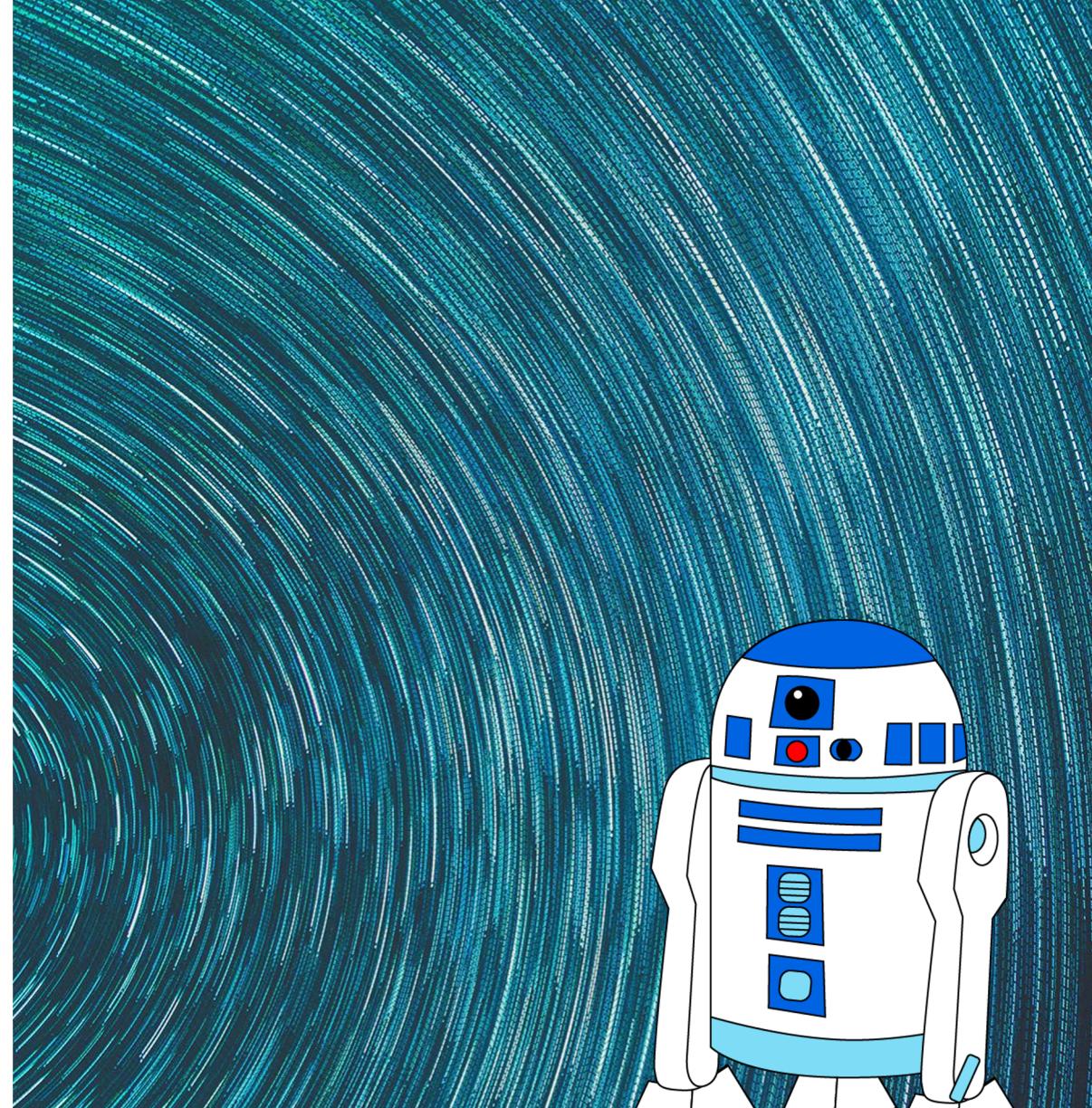


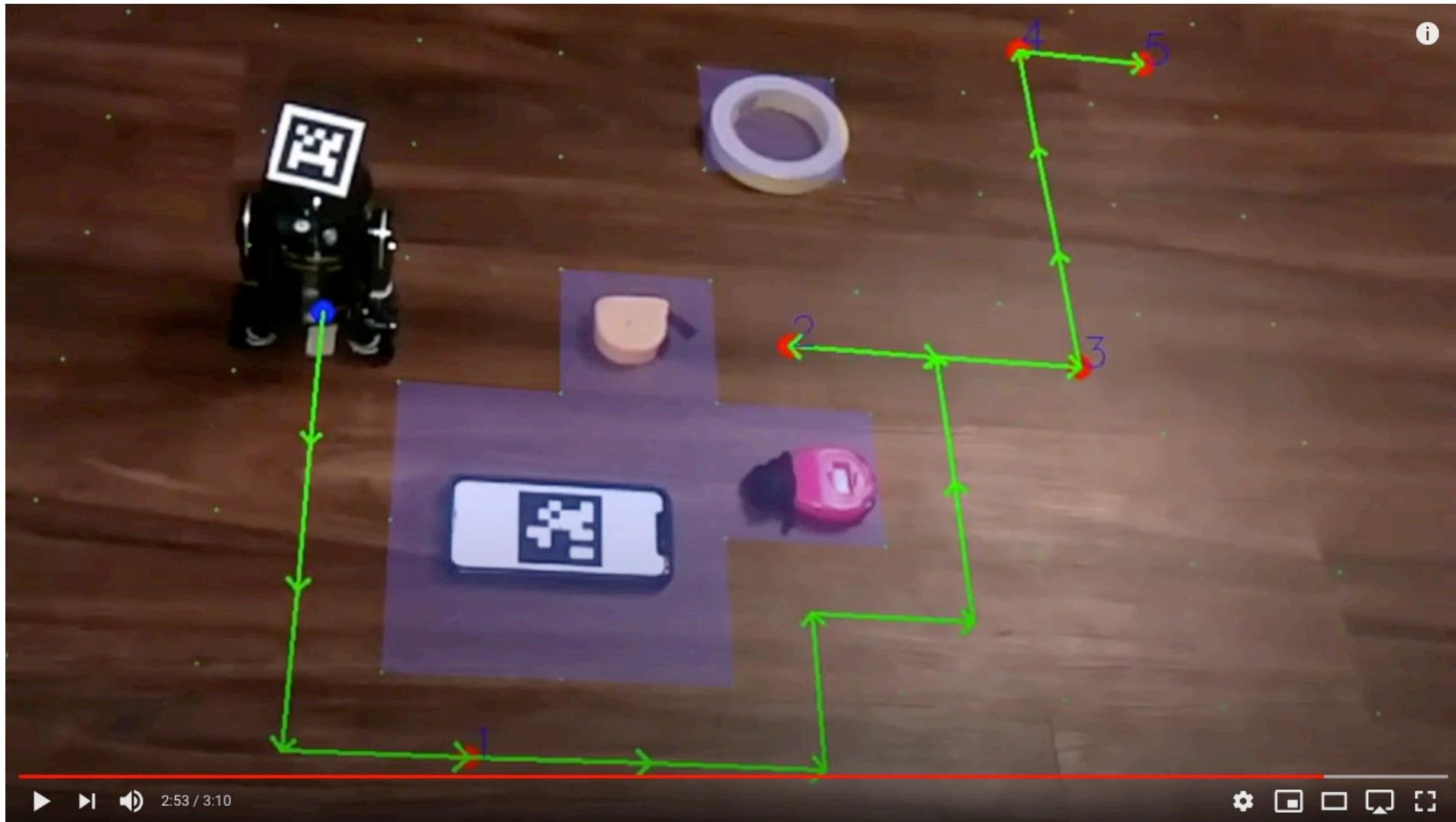
CIS 421/521:
ARTIFICIAL INTELLIGENCE

Logistic Regression

Jurafsky and Martin Chapter 5



R2D2 Extra Credit Highlight Reel



Classifier components

Machine learning classifiers require a training corpus of M observations input/output pairs $(x^{(i)}, y^{(i)})$.

1. A **feature representation** of the input. For each input observation $x^{(i)}$, this will be a vector of features $[x_1, x_2, \dots, x_n]$.
2. A **classification function** that computes the estimated class \hat{y} via $p(y|x)$.
3. An **objective function** for learning, usually involving minimizing error on training examples.
4. An algorithm for **optimizing** the objective function.



Sentiment classifier

Input: "Spiraling away from narrative control as its first three episodes unreel, this series, about a post-apocalyptic future in which nearly everyone is blind, wastes the time of Jason Momoa and Alfre Woodard, among others, on a story that starts from a position of fun, giddy strangeness and drags itself forward at a lugubrious pace."

Output: positive (1) or negative (0)

Sentiment classifier

For sentiment classification, consider an input observation x , represented by a vector of **features** $[x_1, x_2, \dots, x_n]$. The classifier output y can be 1 (positive sentiment) or 0 (negative sentiment). We want to estimate $P(y = 1 | x)$.

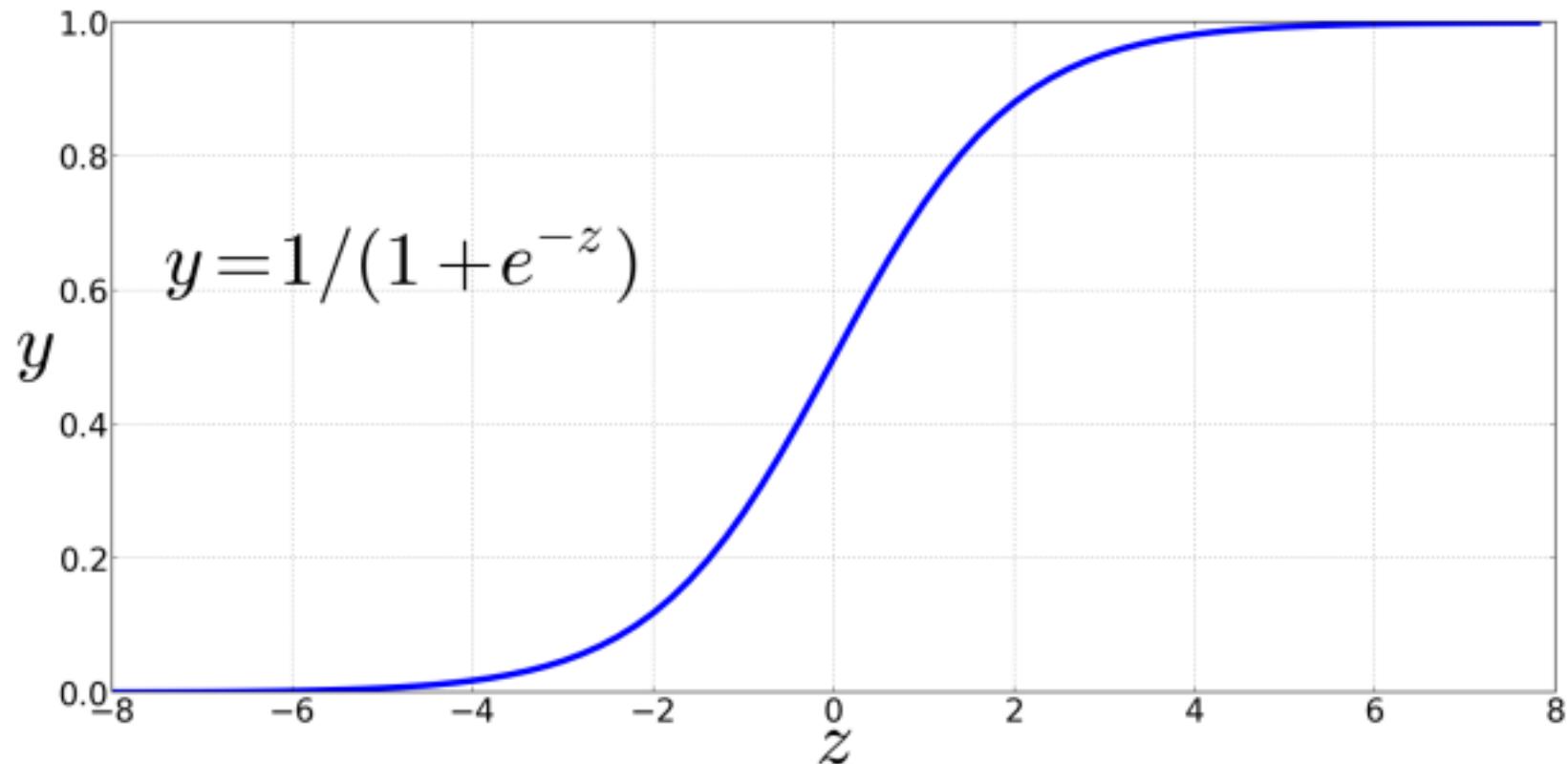
Logistic regression solves this task by learning, from a training set, a vector of **weights** and a **bias term**.

$$z = \sum_i w_i x_i + b$$

We can also write this as a dot product:

$$z = w \cdot x + b$$

Sigmoid function



Probabilities

$$P(y = 1) = \sigma(w \cdot x + b) = \frac{1}{1 + e^{-(w \cdot x + b)}}$$

Decision boundary

Now we have an algorithm that given an instance x computes the probability $P(y = 1 | x)$. How do we make a decision?

$$\hat{y} = \begin{cases} 1 & \text{if } P(y = 1 | x) > 0.5 \\ 0 & \text{otherwise} \end{cases}$$

For a test instance x , we say **yes** if the probability $P(y = 1 | x)$ is more than .5, and **no** otherwise. We call .5 the decision boundary

Extracting Features

It's hokey. There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

| Var | Definition | Value |
|-------|---|-------|
| x_1 | Count of positive lexicon words | |
| x_2 | Count of negative lexicon words | |
| x_3 | Does no appear? (binary feature) | |
| x_4 | Number of 1 st and 2nd person pronouns | |
| x_5 | Does ! appear? (binary feature) | |
| x_6 | Log of the word count for the document | |

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| Var | Definition | Value |
|-------|---|-------|
| x_1 | Count of positive lexicon words | 3 |
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Word count = 64, $\ln(64) = 4.15$

| Var | Definition | Value |
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| x_2 | Count of negative lexicon words | 2 |
| x_3 | Does no appear? (binary feature) | 1 |
| x_4 | Number of 1 st and 2nd person pronouns | 3 |
| x_5 | Does ! appear? (binary feature) | 0 |
| x_6 | Log of the word count for the document | 4.15 |

| Var | Definition | Value | Weight | Product |
|-------|---|-------|--------|---------|
| x_1 | Count of positive lexicon words | 3 | 2.5 | |
| x_2 | Count of negative lexicon words | 2 | -5.0 | |
| x_3 | Does no appear? (binary feature) | 1 | -1.2 | |
| x_4 | Num 1 st and 2 nd person pronouns | 3 | 0.5 | |
| x_5 | Does ! appear? (binary feature) | 0 | 2.0 | |
| x_6 | Log of the word count for the doc | 4.15 | 0.7 | |
| b | bias | 1 | 0.1 | |

$$z = \sum_i w_i x_i + b$$

Computing Z

| Var | Definition | Value | Weight | Product |
|-------|---|-------|--------|---------|
| x_1 | Count of positive lexicon words | 3 | 2.5 | 7.5 |
| x_2 | Count of negative lexicon words | 2 | -5.0 | -10 |
| x_3 | Does no appear? (binary feature) | 1 | -1.2 | -1.2 |
| x_4 | Num 1 st and 2 nd person pronouns | 3 | 0.5 | 1.5 |
| x_5 | Does ! appear? (binary feature) | 0 | 2.0 | 0 |
| x_6 | Log of the word count for the doc | 4.1 | 0.7 | 2.905 |
| b | bias | 1 | 0.1 | 0.1 |

$$z = \sum_i w_i x_i + b$$

z=0.805

Sigmoid(Z)

| Var | Definition | Value | Weight | Product |
|-------|---|-------|--------|---------|
| x_1 | Count of positive lexicon words | 3 | 2.5 | 7.5 |
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| x_4 | Num 1 st and 2 nd person pronouns | 3 | 0.5 | 1.5 |
| x_5 | Does ! appear? (binary feature) | 0 | 2.0 | 0 |
| x_6 | Log of the doc | 7 | | 2.905 |
| b | bias | 1 | | 0.1 |



$$\sigma(0.805) = 0.69$$

Learning in logistic regression

How do we get the weights of the model? We learn the parameters (weights + bias) via learning. This requires 2 components:

1. An objective function or **loss function** that tells us *distance* between the system output and the gold output. We will use **cross-entropy loss**.
2. An algorithm for optimizing the objective function. We will use stochastic gradient descent to **minimize** the **loss function**.

Loss functions

We need to determine for some observation x how close the classifier output ($\hat{y} = \sigma(w \cdot x + b)$) is to the correct output (y , which is 0 or 1).

$L(\hat{y}, y)$ = how much \hat{y} differs from the true y

One example is mean squared error

$$L_{MSE}(\hat{y}, y) = \frac{1}{2}(\hat{y} - y)^2$$

Loss functions for probabilistic classification

We use a loss function that prefers the correct class labels of the training example to be more likely.

Conditional maximum likelihood estimation: Choose parameters w , b that maximize the (log) probabilities of the true labels in the training data.

The resulting loss function is the negative log likelihood loss, more commonly called the **cross entropy loss**.

Loss functions for probabilistic classification

For one observation x , let's **maximize** the probability of the correct label $p(y|x)$.

$$p(y|x) = \hat{y}^y(1 - \hat{y})^{1-y}$$

If $y = 1$, then $p(y|x) = \hat{y}$.

If $y = 0$, then $p(y|x) = 1 - \hat{y}$.

Loss functions for probabilistic classification

Change to logs (still maximizing)

$$\begin{aligned}\log p(y|x) &= \log[\hat{y}^y(1 - \hat{y})^{1-y}] \\ &= y \log \hat{y} + (1 - y) \log(1 - \hat{y})\end{aligned}$$

This tells us what log likelihood should be maximized. But for loss functions, we want to minimize things, so we'll flip the sign.

Cross-entropy loss

The result is cross-entropy loss:

$$L_{CE}(\hat{y}, y) = -\log p(y|x) = -[y \log \hat{y} + (1 - y) \log(1 - \hat{y})]$$

Finally, plug in the definition for $\hat{y} = \sigma(w \cdot x + b)$

$$L_{CE}(\hat{y}, y) = -[y \log \sigma(w \cdot x + b) + (1 - y) \log(1 - \sigma(w \cdot x + b))]$$

Cross-entropy loss

Why does minimizing this negative log probability do what we want? We want the **loss** to be **smaller** if the model's estimate is **close to correct**, and we want the **loss** to be **bigger** if it is **confused**.

It's hokey. There are virtually no surprises , and the writing is second-rate . So why was it so enjoyable? For one thing , the cast is great . Another nice touch is the music . I was overcome with the urge to get off the couch and start dancing . It sucked me in , and it'll do the same to you .

$P(\text{sentiment}=1 | \text{It's hokey...}) = 0.69.$ Let's say $y=1.$

$$\begin{aligned} L_{CE}(\hat{y}, y) &= -[y \log \sigma(w \cdot x + b) + (1 - y) \log(1 - \sigma(w \cdot x + b))] \\ &= -[\log \sigma(w \cdot x + b)] \\ &= -\log(0.69) = 0.37 \end{aligned}$$

Cross-entropy loss

Why does minimizing this negative log probability do what we want? We want the **loss** to be **smaller** if the model's estimate is **close to correct**, and we want the **loss** to be **bigger** if it is **confused**.

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$P(\text{sentiment}=1 | \text{It's hokey...}) = 0.69$. Let's pretend $y=0$.

$$\begin{aligned} L_{CE}(\hat{y}, y) &= -[y \log \sigma(w \cdot x + b) + (1 - y) \log(1 - \sigma(w \cdot x + b))] \\ &= -[\log(1 - \sigma(w \cdot x + b))] \\ &= -\log(0.31) = 1.17 \end{aligned}$$

Cross-entropy loss

Why does minimizing this negative log probability do what we want? We want the **loss** to be **smaller** if the model's estimate is **close to correct**, and we want the **loss** to be **bigger** if it is **confused**.

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If our prediction is **correct**, then our CE loss is **lower**

$$= -\log (0.69) = \mathbf{0.37}$$

If our prediction is **incorrect**, then our CE loss is **higher**

$$-\log (0.31) = \mathbf{1.17}$$

Loss on all training examples

$$\begin{aligned}\log p(\text{training labels}) &= \log \prod_{i=1}^m p(y^{(i)}|x^{(i)}) \\ &= \sum_{i=1}^m \log p(y^{(i)}|x^{(i)}) \\ &= -\sum_{i=1}^m \text{L}_{\text{CE}}(\hat{y}^{(i)}|y^{(i)})\end{aligned}$$

Finding good parameters

We use gradient descent to find good settings for our weights and bias by minimizing the loss function.

$$\hat{\theta} = \operatorname{argmin}_{\theta} \frac{1}{m} \sum_{i=1}^m L_{CE}(y^{(i)}, x^{(i)}; \theta)$$

Gradient descent is a method that finds a minimum of a function by figuring out in which direction (in the space of the parameters θ) the function's slope is rising the most steeply, and moving in the opposite direction.

Gradient descent



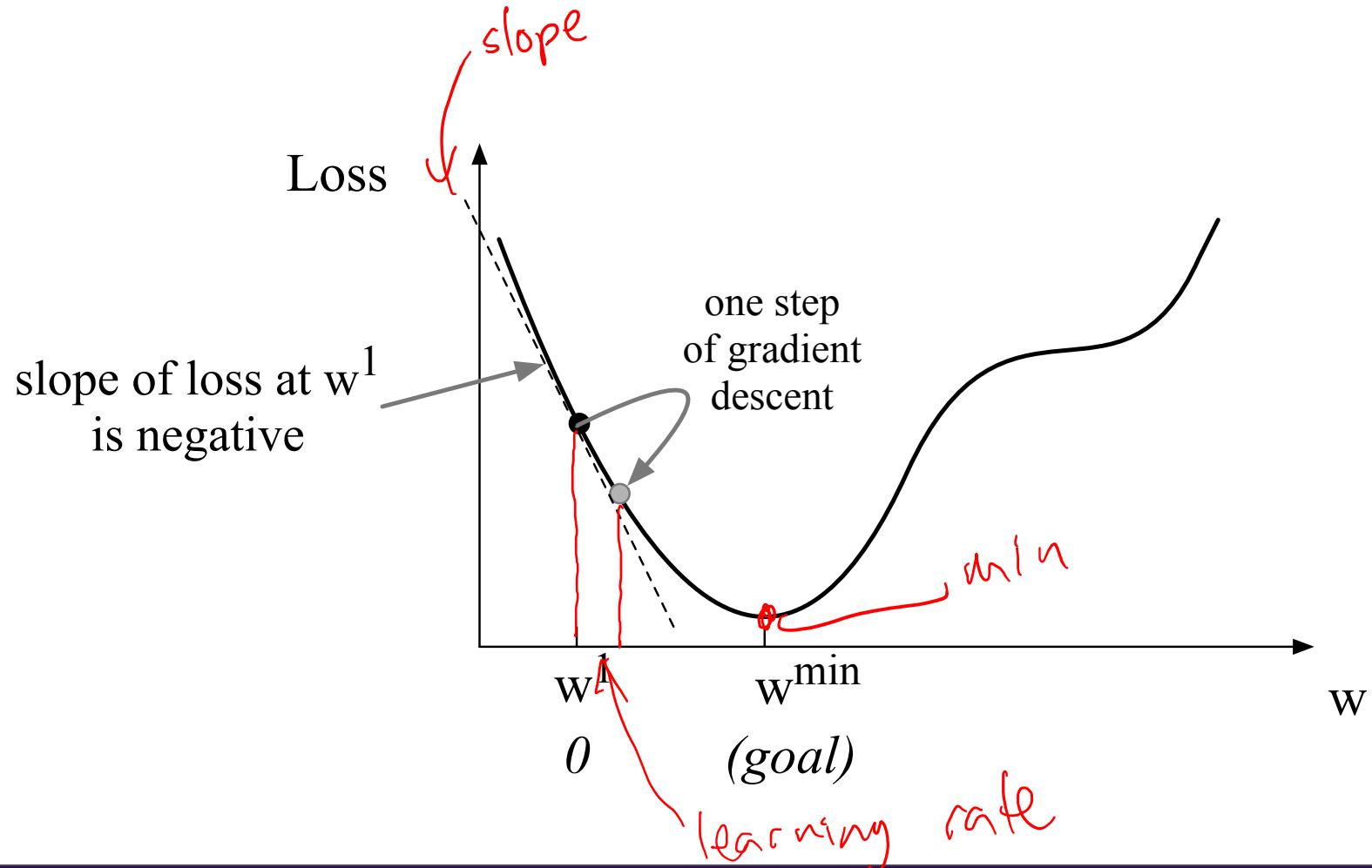
Global v. Local Minimums

For logistic regression, this loss function is conveniently **convex**.

A convex function has just **one minimum**, so there are no local minima to get stuck in.

So gradient descent starting from any point is guaranteed to find the minimum.

Iteratively find minimum



How much should we update the parameter by?

The magnitude of the amount to move in gradient descent is the value of the slope weighted by a learning rate η .

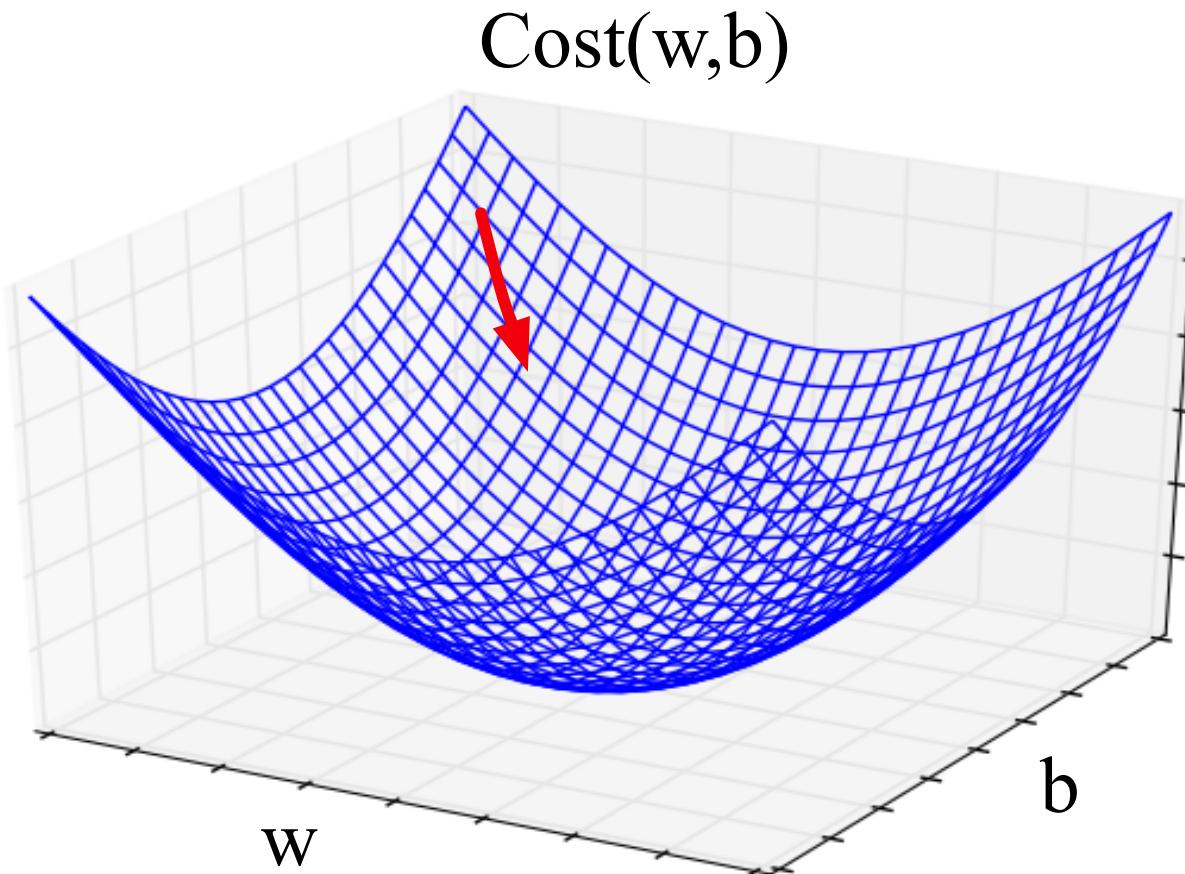
A higher/faster learning rate means that we should move w more on each step.

$$w^{t+1} = w^t - \eta \frac{d}{dw} f(x; w)$$

Annotations in red:

- time $t + 1$ above w^{t+1}
- "new weight" next to w^{t+1}
- "old weight" next to w^t
- "minus" between w^t and the derivative term
- "slope" above the derivative term $\frac{d}{dw} f(x; w)$
- "learning rate" next to η
- "derivative of f_a " next to the derivative term $\frac{d}{dw} f(x; w)$

Many dimensions



Updating each dimension w_i

$$\nabla_{\theta} L(f(x; \theta), y) = \begin{bmatrix} \frac{\partial}{\partial w_1} L(f(x; \theta), y) \\ \frac{\partial}{\partial w_2} L(f(x; \theta), y) \\ \vdots \\ \frac{\partial}{\partial w_n} L(f(x; \theta), y) \end{bmatrix}$$

our model's prediction for input x given parameters θ

The final equation for updating θ based on the gradient is $\theta_{t+1} = \theta_t - \eta \nabla L(f(x; \theta), y)$

Learning rate

The Gradient

To update θ , we need a definition for the gradient $\nabla L(f(x; \theta), y)$.

For logistic regression, the cross-entropy loss function is:

$$L_{CE}(w, b) = -[y \log \sigma(w \cdot x + b) + (1 - y) \log (1 - \sigma(w \cdot x + b))]$$

The derivative of this function for one observation vector x for a single weight w_j is

$$\frac{\partial L_{CE}(w, b)}{\partial w_j} = [\underbrace{\sigma(w \cdot x + b) - y}_{\text{our model's prediction}}] x_j \quad \begin{matrix} \leftarrow & * \text{ value of } \\ & \text{feature } j \end{matrix}$$

The gradient is a very intuitive value: the difference between the true y and our estimate for x , multiplied by the corresponding input value x_j .

Average Loss

$$\begin{aligned}Cost(w, b) &= \frac{1}{m} \sum_{i=1}^m L_{CE}(\hat{y}^{(i)}, y^{(i)}) \\&= -\frac{1}{m} \sum_{i=1}^m y^{(i)} \log \sigma(w \cdot x^{(i)} + b) + (1 - y^{(i)}) \log(1 - \sigma(w \cdot x^{(i)} + b))\end{aligned}$$

This is what we want to minimize!!

The Gradient

The loss for a batch of data or an entire dataset is just the average loss over the m examples

$$Cost(w, b) = -\frac{1}{m} \sum_{i=1}^m y^{(i)} \log \sigma(w \cdot x^{(i)} + b) + (1 - y^{(i)}) \log(1 - \sigma(w \cdot x^{(i)} + b))$$

The gradient for multiple data points is the sum of the individual gradients:

$$\frac{\partial Cost(w, b)}{\partial w_j} = \sum_{i=1}^m [\sigma(w \cdot x^{(i)} + b) - y^{(i)}] x_j^{(i)}$$

Stochastic gradient descent algorithm

"

```
function STOCHASTIC GRADIENT DESCENT( $L()$ ,  $f()$ ,  $x$ ,  $y$ ) returns  $\theta$ 
    # where: L is the loss function
    #      f is a function parameterized by  $\theta$ 
    #      x is the set of training inputs  $x^{(1)}$ ,  $x^{(2)}$ , ...,  $x^{(n)}$ 
    #      y is the set of training outputs (labels)  $y^{(1)}$ ,  $y^{(2)}$ , ...,  $y^{(n)}$ 

     $\theta \leftarrow 0$ 
    repeat T times
        For each training tuple  $(x^{(i)}, y^{(i)})$  (in random order)
            Compute  $\hat{y}^{(i)} = f(x^{(i)}; \theta)$  # What is our estimated output  $\hat{y}$ ?
            Compute the loss  $L(\hat{y}^{(i)}, y^{(i)})$  # How far off is  $\hat{y}^{(i)}$  from the true output  $y^{(i)}$ ?
             $g \leftarrow \nabla_{\theta} L(f(x^{(i)}; \theta), y^{(i)})$  # How should we move  $\theta$  to minimize loss ?
             $\theta \leftarrow \theta - \eta g$  # go the other way instead
    return  $\theta$ 
```

Worked example

Let's walk through a single step of the gradient descent algorithm. We'll use a simple sentiment classifier with just 2 features, and 1 training instance where the correct value is $y = 1$ (this is a positive review).

$x_1 = 3$ (count of positive lexicon words)

$x_2 = 2$ (count of positive negative words)

The initial weights and bias in θ^0 are all set to 0, and the initial learning rate η is 0.1:

$w_1 = w_2 = b = 0$

$\eta = 0.1$

The single update step requires that we compute the gradient, multiplied by the learning rate:

$$\theta^{t+1} = \theta^t - \eta \nabla_{\theta} L(f(x^{(i)}; \theta), y^{(i)})$$

Worked example

The derivative of this function for **a single training example** x for a single weight w_j is

$$\frac{\partial L_{CE}(w, b)}{\partial w_j} = [s(w \cdot x + b) - y]x_j$$

The gradient vector has 3 dimensions, for w_1 , w_2 , and b .

For our input, $x_1 = 3$ and $x_2 = 2$

$$x_2 = 2$$

$$\nabla_{w,b} = \begin{bmatrix} \frac{\partial L_{CE}(w,b)}{\partial w_1} \\ \frac{\partial L_{CE}(w,b)}{\partial w_2} \\ \frac{\partial L_{CE}(w,b)}{\partial b} \end{bmatrix} = \begin{bmatrix} (\sigma(w \cdot x + b) - y)x_1 \\ (\sigma(w \cdot x + b) - y)x_2 \\ \sigma(w \cdot x + b) - y \end{bmatrix} = \begin{bmatrix} (\sigma(0) - 1)x_1 \\ (\sigma(0) - 1)x_2 \\ \sigma(0) - 1 \end{bmatrix} = \begin{bmatrix} -0.5x_1 \\ -0.5x_2 \\ -0.5 \end{bmatrix} = \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix}$$

Worked example

Now that we have a gradient $\nabla_{w,b}$, we compute the new parameter vector θ^1 by moving θ^0 in the opposite direction from the gradient:

$$\theta^1 = \begin{bmatrix} w_1 \\ w_2 \\ b \end{bmatrix} - \eta \begin{bmatrix} -1.5 \\ -1.0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} .15 \\ .1 \\ .05 \end{bmatrix}$$

So after one step of gradient descent, the weights have shifted to be:

$$w_1 = 0.15, w_2 = 0.1, \text{ and } b = .05$$

Mini-batch training

Stochastic gradient descent chooses a **single random example** at a time and updates its weights on that example. As a result the updates can fluctuate.

An alternate is **batch training**, which computes the gradient over the **entire dataset**. This gives a much better estimate of which direction to move the weights but takes a long time to compute.

A commonly used compromise is **mini-batch training**, where we train on a small batch. The batch size can be 512 or 1024, often selected based on computational resources, so that all examples in the mini-batch can be processed in parallel. The loss is then accumulated.

Regularization

Overfitting is a problem with many machine learning models. Overfitting results in poor generalization and poor performance on unseen test set.

In logistic regression, if a feature only occurs in one class then it will get a **high weight**. Sometimes we are just modelling noisy factors that just accidentally correlate with the class.

Regularization is a way to penalize large weights. A regularization term is added to the loss function.

Lasso regression uses L1 regularization

Ridge regression uses L2 regularization

Multinomial logistic regression

Instead of binary classification, we often want more than two classes. For sentiment classification we might extend the class labels to be **positive**, **negative**, and **neutral**.

We want to know the probability of y for each class $c \in C$, $p(y = c | x)$.

To get a proper probability, we will use a **generalization of the sigmoid function** called the **softmax function**.

$$\text{softmax}(z_i) = \frac{e^{z_j}}{\sum_{j=1}^k e^{z_j}} \quad 1 \leq i \leq k$$

Softmax

The softmax function takes in an input vector $z = [z_1, z_2, \dots, z_k]$ and outputs a vector of values normalized into probabilities.

$$\text{softmax}(z) = \left[\frac{e^{z_1}}{\sum_{i=1}^k e^{z_i}}, \frac{e^{z_2}}{\sum_{i=1}^k e^{z_i}}, \dots, \frac{e^{z_k}}{\sum_{i=1}^k e^{z_i}} \right]$$

For example, for this input:

$$z = [0.6, 1.1, -1.5, 1.2, 3.2, -1.1]$$

Softmax will output:

$$[0.056, 0.090, 0.007, 0.099, 0.74, 0.010]$$

Next time: Neural Nets

