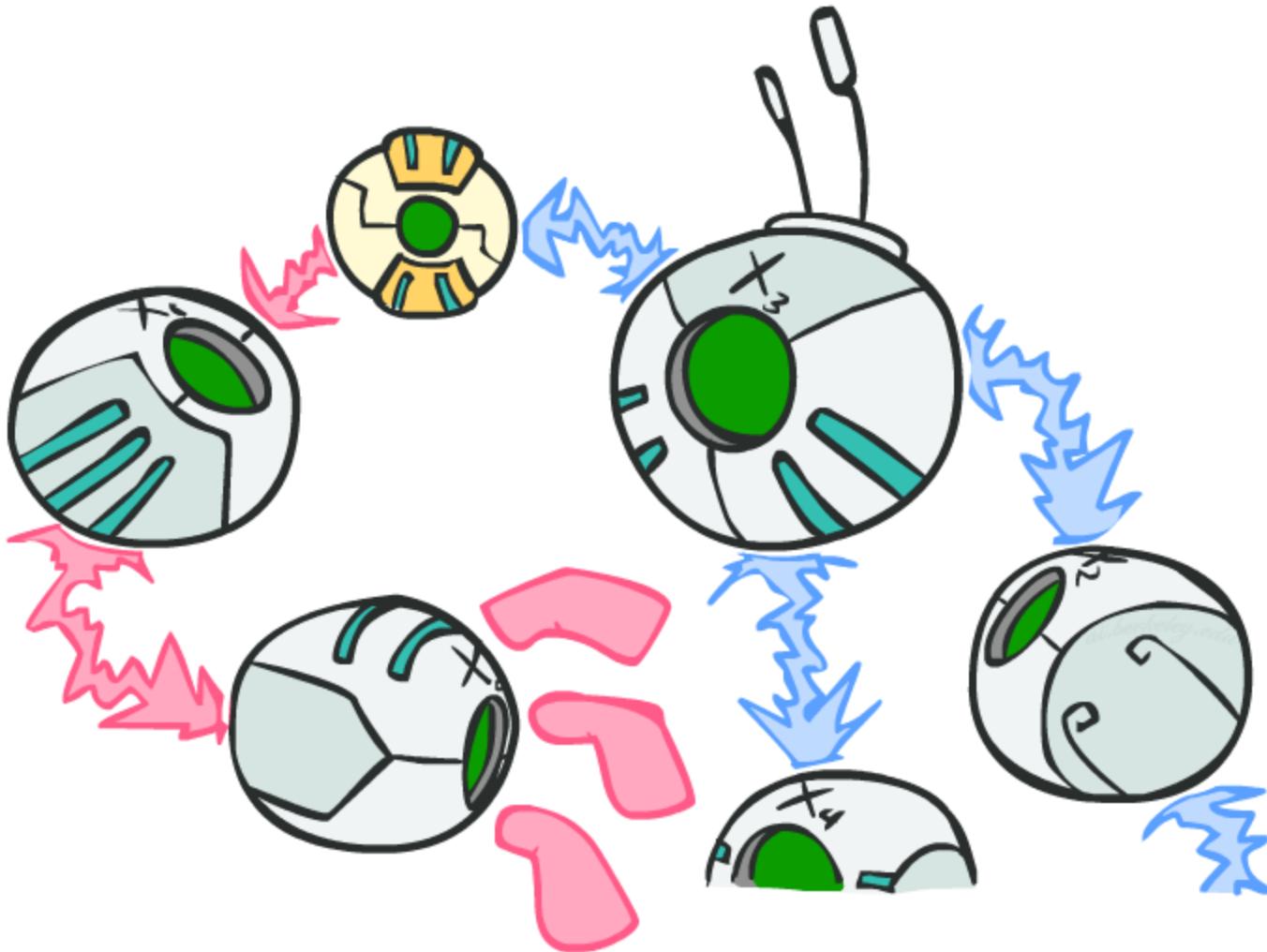
Independence in a BN

- Important question about a BN:
 - Are two nodes independent given certain evidence?
 - If yes, can prove using algebra (sometimes tedious)
 - If no, can prove with a counter example
 - Example:

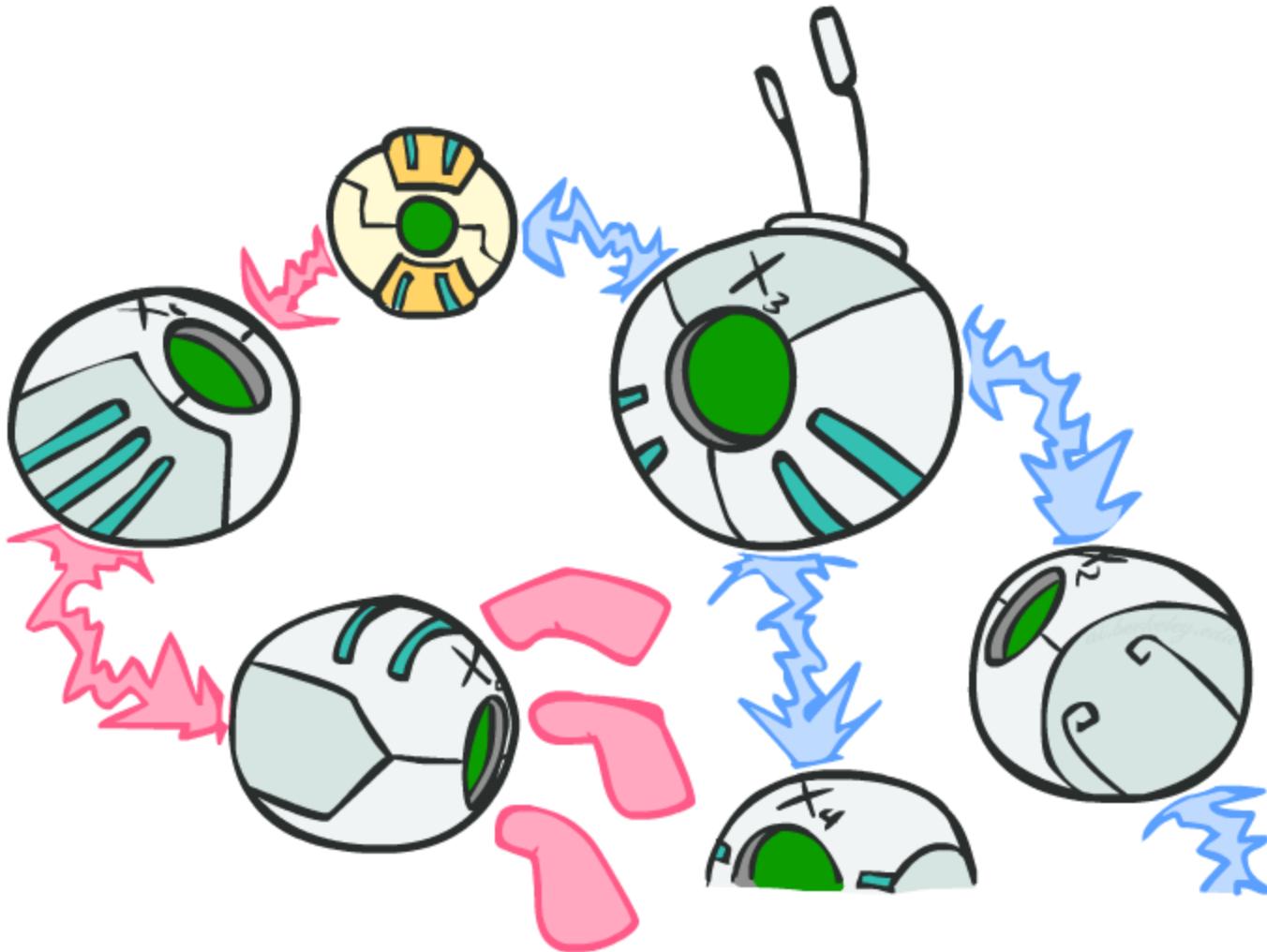


- Question: are X and Z necessarily independent?
 - Answer: no. Example: low pressure causes rain, which causes traffic.
 - X can influence Z, Z can influence X (via Y)
 - Addendum: they *could* be independent: how?

D-separation: Outline



D-separation: Outline



D-separation: Outline

- Study independence properties for triples
- Analyze complex cases in terms of member triples
- D-separation: a condition / algorithm for answering such queries

Causal Chains

- This configuration is a “causal chain”



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.
- Example:
 - Low pressure causes rain causes traffic, high pressure causes no rain causes no traffic
- In numbers:

$$\begin{aligned}P(+y | +x) &= 1, P(-y | -x) = 1, \\P(+z | +y) &= 1, P(-z | -y) = 1\end{aligned}$$

Causal Chains

- This configuration is a “causal chain”
- Guaranteed X independent of Z given Y?



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

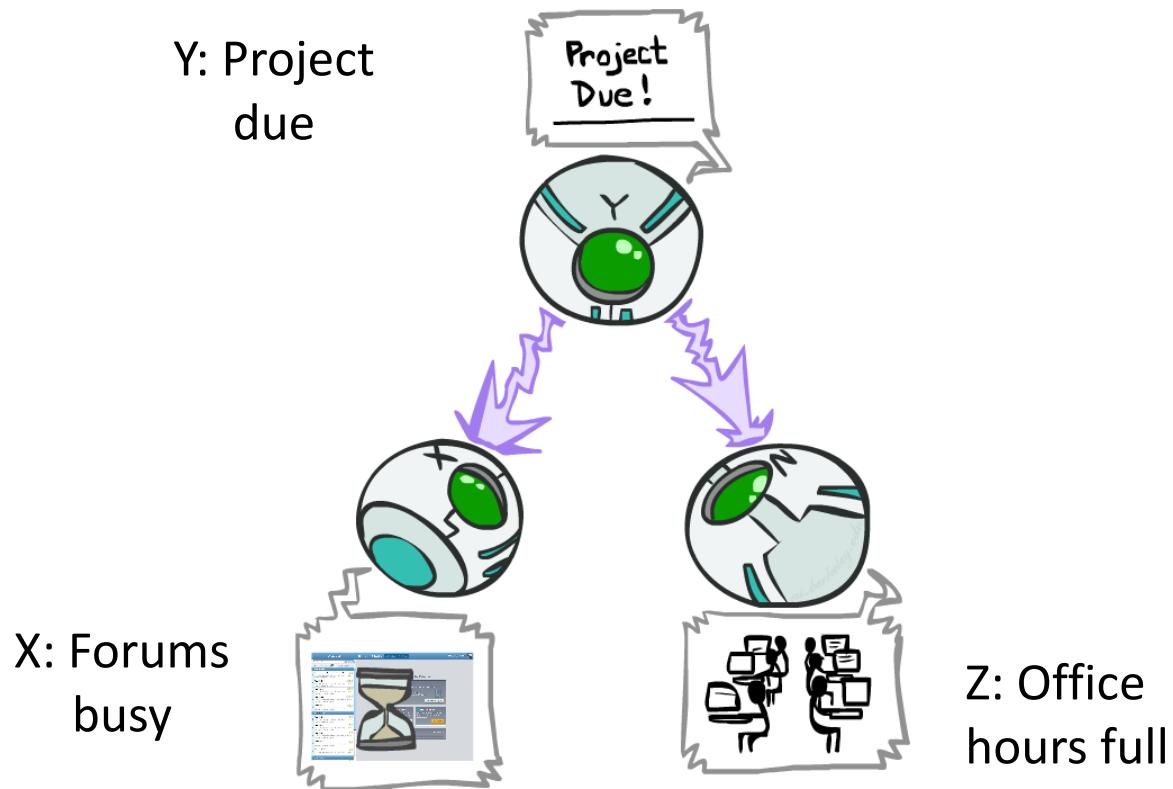
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Evidence along the chain “blocks” the influence

Common Cause

- This configuration is a “common cause”



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

- Guaranteed X independent of Z ? **No!**

- One example set of CPTs for which X is not independent of Z is sufficient to show this independence is not guaranteed.

- Example:

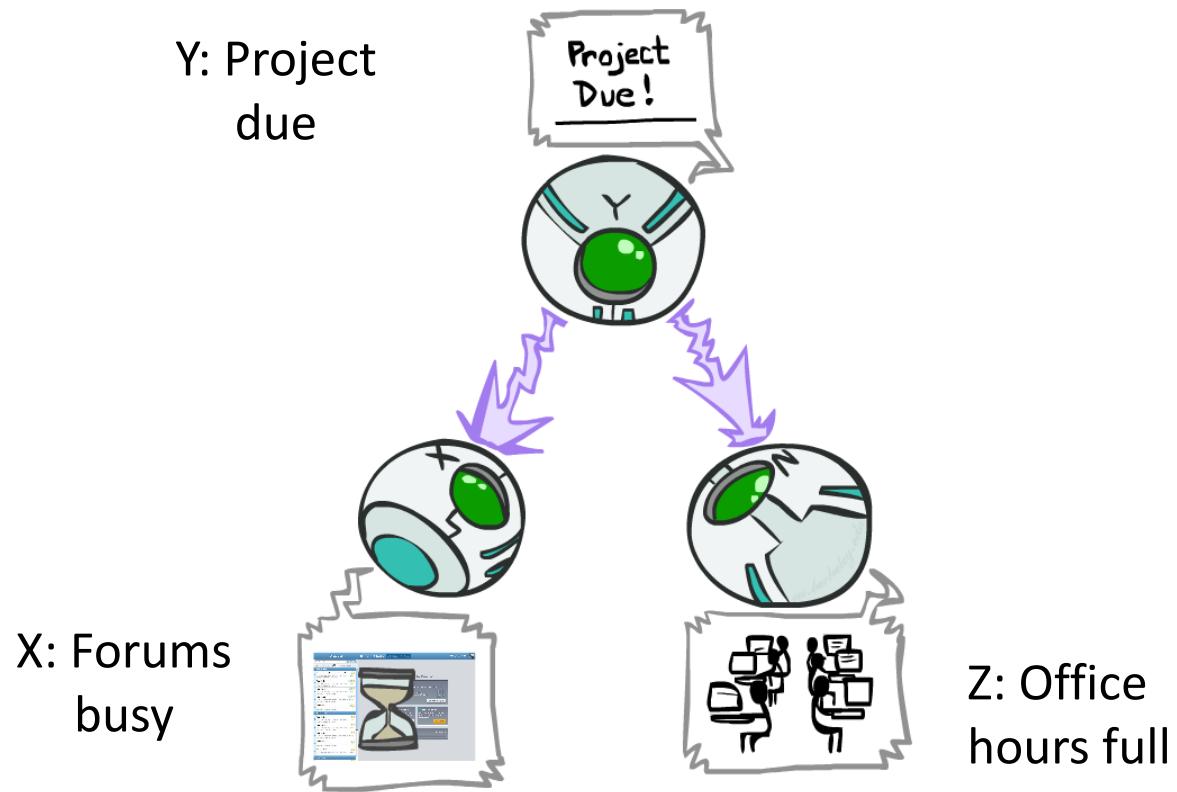
- Project due causes both forums busy and office hours to be full

- In numbers:

$$\begin{aligned}P(+x | +y) &= 1, P(-x | -y) = 1, \\P(+z | +y) &= 1, P(-z | -y) = 1\end{aligned}$$

Common Cause

- This configuration is a “common cause”
- Guaranteed X and Z independent given Y?



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

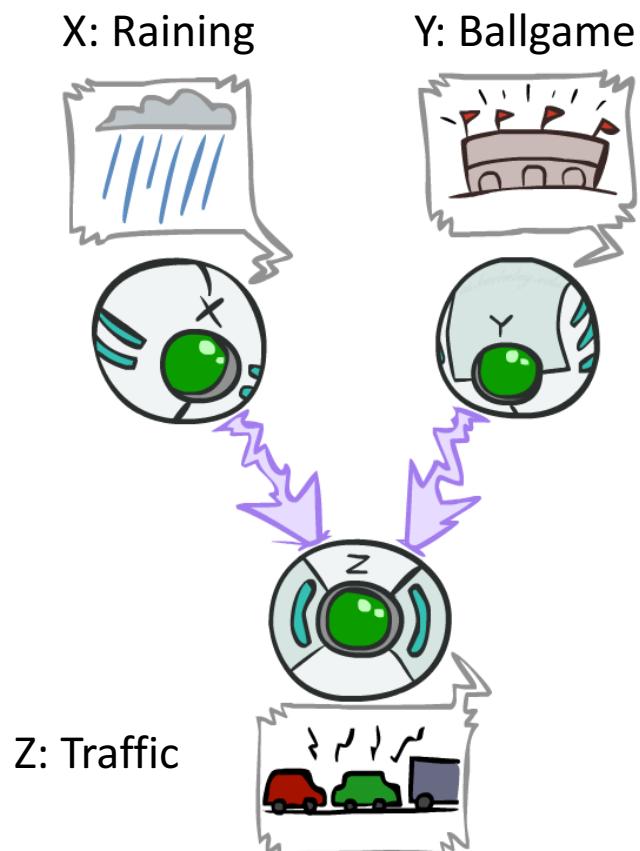
$$\begin{aligned} P(z|x, y) &= \frac{P(x, y, z)}{P(x, y)} \\ &= \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)} \\ &= P(z|y) \end{aligned}$$

Yes!

- Observing the cause blocks influence between effects.

Common Effect

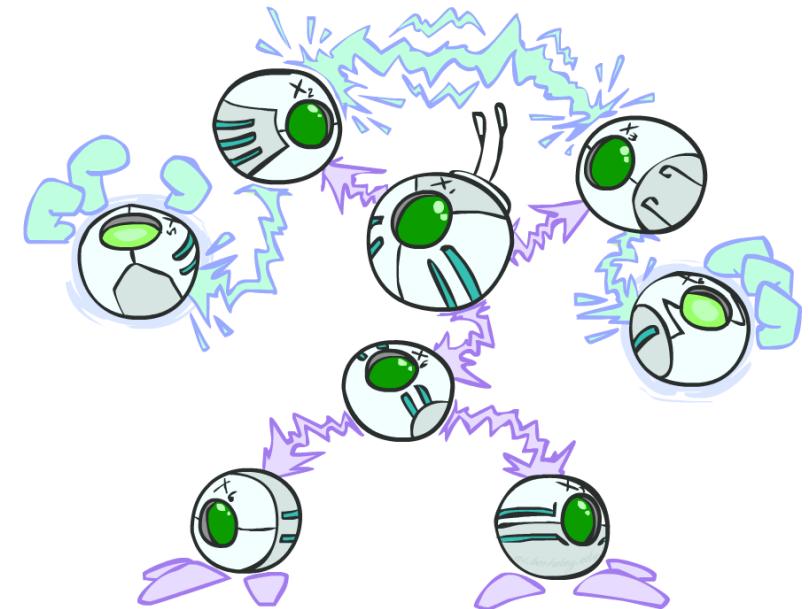
- Last configuration: two causes of one effect (v-structures)



- Are X and Y independent?
 - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
 - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
 - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is backwards from the other cases
 - Observing an effect **activates** influence between possible causes.

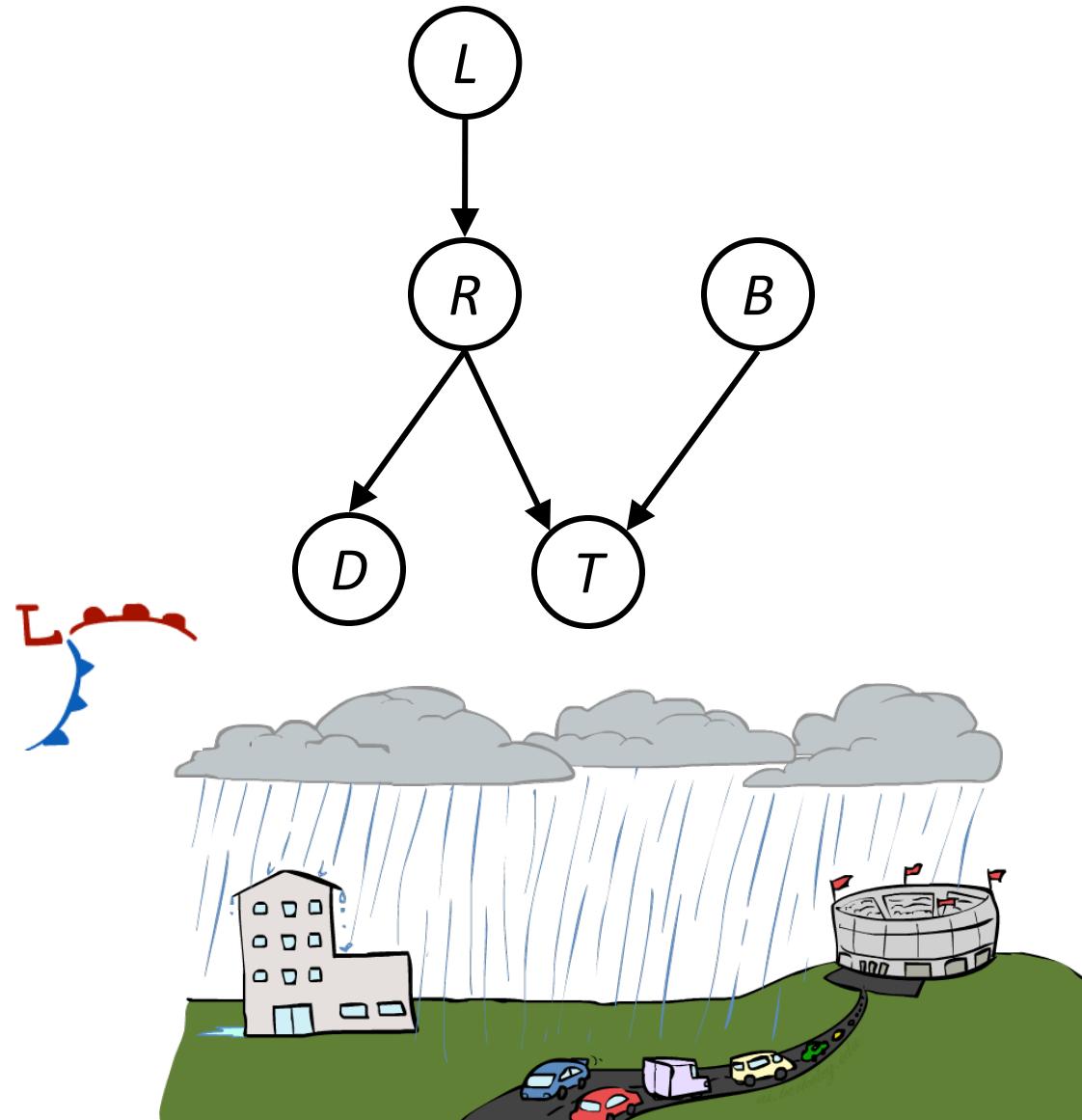
Are your variables in a BN independent?

- General question: in a given BN, are two variables independent (given some evidence)?
- Solution: analyze the graph
- Any complex example can be broken into repetitions of the three canonical cases



Reachability

- Recipe: shade evidence nodes, look for paths in the resulting graph
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
 - Where does it break?
 - Answer: the v-structure at T doesn't count as a link in a path unless "active"



Active / Inactive Paths

- Question: Are X and Y conditionally independent given evidence variables $\{Z\}$?

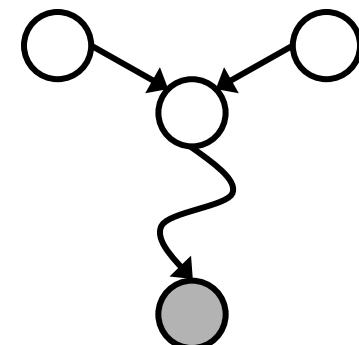
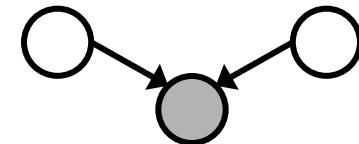
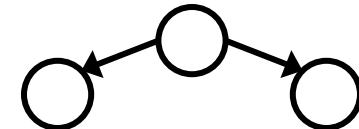
- Yes, if X and Y “d-separated” by Z
- Consider all (undirected) paths from X to Y
- No active paths = independence!

- A path is active if each triple is active:

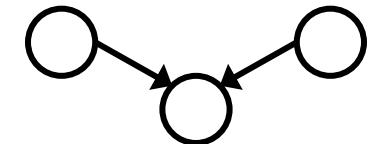
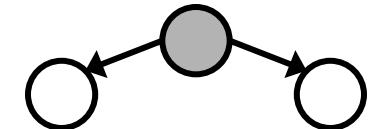
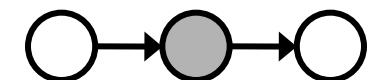
- Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
- Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
- Common effect (aka v-structure)
 $A \rightarrow B \leftarrow C$ where B or one of its descendants is observed

- All it takes to block a path is a single inactive segment

Active Triples



Inactive Triples



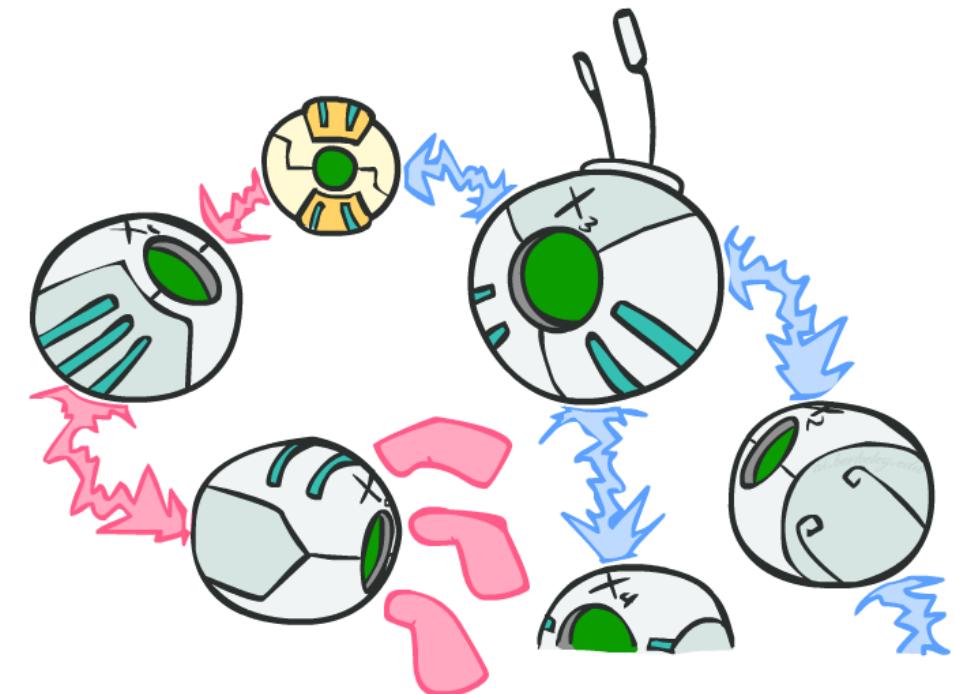
D-Separation

- Query: $X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$?
- Check all (undirected!) paths between X_i and X_j
 - If one or more active, then independence not guaranteed

$X_i \not\perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$

- Otherwise (i.e. if all paths are inactive),
then independence is guaranteed

$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$



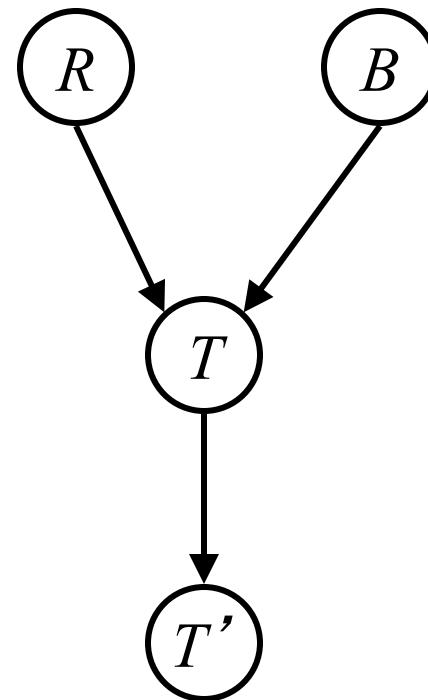
Example

$R \perp\!\!\!\perp B$

Yes

$R \perp\!\!\!\perp B | T$

$R \perp\!\!\!\perp B | T'$



Example

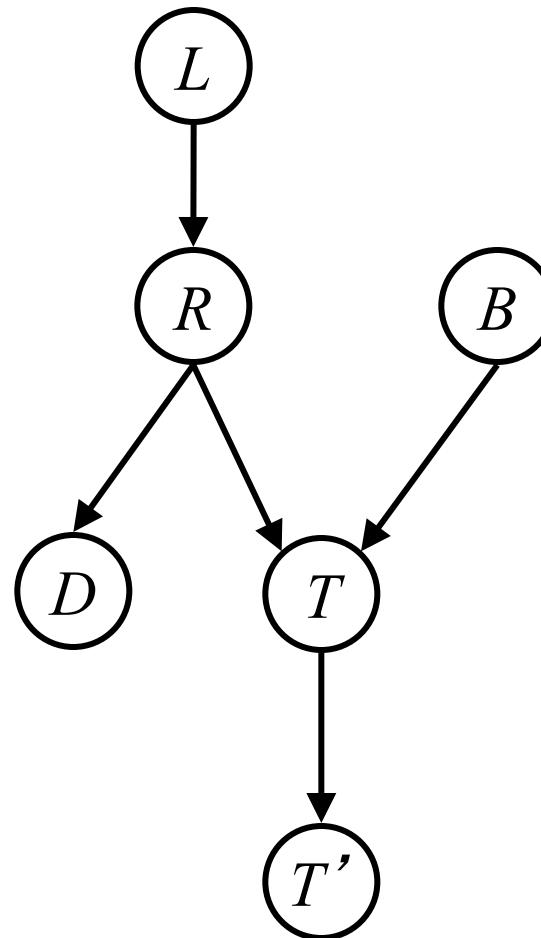
$L \perp\!\!\!\perp T' | T$ Yes

$L \perp\!\!\!\perp B$ Yes

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ Yes

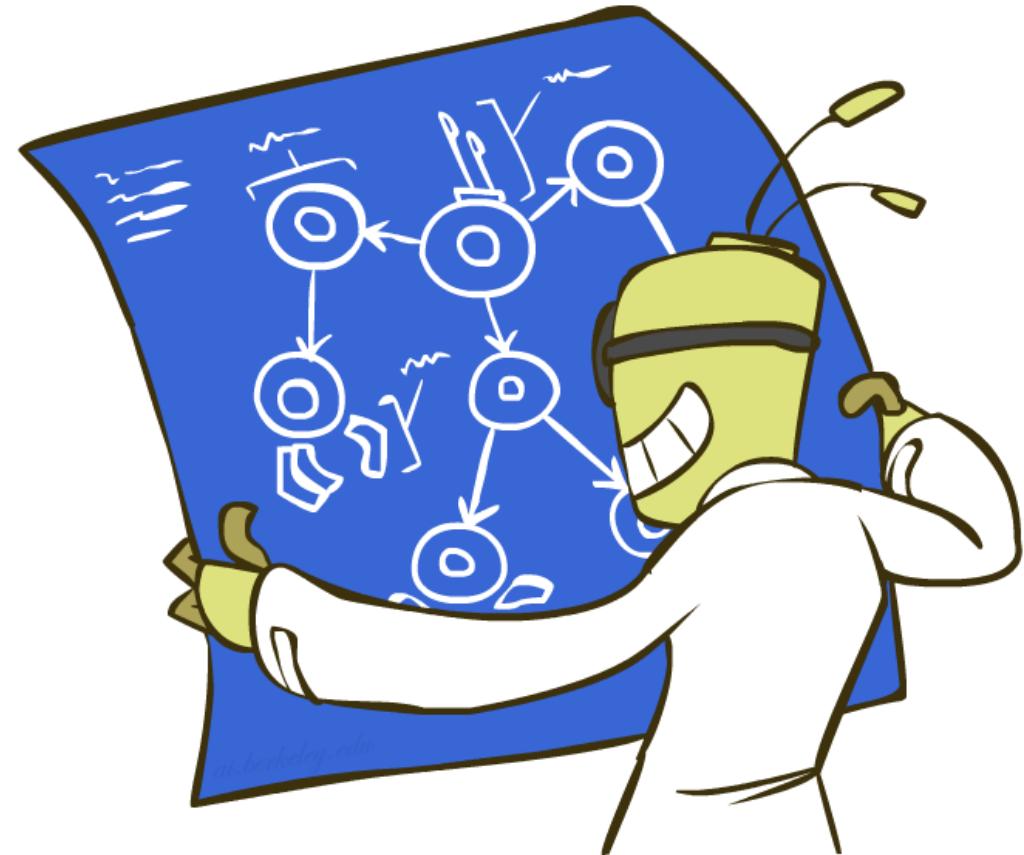


Structure Implications

- Given a Bayes net structure, can run d-separation algorithm to build a complete list of conditional independences that are necessarily true of the form

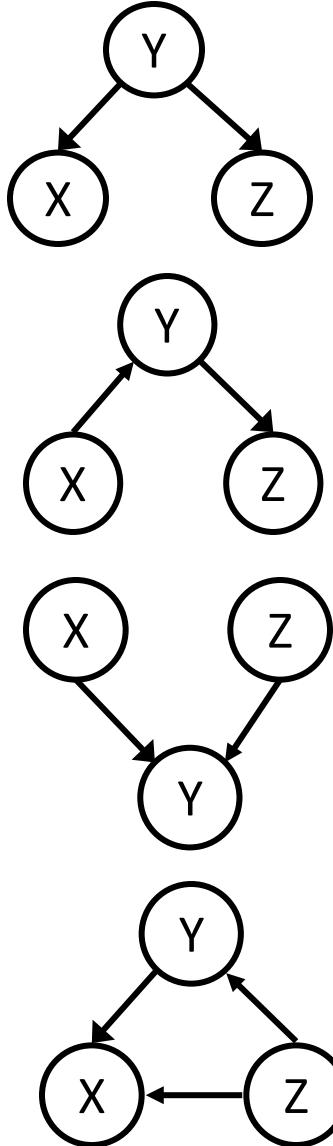
$$X_i \perp\!\!\!\perp X_j | \{X_{k_1}, \dots, X_{k_n}\}$$

- This list determines the set of probability distributions that can be represented



Computing All Independences

COMPUTE ALL THE INDEPENDENCES!



Inference

- Inference: calculating some useful quantity from a joint probability distribution

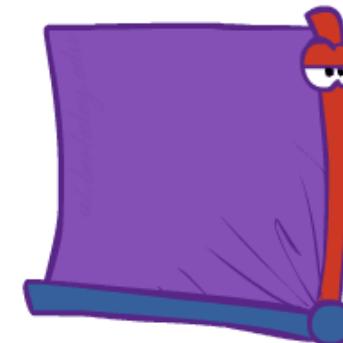
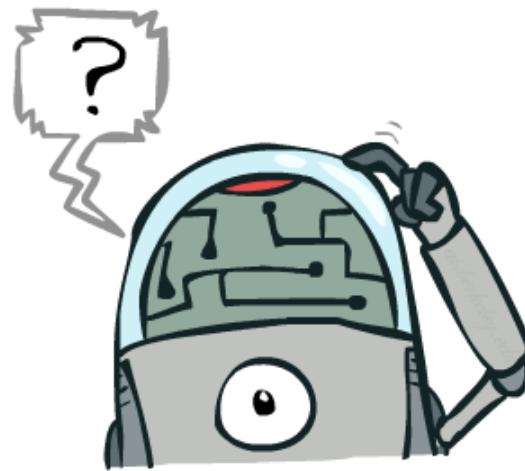
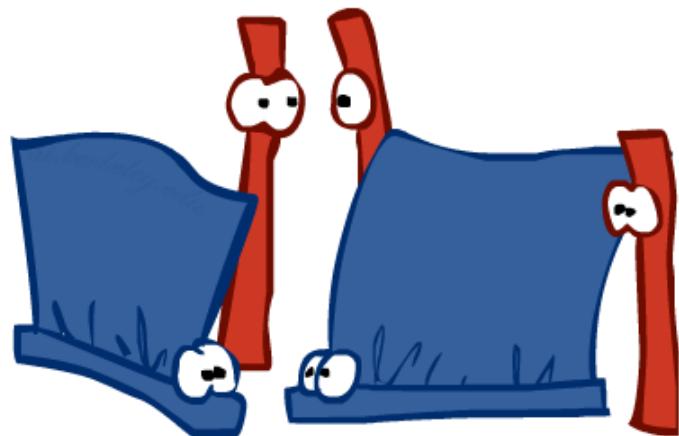
- Examples:

- Posterior probability

$$P(Q|E_1 = e_1, \dots, E_k = e_k)$$

- Most likely explanation:

$$\operatorname{argmax}_q P(Q = q|E_1 = e_1 \dots)$$



Inference by Enumeration

- General case:

- Evidence variables: $E_1 \dots E_k = e_1 \dots e_k$
- Query* variable: Q
- Hidden variables: $H_1 \dots H_r$

$$\left. \begin{array}{l} E_1 \dots E_k = e_1 \dots e_k \\ Q \\ H_1 \dots H_r \end{array} \right\} X_1, X_2, \dots, X_n$$

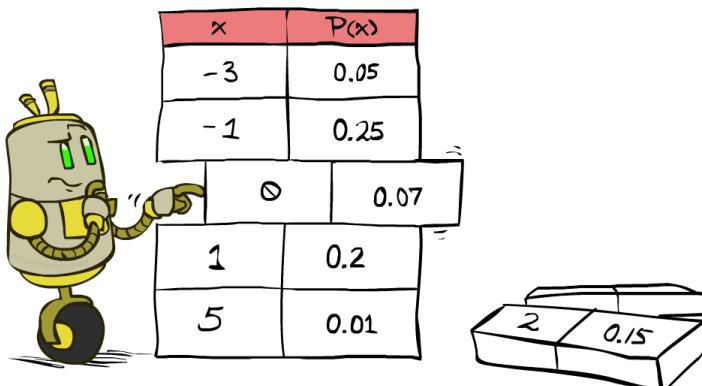
All variables

- We want:

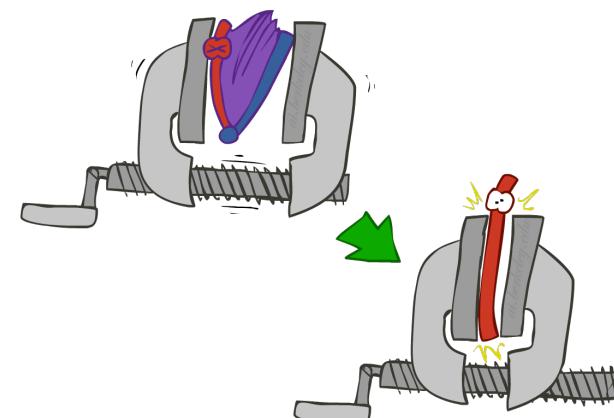
$$P(Q|e_1 \dots e_k)$$

* Works fine with multiple query variables, too

- Step 1: Select the entries consistent with the evidence



- Step 2: Sum out H to get joint of Query and evidence



- Step 3: Normalize

$$\times \frac{1}{Z}$$

$$Z = \sum_q P(Q, e_1 \dots e_k)$$

$$P(Q, e_1 \dots e_k) = \sum_{h_1 \dots h_r} \underbrace{P(Q, h_1 \dots h_r, e_1 \dots e_k)}_{X_1, X_2, \dots, X_n}$$

$$P(Q|e_1 \dots e_k) = \frac{1}{Z} P(Q, e_1 \dots e_k)$$

Inference by Enumeration in Bayes' Net

- Given unlimited time, inference in BNs is easy
- Reminder of inference by enumeration by example:

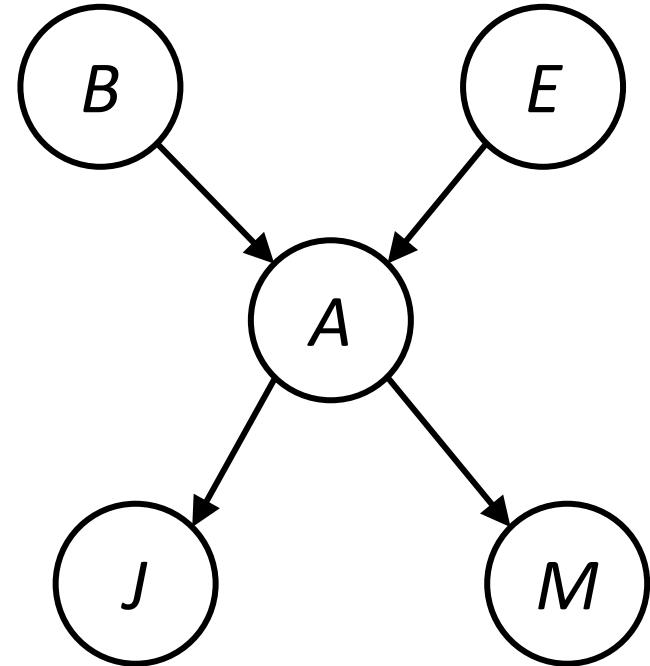
$$P(B \mid +j, +m) \propto_B P(B, +j, +m)$$

$$= \sum_{e,a} P(B, e, a, +j, +m)$$

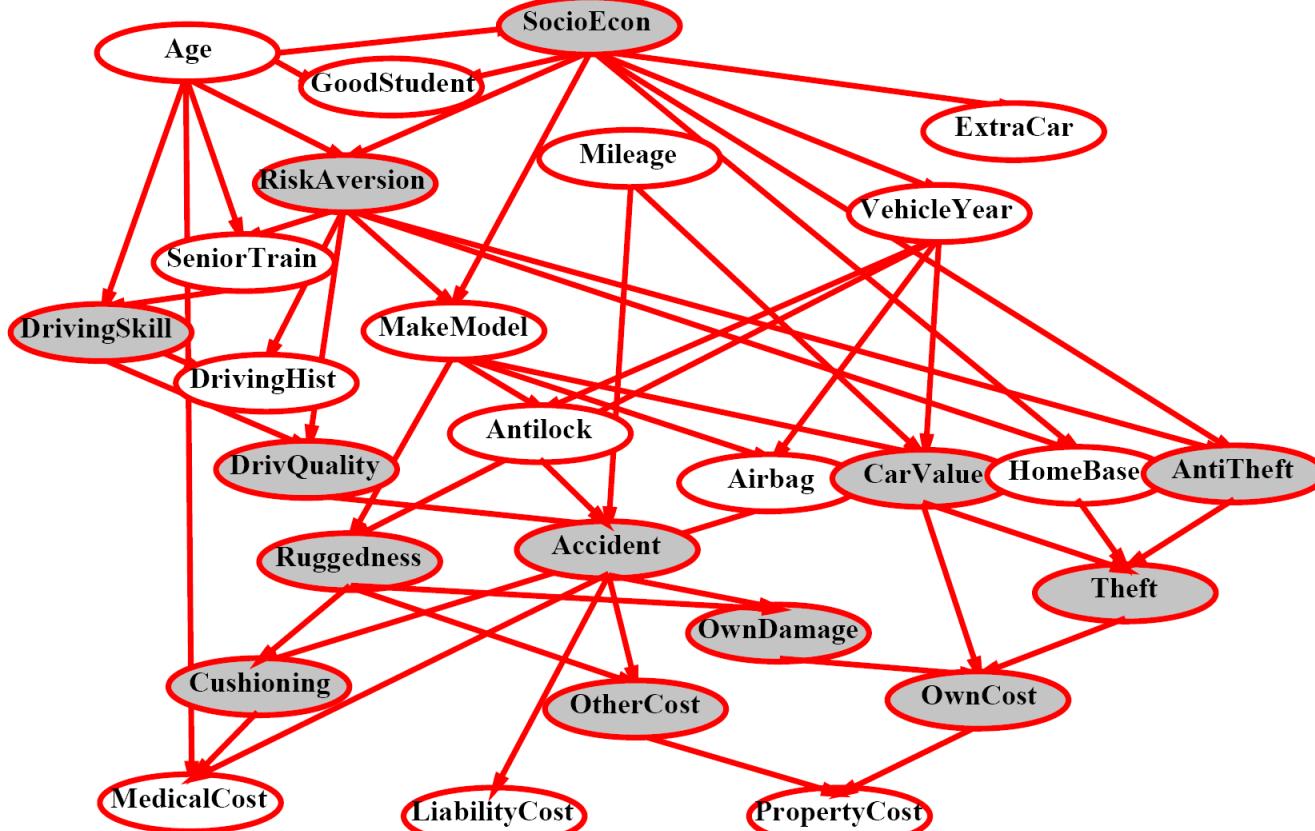
$$= \sum_{e,a} P(B)P(e)P(a|B, e)P(+j|a)P(+m|a)$$

$$= P(B)P(+e)P(+a|B, +e)P(+j|+a)P(+m|+a) + P(B)P(+e)P(-a|B, +e)P(+j|-a)P(+m|-a)$$

$$P(B)P(-e)P(+a|B, -e)P(+j|+a)P(+m|+a) + P(B)P(-e)P(-a|B, -e)P(+j|-a)P(+m|-a)$$



Inference by Enumeration?


$$P(\text{Antilock} | \text{observed variables}) = ?$$

Inference by Enumeration vs. Variable Elimination

- Why is inference by enumeration so slow?
 - You join up the whole joint distribution before you sum out the hidden variables
- Advanced technique: Variable Elimination
 - Interleave joining and marginalizing
 - Still NP-hard, but usually much faster than inference by enumeration
 - See the textbook for a description.