On the Functional Derivative of the Scalar Field Action and the Associated Global Functional

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Abstract

En el marco de la Teoría Cuántica de Campos Algebraica (AQFT), los observables se asocian a regiones del espacio-tiempo en lugar de puntos individuales, lo que permite una descripción algebraica consistente con la relatividad especial. Se introduce el concepto de *net de álgebras locales*, cumpliendo los axiomas de Haag–Kastler: isotonía, covariancia, localidad y normalización. La localidad asegura que los observables asociados a regiones acausalmente desconectadas conmutan, garantizando que no exista influencia causal entre ellas. Como ejemplo, se analiza el campo escalar libre de Klein–Gordon, demostrando que su conmutador se anula para separaciones acausales debido a la invariancia de Lorentz y las relaciones de conmutación canónicas, lo que motiva y respalda la estructura algebraica de AQFT. Este enfoque muestra cómo la causalidad se incorpora de manera natural en la teoría cuántica de campos mediante la formalización algebraica de regiones acausales.

Abstract

We summarize the derivation of the functional derivative of the scalar field action on a pseudo-Riemannian manifold, emphasizing the relation with the Klein–Gordon operator, and compute the global quantity

$$G[\phi] := \int_M \frac{\delta S[\phi]}{\delta \phi}.$$

1 Covariant Action for a Real Scalar Field

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Consider the standard covariant action for a real scalar field ϕ:

$$S[\phi] = \int_M d^n x \, \sqrt{|g|} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \, \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{1}$$

2 Functional Derivative

Varying the action with respect to ϕ and integrating by parts gives the well-known result:

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -\sqrt{|g(x)|}\,(\Box_g + m^2)\phi(x), \tag{2}$$

where $\Box_g = \nabla^\mu\nabla_\mu$ is the covariant d'Alembertian. Thus the Euler–Lagrange equation $\delta S/\delta\phi = 0$ yields the Klein–Gordon equation

$$(\Box_g + m^2)\phi = 0.$$

3 The Global Functional $G[\phi]$

Define

$$G[\phi] := \int_M \frac{\delta S[\phi]}{\delta\phi}. \tag{3}$$

Substituting (2) gives

$$G[\phi] = -\int_M d^n x\,\sqrt{|g|}\,(\Box_g + m^2)\phi. \tag{4}$$

3.1 Boundary Decomposition

Using the identity

$$\sqrt{|g|}\,\Box_g\phi = \nabla_\mu(\sqrt{|g|}\,\nabla^\mu\phi),$$

we integrate by parts:

$$G[\phi] = -\int_M d^n x\,\sqrt{|g|}\,m^2\phi \;-\; \int_M \nabla_\mu(\sqrt{|g|}\,\nabla^\mu\phi)\,d^n x.$$

Applying Stokes' theorem yields

$$G[\phi] = -m^2\int_M d^n x\,\sqrt{|g|}\,\phi \;-\; \int_{\partial M} d^{n-1}y\,\sqrt{|h|}\,n_\mu\nabla^\mu\phi, \tag{5}$$

where n_μ is the outward unit normal and h the induced metric on ∂M.

3.2 Closed Manifolds or Vanishing Boundary Terms

If M is closed or boundary terms vanish (e.g. Dirichlet or Neumann conditions), then

$$G[\phi] = -m^2\int_M d^n x\,\sqrt{|g|}\,\phi. \tag{6}$$

4 Conclusion

Starting from the covariant scalar field action, the functional derivative naturally reproduces the Klein–Gordon operator. Integrating this functional derivative produces the global quantity $G[\phi]$, which consists of a bulk term proportional to the mass and a boundary flux term involving the normal derivative of ϕ.