# Composite Quantum Systems and Their Wave Functions

Omar Corona Tejeda

omarct1989@ciencias.unam.mx

December 19, 2025

**Abstract**

We review the mathematical structure used to describe composite quantum systems. In particular, we explain how the tensor product of Hilbert spaces naturally arises and clarify the meaning of the wave function associated with a bipartite system. Special attention is paid to the distinction between separable states and entangled states, as well as to the role of interactions in generating correlations.

## 1 Single Quantum Systems

In standard quantum mechanics, a physical system is described by a complex Hilbert space $\mathcal{H}$. A pure state is represented by a normalized vector $|\psi\rangle \in \mathcal{H}$. In the position representation, this state is described by a wave function:

$$\boxed{\langle x_0|\psi(x)\rangle := \int_{-\infty}^{\infty} dx\delta(x-x_0)\psi(x) = \psi(x_0)} \tag{1}$$

which satisfies the normalization condition

$$\int_{-\infty}^{\infty} dx|\psi(x)|^2 = 1. \tag{2}$$

This identity follows from the definition of the position eigenstates $|x\rangle$ as generalized eigenvectors which represents *the particle being positioning at the point $x$ that we can identify with* $\delta(x)$. They satisfy

$$\boxed{\langle x_0|x_1\rangle = \int_{-\infty}^{\infty} dx\delta(x-x_0)\delta(x-x_1) = \delta(x_0-x_1)} \tag{3}$$

So the action of the bra $\langle x_0|$ on a state $|\psi\rangle$ therefore corresponds to evaluating the wave function $\psi(x_0) = \langle x_0|\psi\rangle$ at the point $x_0$.

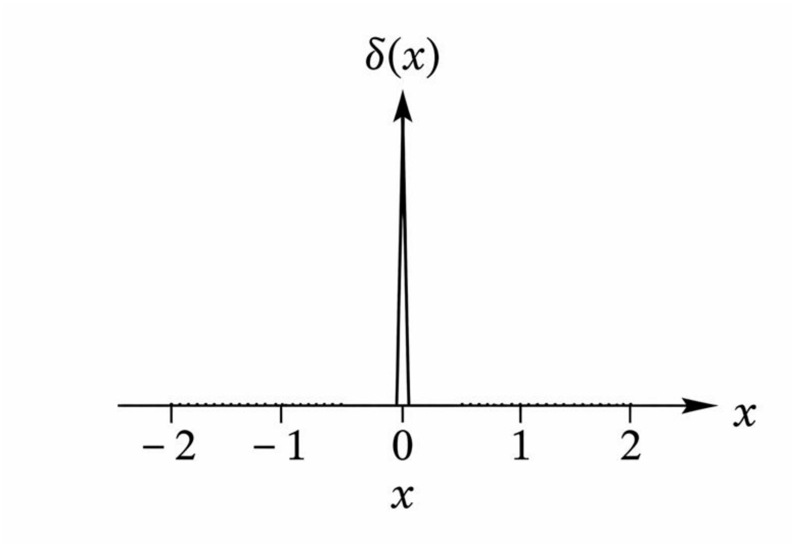


Figure 1: Schematic representation of the Dirac delta distribution $\delta(x)$.

## 2 Composite Quantum Systems

Consider two quantum systems, labeled $A$ and $B$, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_\mathcal{B}$, respectively. The Hilbert space describing the composite system is defined as the tensor product

$$\boxed{\mathcal{H}_{\mathcal{AB}} := \mathcal{H}_A \otimes \mathcal{H}_B.} \tag{4}$$

This construction reflects the fact that the degrees of freedom of the two systems are independent. Any state of the combined system is represented by a vector

$$|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B. \tag{5}$$

## 3 Wave Function of the Composite System

If both systems admit a position representation, with coordinates $x$ for system $A$ and $y$ for system $B$, the wave function of the composite system is defined by

$$\Psi(x, y) = \langle x, y|\Psi\rangle \,. \tag{6}$$

In general, $\Psi(x, y)$ is a complex-valued function of both variables and encodes all physical information about the joint system.

## 4 Product States

If the two systems are prepared independently, with system $A$ in a state $|\psi_A\rangle$ and system $B$ in a state $|\psi_B\rangle$, the joint state is given by the tensor product

$$|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \,. \tag{7}$$

In the position representation, the corresponding wave function factorizes as

$$\Psi(x, y) = \psi_A(x), \psi_B(y). \tag{8}$$

States of this form are called *separable* or *product states*. In such cases, there are no quantum correlations between the two subsystems.

# 5 Entangled States

Most states in the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$ cannot be written as a simple product of states of the individual subsystems. A general state may be expressed as a superposition of product states,

$$|\Psi\rangle = \sum_n c_n, \left|\psi_n^{(A)}\right\rangle \otimes \left|\psi_n^{(B)}\right\rangle , \tag{9}$$

where the coefficients $c_n$ are complex numbers.

When no factorization of the form $|\Psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ exists, the state is said to be *entangled*. In the position representation, this means that the wave function cannot be written as

$$\Psi(x, y) = \psi_A(x), \psi_B(y). \tag{10}$$

Entanglement is a uniquely quantum phenomenon and leads to strong correlations between measurement outcomes on the two subsystems.

# 6 Role of Interactions

The tensor product structure of the Hilbert space is independent of whether the systems interact. Interactions enter through the Hamiltonian of the composite system, which typically takes the form

$$\boxed{H = H_A \otimes I + I \otimes H_B + H_{\text{int}},} \tag{11}$$

where $H_{\text{int}}$ describes the interaction between the subsystems.

Even if the initial state is a product state, the presence of an interaction term generally leads to the dynamical generation of entanglement during time evolution.

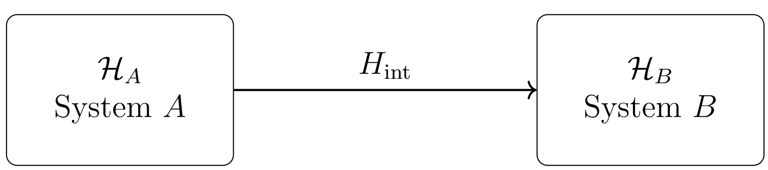


Figure 2: Schematic representation of two quantum systems with an interaction.

# 7 Clarifying the notion of interaction

Any two quantum systems $A$ and $B$, regardless of whether they interact physically, can be described as a single composite system whose Hilbert space is given by the tensor product

$$\boxed{\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.} \tag{12}$$

This kinematical construction alone does not imply the presence of an interaction between the systems.

The distinction between interacting and non-interacting systems is instead determined by the form of the total Hamiltonian. If the dynamics of the composite system is governed by a Hamiltonian of the form

$$\boxed{H = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B,} \tag{13}$$

where $H_A$ and $H_B$ act only on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, then the systems are said to be non-interacting and their dynamics is independent.

An interaction is present when the total Hamiltonian cannot be written in this separable form and instead takes the general structure

$$H = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B + H_{\text{int}}, \tag{14}$$

with

$$H_{\text{int}} \neq 0. \tag{15}$$

The interaction term $H_{\text{int}}$ couples the degrees of freedom of both subsystems and cannot be eliminated by any change of variables or basis transformation.

In such cases, the Schrödinger equation

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle \tag{16}$$

generally leads to states $|\Psi(t)\rangle \in \mathcal{H}_{AB}$ that cannot be factorized as

$$|\Psi(t)\rangle \neq |\psi_A(t)\rangle \otimes |\psi_B(t)\rangle\,, \tag{17}$$

reflecting the emergence of correlations and, in general, quantum entanglement between the subsystems.

From this viewpoint, the appearance of cross terms in the Hamiltonian is not the definition of interaction but rather its mathematical signature, encoding the intrinsic dynamical dependence between the two quantum systems.

## 7.1 Example of interacting systems in classical mechanics

A simple and illustrative example of interacting systems in classical mechanics is provided by two particles of masses $m_1$ and $m_2$ moving along a single spatial dimension, with positions $x_1(t)$ and $x_2(t)$, coupled by an ideal spring of elastic constant $k$.

The Lagrangian of the system is given by

$$\boxed{L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k (x_1 - x_2)^2.} \tag{18}$$

The interaction between the two systems is encoded in the potential energy term

$$\boxed{V_{\text{int}} = \frac{1}{2} k (x_1 - x_2)^2,} \tag{19}$$

which couples the degrees of freedom of both particles. As a consequence, the force acting on each particle depends explicitly on the position of the other one.

Applying the Euler–Lagrange equations yields the equations of motion

$$m_1 \ddot{x}_1 = -k(x_1 - x_2), \qquad m_2 \ddot{x}_2 = -k(x_2 - x_1). \tag{20}$$

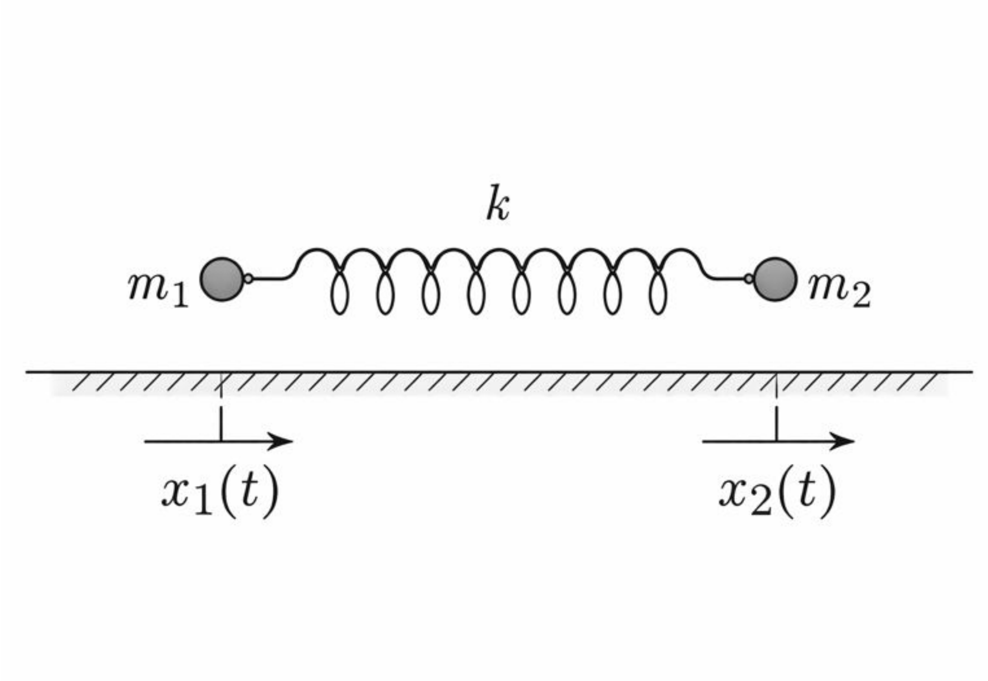


Figure 3: Two point masses $m_1$ and $m_2$ interacting through an ideal spring of elastic constant $k$.

These equations clearly show that the dynamics of each particle cannot be described independently.

Expanding the interaction potential,

$$(x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2, \tag{21}$$

reveals the presence of a cross term proportional to $x_1x_2$, which provides a mathematical signature of the interaction in these coordinates.

Although a change of variables to center-of-mass and relative coordinates allows the equations of motion to be decoupled, the interaction remains physically present and manifests itself through the normal mode frequencies of the system. This example illustrates that interaction is not defined by the appearance of cross terms in a particular set of variables, but by the intrinsic coupling of the dynamical degrees of freedom.

## 7.2 Decoupling of the two-mass system

Consider a system of two point masses $m_1$ and $m_2$ moving along a line and coupled by an ideal spring of elastic constant $k$. Let $x_1(t)$ and $x_2(t)$ denote their positions. The Lagrangian is

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}k(x_1 - x_2)^2. \tag{22}$$

The corresponding equations of motion are

$$m_1\ddot{x}_1 = -k(x_1 - x_2), \qquad m_2\ddot{x}_2 = -k(x_2 - x_1), \tag{23}$$

which are manifestly coupled in the coordinates $x_1$ and $x_2$.

Despite the presence of an interaction, the system can be decoupled by introducing the center-of-mass and relative coordinates

$$X = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, \qquad x = x_1 - x_2. \tag{24}$$

In terms of these variables, the Lagrangian takes the form

$$L = \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}kx^2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}, \tag{25}$$

and the equations of motion become

$$(m_1 + m_2)\ddot{X} = 0, \qquad \mu\ddot{x} + kx = 0. \tag{26}$$

The dynamics thus separates into a free motion of the center of mass and a harmonic oscillation of the relative coordinate with frequency

$$\omega = \sqrt{\frac{k}{\mu}}. \tag{27}$$

This example illustrates an important conceptual point: although the system is genuinely interacting, the equations of motion can be decoupled due to the linear nature of the interaction. Decoupling is therefore a property of the chosen variables and does not imply the absence of interaction.

# 8 Conclusion

The correct mathematical description of a composite quantum system is obtained by taking the tensor product of the Hilbert spaces of the individual subsystems. While separable states correspond to simple products of wave functions, generic states are entangled and cannot be factorized. Interactions play a crucial role in the creation of such entangled states, highlighting the fundamentally non-classical nature of composite quantum systems.