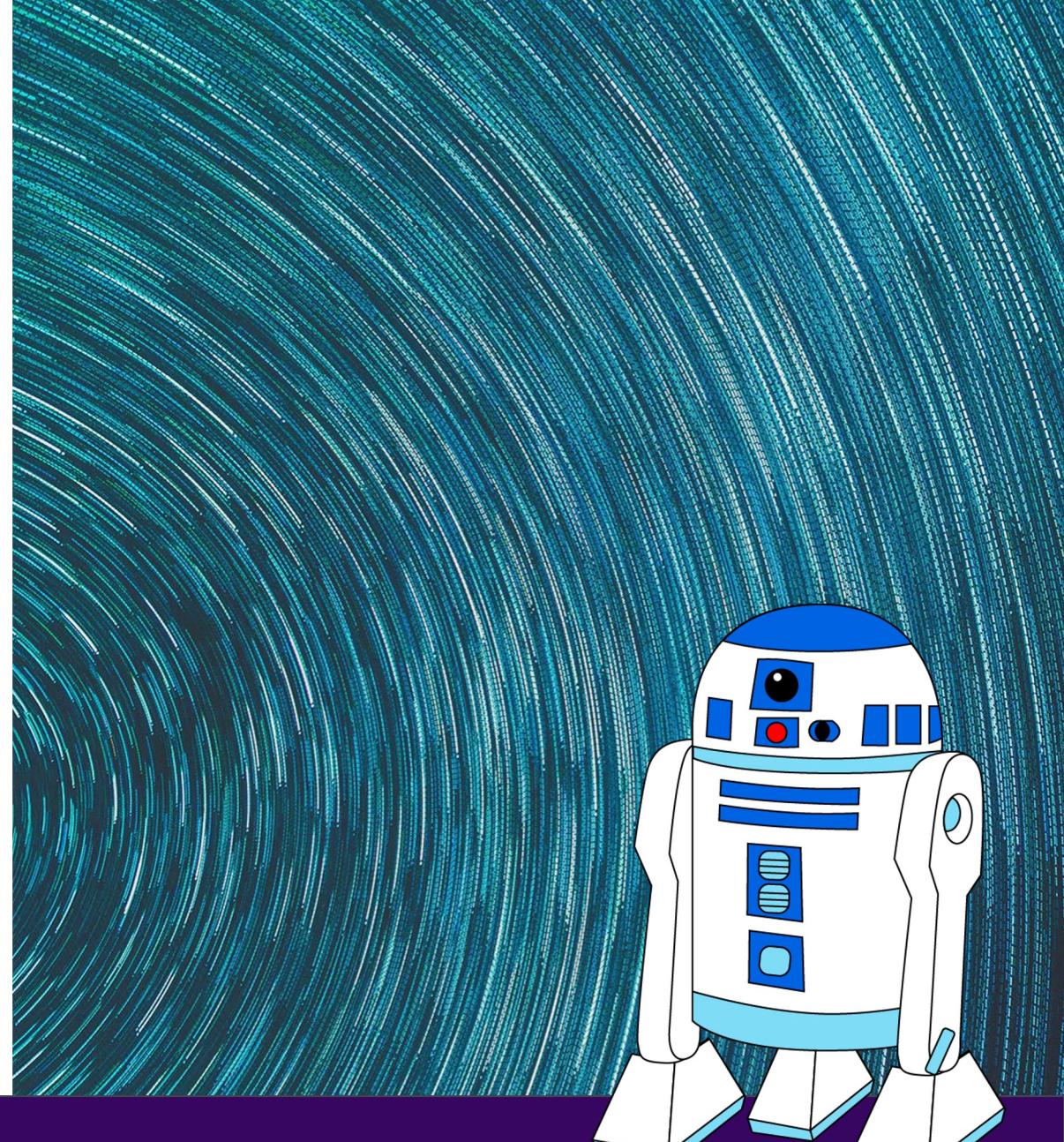


CIS 4210/5210:  
ARTIFICIAL INTELLIGENCE

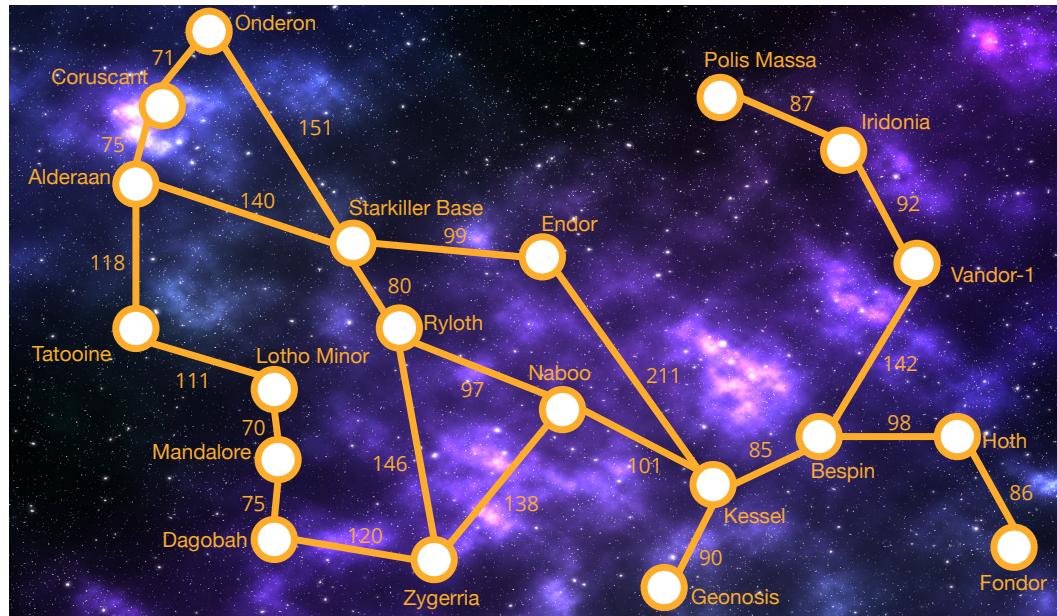
# Logical Agents



# Knowledge-based Agents

**Knowledge-based agents** use a process of **reasoning** over an internal **representation** of knowledge to decide what action to take.

So far, our **problem-solving** agents have performed a **search** over **states** in order to find a plan. The representation of states has been **atomic**.



Limited to commands like  
“Navigate to Kessel”

“Take me to the nearest  
habitable planet where I can  
store my perishable cargo”

# Knowledge-based Agents

A central component of a knowledge-based agent is a **knowledge base** or KB.

A KB contains a set of **sentence** that are written in a **knowledge representation language**. The sentence contains some assertion about the world.

Natural language sentences	Knowledge representation language sentence
<i>Hoth is a planet</i>	planet(hoth)
<i>Hoth is habitable</i>	habitable(hoth)
<i>Hoth is far from its sun</i>	far_from(hoth, sol)
<i>If a planet is far from its sun then it is cold</i>	planet(x) and sun(y) and far_from(x,y) → cold(x)

# Knowledge-based Agents

There are two kinds of sentences:

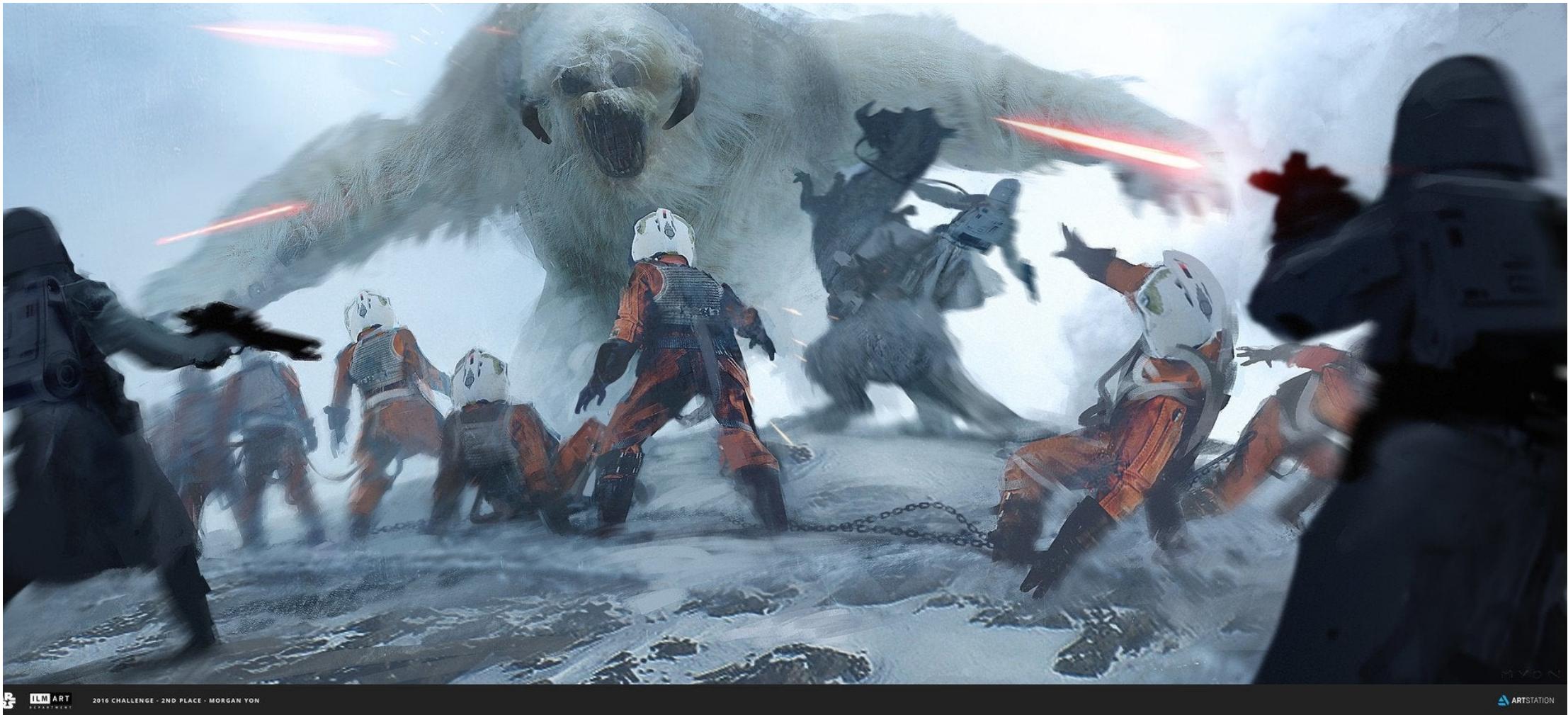
- **Axioms** – a sentence that is given
- **Derived sentences** – a new sentence that is derived from others sentences

The process of deriving new sentences from old sentences is called **inference**.

A KB can initially contain some **background information** about the world, and a knowledge-based agent can add to the information in the KB through its observations of the world.

In addition to asserting new knowledge into its KB, a knowledge-based agent can also query the KB and ask it to derive new knowledge in order to select what action it should take.

# Hunt the Wampas



# Wampa World

Our **knowledge-based agent**, R2D2, explores **a cave** consisting of **rooms** connected by passageways.

Lurking somewhere in the cave is the **Wampa**, a beast that eats any agent that enters its room.

Some rooms contain bottomless **pits** that trap any agent that wanders into the room.

In one room is master **Luke**.

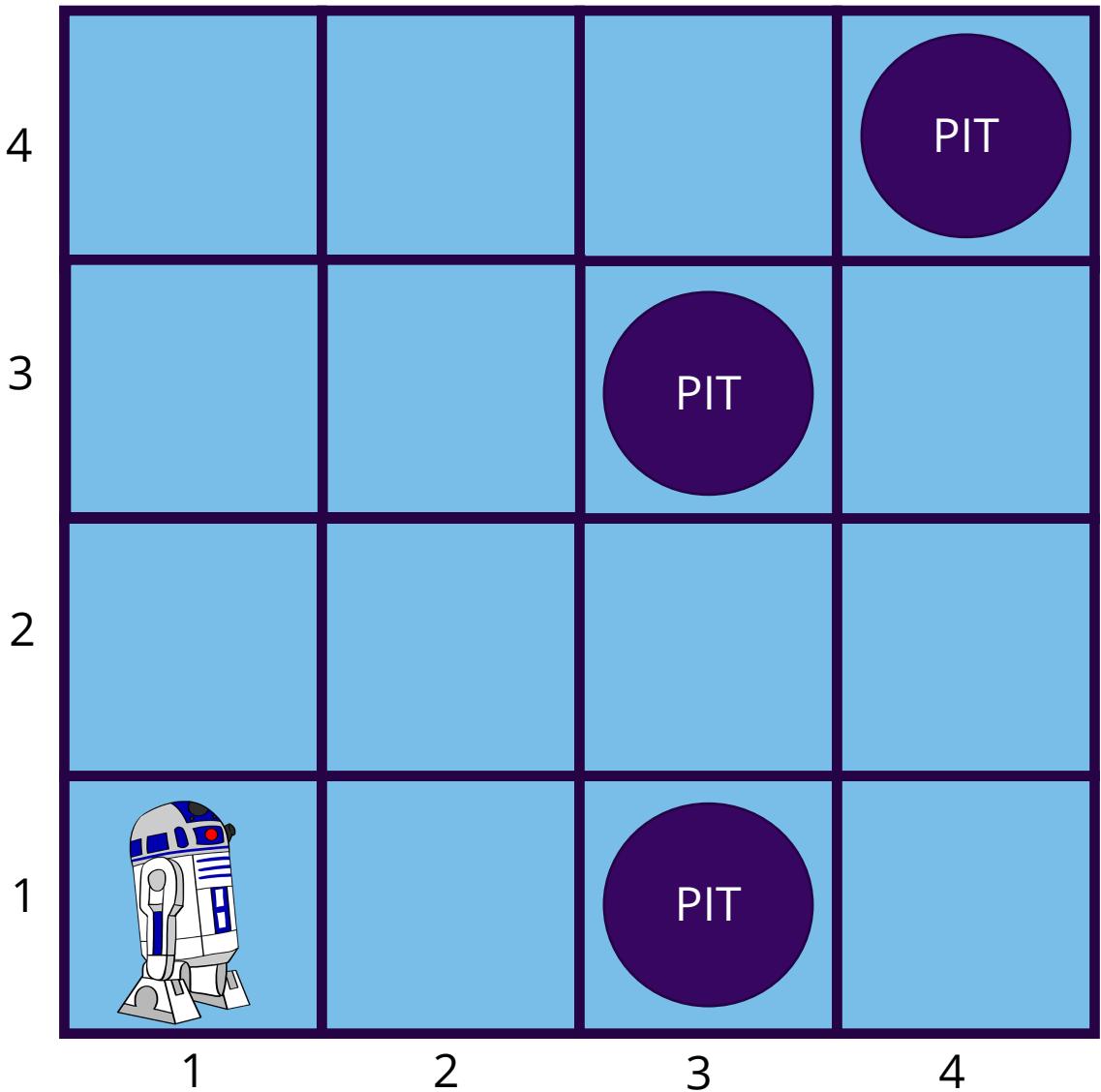
The goal is:

- collect Luke
- exit the world
- without being eaten



# Wampa World

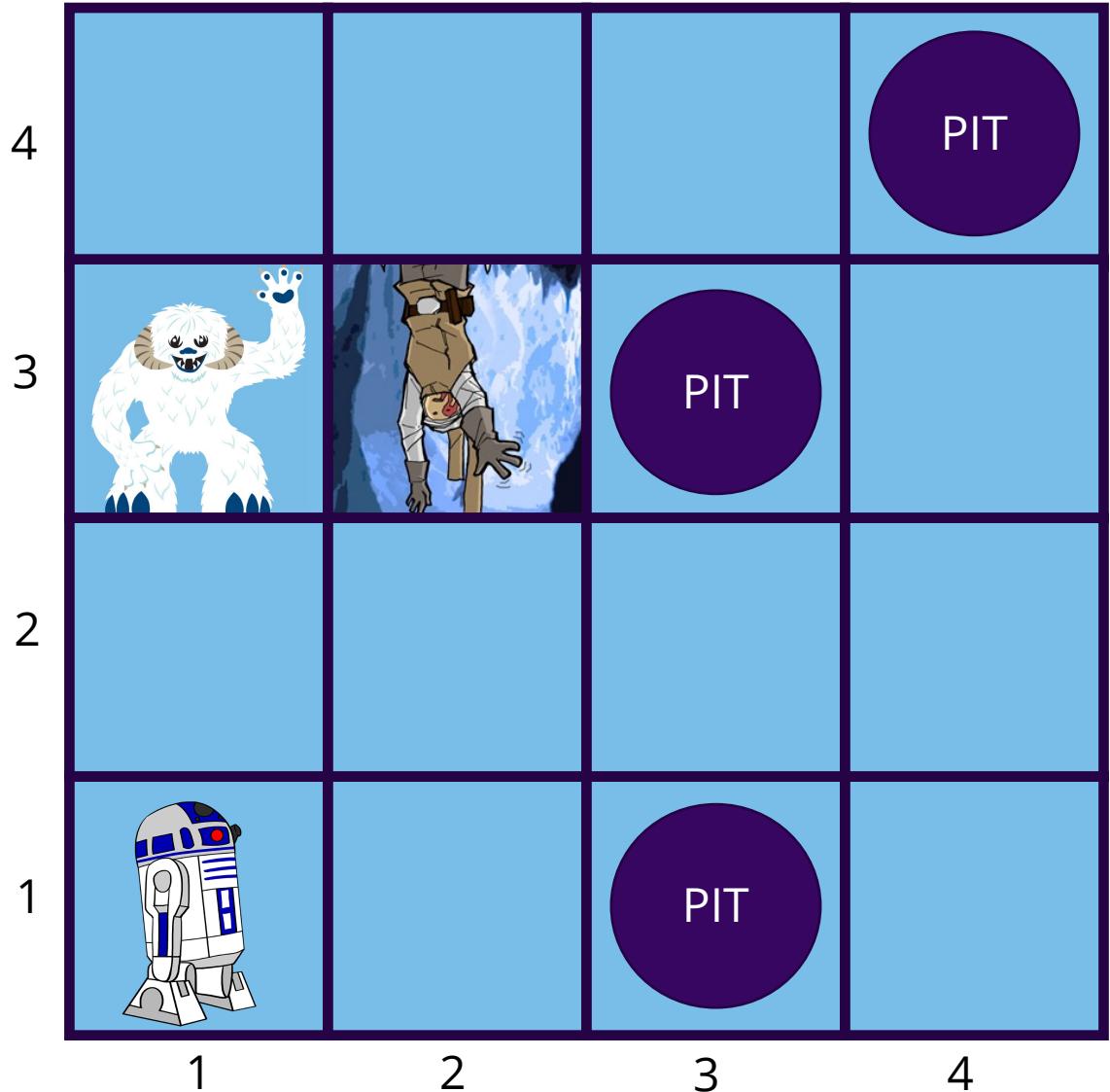
**Environment:** A 4x4 grid of rooms. The agent starts in the square [1,1]. Wampa and Luke are randomly placed in other squares. Each square can be pit with 20% probability.



# Wampa World

## Performance measure:

- +1000 points for rescuing Luke and leaving the cave
- 1000 for falling into a pit or being eaten by the Wampa
- 1 for each action taken
- 10 for using up your blaster fire



# Wampa World

## Actuators:

R2 can move *Forward*, *TurnLeft*, *Turn right*.

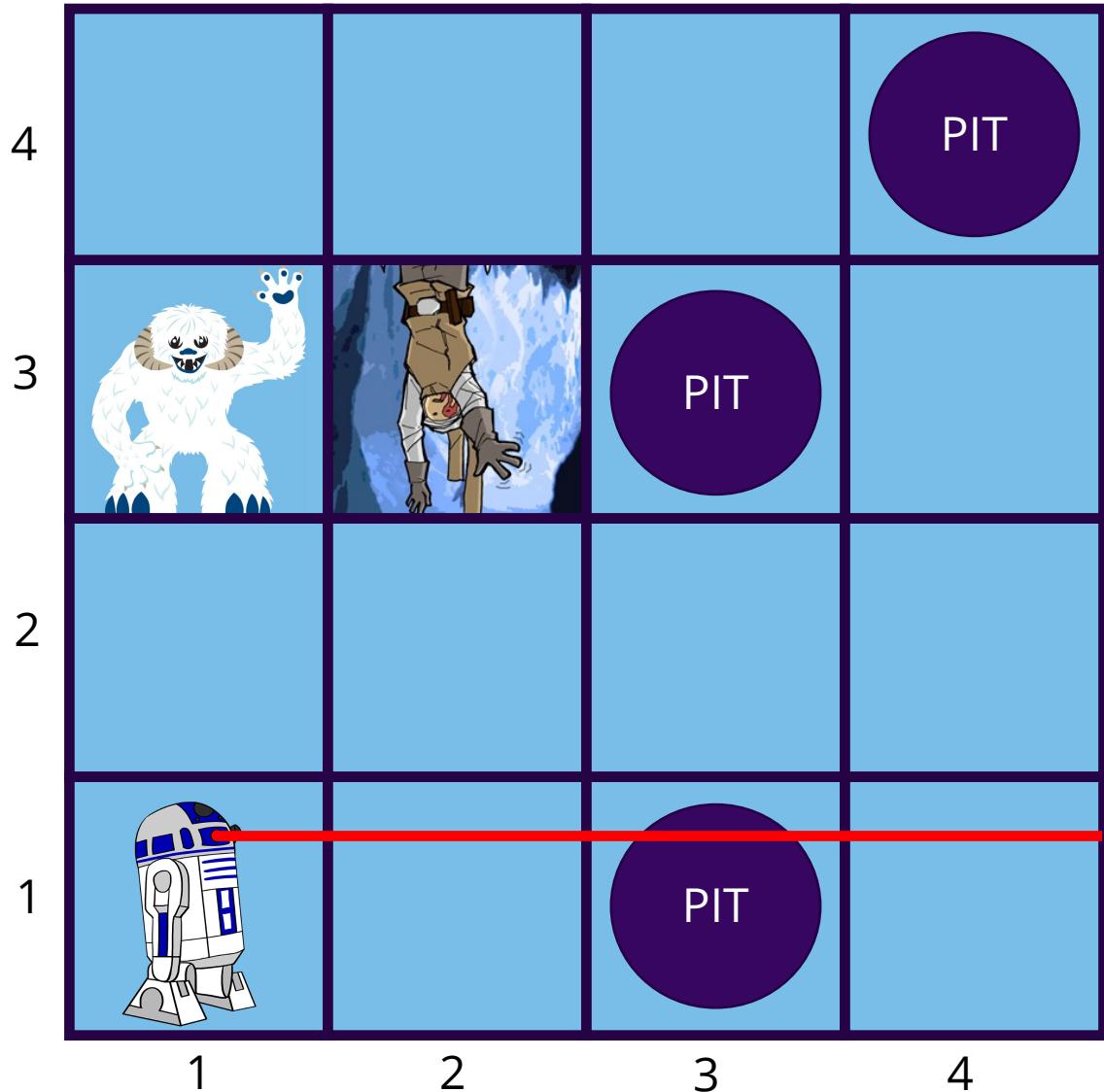
Agent dies if it moves into a pit or a Wumpus square.

*Grab* can pick up Luke.

*Shoot* fires blaster bolt in a straight line in the direction that R2D2 is facing.

If the blaster hits the Wampa, it dies. R2 only has enough power for one shot.

*Climb* gets R2 out of the cave but only works in [1, 1]



# Wampa World

## Sensors:

In each square adjacent to the Wampa, R2D2's olfactory sensor perceives a *Stench*

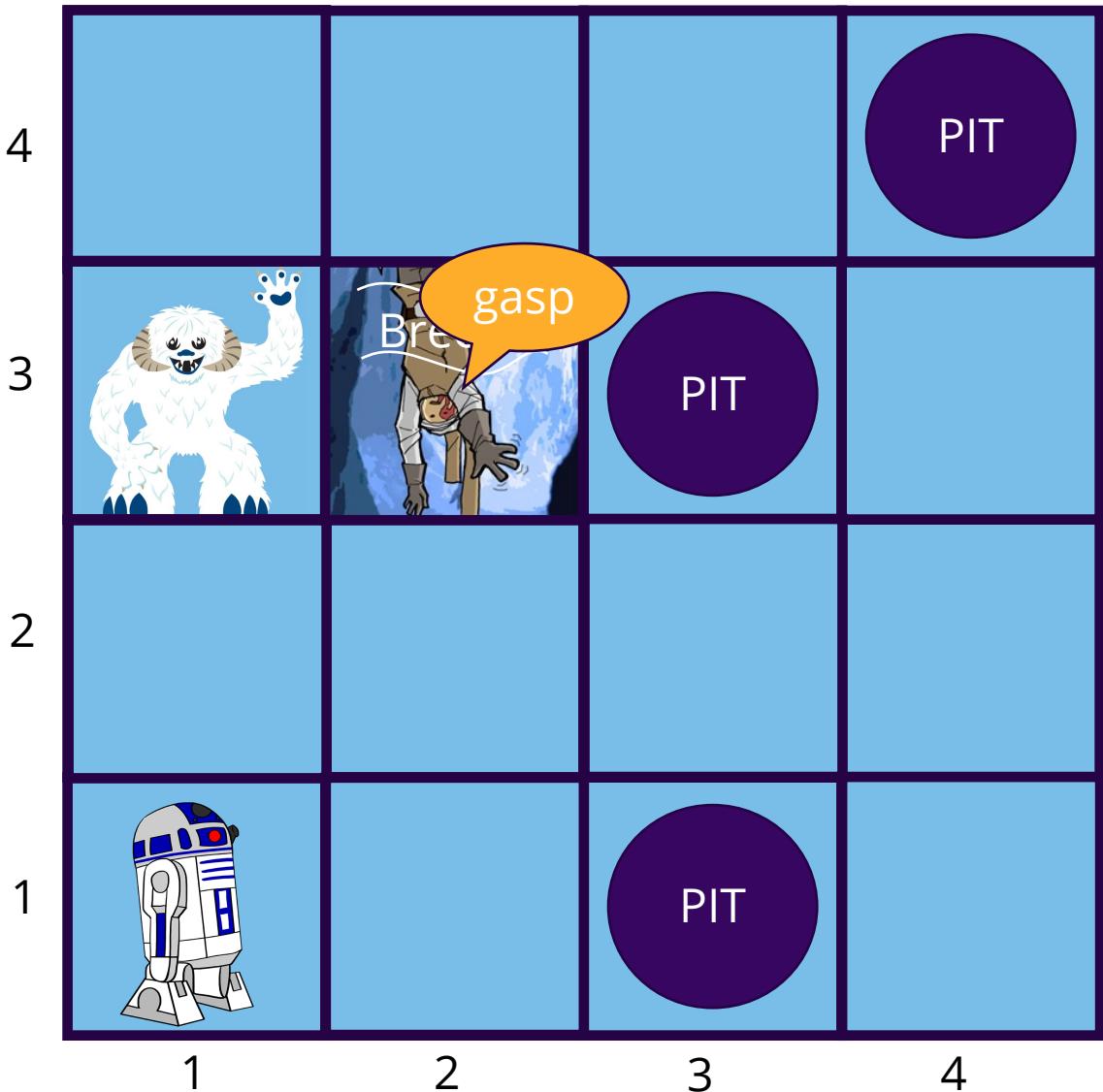
In each square adjacent to a pit, R2D2's wind sensor perceives a *Breeze*

In the square with Luke, R2D2's audio sensor perceives a *Gasp*

When R2D2 walks into a wall it perceives a *Bump*

When the Wampa is killed , R2D2's audio sensor perceives a *Scream*

Percept=[*Stench*, *Breeze*, *Gasp*, *None*, *None*]



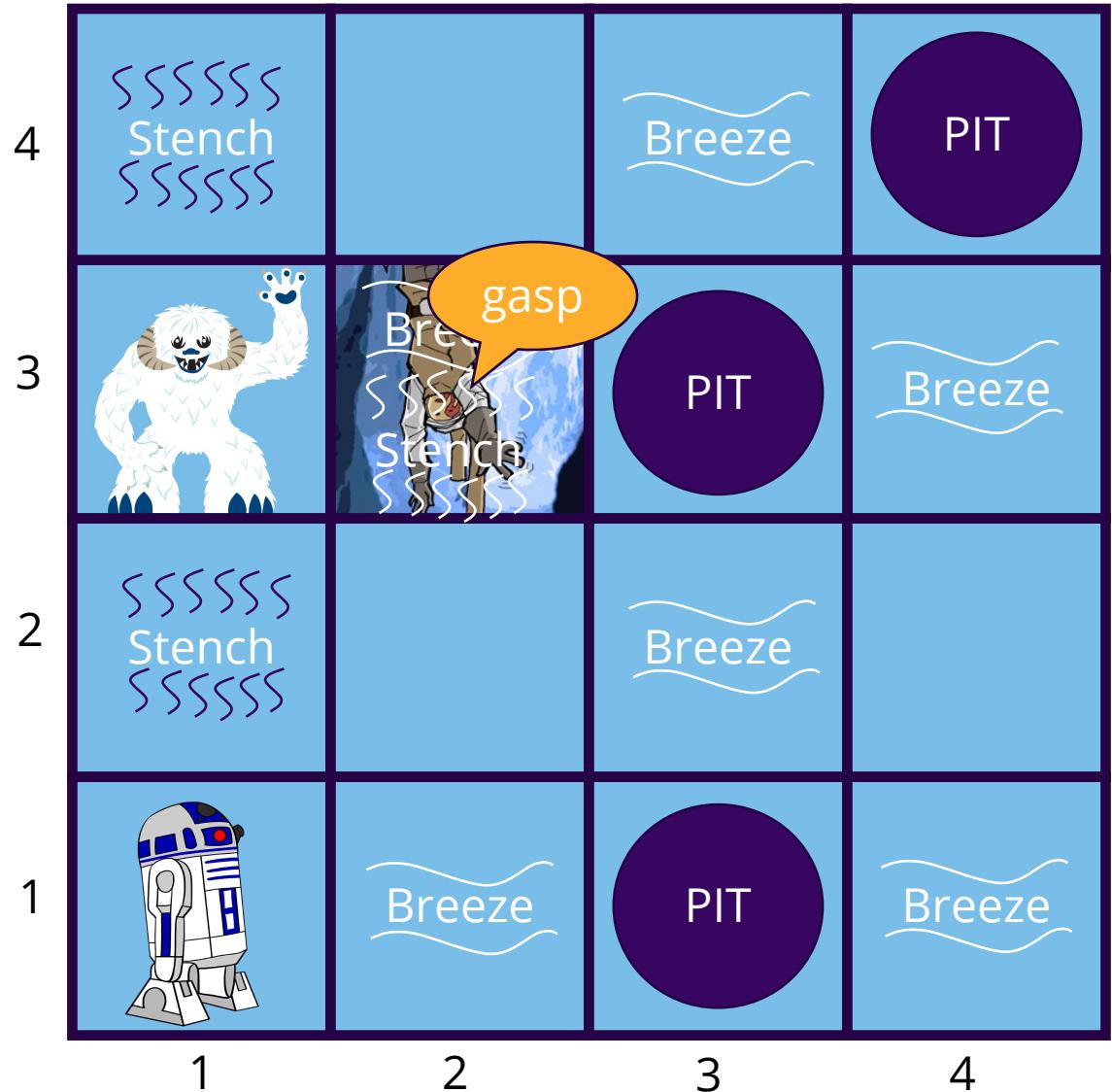
# Wampa World

**Deterministic, discrete, static, single-agent** (Wampa doesn't move)

**Sequential** because reward doesn't come for many steps

**Partially observable** because some parts of the state are not directly perceptible:

- Location of Luke, Wampa, and pits aren't directly observable.

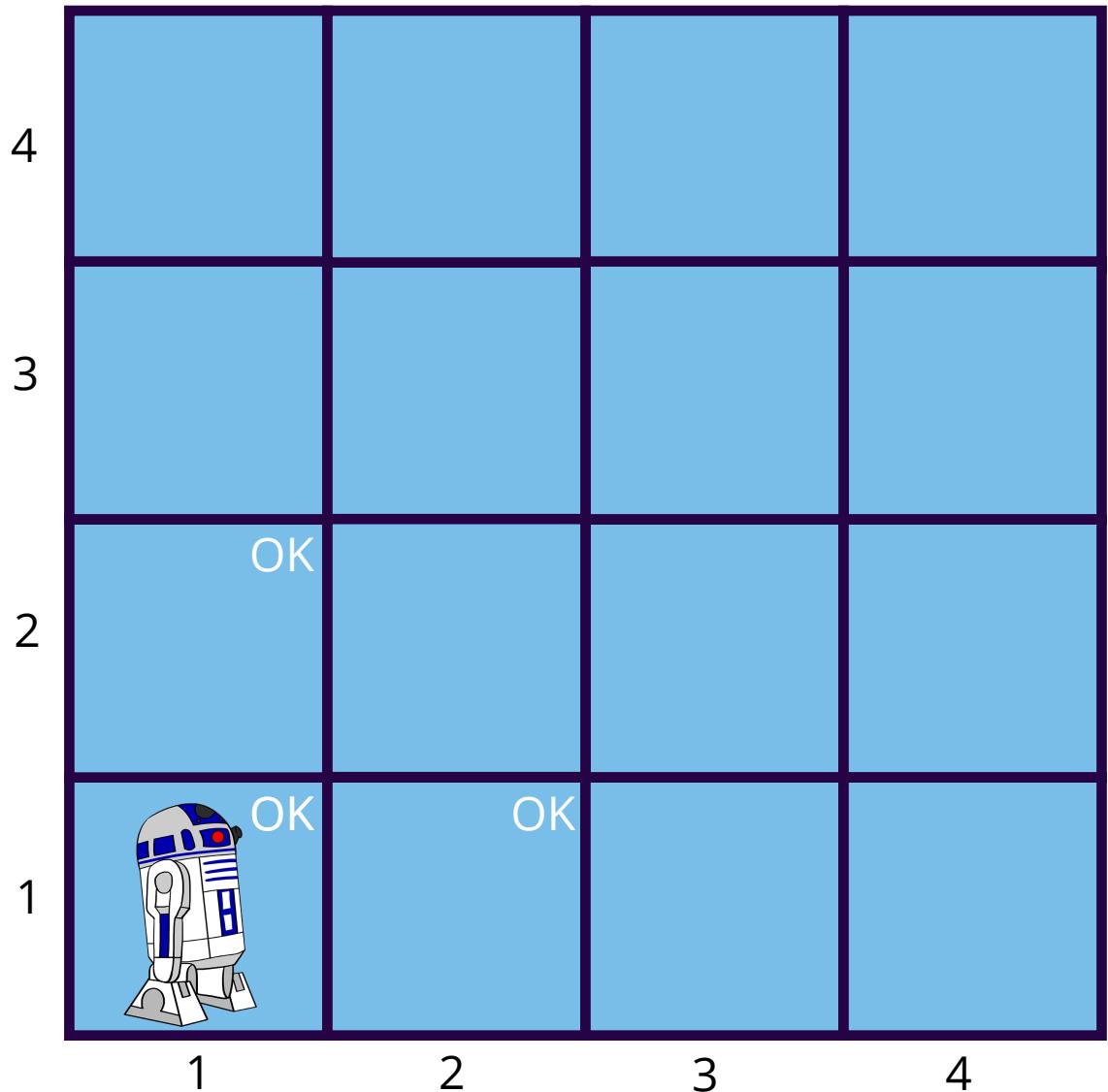


# Wampa World Walkthrough

R2D2 starts in [1,1]

Percept=[None, None, None, None, None]

What can we conclude about [1,2] and [2,1]?

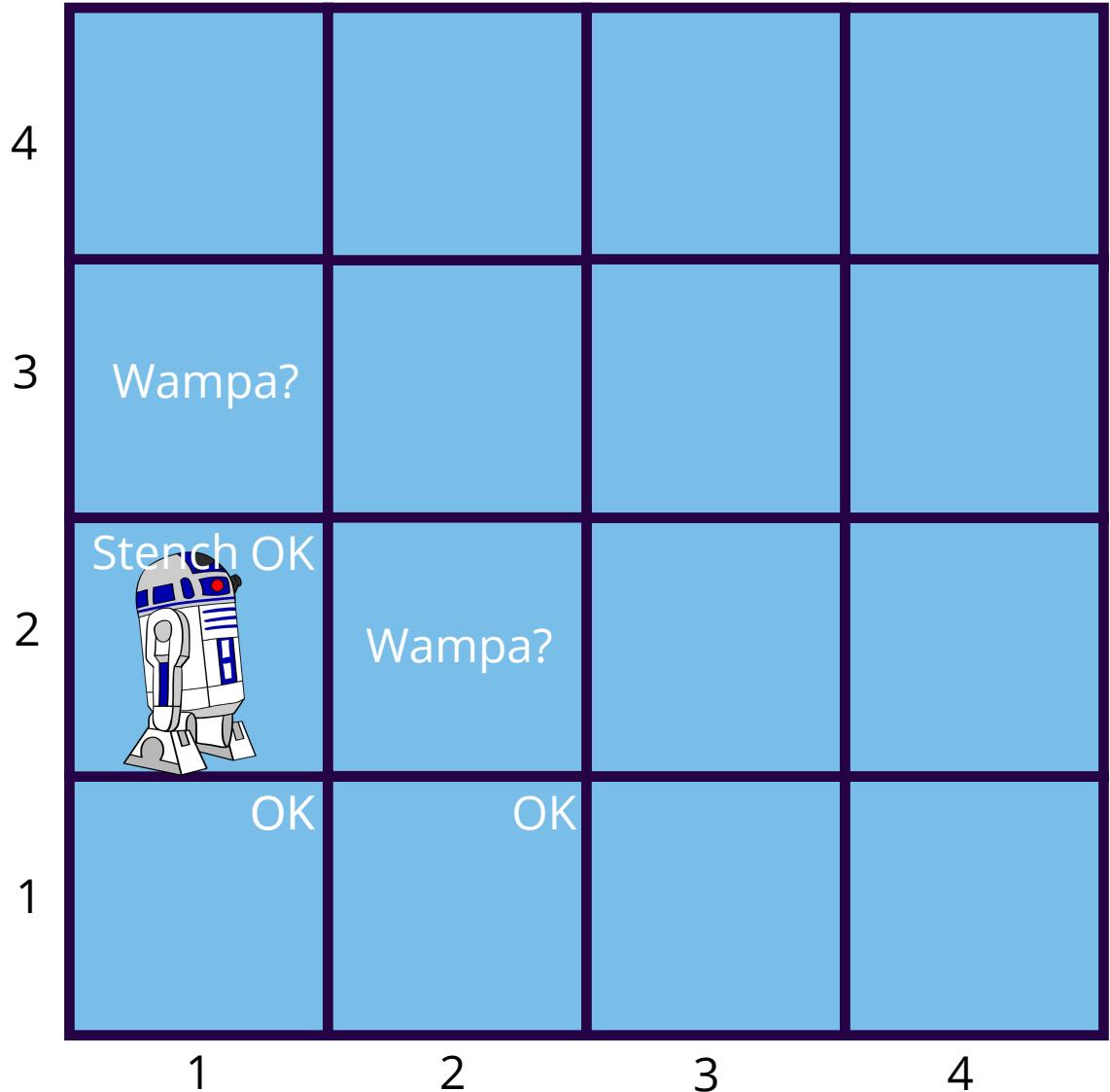


# Wampa World Walkthrough

R2D2 moves to [1,2]

Percept=[*Stench, None, None, None None*]

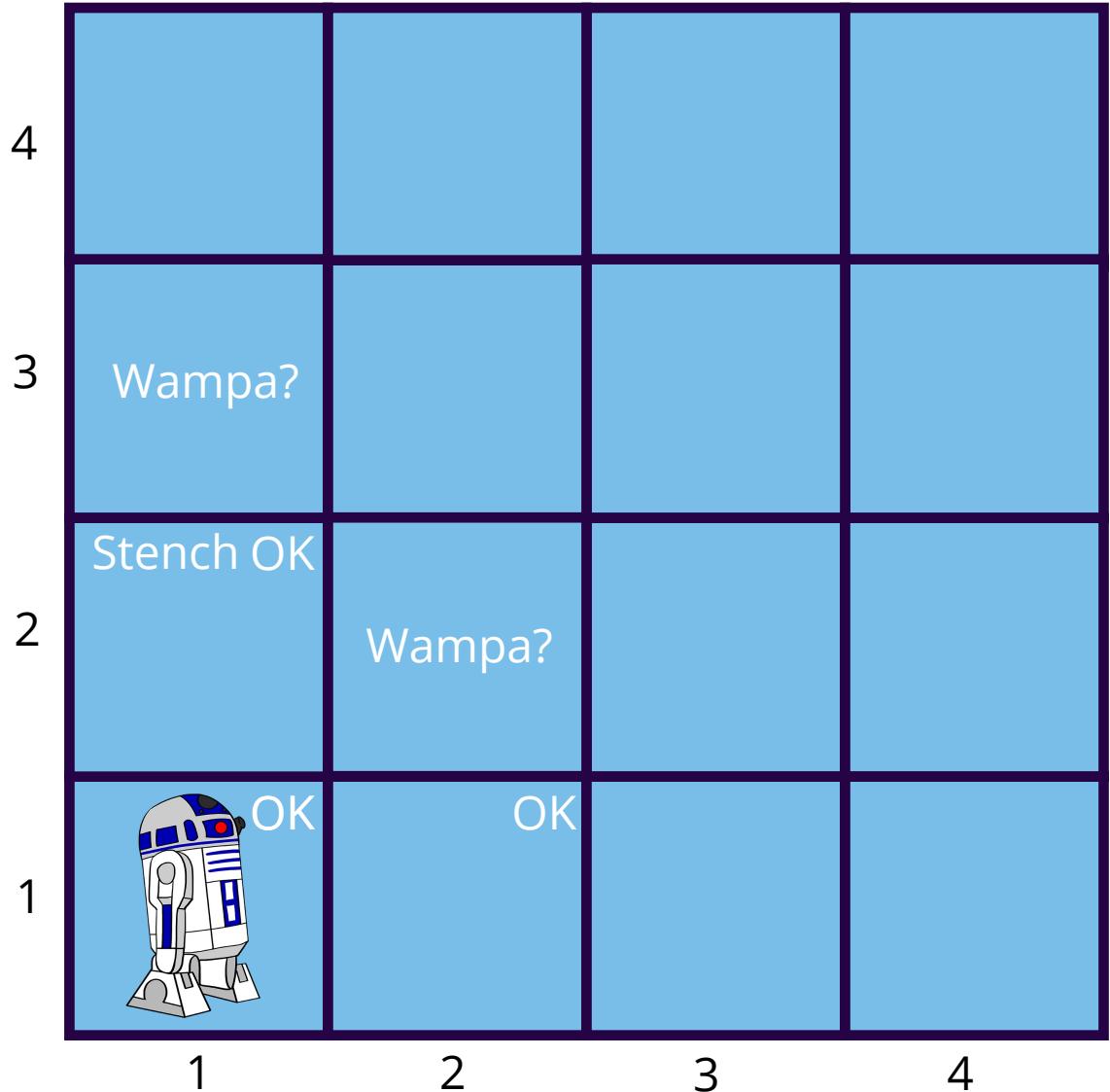
What can we conclude about [3,1] and [2,2] from the Stench?



# Wampa World Walkthrough

R2D2 moves back to [1,1] and gets the same percept vector as before

Percept=[None, None, None, None None]

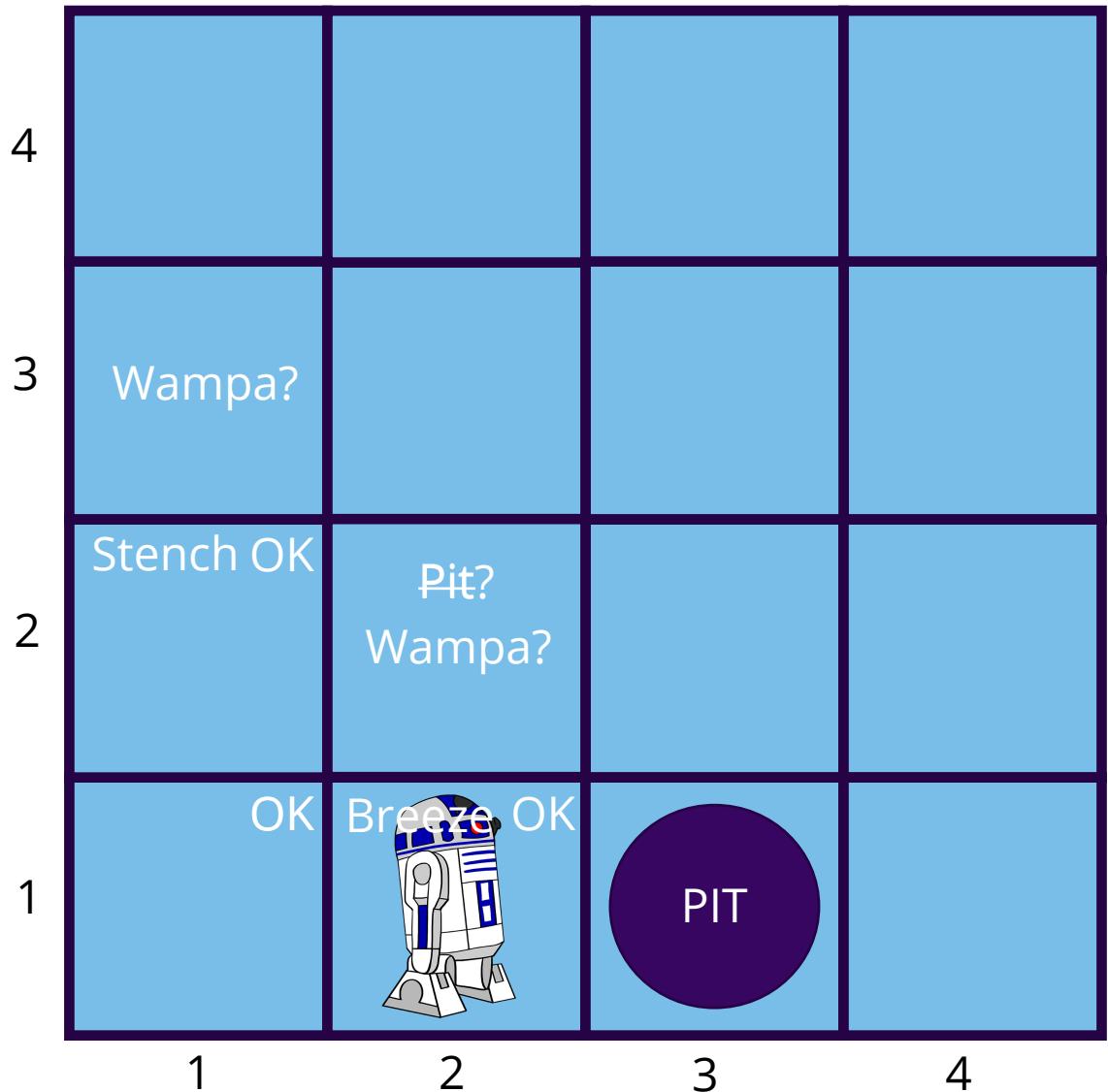


# Wampa World Walkthrough

R2D2 moves to [2,1]

Percept=[None, Breeze, None, None None]

What can we conclude about [3,1] and [2,2] based on the Breeze?

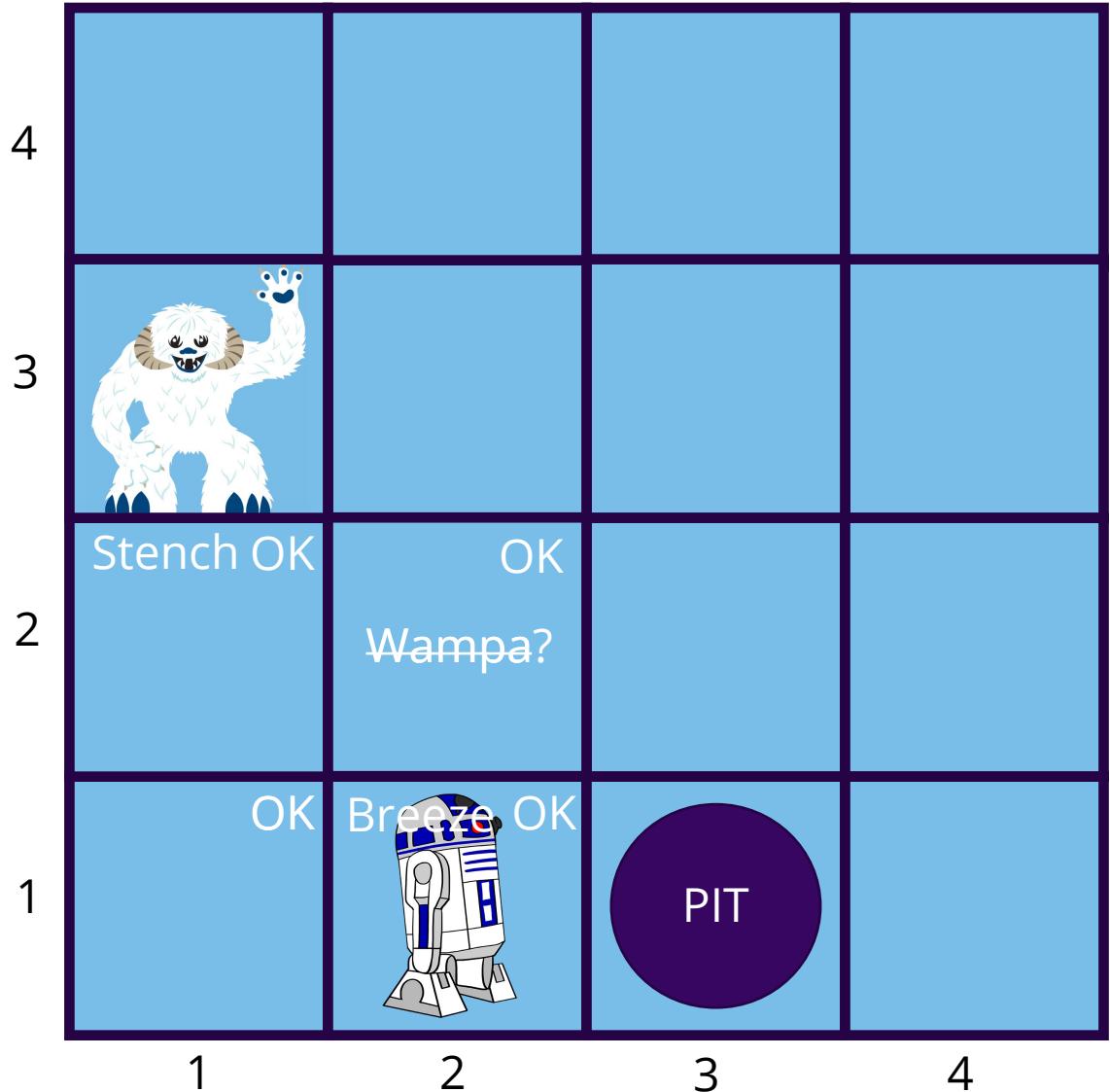


# Wampa World Walkthrough

R2D2 moves to [2,1]

Percept=[*None, Breeze, None, None None*]

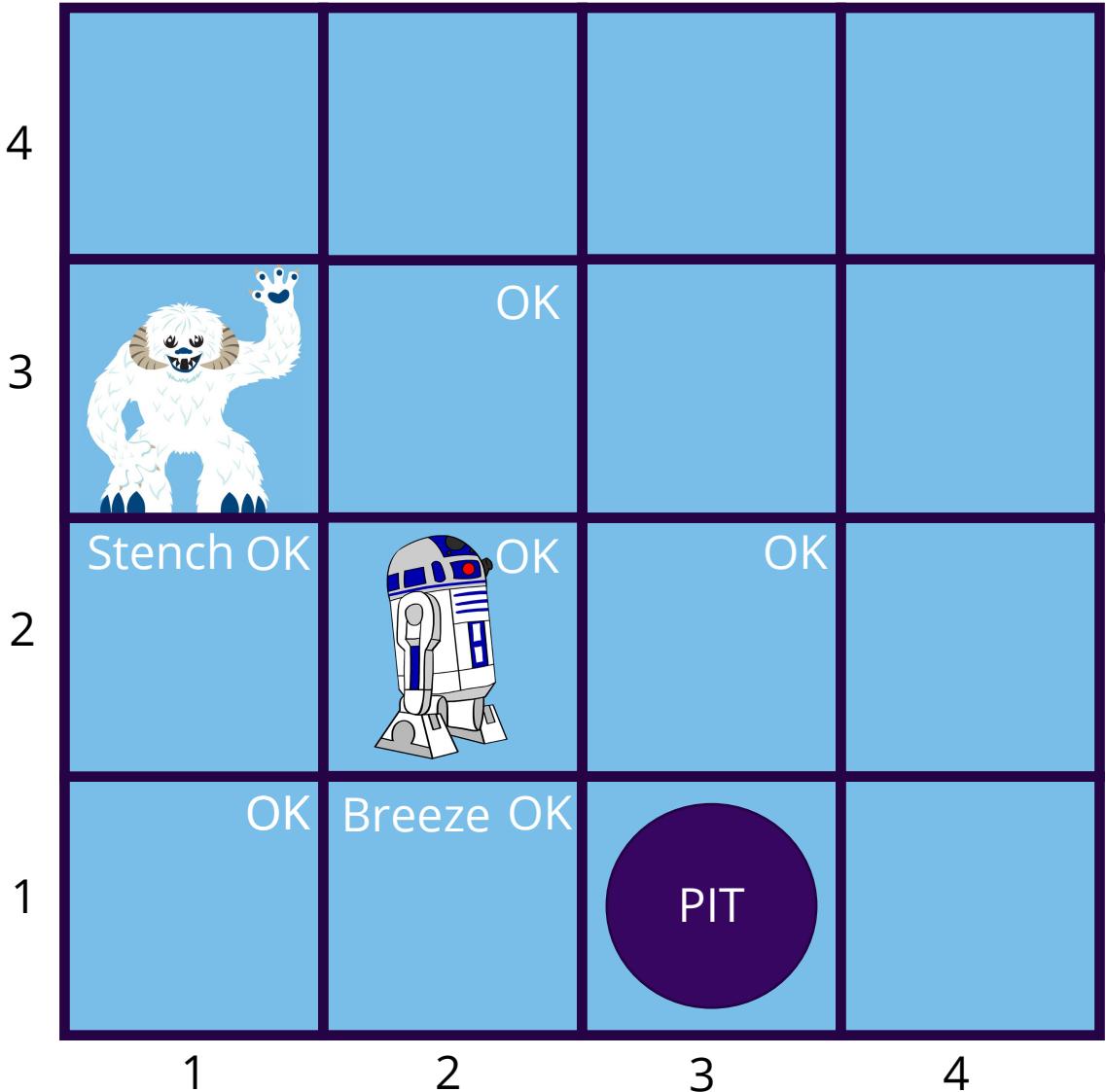
What can we conclude about [2,2] and [1,3] based on the lack of a Stench here?



# Wampa World Walkthrough

R2D2 moves to [2,2]

Percept=[None, None, None, None, None]

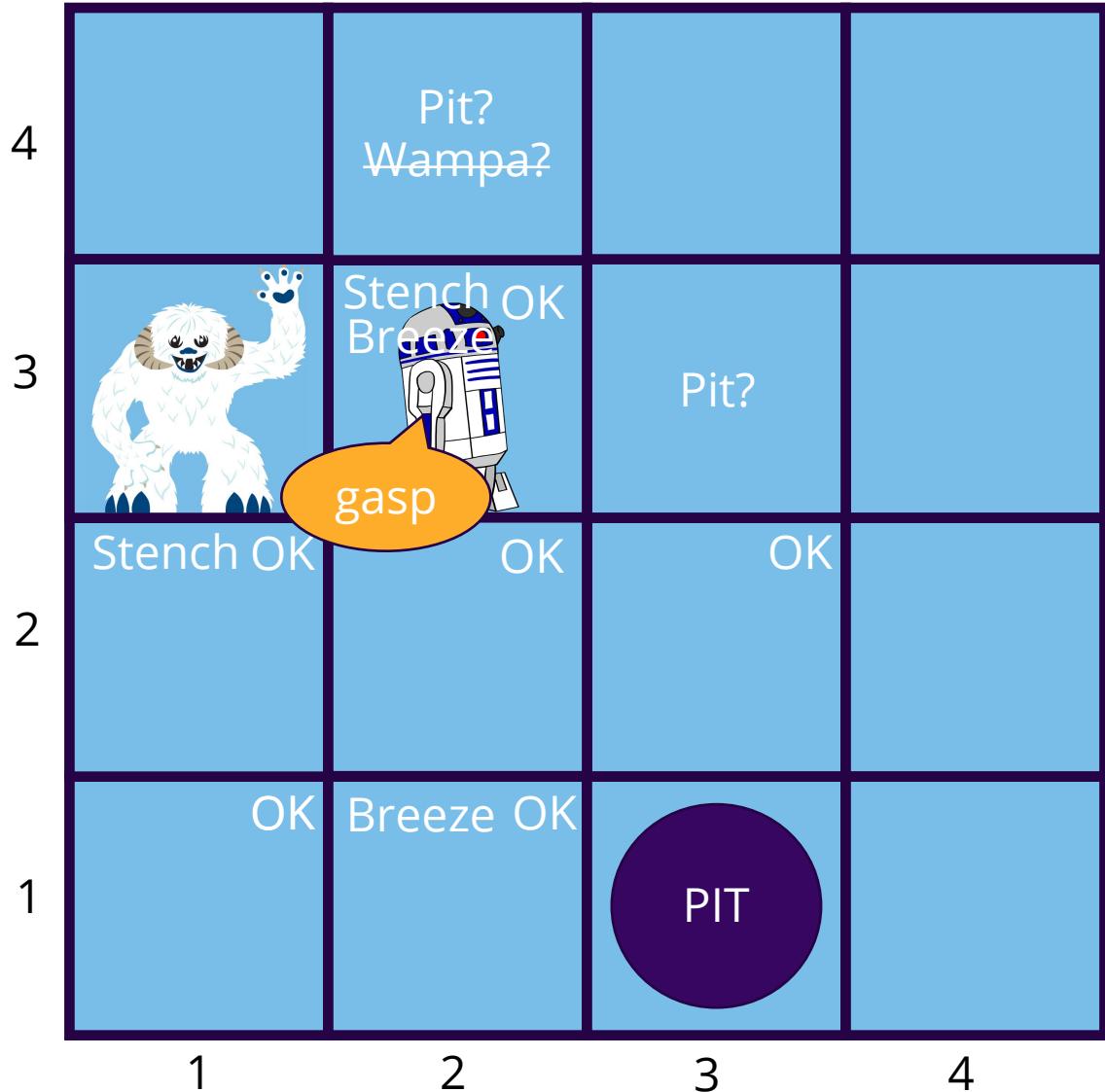


# Wampa World Walkthrough

R2D2 moves to [2,3]

Percept=[*Stench, Breeze, Gasp, None, None*]

What can we conclude about [2,4] or [3,3]?

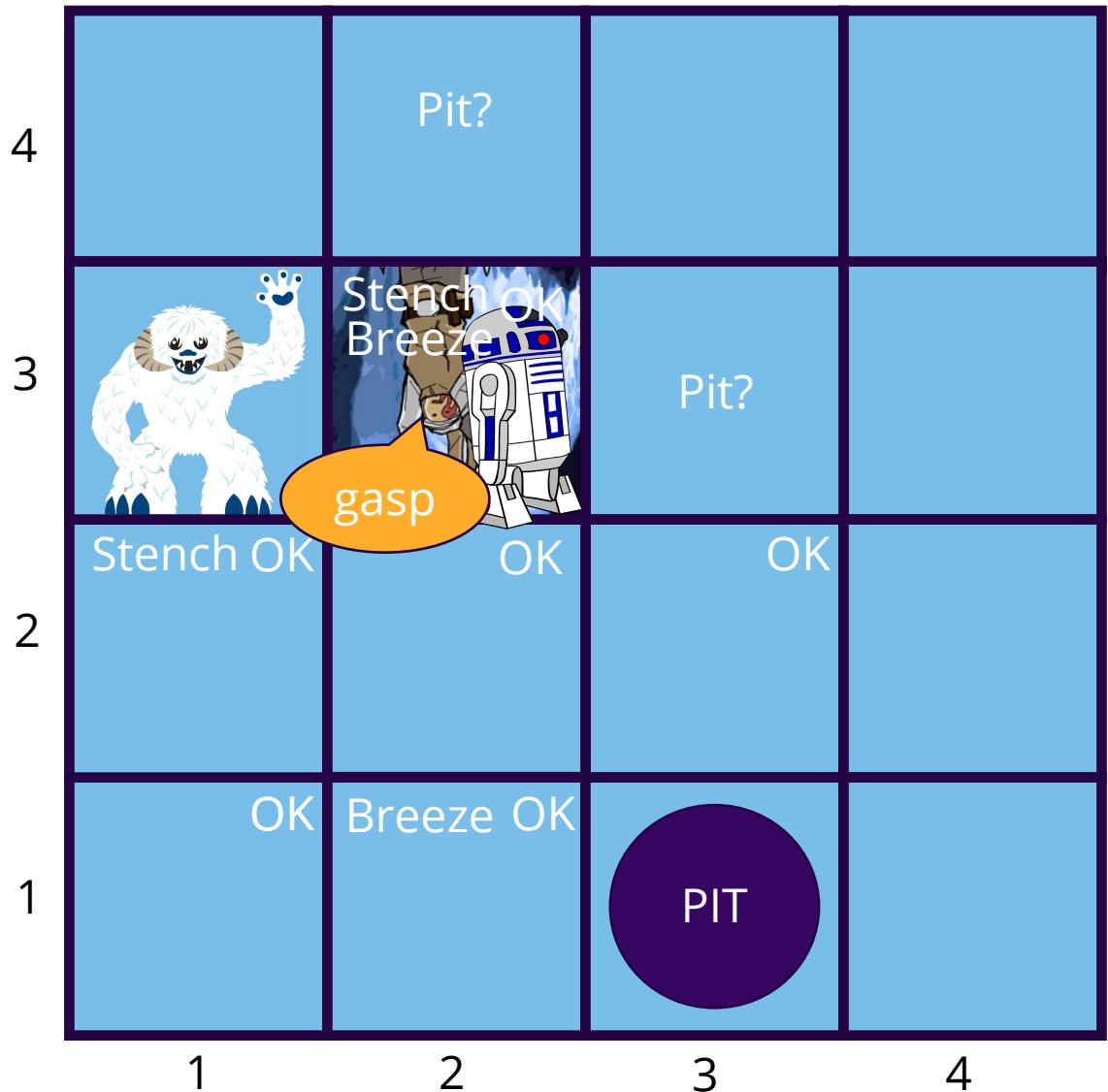


# Wampa World Walkthrough

R2D2 moves to [2,3]

Percept=[*Stench, Breeze, Gasp, None, None*]

We heard a Gasp, so Luke is here!



# Logic

**Logic** can serve as a general class of **representations** for knowledge-based agents. Here we are going to examine **Propositional Logic**.

- KB consists of **sentences** in the **representation language**
- Representation language has a **syntax** that specifies sentences that are well-formed
- A logic defines the **semantics** of the sentences, which is their **meaning**
- The semantics defines the **truth** of each sentence with respect to a **possible world**, which we will often call a **model**.

# Possible worlds and models

- Models are mathematical abstractions that have a fixed set of **truth values** which are **{true, false}** for each sentence.
- If sentence  $\alpha$  is true in model  $m$  then we say
  - $m$  satisfies  $\alpha$ , or
  - $m$  is a model of  $\alpha$
- We use the notation  $M(\alpha)$  to mean the **set of all models** of  $\alpha$ .

For instance,  $\alpha$  could be a sentence that means “there is no pit in [2,2]”. In that case,  $M(\alpha)$  would be all instances of Wampa World where [2,2] doesn’t have a pit.

# Logical Entailment

Once we have a notion of truth, we can start to define **logical reasoning**. Logical reasoning involves the **entailment** relation between sentence.

In plain English, entailment is the idea that a sentence **follows logically** from another sentence.

To write sentence  $\alpha$  entails sentence  $\beta$  in mathematical notation we use the  **$\models$  symbol**:

$$\alpha \models \beta$$

The definition is

$$\alpha \models \beta \text{ if and only if } M(\alpha) \subseteq M(\beta)$$

This means that  $\alpha$  is more specific, or stronger than,  $\beta$ . For instances,  $\beta$  could mean that “The agent is a robot” and  $\alpha$  could mean “The agent is an astromech”.

# Knowledge Base

The KB can be thought of as a set of sentences.

$\alpha_1$  = "There is no pit in [1,2]"

$\alpha_2$  = "There is a pit in [3,1]"

$\alpha_3$  = "There is a wampa in [1,3]"

**The KB is false** in models that contradict what the agent knows. For example, the KB is false in any model  $m$  where [1,2] contains a pit.

**Possible Worlds** is the process of enumerating all Possible Worlds that are compatible with the KB.  $M(KB) \subseteq M(\alpha_1)$

		Pit?	
4			
3	Stench OK Visited	Stench OK Visited	Pit?
2	Stench OK Visited	OK Visited	OK
1	OK Visited	OK Visited	PIT
	1	2	3

# Logical inference

Entailment can be applied to derive conclusions, which is the process of **logical inference**.

We can think about the consequences of a KB as a large set of additional sentences that are entailed given the sentences that have been added to the KB.

We would like to design **inference algorithms** to enumerate these sentences.

When an inference algorithm  $i$  allows us to conclude that  $\alpha$  is true, then we write

$$\mathbf{KB} \vdash_i \alpha$$

“ $\alpha$  is derived from  $\mathbf{KB}$  by  $i$ ”

# Propositional Logic

# Propositional Logic

**Atomic sentences** are represented with a single **propositional symbols**.

Propositional symbols **stand for a statement** that can be true or false.

For example, **W<sub>1,3</sub>** is a propositional symbol that we choose to stand for

“There is a Wampa at location [1,3]”

That statement can be true or false.

The symbol **FacingEast** could stand for “The agent is currently facing East”.

# Propositional Logic

**Complex sentences** are constructed from simpler ones using parentheses and **logical connectives**.

Logical Connective	Meaning
$\neg$ (not)	$\neg W_{1,3}$ is the negation of $W_{1,3}$
$\wedge$ (and)	$W_{1,3} \wedge P_{3,1}$ is called a conjunction
$\vee$ (or)	$W_{1,3} \vee P_{3,1}$ is called a disjunction
$\Rightarrow$ (implies)	$W_{1,3} \Rightarrow S_{1,2}$ is called an implication. $W_{1,3}$ is its <b>premise or antecedent</b> and $S_{1,2}$ is its <b>conclusion or consequence</b>
$\Leftrightarrow$ (if and only if)	$W_{1,3} \Leftrightarrow \neg W_{3,4}$ is called an biconditional

# Truth Tables

## Negation

P	$\neg P$
True	False
False	True

**"It is not the case that** the Death Star is a moon" is **true** because "the Death Star is a moon" is **false**.

**"It is not the case that Wampas** smell bad" is **false** because "Wampas smell bad" is **true**.

# Truth Tables

## Conjunction

P	Q	$P \wedge Q$
True	True	True
True	False	False
False	True	False
False	False	False

P and Q are both true:

"Wampas smell bad **and** Tauntauns smell bad."      (This sentence is true)

P is true and Q is false:

"Wampas smell bad **and** Tauntauns are robots."      (This sentence is false)

P is false and Q is false:

"Wampas smell good **and** Tauntauns are robots."      (This sentence is false)

# Truth Tables

## Disjunction

P	Q	$P \vee Q$
True	True	True
True	False	True
False	True	True
False	False	False

P and Q are true:

"Wampas smell bad **or** Tauntauns smell bad." (This sentence is true)

P is true and Q is false:

"Wampas smell bad **or** Tauntauns are robots." (This sentence is true)

P is false and Q is false:

"Wampas smell good **or** Tauntauns are robots." (This sentence is false)

# Truth Tables

## Conditional

P	Q	$P \Rightarrow Q$
True	True	True
True	False	False
False	True	True
False	False	True

Which of these scenarios did break my promise to you?

To understand why the conditional is defined this way assume that I tell you this promise  $P \Rightarrow Q$ :  
**If you join the dark side then we will rule the galaxy together.**

In which of these four scenario did I tell a lie?

1. **You join the dark side, and we rule the galaxy together.** (Both P and Q are True)
2. You join the dark side, **but** we **don't** rule the galaxy together. (P is True, Q is False)
3. You **don't** join the dark side, **but** we **still** rule the galaxy together. (P is False, Q is True)
4. You **don't** join the dark side, **and** we **don't** rule the galaxy together. (P is False, Q is True)

This one.

# Truth Tables

Shorthand for  
 $P \Rightarrow Q \wedge Q \Rightarrow P$

Biconditional

P	Q	$P \Leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

I tell you: The Death Star can be destroyed **if and only if** your missile hits its vulnerable spot.

1. **The Death Star is destroyed, and you hit the vulnerable spot.** (Both P and Q are True)
2. The Death Star **is** destroyed, **but** you **didn't** hit its vulnerable spot. (P is True, Q is False)
3. The Death Star **isn't** destroyed, **but** you **did** hit its vulnerable spot. (P is False, Q is True)
4. The Death Star **isn't** destroyed, **and** you **also didn't** hit its vulnerable spot. (P is False, Q is False)

# Propositional Logic

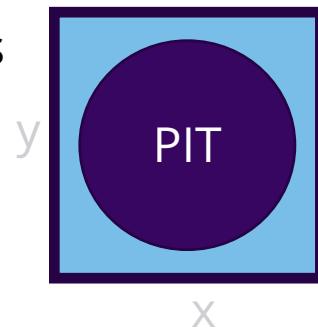
- Atomic sentences and logical connectives define the **syntax** of propositional logic and specifies which sentences are well-formed
- Truth tables define the **semantics** of the sentences, which is their **meaning**
- The semantics defines the **truth** of each sentence with respect to a **possible world**, which we will often call a **model**.
- Our agent can maintain a **knowledge base** which contains a set sentences whose truth value is known.

# Knowledge Bases

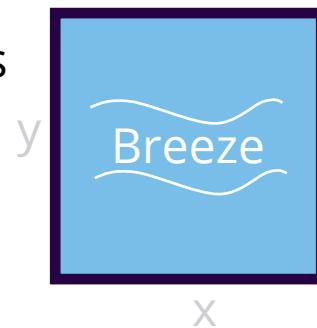
# A Simple Knowledge Base

We can construct a knowledge base for Wampa World.  
Let's start with a set of symbols for each location  $[x,y]$ :

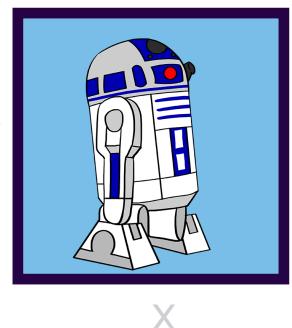
$P_{x,y}$  is true if there is  
a pit in  $[x,y]$



$B_{x,y}$  is true if there is  
a breeze in  $[x,y]$



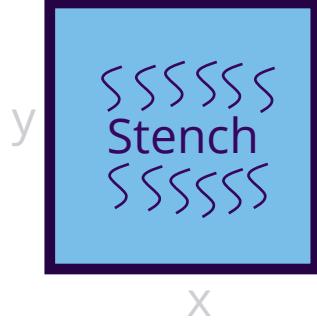
$L_{x,y}$  is true  
if the agent  
is in  $[x,y]$



$W_{x,y}$  is true if there is  
a Wampa in  $[x,y]$



$S_{x,y}$  is true if there is  
a stench in  $[x,y]$



## Logical Connectives

$\neg$  (not)

$\wedge$  (and)

$\vee$  (or)

$\Rightarrow$  (implies)

$\Leftrightarrow$  (if and only if)

# A Simple Knowledge Base

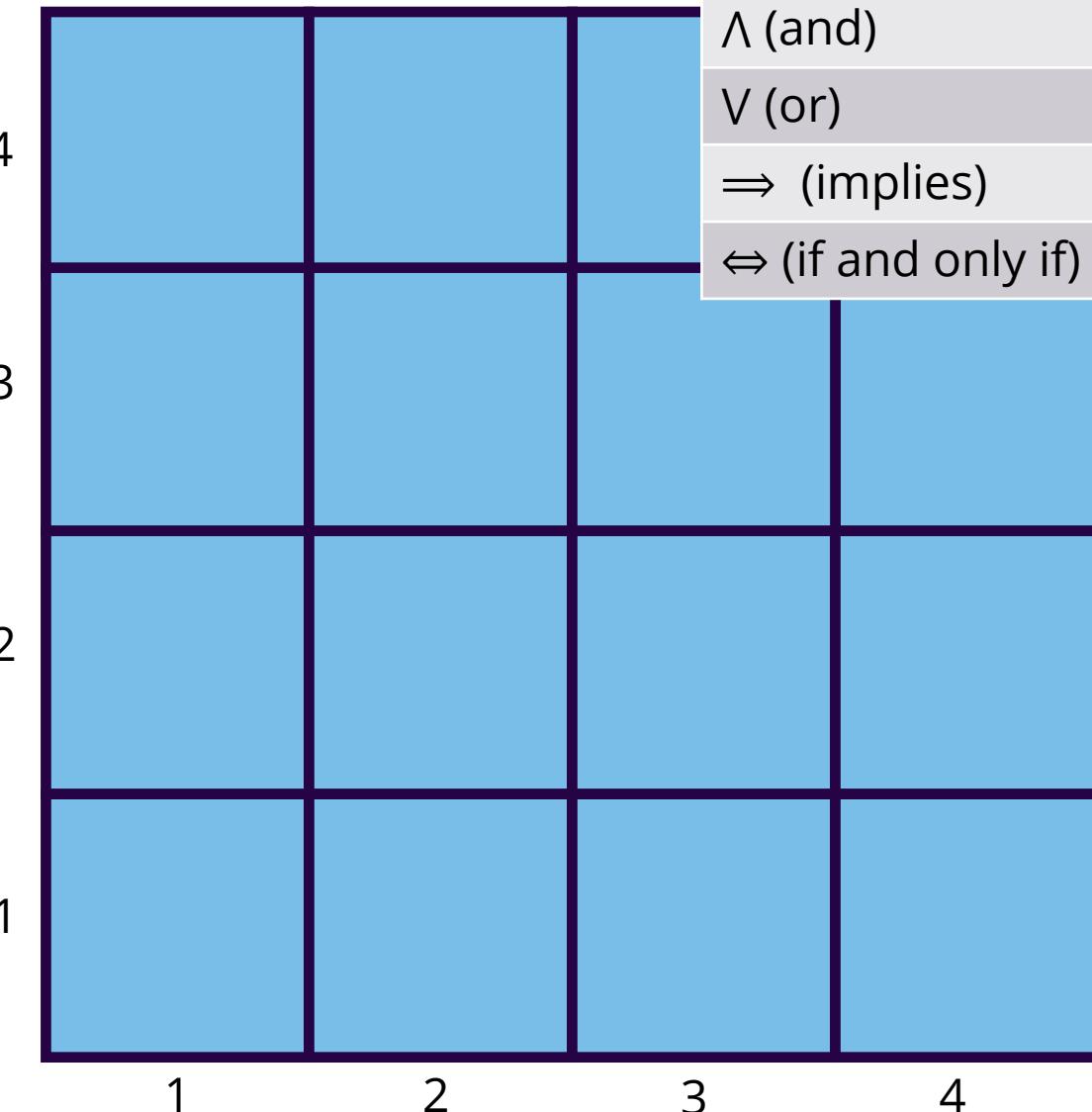
We can construct sentences out of these using our logical connectors. We'll label each sentence.

$$R1: \neg P_{1,1}$$

$$R2: B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$R3: B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

**These are true of all Wampa Worlds.**



$\neg$  (not) $\wedge$  (and) $\vee$  (or) $\Rightarrow$  (implies) $\Leftrightarrow$  (if and only if)

# A Simple Knowledge Base

We can construct sentences out of these using our logical connectors. We'll label each sentence.

$$R1: \neg P_{1,1}$$

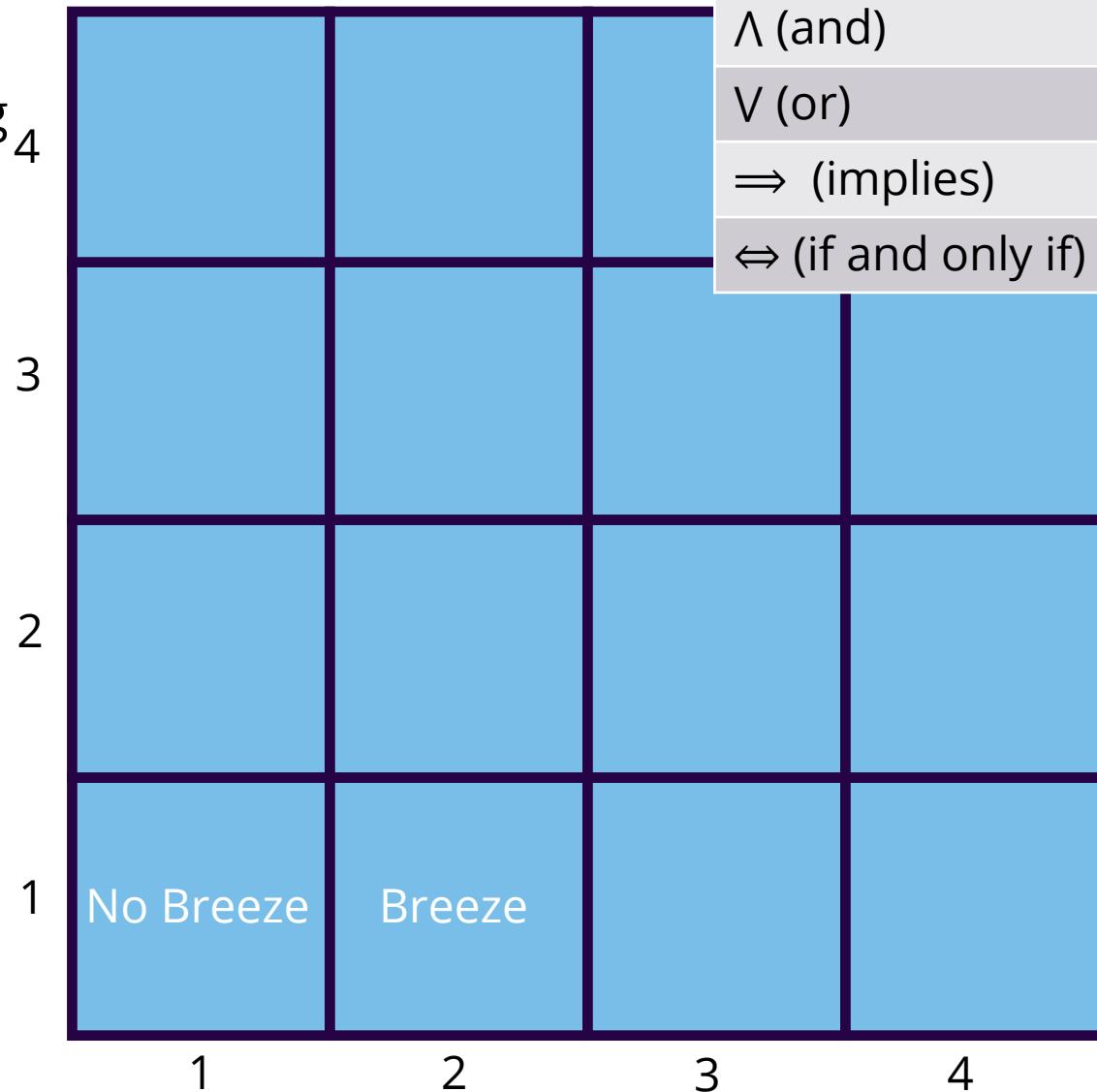
$$R2: B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$R3: B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

What if we perceive the presence or absence of breeze in [1,1], [2,1]?

$$R4: \neg B_{1,1}$$

$$R5: B_{2,1}$$



$\neg$  (not) $\wedge$  (and) $\vee$  (or) $\Rightarrow$  (implies) $\Leftrightarrow$  (if and only if)

# A Simple Knowledge Base

We can construct sentences out of these using our logical connectors. We'll label each sentence.

$$R1: \neg P_{1,1}$$

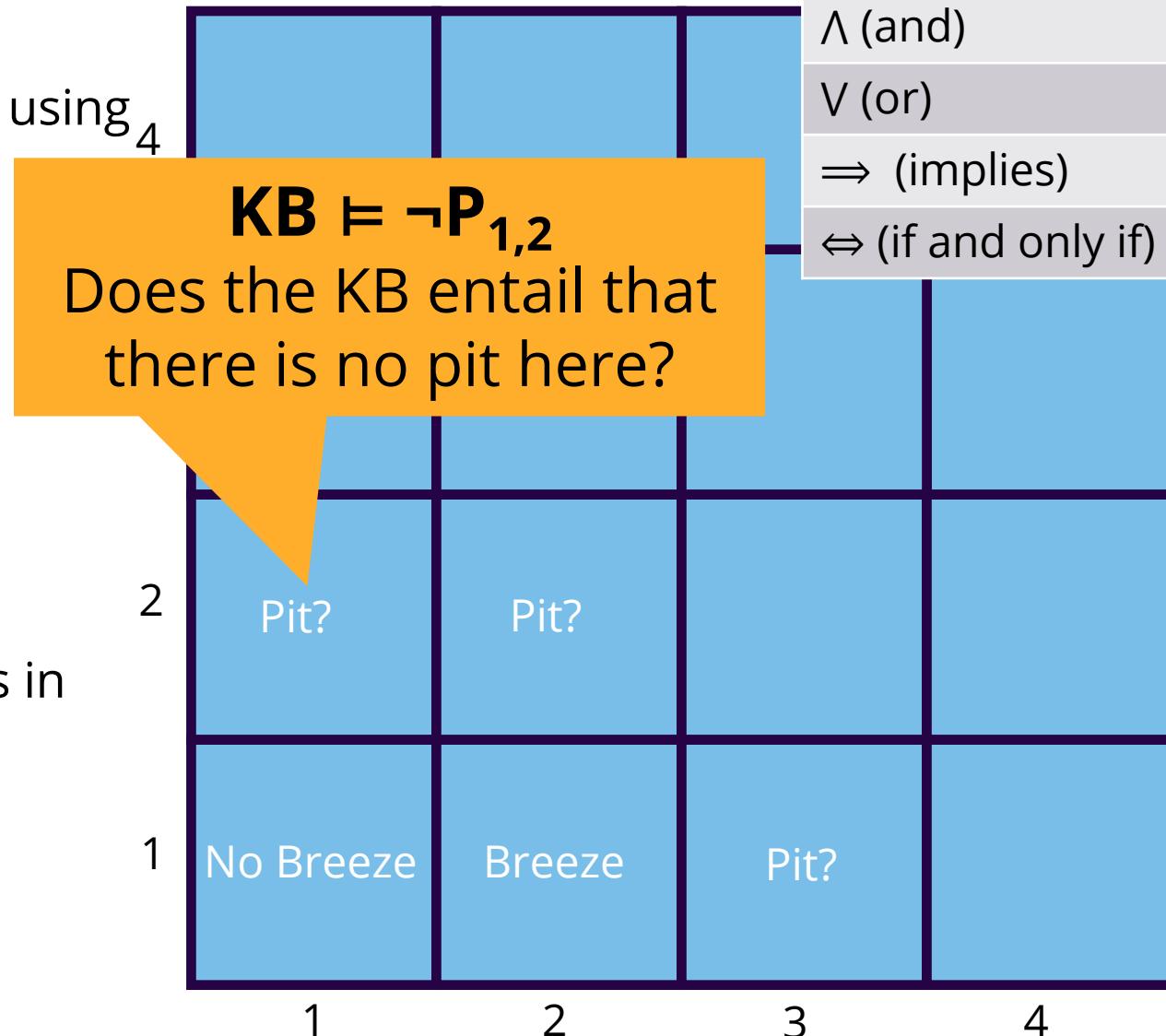
$$R2: B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$$

$$R3: B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$$

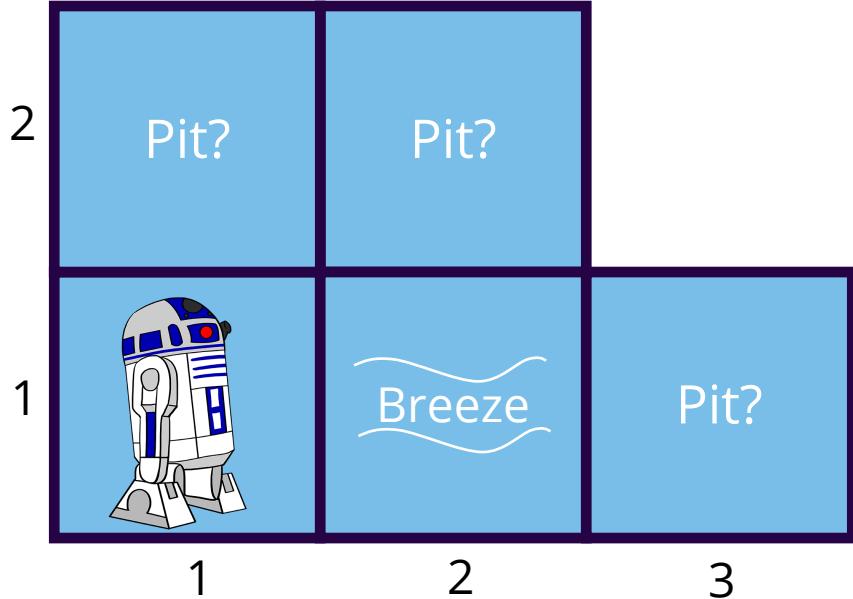
$$R4: \neg B_{1,1}$$

$$R5: B_{2,1}$$

Can mechanically combine the sentences in our KB to prove that a pit exists at (or is absent from) any location?



# Possible Worlds



Models are mathematical abstractions that have a fixed set of **truth values** which are **{true, false}** for each sentence.

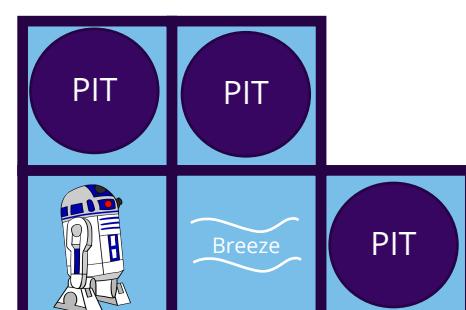
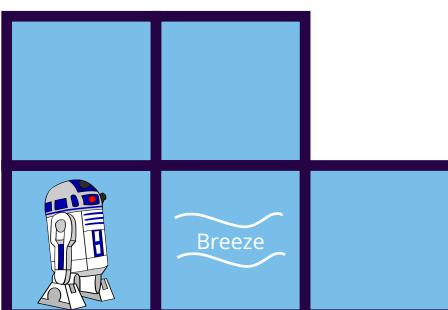
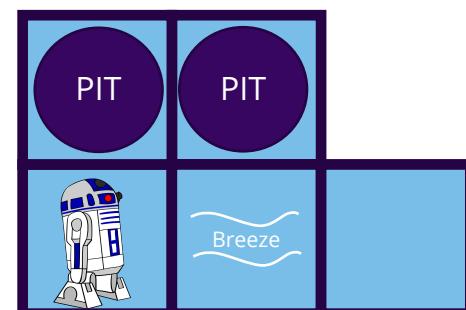
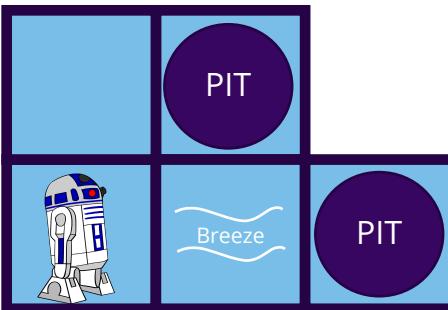
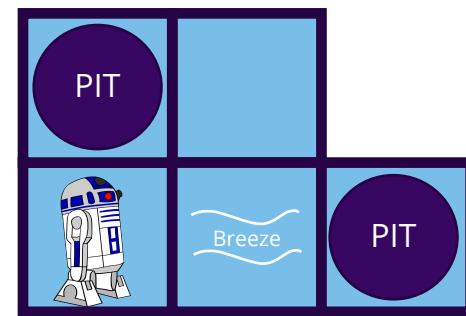
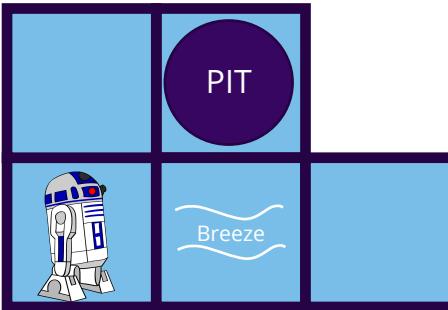
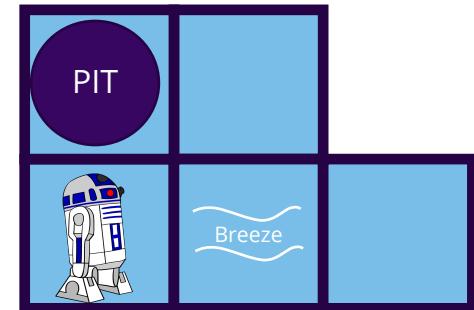
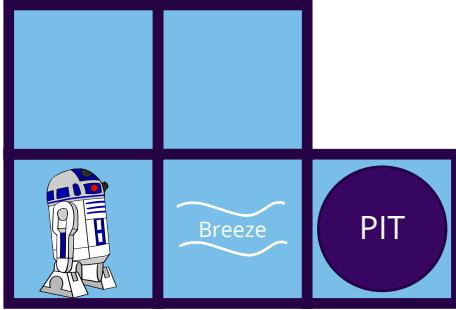
If sentence  $\alpha$  is true in model  $m$  then we say

- $m$  satisfies  $\alpha$ , or
- $m$  is a model of  $\alpha$

We use the notation  $M(\alpha)$  to mean the **set of all models** of  $\alpha$ .

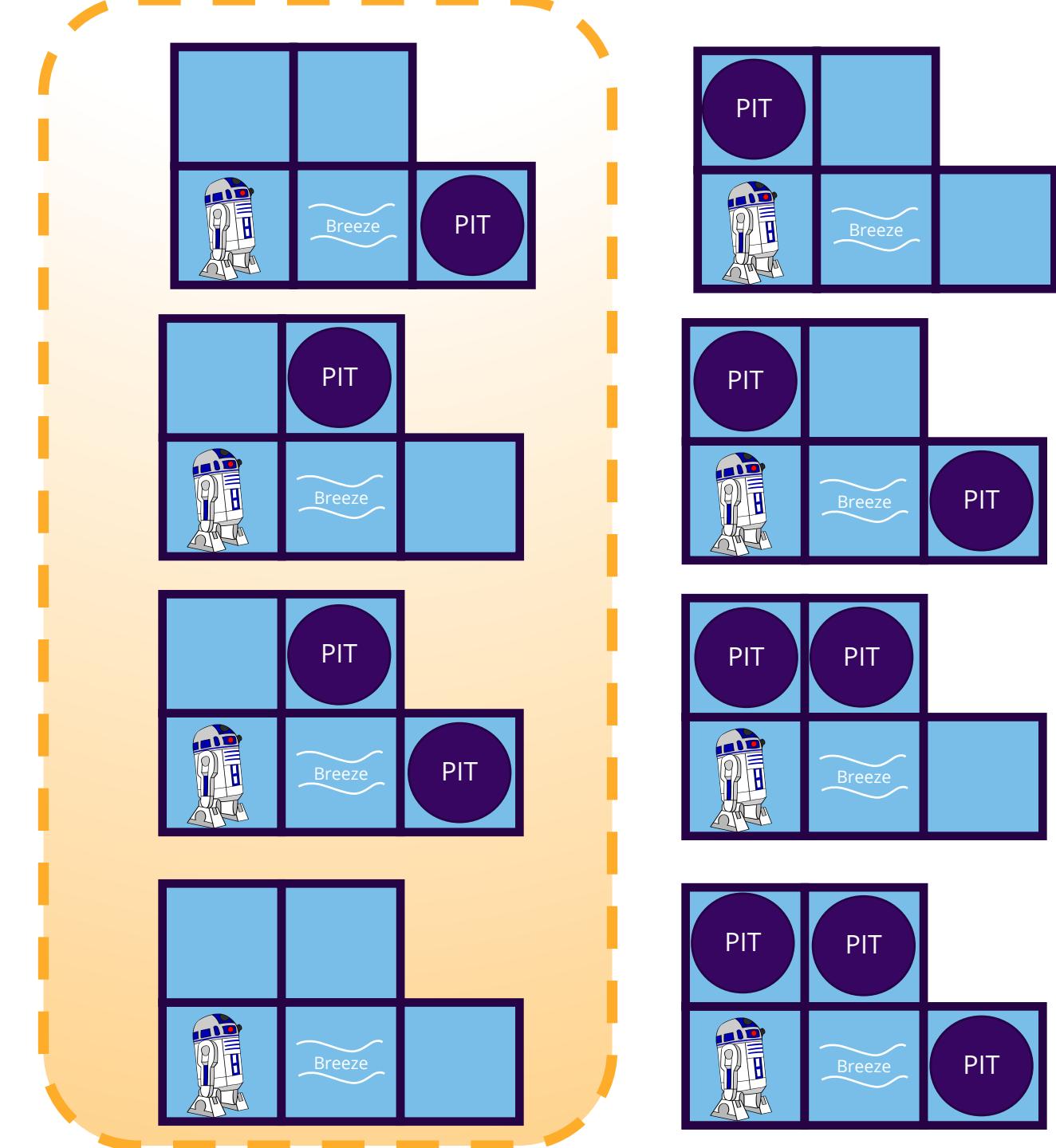
# Possible Worlds

$\alpha_1$  = "There is no pit in [1,2]"



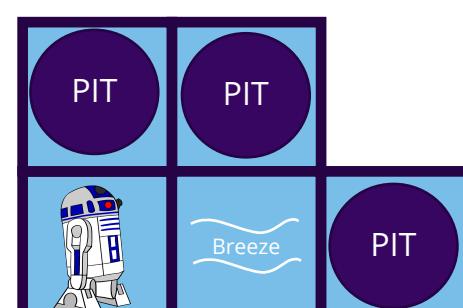
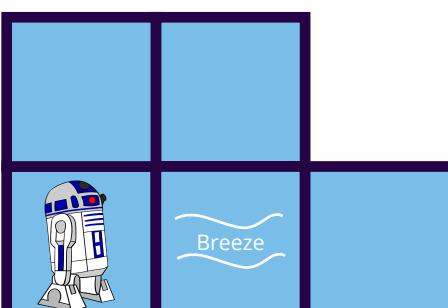
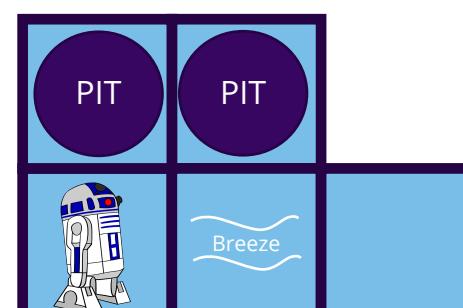
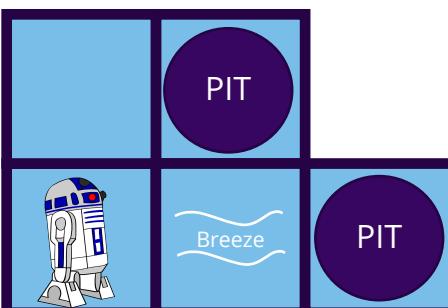
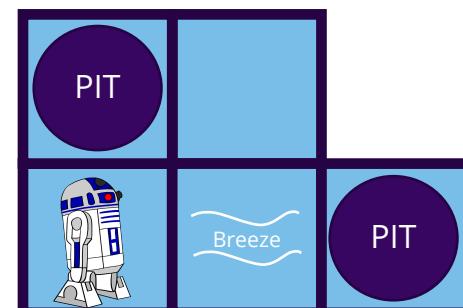
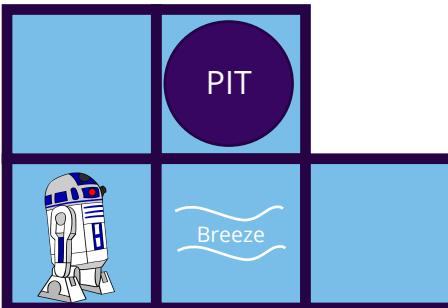
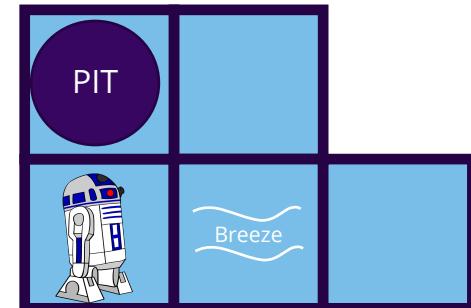
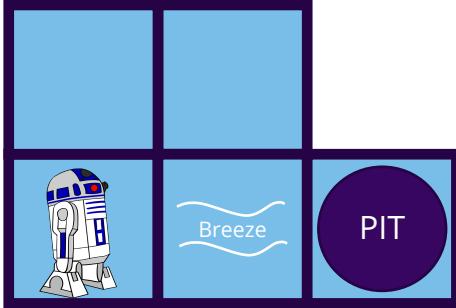
# Possible Worlds

$\alpha_1$  = "There is no pit in [1,2]"



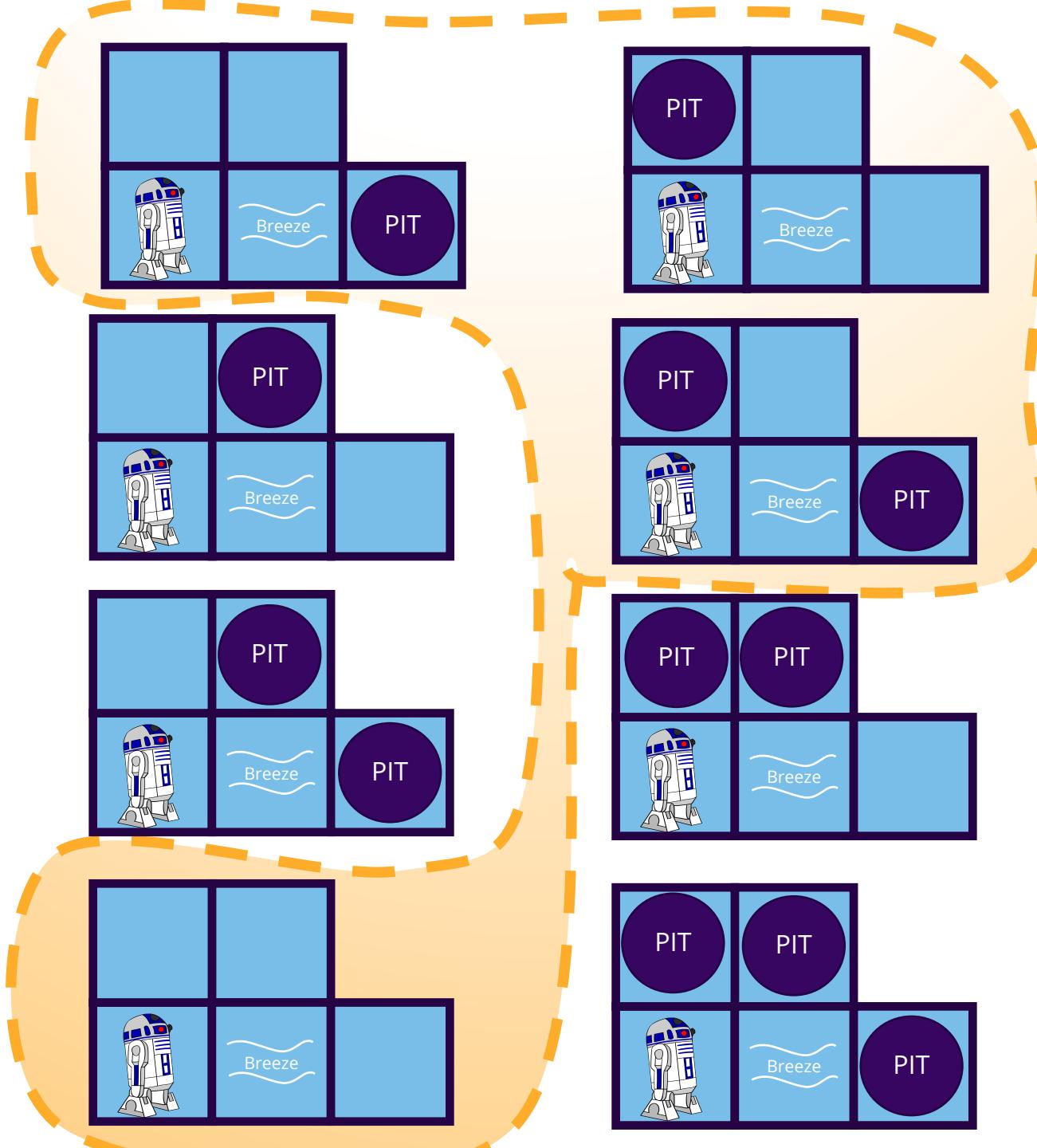
# Possible Worlds

$\alpha_2$  = "There is no pit in [2,2]"



# Possible Worlds

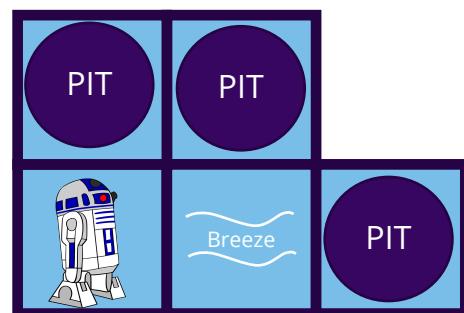
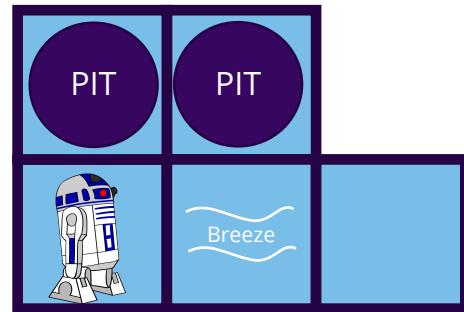
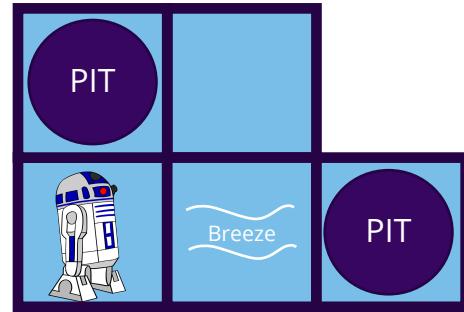
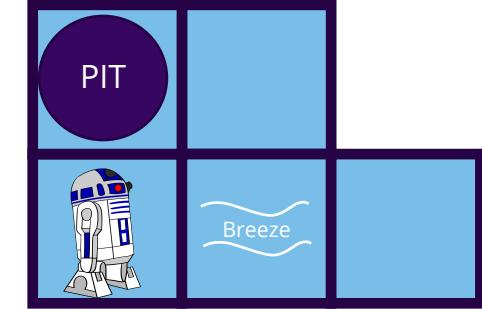
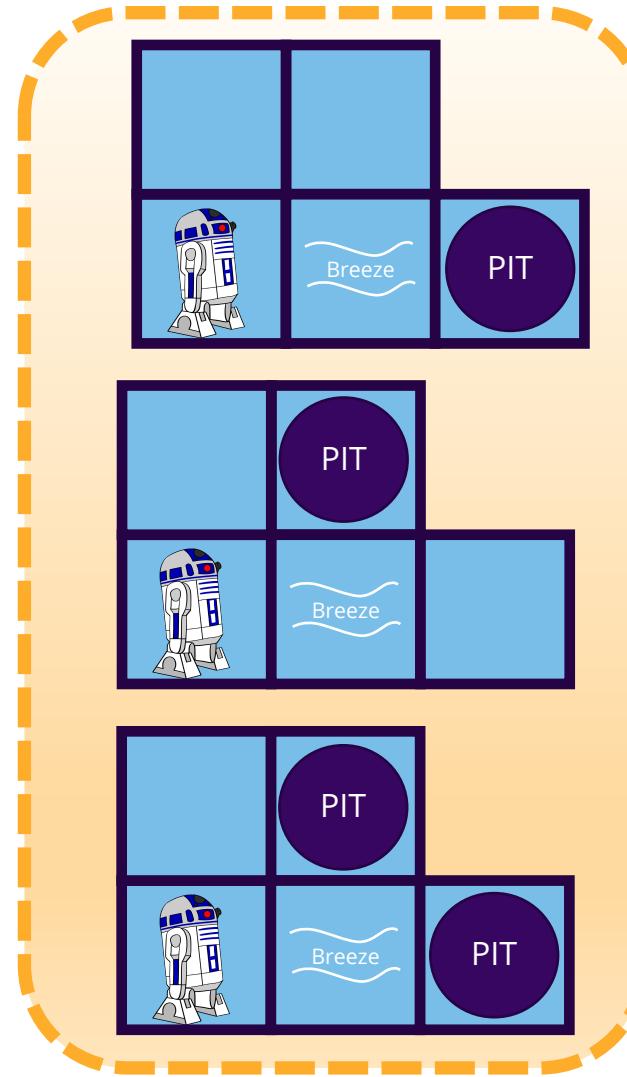
$\alpha_2$  = "There is no pit in [2,2]"



# Possible Worlds

**KB =**

- R1:  $\neg P_{1,1}$
- R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
- R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$
- R4:  $\neg B_{1,1}$
- R5:  $B_{2,1}$

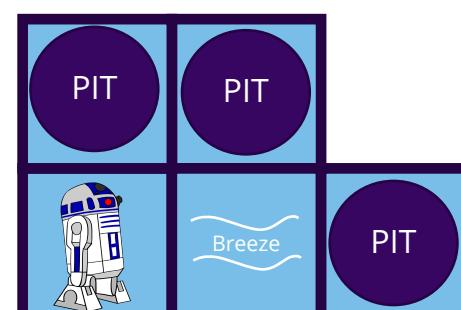
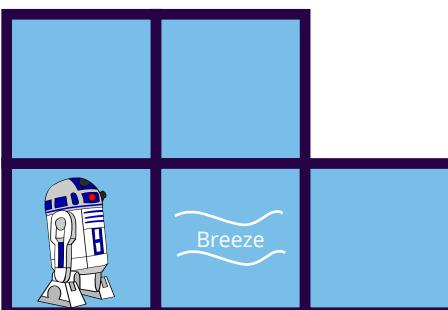
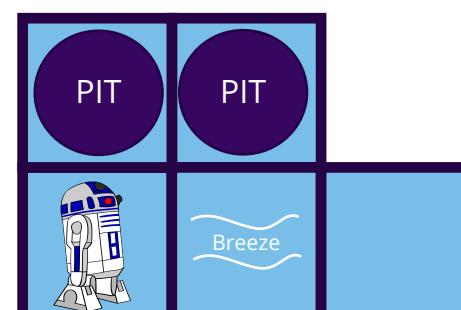
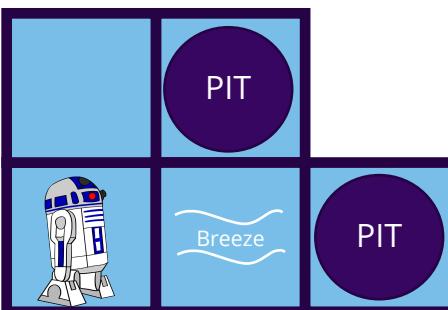
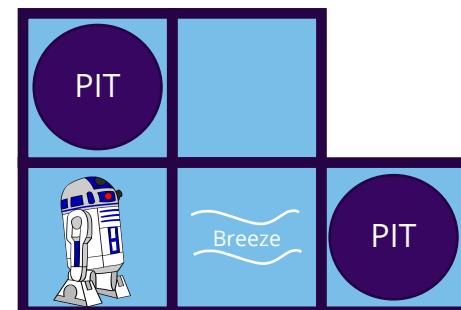
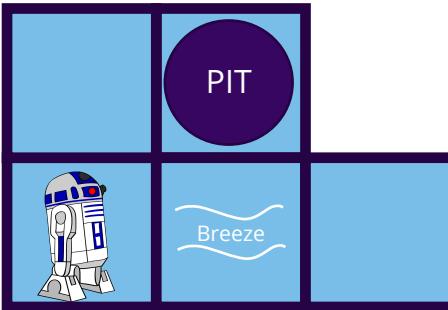
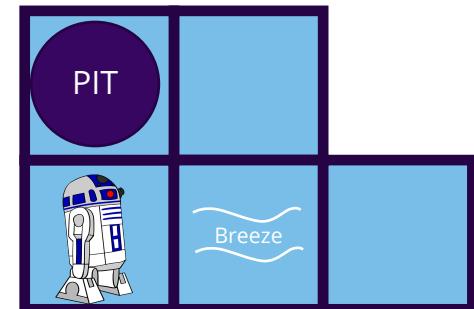
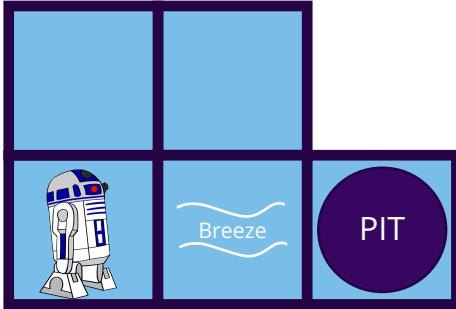


# Possible Worlds

$\beta \models \alpha$  if and only if  $M(\beta) \subseteq M(\alpha)$

" $\beta$  entails  $\alpha$  if and only if every model in which  $\beta$  is true,  $\alpha$  is also true"

Does our KB entail that there is no pit in [1,2]?  
 $KB \models \alpha_1$  if and only if  $M(KB) \subseteq M(\alpha_1)$



# Possible Worlds

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" $\beta$  entails  $\alpha$  if and only if every model in which  $\beta$  is true,  $\alpha$  is also true"

Does our KB entail that there is no pit in [1,2]?

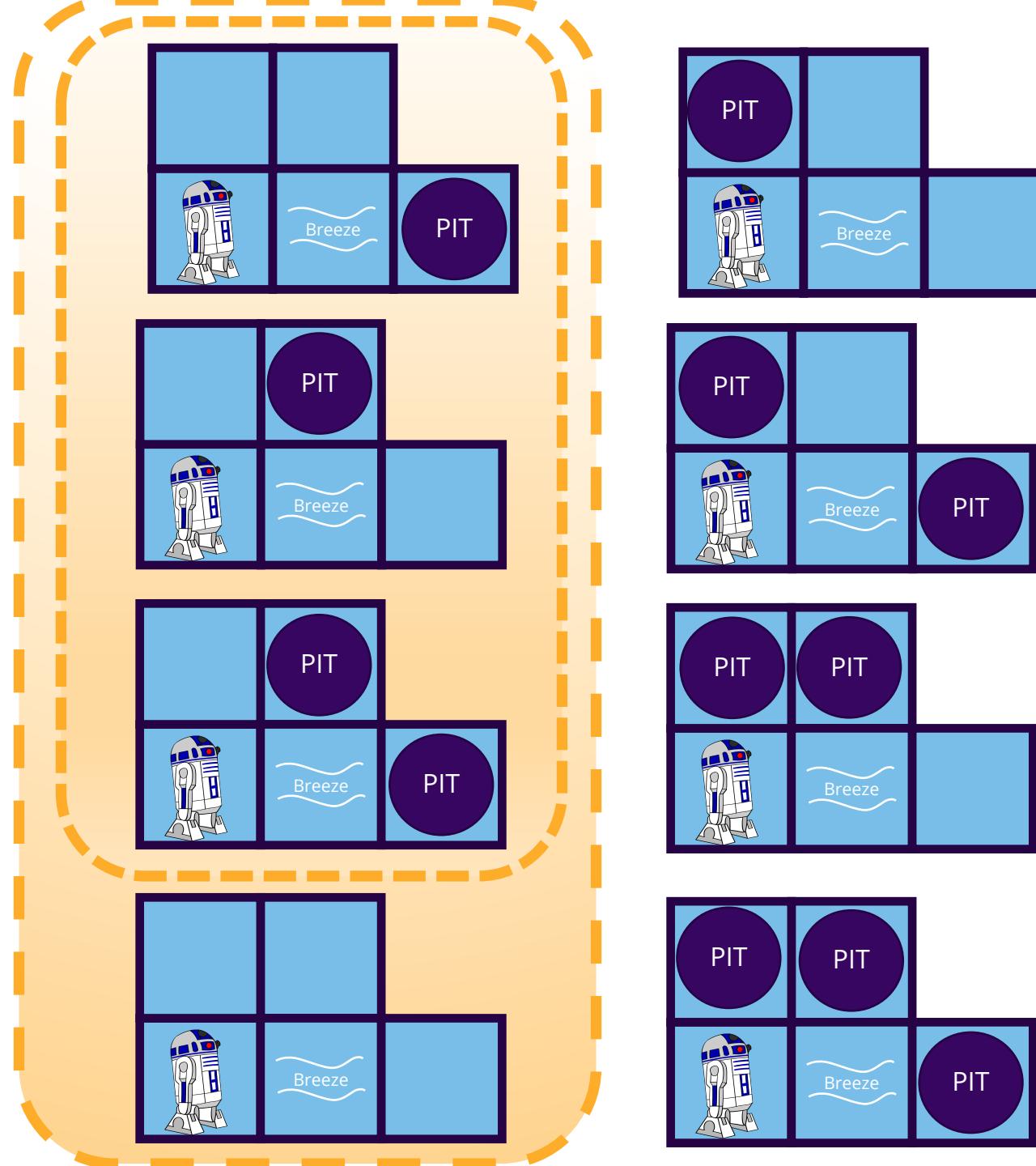
$KB \models \alpha_1$  if and only if  $M(KB) \subseteq M(\alpha_1)$

$KB =$

- R1:  $\neg P_{1,1}$
- R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
- R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$
- R4:  $\neg B_{1,1}$
- R5:  $B_{2,1}$

$\alpha_1$  = "There is no pit in [1,2]"

**$KB \models \alpha_1$**



# Possible Worlds

$\beta \models \alpha$  if and only if  $M(\beta) \subseteq M(\alpha)$

" $\beta$  entails  $\alpha$  if and only if every model in which  $\beta$  is true,  $\alpha$  is also true"

Does our KB entail that there is  
**no pit in [2,2]**?

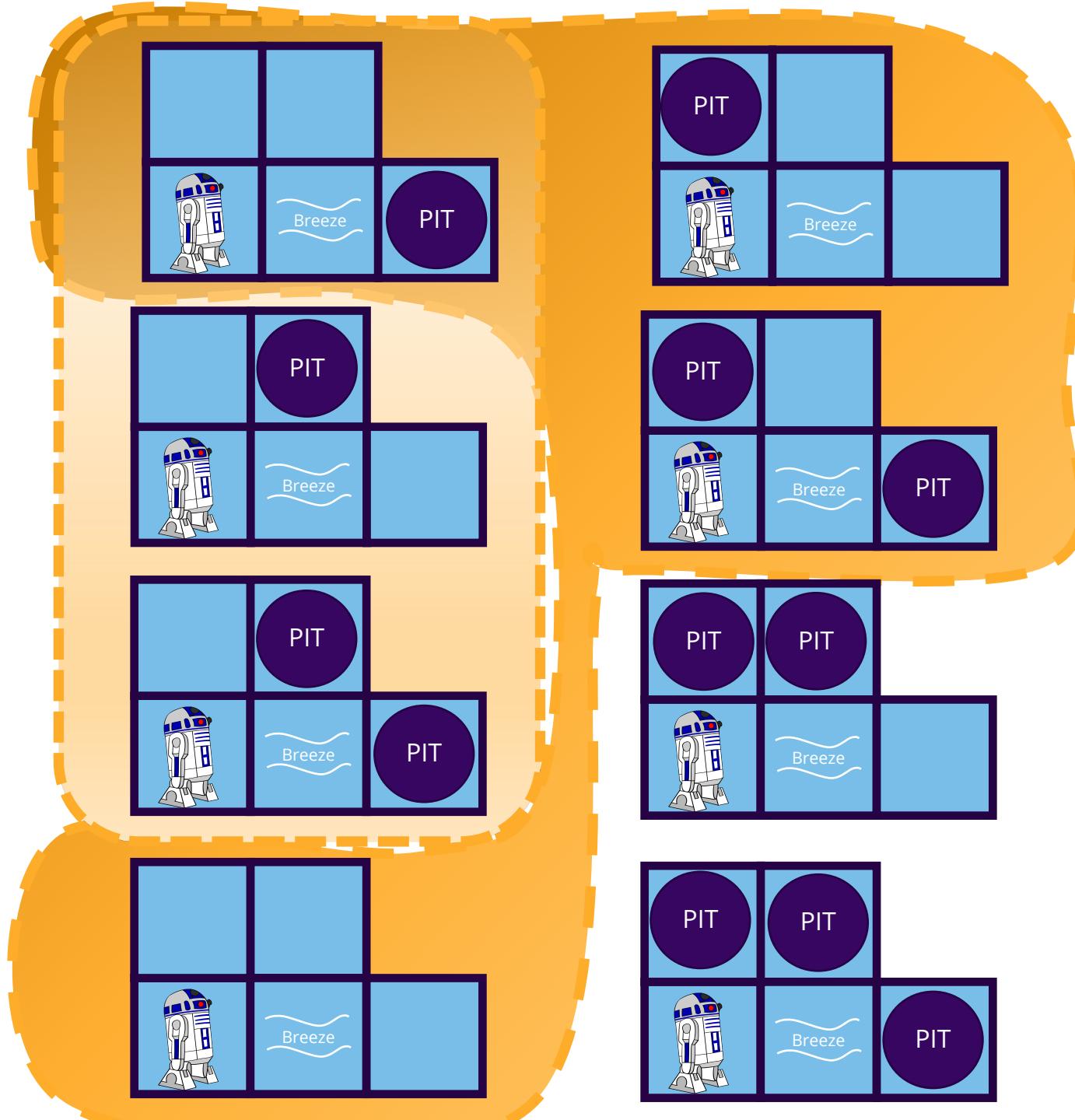
$KB \models \alpha_2$  if and only if  $M(KB) \subseteq M(\alpha_2)$

$KB =$

- R1:  $\neg P_{1,1}$
- R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$
- R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$
- R4:  $\neg B_{1,1}$
- R5:  $B_{2,1}$

$\alpha_2$  = "There is no pit in [2,2]"

**KB does not entail  $\alpha_2$  in some models where KB is true  $\alpha_2$  is false**



# Entailment and Inference

Entailment can be applied to derive conclusions, which is the process of **logical inference**.

We can think about the consequences of a KB as a large set of additional sentences that are entailed given the sentences that have been added to the KB.

We would like to design **inference algorithms** to enumerate these sentences.

When an inference algorithm  $i$  allows us to conclude that  $\alpha$  is true, then we write

$$\mathbf{KB} \vdash_i \alpha$$

“ $\alpha$  is derived from  $\mathbf{KB}$  by  $i$ ”

# Soundness and Completeness

- **Sound** – the inference algorithm should **only** derive entailed sentences.
- **Complete** – an inference algorithm is complete if it can derive **all** sentences that are entailed

# Theorem Proving

# Review: Propositional Logic

**Complex sentences** are constructed from simpler ones using parentheses and **logical connectives**.

Logical Connective	Meaning
$\neg$ (not)	$\neg W_{1,3}$ is the negation of $W_{1,3}$
$\wedge$ (and)	$W_{1,3} \wedge P_{3,1}$ is called a conjunction
$\vee$ (or)	$W_{1,3} \vee P_{3,1}$ is called a disjunction
$\Rightarrow$ (implies)	$W_{1,3} \Rightarrow S_{1,2}$ is called an implication. $W_{1,3}$ is its <b>premise or antecedent</b> and $S_{1,2}$ is its <b>conclusion or consequence</b>
$\Leftrightarrow$ (if and only if)	$W_{1,3} \Leftrightarrow \neg W_{3,4}$ is called an biconditional

# Logical Equivalence

Two sentences  $\alpha$  and  $\beta$  are **logically equivalent** if they are true in the same set of models. We write this as

$$\alpha \equiv \beta$$

An alternate definition of logical equivalence is that two sentences  $\alpha$  and  $\beta$  are logically equivalent if and only if they entail each other.

$$\alpha \equiv \beta \text{ if and only if } \alpha \vDash \beta \text{ and } \beta \vDash \alpha.$$

# Logical Equivalence

$(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$	Commutativity of $\wedge$
$(\alpha \vee \beta) \equiv (\beta \vee \alpha)$	Commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma) \equiv (\alpha \wedge (\beta \wedge \gamma))$	Associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma) \equiv (\alpha \vee (\beta \vee \gamma))$	Associativity of $\vee$
$\neg(\neg \alpha) \equiv \alpha$	Double-negation elimination
$(\alpha \Rightarrow \beta) \equiv (\neg \beta \Rightarrow \neg \alpha)$	Contraposition
$(\alpha \Rightarrow \beta) \equiv (\neg \alpha \vee \beta)$	Implication elimination
$(\alpha \Leftrightarrow \beta) \equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$	Biconditional elimination
$\neg(\alpha \wedge \beta) \equiv (\neg \alpha \vee \neg \beta)$	De Morgan
$\neg(\alpha \vee \beta) \equiv (\neg \alpha \wedge \neg \beta)$	De Morgan
$((\alpha \wedge \beta) \vee \gamma) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	Distributivity of $\wedge$ over $\vee$
$((\alpha \vee \beta) \wedge \gamma) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	Distributivity of $\vee$ over $\wedge$

# Inference Rules

**Inference Rules** can be used to derive proofs. Here's the notation:

Whenever sentences of the form  $\alpha \Rightarrow \beta$  and  $\alpha$  are given...

$$\frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

...then the sentence  $\beta$  can be inferred.

# Inference Rules

Modus Ponens:

$$\frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

And Elimination:

$$\frac{\alpha \wedge \beta}{\alpha}$$

Biconditional Elimination:

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}$$

Biconditional Elimination:

$$\frac{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}{\alpha \Leftrightarrow \beta}$$

All of the logical equivalence rules can be re-written as inference rules.

# Inference Example

## Our KB

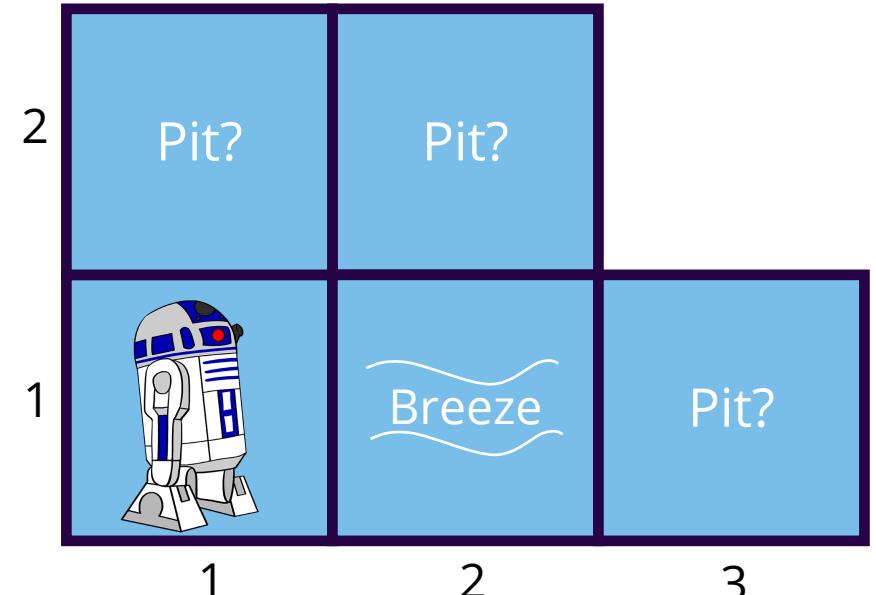
R1:  $\neg P_{1,1}$

R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

R4:  $\neg B_{1,1}$

R5:  $B_{2,1}$



# Inference Example

## Our KB

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

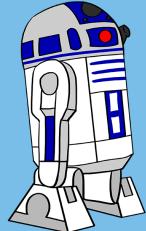
Biconditional Elimination:

$$\frac{\alpha \Leftrightarrow \beta}{(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)}$$

Apply biconditional Elimination to R2 to get R6.

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

**Monotonicity:** if  $\text{KB} \models \alpha$  then  $\text{KB} \wedge \beta \models \alpha$   
We can safely add to the KB, without invalidating anything else that we inferred.

2	Pit?	Pit?
1		Breeze
3		Pit?

# Inference Example

## Our KB

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

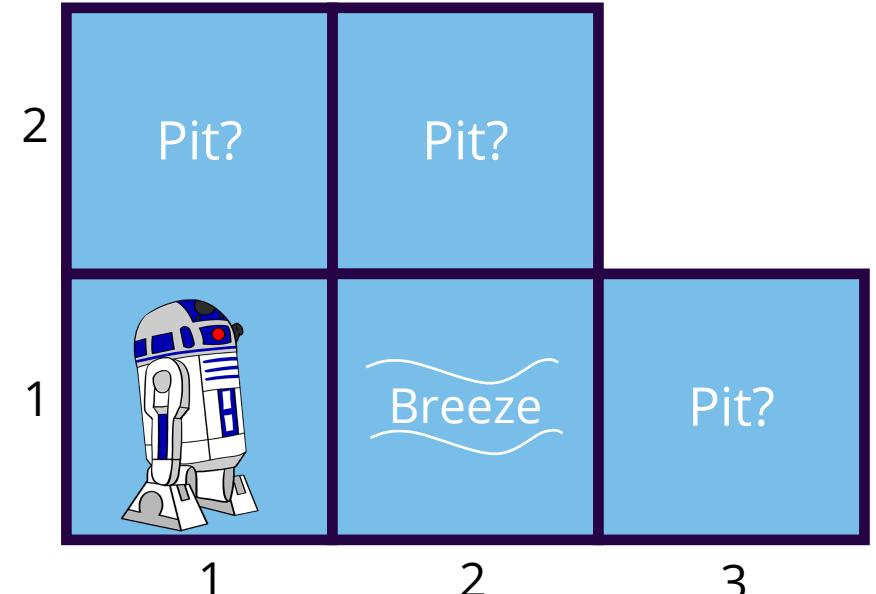
And Elimination:

$$\underline{\alpha \Delta \beta}$$
  
 $\alpha$

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

Apply And-Elimination to R6 to get R7.

R7:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1}))$



# Inference Example

## Our KB

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

Logical equivalence for contrapositives:

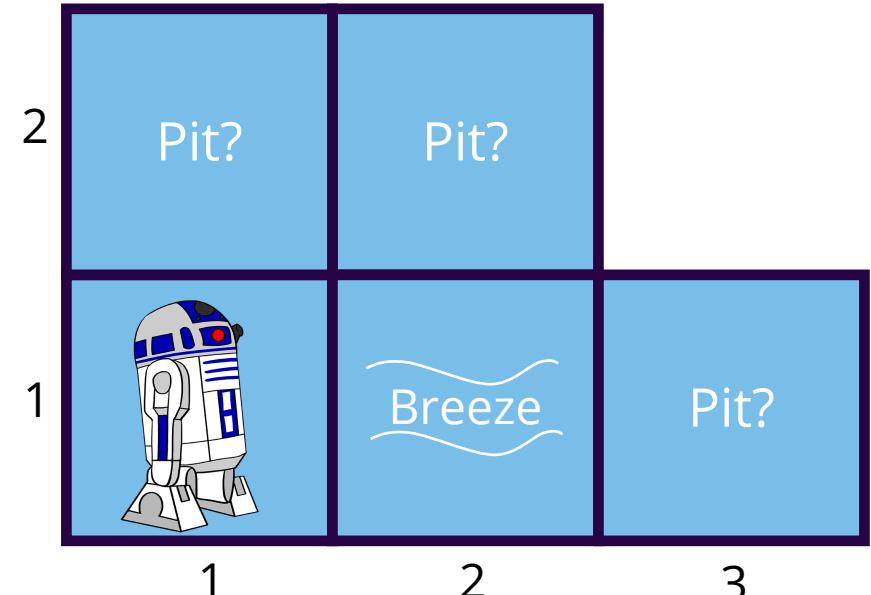
$$\begin{aligned} & (\underline{\alpha \Rightarrow \beta}) \\ & (\neg \beta \Rightarrow \neg \alpha) \end{aligned}$$

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

Logical equivalence for contrapositives applied to R7 gives R8.

R8:  $(\neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1}))$



# Inference Example

## Our KB

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

Modus Ponens:

$$\frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

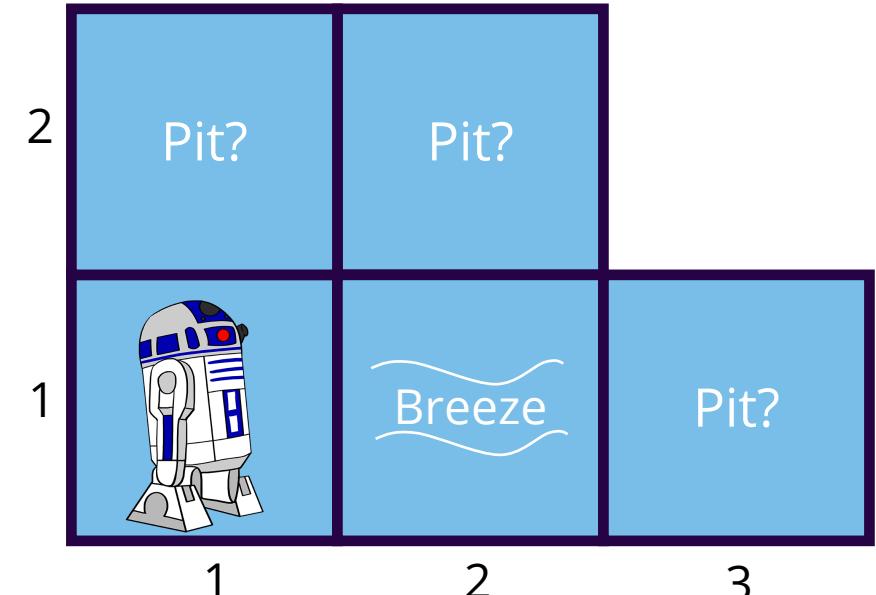
R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R8:  $(\neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1}))$

Apply Modus Ponens to R4 and R8 to get:

R9:  $\neg (P_{1,2} \vee P_{2,1})$



# Inference Example

## Our KB

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

De Morgan's Rule

$$\begin{array}{c} \underline{\neg(\alpha \vee \beta)} \\ (\neg\alpha \wedge \neg\beta) \end{array}$$

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

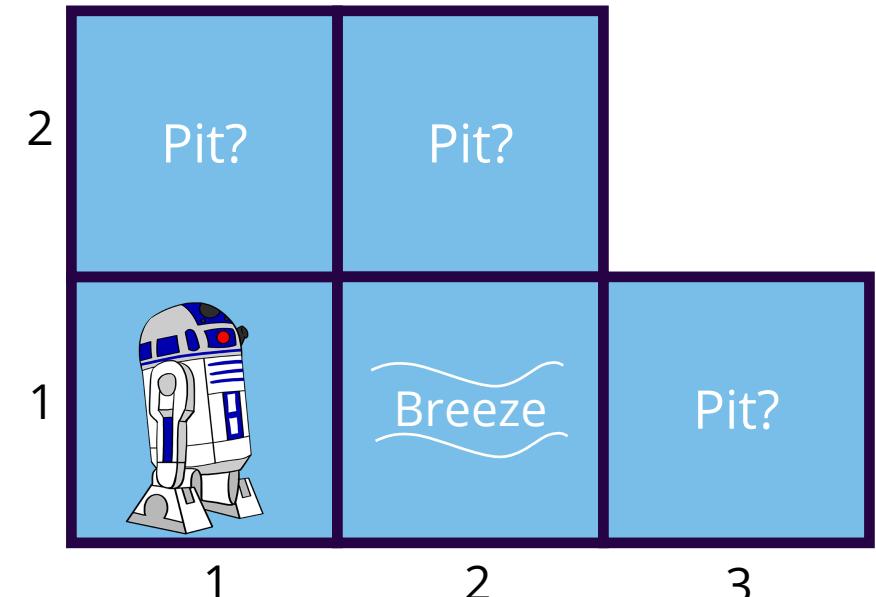
R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R8:  $(\neg B_{1,1} \Rightarrow \neg(P_{1,2} \vee P_{2,1}))$

R9:  $\neg(P_{1,2} \vee P_{2,1})$

Apply De Morgan's Rule to R9:

R10:  $\neg P_{1,2} \wedge \neg P_{2,1}$



# Inference Example

## Our KB

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

And Elimination:

$$\underline{\alpha \Delta \beta}$$
  
 $\alpha$

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

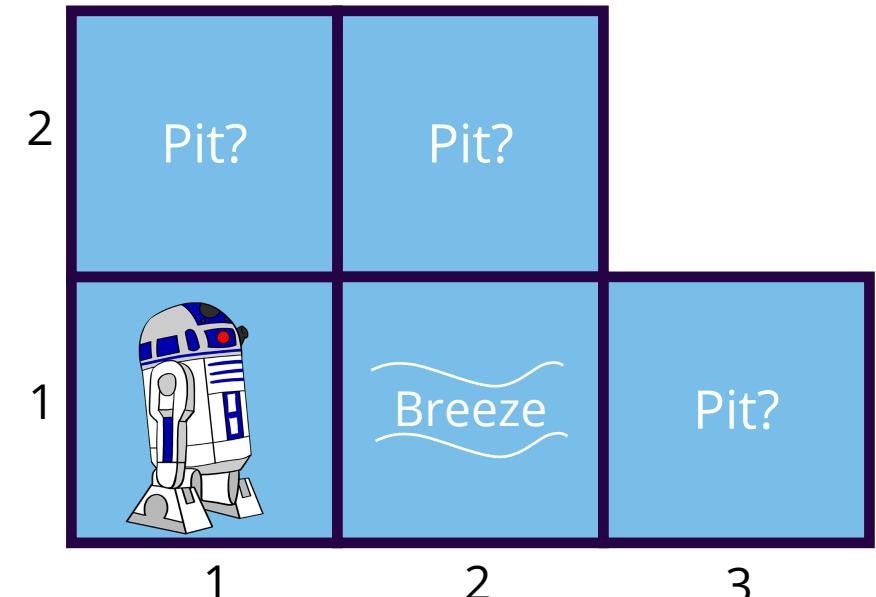
R8:  $(\neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1}))$

R9:  $\neg (P_{1,2} \vee P_{2,1})$

R10:  $\neg P_{1,2} \wedge \neg P_{2,1}$

R11:  $\neg P_{1,2}$

R12:  $\neg P_{2,1}$



# Inference Example

## Our KB

R1:  $\neg P_{1,1}$

R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$

R4:  $\neg B_{1,1}$

R5:  $B_{2,1}$

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R8:  $(\neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1}))$

R9:  $\neg (P_{1,2} \vee P_{2,1})$

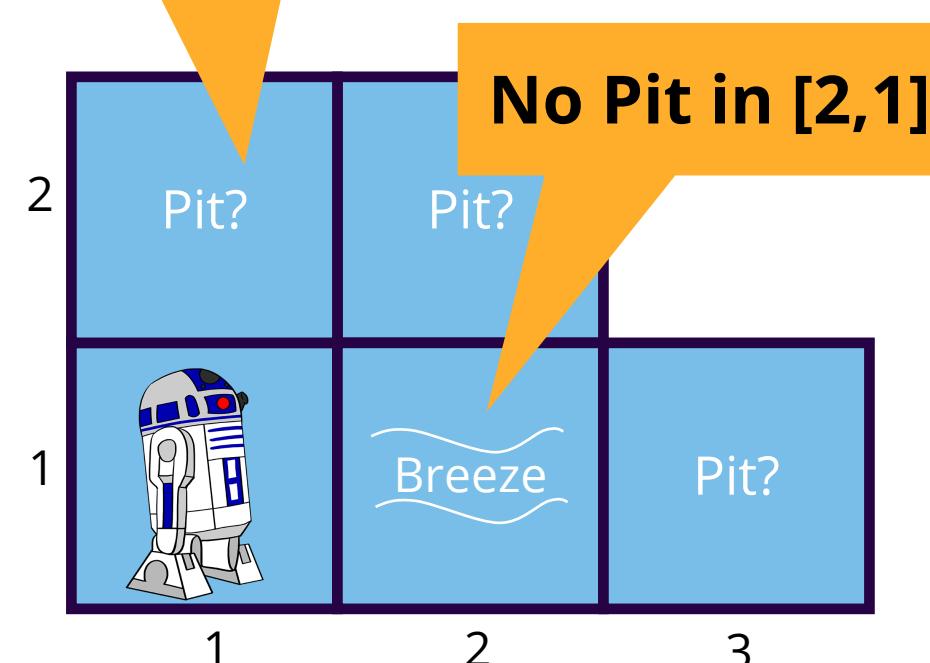
R10:  $\neg P_{1,2} \wedge \neg P_{2,1}$

R11:  $\neg P_{1,2}$

R12:  $\neg P_{2,1}$

No Pit in [1,2].

No Pit in [2,1].



# Search for a Proof

We can use search algorithms to find a sequence of steps that constitutes a proof!

- **Initial State:** the initial knowledge base.
- **Actions:** All inference rules applied to all sentence that match the top half of the inference rule.
- **Result:** Add the sentence on the bottom half to the KB.
- **Goal:** The goal is a state that contains the sentence that we are trying to prove.

## Initial State

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$

## Action

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

---

$(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

## Result

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R6:  $B_{2,1}$   
R7:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

## Goal

$\neg P_{1,2}$

# Monotonicity implies Soundness

Monotonicity says that the set of entailed sentence can only increase as information is assed to the KB.

$$\text{if } \mathbf{KB} \vDash \alpha \text{ then } \mathbf{KB} \wedge \beta \vDash \alpha$$

If we add additional information to the KB, it doesn't invalidate any other info that we've already inferred.

Therefore it is **sound** it will only derive entailed sentences.

# Proof by Resolution

But is it **complete**? Can this method derive all sentences that are entailed? If we used iterative deepening search, then we would find any reachable goal. But what if we were missing an important inference rule?

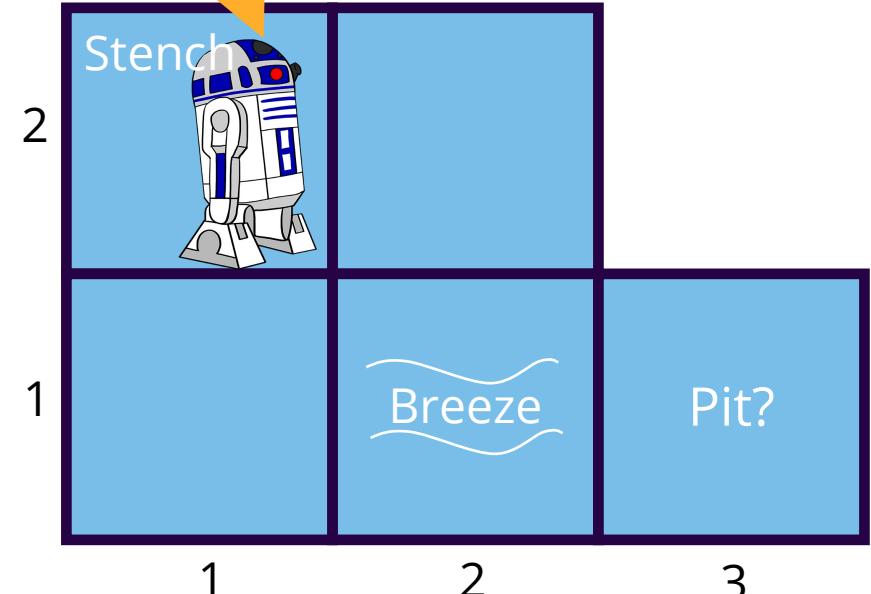
There is a single inference rule, **resolution**, which yields a complete inference algorithm when coupled with any complete search algorithm.

R1:  $\neg P_{1,1}$   
R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
R4:  $\neg B_{1,1}$   
R5:  $B_{2,1}$   
R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$   
R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$   
R8:  $(\neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1}))$   
R9:  $\neg (P_{1,2} \vee P_{2,1})$   
R10:  $\neg P_{1,2} \wedge \neg P_{2,1}$   
R11:  $\neg P_{1,2}$   
R12:  $\neg P_{2,1}$

R13:  $\neg B_{2,1}$   
R14:  $B_{1,2} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{1,3})$   
R15:  $\neg P_{2,2}$   
R16:  $\neg P_{1,3}$   
R17:  $P_{1,1} \vee P_{2,2} \vee P_{3,1}$

## New Rules

**Stench, but no Breeze**



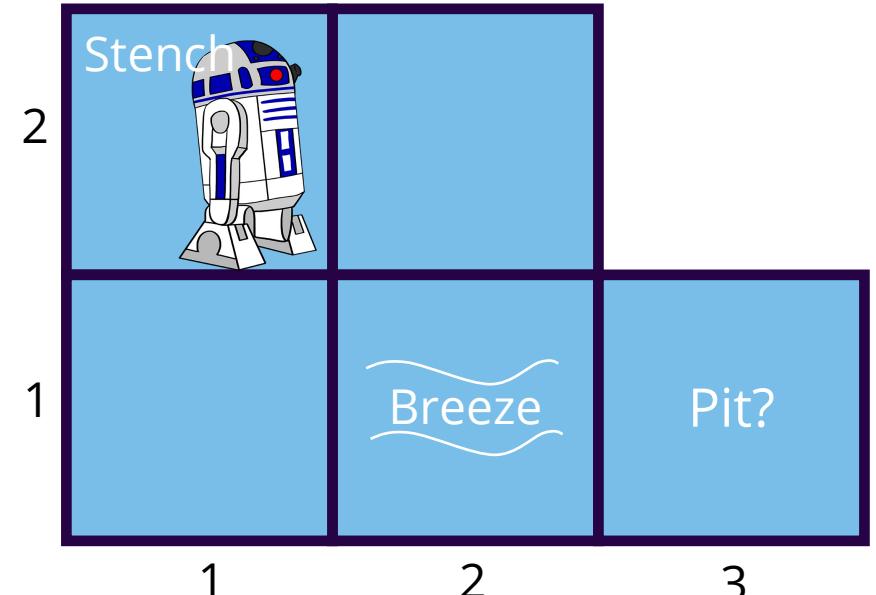
R1:  $\neg P_{1,1}$   
 R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$   
 R3:  $B_{2,1} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})$   
 R4:  $\neg B_{1,1}$   
 R5:  $B_{2,1}$   
 R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$   
 R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$   
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 R9:  $\neg (P_{1,2} \vee P_{2,1})$   
 R10:  $\neg P_{1,2} \wedge \neg P_{2,1}$   
 R11:  $\neg P_{1,2}$   
 R12:  $\neg P_{2,1}$

R13:  $\neg B_{2,1}$   
 R14:  $B_{1,2} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{1,3})$   
 R15:  $\neg P_{2,2}$   
 R16:  $\neg P_{1,3}$   
 R17:  $P_{1,1} \vee P_{2,2} \vee P_{3,1}$

R18:  $(P_{1,1} \vee P_{3,1})$

First application of **Resolution** using R17 and R15:

$$\frac{(P_{1,1} \vee P_{2,2} \vee P_{3,1}) \wedge \neg P_{2,2}}{(P_{1,1} \vee P_{3,1})}$$



R1:  $\neg P_{1,1}$

R2:  $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R3:  $B_{2,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

R4:  $\neg B_{1,1}$

R5:  $B_{2,1}$

R6:  $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R7:  $((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

R8:  $(\neg B_{1,1} \Rightarrow \neg (P_{1,2} \vee P_{2,1}))$

R9:  $\neg (P_{1,2} \vee P_{2,1})$

R10:  $\neg P_{1,2} \wedge \neg P_{2,1}$

R11:  $\neg P_{1,2}$

R12:  $\neg P_{2,1}$

R13:  $\neg B_{2,1}$

R14:  $B_{1,2} \Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{1,3})$

R15:  $\neg P_{2,2}$

R16:  $\neg P_{1,3}$

R17:  $P_{1,1} \vee P_{2,2} \vee P_{3,1}$

R18:  $(P_{1,1} \vee P_{3,1})$

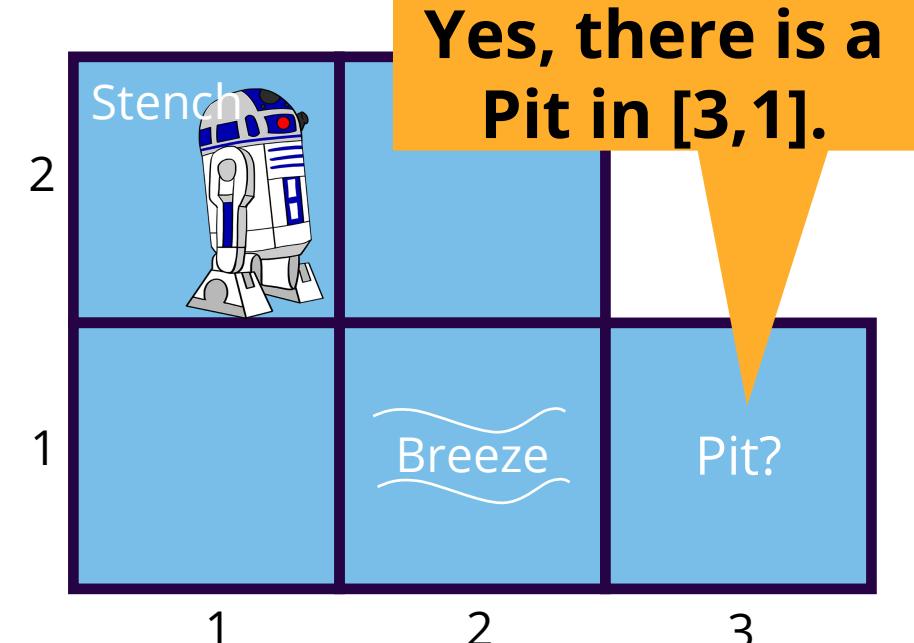
R19:  $P_{3,1}$

And there's no pit in [1,1]

First application of Resolution using R18 and R1:

$$\frac{(P_{1,1} \vee P_{3,1}) \wedge \neg P_{1,1}}{P_{3,1}}$$

Then there must be a pit in [3,1]



# Resolution

If  $\ell_i$  and  $m$  are complementary literals, which means that one is the negation of another, then we can eliminate them via **unit resolution**.

$$\frac{(\ell_1 \vee \dots \vee \ell_k), m}{(\ell_1 \vee \dots \vee \ell_{i-1} \vee \ell_{i+1} \vee \dots \vee \ell_k)}$$

# Resolution

If  $\ell_i$  and  $m$  are complementary literals, which means that one is the negation of another, then we can eliminate them via **unit resolution**.

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Full **resolution rule** for complementary literals  $\ell_i$  and  $m_j$

$$\frac{\ell_1 \vee \dots \vee \ell_k, \quad m_1 \vee \dots \vee m_n}{\ell_1 \vee \ell_{i-1} \vee \dots \vee \ell_{i+1} \vee \ell_k \vee m_1 \vee \dots \vee m_{j-1} \vee \dots \vee m_{j+1} \vee m_k}$$

# Resolution

If  $\ell_i$  and  $m$  are complementary literals,  
which means that one is the negation of  
another, then they can be eliminated from the union.

**The resolution rule only applies to clauses (= disjunctions of literals).**

$$\frac{\ell_1 \vee \dots \vee \ell_k, m}{\ell_1 \vee \ell_{i-1} \vee \dots \vee \ell_{i+1} \vee \ell_k \vee m_1 \vee \dots \vee m_n}$$

complementary literals

$$\frac{m_1 \vee \dots \vee m_n}{m_1 \vee \dots \vee m_{j-1} \vee \dots \vee m_{j+1} \vee m_k}$$

Our KB has full set of logical sentences.

$$\ell_1 \vee \dots \vee \ell_k, m$$

**Answer: Every sentence can be equivalent to a conjunction of clauses.**

How can it lead to a complete inference procedure for all of propositional logic?

# Conjunctive Normal Form (CNF)

Every sentence of propositional logic is logically equivalent to a conjunction of clauses.  
Clauses are disjunctions of literals.

1. Eliminate  $\Leftrightarrow$  by replacing  $\alpha \Leftrightarrow \beta$  with  $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
2. Eliminate  $\Rightarrow$  by replacing  $\alpha \Rightarrow \beta$  with  $\neg \alpha \vee \beta$
3. Move  $\neg$  inwards (eliminate double negation, and apply De Morgan's rules)
4. Now the sentence will contain nested  $\wedge$  and  $\vee$  operators applied to literals. Apply distributivity law to distribute  $\vee$  over  $\wedge$  wherever possible.

The sentence is now in CNF!

# Resolution Algorithm

Inference procedures based on resolution work using the principle of **proof by contradiction**.

To show  $\text{KB} \models \alpha$  then we can show that  $\text{KB} \wedge \neg\alpha$  is **unsatisfiable**.

Here's how:

- First, convert  $(\text{KB} \wedge \neg\alpha)$  into CNF
- Then apply the resolution rule to the resulting clauses.
- Resolve every pair that contains complementary literals to produce a new clause
- If that new clause isn't already present, then add it to the set.
- Continue the process until one of two things happens:
  1. There are no new clauses that can be added. In this case, the KB does not entail  $\alpha$
  2. Two clauses resolve to yield the **empty clause**, in which case the KB does entail  $\alpha$

# Resolution Algorithm

Inference procedures based on resolution work using the principle of **proof by contradiction**.

To show  $\text{KB} \models \alpha$  then we can show that  $\text{KB} \wedge \neg\alpha \vdash \bot$ .

Here's how:

- First, convert  $(\text{KB} \wedge \neg\alpha)$  into **CNF**
- Then apply the **resolution rule** to the resulting clauses.
- Resolve every pair that contain **complements**.
- If that new clause isn't already present, then **add it to the set**.
- Continue the process until one of two things happens:
  1. There are **no new clauses** that can be added. In this case, the KB does not entail  $\alpha$
  2. Two clauses resolve to yield the **empty clause**, in which case the KB does entail  $\alpha$

The empty clause arises from resolving two contradictory unit clauses like  $P$  and  $\neg P$ .

# Resolution Algorithm

**function** PL-RESOLUTION( $KB, \alpha$ ) **returns** *true* or *false*

**inputs:**  $KB$ , the knowledge base, a sentence in propositional logic  
 $\alpha$ , the query, a sentence in propositional logic

$clauses \leftarrow$  the set of clauses in the CNF representation of  $KB \wedge \neg\alpha$

$new \leftarrow \{\}$

**while** *true* **do**

**for each** pair of clauses  $C_i, C_j$  **in**  $clauses$  **do**

$resolvents \leftarrow$  PL-RESOLVE( $C_i, C_j$ )

**if**  $resolvents$  contains the empty clause **then return** *true*

$new \leftarrow new \cup resolvents$

**if**  $new \subseteq clauses$  **then return** *false*

$clauses \leftarrow clauses \cup new$

# Horn Clauses and Definite Clauses

The resolution algorithm is **complete!** This makes it a very important inference method.

However, we don't always need the full power of resolution. Some KBs are formulated in a **more restrictive** form which enables them to use **more efficient** inference algorithms.

- **Definite clause** – disjunction of literals where **exactly one is positive**
- **Horn clause** – disjunction of literals where **at most one is positive**
  - **This includes definite clauses**
  - **And clauses with no positive literals**

Horn clauses are closed under resolution. If you apply resolution to a horn clause, you get back a horn clause.

# KBs with only definite clauses

1. Every definite clause can be written as an implication whose premise is a conjunction of positive literals whose conclusion is a positive literal.  
 $\neg L_{1,1} \vee \neg \text{Breeze} \vee B_{1,1}$  can be written as the implication  $L_{1,1} \wedge \text{Breeze} \Rightarrow B_{1,1}$
2. Inference with Horn clauses can be done through the forward-chaining and backward-chaining algorithms
3. Deciding entailment with Horn clauses can be done in **linear time** with respect to the size of the KB.

# Forward Chaining Algorithm

```
function PL-FC-ENTAILS?(KB, q) returns true or false
    inputs: KB, the knowledge base, a set of propositional definite clauses
            q, the query, a proposition symbol
    count  $\leftarrow$  a table, where  $count[c]$  is initially the number of symbols in clause  $c$ 's premise
    inferred  $\leftarrow$  a table, where  $inferred[s]$  is initially false for all symbols
    queue  $\leftarrow$  a queue of symbols, initially symbols known to be true in KB

    while queue is not empty do
         $p \leftarrow \text{POP}(queue)$ 
        if  $p = q$  then return true
        if  $inferred[p] = \text{false}$  then
             $inferred[p] \leftarrow \text{true}$ 
            for each clause  $c$  in KB where  $p$  is in  $c.\text{PREMISE}$  do
                decrement  $count[c]$ 
                if  $count[c] = 0$  then add  $c.\text{CONCLUSION}$  to queue
    return false
```

# Forward Chaining and Backward Chaining

**Forward Chaining** is a kind of **data-driven reasoning**. An agent can start with known data, adding premises as new percepts come in, and then incrementally apply the forward chaining algorithm to derive new conclusions. It doesn't have to have a specific query in mind to do so.

**Backward chaining** works backwards from a query. If the query is known to be true, then no work is needed. Otherwise, the algorithm finds implications in the KB whose conclusions is the query. If all the premises of one of those can be proved to be true (via backward chaining), then the query is true.

**Backward chaining is a form of goal-directed reasoning.**