Machine Learning Lecture Notes

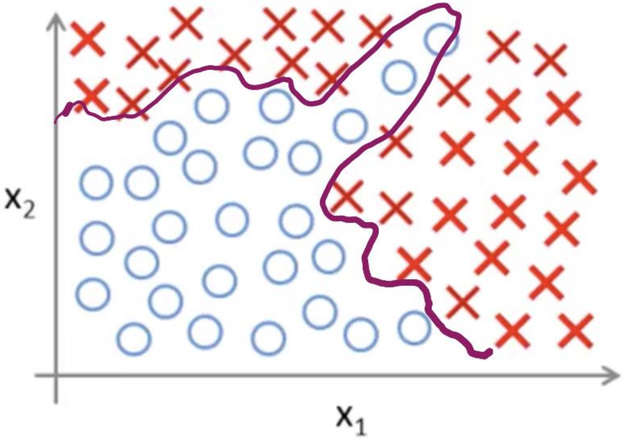
Lecture Notes by Şükrü Ozan, Ph.D.

LECTURE 08

Motivations

Non-linear Hypotheses

Non-linear Classification



$$g(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1 x_2 + \theta_4 x_1^2 x_2 + \theta_5 x_1^3 x_2 + \theta_6 x_1 x_2^2 + \ldots)$$

$x_1, \ldots, x_{100}, x_1^2, \ldots, x_{100}^2, x_1 x_2, x_1 x_3, \ldots, x_{99} x_{100}$

≈ 5000 feature $\quad O(n^2)$

$x_1 = $ size

$x_2 = $ # bedrooms

$x_3 = $ # floors

$x_4 = $ age

$\ldots$

x_{100}

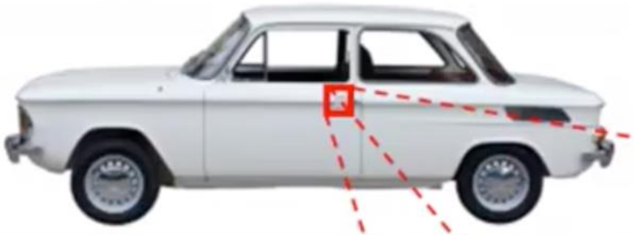
$x_1, \ldots, x_{100}, x_1^2, \ldots, x_{100}^2, x_1^3, \ldots, x_{100}^3, x_1 x_2 x_3, \ldots$

$x_{99} x_{98} x_{100}$

≈ 170.000 feature $\quad O(n^3)$

What is this?

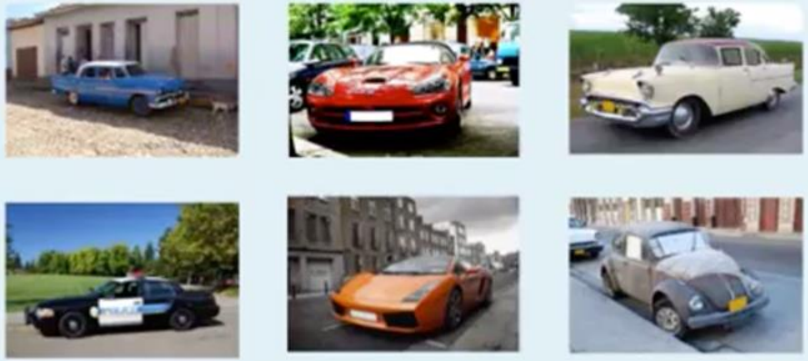
You see this:



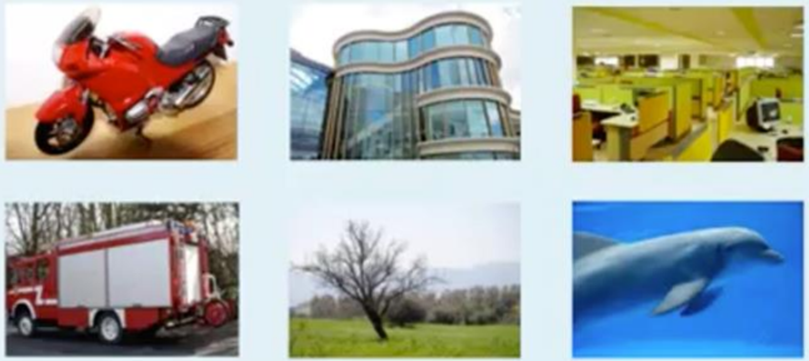
But the camera sees this:

194	210	201	212	199	213	215	195	178	158	182	209
180	189	190	221	209	205	191	167	147	115	129	163
114	126	140	188	176	165	152	140	170	106	78	88
87	103	115	154	143	142	149	153	173	101	57	57
102	112	106	131	122	138	152	147	128	84	58	66
94	95	79	104	105	124	129	113	107	87	69	67
68	71	69	98	89	92	98	95	89	88	76	67
41	56	68	99	63	45	60	82	58	76	75	65
20	43	69	75	56	41	51	73	55	70	63	44
50	50	57	69	75	75	73	74	53	68	59	37
72	59	53	66	84	92	84	74	57	72	63	42
67	61	58	65	75	78	76	73	59	75	69	50

Computer Vision: Car detection



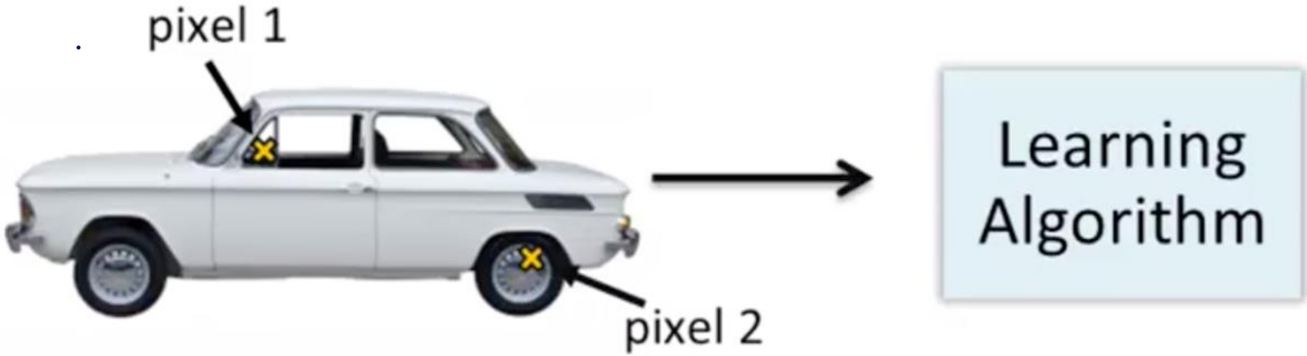
Cars

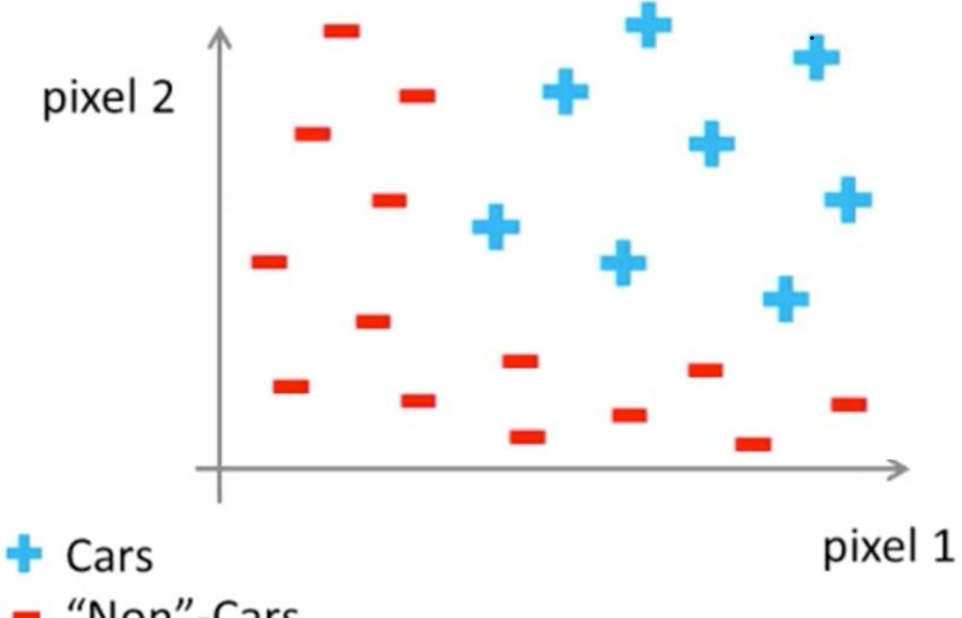


Not a car

Testing: 

What is this?





50 x 50 pixel images→ 2500 pixels

$n = 2500$ (7500 if RGB)

$$x = \begin{bmatrix} \text{pixel 1 intensity} \\ \text{pixel 2 intensity} \\ \vdots \\ \text{pixel 2500 intensity} \end{bmatrix}$$

Quadratic features ($x_i \times x_j$): ≈3 million features

Suppose you are learning to recognize cars from 100×100 pixel images (grayscale, not RGB). Let the features be pixel intensity values. If you train logistic regression including all the quadratic terms ($x_i x_j$) as features, about how many features will you have?

- ○ 5,000
- ○ 100,000
- ● 50 million (5×10^7)
- ○ 5 billion (5×10^9)

$10^4 \quad x_i x_j$

$\frac{10^4 \, 10^4}{2} \approx 50$ mil.

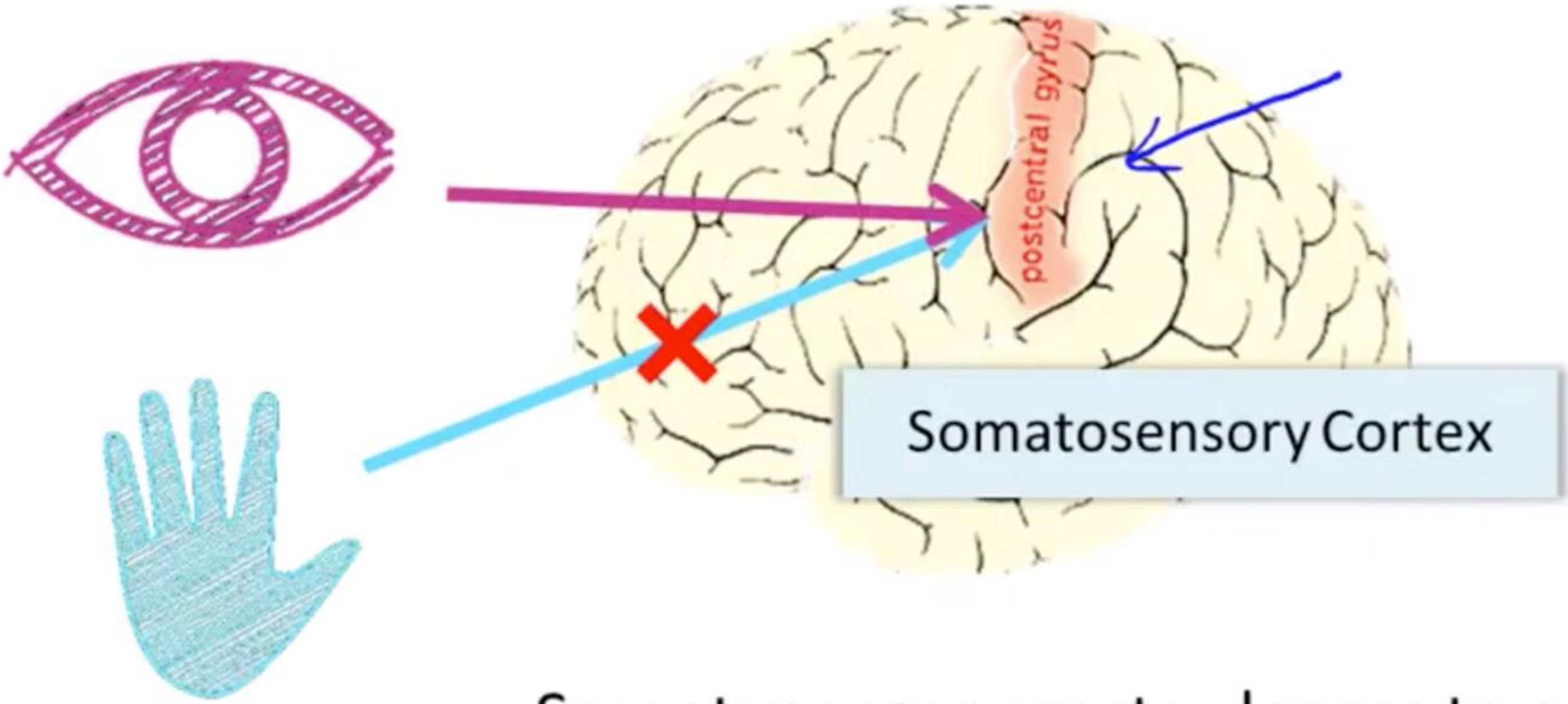
Motivations

Neurons and Brain

Neural Networks

Origins: Algorithms that try to mimic the brain.
Was very widely used in 80s and early 90s; popularity diminished in late 90s.
Recent resurgence: State-of-the-art technique for many applications

The “one learning algorithm” hypothesis



Somatosensory cortex learns to see

[Roe et al. 1992]
[Metin & Frost, 1989]

Sensor representations in the brain

Seeing with your tongue

Human echolocation (sonar)

Haptic belt: Direction sense

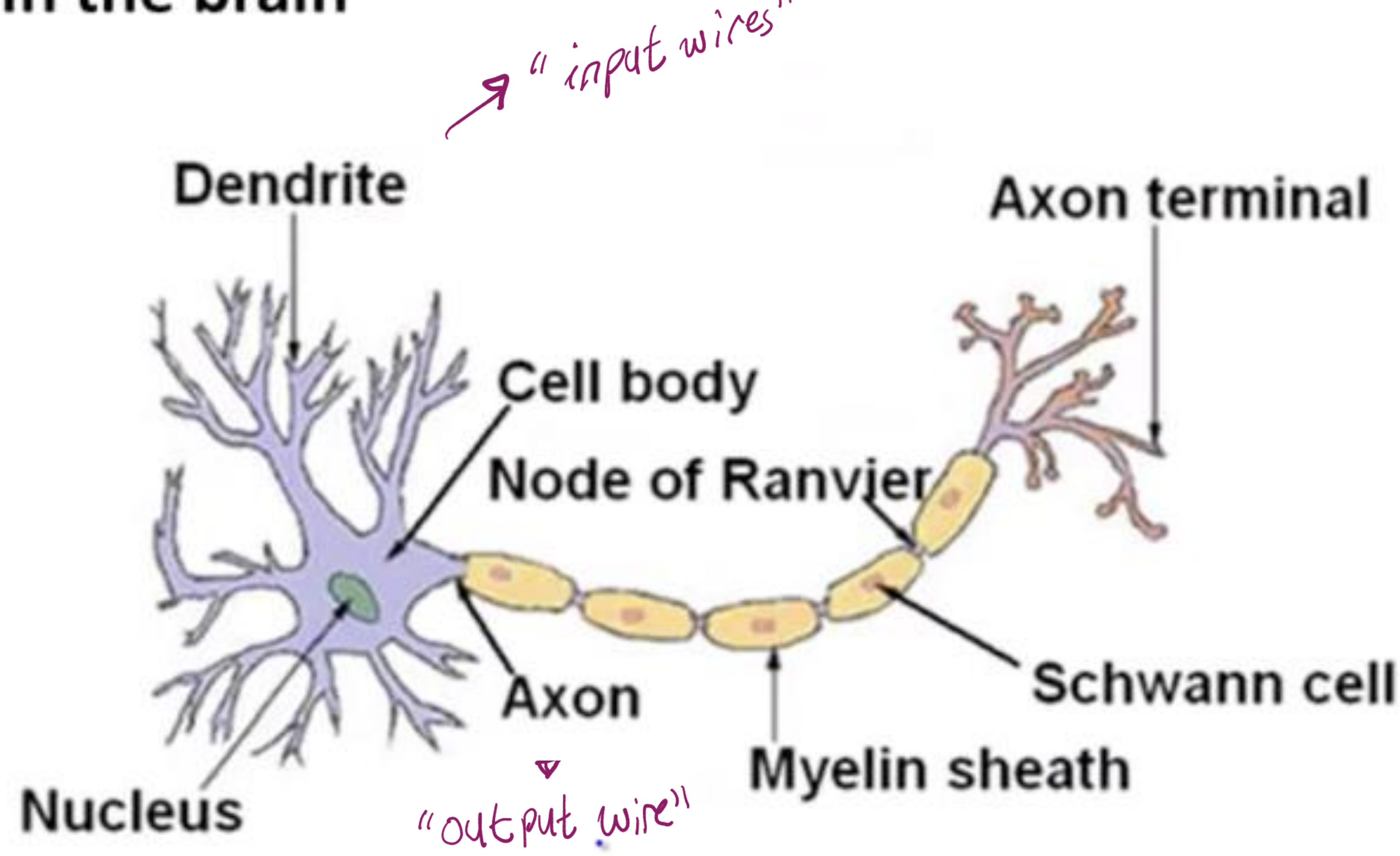
Implanting a 3rd eye

[BrainPort; Welsh & Blasch, 1997; Nagel et al., 2005; Constantine-Paton & Law, 2009]

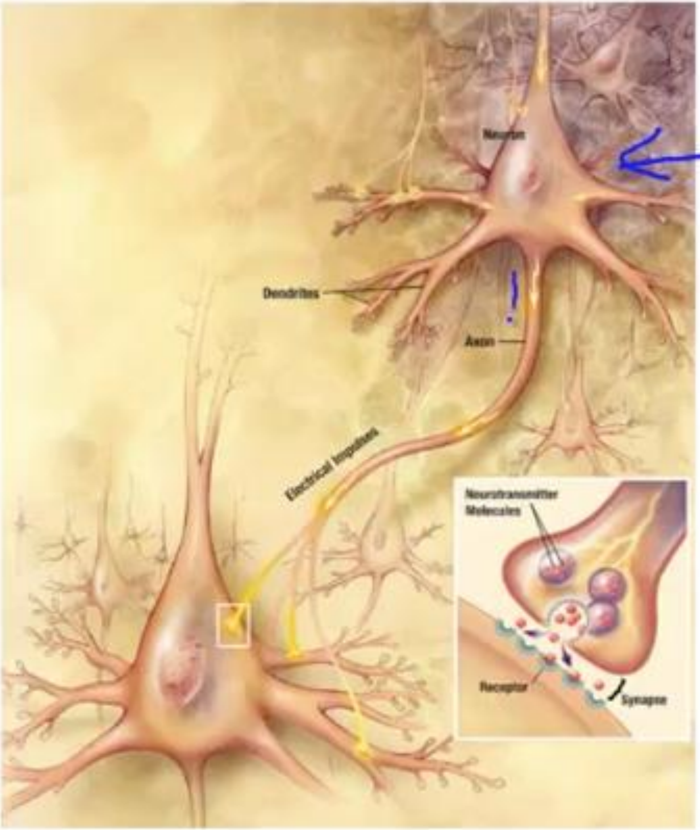
Neural Networks

Model Representation I

Neuron in the brain

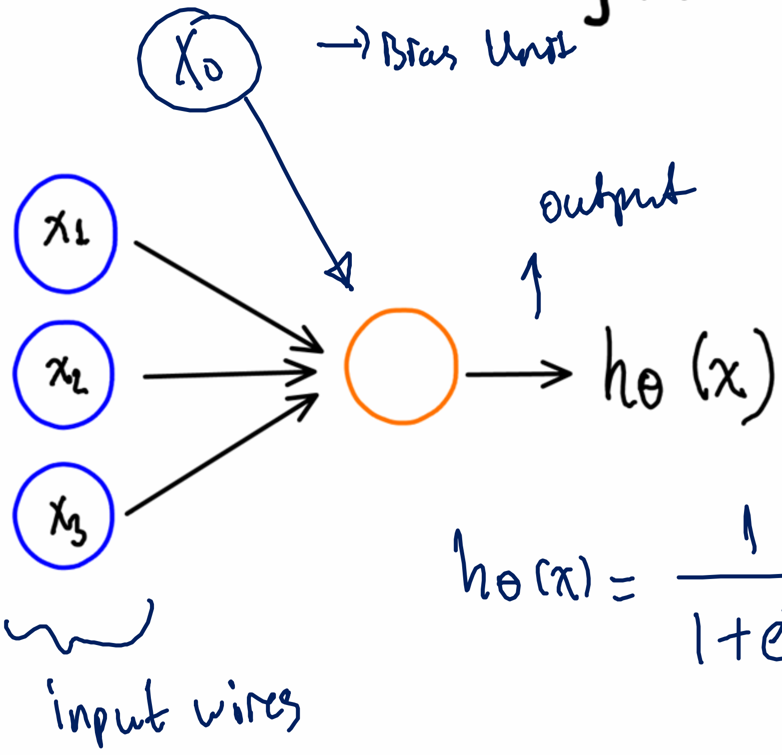


Neurons in the brain



[Credit: US National Institutes of Health, National Institute on Aging]

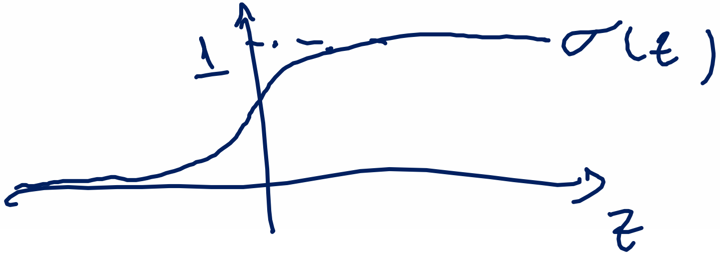
Neuron Model: Logistic Unit



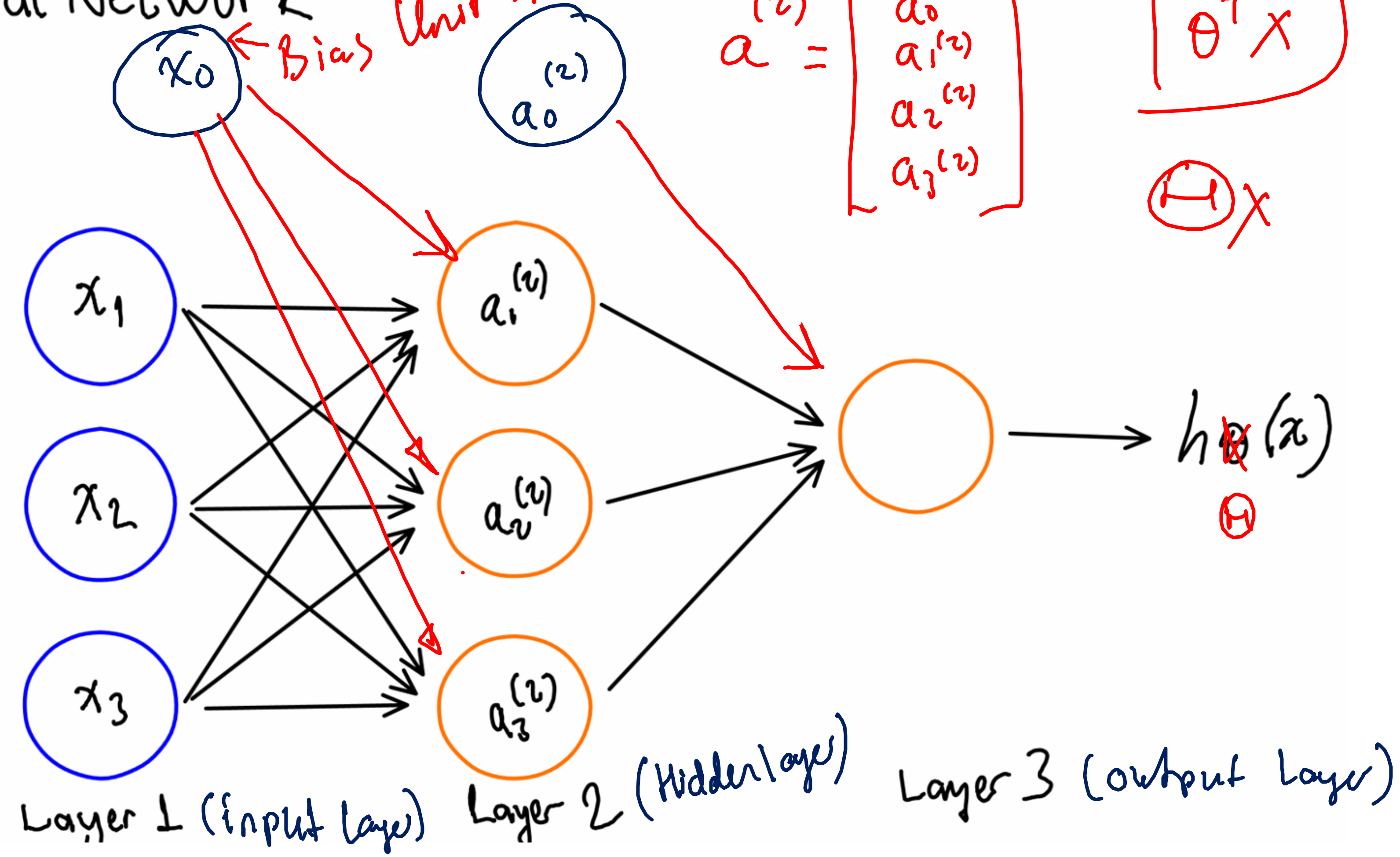
$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

$$h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}$$

(Sigmoid (logistic) activation function)



Neural Network

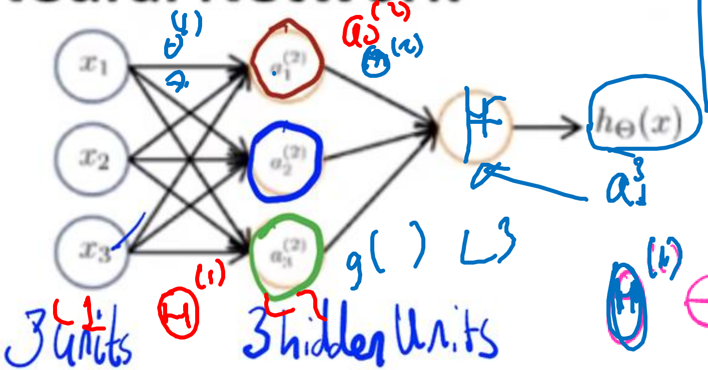


$$a^{(2)} = \begin{bmatrix} a_0^{(2)} \\ a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \end{bmatrix}$$

$$\boxed{\theta^T x}$$

$$\Theta x$$

Neural Network



$X = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$a_i^{(j)}$ = "activation" of unit i in layer j

$\Theta^{(j)}$ = matrix of weights controlling function mapping from layer j to layer $j+1$

$\Theta^{(1)} \in \mathbb{R}^{3\times 4}$

$g(\Theta x)$

$g(z) = \frac{1}{1+e^{-z}}$

$\Theta^{2} \in \mathbb{R}^{1\times 4}$

$$a_1^{(2)} = g(\Theta_{10}^{(1)}x_0 + \Theta_{11}^{(1)}x_1 + \Theta_{12}^{(1)}x_2 + \Theta_{13}^{(1)}x_3)$$

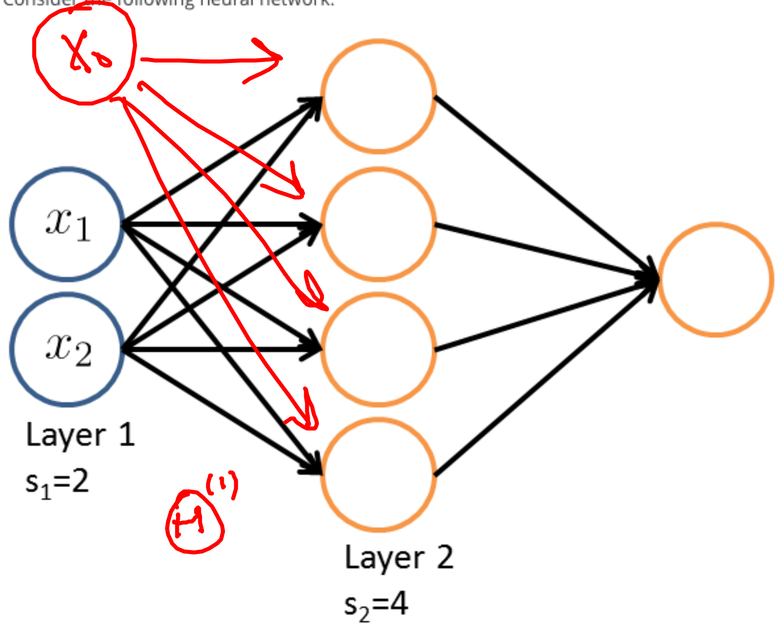
$$a_2^{(2)} = g(\Theta_{20}^{(1)}x_0 + \Theta_{21}^{(1)}x_1 + \Theta_{22}^{(1)}x_2 + \Theta_{23}^{(1)}x_3)$$

$$a_3^{(2)} = g(\Theta_{30}^{(1)}x_0 + \Theta_{31}^{(1)}x_1 + \Theta_{32}^{(1)}x_2 + \Theta_{33}^{(1)}x_3)$$

$$h_\Theta(x) = a_1^{(3)} = g(\boxed{\Theta_{10}^{(2)}a_0^{(2)} + \Theta_{11}^{(2)}a_1^{(2)} + \Theta_{12}^{(2)}a_2^{(2)} + \Theta_{13}^{(2)}a_3^{(2)}})$$

If network has s_j units in layer j, s_{j+1} units in layer $j+1$, then $\Theta^{(j)}$ will be of dimension $s_{j+1} \times (s_j + 1)$.

Consider the following neural network:



What is the dimension of $\Theta^{(1)}$ (Hint: add a bias unit to the input and hidden layers)?

$S_{j+1} \times (S_j + 1)$

$S_2 \times (S_1 + 1)$

4×3

- ○ 2×4
- ○ 4×2
- ○ 3×4
- ● $\boxed{4 \times 3}$

Model Representation I

Let's examine how we will represent a hypothesis function using neural networks. At a very simple level, neurons are basically computational units that take inputs (**dendrites**) as electrical inputs (called "spikes") that are channeled to outputs (**axons**). In our model, our dendrites are like the input features $x_1 \cdots x_n$, and the output is the result of our hypothesis function. In this model our x_0 input node is sometimes called the "bias unit." It is always equal to 1. In neural networks, we use the same logistic function as in classification, $\frac{1}{1+e^{-\theta^T x}}$, yet we sometimes call it a sigmoid (logistic) **activation** function. In this situation, our "theta" parameters are sometimes called "weights".

Visually, a simplistic representation looks like:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \rightarrow [\quad] \rightarrow h_\theta(x)$$

Our input nodes (layer 1), also known as the "input layer", go into another node (layer 2), which finally outputs the hypothesis function, known as the "output layer".

We can have intermediate layers of nodes between the input and output layers called the "hidden layers."

In this example, we label these intermediate or "hidden" layer nodes $a_0^2 \cdots a_n^2$ and call them "activation units."

$$
\begin{aligned}
a_i^{(j)} &= \text{"activation" of unit } i \text{ in layer } j \\
\Theta^{(j)} &= \text{matrix of weights controlling function mapping from layer } j \text{ to layer } j+1
\end{aligned}
$$

If we had one hidden layer, it would look like:

$$
\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \\ a_3^{(2)} \end{bmatrix} \rightarrow h_\theta(x)
$$

The values for each of the "activation" nodes is obtained as follows:

$$
\begin{aligned}
a_1^{(2)} &= g(\Theta_{10}^{(1)}x_0 + \Theta_{11}^{(1)}x_1 + \Theta_{12}^{(1)}x_2 + \Theta_{13}^{(1)}x_3) \\
a_2^{(2)} &= g(\Theta_{20}^{(1)}x_0 + \Theta_{21}^{(1)}x_1 + \Theta_{22}^{(1)}x_2 + \Theta_{23}^{(1)}x_3) \\
a_3^{(2)} &= g(\Theta_{30}^{(1)}x_0 + \Theta_{31}^{(1)}x_1 + \Theta_{32}^{(1)}x_2 + \Theta_{33}^{(1)}x_3) \\
h_\Theta(x) = a_1^{(3)} &= g(\Theta_{10}^{(2)}a_0^{(2)} + \Theta_{11}^{(2)}a_1^{(2)} + \Theta_{12}^{(2)}a_2^{(2)} + \Theta_{13}^{(2)}a_3^{(2)})
\end{aligned}
$$

This is saying that we compute our activation nodes by using a 3×4 matrix of parameters. We apply each row of the parameters to our inputs to obtain the value for one activation node. Our hypothesis output is the logistic function applied to the sum of the values of our activation nodes, which have been multiplied by yet another parameter matrix $\Theta^{(2)}$ containing the weights for our second layer of nodes.

Each layer gets its own matrix of weights, $\Theta^{(j)}$.

The dimensions of these matrices of weights is determined as follows:

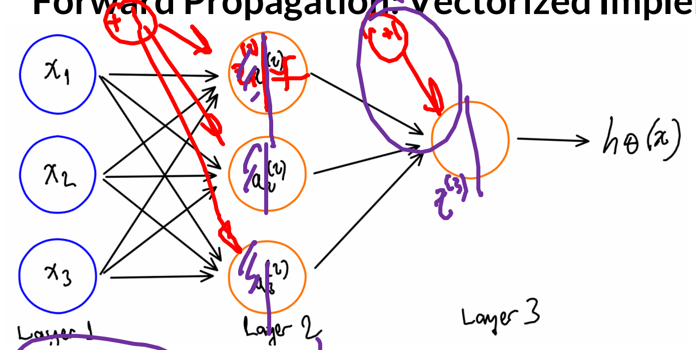
$$\text{If network has } s_j \text{ units in layer } j \text{ and } s_{j+1} \text{ units in layer } j+1, \text{ then } \Theta^{(j)} \text{ will be of dimension } s_{j+1} \times (s_j + 1).$$

The +1 comes from the addition in $\Theta^{(j)}$ of the "bias nodes," x_0 and $\Theta_0^{(j)}$. In other words the output nodes will not include the bias nodes while the inputs will. The following image summarizes our model representation:

Neural Networks

Model Representation II

Forward Propagation: Vectorized Implementation



$$\boxed{a^{(1)} = X}$$

$$X = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_0^{(1)} \\ a_1^{(1)} \\ a_2^{(1)} \\ a_3^{(1)} \end{bmatrix} \qquad z^{(2)} = \begin{bmatrix} z_1^{(2)} \\ z_2^{(2)} \\ z_3^{(2)} \end{bmatrix}$$

$$a_1^{(2)} = g(\underbrace{\Theta_{10}^{(1)} x_0 + \Theta_{11}^{(1)} x_1 + \Theta_{12}^{(1)} x_2 + \Theta_{13}^{(1)} x_3}_{z_1^{(2)}})$$

$$a_2^{(2)} = g(\underbrace{\Theta_{20}^{(1)} x_0 + \cdots + \Theta_{23}^{(1)} x_3}_{z_2^{(2)}})$$

$$a_3^{(2)} = g(\underbrace{\Theta_{30}^{(1)} x_0 + \cdots + \Theta_{33}^{(1)} x_3}_{z_3^{(2)}})$$

$$z^{(2)} = \Theta^{(1)} X = \boxed{\Theta^{(1)} a^{(1)}}$$

$$\boxed{a^{(2)} = g(z^{(2)})}$$

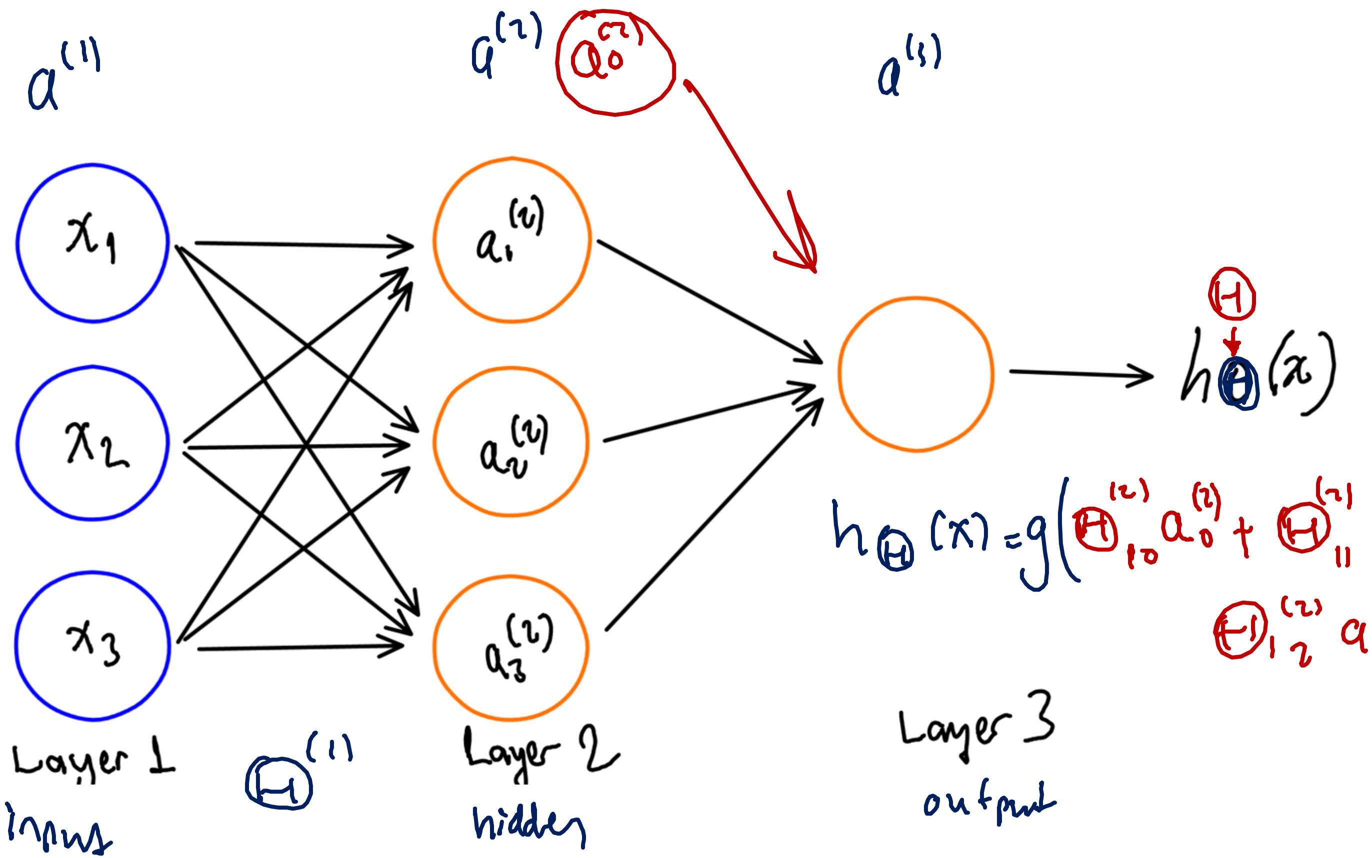
add

$$a_0^{(2)} = 1 \Rightarrow a^{(2)} \in \mathbb{R}^4$$

$$z^{(3)} = \Theta^{(2)} a^{(2)}$$

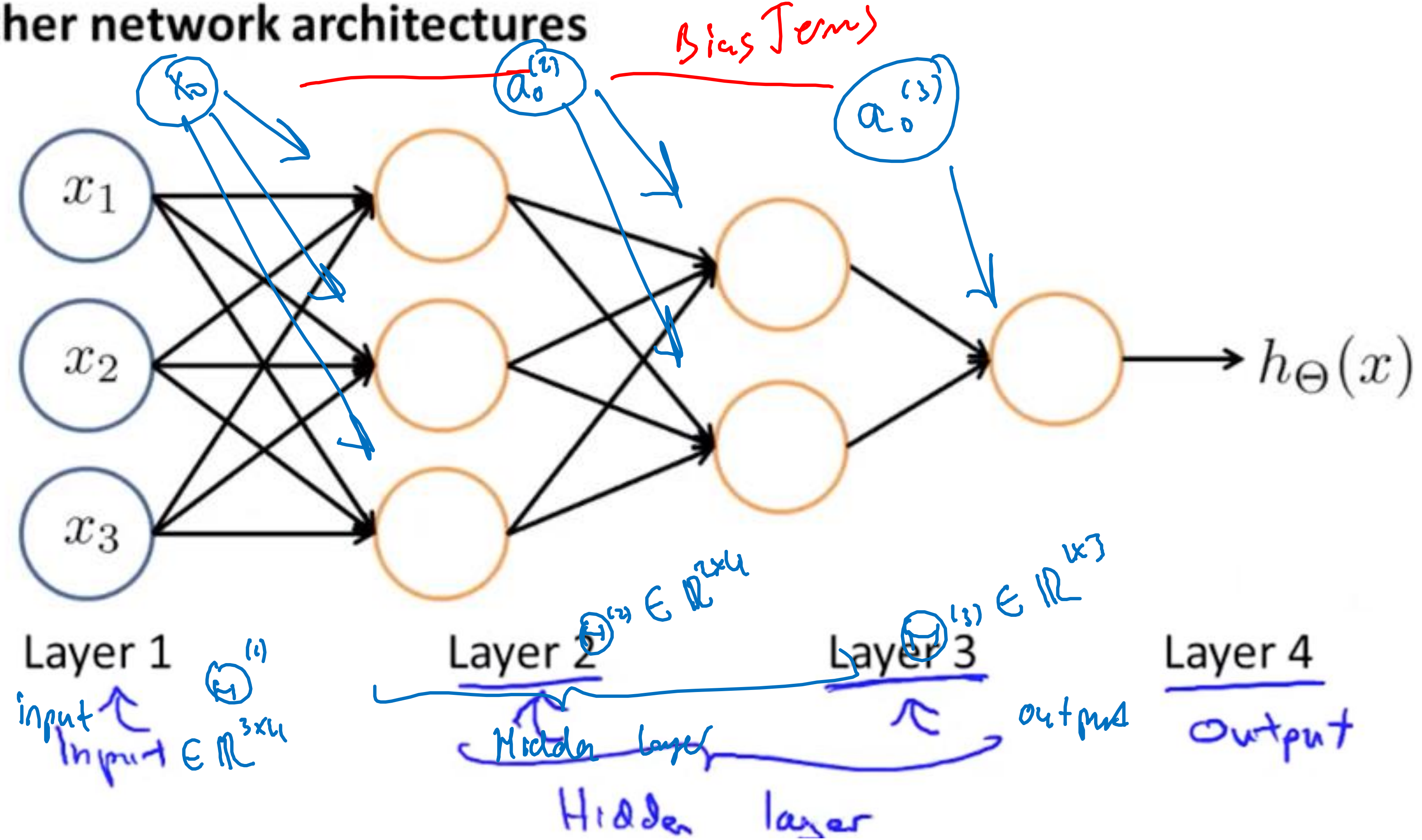
$$h_\Theta(X) = a^{(3)} = g(z^{(3)})$$

Neural Network learning its own features

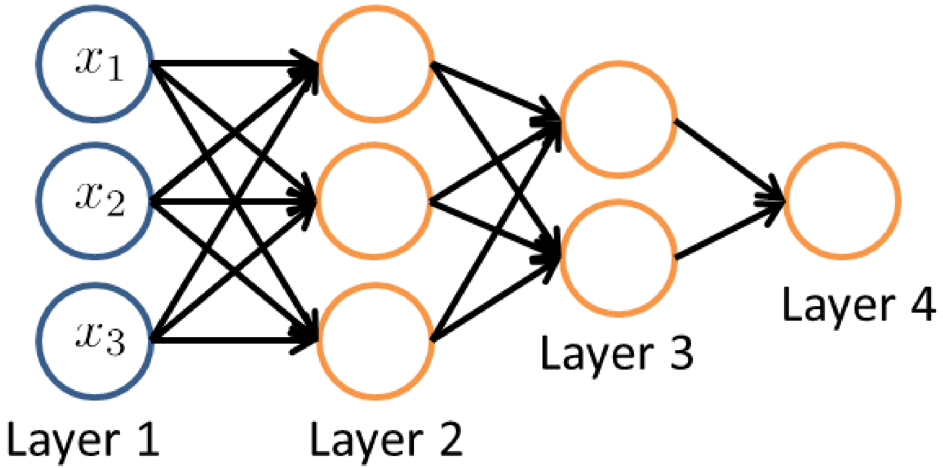


$$h_\Theta(x) = g\left(\Theta^{(2)}_{10} a^{(2)}_0 + \Theta^{(2)}_{11} a^{(2)}_1 + \Theta^{(2)}_{12} a^{(2)}_2 + \Theta^{(2)}_{13} a^{(2)}_3\right)$$

Other network architectures



Consider the network:



Let $a^{(1)} = x \in \mathbb{R}^{n+1}$ denote the input (with $a_0^{(1)} = 1$).

How would you compute $a^{(2)}$?

- ○ $a^{(2)} = \Theta^{(1)} a^{(1)}$
- ○ $z^{(2)} = \Theta^{(2)} a^{(1)};\ a^{(2)} = g(z^{(2)})$
- ● $z^{(2)} = \Theta^{(1)} a^{(1)};\ a^{(2)} = g(z^{(2)})$
- ○ $z^{(2)} = \Theta^{(2)} g(a^{(1)});\ a^{(2)} = g(z^{(2)})$

Model Representation II

To re-iterate, the following is an example of a neural network:

$$
\begin{aligned}
a_1^{(2)} &= g(\Theta_{10}^{(1)}x_0 + \Theta_{11}^{(1)}x_1 + \Theta_{12}^{(1)}x_2 + \Theta_{13}^{(1)}x_3) \\
a_2^{(2)} &= g(\Theta_{20}^{(1)}x_0 + \Theta_{21}^{(1)}x_1 + \Theta_{22}^{(1)}x_2 + \Theta_{23}^{(1)}x_3) \\
a_3^{(2)} &= g(\Theta_{30}^{(1)}x_0 + \Theta_{31}^{(1)}x_1 + \Theta_{32}^{(1)}x_2 + \Theta_{33}^{(1)}x_3) \\
h_\Theta(x) = a_1^{(3)} &= g(\Theta_{10}^{(2)}a_0^{(2)} + \Theta_{11}^{(2)}a_1^{(2)} + \Theta_{12}^{(2)}a_2^{(2)} + \Theta_{13}^{(2)}a_3^{(2)})
\end{aligned}
$$

In this section we'll do a vectorized implementation of the above functions. We're going to define a new variable $z_k^{(j)}$ that encompasses the parameters inside our g function. In our previous example if we replaced by the variable z for all the parameters we would get:

$$
\begin{aligned}
a_1^{(2)} &= g(z_1^{(2)}) \\
a_2^{(2)} &= g(z_2^{(2)}) \\
a_3^{(2)} &= g(z_3^{(2)})
\end{aligned}
$$

The vector representation of x and z^j is:

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \cdots \\ x_n \end{bmatrix} z^{(j)} = \begin{bmatrix} z_1^{(j)} \\ z_2^{(j)} \\ \cdots \\ z_n^{(j)} \end{bmatrix}$$

Setting $x = a^{(1)}$, we can rewrite the equation as:

$$z^{(j)} = \Theta^{(j-1)} a^{(j-1)}$$

We are multiplying our matrix $\Theta^{(j-1)}$ with dimensions $s_j \times (n+1)$ (where s_j is the number of our activation nodes) by our vector $a^{(j-1)}$ with height (n+1). This gives us our vector $z^{(j)}$ with height s_j. Now we can get a vector of our activation nodes for layer j as follows:

$$a^{(j)} = g(z^{(j)})$$

Where our function g can be applied element-wise to our vector $z^{(j)}$.

We can then add a bias unit (equal to 1) to layer j after we have computed $a^{(j)}$. This will be element $a_0^{(j)}$ and will be equal to 1. To compute our final hypothesis, let's first compute another z vector:

$$z^{(j+1)} = \Theta^{(j)} a^{(j)}$$

We are multiplying our matrix $\Theta^{(j-1)}$ with dimensions $s_j \times (n+1)$ (where s_j is the number of our activation nodes) by our vector $a^{(j-1)}$ with height (n+1). This gives us our vector $z^{(j)}$ with height s_j. Now we can get a vector of our activation nodes for layer j as follows:

$$a^{(j)} = g(z^{(j)})$$

Where our function g can be applied element-wise to our vector $z^{(j)}$.

We can then add a bias unit (equal to 1) to layer j after we have computed $a^{(j)}$. This will be element $a_0^{(j)}$ and will be equal to 1. To compute our final hypothesis, let's first compute another z vector:

$$z^{(j+1)} = \Theta^{(j)} a^{(j)}$$

We get this final z vector by multiplying the next theta matrix after $\Theta^{(j-1)}$ with the values of all the activation nodes we just got. This last theta matrix $\Theta^{(j)}$ will have only **one row** which is multiplied by one column $a^{(j)}$ so that our result is a single number. We then get our final result with:

$$h_\Theta(x) = a^{(j+1)} = g(z^{(j+1)})$$

Notice that in this **last step**, between layer j and layer j+1, we are doing **exactly the same thing** as we did in logistic regression. Adding all these intermediate layers in neural networks allows us to more elegantly produce interesting and more complex non-linear hypotheses.