August 23, 2020

Given the following elliptic problem (1) with Neumann boundary conditions (2) and holding the Integral compatibility condition (3)

$$\nabla \cdot (H \nabla \phi) = \nabla \cdot F \text{ on } \Omega \tag{1}$$

$$\nabla \phi \cdot \vec{n} = 0 \text{ on } \partial\Omega \tag{2}$$

$$F \cdot \vec{n} = 0 \text{ on } \partial\Omega \tag{3}$$

$$\Omega = [0, 0] \times [\pi, \pi]$$

Let

$$\nabla \cdot F = divF$$

.

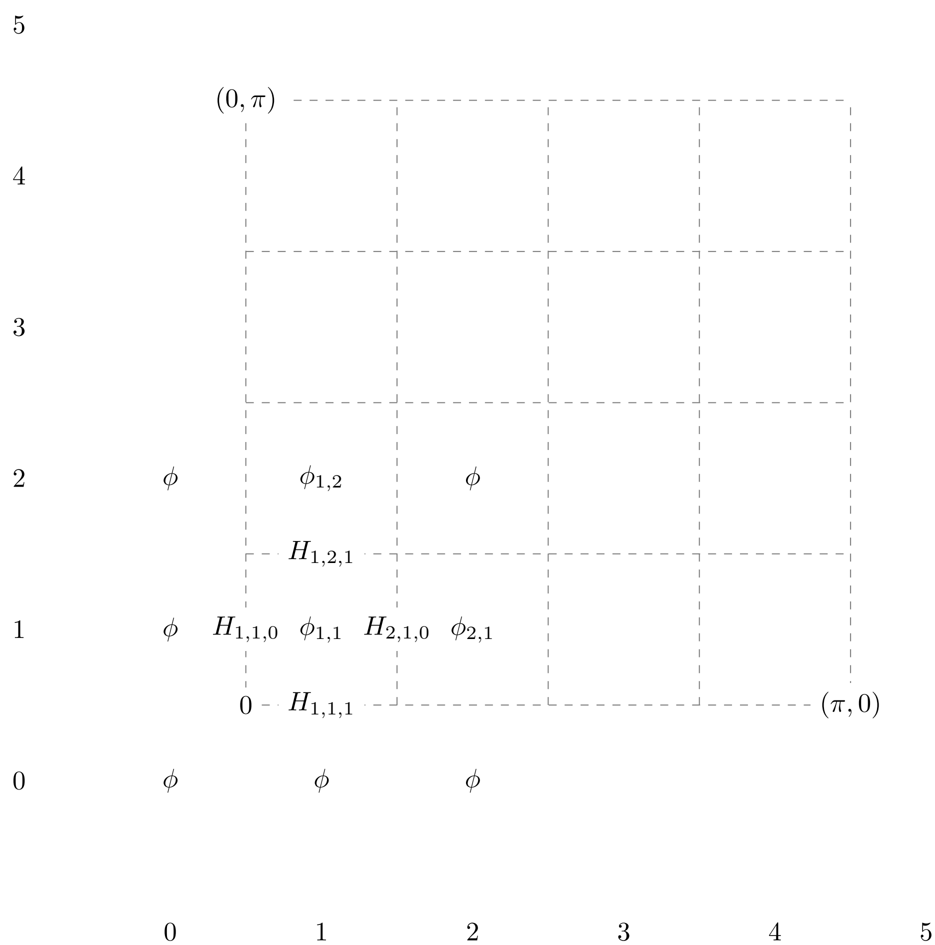
Then the approximate version of the above is

$$\begin{aligned}\nabla \cdot (H \nabla \phi)(x, y) \approx \frac{1}{h^2} \Bigg\{ & \phi(x + h, y) H\left(x + \frac{h}{2}, y\right) + \phi(x - h, y) H\left(x - \frac{h}{2}, y\right) \\ & + \phi(x, y + h) H\left(x, y + \frac{h}{2}\right) + \phi(x, y - h) H\left(x, y - \frac{h}{2}\right) \\ & - \phi(x, y) \left[H\left(x + \frac{h}{2}, y\right) + H\left(x - \frac{h}{2}, y\right) + H\left(x, y + \frac{h}{2}\right) + H\left(x, y - \frac{h}{2}\right) \right] \Bigg\}\end{aligned}$$

and

$$\begin{aligned}\nabla \cdot F(x, y) = divF(x, y) \approx \frac{1}{h} \Bigg\{ & F\left(x + \frac{h}{2}, y\right) - F\left(x - \frac{h}{2}, y\right) \\ & + F\left(x, y + \frac{h}{2}\right) - F\left(x, y + \frac{h}{2}\right) \Bigg\}\end{aligned} \tag{4}$$

Then by discretizing the space Ω with n=4, we have the grid below.



Ω is the area inside the dashed grid. Note that ϕ and $divF$ are situated in the same place and indexed in the same way. The same is true for H and F.

Thus (1) is equivalent to (5)

$$\frac{1}{h^2}\Big[\phi_{i+1,j}H_{i+1,j,0}+\phi_{i-1,j}H_{i,j,0}+\phi_{i,j+1}H_{i,j+1,1}+\phi_{i,j-1}H_{i,j,1} - \phi_{i,j}\big(H_{i+1,j,0}+H_{i,j,0}+H_{i,j+1,1}+H_{i,j,1}\big)\Big] = divF_{i,j} \quad (5)$$

where

$$divF_{i,j} = \frac{1}{h}\big[F_{i+1,j,0}-F_{i,j,0}+F_{i,j+1,1}-F_{i,j,1}\big]$$

Equation (2) is equivalent to the 4 equations below

$$
\begin{aligned}
\phi_{i,0} &= \phi_{i,1} \\
\phi_{i,n+1} &= \phi_{i,n} \\
\phi_{0,j} &= \phi_{1,j} \\
\phi_{n+1,j} &= \phi_{n,j}
\end{aligned}
$$

And equation (3) becomes simply

$$
\begin{aligned}
F_{1,j,0} &= 0 \\
F_{n+1,j,0} &= 0 \\
F_{i,1,1} &= 0 \\
F_{i,n+1,1} &= 0
\end{aligned}
$$