On the Functional Derivative of the Scalar Field Action and the Associated Global Functional

Omar Corona Tejeda

omarct1989@ciencias.unam.mx

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Abstract

We summarize the derivation of the functional derivative of the scalar field action on a pseudo-Riemannian manifold, emphasizing the relation with the Klein–Gordon operator, and compute the global quantity

$$G[\phi] := \int_M \frac{\delta S[\phi]}{\delta \phi}.$$

1 Covariant Action for a Real Scalar Field

Let (M, g) be an n-dimensional pseudo-Riemannian manifold. Consider the standard covariant action for a real scalar field ϕ:

$$S[\phi] = \int_M d^n x \, \sqrt{|g|} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \, \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{1}$$

2 Functional Derivative

Varying the action with respect to ϕ and integrating by parts gives the well-known result:

$$\frac{\delta S[\phi]}{\delta \phi(x)} = -\sqrt{|g(x)|} \, (\Box_g + m^2) \phi(x), \tag{2}$$

where $\Box_g = \nabla^\mu \nabla_\mu$ is the covariant d'Alembertian. Thus the Euler–Lagrange equation $\delta S / \delta \phi = 0$ yields the Klein–Gordon equation

$$(\Box_g + m^2) \phi = 0.$$

3 The Global Functional $G[\phi]$

Define

$$G[\phi] := \int_M \frac{\delta S[\phi]}{\delta \phi}. \tag{3}$$

Substituting (2) gives

$$G[\phi] = -\int_M d^n x \sqrt{|g|}\,(\Box_g + m^2)\phi. \tag{4}$$

3.1 Boundary Decomposition

Using the identity

$$\sqrt{|g|}\,\Box_g \phi = \nabla_\mu(\sqrt{|g|}\,\nabla^\mu \phi),$$

we integrate by parts:

$$G[\phi] = -\int_M d^n x \sqrt{|g|}\, m^2 \phi \;-\; \int_M \nabla_\mu(\sqrt{|g|}\,\nabla^\mu \phi)\, d^n x.$$

Applying Stokes' theorem yields

$$G[\phi] = -m^2 \int_M d^n x \sqrt{|g|}\,\phi \;-\; \int_{\partial M} d^{n-1} y \sqrt{|h|}\, n_\mu \nabla^\mu \phi, \tag{5}$$

where n_μ is the outward unit normal and h the induced metric on ∂M.

3.2 Closed Manifolds or Vanishing Boundary Terms

If M is closed or boundary terms vanish (e.g. Dirichlet or Neumann conditions), then

$$G[\phi] = -m^2 \int_M d^n x \sqrt{|g|}\,\phi. \tag{6}$$

4 Conclusion

Starting from the covariant scalar field action, the functional derivative naturally reproduces the Klein–Gordon operator. Integrating this functional derivative produces the global quantity $G[\phi]$, which consists of a bulk term proportional to the mass and a boundary flux term involving the normal derivative of ϕ.