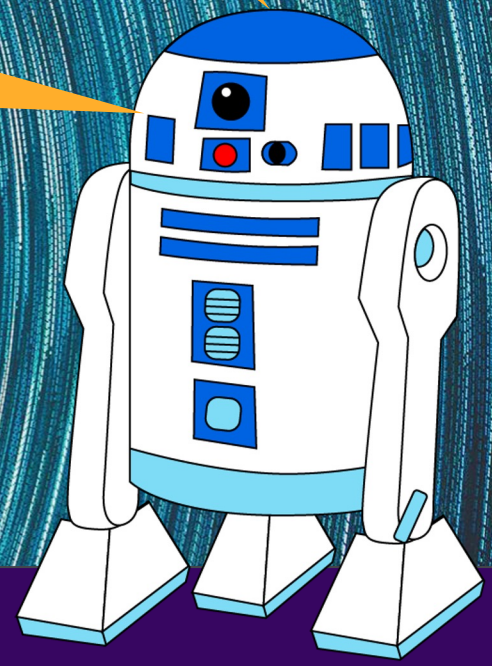
CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

Markov Decision Processes

The date for Midterm 2 has been set. It is Dec 19.

HW5 is due Oct 25



Navigating an Asteroid Field

Suppose we have a **fully-observable** 4x3 environment with goal states.

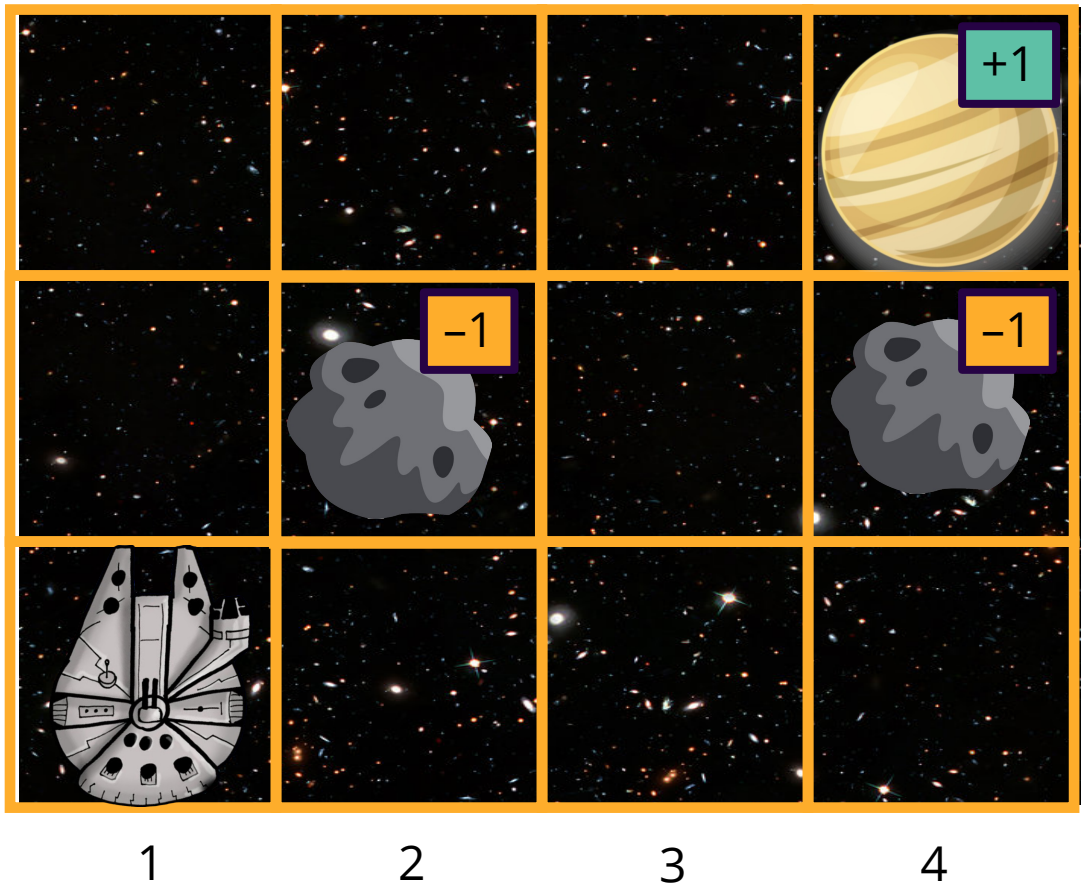
The millennium falcon begins in the start state and **picks an action at each time step**.

Actions: *Up, Down, Left, Right*

The game **terminates when it reaches a goal state** (+1 or -1).

If the environment were **deterministic**, the solution would be easy:

[*Up, Up, Right, Right, Right*]



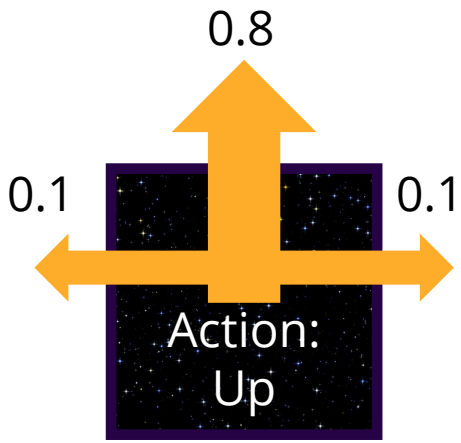
Navigating an Asteroid Field

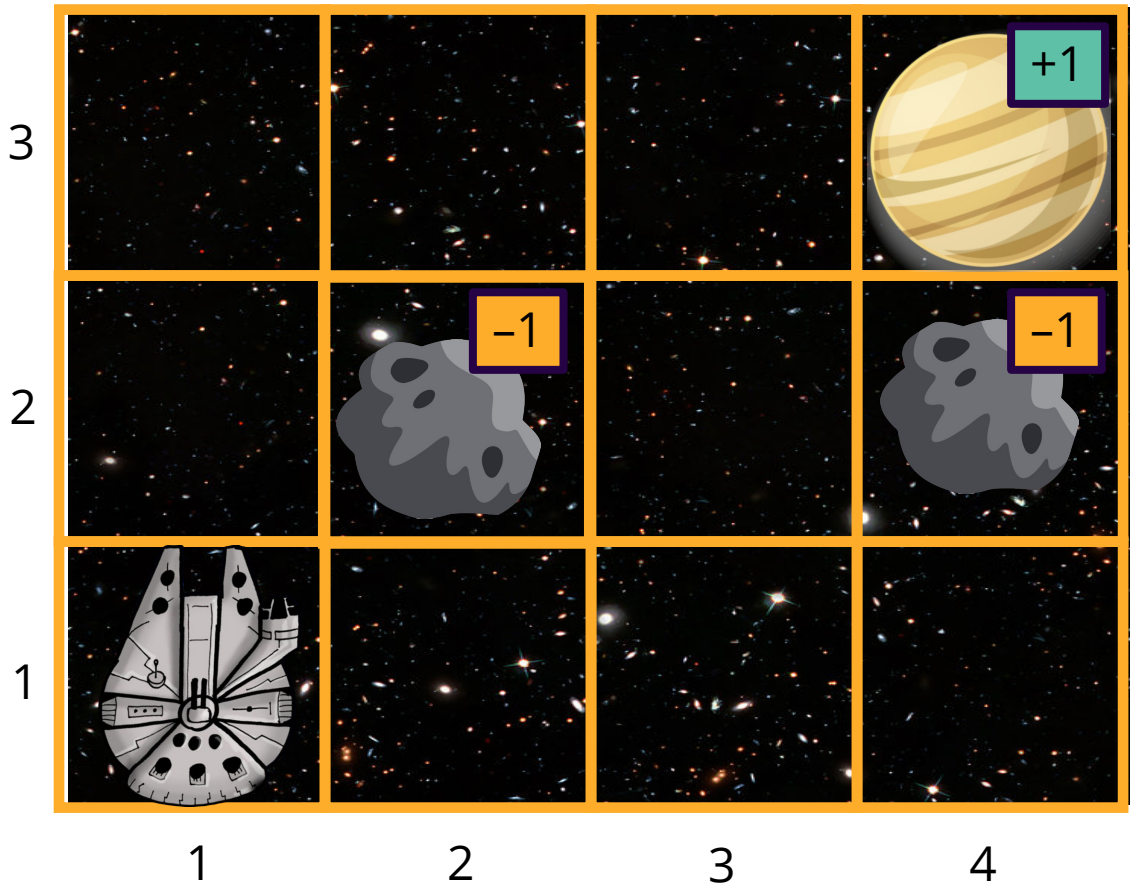
Instead of making the environment deterministic, we will make it **stochastic**.

If the Falcon selects the action *Up* then it only moves up 80% of the time.

10% of the time the weird gravity fields cause it to veer off to the left or right.

Transition Model:





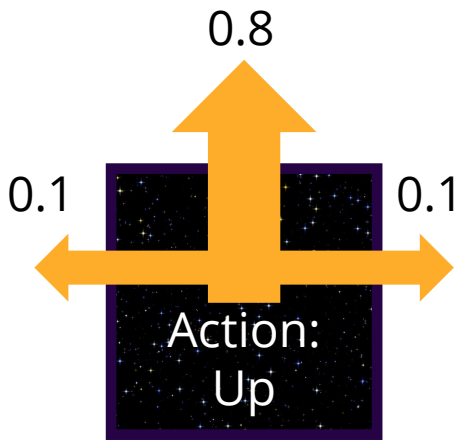
Navigating an Asteroid Field

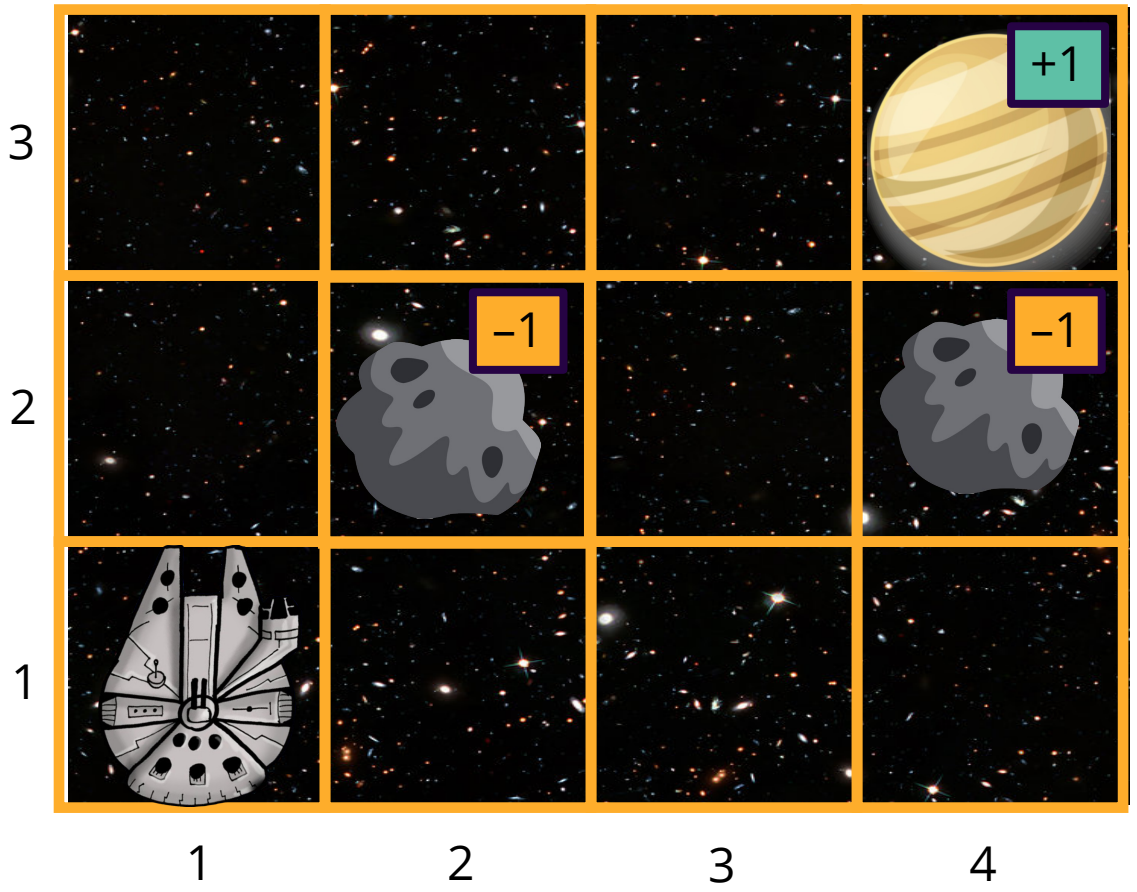
For action sequence

[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

Transition Model:





Navigating an Asteroid Field

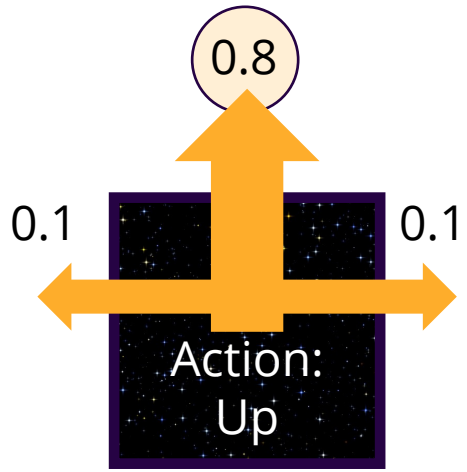
For action sequence

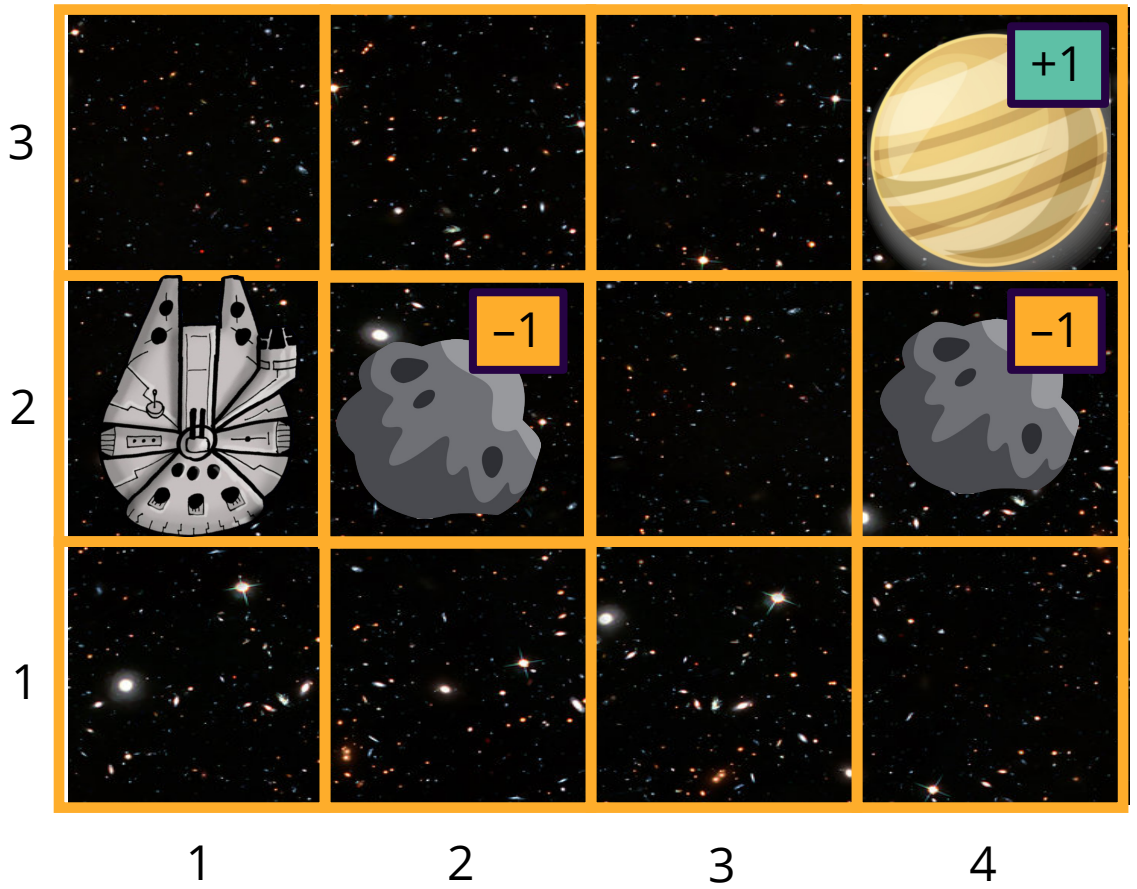
[*Up*, *Up*, *Right*, *Right*, *Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.8

Transition Model:





Navigating an Asteroid Field

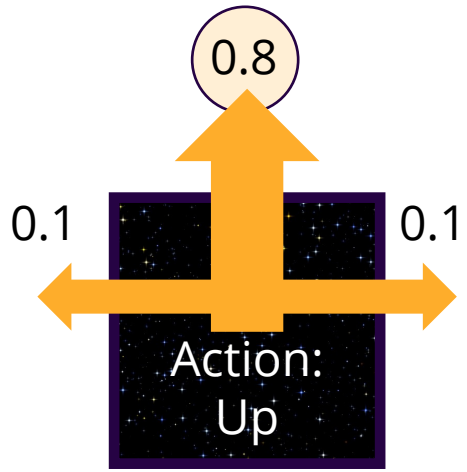
For action sequence

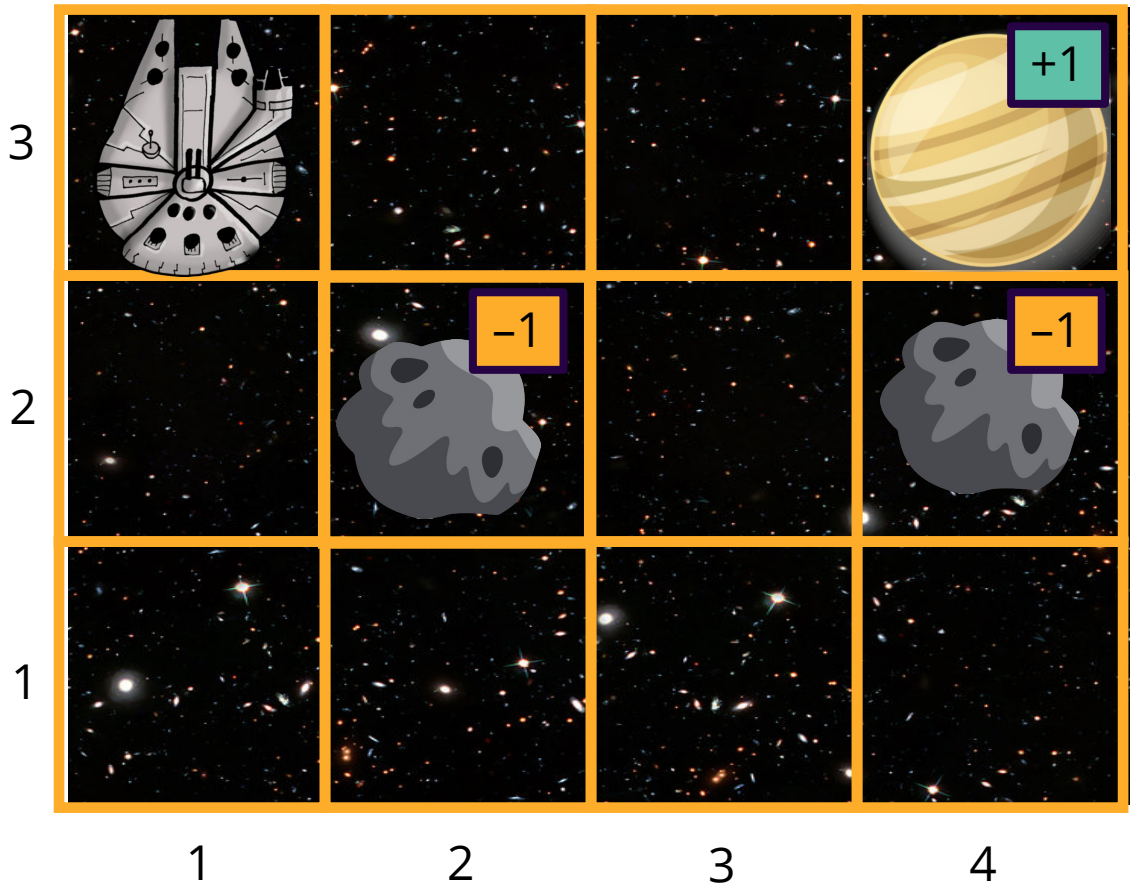
[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.8 * 0.8

Transition Model:





Navigating an Asteroid Field

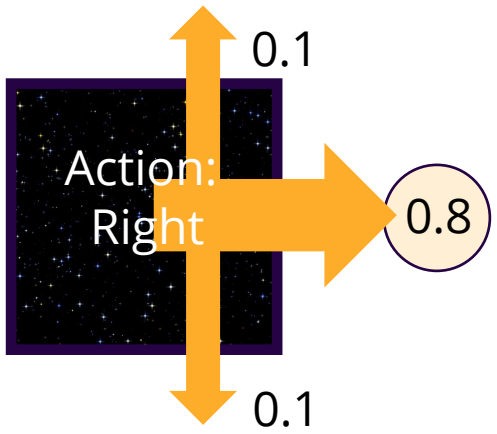
For action sequence

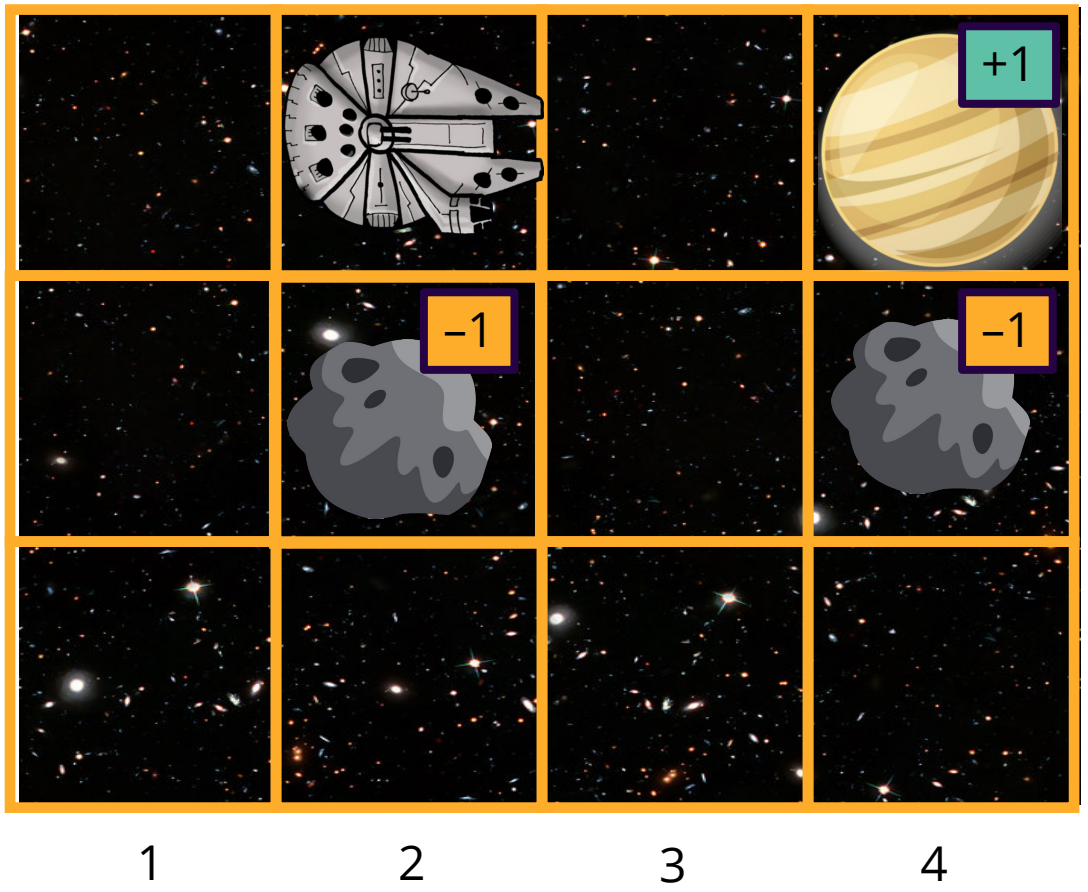
[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.8 * 0.8 * 0.8

Transition Model:





Navigating an Asteroid Field

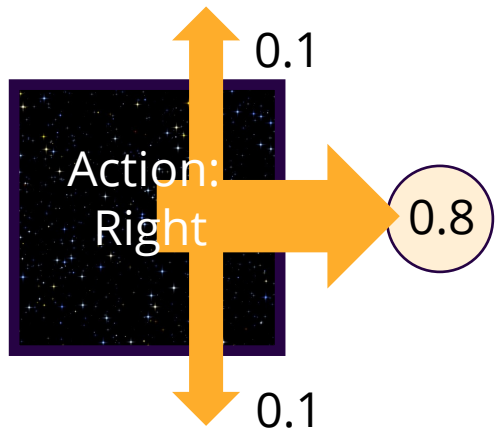
For action sequence

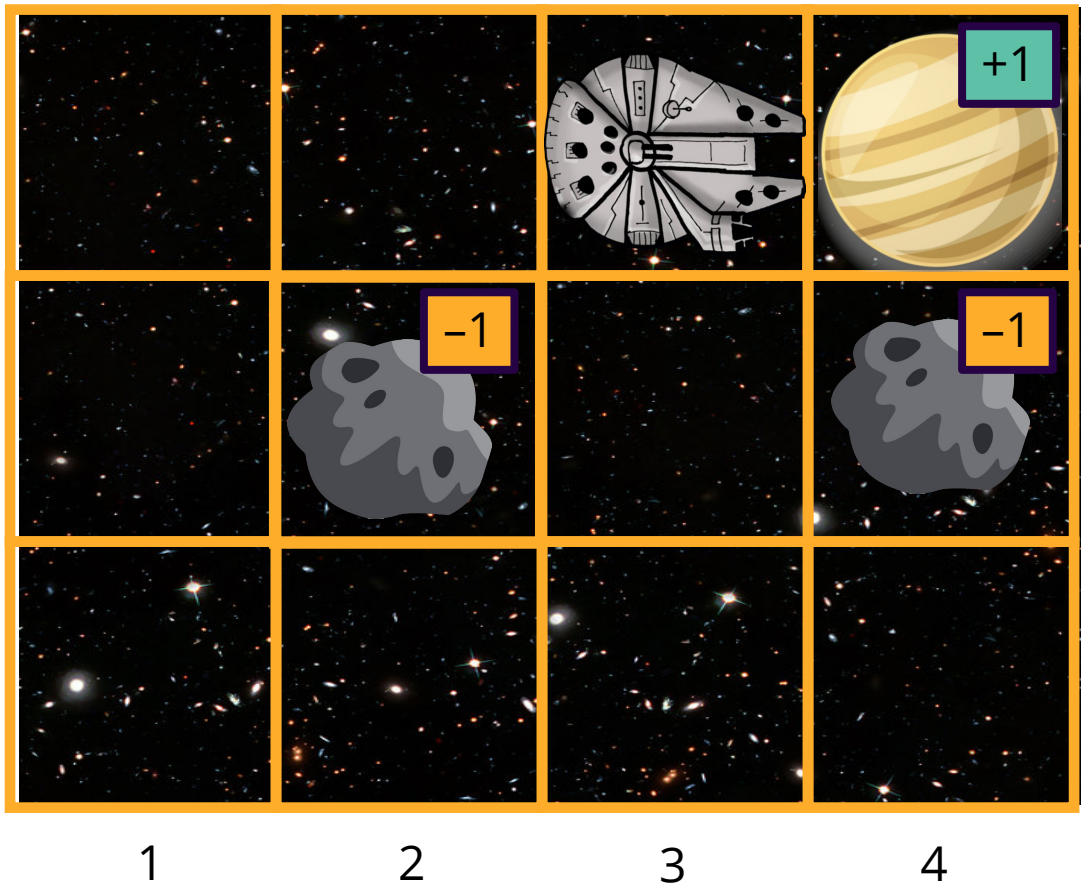
[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.8 * 0.8 * 0.8 * 0.8

Transition Model:





Navigating an Asteroid Field

For action sequence

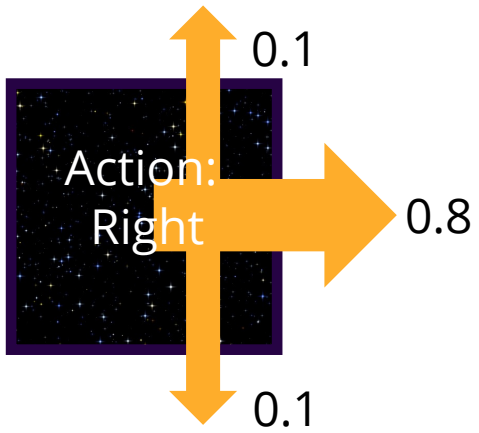
[*Up, Up, Right, Right, Right*],

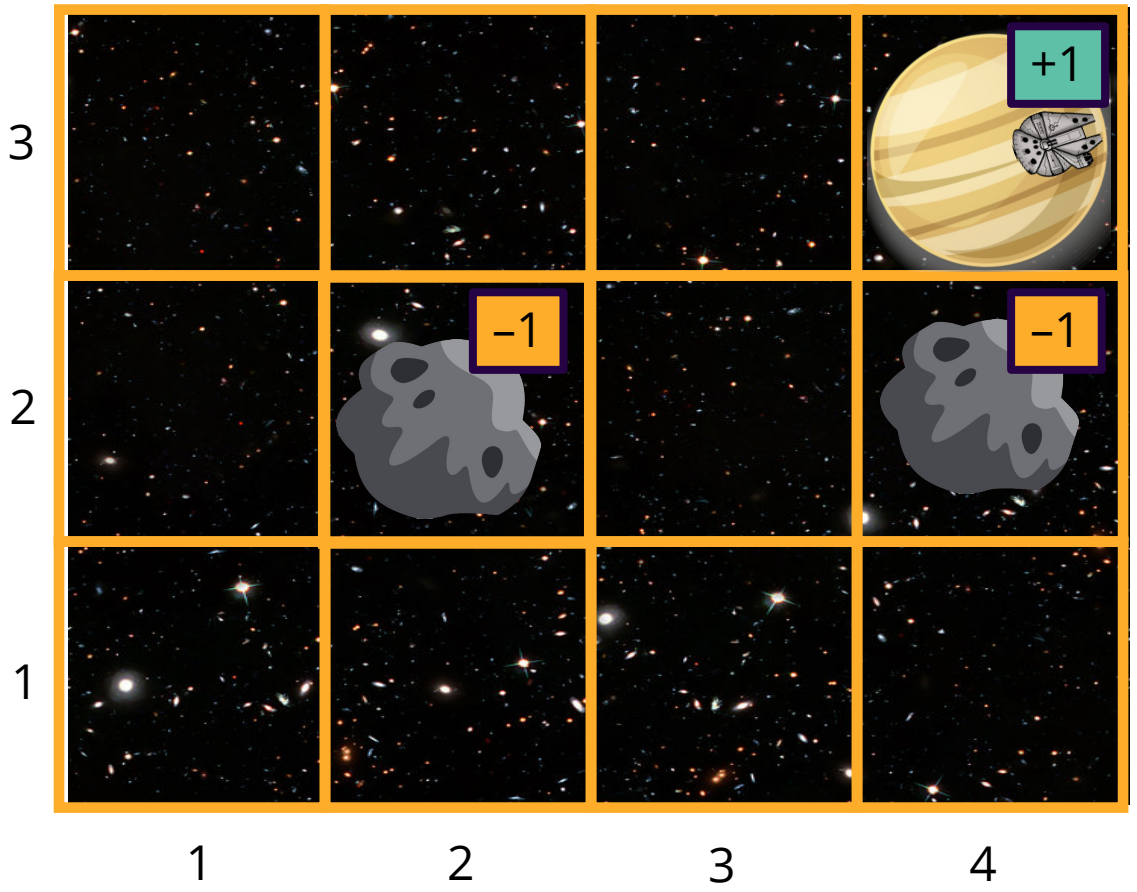
what's the probability that the millennium falcon reaches the intended goal?

0.8 * 0.8 * 0.8 * 0.8 * 0.8

= 0.32768

Transition Model:





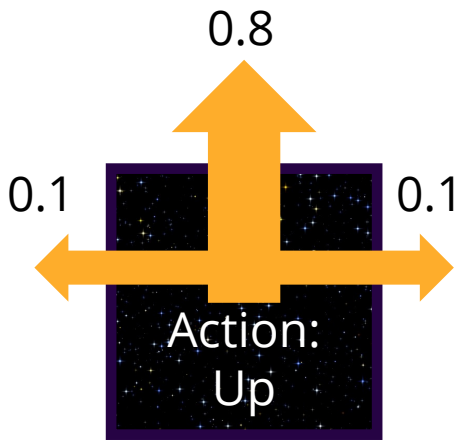
Navigating an Asteroid Field

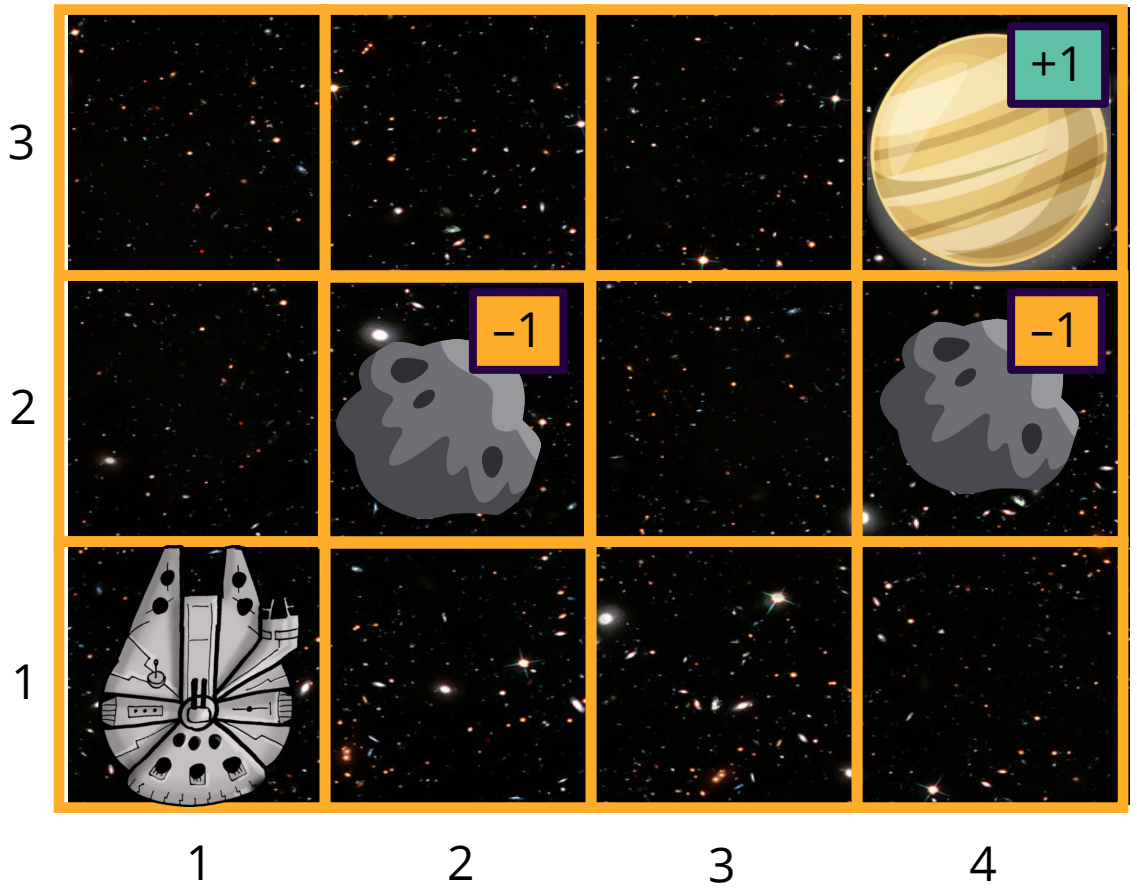
For action sequence

[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

Transition Model:





Navigating an Asteroid Field

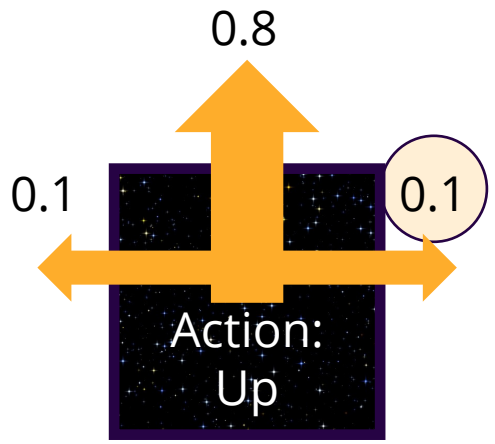
For action sequence

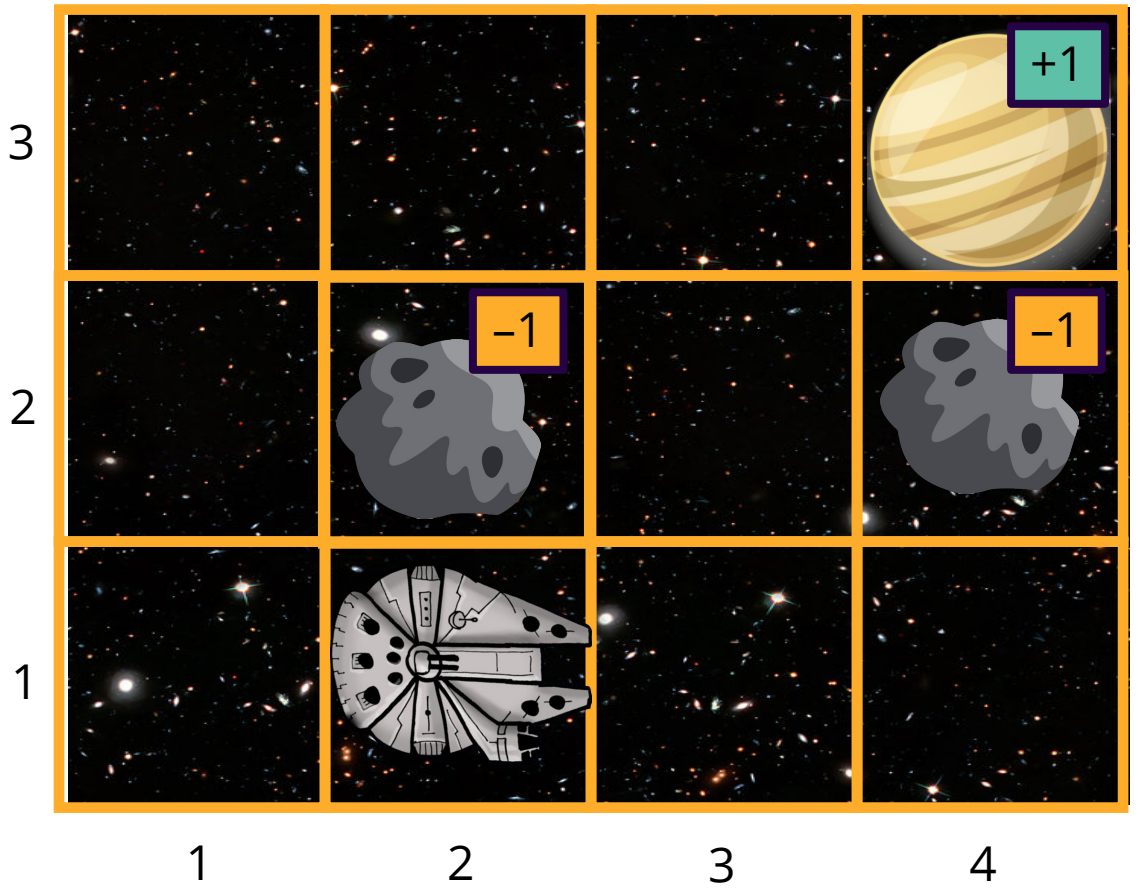
[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.1

Transition Model:





Navigating an Asteroid Field

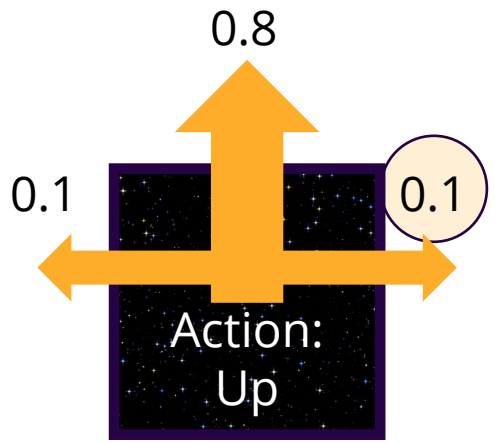
For action sequence

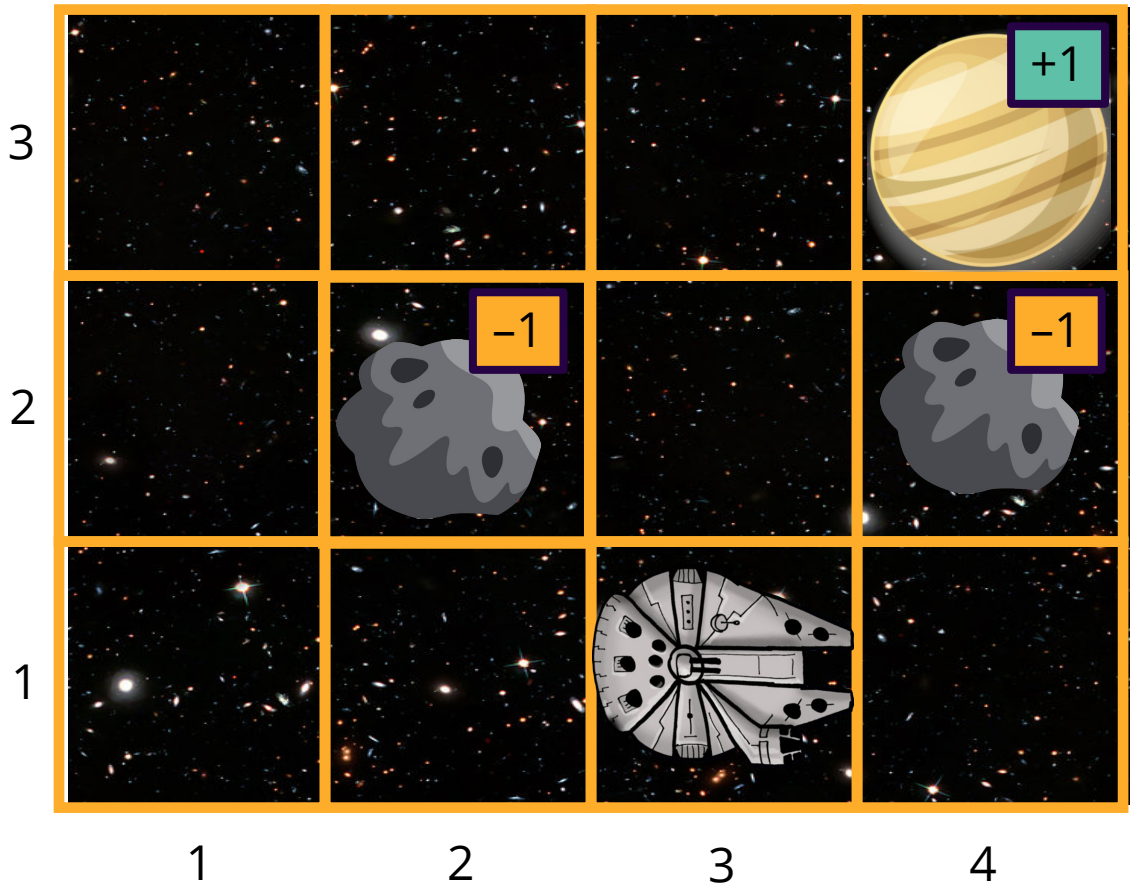
[*Up*, *Up*, *Right*, *Right*, *Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.1 * 0.1

Transition Model:





Navigating an Asteroid Field

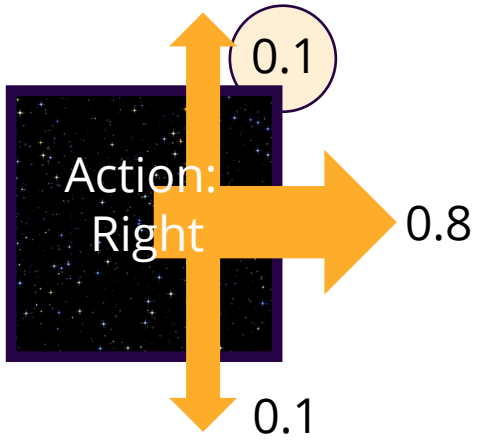
For action sequence

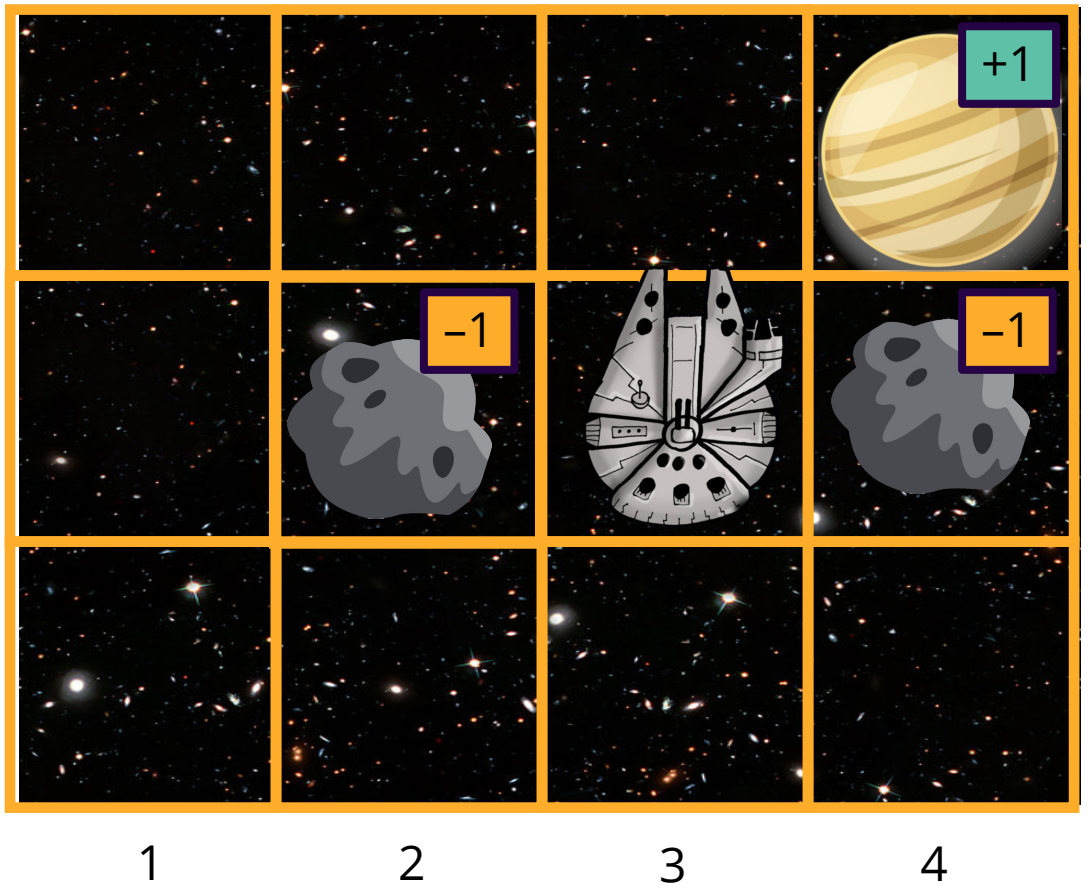
[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.1 * 0.1 * 0.1

Transition Model:





Navigating an Asteroid Field

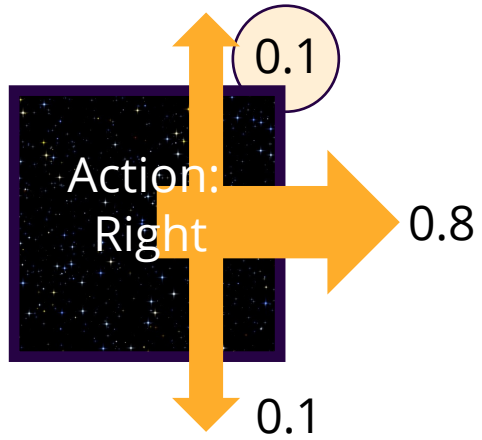
For action sequence

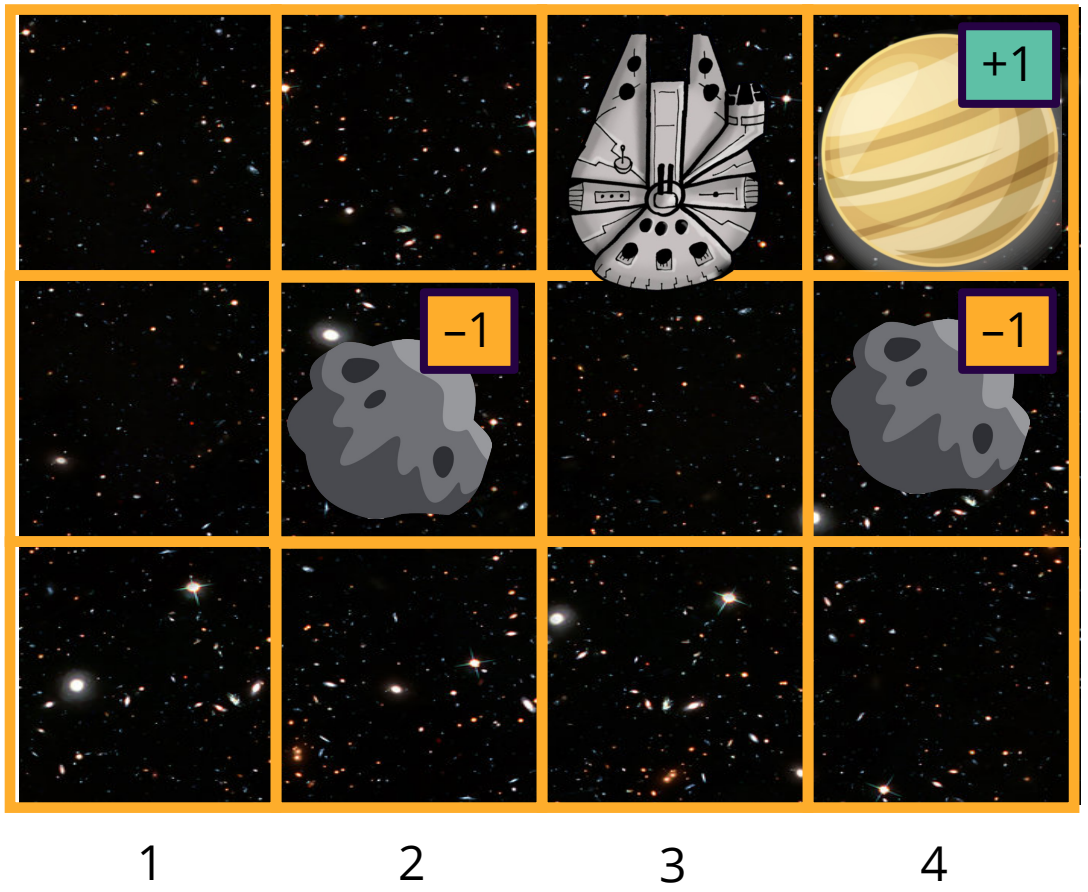
[*Up, Up, Right, Right, Right*],

what's the probability that the millennium falcon reaches the intended goal?

0.1 * 0.1 * 0.1 * 0.1

Transition Model:





Navigating an Asteroid Field

For action sequence

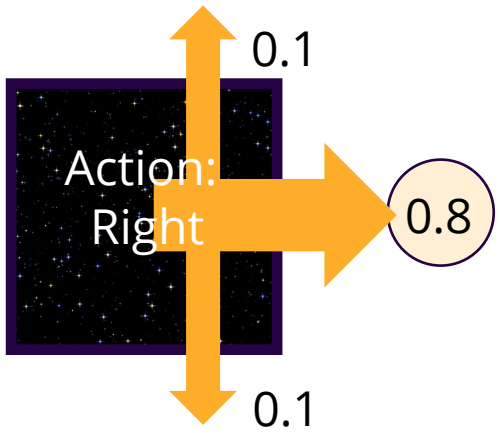
[*Up, Up, Right, Right, Right*],

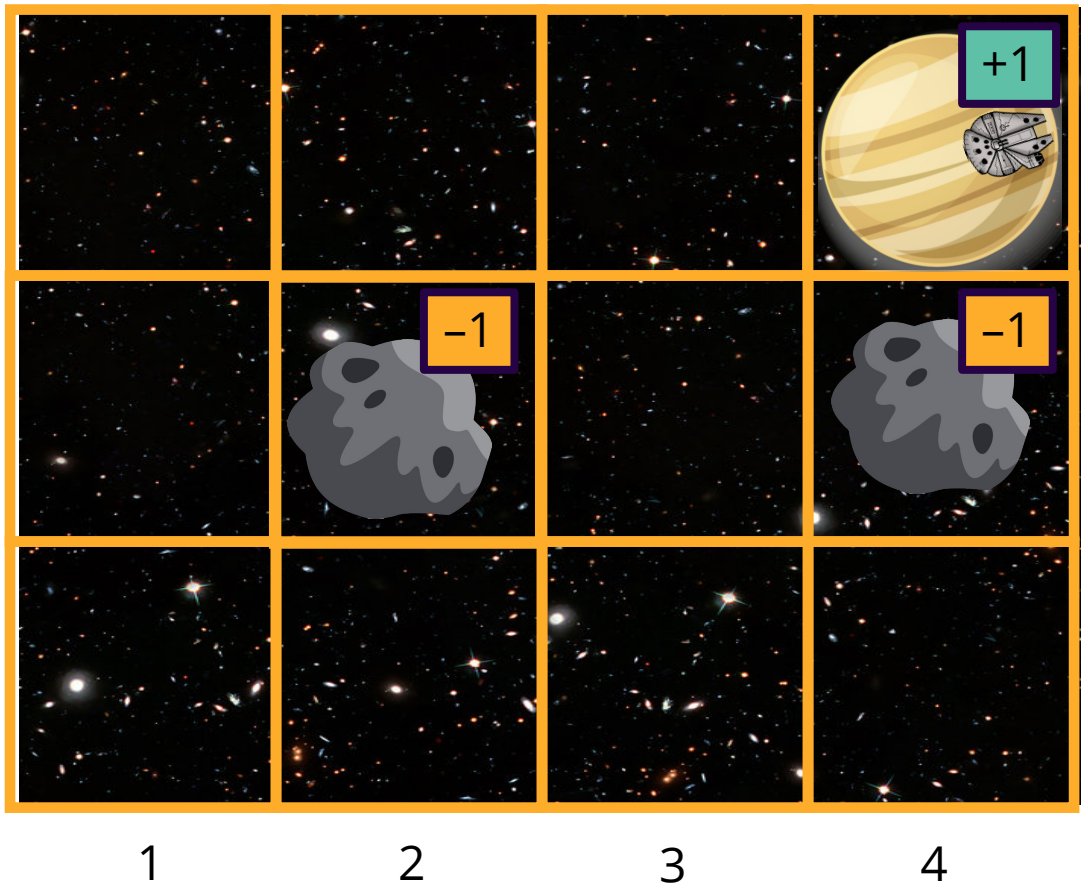
what's the probability that the millennium falcon reaches the intended goal?

0.1 * 0.1 * 0.1 * 0.1 * 0.8

= 0.00008

Transition Model:





Navigating an Asteroid Field

For action sequence

[*Up, Up, Right, Right, Right*],

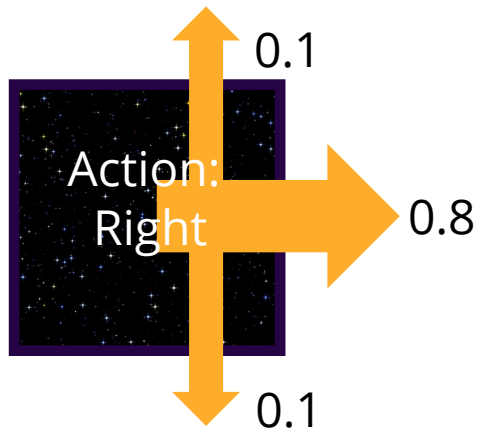
what's the probability that the millennium falcon reaches the intended goal?

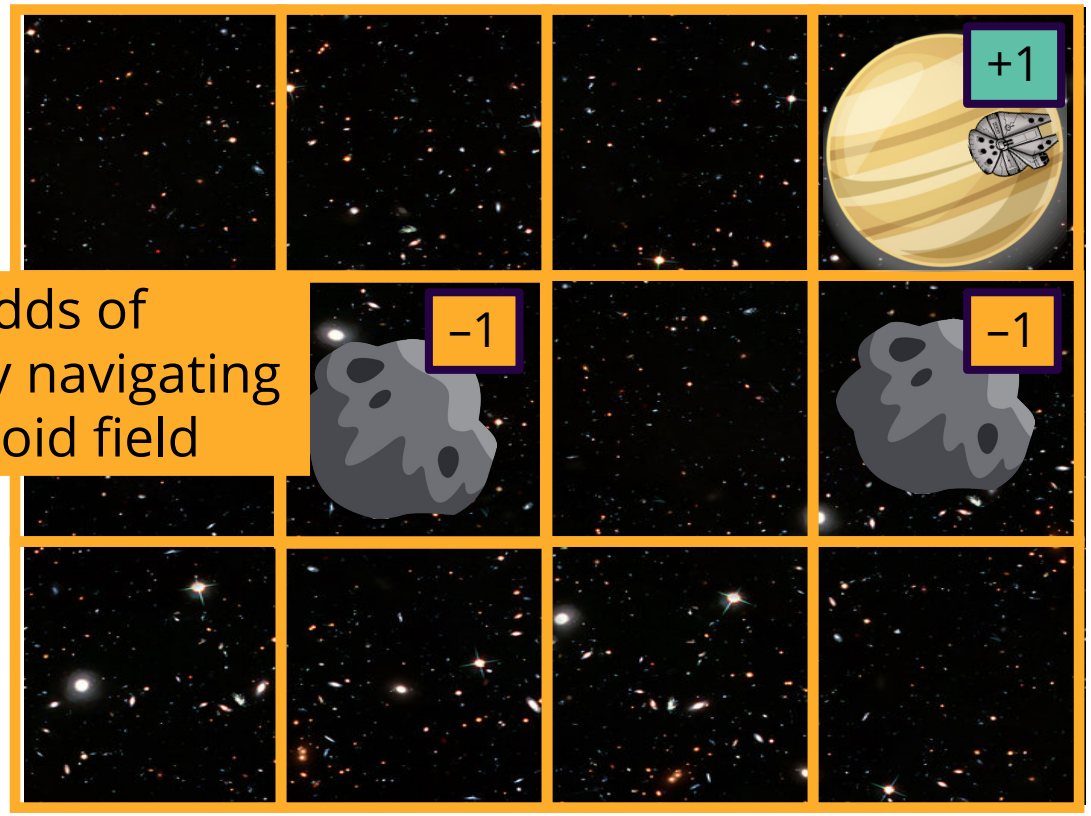
0.32768 + 0.00008

= 0.32776

The odds of successfully navigating an asteroid field

Transition Model:





Stochastic Transition Model

In our search algorithms so far, the transition model was deterministic and described the outcome of each action in each state.

The transition function is sometimes written as T(s, a, s'), or explicitly as a probability:

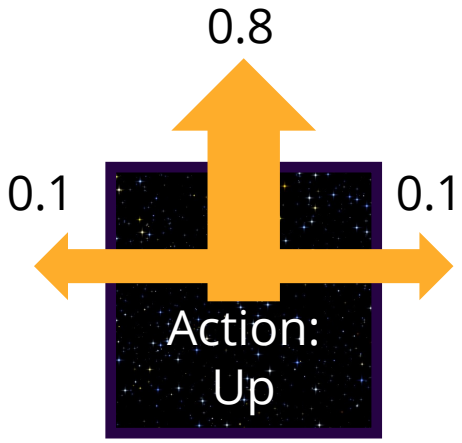
The probability of

arriving in state **s'**

$$p(s' \mid s, a)$$

given that

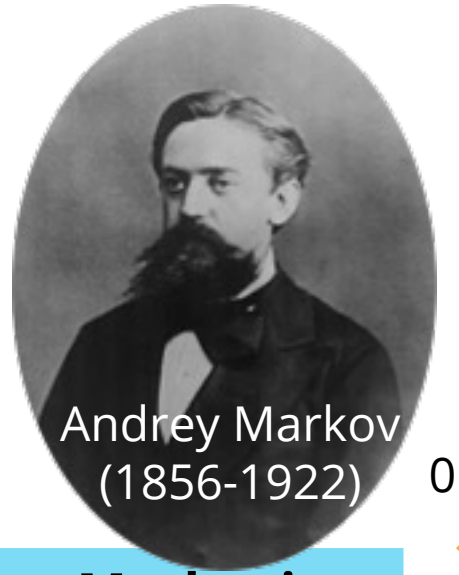
we are in state **s** and we selected action **a**

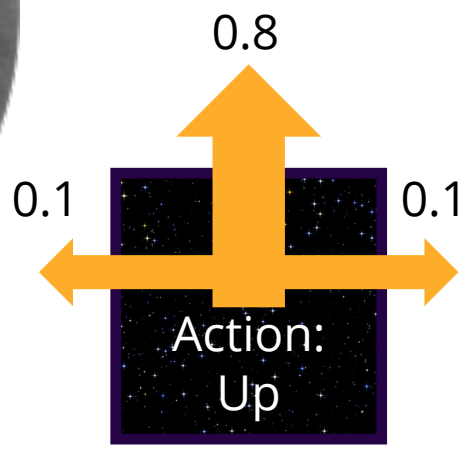


Stochastic Transition Model

In our search algorithms so far, the transition model was deterministic and described the outcome of each action in each state.

The transition function is sometimes written as T(s, a, s'), or explicitly as a probability:





$p(s' \mid s, a)$

Transitions are **Markovian**: the probability of arriving in s' only depends on s and not the history of earlier states.

Reward function

We will specify a **utility or reward function** for the agent.

The “rewards“ can be **positive** or **negative** but are bounded by some maximum value.

Because the decision process is **sequential,** we must specify the utility function on a sequence of states and actions.

Instead of only giving a reward at the goal states, the agent can **receive a reward at each time step**, based on its transition from **s** to **s’** via action **a**.

This is defined by a reward function

R(s, a, s’)

For example, we could give the Millennium Falcon a small negative reward of -0.04 for every transition except for entering the terminal states (+1 for entering the planet’s orbit or -1 for smashing into an asteroid).

The **rewards are additive**, so if the Millennium Falcon takes 4 steps before entering the planet's orbit, it gets *-0.04 + -0.04 + -0.04 + -0.04 + 1 = 0.84* for that solution.

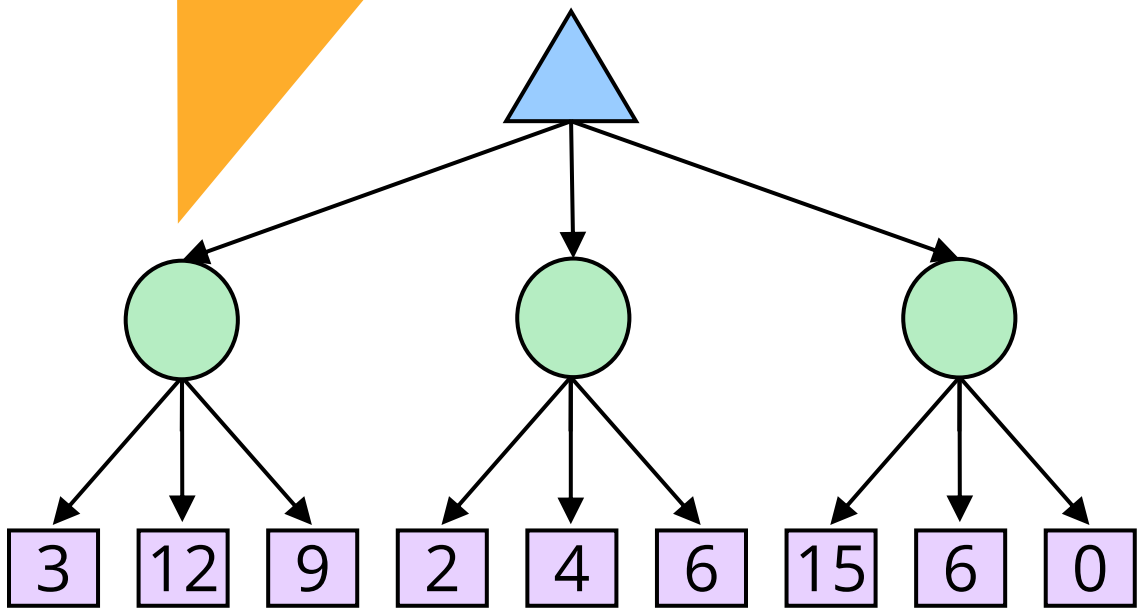
Markov Decision Process

A **Markov decision process** or **MDP** is

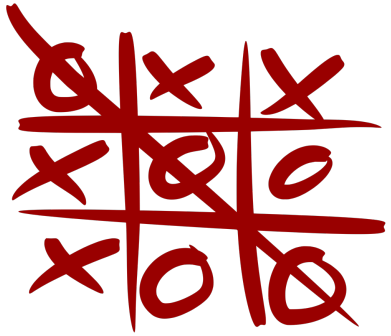
- a **sequential** decision problem
- for a **fully observable** environment
- with a **stochastic** transition model
- that has **additive rewards**

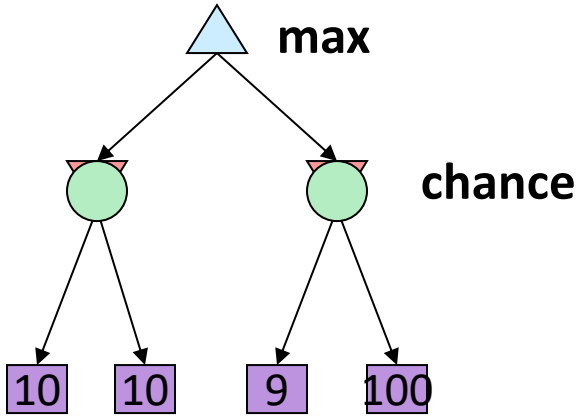
MDPs are non-deterministic search problems. One way of solving them is via **expectimax** search.

Expectimax node: outcome is uncertain. In expectimax search we calculate their expected utilities.



Review: Expectimax

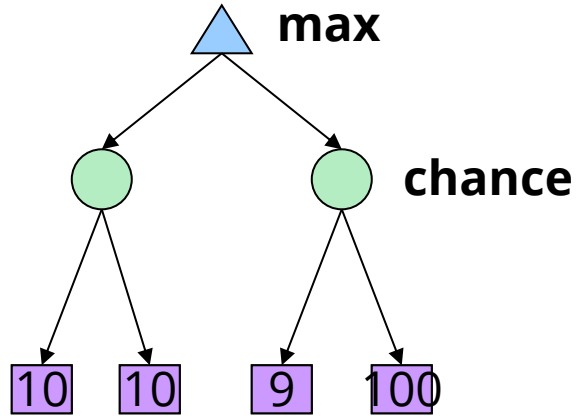




Idea: Uncertain outcomes controlled by chance, not an adversary!

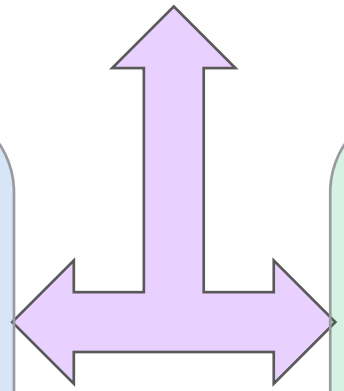
Review: Expectimax Search

- Why wouldn't we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Actions can fail: when moving a robot, wheels might slip
- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes
- Expectimax search: compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their expected utilities
  - Take weighted average (expectation) of children
- In this module, we'll learn how to formalize the underlying uncertain-result problems as Markov Decision Processes



Review: Expectimax Pseudocode

```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

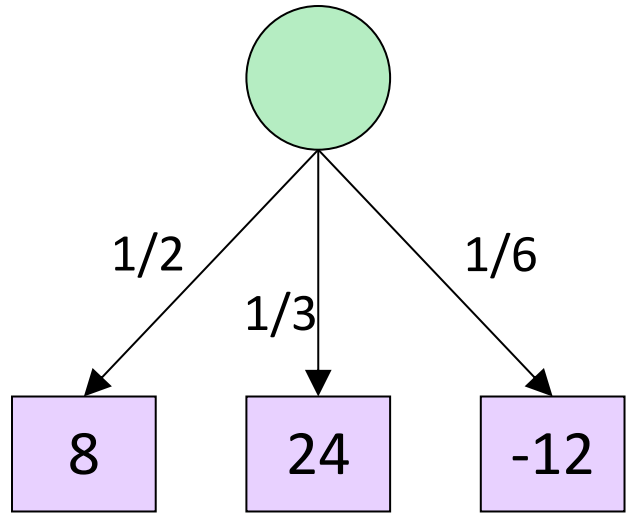


```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

Review: Expectimax Pseudocode

```
def exp-value(state):
  initialize v = 0
  for each successor of state:
      p = probability(successor)
      v += p * value(successor)
  return v
```



$$v = \frac{1}{2} \cdot (8) + \frac{1}{3} \cdot (24) + \frac{1}{6} \cdot (-12)$$

Markov Decision Process

To find a solution to an MDP, you need to define the following things:

- **A set of states** $s \in S$
- **A set of actions** $a \in A$
- A transition function **T(s, a, s')**
  - Probability that executing action **a** in **s** will lead to **s' P(s' | s, a)**
  - The probability is called **the model**
- A reward function **R(s, a, s')**
  - Sometimes just R(s) or R(s')
- An **initial state** s_0
- Optionally, one or more **terminal states**

Solution == Policy

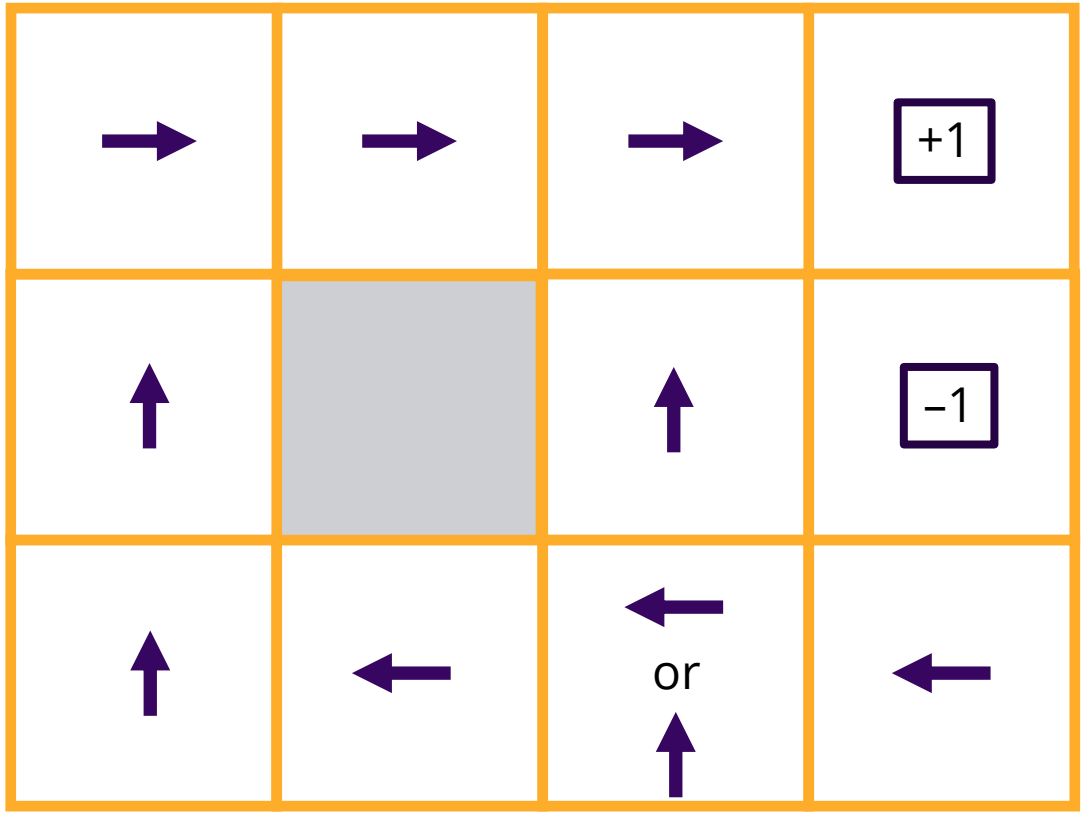
In search problems a solution was a sequence of action that corresponded to the shortest path.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$

Policy **π** tells the agent what action to take at state s.



This is an example policy for a grid world

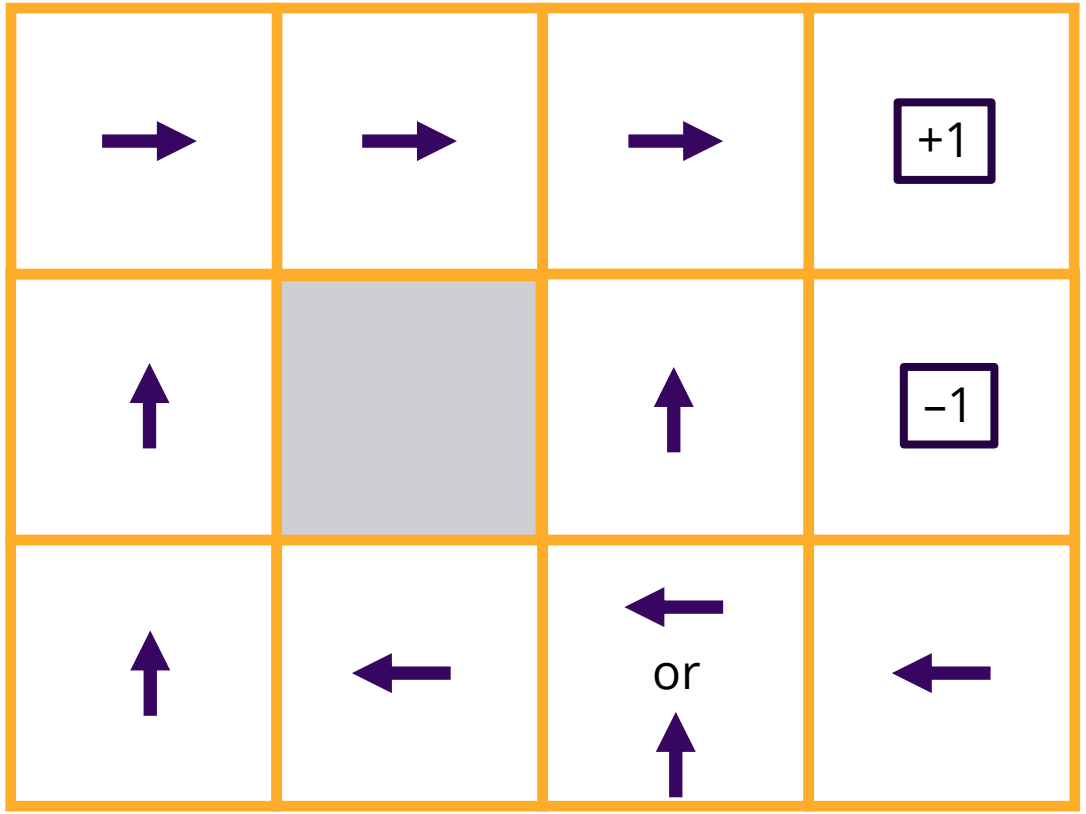
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



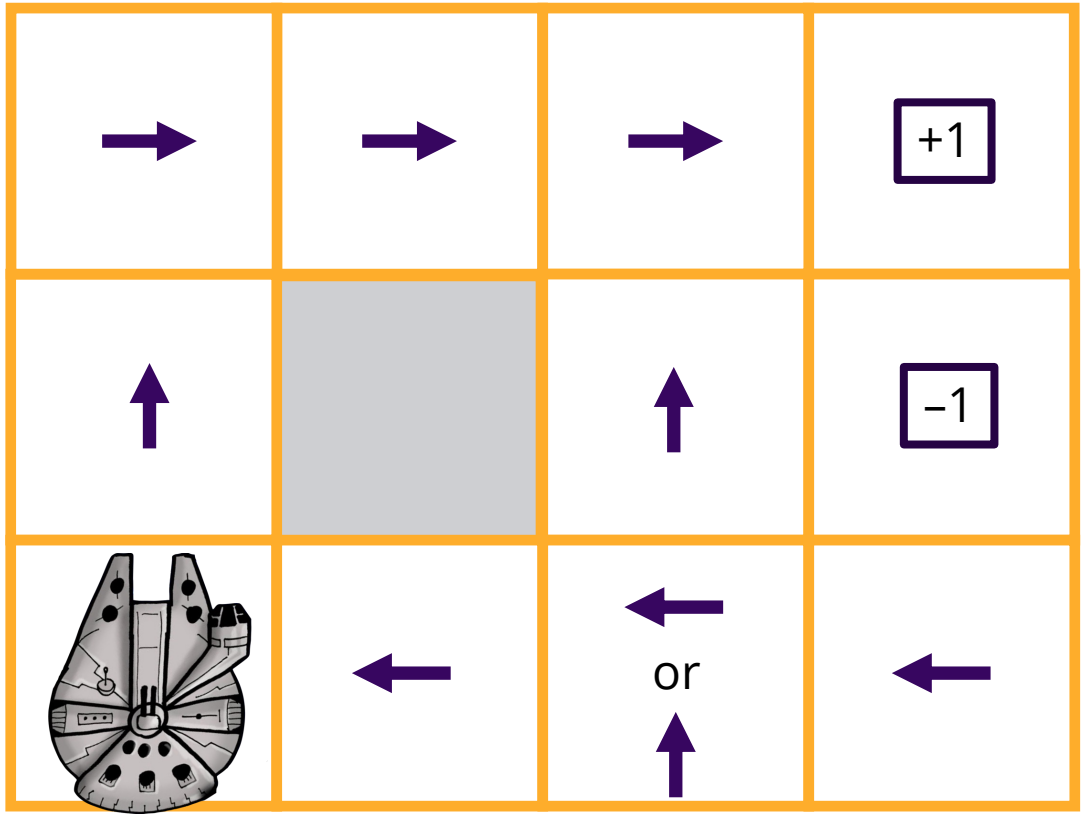
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



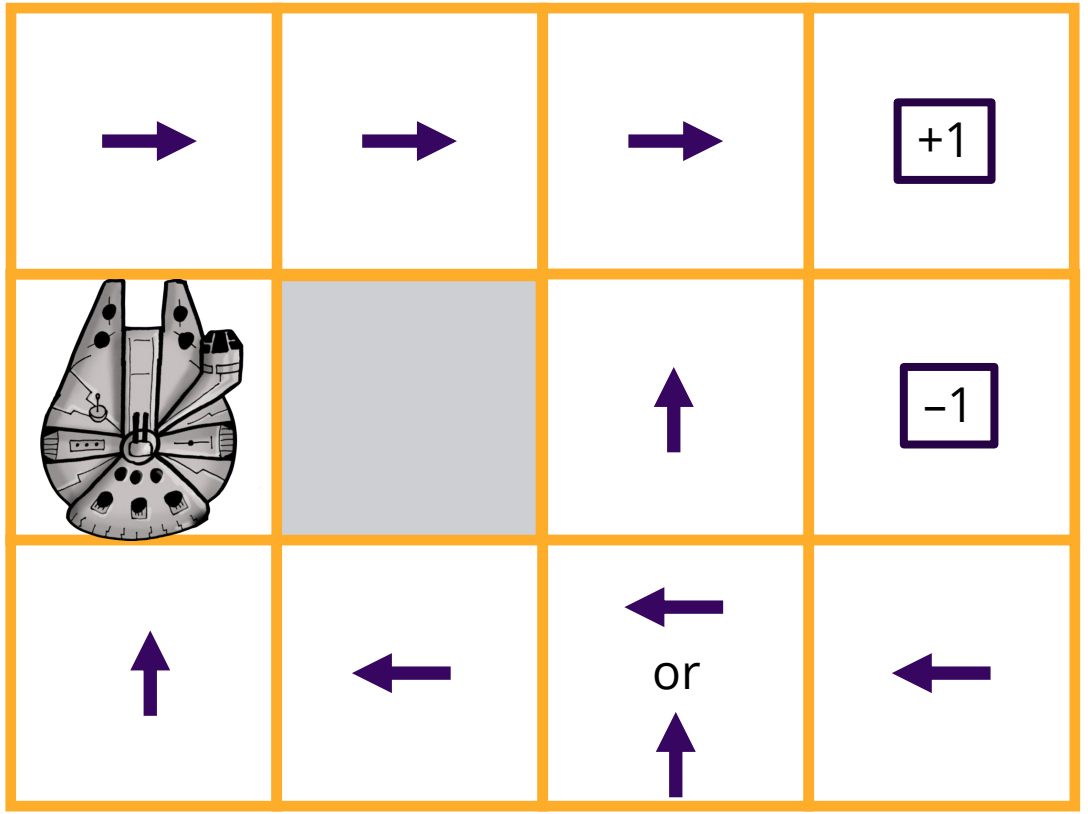
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



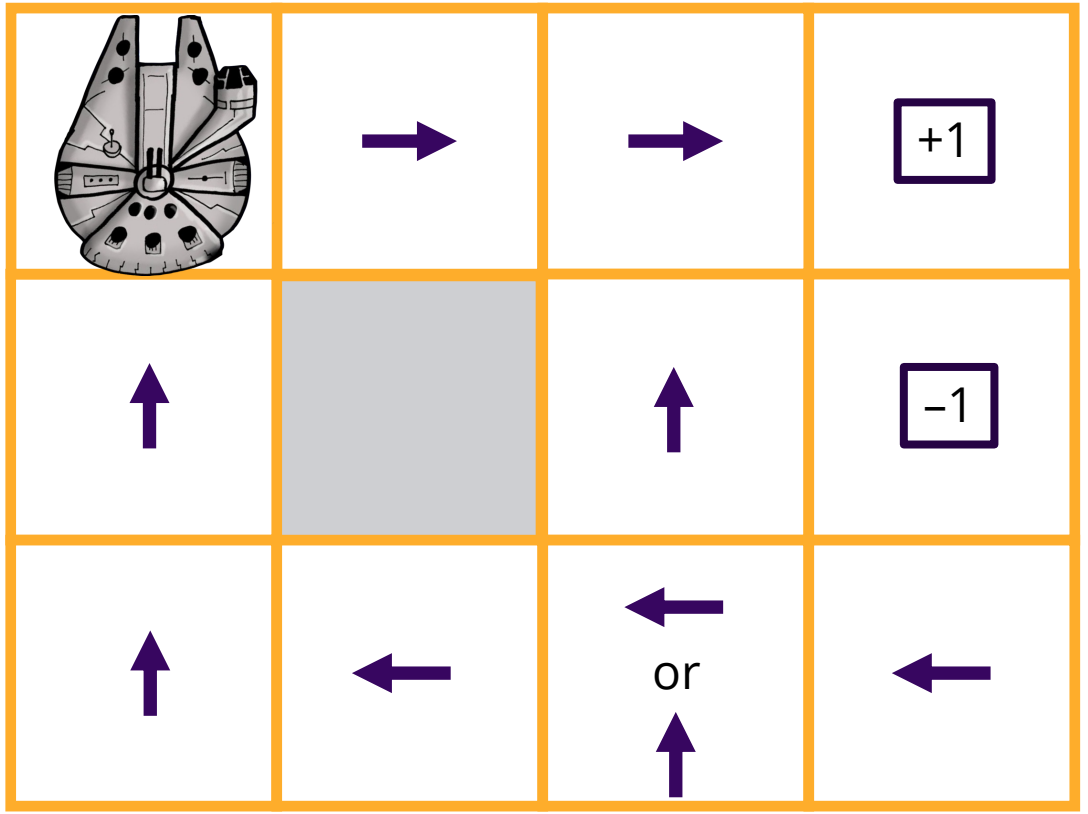
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



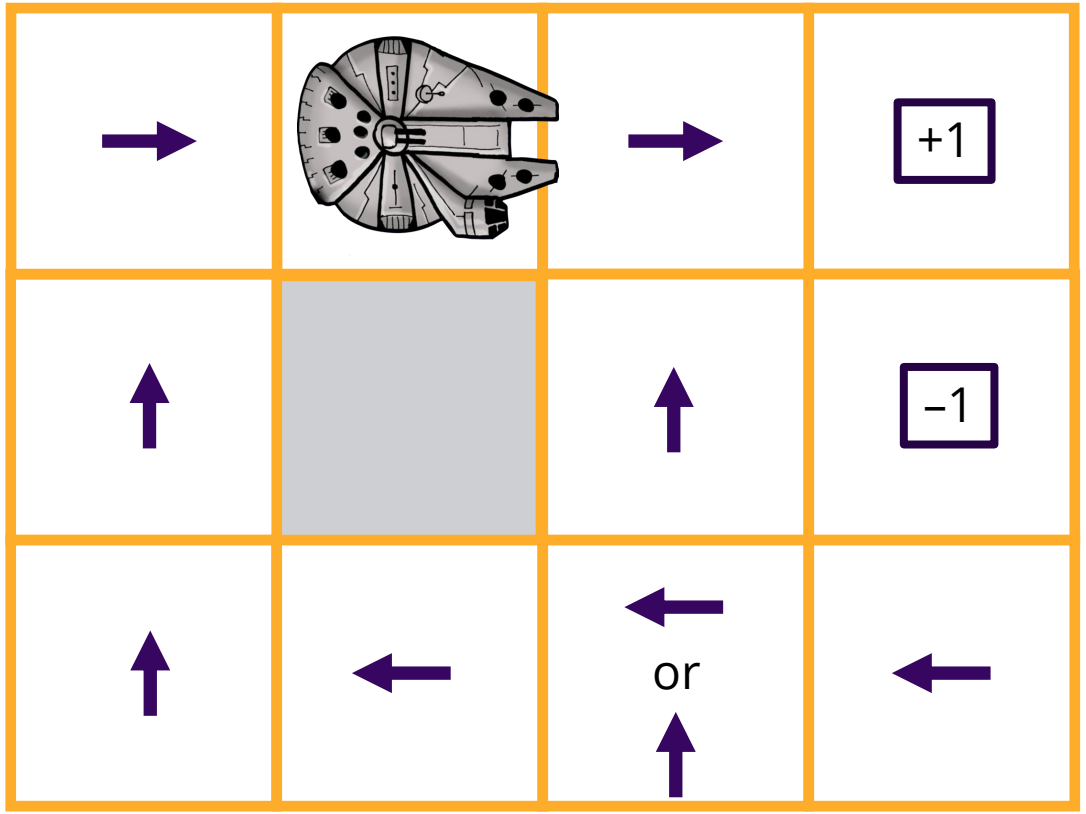
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



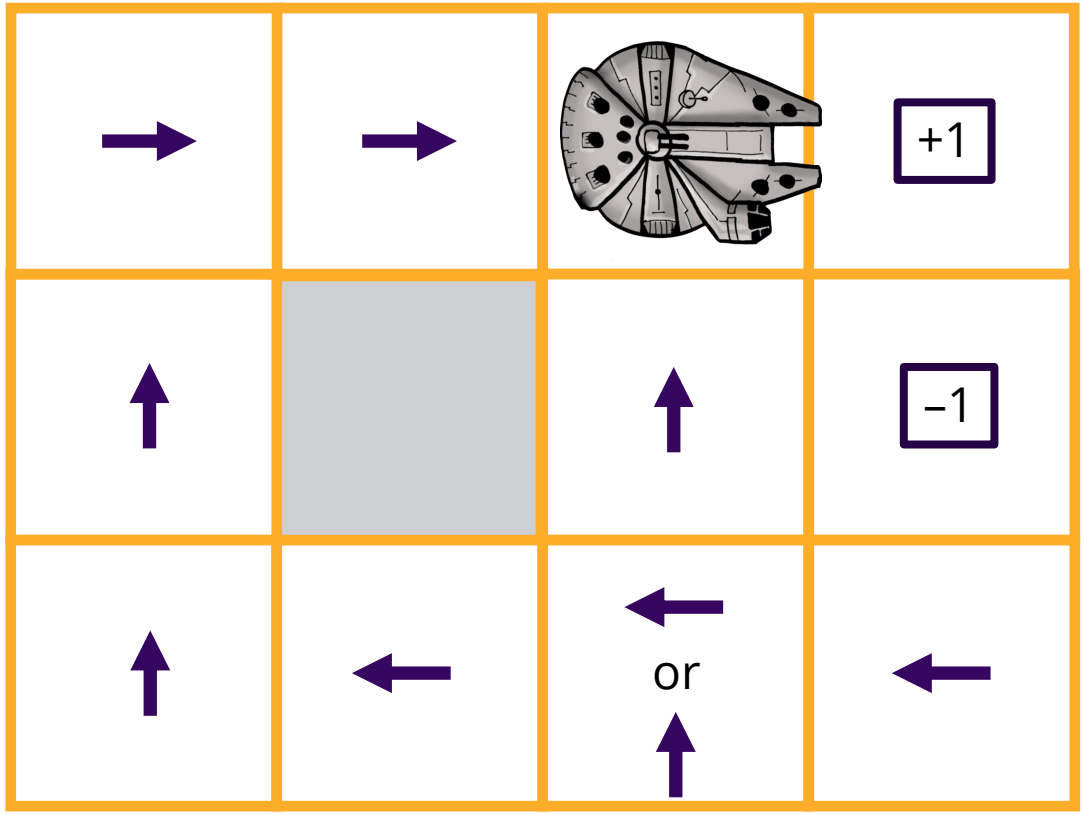
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



Solution == Policy

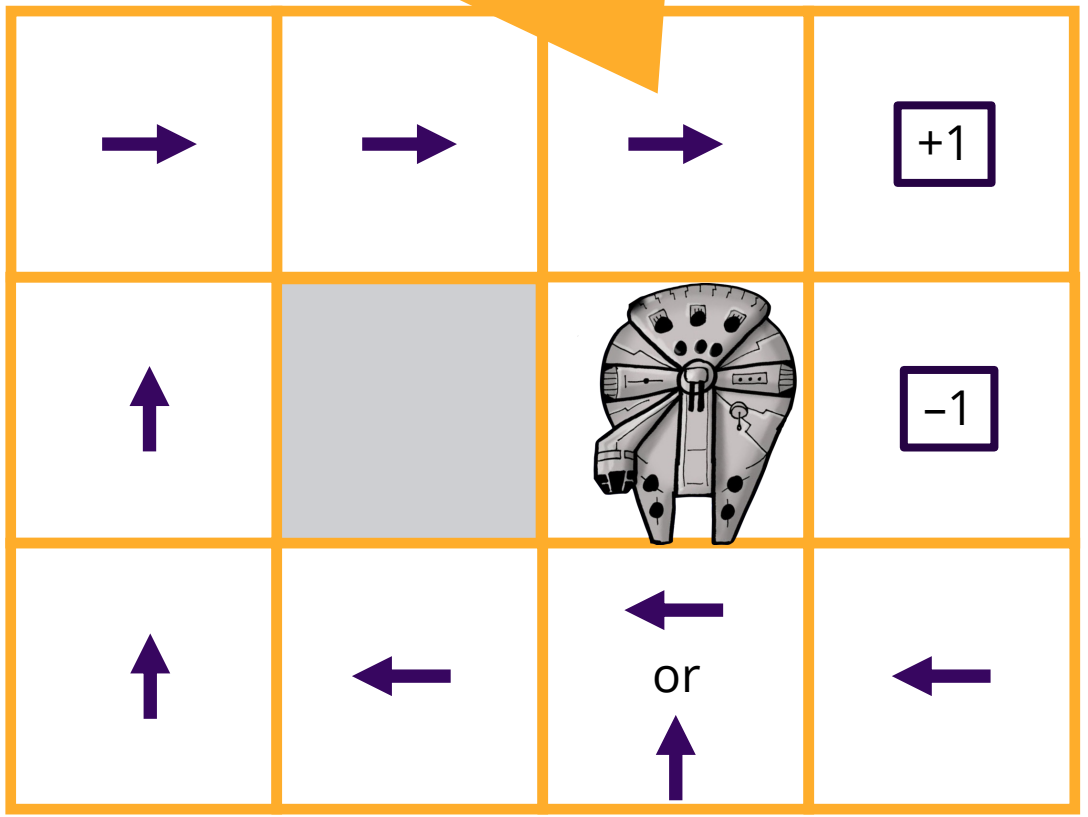
Even though the policy told me to go right here, there's no guarantee that me picking the action Right will result in me moving right. It's stochastic!

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



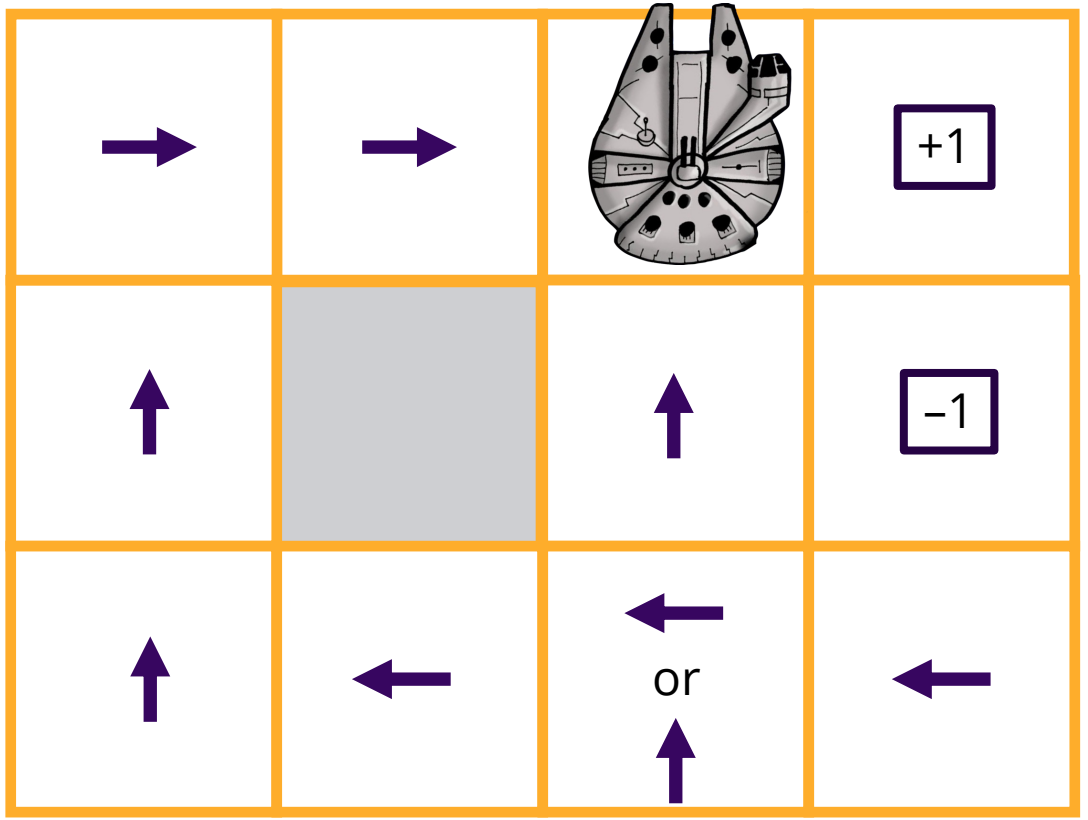
Solution == Policy

In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$



Solution == Policy

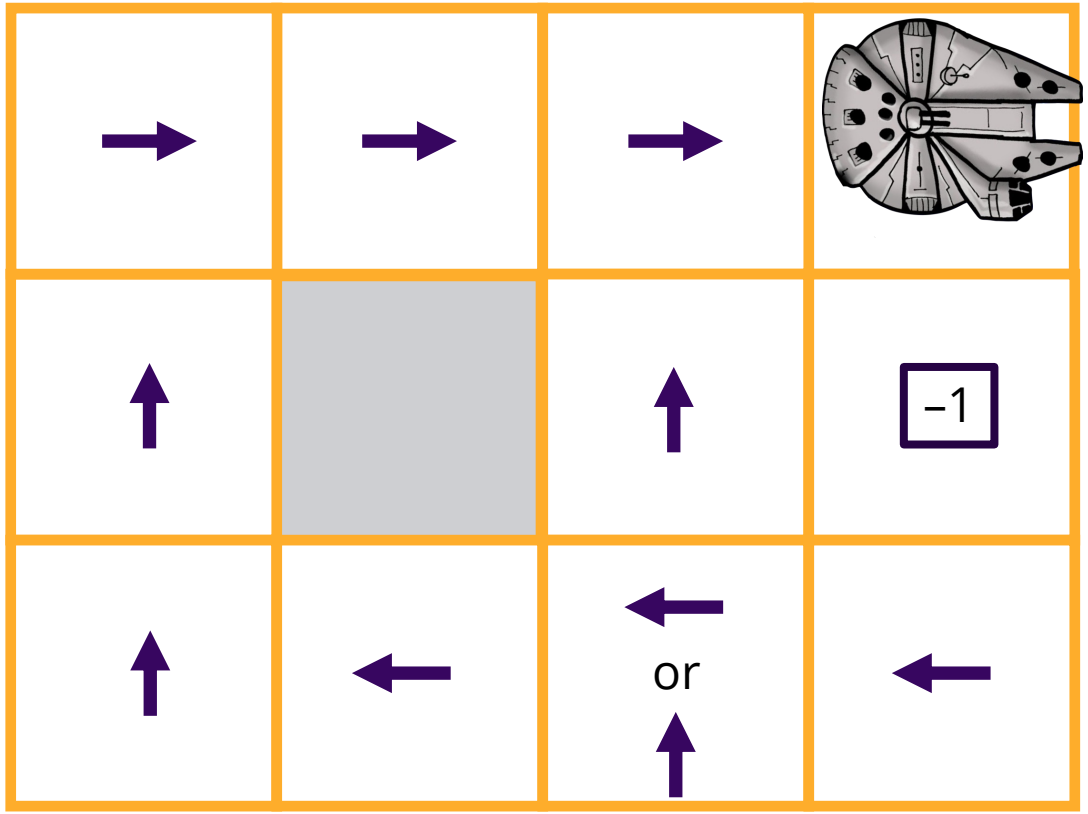
In search problems a solution was **a plan**: a sequence of action that corresponded to the shortest path from the start to a goal.

Because of the non-determinism in MDPs we cannot simply give a sequence of actions.

Instead, the solution to an MDP is a **policy.** A policy maps from a state onto the action to take if the agent is in that state.

$$\pi(s) = a$$

We will use π^* to denote the **optimal policy**.

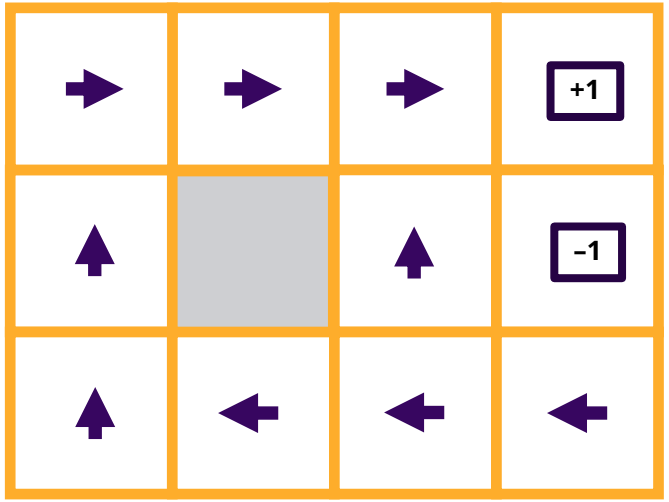


Policies and Rewards

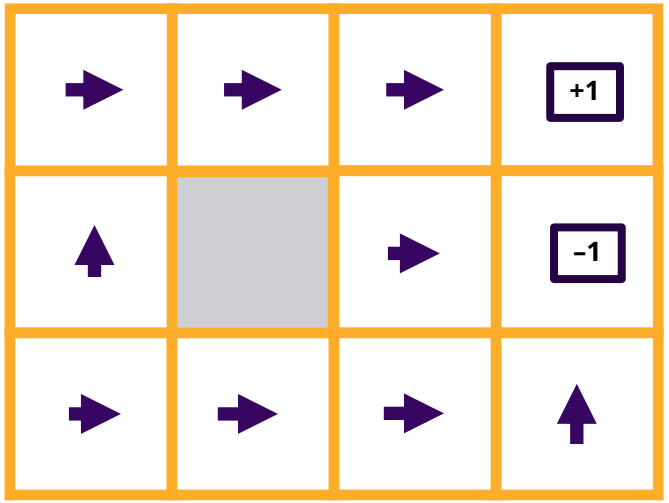
Even if the **same policy** is executed multiple times by the agent, this may lead to different sequence of states and actions (**environment history**), and thus a **different score** under the reward function.

Therefore we need to compute the **expected utility** of all the possible paths generated by a policy.

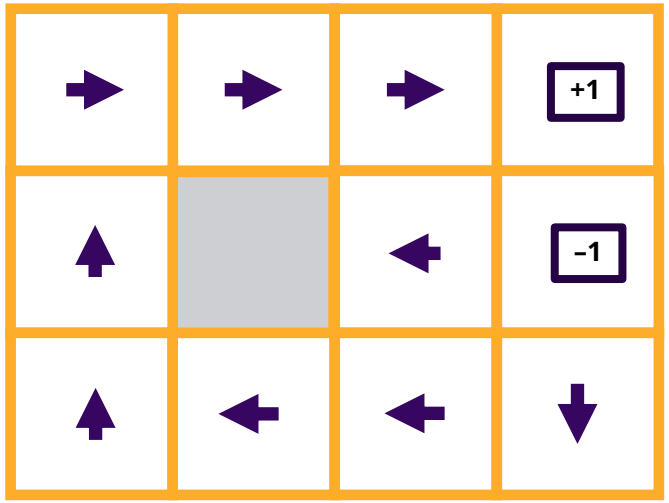
r = -0.04



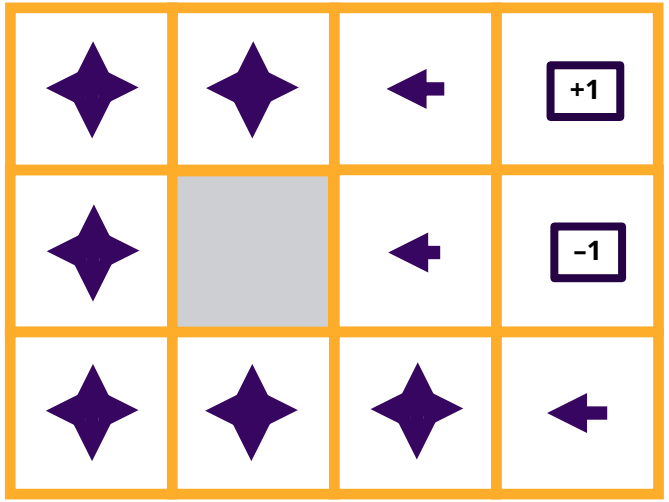
r < -1.65



-0.03 < r < 0



r > 0



Sequences of Rewards

The performance of an agent in an MDP is the sum of the rewards for the transitions it takes.

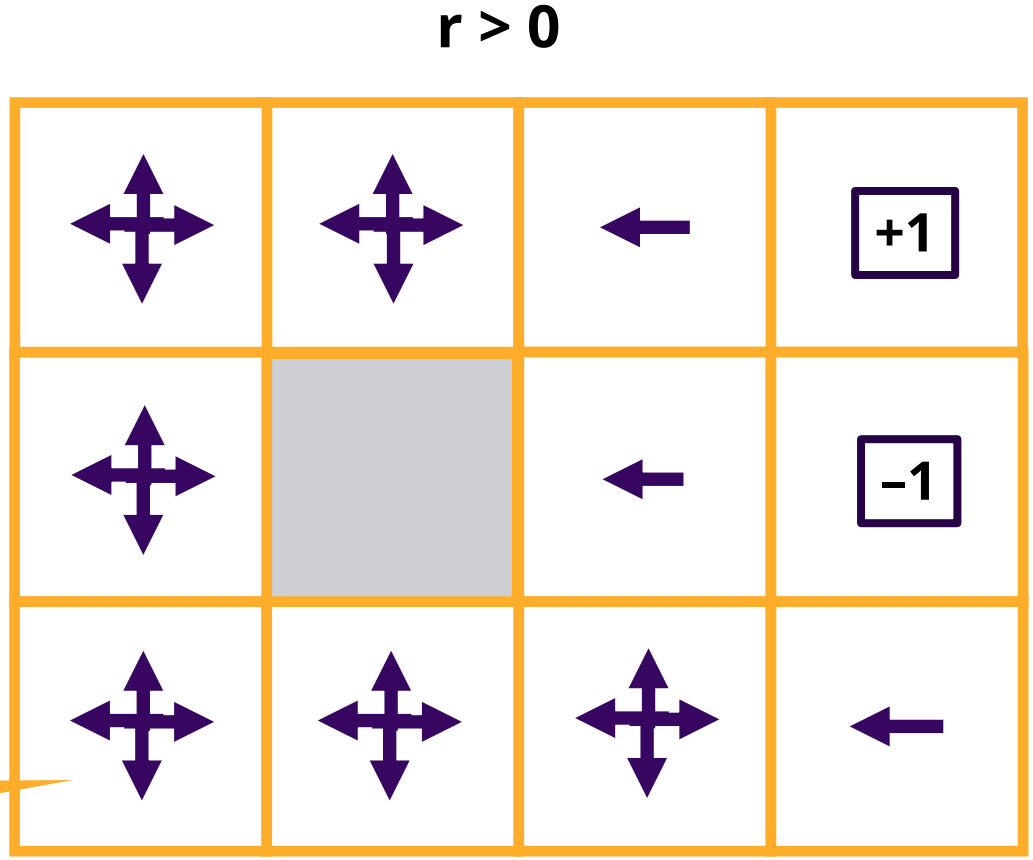
$$U_h([s_0, a_0, s_1, a_1 \ldots, s_n])$$

$$= R(s_0, a_0, s_1) + R(s_1, a_1, s_2) + \ldots + R(s_{n-1}, a_{n-1}, s_n)$$

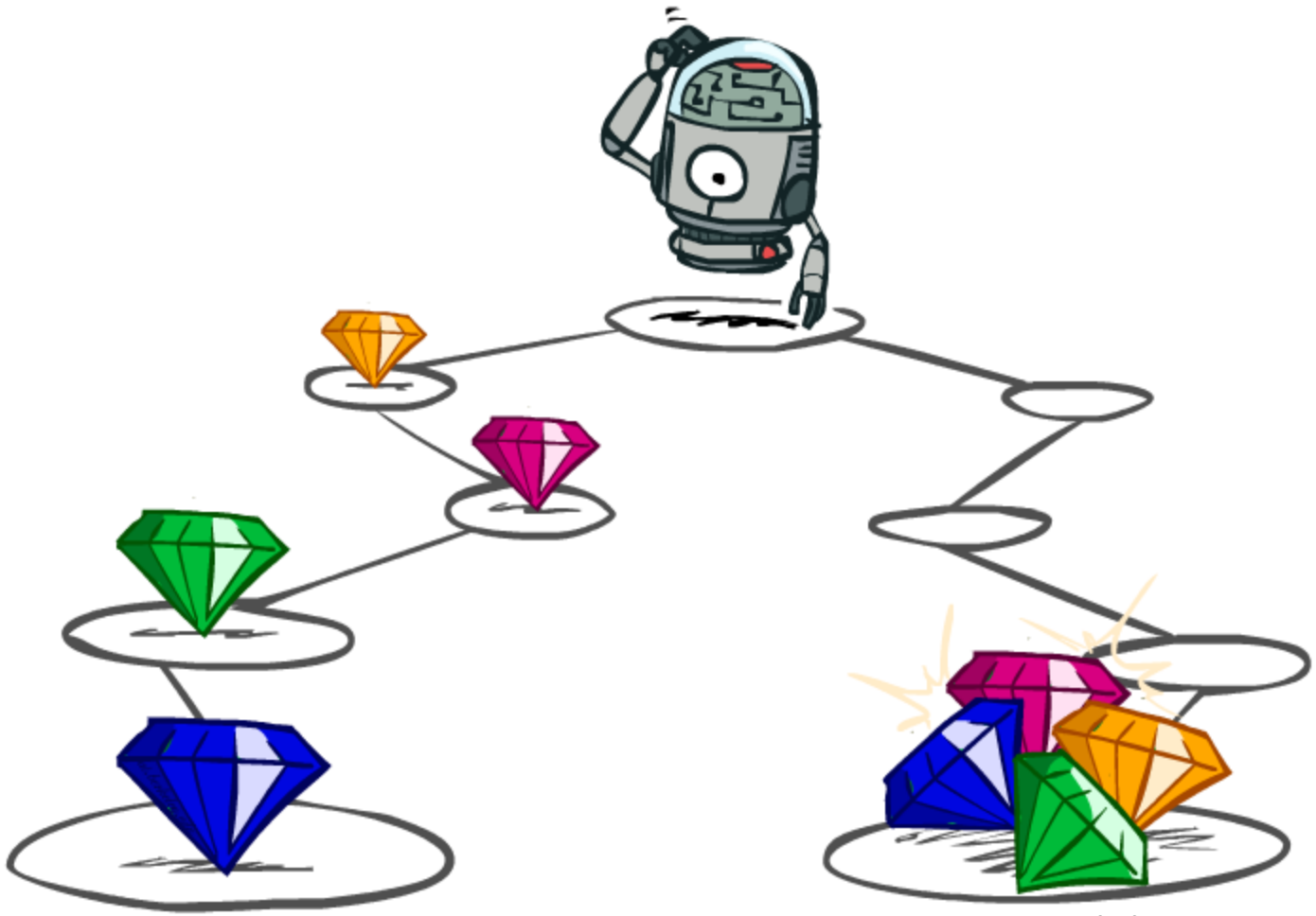
Utility function on an environment history.

Sequence of states and actions

Bounce around forever, and avoid the exits … **infinite rewards!!**



Utilities of Sequences



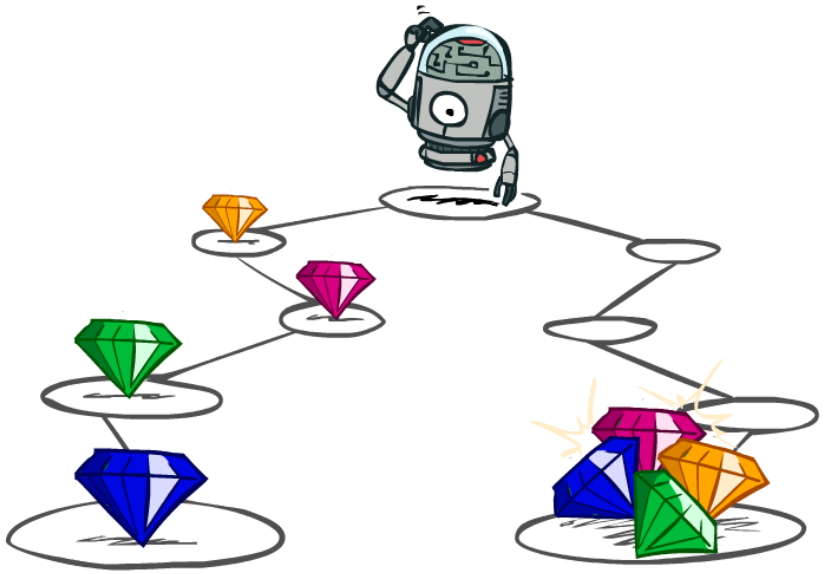
Slides courtesy of Dan Klein and Pieter Abbee
University of California, Berkeley

Utilities of Sequences

What preferences should an agent have over reward sequences?

More or less? [1, 2, 2] or [2, 3, 4]

Now or later? [0, 0, 1] or [1, 0, 0]



Discounting

It’s reasonable to maximize the sum of rewards

It’s also reasonable to prefer rewards now to rewards later

One solution: values of rewards decay exponentially

1

Worth Now

γ

Worth Next Step

γ^2

Worth In Two Steps

Discounting

How to discount?

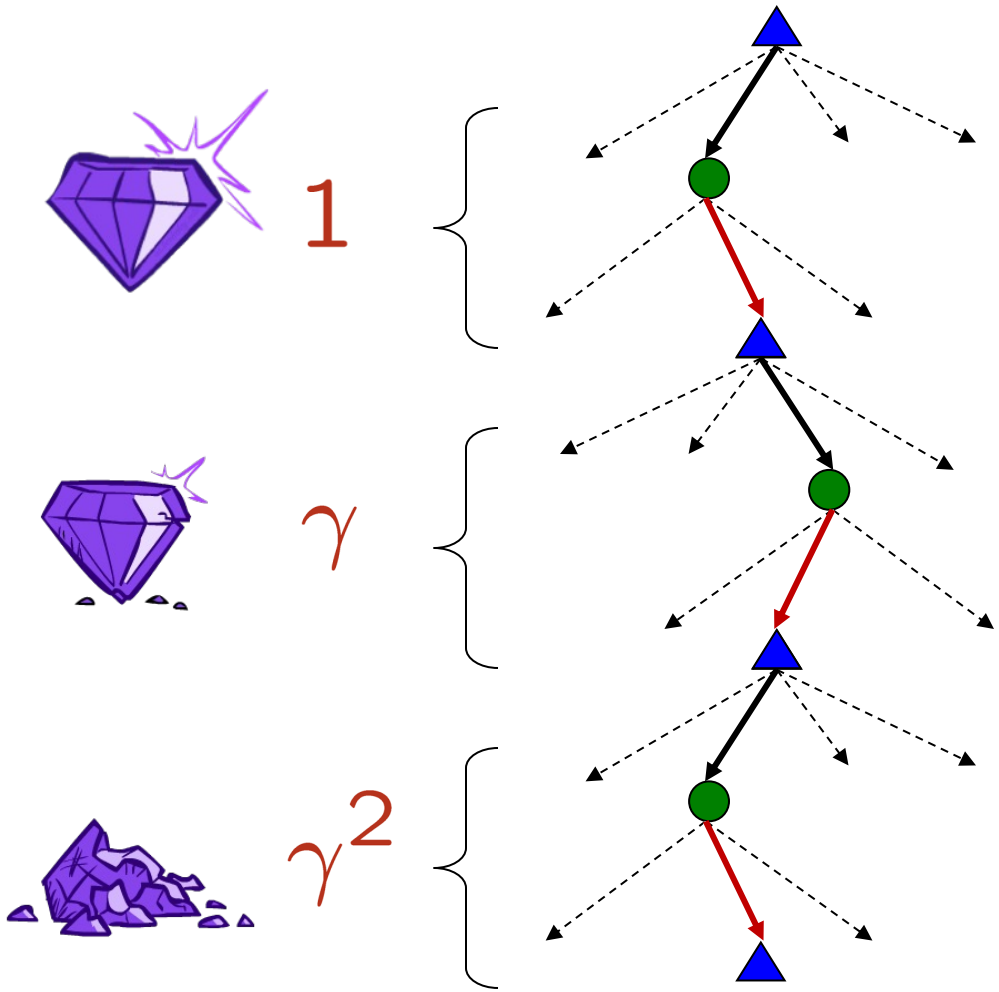
- Each time we descend a level, we multiply in the discount once

Why discount?

- Sooner rewards probably do have higher utility than later rewards
- Also helps our algorithms converge

Example: discount of 0.5

- $U([1,2,3]) = 0.5^0*1 + 0.5^1*2 + 0.5^2*3$
  $= 1*1 + 0.5*2 + 0.25*3$
- $U([1,2,3]) < U([3,2,1])$



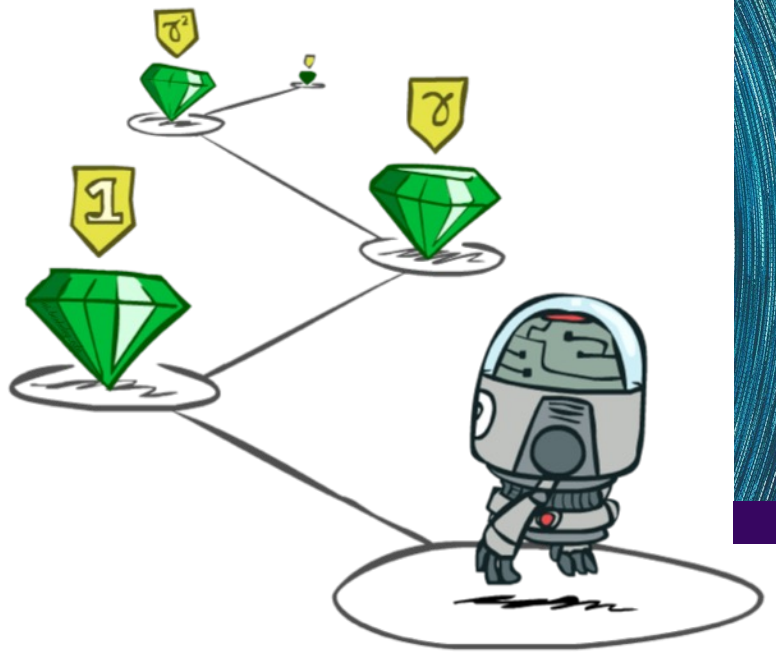
Stationary Preferences

Theorem: if we assume stationary preferences:

$$[a_1, a_2, \ldots] \succ [b_1, b_2, \ldots]$$

$$\Updownarrow$$

$$[r, a_1, a_2, \ldots] \succ [r, b_1, b_2, \ldots]$$



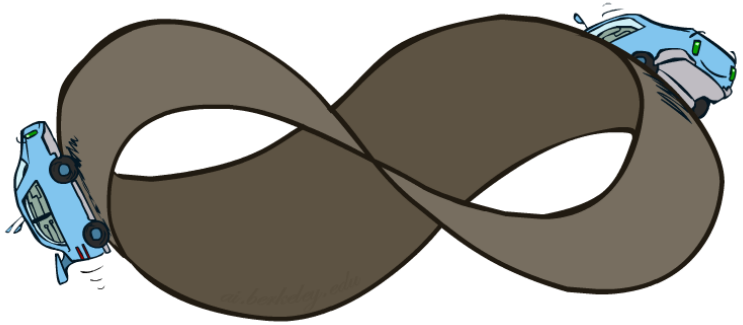
Then: there are only two ways to define utilities

- Additive utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$
- Discounted utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?

- Solutions:
  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)



  - Discounting: use $0 < \gamma < 1$

$$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\text{max}}/(1-\gamma)$$

    - Smaller γ means smaller "horizon" – shorter term focus

  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

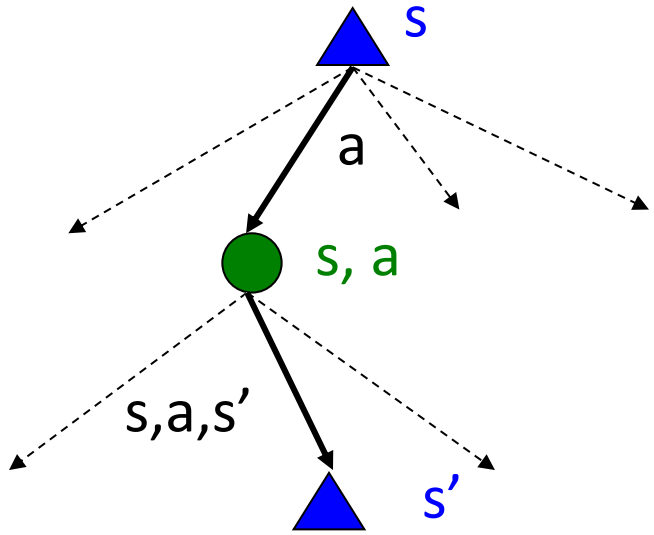
Recap: Defining MDPs

Markov decision processes:

- Set of states S
- Start state s_0
- Set of actions A
- Transitions P(s'|s,a) (or T(s,a,s'))
- Rewards R(s,a,s') (and discount γ)

MDP quantities so far:

- Policy = Choice of action for each state
- Utility = sum of (discounted) rewards

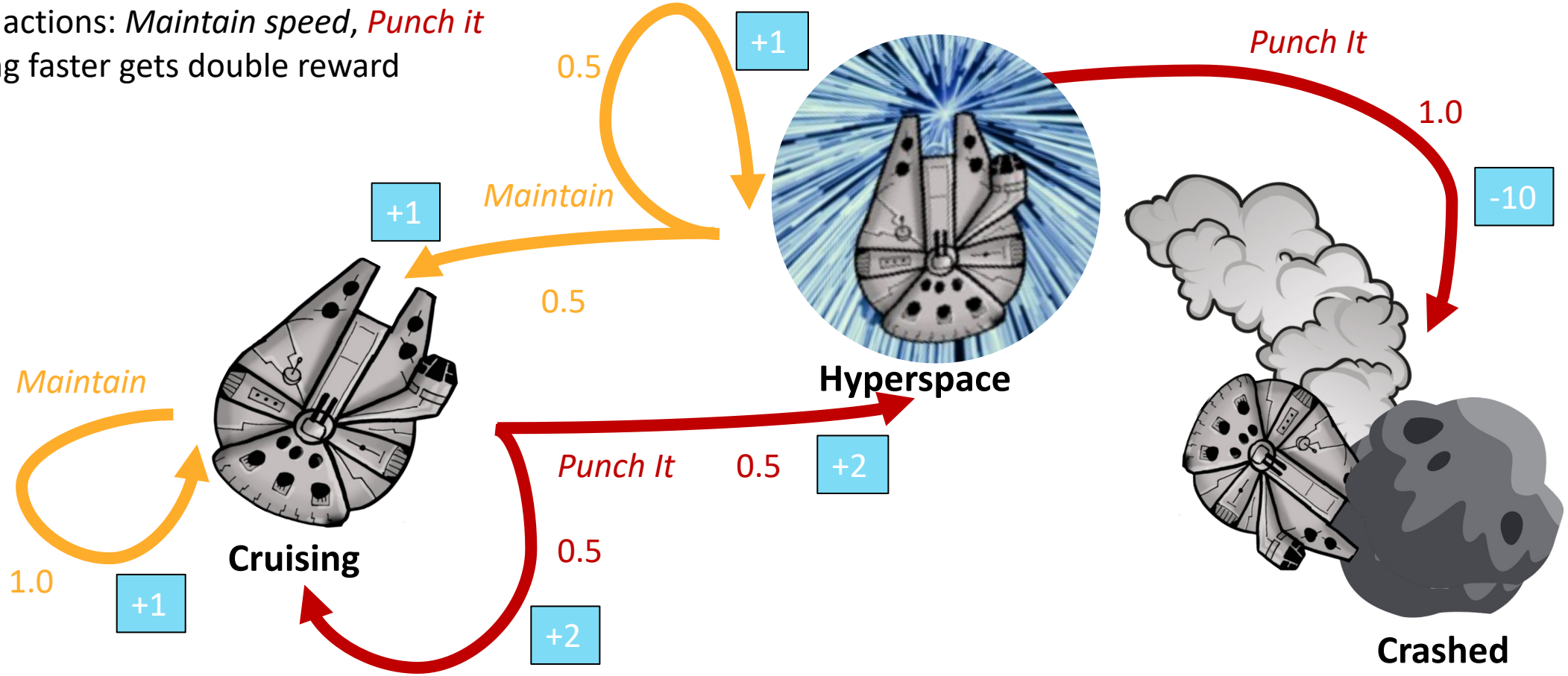


CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

Markov Decision Processes part 2

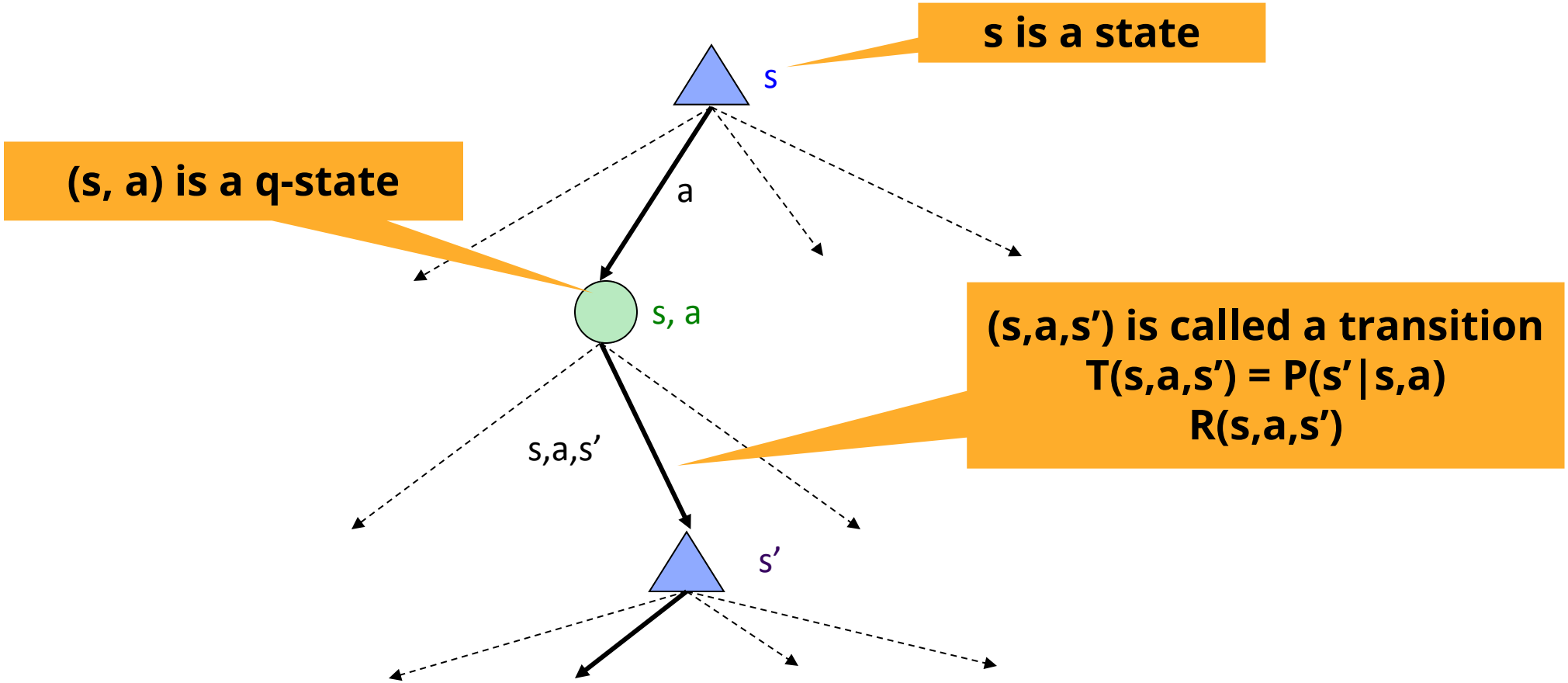
Example Hyperdrive MDP

The Millennium Falcon needs to travel far far away, quickly
Three states: Cruising, Hyperspace, Crashed
Two actions: *Maintain speed*, *Punch it*
Going faster gets double reward

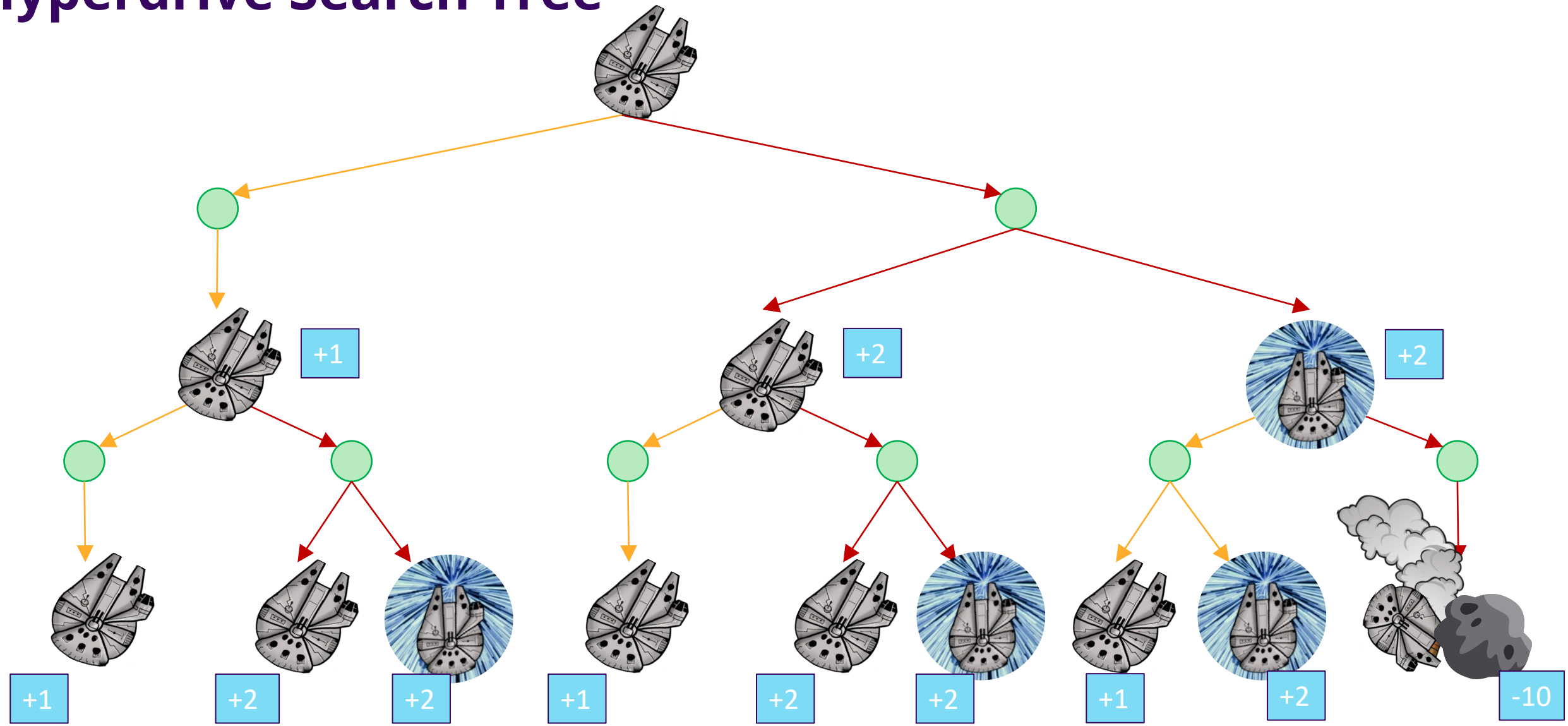


MDP Search Trees

Each MDP state projects an expectimax-like search tree



Hyperdrive Search Tree



Optimal Quantities

- The value (utility) of a state s:

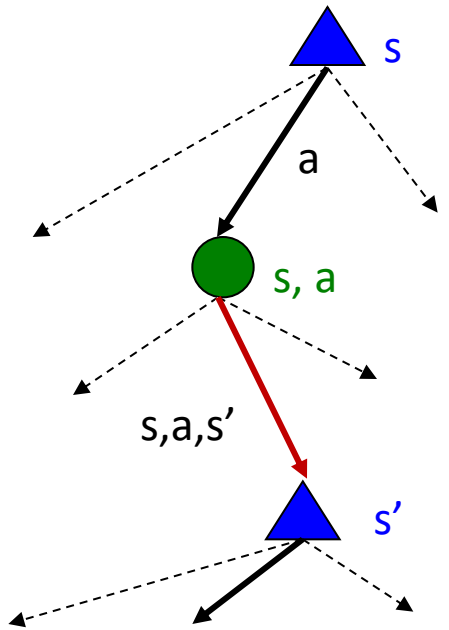
  $V^*(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):

  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally

- The optimal policy:

  $\pi^*(s)$ = optimal action from state s



s is a *state*

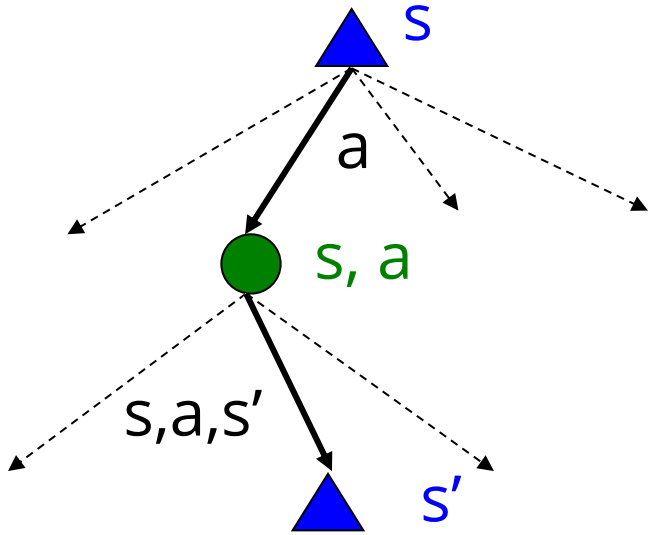
(s, a) is a *q-state*

(s,a,s') is a *transition*

Values of States

Fundamental operation: compute the (expectimax) value of a state

- Expected utility under optimal action
- Average sum of (discounted) rewards
- This is just what expectimax computed!



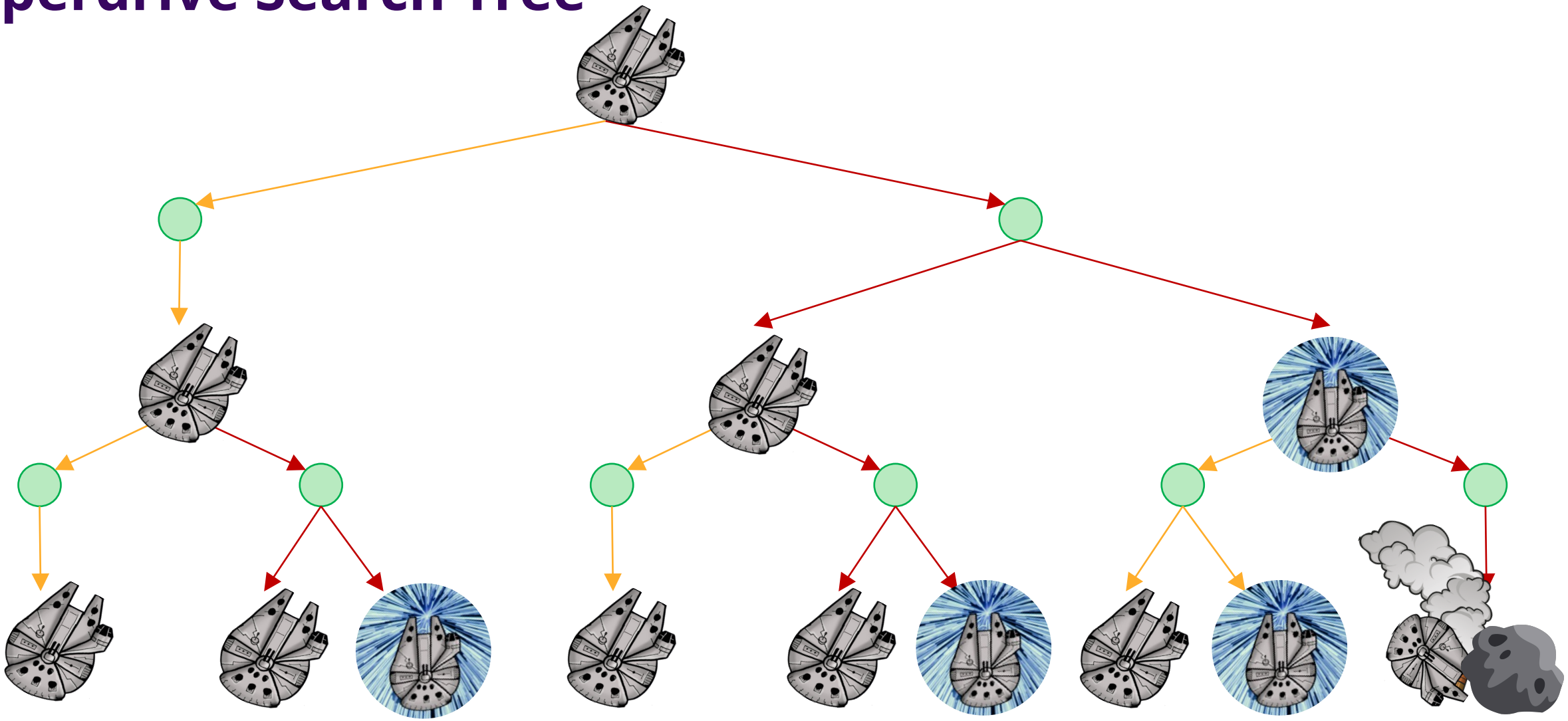
Recursive definition of value:

$$V^*(s) = \max_a Q^*(s,a)$$

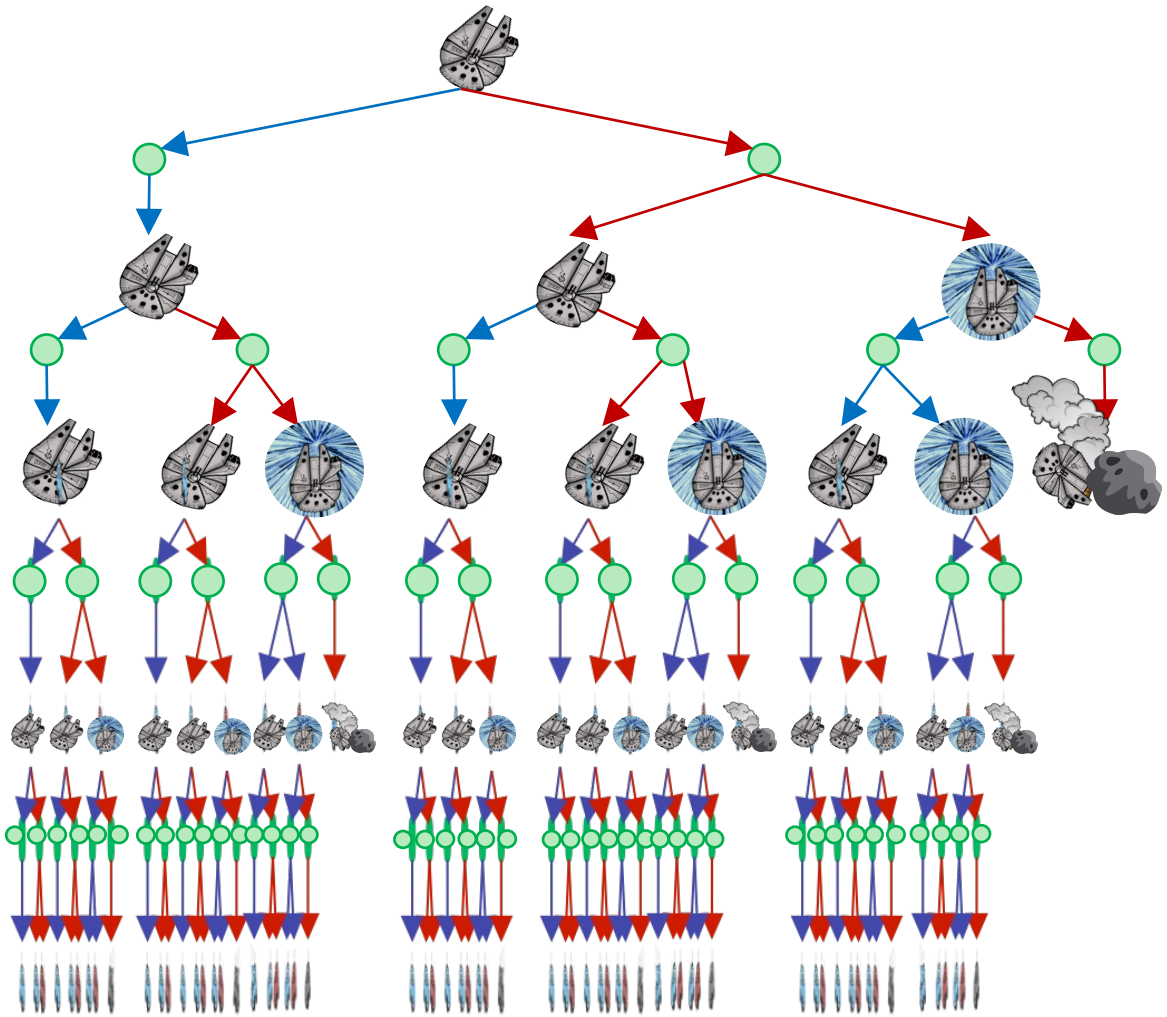
$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

Hyperdrive Search Tree



Hyperdrive Search Tree



Hyperdrive Search Tree

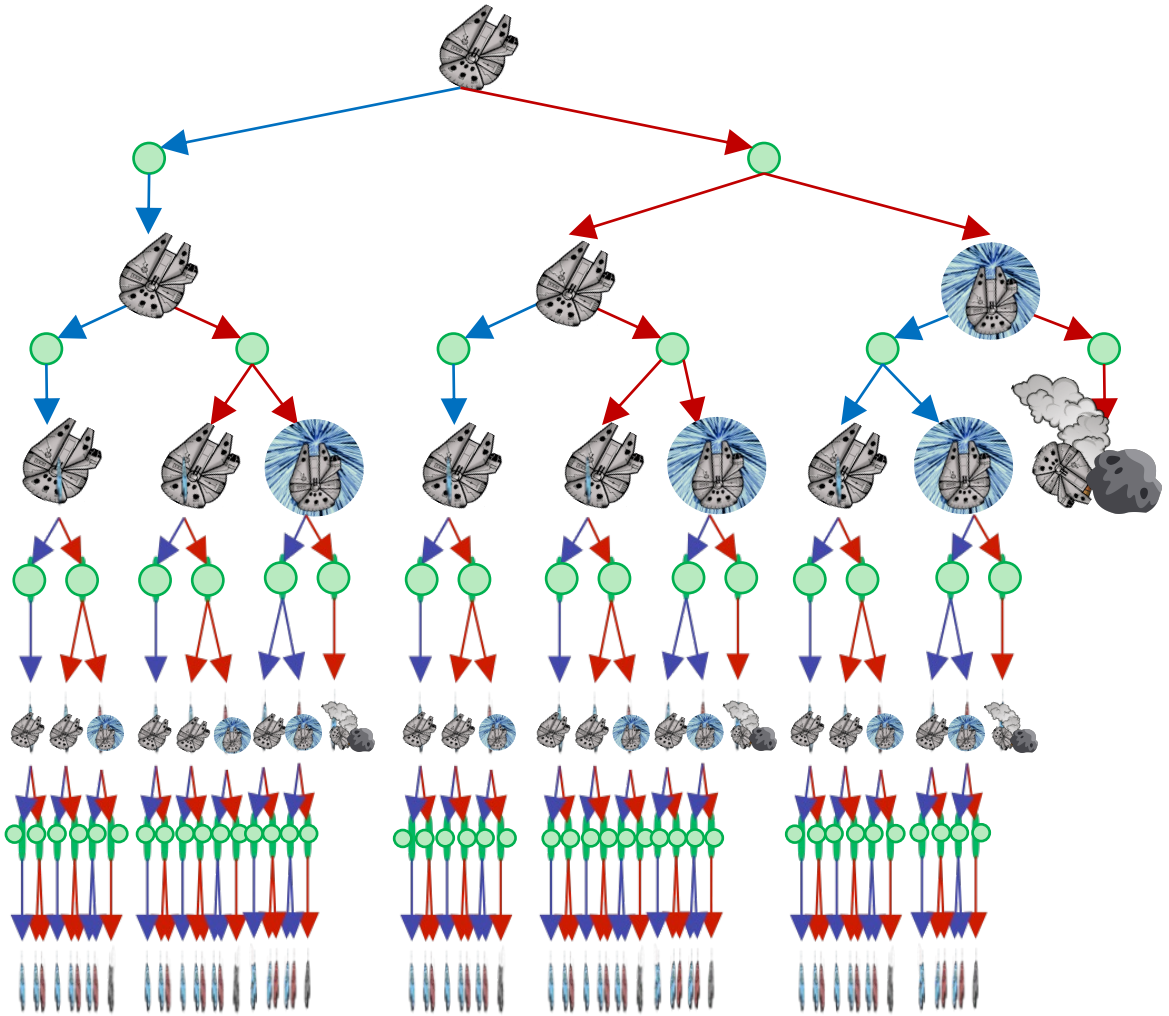
We're doing way too much work with expectimax!

Problem: States are repeated

- Idea: Only compute needed quantities once

Problem: Tree goes on forever

- Idea: Do a depth-limited computation, but with increasing depths until change is small
- Note: deep parts of the tree eventually don't matter if $\gamma < 1$

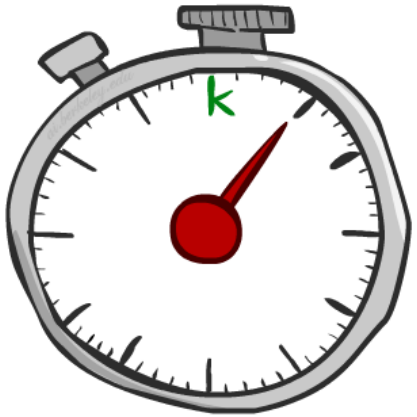


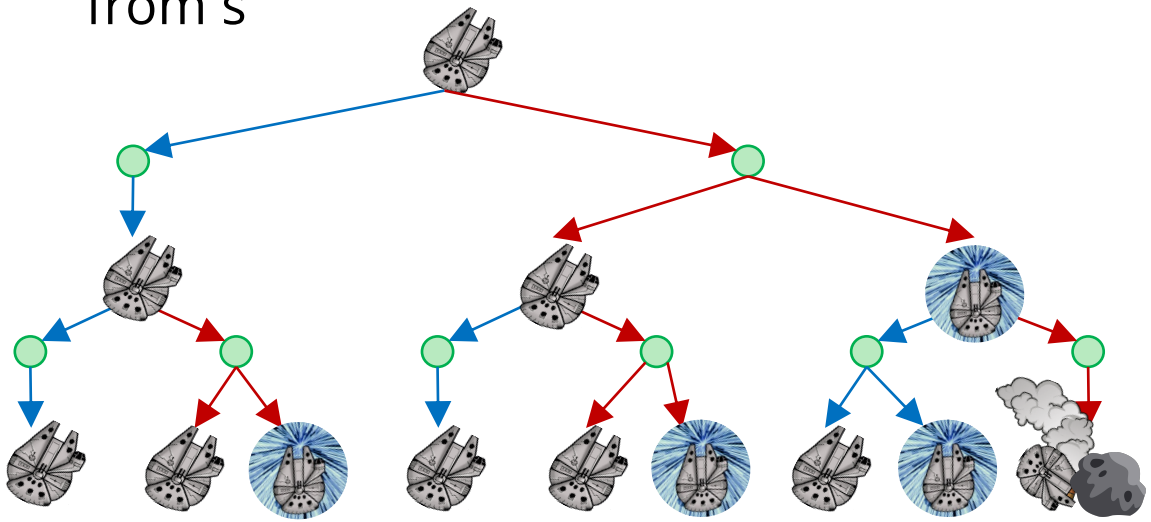
Time-Limited Values

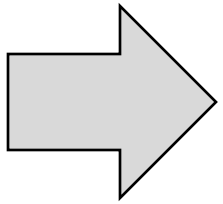
Key idea: time-limited values

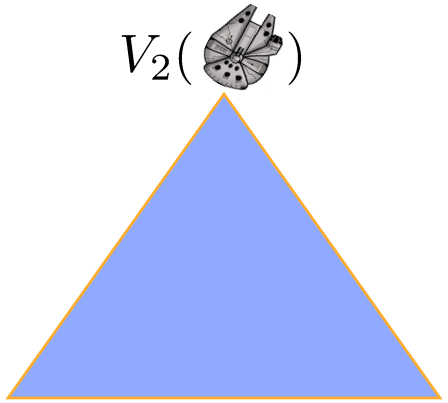
Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps

- Equivalently, it's what a depth-k expectimax would give from s

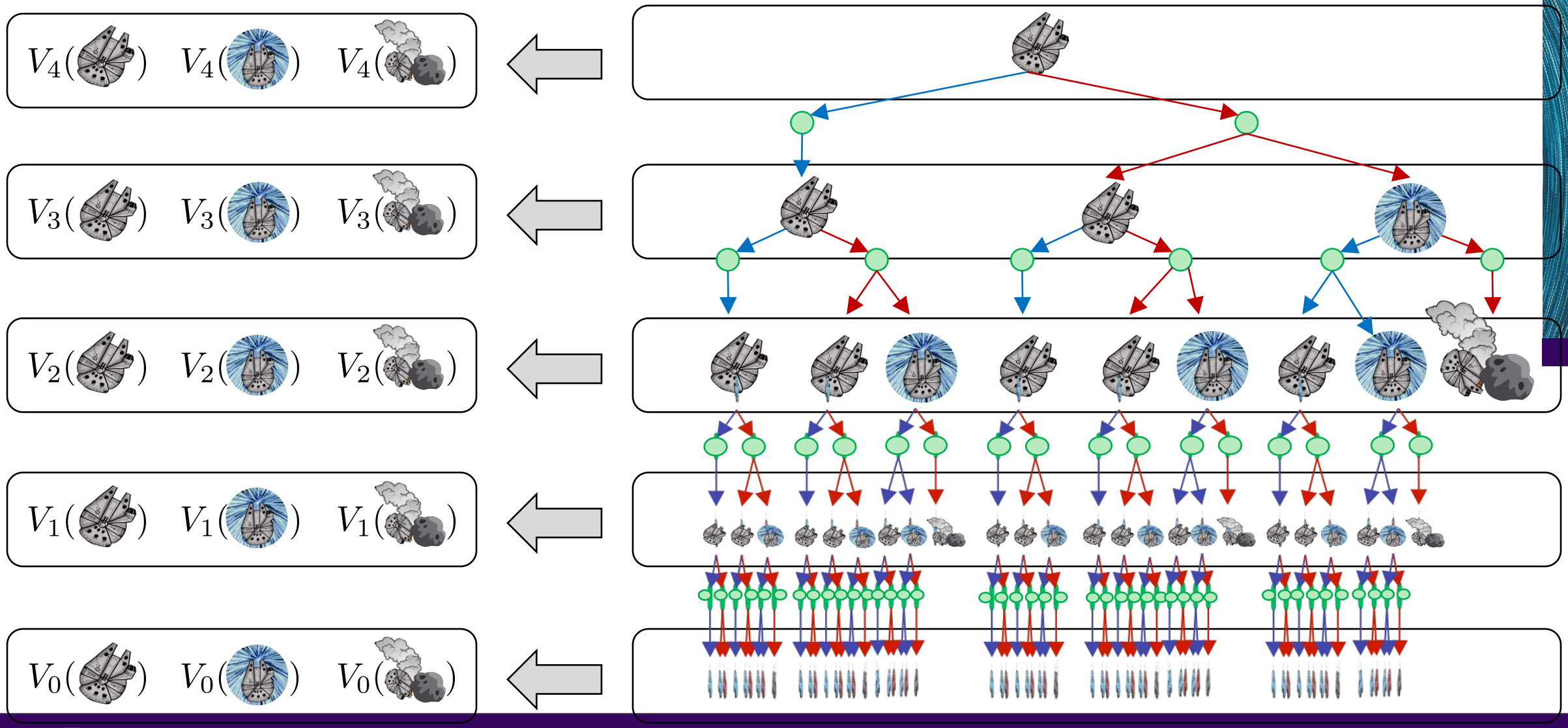




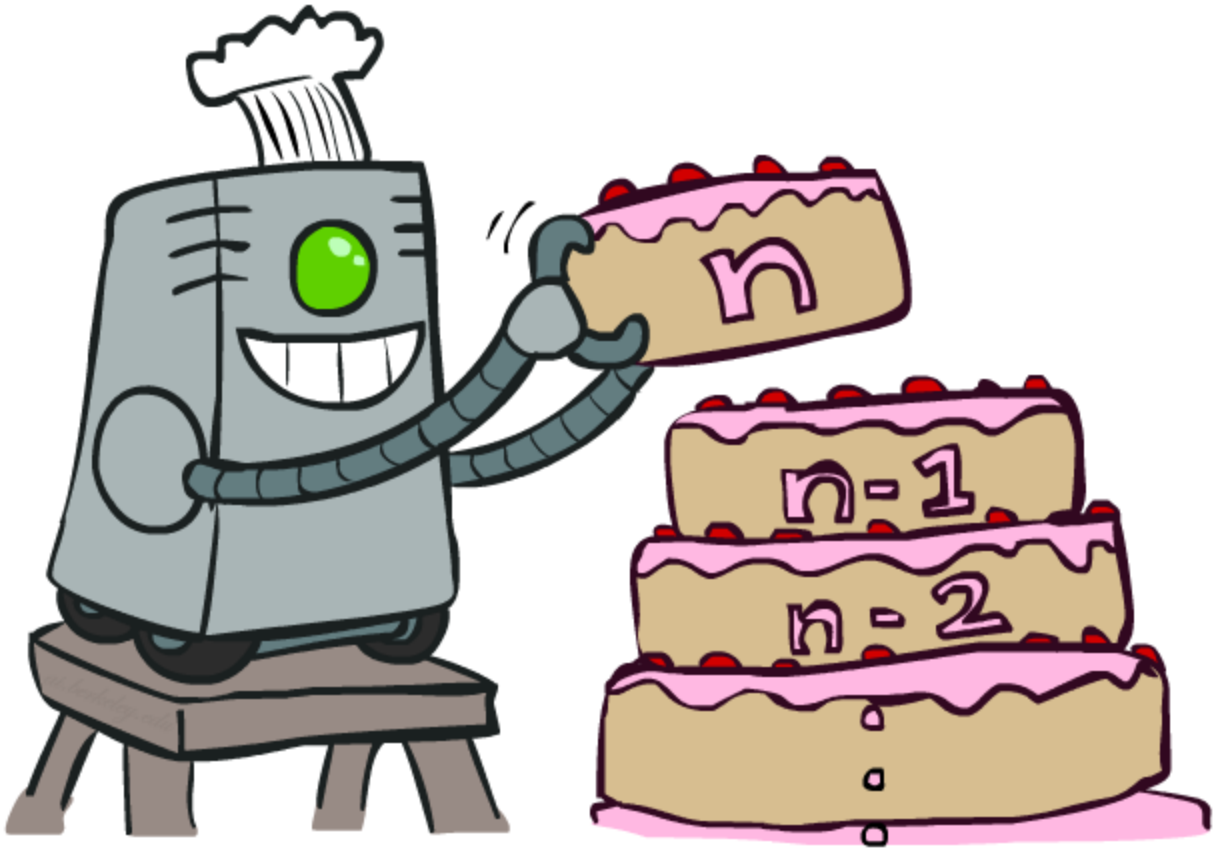




Computing Time-Limited Values



Value Iteration

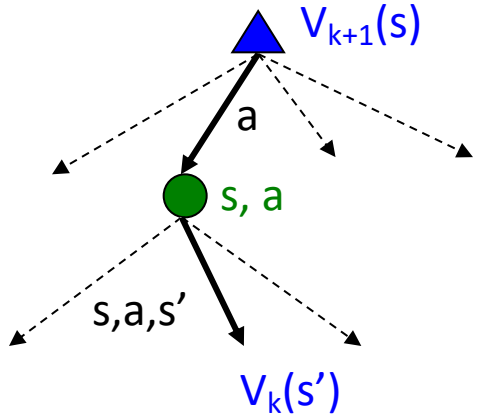


Value Iteration

Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero

Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

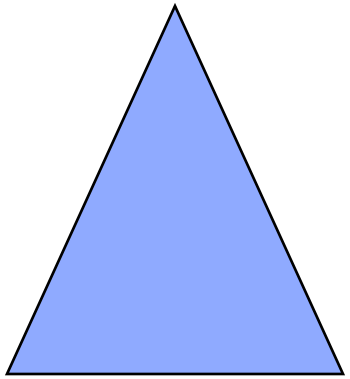


Repeat until convergence

Complexity of each iteration: $O(S^2A)$

Theorem: will converge to unique optimal values

- Basic idea: approximations get refined towards optimal values
- Policy may converge long before values do

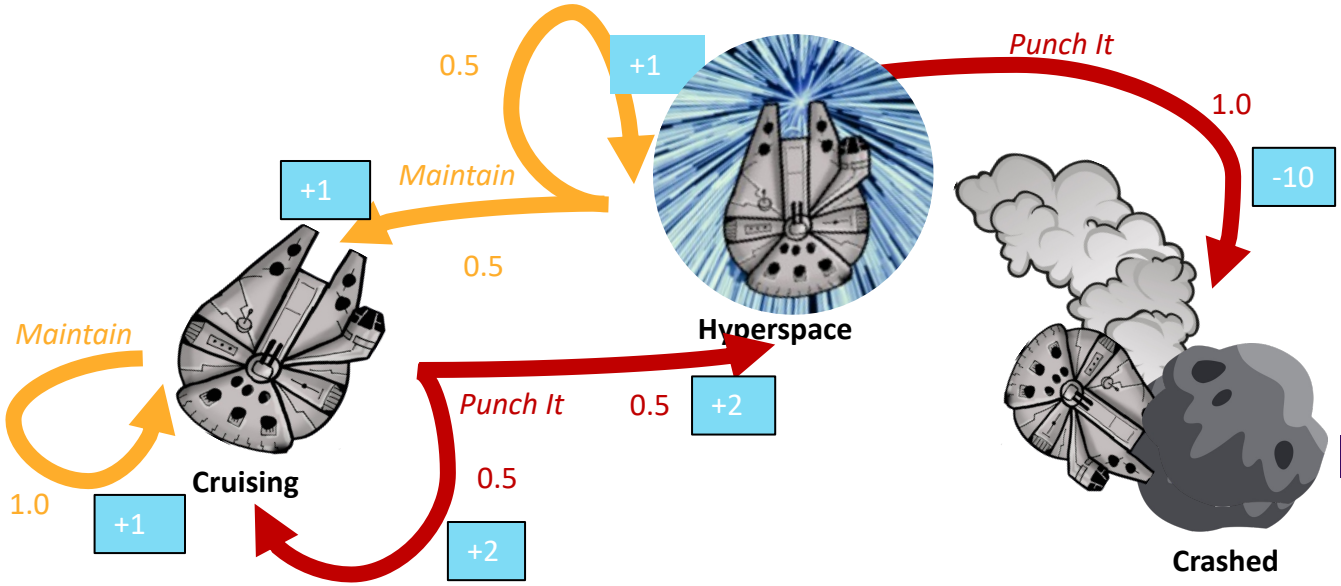


Example: Value Iteration

V_2

V_1

V_0

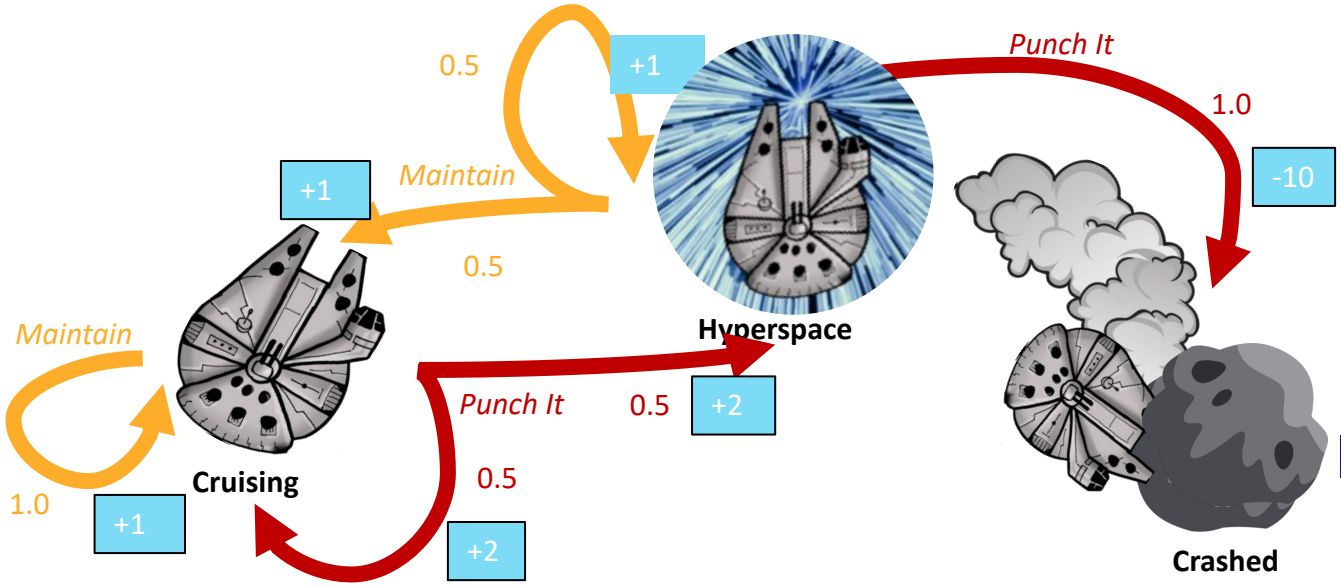


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

Example: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

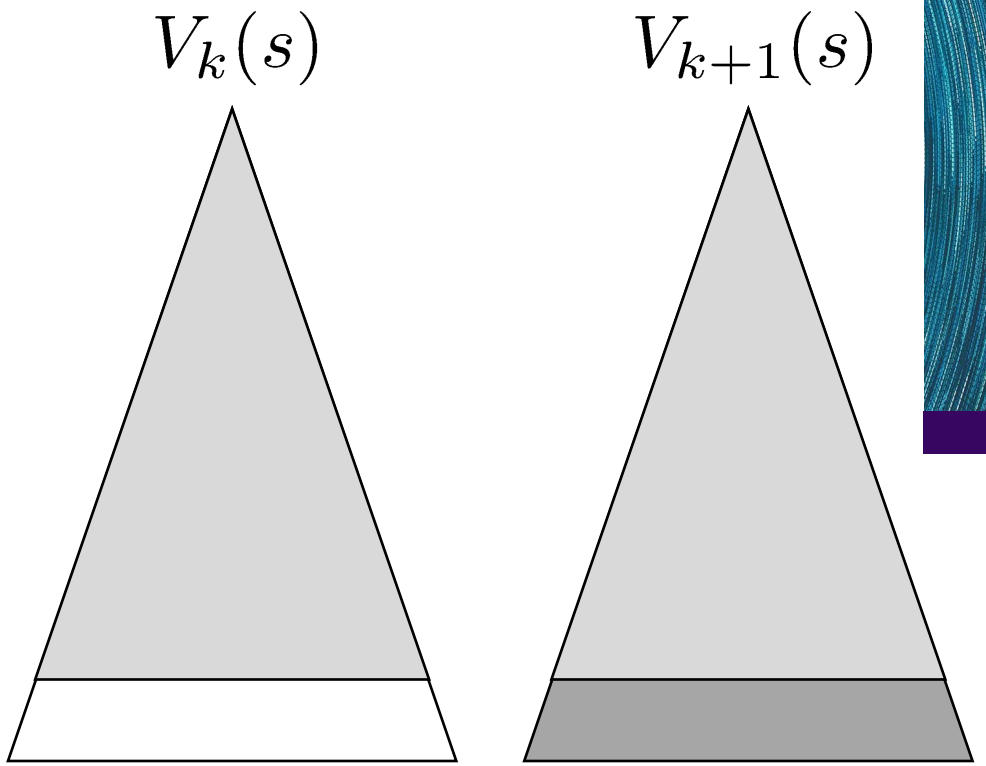
Convergence*

How do we know the V_k vectors are going to converge?

Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values

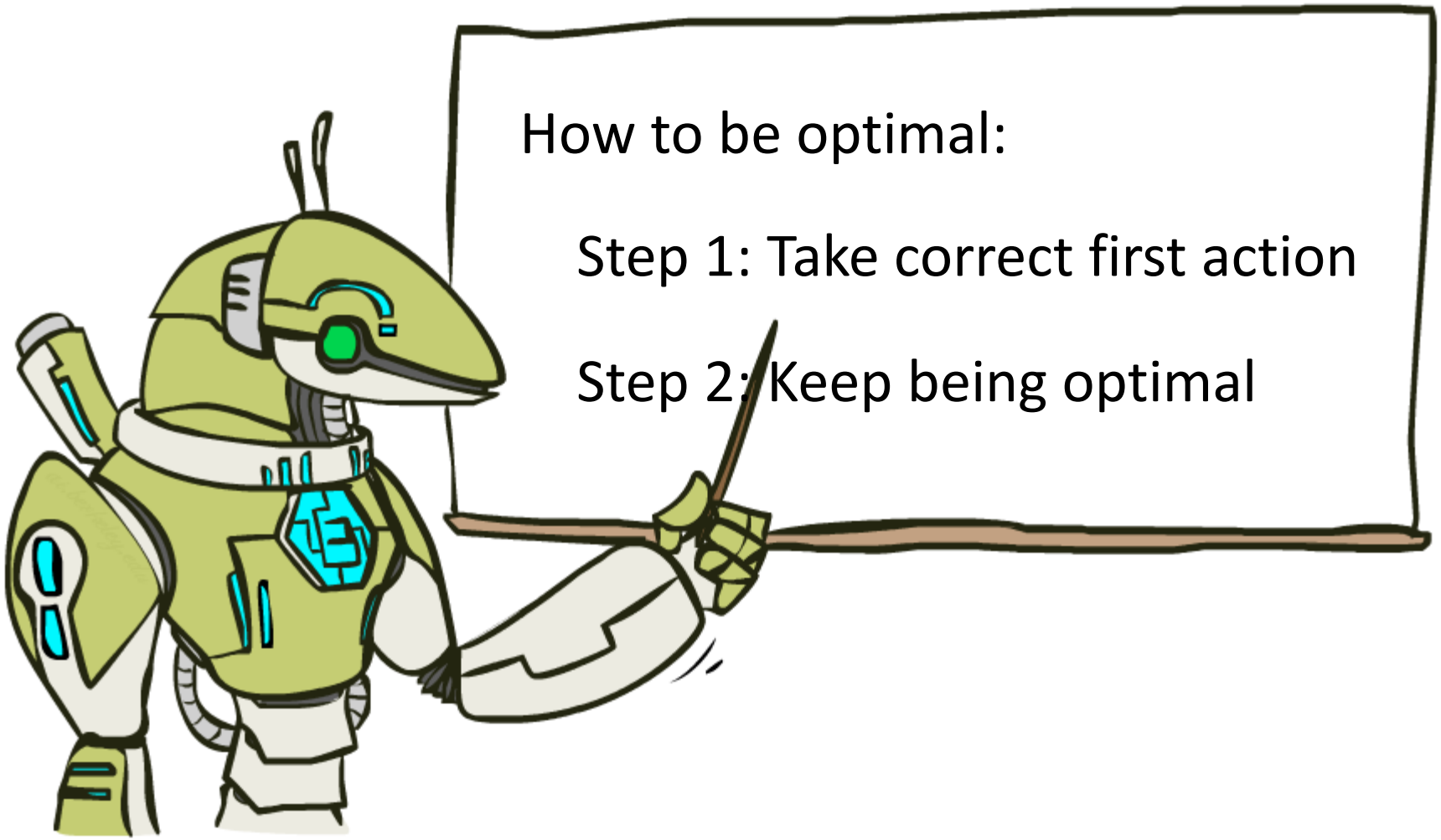
Case 2: If the discount is less than 1

- Sketch: For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
- The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
- That last layer is at best R_{MAX}
- It is at worst R_{MIN}
- But everything is discounted by γ^k that far out
- So V_k and V_{k+1} are at most $\gamma^k R_{MAX}$ different
- So as k increases, the values converge



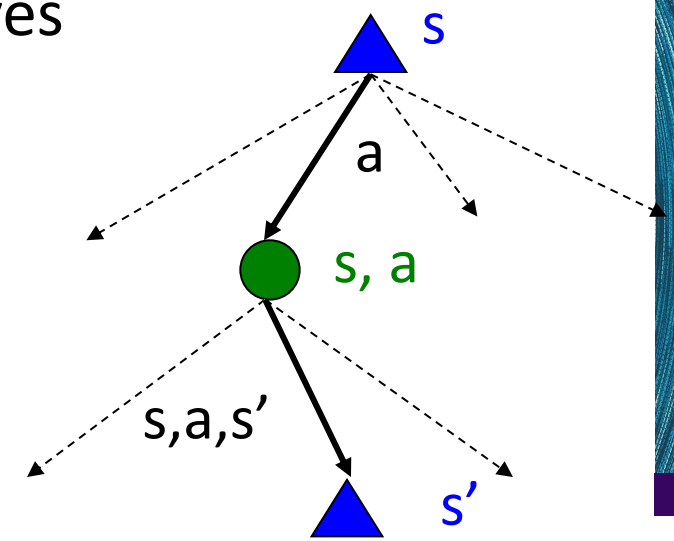
**Assuming a discount of <1*

The Bellman Equations



The Bellman Equations

Definition of "optimal utility" via expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values



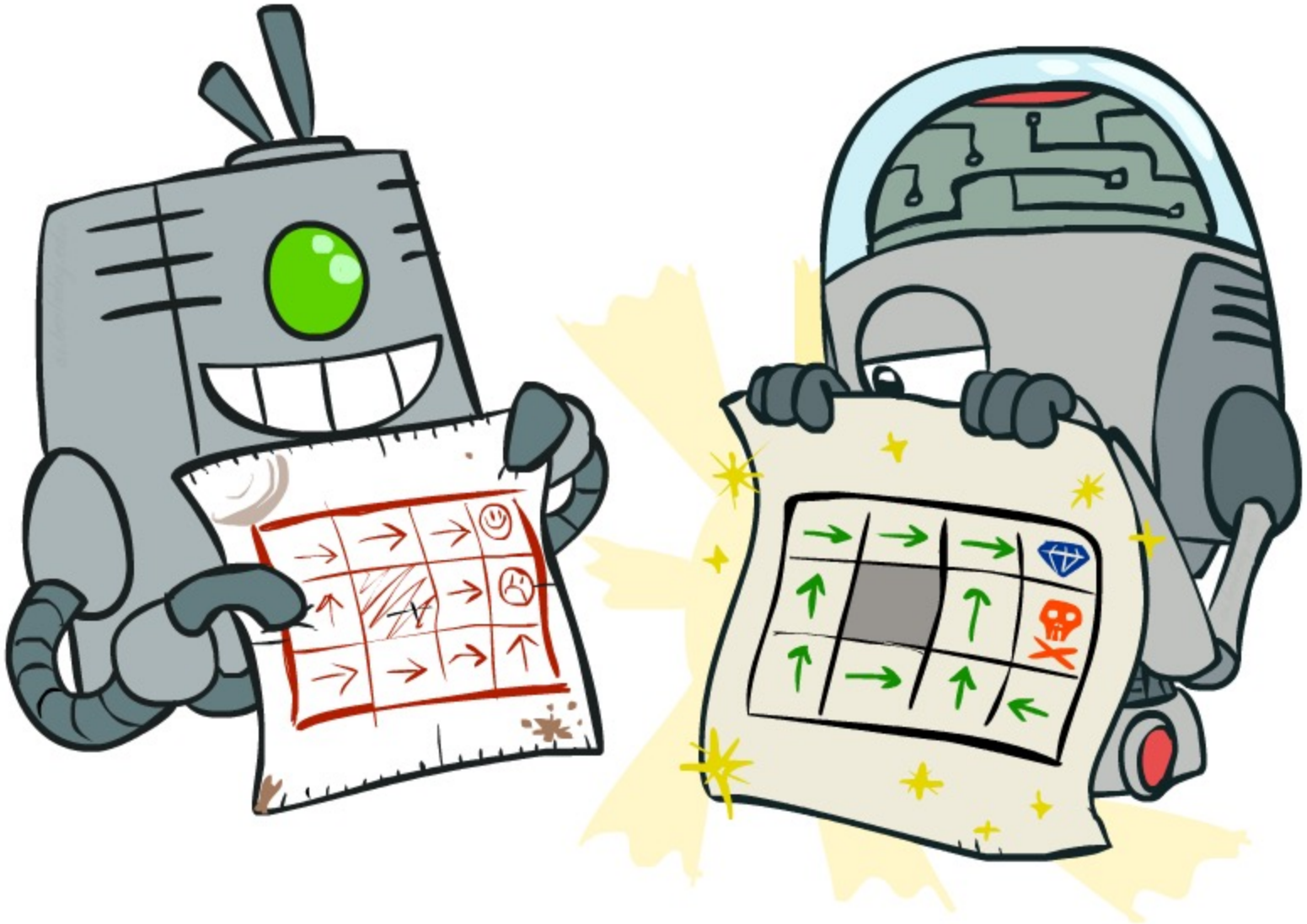
$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

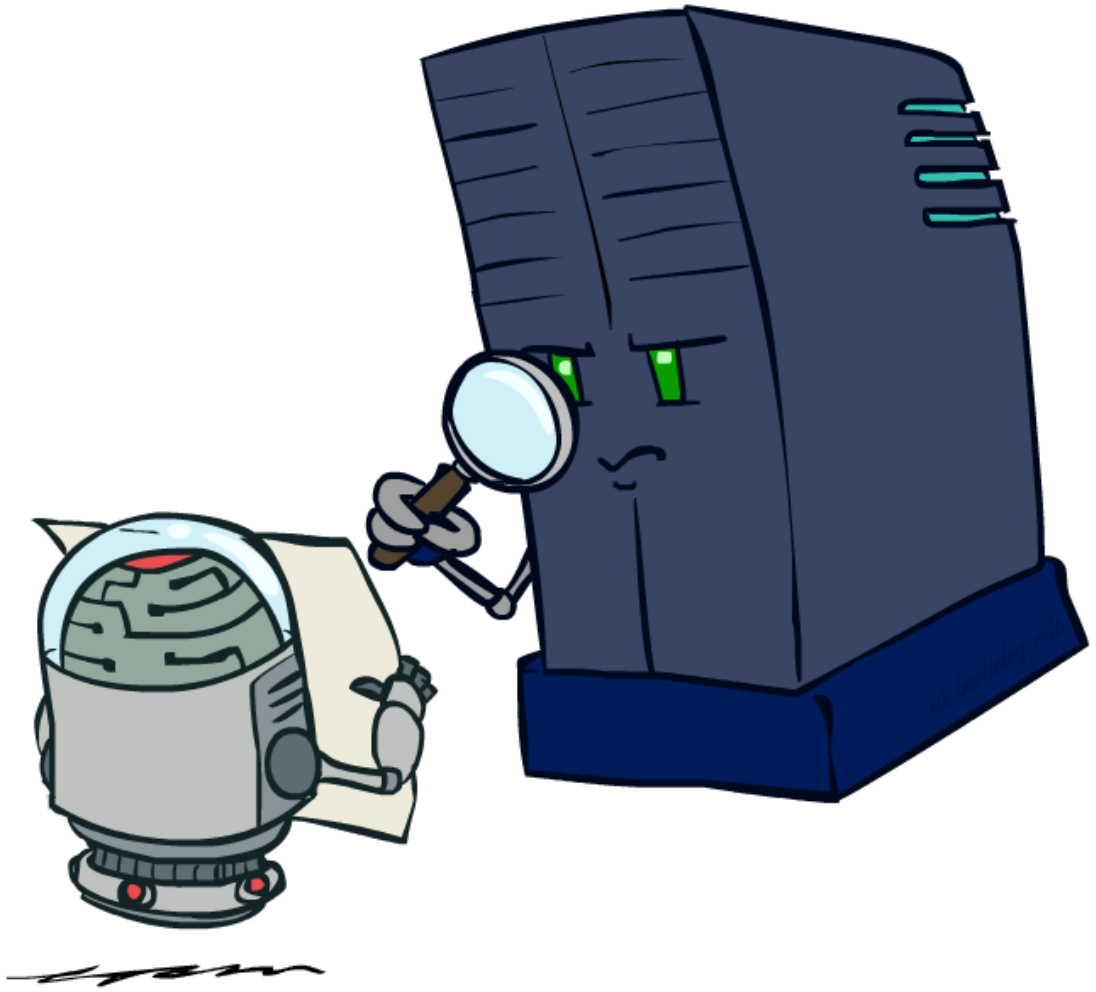
$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

These are the Bellman equations, and they characterize optimal values in a way we'll use over and over

Policy Methods

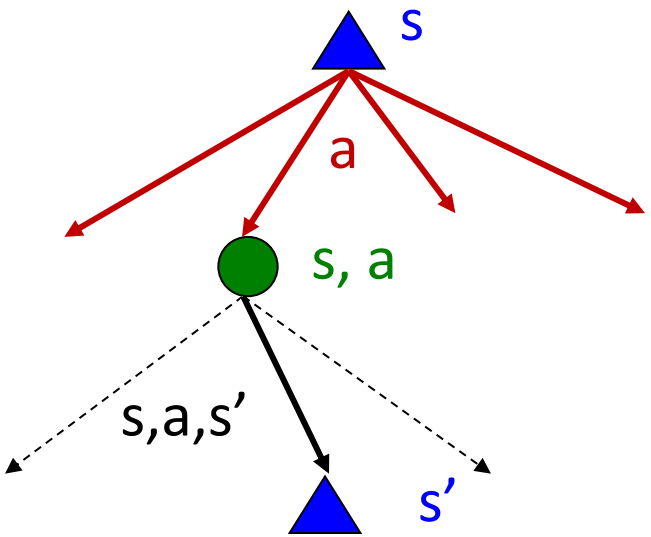


Policy Evaluation

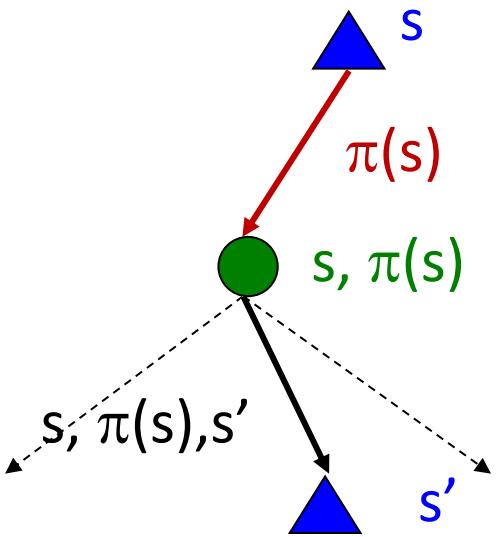


Fixed Policies

Do the optimal action



Do what π says to do



Expectimax trees max over all actions to compute the optimal values

If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state

- … though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy

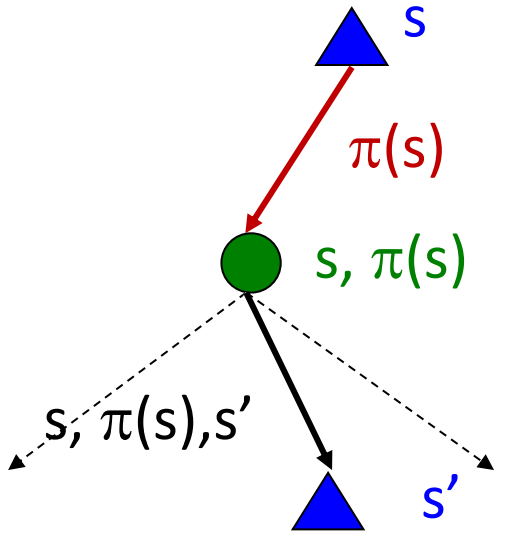
Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy

Define the utility of a state s, under a fixed policy π:

$V^{\pi}(s)$ = expected total discounted rewards starting in s and following π

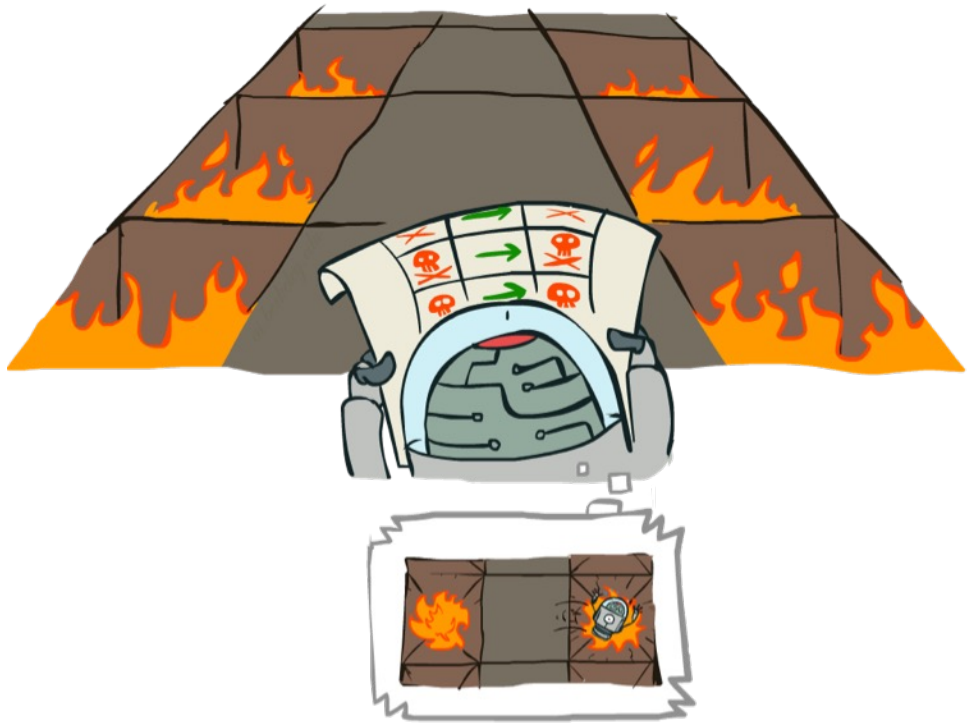
Recursive relation (one-step look-ahead / Bellman equation):

$$V^{\pi}(s) = \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

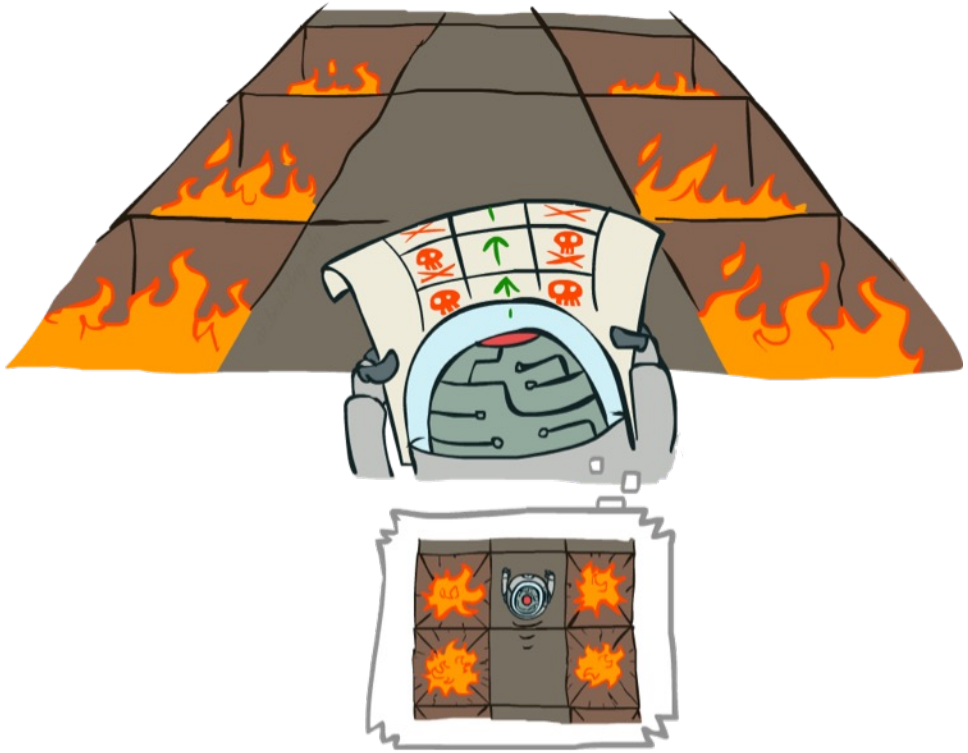


Example: Policy Evaluation

Always Go Right



Always Go Forward



Example: Policy Evaluation

Always Go Right

Always Go Forward

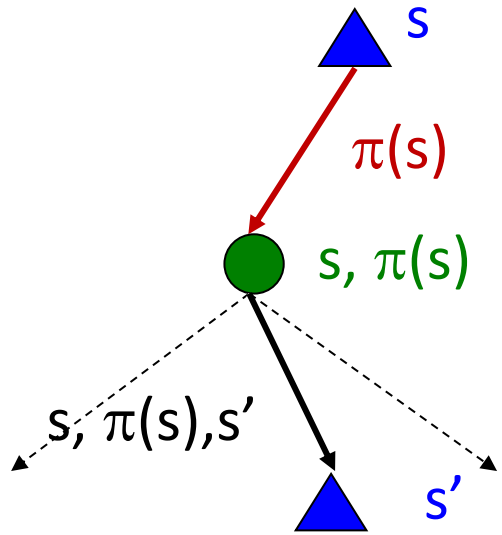
Policy Evaluation

How do we calculate the V's for a fixed policy π?

Idea 1: Turn recursive Bellman equations into updates
(like value iteration)

$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

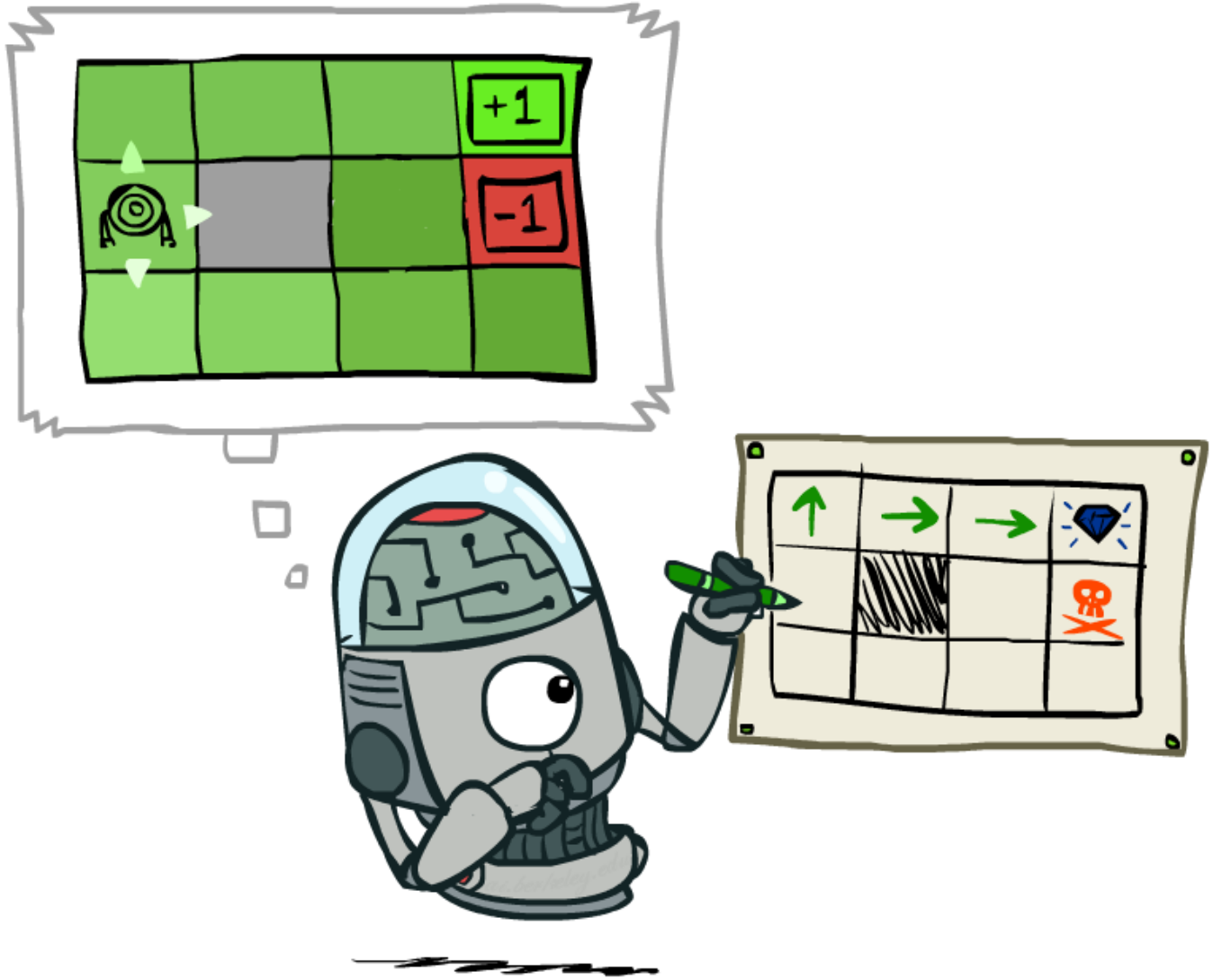


Efficiency: $O(S^2)$ per iteration

Idea 2: Without the maxes, the Bellman equations are just a linear system

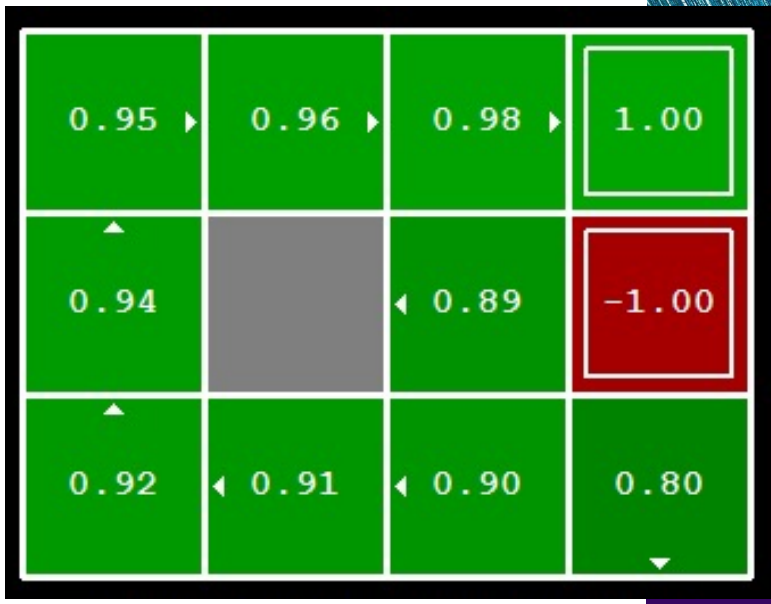
- Solve with Matlab (or your favorite linear system solver)

Policy Extraction



Computing Actions from Values

Let's imagine we have the optimal values V*(s)



How should we act?

- It's not obvious!

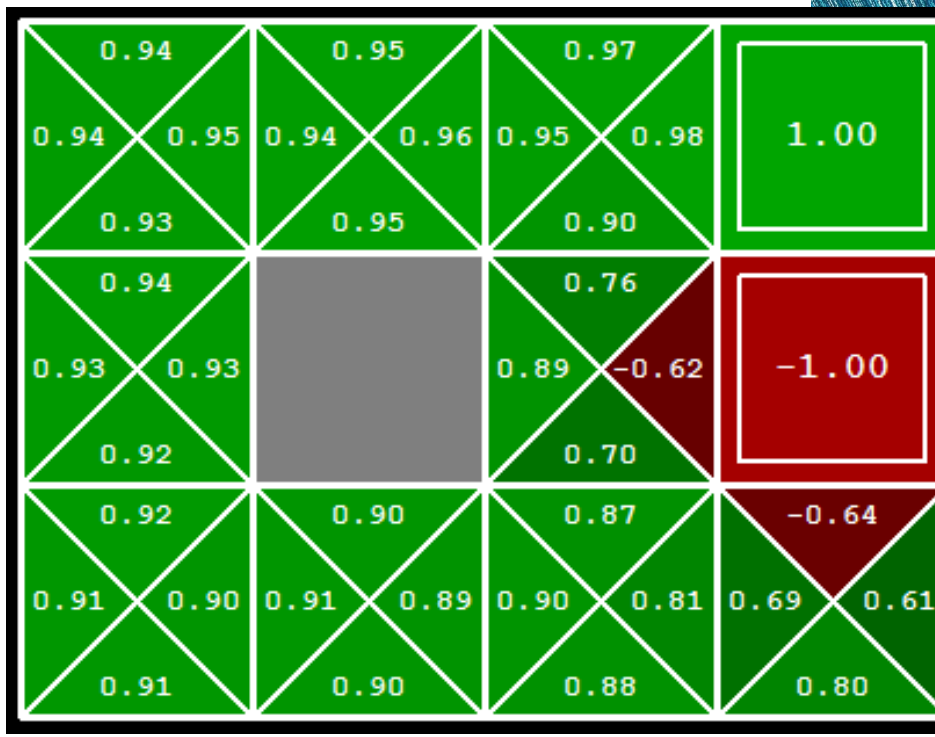
We need to do a mini-expectimax (one step)

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

This is called policy extraction, since it gets the policy implied by the values

Computing Actions from Q-Values

Let's imagine we have the optimal q-values:



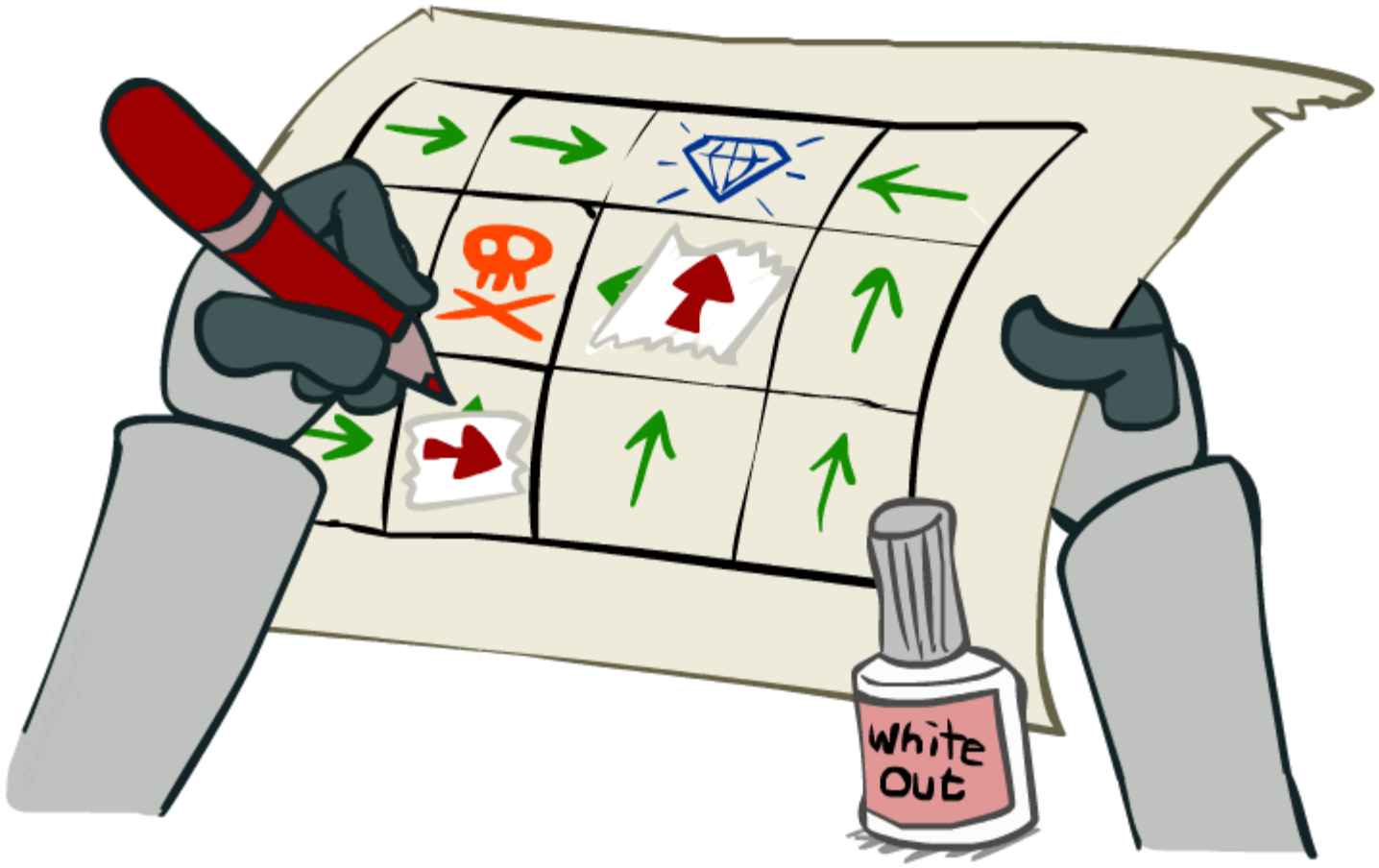
How should we act?

- Completely trivial to decide!

$$\pi^*(s) = \arg\max_a Q^*(s, a)$$

Important lesson: actions are easier to select from q-values than values!

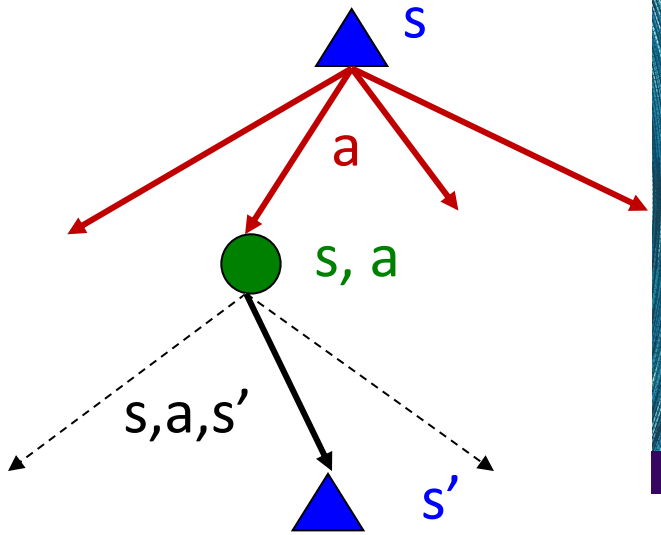
Policy Iteration



Problems with Value Iteration

Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$



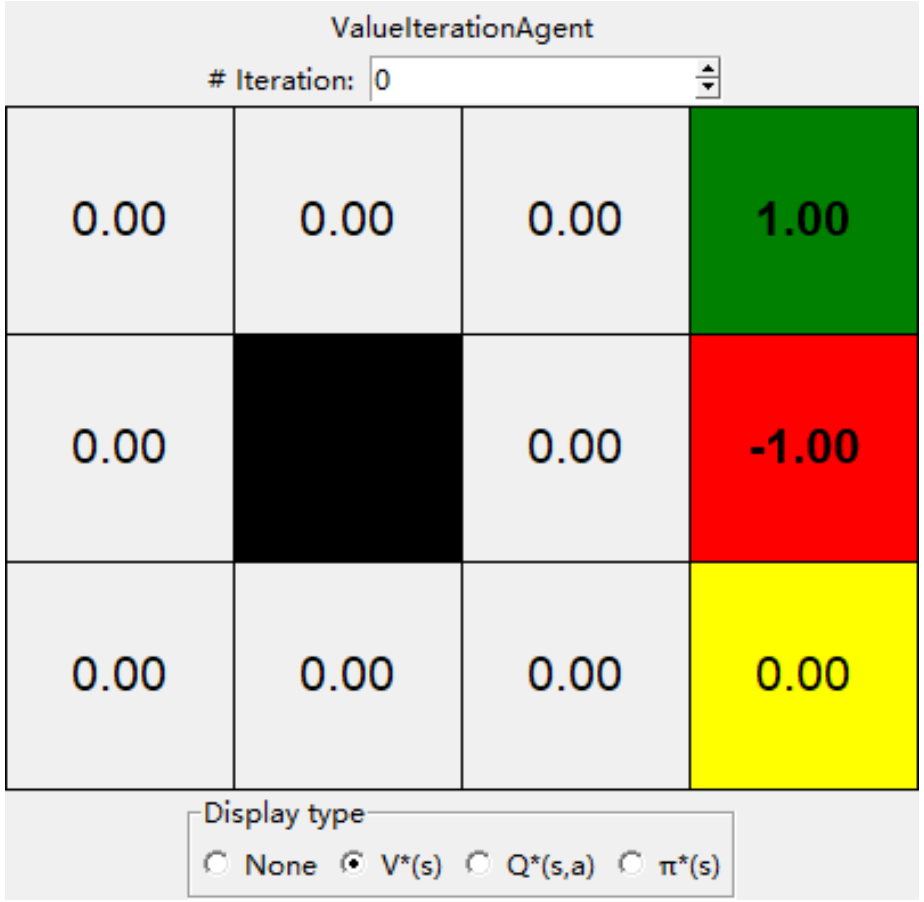
Problem 1: It's slow – $O(S^2A)$ per iteration

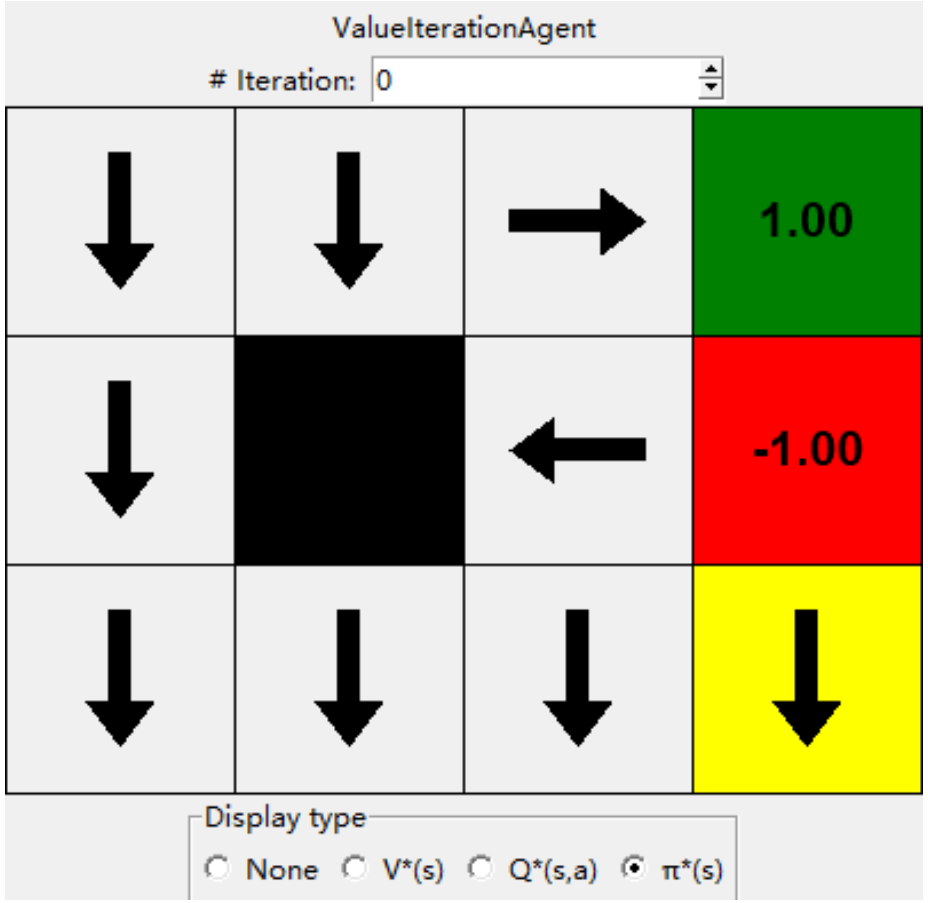
Problem 2: The "max" at each state rarely changes

Problem 3: The policy often converges long before the values

[Demo: value iteration (L9D2)]

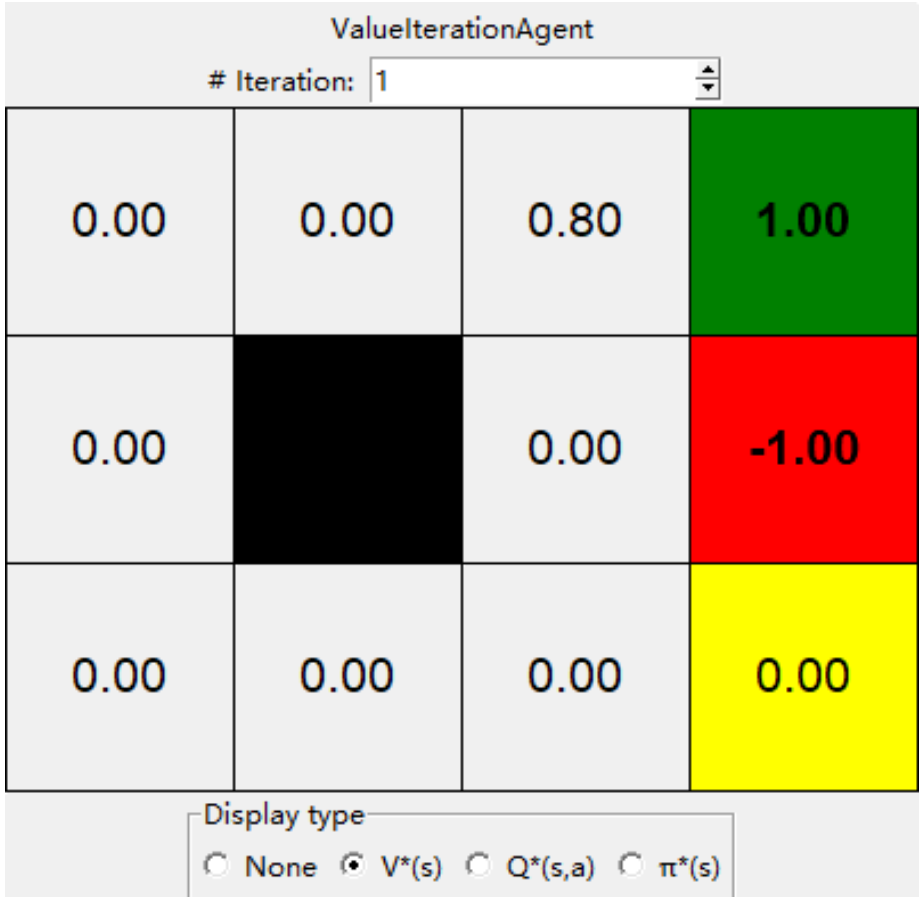
k=0

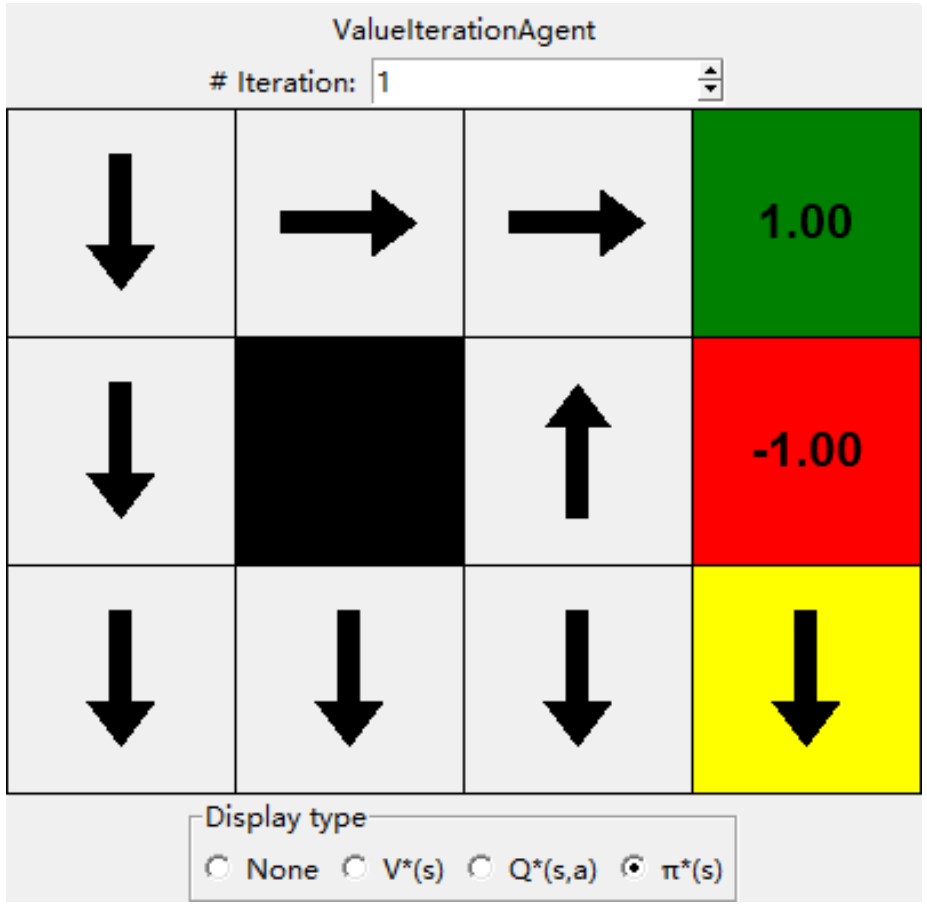




Noise = 0.2
Discount = 0.9
Living reward = 0

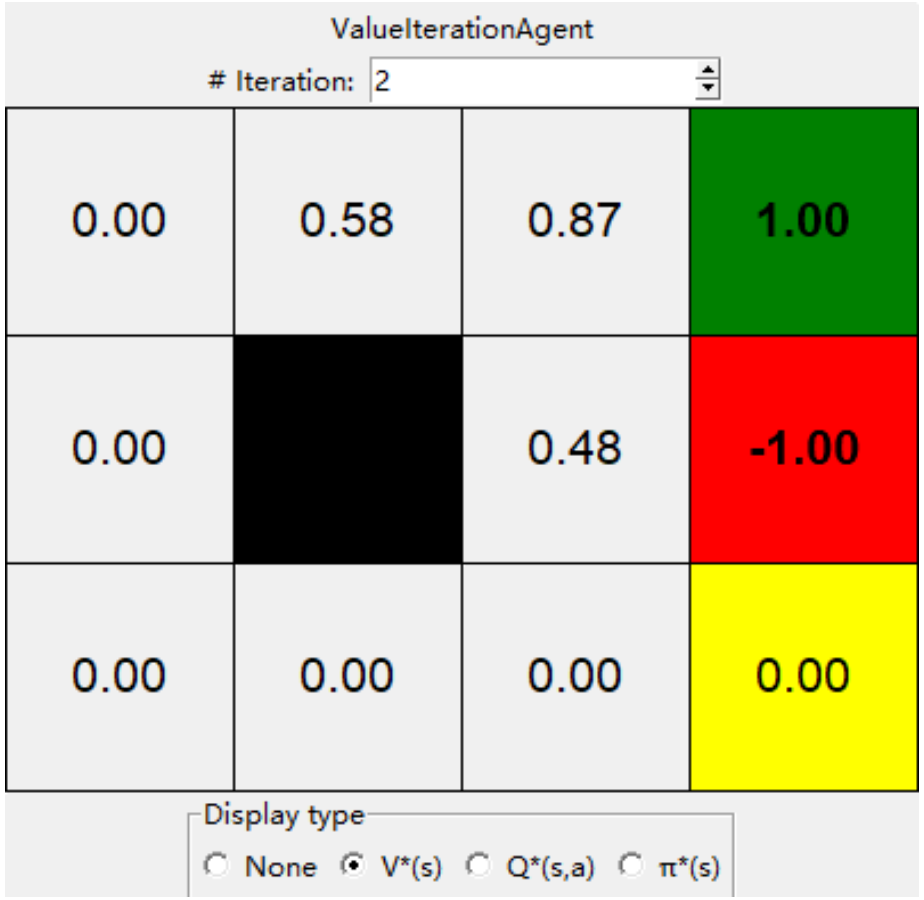
k=1

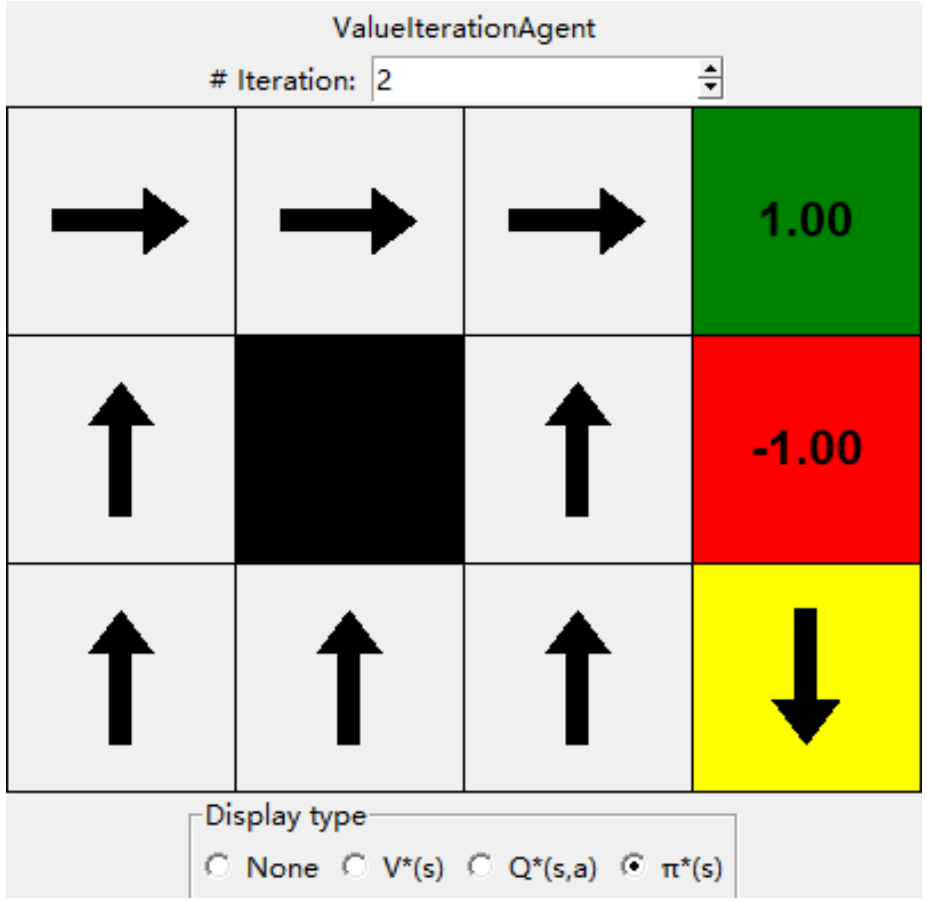




Noise = 0.2
Discount = 0.9
Living reward = 0

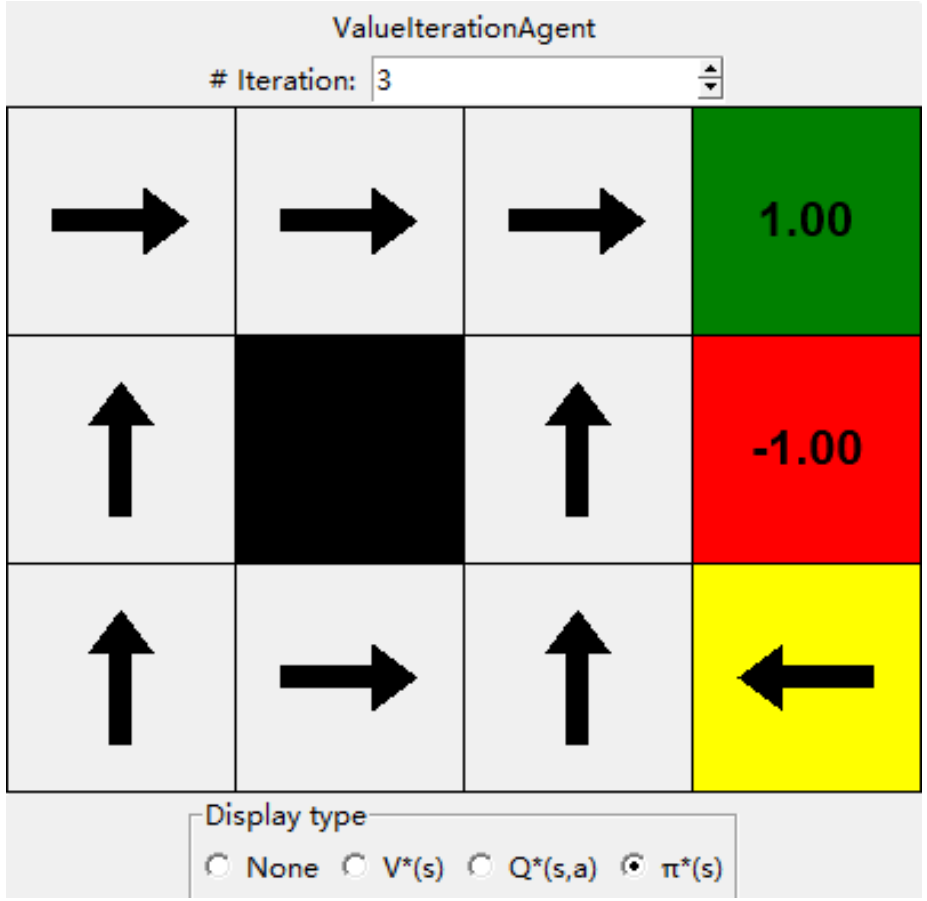
k=2





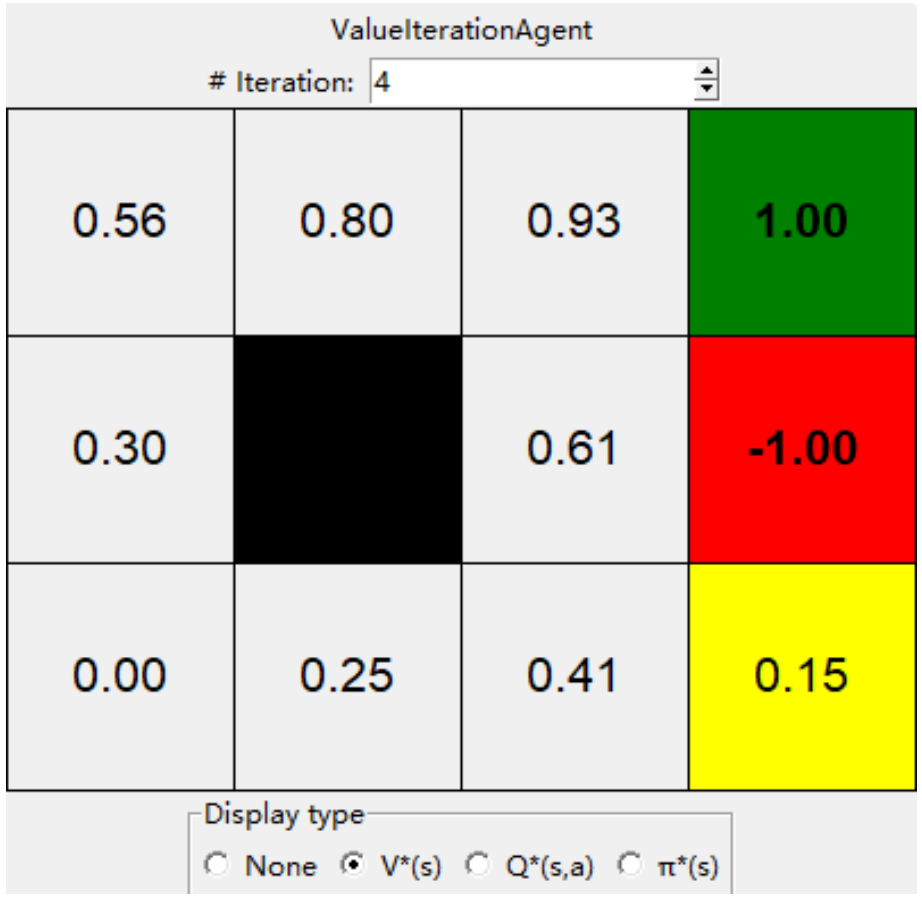
Noise = 0.2
Discount = 0.9
Living reward = 0

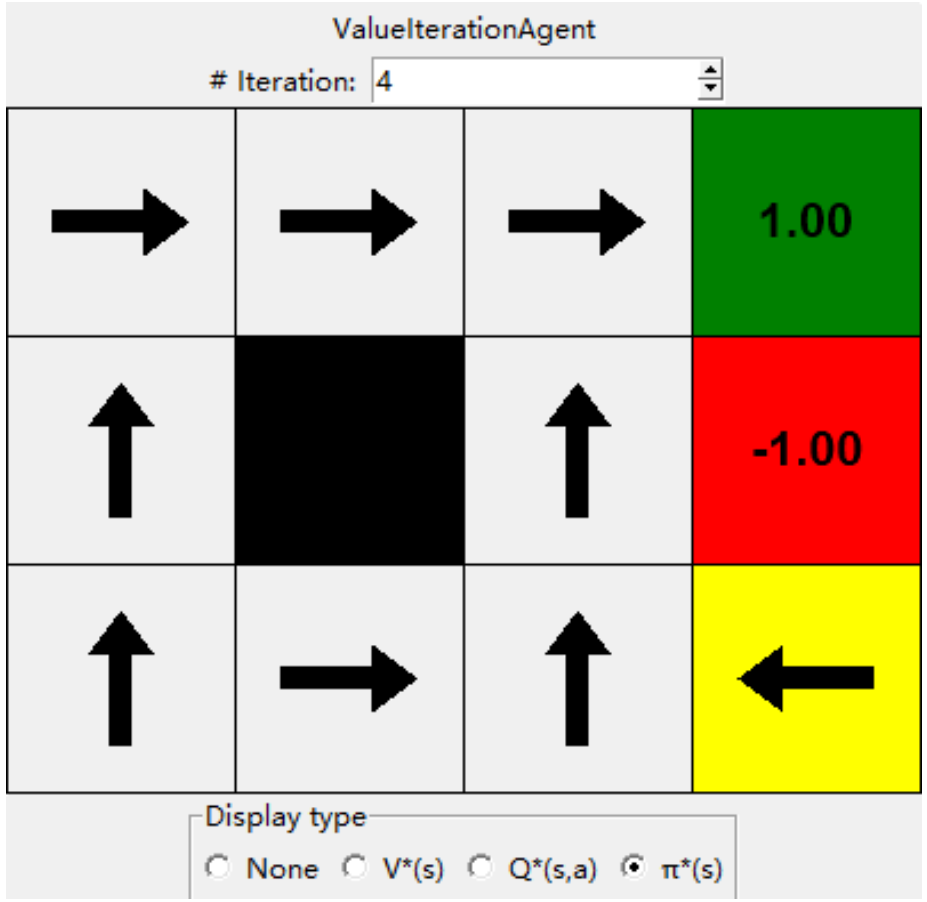
k=3



Noise = 0.2
Discount = 0.9
Living reward = 0

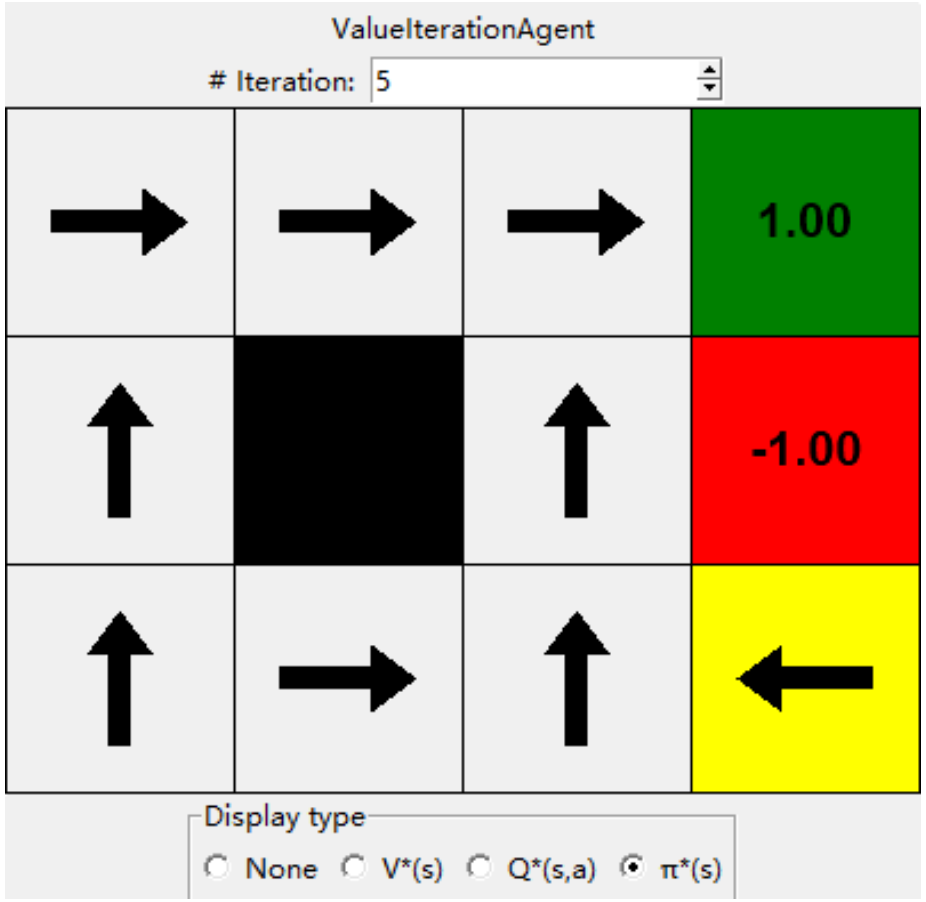
k=4





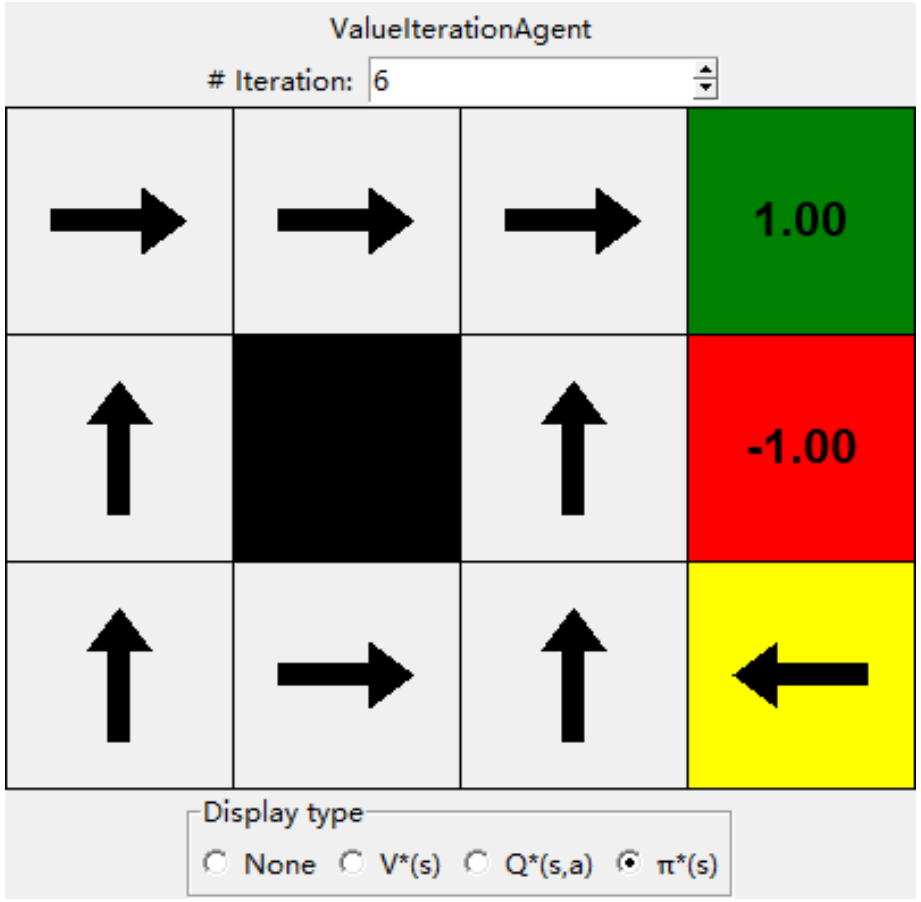
Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



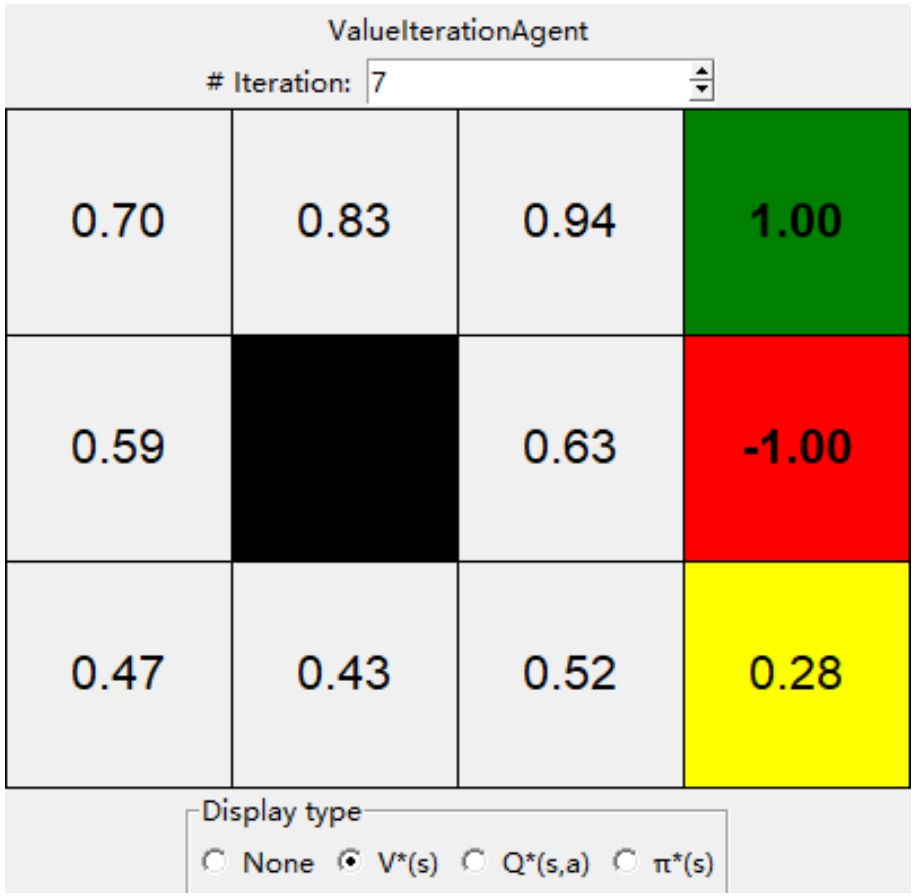
Noise = 0.2
Discount = 0.9
Living reward = 0

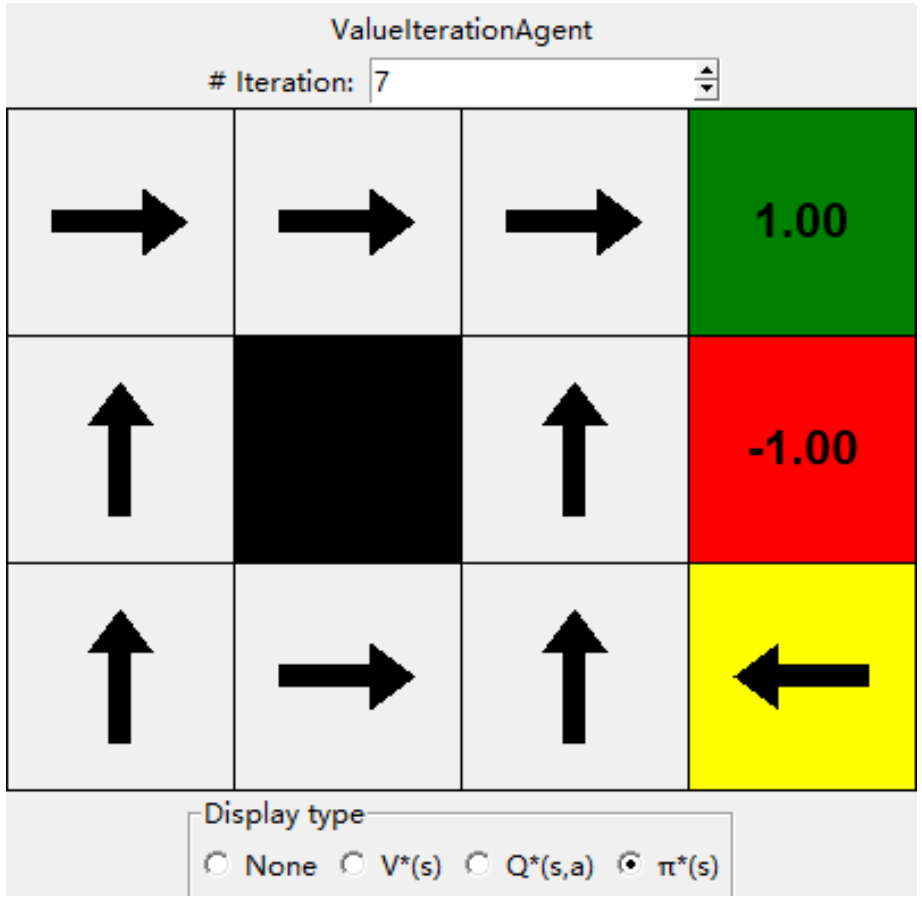
k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

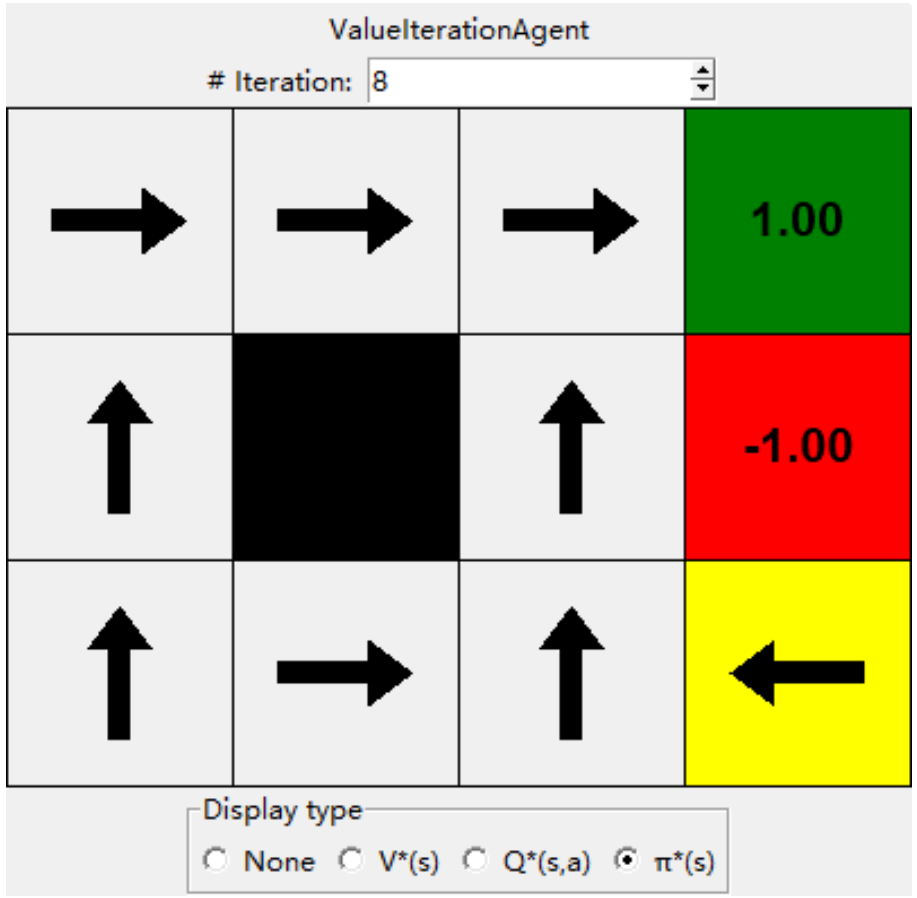
k=7





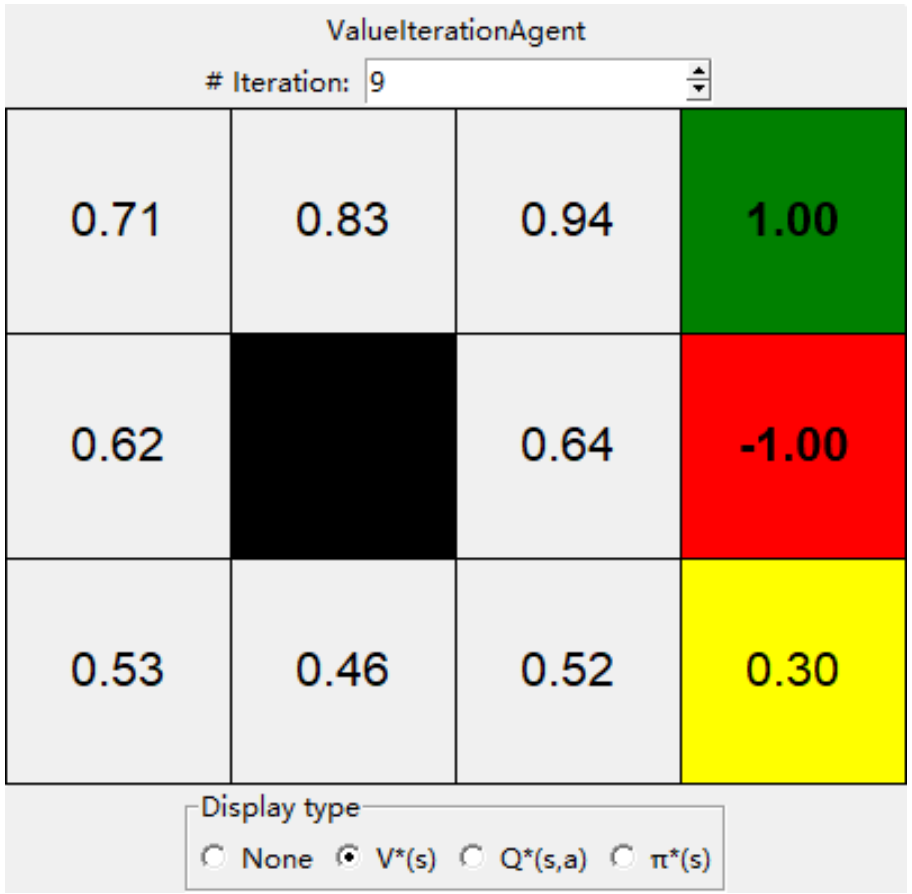
Noise = 0.2
Discount = 0.9
Living reward = 0

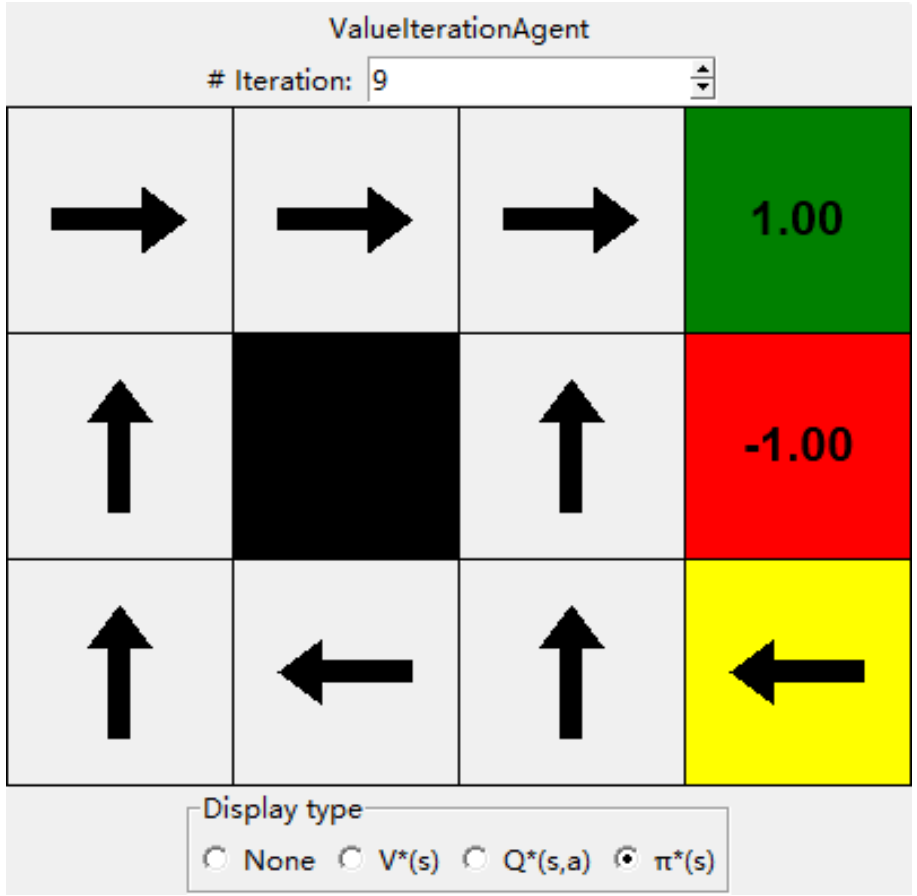
k=8



Noise = 0.2
Discount = 0.9
Living reward = 0

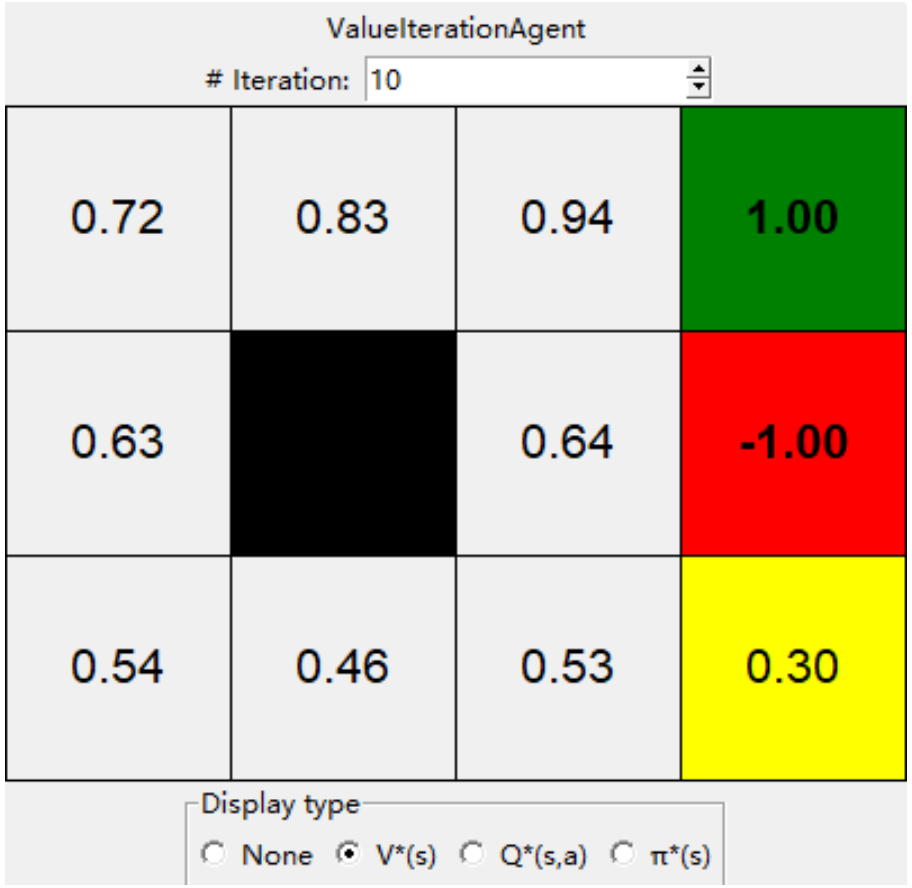
k=9

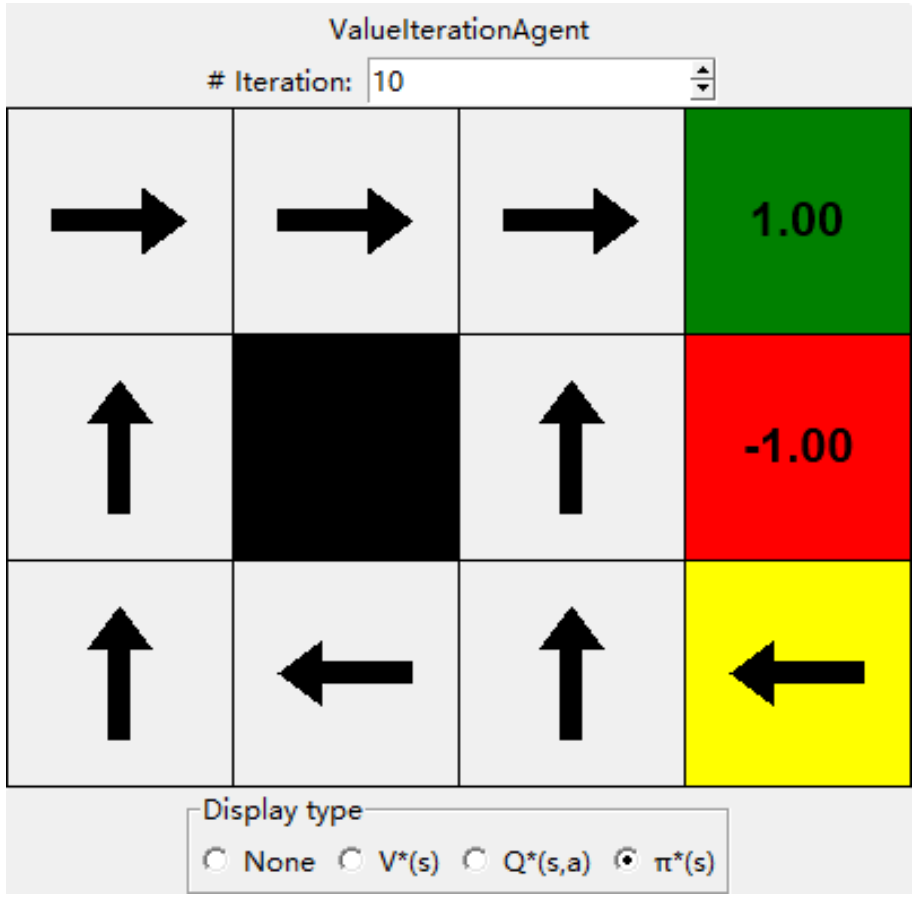




Noise = 0.2
Discount = 0.9
Living reward = 0

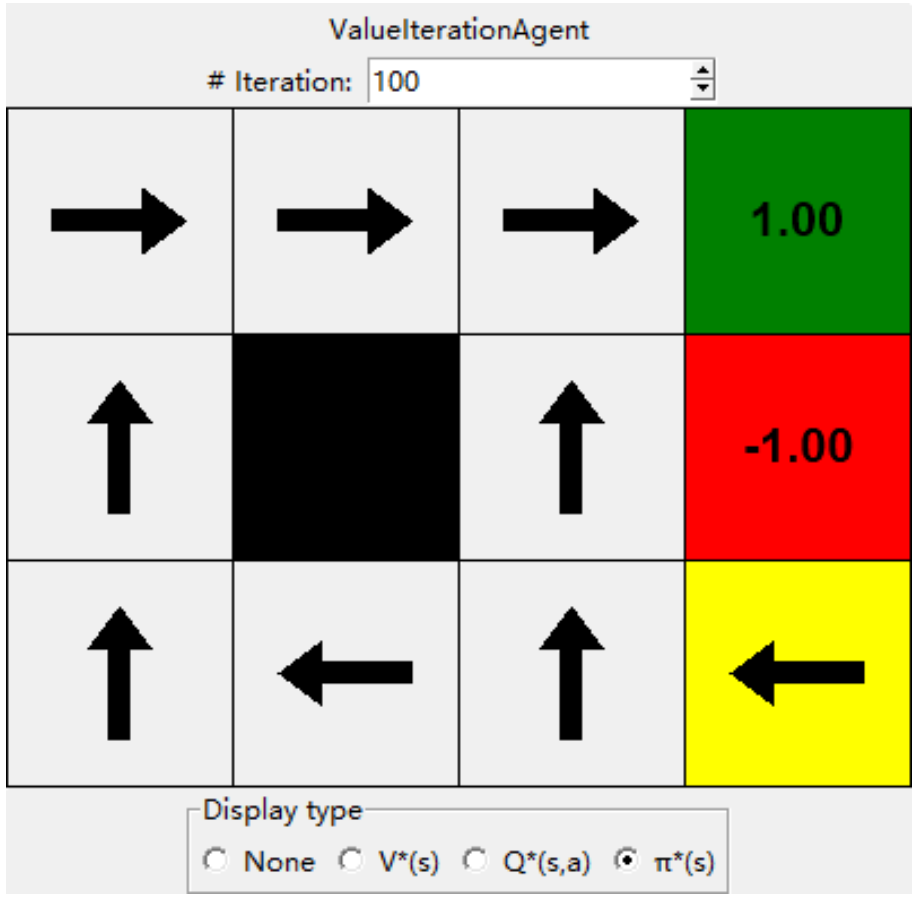
k=10





Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

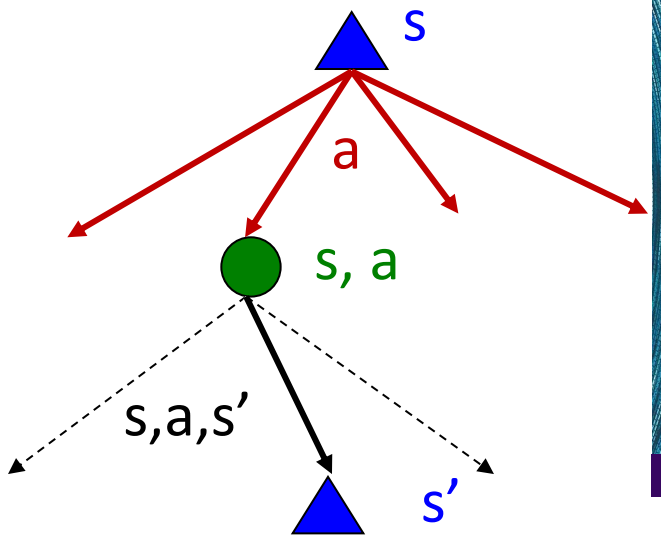


Noise = 0.2
Discount = 0.9
Living reward = 0

Problems with Value Iteration

Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$



Problem 1: It's slow – $O(S^2A)$ per iteration

Problem 2: The "max" at each state rarely changes

Problem 3: The policy often converges long before the values

Policy Iteration

Alternative approach for optimal values:

- Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
- Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
- Repeat steps until policy converges

This is policy iteration

- It's still optimal!
- Can converge (much) faster under some conditions

Policy Iteration

Evaluation: For fixed current policy π, find values with policy evaluation:

- Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s') \right]$$

Improvement: For fixed values, get a better policy using policy extraction

- One-step look-ahead:

$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s') \right]$$

Comparison

Both value iteration and policy iteration compute the same thing (all optimal values)

In value iteration:

- Every iteration updates both the values and (implicitly) the policy
- We don’t track the policy, but taking the max over actions implicitly recomputes it

In policy iteration:

- We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
- After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
- The new policy will be better (or we’re done)

Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

So you want to….

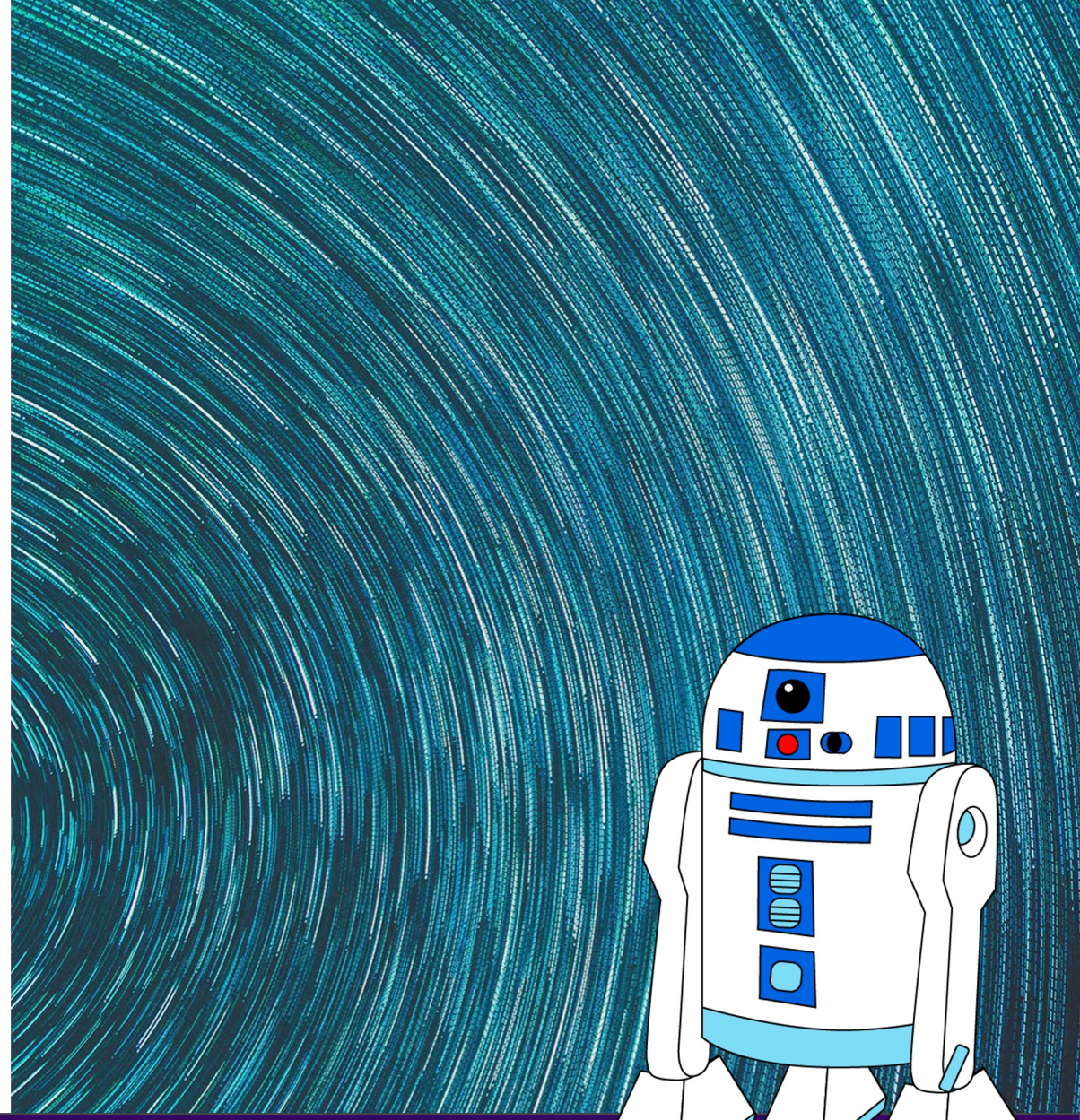
- Compute optimal values: use value iteration or policy iteration
- Compute values for a particular policy: use policy evaluation
- Turn your values into a policy: use policy extraction (one-step lookahead)

These all look the same!

- They basically are – they are all variations of Bellman updates
- They all use one-step lookahead expectimax fragments
- They differ only in whether we plug in a fixed policy or max over actions

CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

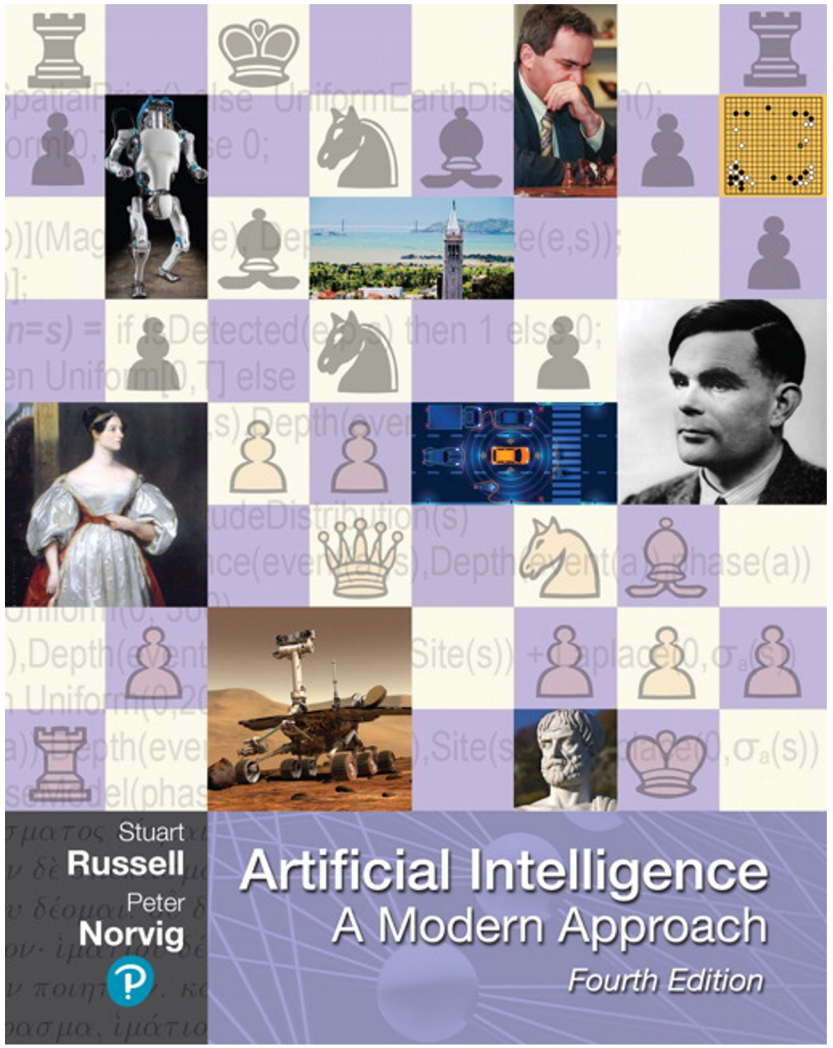
Maximum Expected Utilities



Penn Engineering
University of Pennsylvania

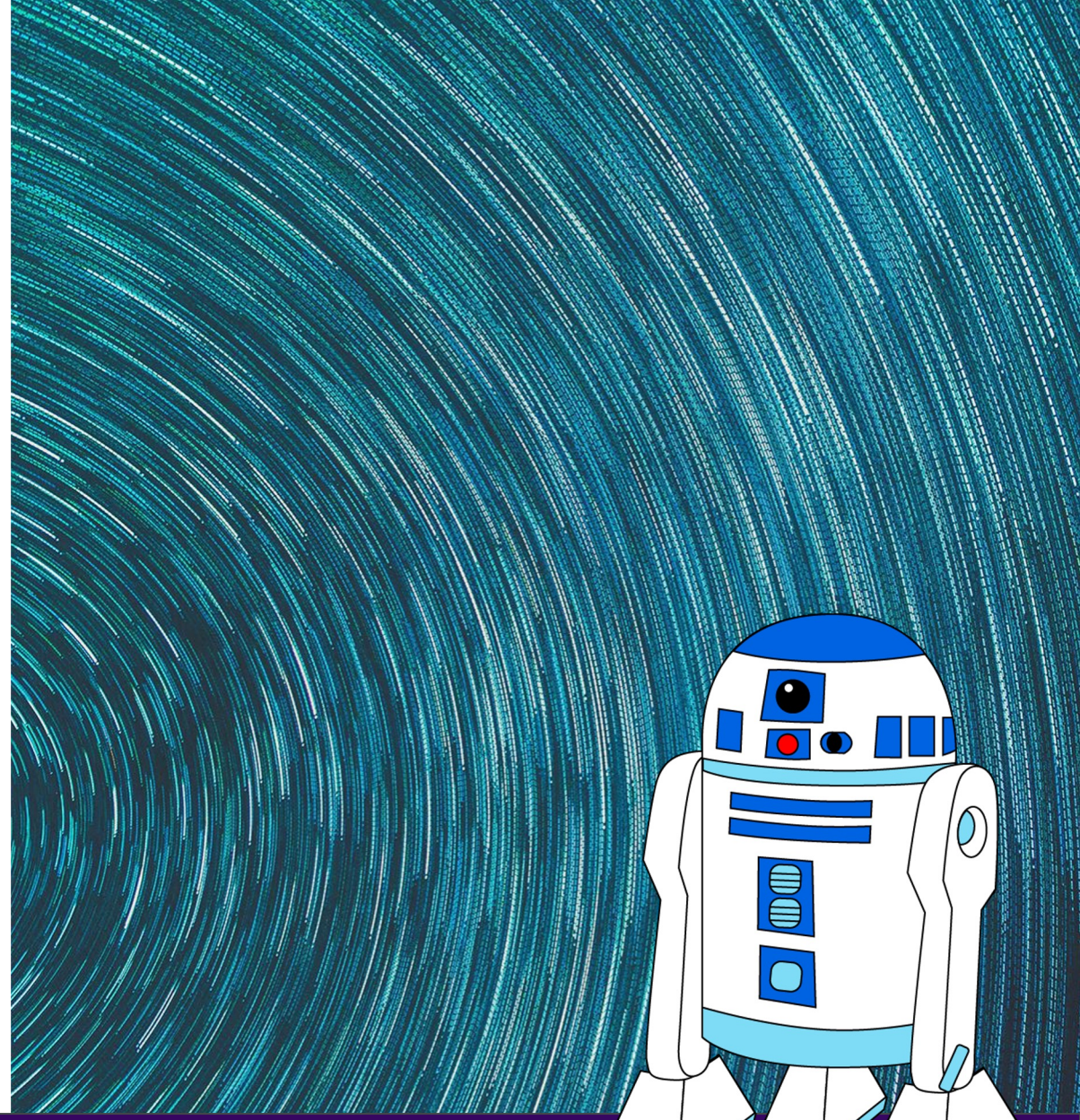
Utilities

- Read Chapter 16 of the textbook (sections 16.1-16.3)



CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

Utilities



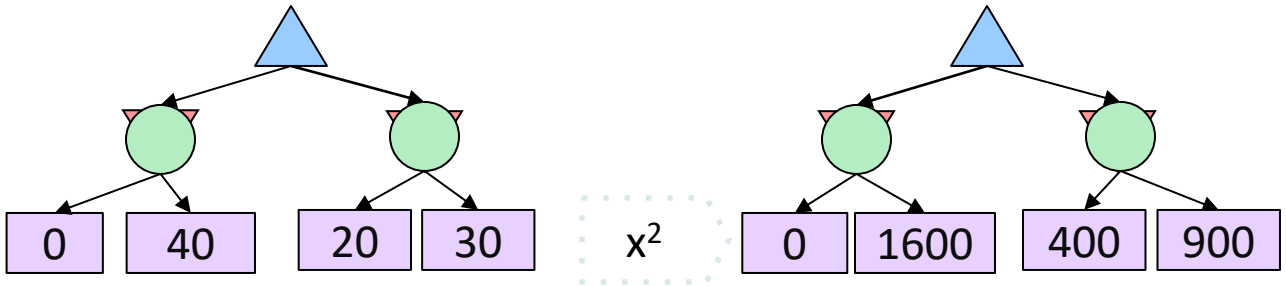
Penn Engineering
University of Pennsylvania

Maximum Expected Utility

- Why should we average utilities? Why not minimax?

- Principle of maximum expected utility:
  - A rational agent should choose the action that maximizes its expected utility, given its knowledge
- Questions:
  - Where do utilities come from?
  - How do we know such utilities even exist?
  - How do we know that averaging even makes sense?
  - What if our behavior (preferences) can't be described by utilities?

Why Expectations?



- For worst-case minimax reasoning, terminal function scale doesn't matter
  - We just want better states to have higher evaluations (get the ordering right)
  - We call this insensitivity to monotonic transformations

- For average-case expectimax reasoning, we need *magnitudes* to be meaningful

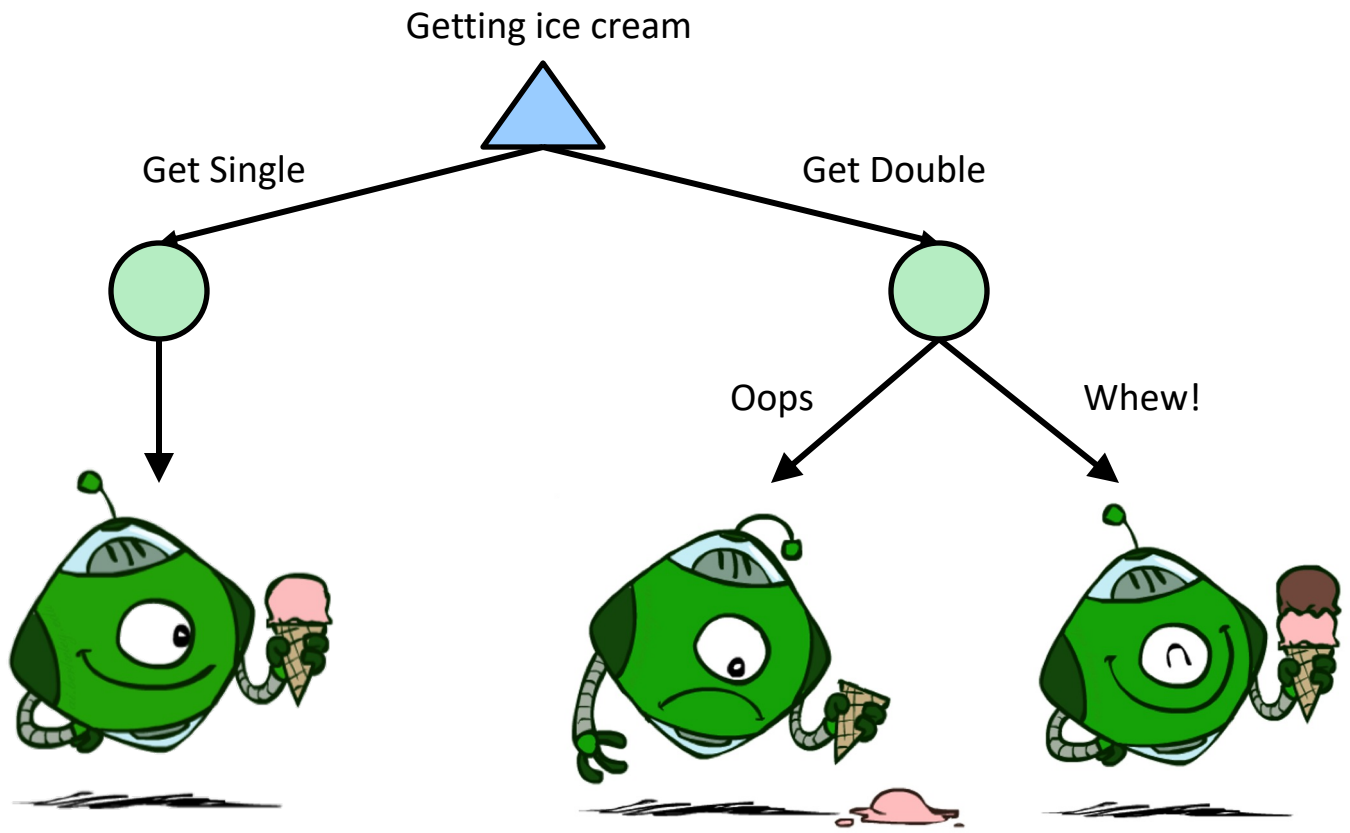
Utilities

- Utilities are functions from outcomes (states of the world) to real numbers that describe an agent's preferences

- Where do utilities come from?
  - In a game, may be simple (+1/-1)
  - Utilities summarize the agent's goals
  - Theorem: any "rational" preferences can be summarized as a utility function

- We hard-wire utilities and let behaviors emerge
  - Why don't we let agents pick utilities?
  - Why don't we prescribe behaviors?

Utilities: Uncertain Outcomes

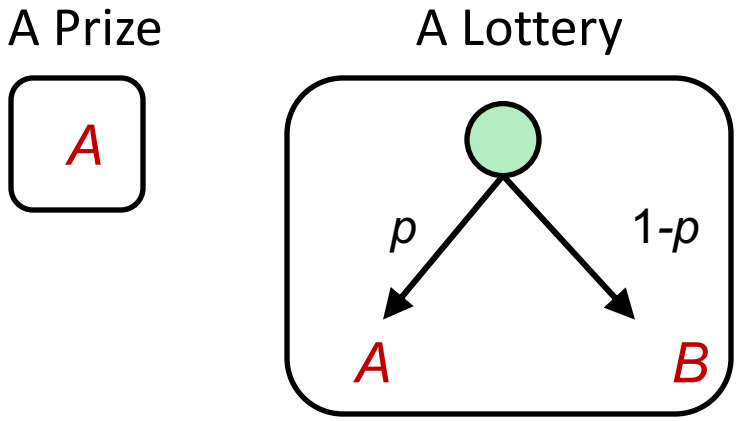


Preferences

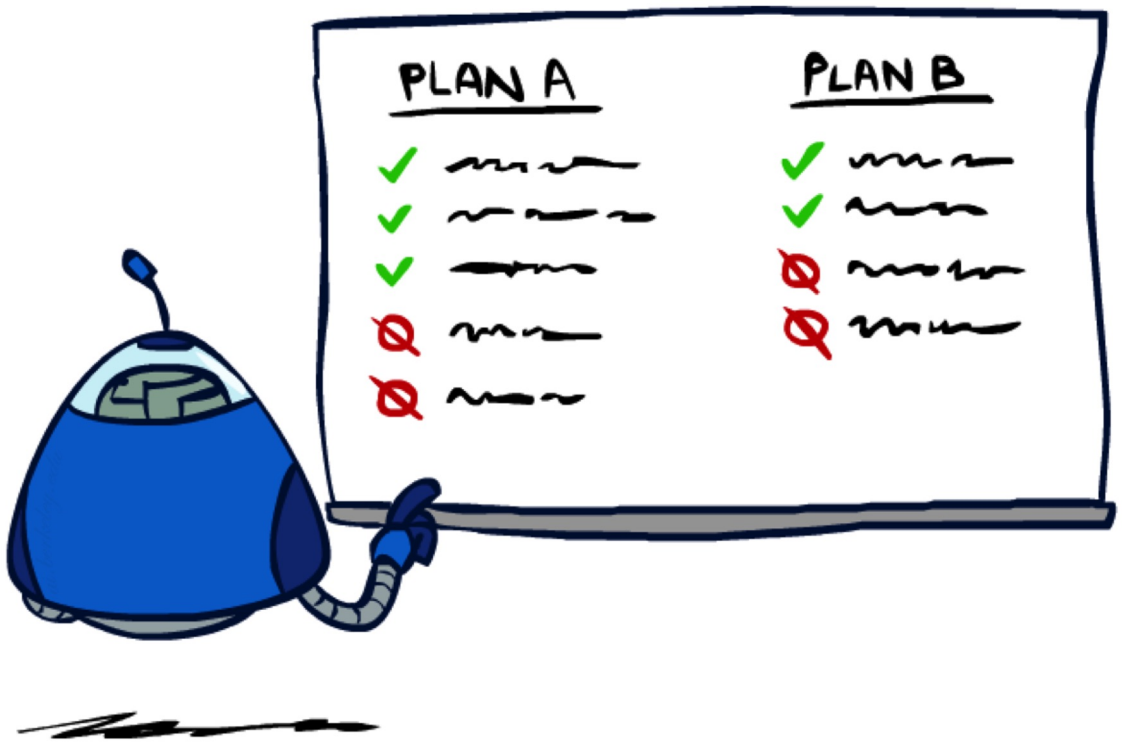
- An agent must have preferences among:
  - Prizes: A, B, etc.
  - Lotteries: situations with uncertain prizes

$$L = [p, A;\ \ (1-p), B]$$

- Notation:
  - Preference: $A \succ B$
  - Indifference: $A \sim B$



Rationality

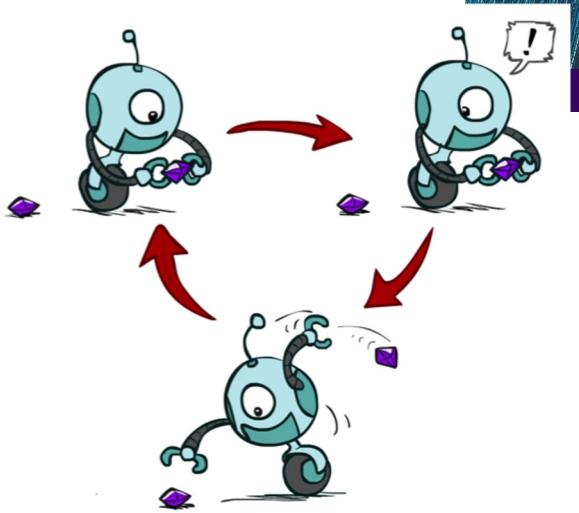


Rational Preferences

- We want some constraints on preferences before we call them rational, such as:

Axiom of Transitivity: (A > B) ^ (B > C) -> (A > C)

- For example: an agent with intransitive preferences can be induced to give away all its money
  - If B > C, then an agent with C would pay (say) 1 cent to get B
  - If A > B, then an agent with B would pay (say) 1 cent to get A
  - If C > A, then an agent with A would pay (say) 1 cent to get C



Rational Preferences

The Axioms of Rationality

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p\ [p, A;\ 1-p, C] \sim B$$

Substitutability

$$A \sim B \Rightarrow [p, A;\ 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A;\ 1-p, B] \succeq [q, A;\ 1-q, B])$$

Theorem: Rational preferences imply behavior describable as maximization of expected utility

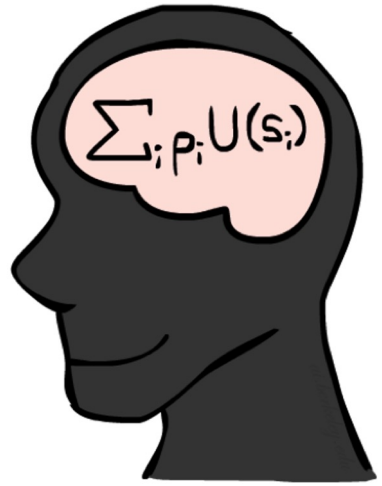
MEU Principle

- Theorem [Ramsey, 1931; von Neumann & Morgenstern, 1944]
  - Given any preferences satisfying these constraints, there exists a real-valued function U such that:

$$U(A) \geq U(B) \;\Leftrightarrow\; A \succeq B$$

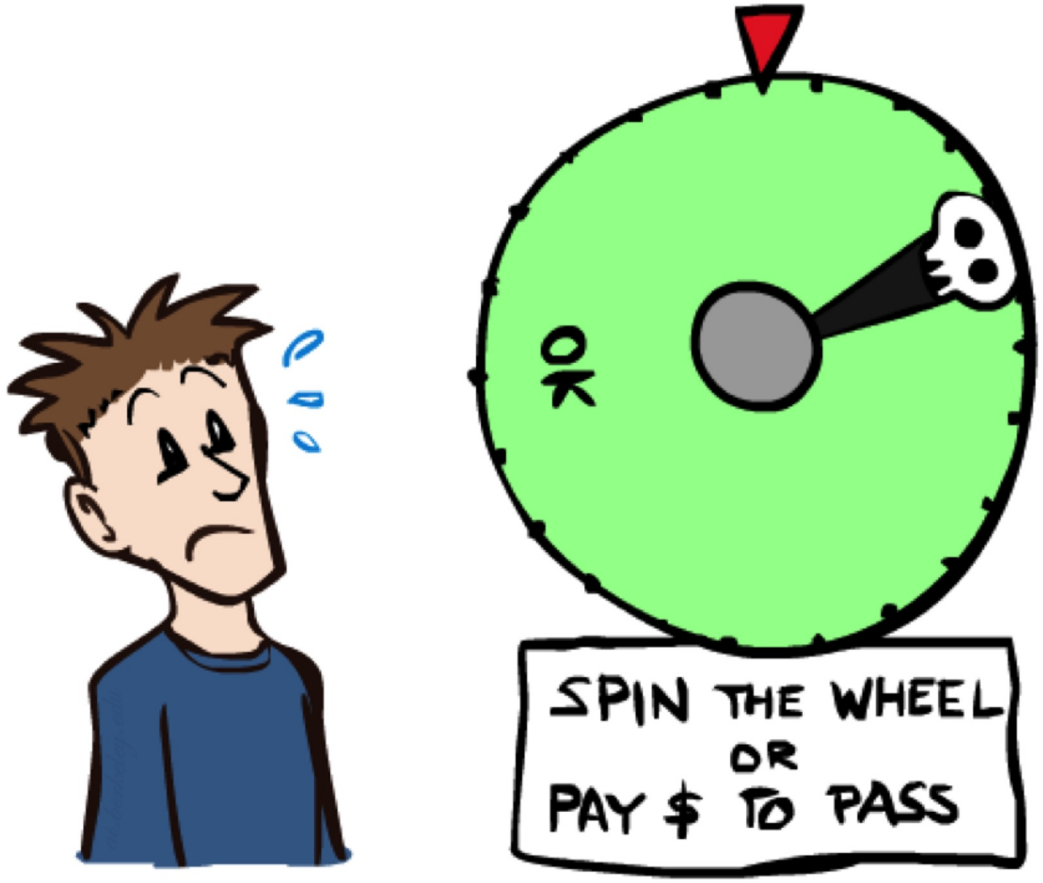
$$U([p_1, S_1;\; \ldots\; ;\; p_n, S_n]) = \sum_i p_i U(S_i)$$

  - I.e. values assigned by U preserve preferences of both prizes and lotteries!



- Maximum expected utility (MEU) principle:
  - Choose the action that maximizes expected utility
  - Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe, a reflex vacuum cleaner

Human Utilities



Utility Scales

- Normalized utilities: $u_+ = 1.0$, $u_- = 0.0$
- Micromorts: one-millionth chance of death, useful for paying to reduce product risks, etc.
- QALYs: quality-adjusted life years, useful for medical decisions involving substantial risk

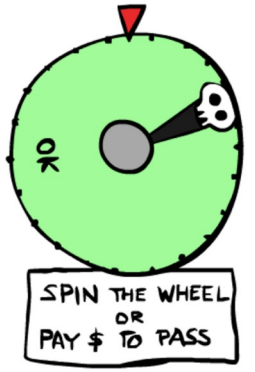
Micromort examples

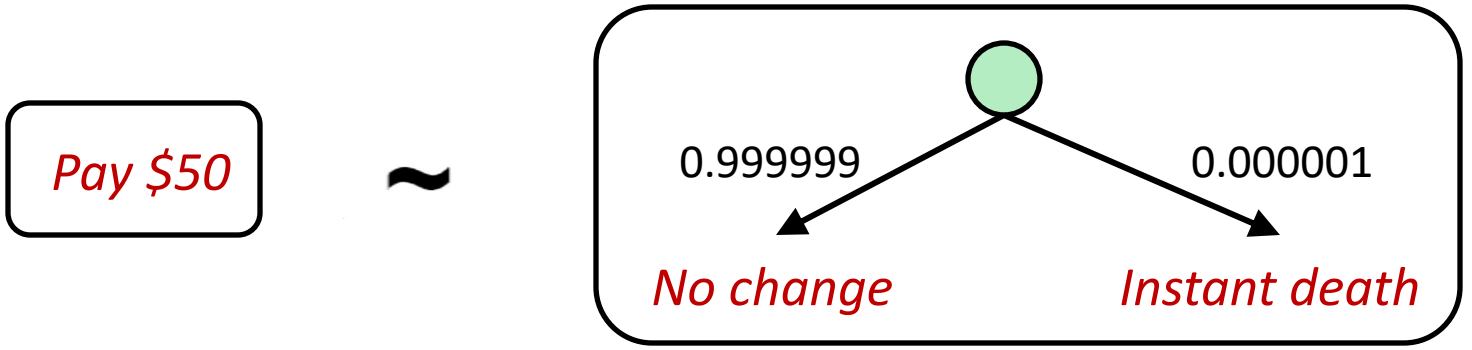
Death from	Micromorts per exposure
Scuba diving	5 per dive
Skydiving	7 per jump
Base-jumping	430 per jump
Climbing Mt. Everest	38,000 per ascent

1 Micromort	
Train travel	6000 miles
Jet	1000 miles
Car	230 miles
Walking	17 miles
Bicycle	10 miles
Motorbike	6 miles

Human Utilities

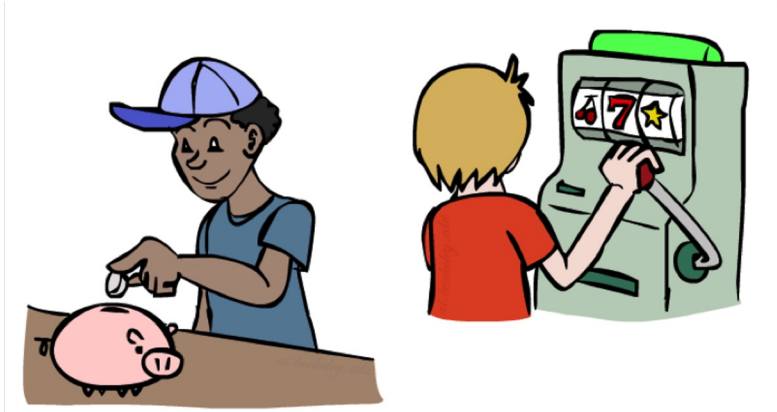
- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment (elicitation) of human utilities:
  - Compare a prize A to a standard lottery L_p between
    - "best possible prize" u_+ with probability p
    - "worst possible catastrophe" u_- with probability 1-p
  - Adjust lottery probability p until indifference: A ~ L_p
  - Resulting p is a utility in [0,1]

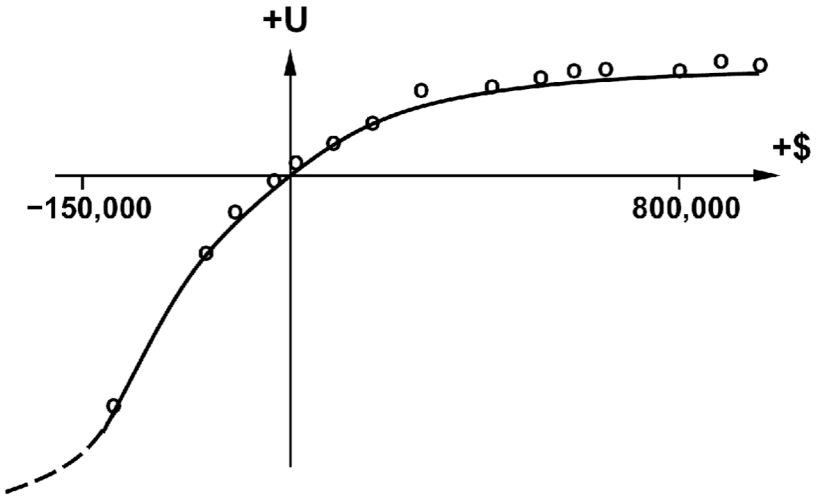




Money

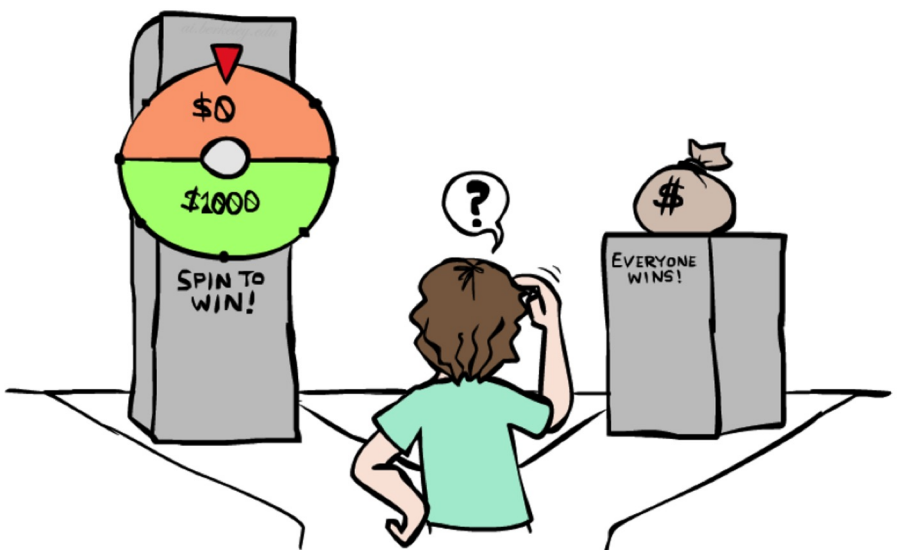
- Money **does not** behave as a utility function, but we can talk about the utility of having money (or being in debt)
- Given a lottery L = [p, $X; (1-p), $Y]
  - The expected monetary value EMV(L) is p*X + (1-p)*Y
  - U(L) = p*U($X) + (1-p)*U($Y)
  - Typically, U(L) < U(EMV(L))
  - In this sense, people are risk-averse
  - When deep in debt, people are risk-prone





Example: Insurance

- Consider the lottery [0.5, $1000; 0.5, $0]
  - What is its expected monetary value? ($500)
  - What is its certainty equivalent?
    - Monetary value acceptable in lieu of lottery
    - $400 for most people
  - Difference of $100 is the insurance premium
    - There's an insurance industry because people will pay to reduce their risk
    - If everyone were risk-neutral, no insurance needed!
  - It's win-win: you'd rather have the $400 and the insurance company would rather have the lottery (their utility curve is linear and they have many lotteries)



Example: Human Rationality?

- Famous example of Allais (1953)
  - A: [0.8, $4k; 0.2, $0]
  - B: [1.0, $3k; 0.0, $0]
  - C: [0.2, $4k; 0.8, $0] ⇦
  - D: [0.25, $3k; 0.75, $0]
- Most people prefer B > A, C > D
- But if U($0) = 0, then
  - B > A ⇒ U($3k) > 0.8 U($4k)
  - C > D ⇒ 0.8 U($4k) > U($3k)