CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

Constraint Satisfaction Problems

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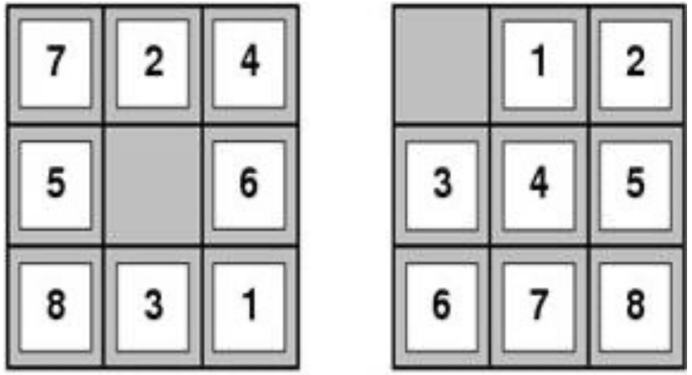
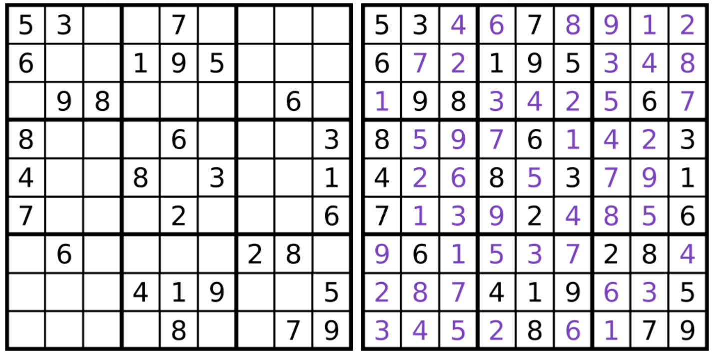
What is Search For?

- **Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space**

- **Planning: sequences of actions**
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- **Identification: assignments to variables**
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems





Big idea

- **Represent the *constraints* that solutions must satisfy in a uniform *declarative* language**
- **Find solutions by *GENERAL PURPOSE* search algorithms with no changes from problem to problem**
  - No hand-built transition functions
  - No hand-built heuristics
- **Just specify the problem in a formal declarative language, and a general-purpose algorithm does everything else!**

Constraint Satisfaction Problems

A CSP consists of:

- *Finite set of variables* $X_1, X_2, \ldots, X_n$
- *Nonempty **domain** of possible values* for each variable $D_1, D_2, \ldots D_n$ *where* $D_i = \{v_1, \ldots, v_k\}$
- *Finite set of constraints* $C_1, C_2, \ldots, C_m$
  - Each *constraint* C_i limits the values that variables can take, e.g., $X_1 \neq X_2$ A *state* is defined as an *assignment* of values to some or all variables.

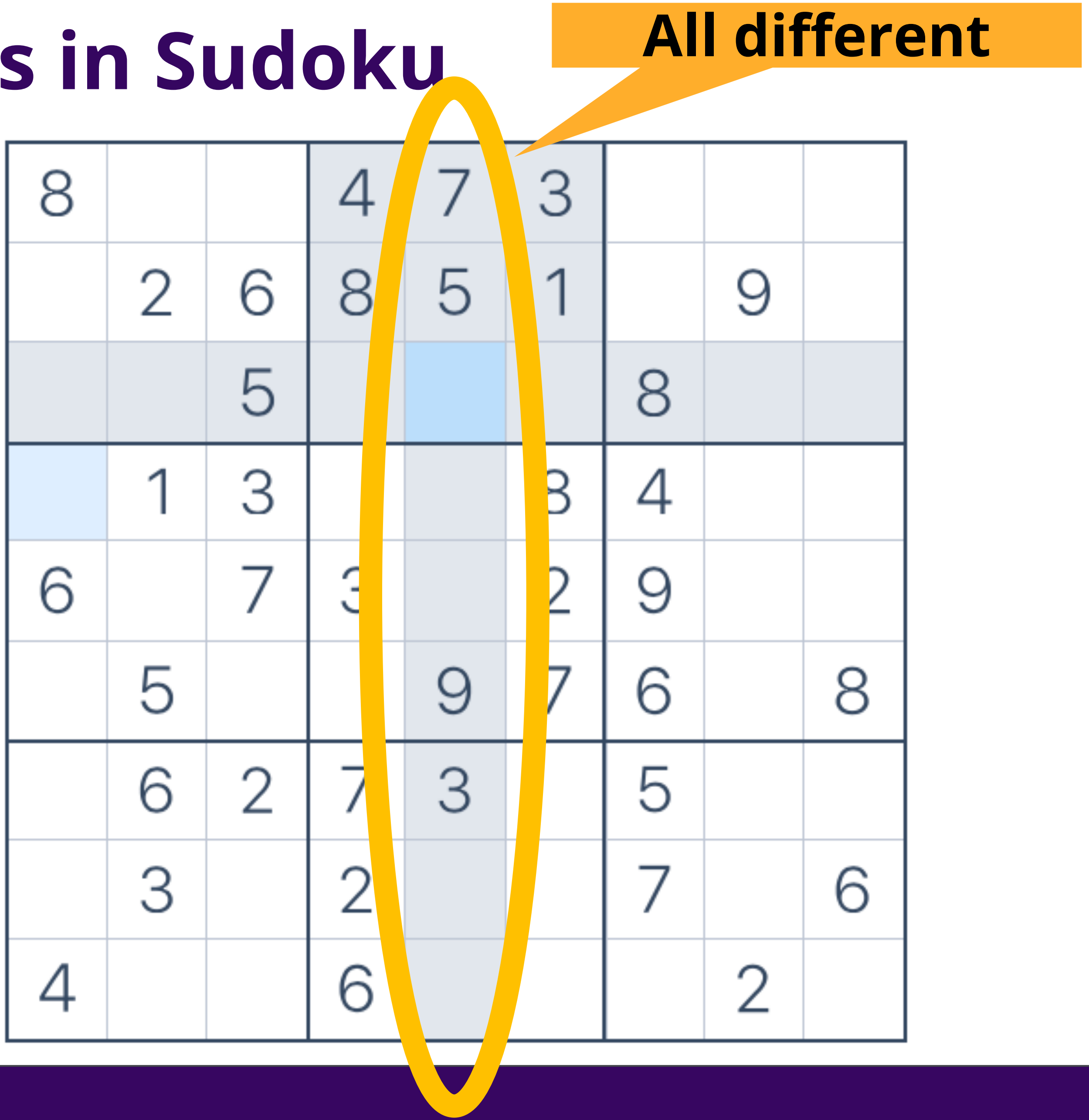
- **A *consistent* assignment does not violate the constraints.**
- **Example problem: Sudoku**

Constraints in Sudoku

8			4	7	3			
	2	6	8	5	1		9	
		5				8		
	1	3			8	4		
6		7	3		2	9		
	5			9	7	6		8
	6	2	7	3		5		
	3		2			7		6
4			6				2	

All different

Constraints in Sudoku



Constraints in Sudoku

All different

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	2	6	8	5	1		9	
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	5			9	7	6		8
	6	2	7	3		5		
	3		2			7		6
4			6				2	

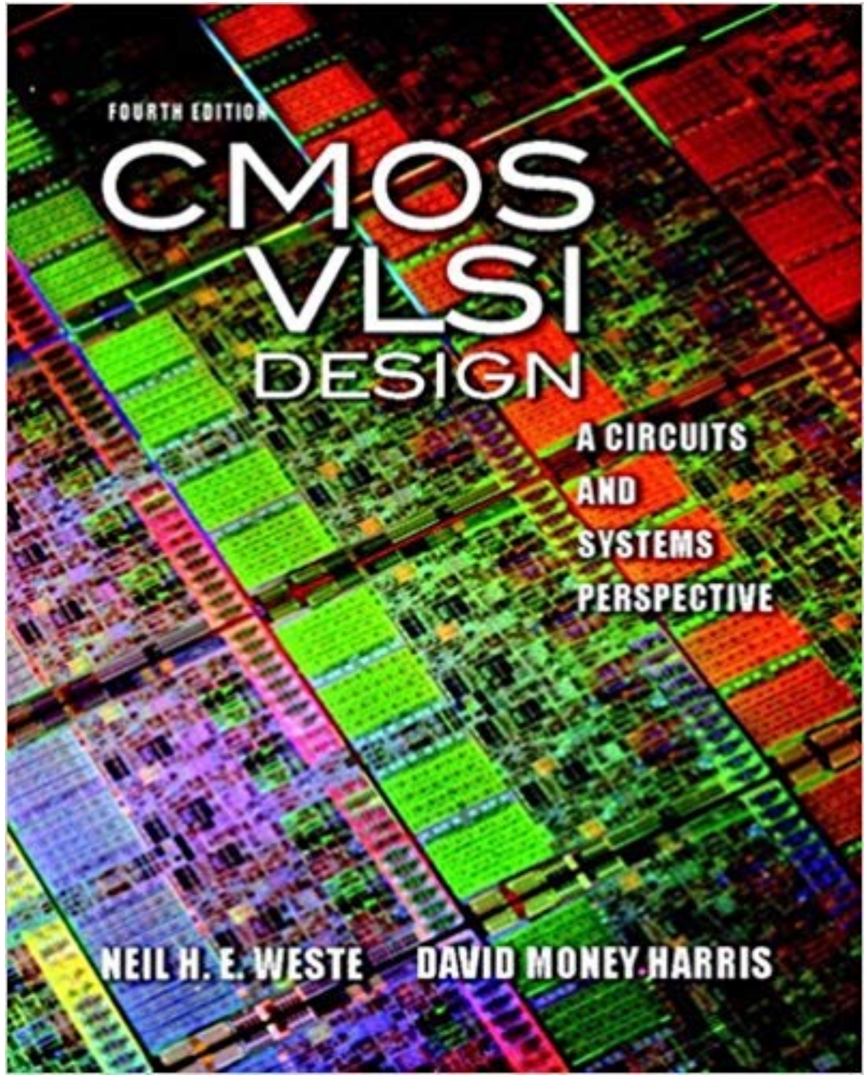
Constraint satisfaction problems

- **An assignment is *complete* when every variable is assigned a value.**
- **A *solution* to a CSP is a *complete, consistent* assignment.**
- **Solutions to CSPs can be found by a completely *general purpose* algorithm, given only the formal specification of the CSP.**

- **Beyond our scope: CSPs that require a solution that maximizes an *objective function*.**

Applications

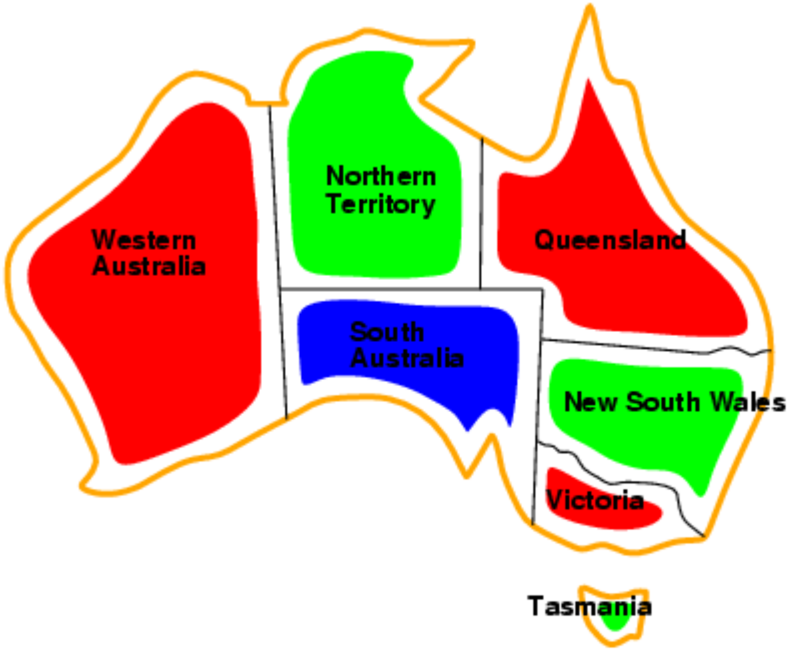
- **Map coloring**
- **Scheduling problems**
  - Job shop scheduling
  - Scheduling the Webb Space Telescope
- **Floor planning for VLSI**
- **Sudoku**
- **…**



Example: Map-coloring

- **Variables:** ***WA, NT, Q, NSW, V, SA, T***
- **Domains:** $\boldsymbol{D_i}$ **= {red,green,blue}**
- **Constraints: adjacent regions must have different colors**
  - e.g., WA ≠ NT
    - So (WA,NT) must be in {(red,green),(red,blue),(green,red), …}

Example: Map-coloring



Solutions: complete and consistent assignments

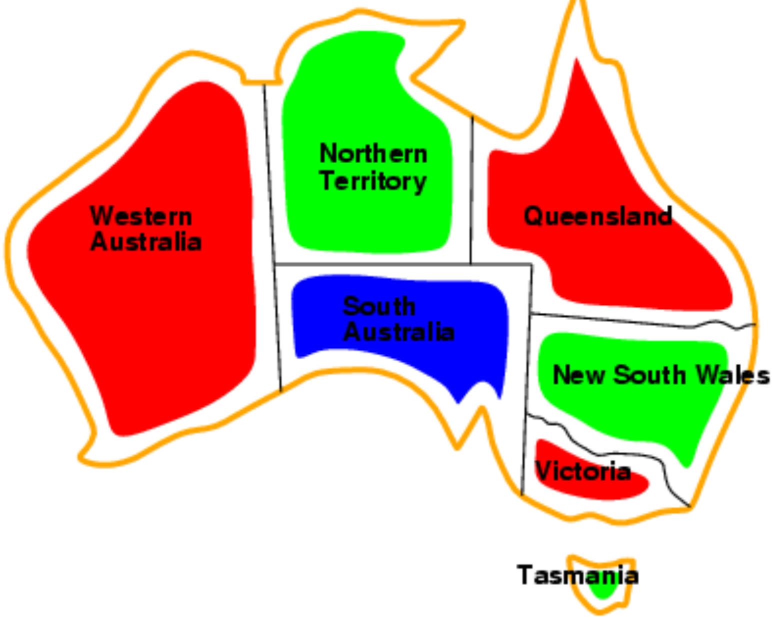
- e.g., WA = red, NT = green,Q = red, NSW = green,
  V = red, SA = blue, T = green

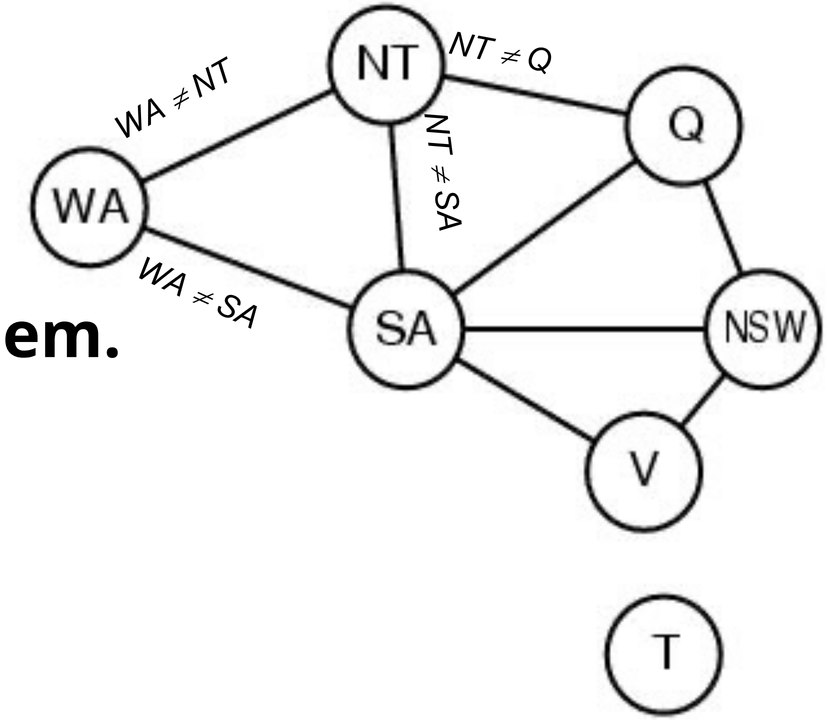
Benefits of CSP

- **Clean specification of many problems, generic goal, successor function & heuristics**
  - Just represent problem as a CSP & solve with general package
- **CSP "knows" which variables violate a constraint**
  - And hence where to focus the search
- ***CSPs:* Automatically prune off all branches that violate constraints**
  - (State space search could do this only by *hand-building constraints into the successor function)*

CSP Representations

- *Constraint graph:*
  - *nodes* are variables
  - *arcs* are (binary) constraints
- *Standard representation pattern:*
  - variables with values
- *Constraint graph* simplifies search.
  - e.g. Tasmania is an independent subproblem.
- *This problem: A binary CSP:*
  - each constraint relates two variables





Varieties of CSPs

- ***Discrete variables***
  - finite domains:
    - n variables, domain size d → $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, includes Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- ***Continuous variables***
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

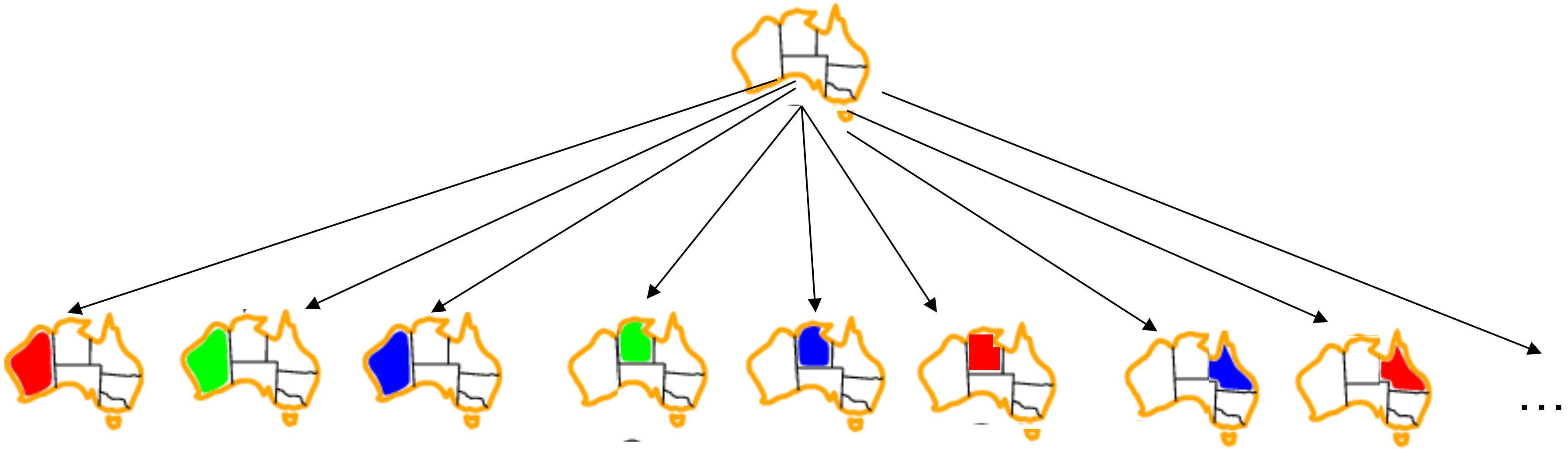
- ***Unary* constraints involve a single variable,**
  - e.g., SA ≠ green
- ***Binary* constraints involve pairs of variables,**
  - e.g., SA ≠ WA
- ***Higher-order* constraints involve 3 or more variables**
  - e.g., crypt-arithmetic column constraints
- ***Preference* (soft constraints) e.g. *red* is better than *green* can be represented by a cost for each variable assignment**
  - Constrained optimization problems.

Idea 1: CSP as a search problem

- **A CSP can easily be expressed as a search problem**
  - *Initial State:* the empty assignment {}.
  - *Successor function:* Assign value to any unassigned variable *provided that there is not a constraint conflict*.
  - *Goal test:* the current assignment is complete.
  - *Path cost:* a constant cost for every step.

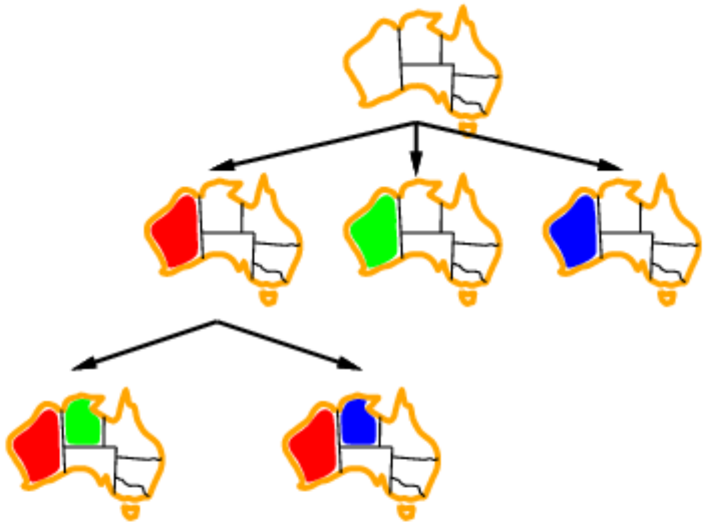
- **Solution is always found at depth *n*, for *n* variables**
  - Hence Depth First Search can be used

Search and branching factor



- n variables of domain size d
- Branching factor at the root is n*d
- Branching factor at next level is (n-1)*d
- Tree has $n!*d^n$ leaves

Search and branching factor



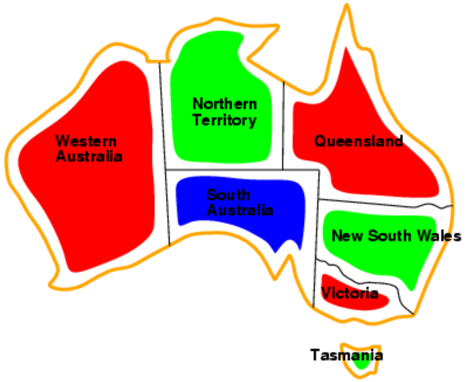
- **The variable assignments are *commutative***
  - Eg *[step 1: **WA = red**; step 2: **NT = green**]* equivalent to *[step 1: **NT = green**; step 2: **WA = red**]*
  - Therefore, a *tree search,* not a *graph search*

- **Only need to consider assignments to a single variable at each node**
  - $b = d$ and there are d^n leaves *(n variables, domain size d)*

Search and *Backtracking*

- **Depth-first search for CSPs with single-variable assignments is called *backtracking* search**
- **The term backtracking search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.**
- **Backtracking search is the basic *uninformed* algorithm for CSPs**

Backtracking example



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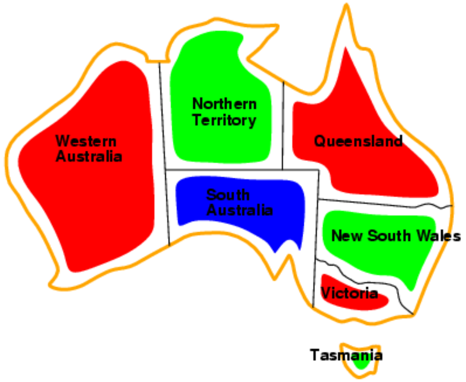
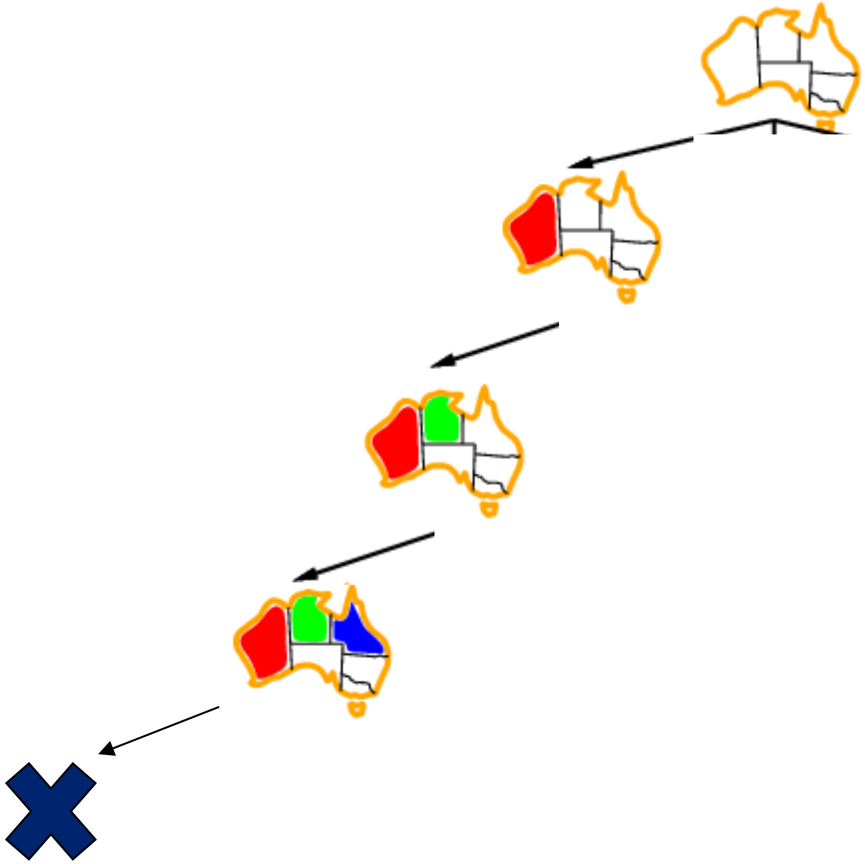


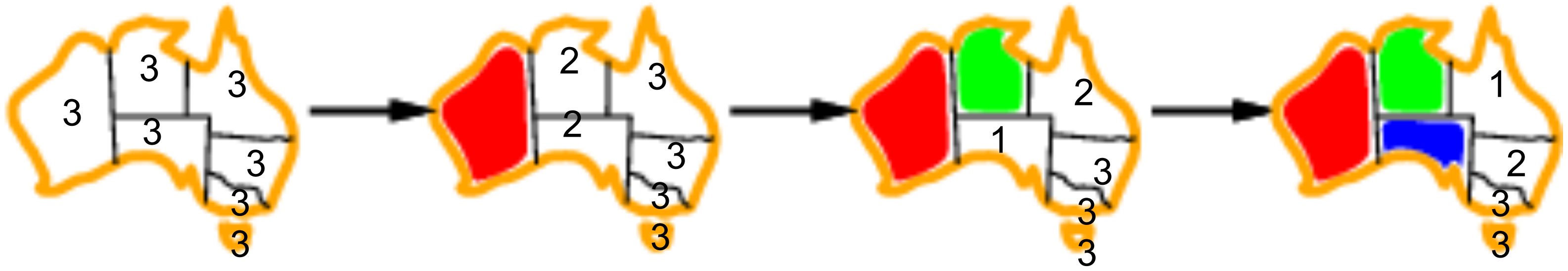


Idea 2: Improving backtracking efficiency

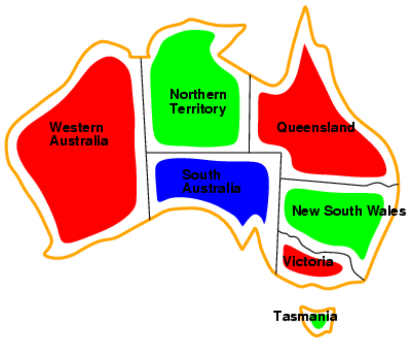
- ***General-purpose*** **methods &** ***general-purpose*** **heuristics can give huge gains in speed,** ***on average***
- **Heuristics:**
  - Q: Which variable should be assigned next?
    1. **Most constrain*ed* variable**
    2. **(if ties:) Most constrain*ing* variable**
  - Q: In what order should that variable's values be tried?
    3. **Least constraining *value***
  - Q: Can we detect inevitable failure early?
    4. **Forward checking**

Heuristic 1: Most constrain*ed* variable

- Choose a variable with the ***fewest legal values***



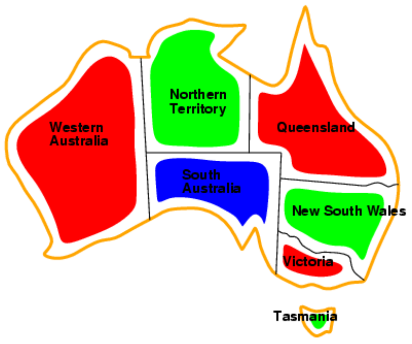
- a.k.a. ***minimum remaining values (MRV)*** heuristic



Heuristic 2: Most constrain*ing* variable

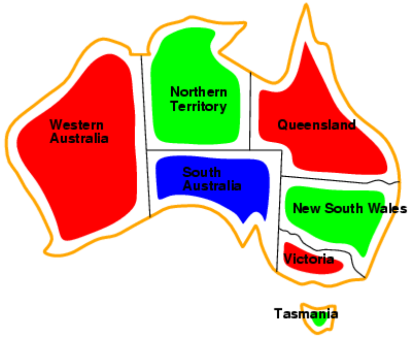
- Tie-breaker among most constrained variables
- Choose the variable with the *most constraints on remaining variables*

These two heuristics together lead to immediate solution of our example problem



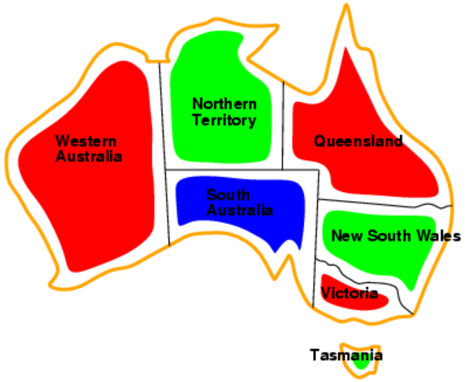
Heuristic 3: Least constraining *value*

- **Given a variable, *choose the least constraining value:***
  - the one that rules out the fewest values in the remaining variables



Note: demonstrated here independent of the other heuristics

Heuristic 4: Forward checking

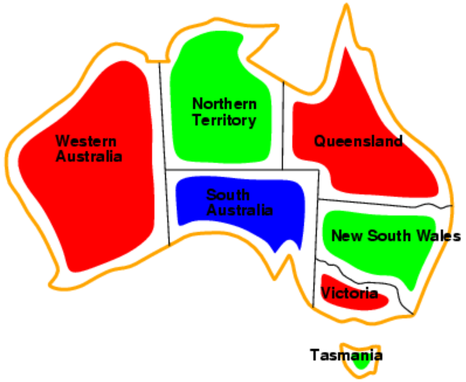


- **Idea**:
  - Keep track of *remaining* legal values for *unassigned* variables
  - Terminate search when any unassigned variable has no remaining legal values

New data structure

(A first step towards Arc Consistency & AC-3)

Forward checking



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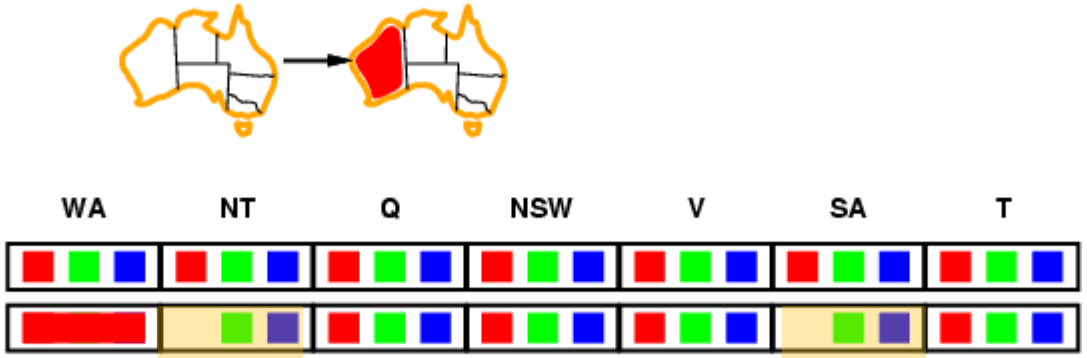
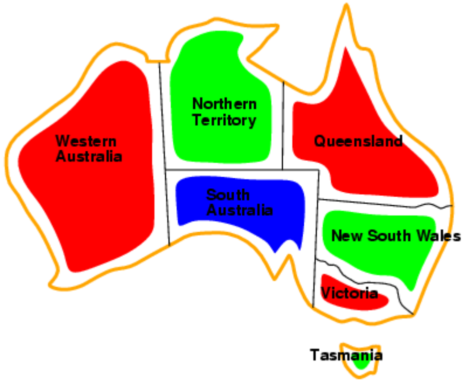


Forward checking



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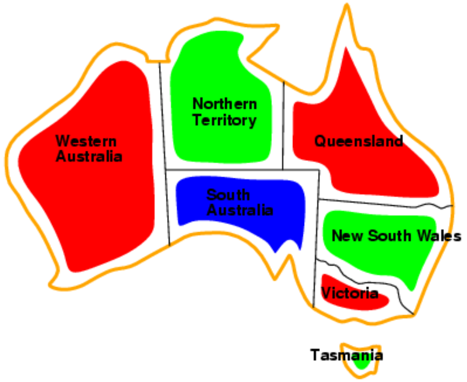

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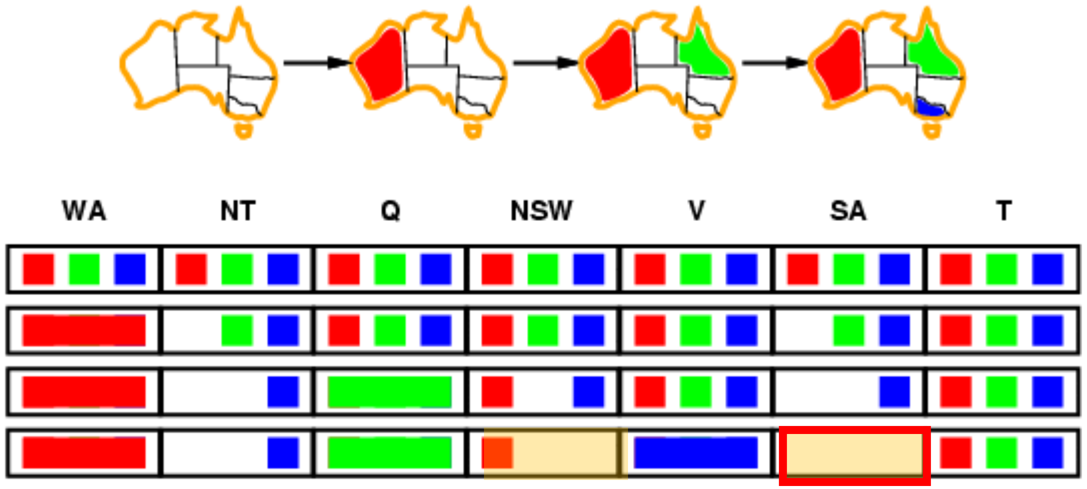
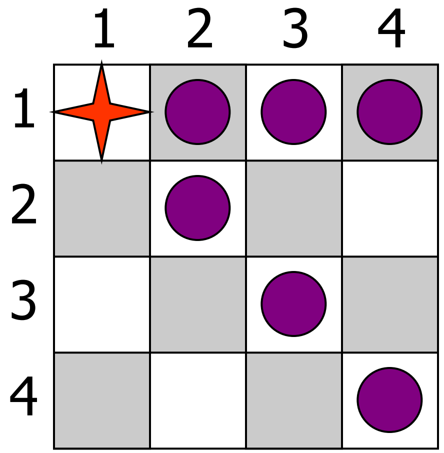
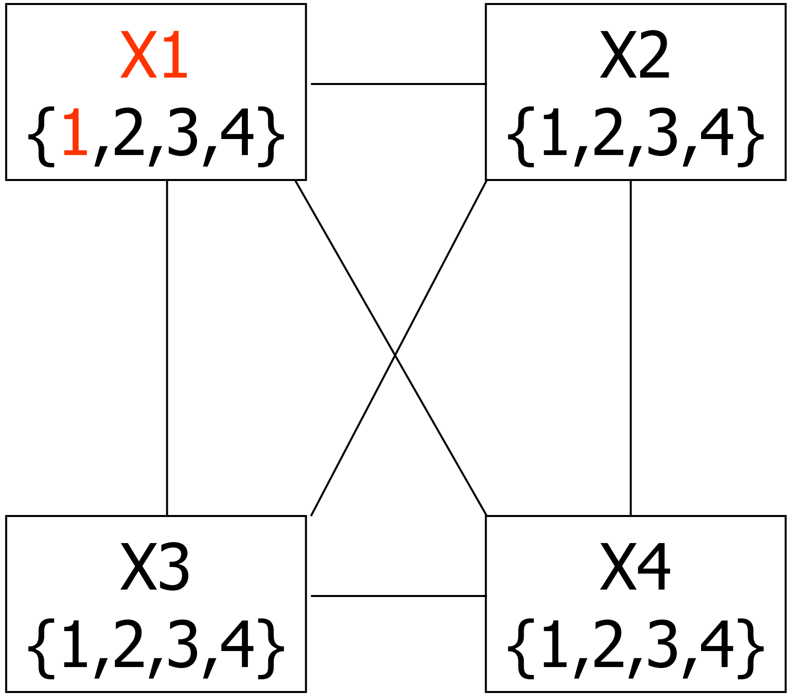


Terminate! No possible value for SA

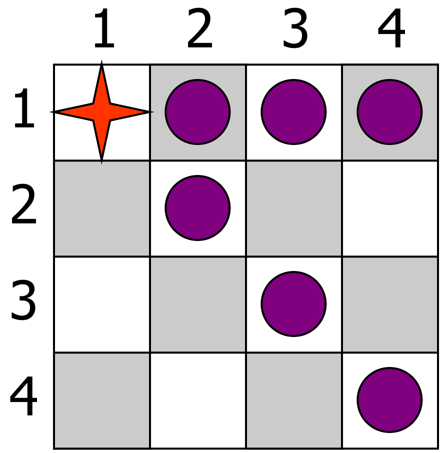
Example: 4-Queens Problem

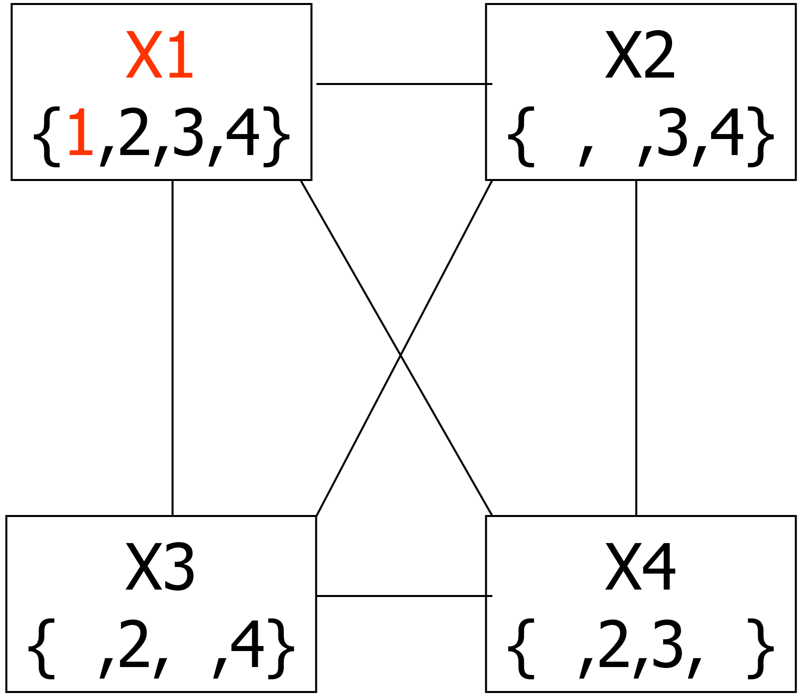




Assign value to unassigned variable

Example: 4-Queens Problem





Forward check!

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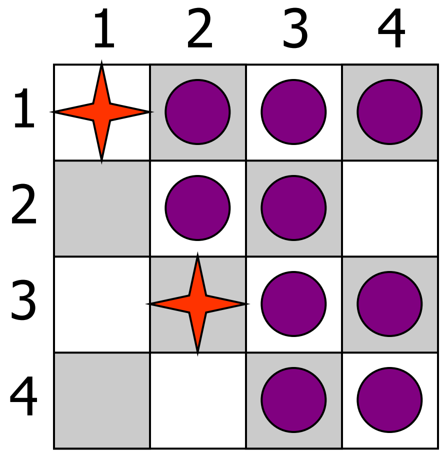

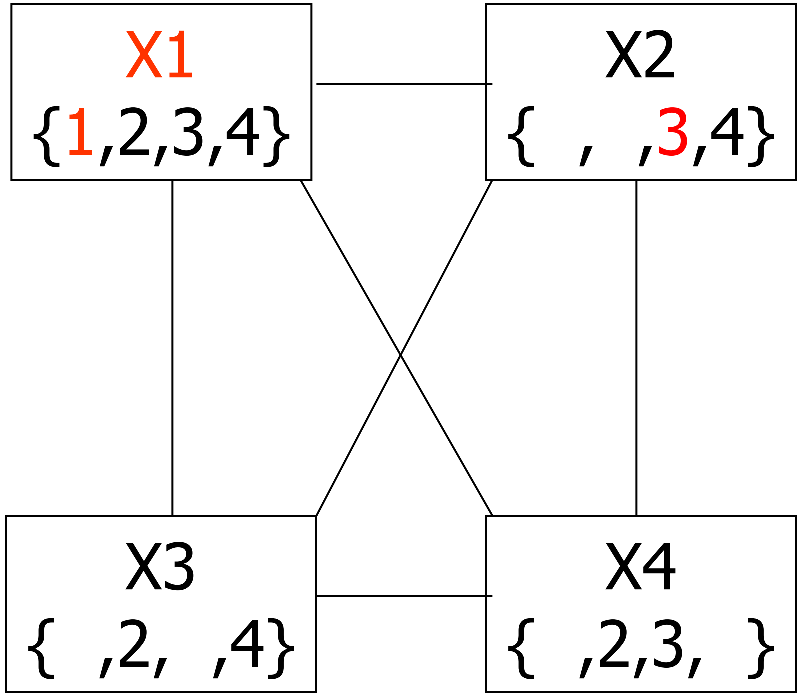

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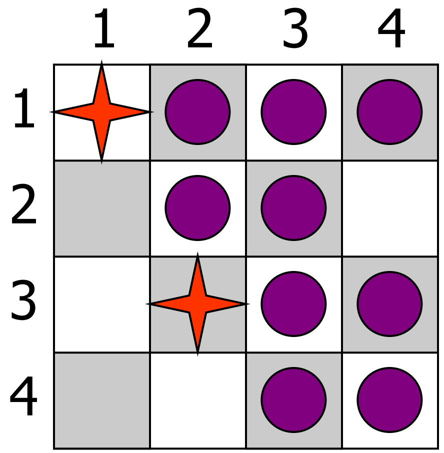
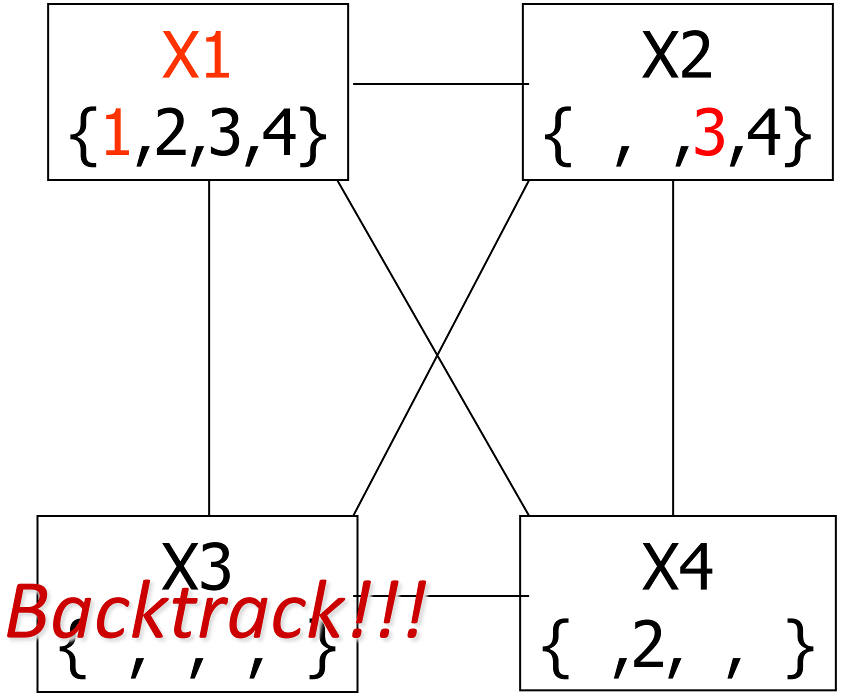
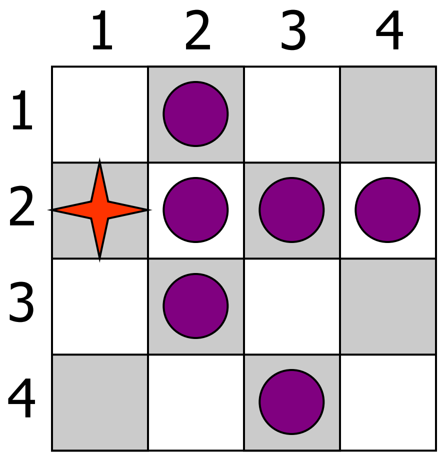


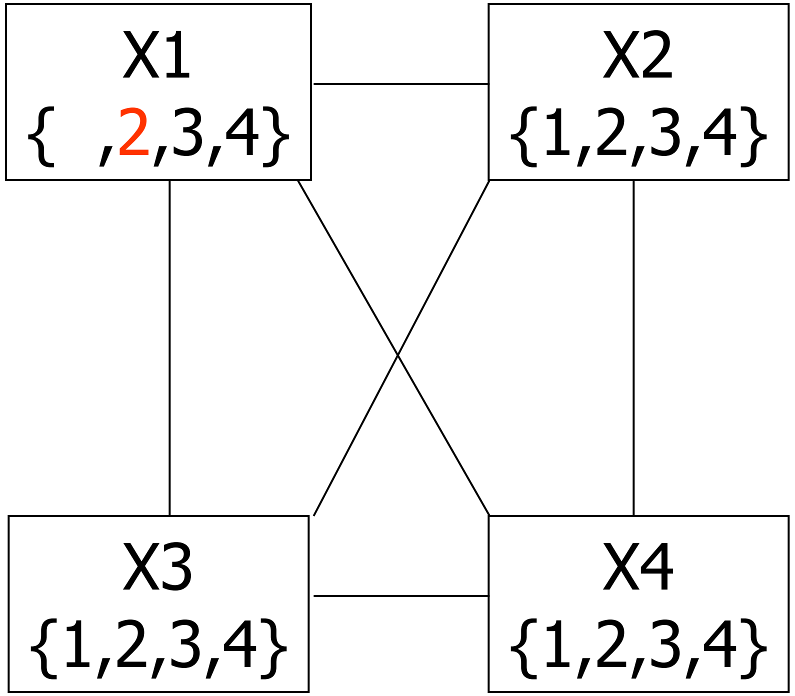


Forward check!

Example: 4-Queens Problem

Picking up a little later after two steps of backtracking....





Assign value to unassigned variable

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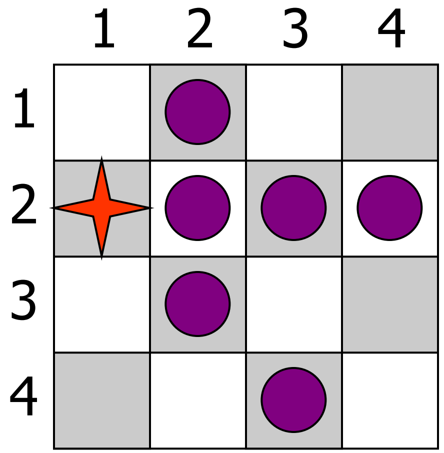
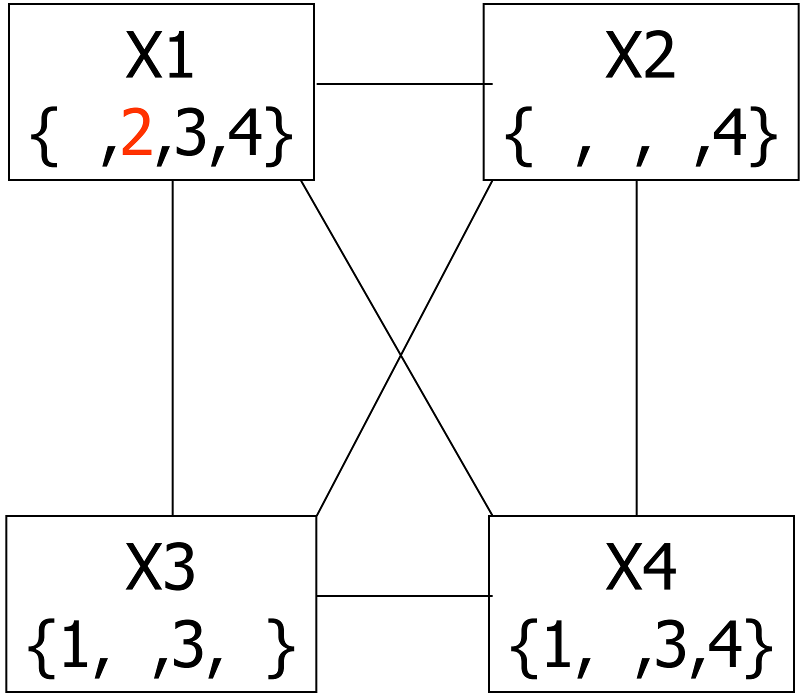
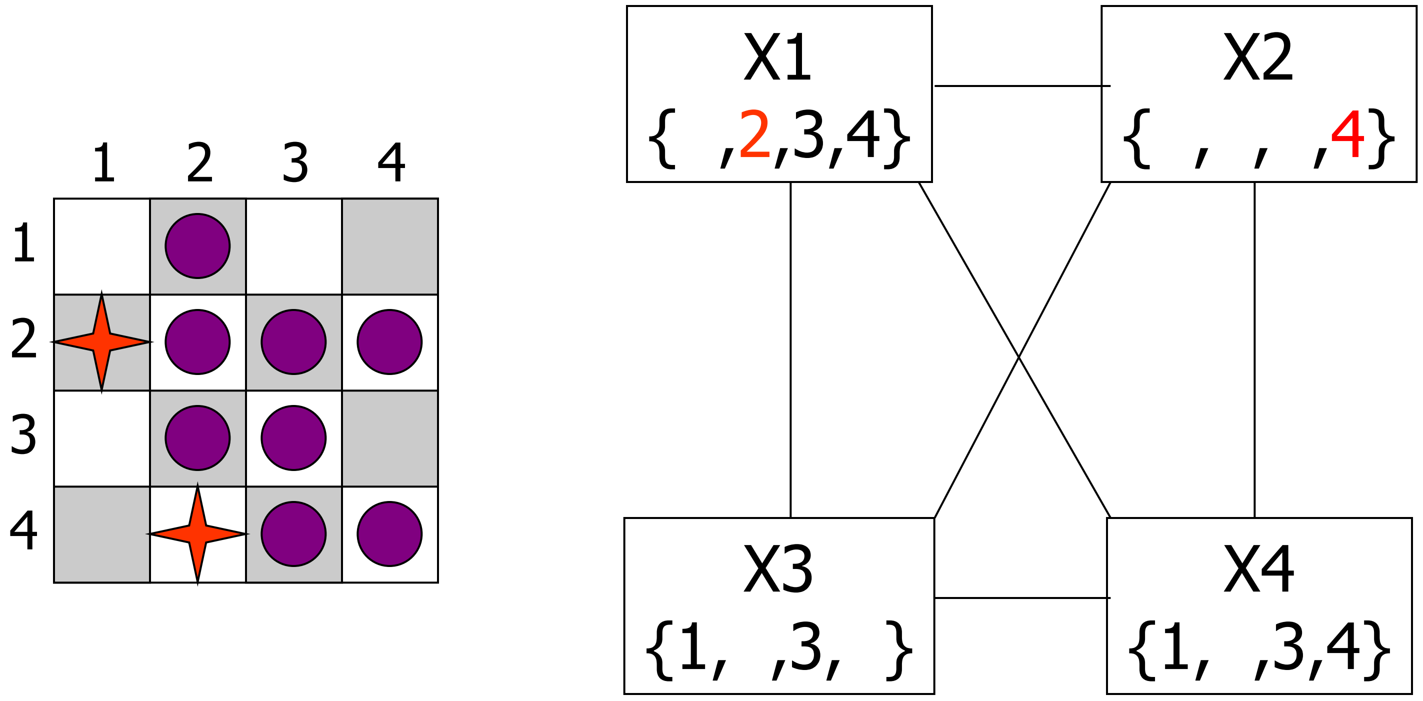




Forward check!

Example: 4-Queens Problem



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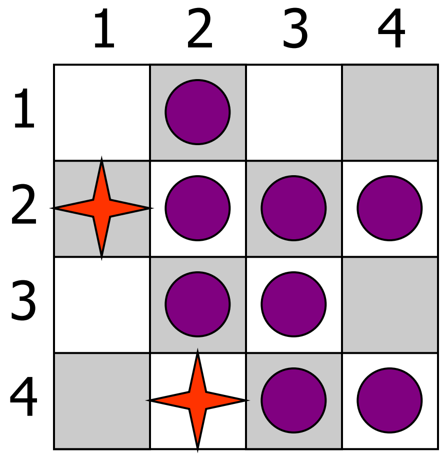

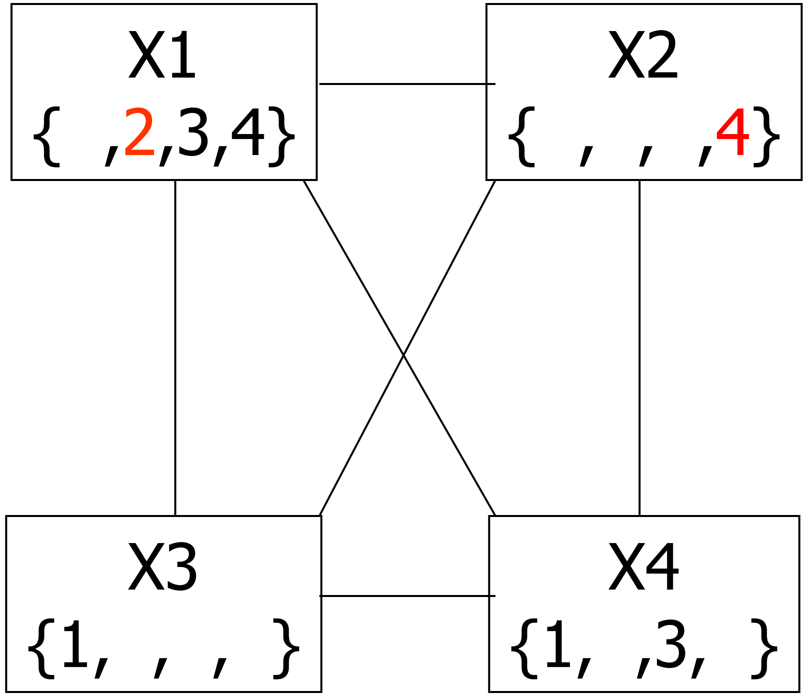

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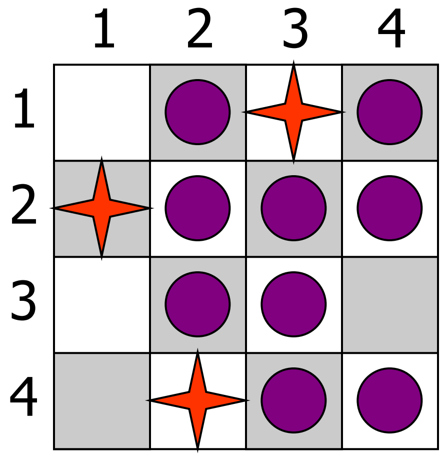

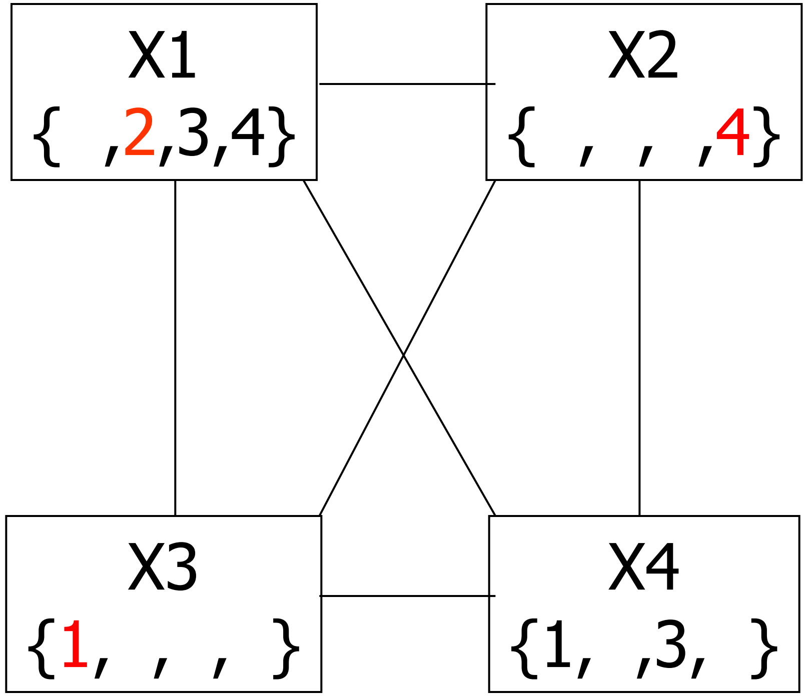

Assign value to unassigned variable

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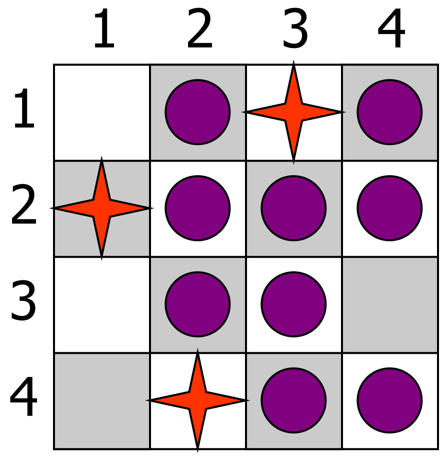

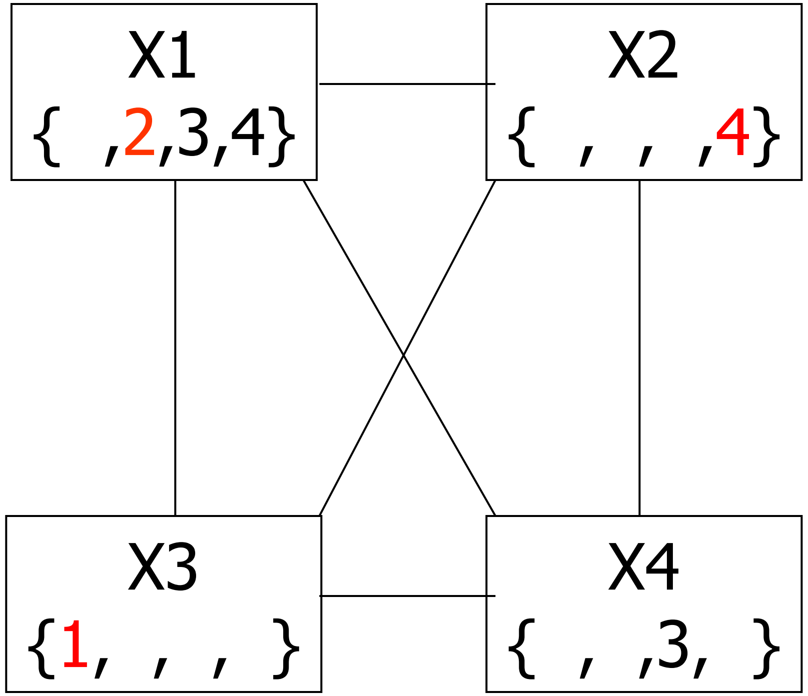

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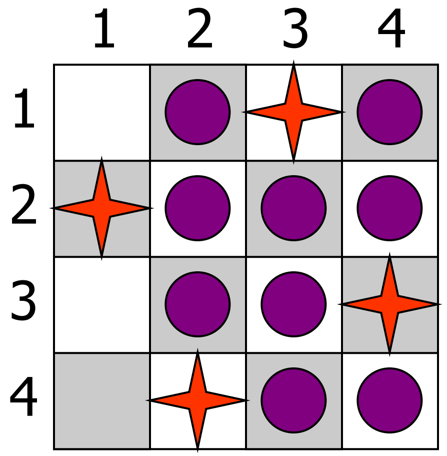
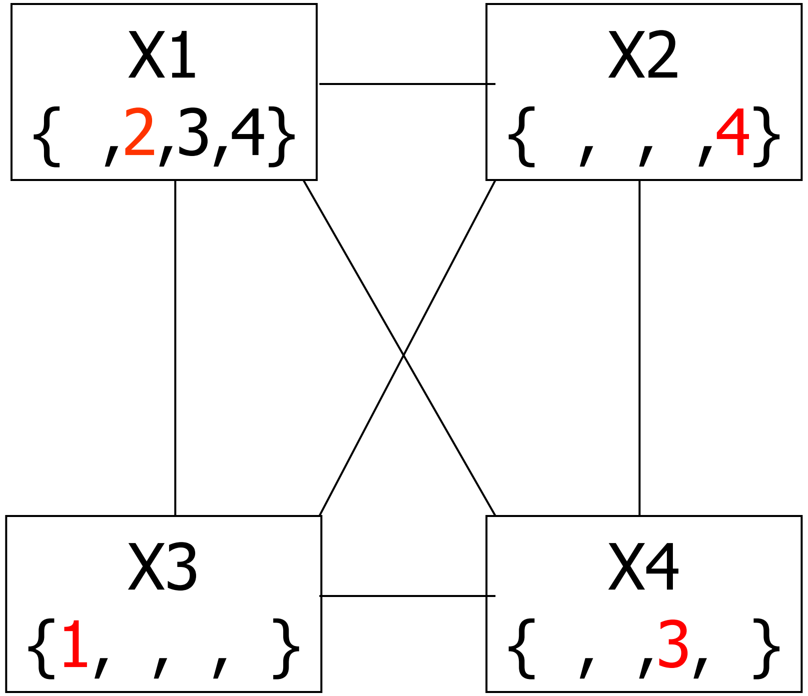
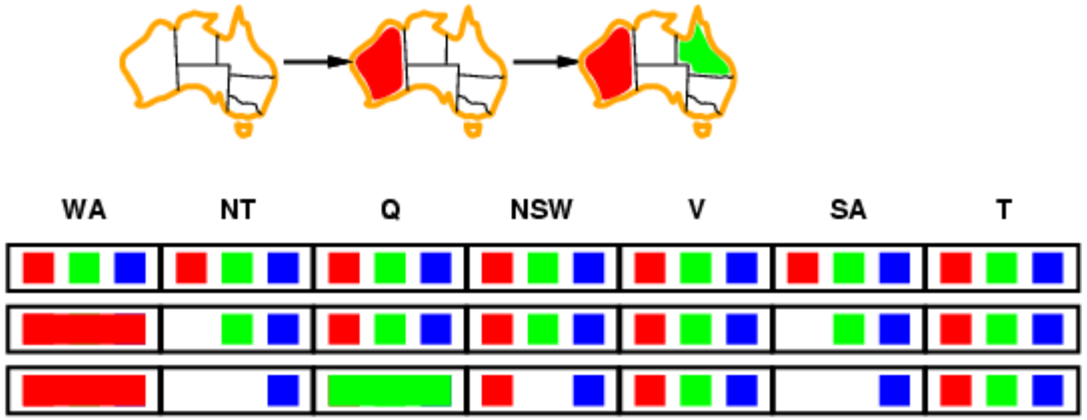




Assign value to unassigned variable

Towards Constraint propagation

- **Forward checking** propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- **Constraint propagation** goes beyond forward checking & repeatedly enforces constraints locally

CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

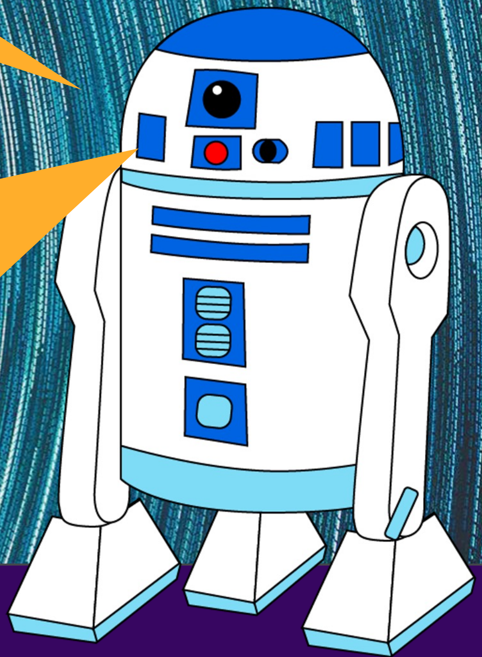
Constraint Satisfaction Problems

Professor Chris Callison-Burch

The OHQs have gotten long! The TAs and I are brainstorming ways of improving this.

Reading: AIMA Sections 6.1-6.5.

Optional Course Content Recitation Tonight from 8-9pm on Games and Adversarial Search



Review: CSPs and Search

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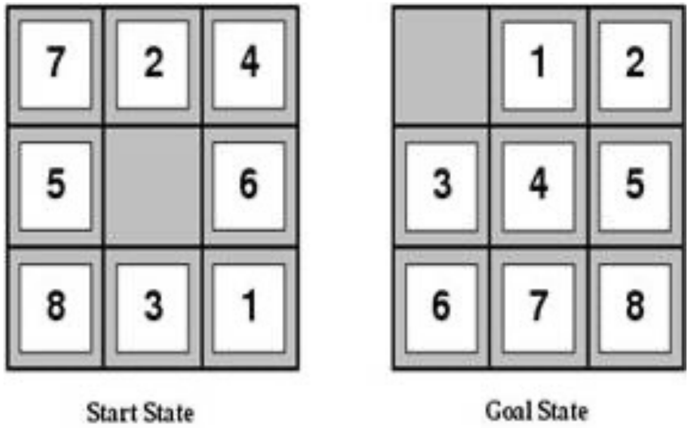

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

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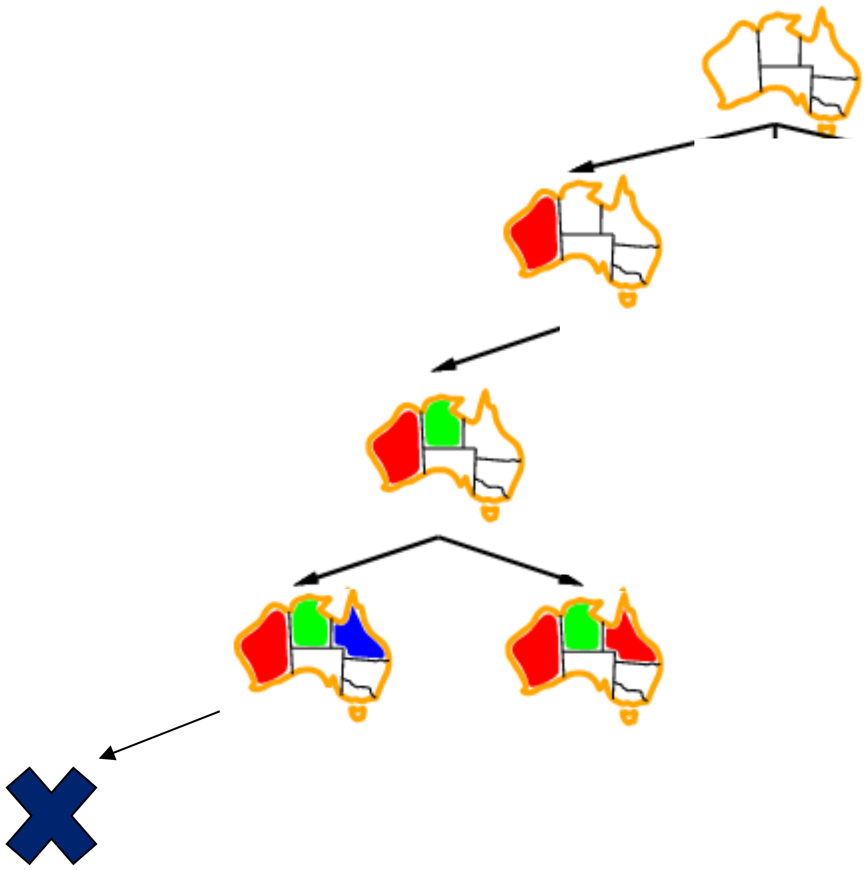
- The term ***backtracking*** search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.
- Backtracking search is the basic ***uninformed*** algorithm for CSPs

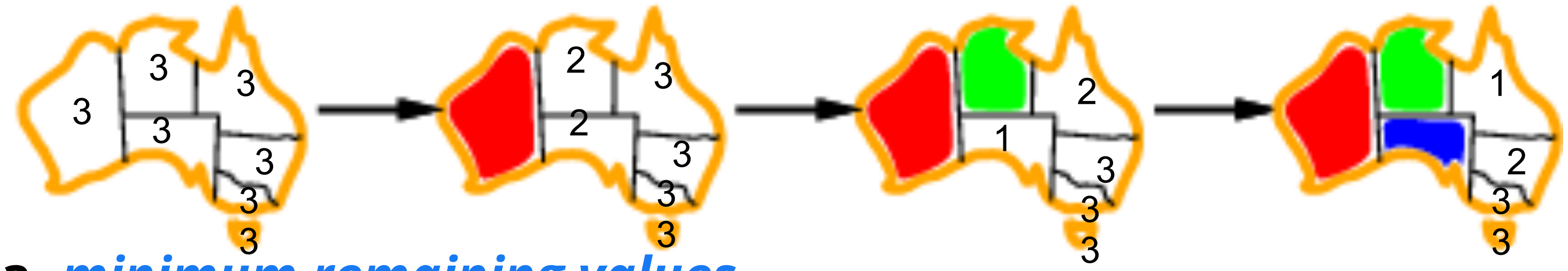


Review: Improving backtracking efficiency

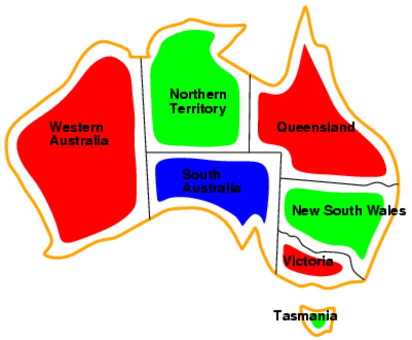
- *General-purpose* methods & *general-purpose* heuristics can give huge gains in speed, *on average*
- Heuristics:
  - Q: Which variable should be assigned next?
    1. **Most constrain*ed* variable**
    2. **(if ties:) Most constrain*ing* variable**
  - Q: In what order should that variable's values be tried?
    3. **Least constraining *value***
  - Q: Can we detect inevitable failure early?
    4. **Forward checking**

Review: Heuristic 1: Most constrain*ed* variable

- Choose a variable with the *fewest legal values*



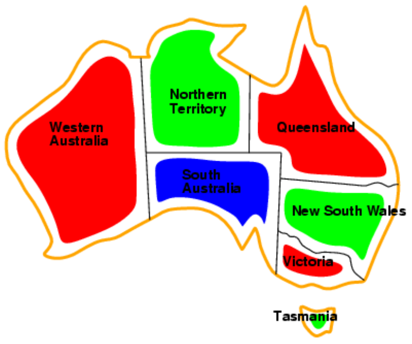
- a.k.a. *minimum remaining values (MRV)* heuristic



Review: Heuristic 2: Most constrain*ing* variable

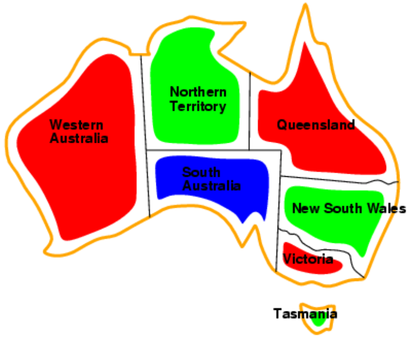
- **Tie-breaker among most constrained variables**
- **Choose the variable with the *most constraints on remaining variables***

These two heuristics together lead to immediate solution of our example problem



Review: Heuristic 3: Least constraining *value*

- **Given a variable, *choose the least constraining value:***
  - the one that rules out the fewest values in the remain



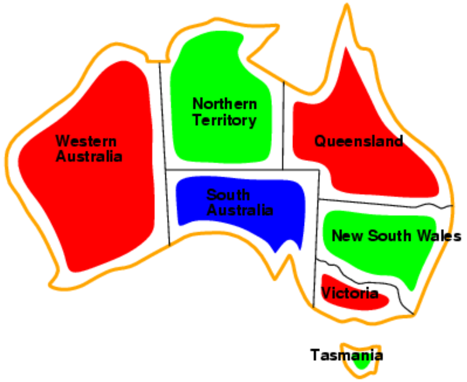
Note: demonstrated here independent of the other heuristics

Review: Heuristic 4: Forward checking

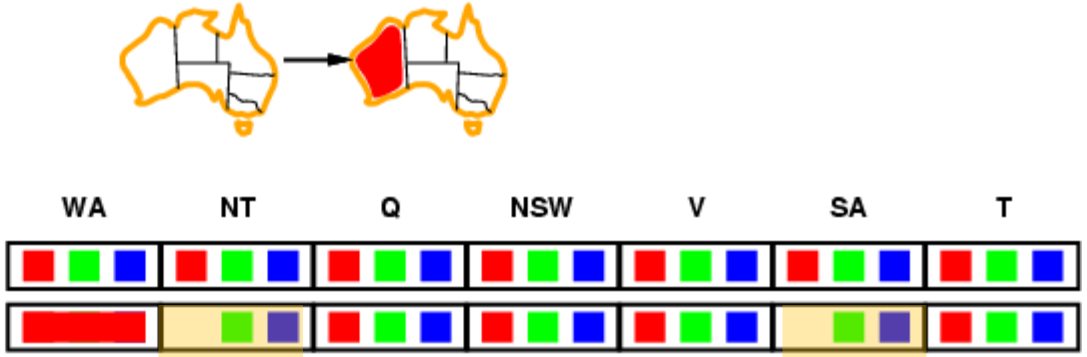
- Idea:
  - Keep track of *remaining* legal values for *unassigned* variables
  - Terminate search when any unassigned variable has no remaining legal values

New data structure

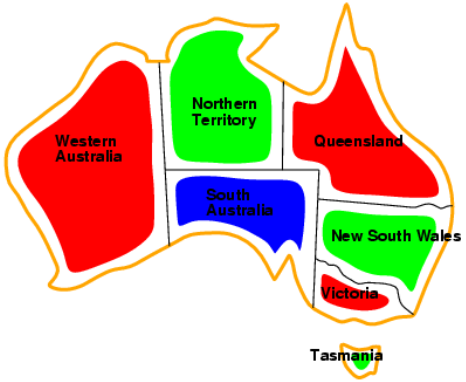
Review: Forward checking



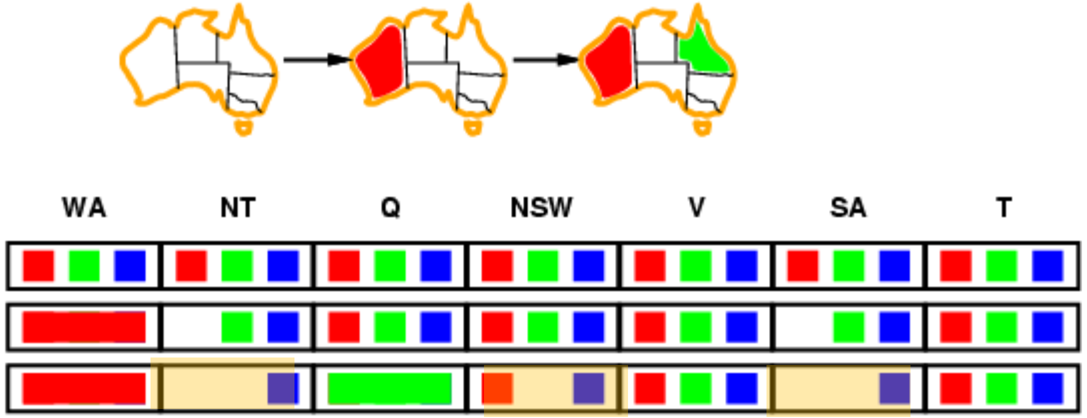
- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any unassigned variable has no remaining legal values



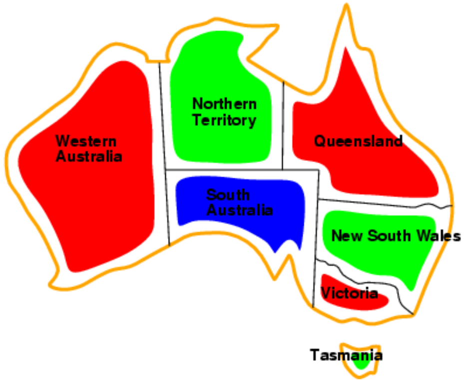
Review: Forward checking



- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any unassigned variable has no remaining legal values



Review: Forward checking



- Idea:
  - Keep track of remaining legal values for unassigned variables
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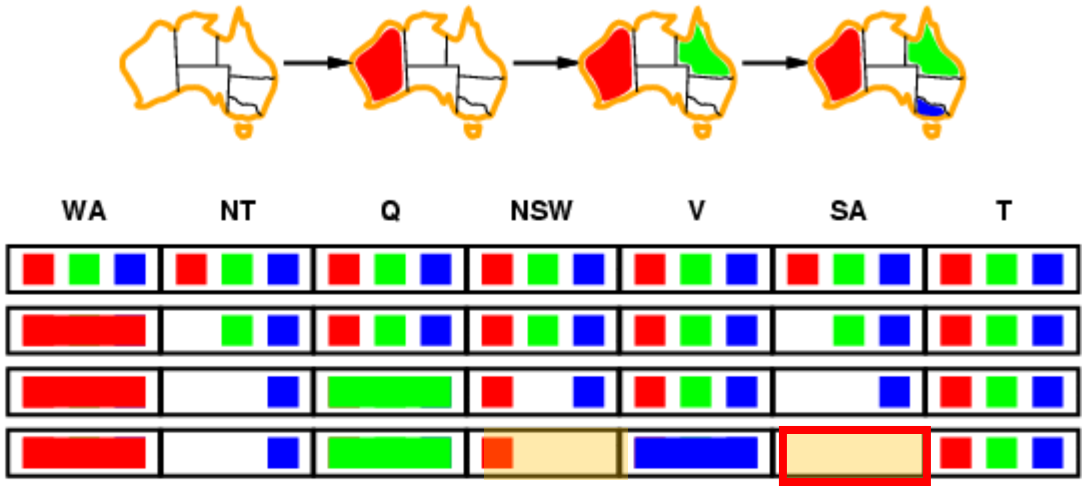
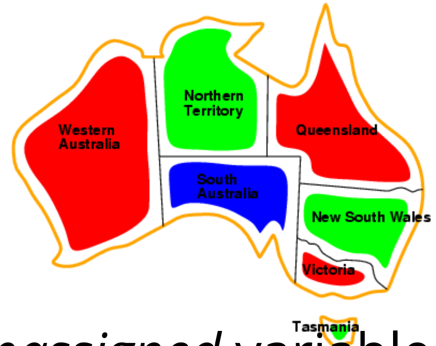
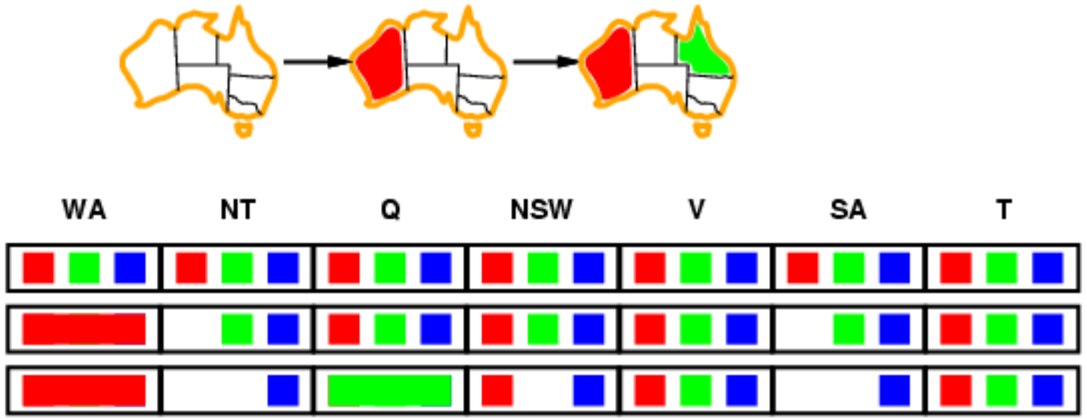


Towards Constraint propagation



- Forward checking propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- Constraint propagation goes beyond forward checking & repeatedly enforces constraints locally

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Arc Consistency, Constraint Propagation & AC-3

Professor Chris Callison-Burch

Idea 3 (*big* idea): *Inference* in CSPs

- **CSP solvers combine search *and inference***
  - Search
  - *Constraint propagation (inference)*
    - Eliminates possible values for a variable if the value would violate local consistency
  - *Can do inference first, or intertwine it with search*
    - You'll investigate this in the Sudoku homework

Search = assign a value to a variable

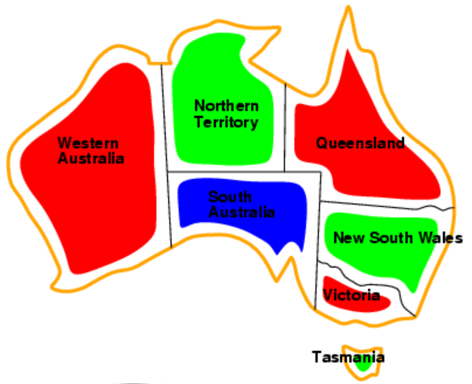
Inference = use constraints to reduce number of legal values for a variable

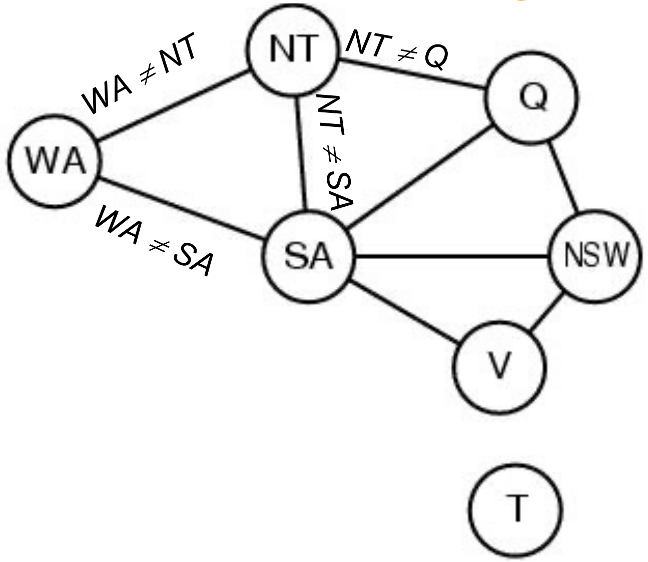
Local Consistency

- **Node consistency: satisfies unary constraints**
  - This is trivial!
- **Arc consistency: satisfies binary constraints**
  - X_i is arc-consistent with respect to X_j
  - If for every value v in D_i
  - There is some value w in D_j that satisfies the binary constraint on the arc between X_i *and* X_j

CSP Representations

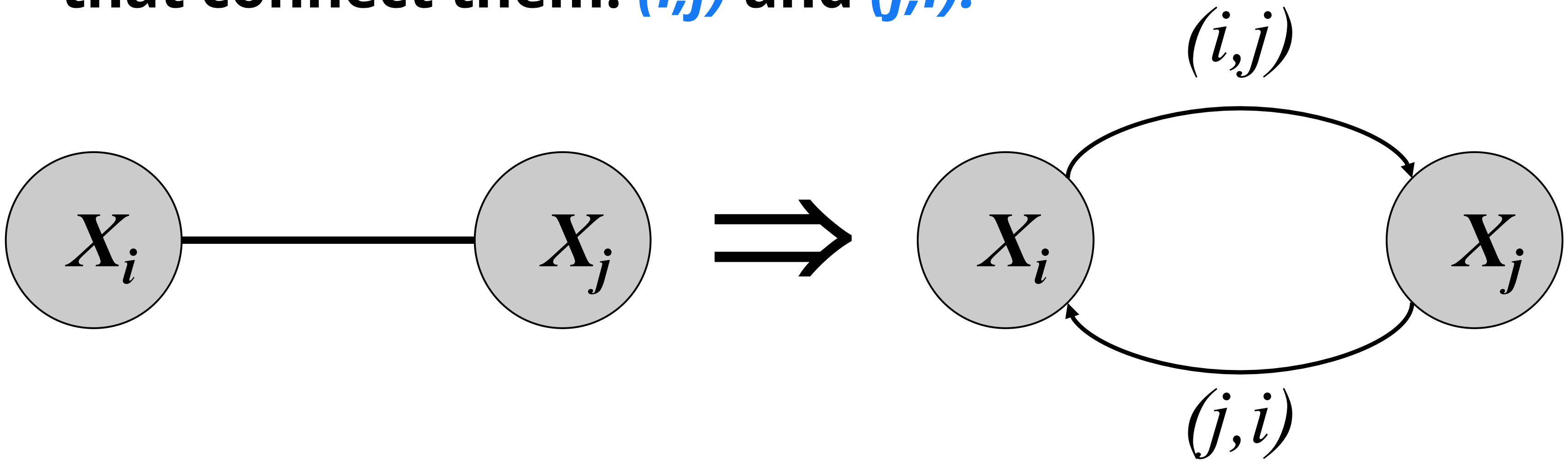
- ***Constraint graph:***
  - *nodes* are variables
  - *edges are constraints*





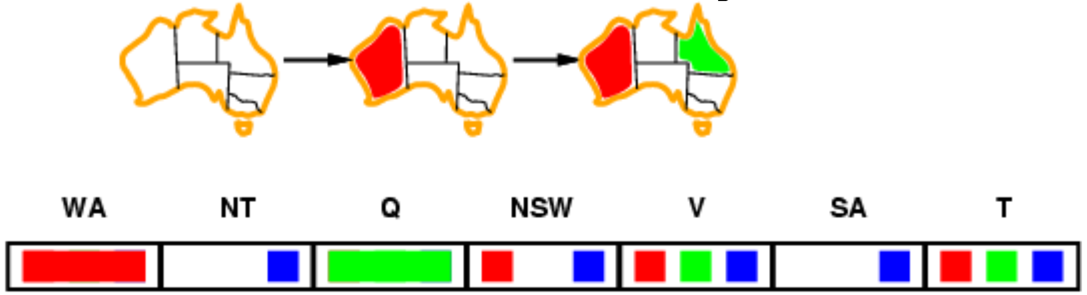
Edges to Arcs: From Constraint Graph to Directed Graph

- Given a pair of nodes X_i and X_j connected by a constraint ***edge***, we represent this not by a single undirected edge, but a ***pair of directed arcs***.
  - For a connected pair of nodes X_i and X_j, there are ***two*** arcs that connect them: *(i,j)* and *(j,i)*.



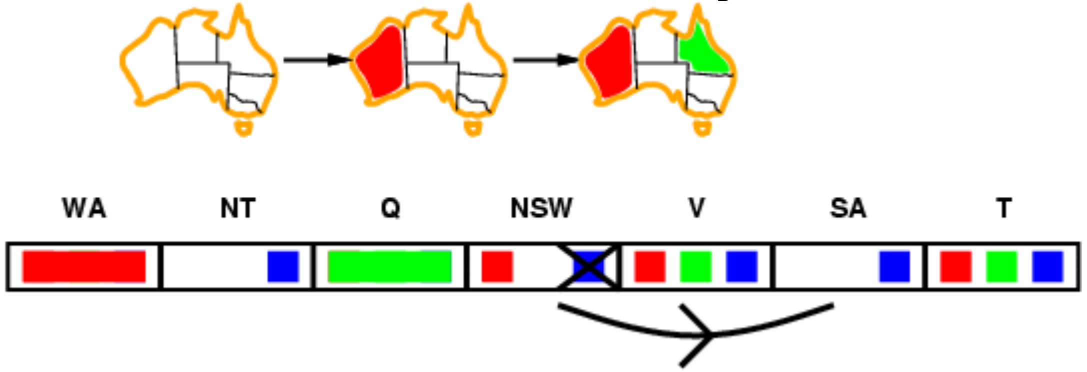
Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* →*Y* is consistent iff
  for every value *x* of *X* there is some allowed *y*



Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value x of X there is some allowed y



Arc consistency

- Simplest form of propagation makes each arc consistent
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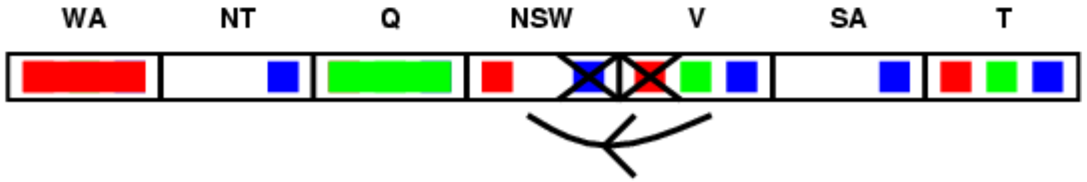
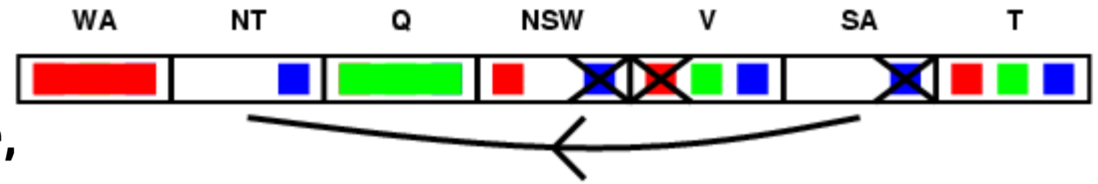


- If *X* loses a value, recheck neighbors of *X*

Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* →*Y* is consistent iff
  for every value *x* of *X* there is some allowed *y*



- If *X* loses a value,
- Detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

Arc consistency

An arc *(i,j) is* arc consistent if and only if every value *v* on X_i is consistent with some label on Y_j.

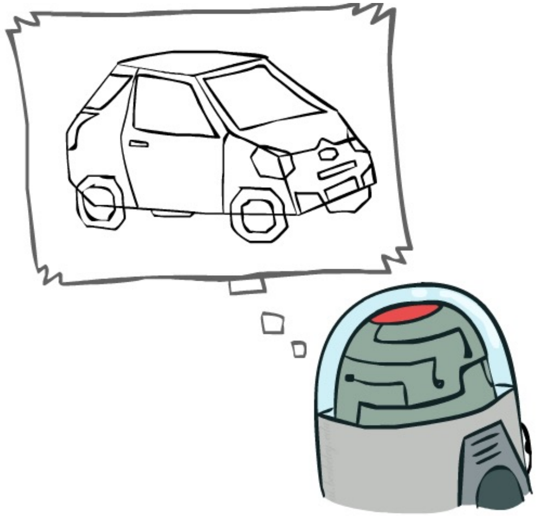
To make an arc *(i,j)* arc consistent,
for each value *v* on X_i ,
if there is no label on Y_j consistent with *v*
then remove *v* from X_i

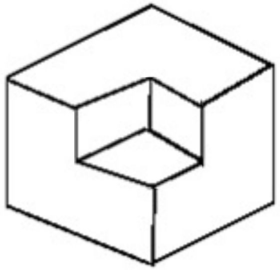
- **Given *d* values, checking arc (i,j) takes *O(d^2)* time worst case**

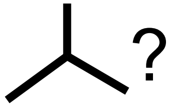
d is the size of the domain – the number of values

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP



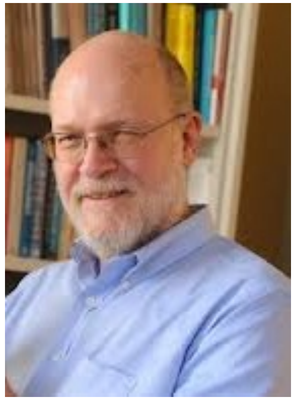




- Approach:
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Slide credit: Dan Klein and Pieter Abbeel
http://ai.berkeley.edu

Replacing Search: Constraint Propagation Invented…



Dave Waltz's insight:

- By ***iterating*** over the graph, the arc-consistency *constraints* can be *propagated* along arcs of the graph.
- *Search*: Use constraints to *add* labels to find *one solution*
- *Constraint Propagation*: Use constraints to *eliminate labels* to simultaneously find *all solutions*

The Waltz/Mackworth Constraint Propagation Algorithm

1. **Assign *every* node in the constraint graph a set of *all* possible values**
2. **Repeat until there is no change in the set of values associated with any node:**
    3. For each node i:
        4. For each neighboring node j in the picture:
            5. Remove any value from i which is not arc consistent with j.

Inefficiencies: Towards AC-3

1. **At each iteration, we only need to examine those X_i *where at least one neighbor of X_i has lost a value* in the previous iteration.**
2. **If X_i loses a value only because of arc inconsistencies with Y_j, we *don't need to check* Y_j on the next iteration.**
3. **Removing a value on X_i can only make Y_j arc-inconsistent with respect to X_i itself. Thus, we only need to check that *(j,i)* is still arc-consistent.**

These insights lead a much better algorithm…

AC-3

function AC-3(*csp*) return the CSP, possibly with reduced domains

inputs: *csp*, a binary csp with variables $\{X_1, X_2, \ldots, X_n\}$

local variables: *queue,* a queue of arcs initially the arcs in *csp*

while queue is not empty do

$(X_i, X_j) \leftarrow$ *queue*.pop()

if REMOVE-INCONSISTENT-VALUES(X_i, X_j) then

for each X_k in NEIGHBORS[X_i] – {X_j} do
add (X_k, X_i) to queue

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) return *true* iff we remove a value

removed ← *false*

for each *x* in DOMAIN[X_i] do

if no value *y* in DOMAIN[X_j] allows (x,y) to satisfy
the constraints between X_i and X_j

then delete x from DOMAIN[X_i]; *removed* ← *true*

return *removed*

Keep track of what arcs we need to process

Add back arcs to neighbors whenever a node had values removed

AC-3: Worst Case Complexity Analysis

- **All nodes can be connected to *every* other node,**
  - so each of ***n*** nodes must be compared against ***n-1*** other nodes,
  - so total # of arcs is ***2*n*(n-1), i.e. O(n²)***
- **If there are *d* values, checking arc (i,j) takes $O(d^2)$ time**
- **Each arc (i,j) can only be inserted into the queue *d* times**
- **Worst case complexity: $O(n^2 d^3)$**

(For *planar* constraint graphs, the number of arcs can only be *linear in N and* the time complexity is only $O(nd^3)$)

When to Iterate, When to Stop?

The crucial principle:

If a value is removed from a node X_i,
then the values on all of X_i's neighbors must be reexamined.

Why? *Removing* a value from a node may result in one of the neighbors becoming arc *inconsistent*, so we need to check…

(but each neighbor X_j can only become inconsistent with respect to the removed values on X_i)

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Other techniques for speeding up finding solutions for CSPs

Professor Chris Callison-Burch

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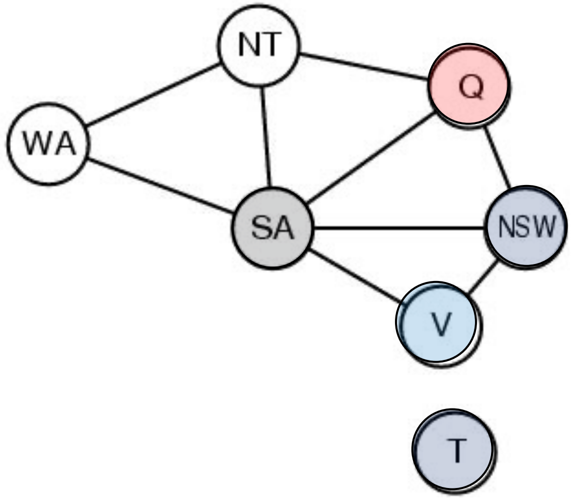
Chronological backtracking

- DFS does Chronological backtracking
  - If a branch of a search fails, backtrack to the most recent variable assignment and try something different
  - But this variable may not be related to the failure
- Example: Map coloring of Australia
  - Variable order
    - Q, NSW, V, T, SA, WA, NT.
  - Current assignment:
    - Q=red, NWS=green, V=blue, T= red
  - SA cannot be assigned anything
  - But reassigning T does not help!

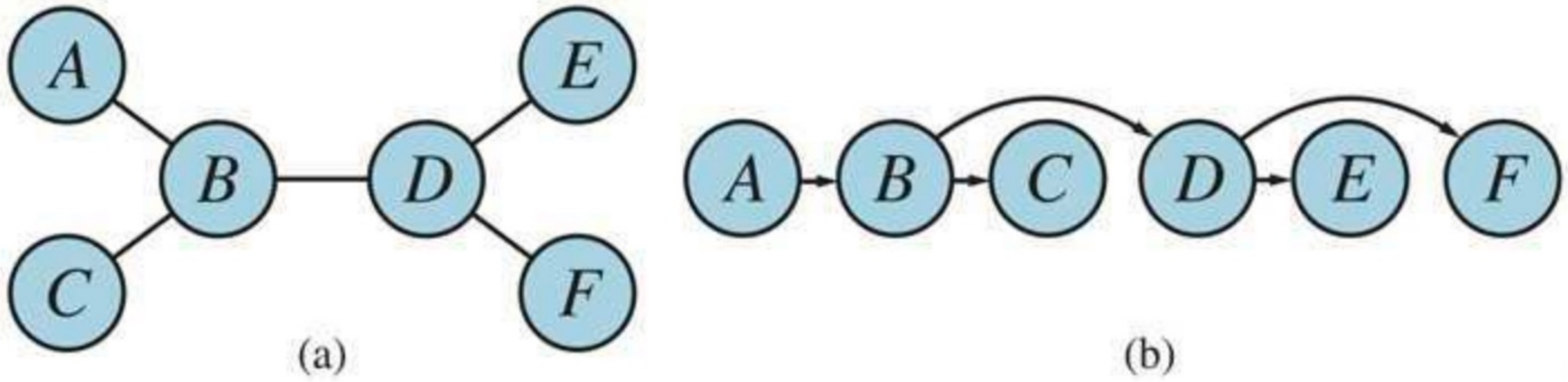


Backjumping: Improved backtracking

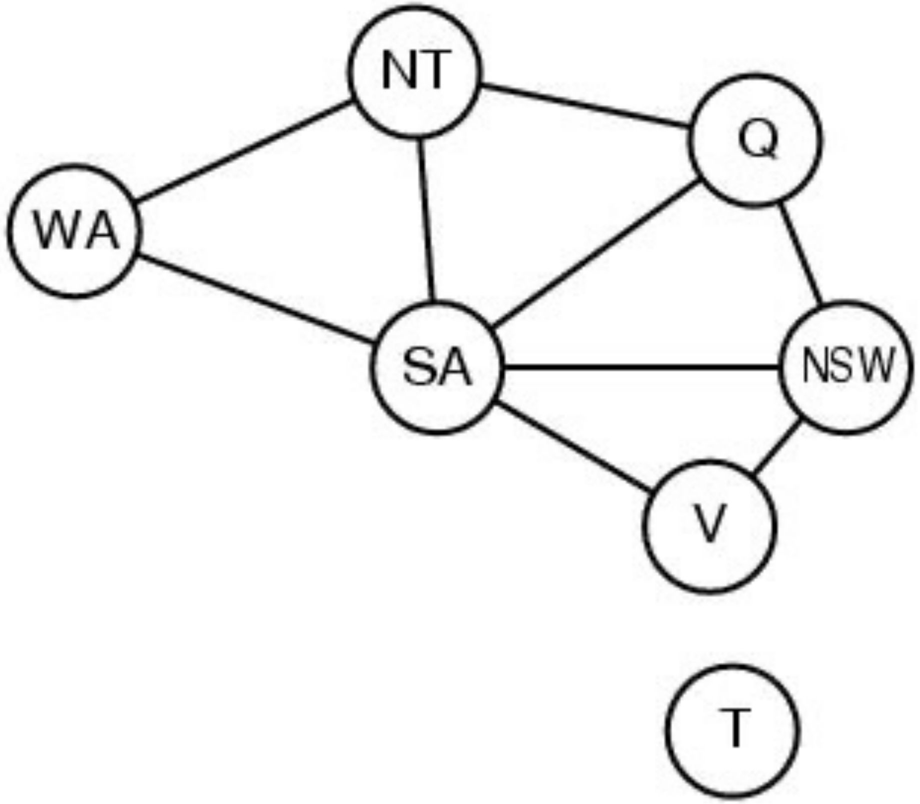
- Find “the conflict set”
  - Those variable assignments that are in conflict
  - Conflict set for SA: {Q=red, NSW=green, V=blue}
- Jump back to reassign one of those conflicting variables
- Forward checking can build the conflict set
  - See textbook for details

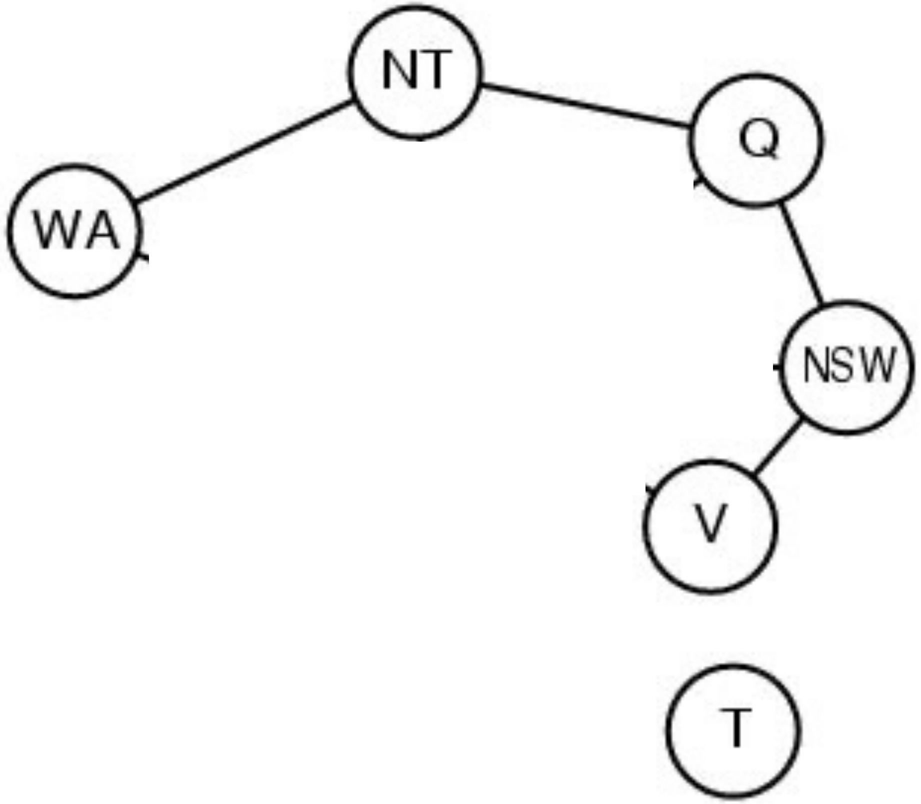
Simple CSPs can be solved quickly

1. Completely independent subproblems
   e.g. Australia & Tasmania
   - Easiest
2. Constraint graph is a tree
   - Any two variables are connected by only a single path
   - Permits solution in time linear in number of variables
   - Do a topological sort and just march down the list



Cutset conditioning





Local search for CSPs

- Local search like hill-climbing search for nearby solutions that improve an objective function.

- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values

- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with *h(n)* = total number of violated constraints

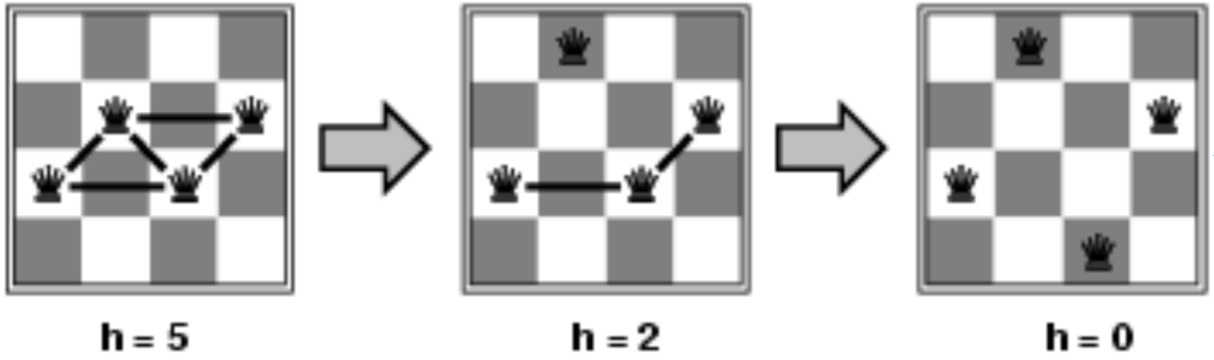
Min-Conflicts algorithm

function MIN-CONFLICTS(*csp, max_steps*) **returns** a solution or failure
        inputs: *csp*, a constraint satisfaction problem
                  max_steps, the number of steps allowed before giving up

current ← an initial complete assignment for *csp*
for i = 1 to max_steps **do**
        if *current* is a solution for *csp* **then return** *current*
        var ← a randomly chosen conflicted variable from cp. VARIABLES
        value ← the value *v* for var that minimizes CONFLICTS(csp, var, *v, current*)
        set *var* = *value in current*
return *failure*

Min-Conflicts Example: n-queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks



Given random initial state, local min-conflicts can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 1,000,000)

Min-conflicts reduced time to scheduled Hubble Space Telescope from 3 weeks to 10 minutes

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Next week: Logical Agents