The Einbein

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1 The Einbein

In theoretical physics, especially in string theory and the formulation of relativistic point particles, the Einbein is an auxiliary field introduced to simplify or generalize the action. The term Einbein comes from German and means "one leg," analogous to the vierbein (four-leg) in general relativity.

1.1 Einbein in the Relativistic Point Particle

The standard action for a relativistic point particle of mass m m moving in a space-time with metric

$$S = -m \int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

Where τ is the *proper time* of the particle. This action describes the relativistic movement of a particle in space-time.

The Lagrangian presents an inconvenience that is not polynomial in velocities $\dot{x}^\mu := \frac{dx^\mu}{d\tau}$ and also that the equation is not manifestly invariant i.e., that under changes of parametrization of time τ the action remains equal in form.

To avoid this complication, one introduces an auxiliary field , called **The Einbein**, leading to the Polyakov-like action for a mass particle:

$$S = \frac{1}{2} \int d\tau (e^{-1} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - em^2)$$

Where:

- $e(\tau)$ is a function and plays a role similar to a 1D metric along the worldline.
- This new action is polynomial in $\dot{x}^\mu$ which simplifies quantization.

- Varying with respect to $e(\tau)$ gives the constraint that relates to the original square-root form.

Also, in string theory, the worldsheet metric h_{ab} plays a similar role to The Einbein. In fact, the Einbein can be viewed as the 1-D analogue of the worldsheet metric.

1.2 Gauge Symmetry

The Einbein formulation introduces a gauge symmetry under worldline reparameterizations:

$$\tau \to \tau'(\tau),\ x^\mu \to x'^\mu(\tau'),\ e(\tau')d\tau' = e(\tau)d\tau$$

Rewriting the action as:

$$S[x^\mu, e] = \frac{m}{2}\int\left(\frac{dx_\mu}{ed\tau}\frac{dx^\mu}{ed\tau} - c^2\right)ed\tau \tag{1}$$

$$= \frac{m}{2}\int\left(\frac{dx'_\mu}{ed\tau'}\frac{dx'^\mu}{ed\tau'} - c^2\right)ed\tau' \tag{2}$$

This symmetry allows one to fix $e(\tau)$ to a convenient value, such as $e = 1$ or a constant.

The Einbein is a powerful tool for writing reparameterization-invariant actions in a polynomial form. It is widely used in:

- Relativistic particle models (massive and massless)
- First-quantized approaches to quantum field theory
- BRST quantization
- Supersymmetric models (e.g., superparticle)

2 Variation of $e(\tau)$

We consider the action for a relativistic point particle of mass m m in a spacetime with metric $g_{\mu\nu}(x)$:

$$S[x^\mu, e] = \frac{1}{2}\int d\tau(e^{-1}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - em^2)$$

We first vary the action with respect to the Einbein $e(\tau)$:

$$\delta S = -\frac{1}{2}\int d\tau \delta e(e^{-2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + m^2)$$

Setting $\delta S = 0$ for arbitrary δe gives the **constrain equation**

$$\boxed{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + e^2 m^2 = 0}$$

This is a constraint that must be satisfied along the trajectory.

These equations tell us that the norm of the velocity is $-e^2 m^2$:

$$\frac{\dot{p}^2}{m^2} + e^2 m^2 = 0$$

where $p := m\dot{x}$ and $p^2 = m^2 \dot{x}^2 := m^2 x^\nu g_{\mu\nu} \dot{x}^\nu$ i.e

$$\boxed{e = \sqrt{-\frac{p^2}{m^4}} = \frac{1}{m}\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}}$$

3 Recover the Standard Square Root Action

We can plug this back into the Einbein action:

$$S = \frac{1}{2}\int d\tau \left(\frac{1}{e} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - em^2\right) \tag{3}$$

$$= \frac{1}{2}\int d\tau \left(m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + m\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\right) \tag{4}$$

$$= -m\int d\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \tag{5}$$

$$= -m\int ds \tag{6}$$

This is the standard action with which we begin.

4 Variation with Respect to x^λ

Now vary the action with respect to the worldline coordinate $x^\lambda(\tau)$:

$$\delta S = \frac{1}{2}\int d\tau (e^{-1}\delta g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + 2e^{-1} g_{\mu\nu}\delta\dot{x}^\mu \dot{x}^\nu)$$

Apply integration by parts to the second term and simplify, and you get the equation of motion

$$\boxed{\frac{d}{d\tau}(e^{-1}g_{\lambda\nu}\dot{x}^{\nu}) - \frac{1}{2}e^{-1}\partial_{\lambda}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = 0}$$

After a little more calculation, it can be written as:

$$\boxed{\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = \frac{\dot{e}}{e}\dot{x}^{\mu}}$$

which is the **equaton of motion of the particle**

5 What is the Einbein?

The Einbein $e(\tau)$ is a **dynamical scalar field** on the 1D worldline of a particle. You can think of it as the 1D analogue of a metric on the worldline, just as the vielbein (vierbein) generalizes the metric in general relativity. It allows you to decouple the coordinate τ from physical proper time and gives you a gauge of freedom under reparameterizations.

6 Purpose and Benefits

The original relativistic particle action is:

$$S = -m\int d\tau\sqrt{-\dot{x}^2}$$

It has a square root, which makes the action non-polynomial and harder to work with (e.g., in canonical quantization).

By introducing The Einbein:

$$S = \frac{1}{2}\int d\tau(e^{-1}\dot{x}^2 - em^2)$$

It removes the square root and gets a quadratic action in velocities. This is essential for path integrals and **BRST quantization**.