**Definition 0.1.** *The distribution function of a random variable $X$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the function $F_X : \mathbb{R} \to [0,1]$ given by*

$$F_X(x) = \mathbb{P}(X \leq x)\,.$$

**Proposition 0.2.** *The distribution function of a random variable has the following properties:*

*(1)* $F_X(\cdot)$ *is non-decreasing and hence has only jump discontinuities.*

*(2)* $\lim_{x\uparrow\infty} F_X(x) = 1$, $\lim_{x\downarrow -\infty} F_X(x) = 0$.

*(3)* $\lim_{h\downarrow 0} F_X(x+h) = F_X(x), \forall x \in \mathbb{R}$, *thus CDF is right continuous.*

*(4)* $\lim_{h\downarrow 0} F_X(x-h) = F_X(x) - P(X = x), \forall x \in \mathbb{R}$.

**Theorem 0.3.** *Let $F$ be a function from $\mathbb{R}$ to $[0,1]$ satisfying the properties of the above proposition, then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X$ defined on it whose distribution function is $F$.*

**Two Special Cases**

• There exists a non-negative function $f$ on $\mathbb{R}$ such that

$$\mu_X[a,b] = \mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx\,.$$

Thus

$$1 = \mathbb{P}(X \in \mathbb{R}) = \lim_{n\to\infty} \mathbb{P}(-n \leq X \leq n) = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx = \int_{-\infty}^{\infty} f(x)dx\,.$$

• X takes only countably many values $x_i$. Define $p_i = \mathbb{P}(X = x_i)$. Then

$$\mu_X(B) = \sum_{\{i: x_i \in B\}} p_i\,.$$

In the first case $X$ is said to have an absolutely continuous distribution with probability density function $f$ and in the second case $X$ is said to have a discrete distribution with probability mass function $\{p_i\}$.

**Example:** Consider the functions:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,, N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy\,.$$

Let $X$ be uniformly distributed on $[0,1]$. Notice that $N$ is a strictly increasing function. So it has an inverse $N^{-1}$. Define the random variable $Z = N^{-1}(X)$. Then

$$\begin{aligned}
\mu_Z[a,b] &= \mathbb{P}(\omega \in \Omega : a \leq Z(\omega) \leq b)\\
&= \mathbb{P}(\omega \in \Omega : a \leq N^{-1}(X)(\omega) \leq b)\\
&= \mathbb{P}(\omega \in \Omega : N(a) \leq NN^{-1}(X)(\omega) \leq N(b))\\
&= \mathbb{P}(\omega \in \Omega : N(a) \leq X(\omega) \leq N(b))\\
&= N(b) - N(a) = \int_a^b \varphi(x)dx\,.
\end{aligned}$$

The measure $\mu_X$ on $\mathbb{R}$ given by this formula is called the standard normal distribution. Any random variable that has this distribution, regardless of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which it is defined, is called a standard normal random variable.

## 0.1 Expectation

Let $X$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We want to compute an "average value" of $X$, where we take the probabilities into account while computing the average.
If $\Omega$ is countable then we can simply define

$$\text{average value" of X} := \mathbb{E}(X) := \sum_{k=0}^{\infty} X(w_k)\mathbb{P}(X = w_k)\,,$$

where $\Omega = \{w_1, w_2, \ldots\}$
But if $\Omega$ is uncountable then we must think in terms of integrals.

# 1 Riemann integration

**Partition**: Let $[a, b]$ be a closed and bounded interval. A partition of $[a, b]$ is a finite sequence $P = (x_0, x_1, \cdots, x_n)$ of points of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$. The family of all partitions of $[a, b]$ is denoted by $\mathcal{P}[a, b]$ and the partition $P = (x_0, x_1, \cdots, x_n)$ is a member of $\mathcal{P}[a, b]$.
For example, $P = (0, 1/4, 1/3, 1/2, 2/3, 3/4, 1)$ is a partition of $[0, 1]$, $Q = (0, 1/4, 3/8, 1/2, 3/4, 7/8, 1)$ is another partition of of $[0, 1]$.
**Riemann sums:-** Let $f : [a, b] \to \mathbb{R}$ be a bounded function on $[a, b]$. Let $P \in \mathcal{P}[a, b]$ (i.e, $P = (x_0, x_1, \cdots, x_n)$, where $a = x_0 < x_1 < \cdots < x_n = b$). Since $f$ is bounded on $[a, b]$, $f$ is bounded on $[x_{r-1}, x_r]$, for $r = 1, 2, \cdots, n$. Let $M = \sup\limits_{x\in[a,b]} f(x)$, $m = \inf\limits_{x\in[a,b]} f(x)$; $M_r = \sup\limits_{x\in[x_{r-1},x_r]} f(x)$, $m_r = \inf\limits_{x\in[x_{r-1},x_r]} f(x)$; for $r = 1, 2, \cdots, n$. Then $m \le m_r \le M_r \le M$, for $r = 1, 2, \cdots, n$. The sum $U(P, f) := \sum\limits_{i=1}^{n} M_r(x_r - x_{r-1})$ is said to be the upper Riemann sum and the sum $L(P, f) := \sum\limits_{i=1}^{n} m_r(x_r - x_{r-1})$ is said to be lower Riemann sum.

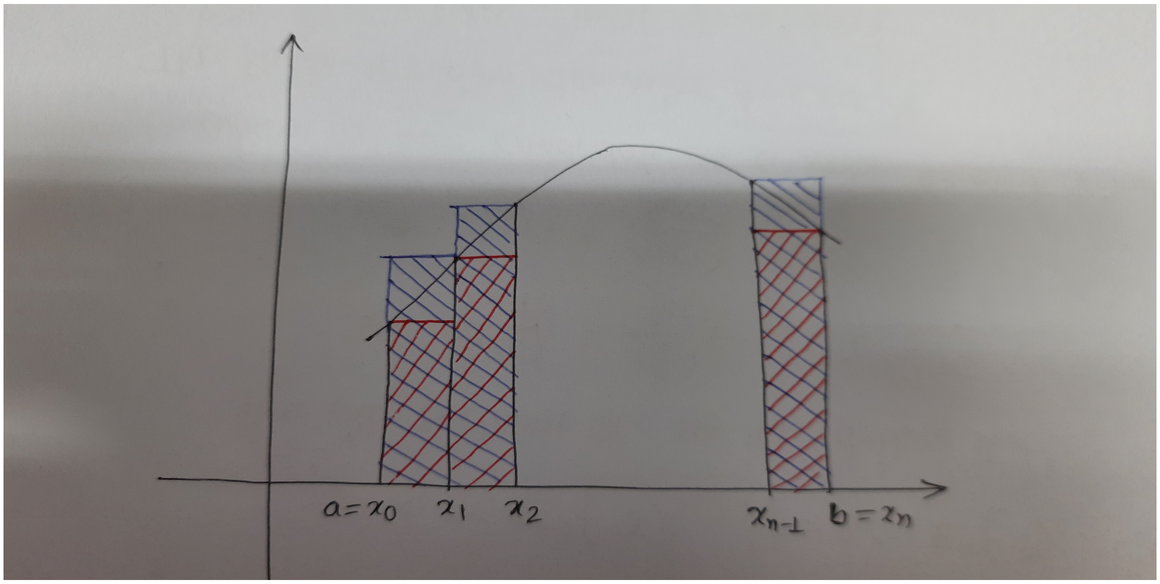


Figure 1:

Here $U(P, f)$ is the blue shaded area (region) and $L(P, f)$ is the red shaded area (region) of Figure 1. Note that $m(x_r - x_{r-1}) \leq m_r(x_r - x_{r-1}) \leq M_r(x_r - x_{r-1}) \leq M(x_r - x_{r-1})$, for $r = 1, 2, \cdots, n$. Therefore,

$$m \sum_{r=1}^{n}(x_r - x_{r-1}) \leq \sum_{r=1}^{n} m_r(x_r - x_{r-1}) \leq \sum_{r=1}^{n} M_r(x_r - x_{r-1}) \leq M \sum_{r=1}^{n}(x_r - x_{r-1}),$$

or, $m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$. We have two sets of real numbers $\{U(P, f) : P \in \mathcal{P}[a, b]\}$ and $\{L(P, f) : P \in \mathcal{P}[a, b]\}$ both sets are bounded. The supremum of the set $\{L(P, f) : P \in P \in \mathcal{P}[a, b]\}$ exists and it is called the lower integral of $f$ on $[a, b]$ and is denoted by $\underline{\int_a^b} f(x)dx$. The infimum of the set $\{U(P, f); P \in P \in \mathcal{P}[a, b]\}$ exists and it is called the upper integral of $f$ on $[a, b]$ and is denoted by $\overline{\int_a^b} f(x)dx$. $f$ is said to be Riemann integral on $[a, b]$ if

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$$

The common value is called the Riemann integral of $f$ on $[a, b]$ and it is denoted by $\int_a^b f(x)dx$.

**Exercise:-**

(1) Let $f(x) = c$, $x \in [a, b]$. Prove that $f$ is Riemann integral on $[a, b]$.

(2) A function $f$ is defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational.} \end{cases}$$

Show that $f$ is not Riemann integral on $[0, 1]$.

(3) Prove that the function $f$ is defined on $[a, b]$ by $f(x) = x$, $x \in [a, b]$ is Riemann integral on $[a, b]$. Evaluate $\int_a^b f(x)dx$.

(4) $f(x) = x^2$.

(5) $f(x) = e^x$.

**Refinement of a partition:-** Let $P = (x_0, x_1, x_2, \cdots, x_n)$ be a partition of $[a, b]$. A partition $Q$ of $[a, b]$ is said to be a refinement of $P$ if $P$ is a proper subset of $Q$. That is $Q$ is obtained by adjoining a finite number of additional points to $P$.

For example, let $P = (0, 1/4, 1/2, 3/4, 1)$ be a partition of $[0, 1]$ and $Q = (0, 1/8, 1/4, 1/2, 3/4, 7/8, 1)$, then $Q$ is a refinement of $P$. If $R = (0, 1/8, 1/4, 3/8, 1/2, 3/4, 1)$, then $R$ is a refinement of $P$ but not a refinement of $Q$.

**Lemma 1.1.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P$ be a partition of $[a, b]$. If $Q$ is a refinement of $P$, then $U(P, f) \geq U(Q, f)$ and $L(P, f) \leq L(Q, f)$.*

**Norm of partition:-** Let $P = (x_0, x_1, \cdots, x_n)$ be a partition of $[a, b]$. Then norm of a partition denoted by $\|P\|$, is defined by

$$\|P\| = \max_{r \in \{1, 2, \cdots, n\}} |x_r - x_{r-1}|.$$

If $Q$ is a refinement of $P$, then $\|Q\| \leq \|P\|$.

**Lemma 1.2.** *Let $f : [a,b] \to \mathbb{R}$ be bounded. If $\{P_n\}$ is a sequence of partition of $[a,b]$ such that $\|P_n\| \to 0$, then*

*(i)* $\lim_{n\to\infty} U(P_n, f) = \overline{\int_a^b} f$

*(ii)* $\lim_{n\to\infty} L(P_n, f) = \underline{\int_a^b} f$.

**Condition for integrability:-** Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f$ is integrable on $[a,b]$ if and only if for each $\varepsilon > 0$, there exists a partition of $P$ of $[a,b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $f$ is integrable on $[a,b]$ iff for each $\varepsilon > 0$ there exists a positive $\delta$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

for every partition $P$ of $[a,b]$ satifying $\|P\| \le \delta$.

**Properties:-**

(1) Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be both Riemann integrable on $[a,b]$. Then $f + g$ is Riemann integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(2) Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable on $[a,b]$ and $c \in \mathbb{R}$. Then $cf$ is integrable on $[a,b]$ and $\int_a^b cf = c\int_a^b f$.

(3) $|f|$, $f^2$, $f \cdot g$ are Riemann integrable. If $g \ge k > 0$ then $1/g$ is also Riemann integrable.

**Ex.** A function $f$ is defined by $f(x) = x^2$, $x \in [a,b]$, where $a > 0$. Find $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$. Deduce that $f$ is integrable on $[a,b]$.

**Ans:-** $f$ is bounded on $[a,b]$. Let $P_n = (a, a+h, a+2h, \cdots, a+nh)$ where $h = \frac{b-a}{n}$. Then $P_n$ is partition of $[a,b]$ with $\|P_n\| = \frac{b-a}{n}$. Since $f$ is increasing function on $[a,b]$,
$M_r = (a+rh)^2$, $m_r = [a+(r-1)h]^2$ for $r = 1, 2, \cdots, n$.

$$\begin{aligned}
U(P_n, f) &= h\Big[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2\Big] \\
&= h\Big[(a^2 + a^2 + \cdots + a^2) + 2ah(1 + 2 + 3 + \cdots + n) + h^2(1^2 + 2^2 + 3^3 + \cdots + n^2)\Big] \\
&= h\Big[na^2 + 2ah\frac{n(n+1)}{2} + h^2\frac{n(n+1)(2n+1)}{6}\Big] \\
&= nha^2 + anh(nh+h) + \frac{nh(nh+h)(2nh+h)}{6} \\
&= (b-a)a^2 + a(b-a)^2(1 + \frac{1}{n}) + \frac{1}{6}(b-a)^3(1 + \frac{1}{n})(2 + \frac{1}{n})
\end{aligned}$$

and

$$\begin{aligned}
L(P_n, f) &= h\Big[a^2 + (a+h)^2 + (a+2h)^2 + \cdots + (a+(n-1)h)^2\Big] \\
&= h\Big[na^2 + 2ah\frac{n(n-1)}{2} + h^2\frac{n(n-1)(2n-1)}{6}\Big] \\
&= nha^2 + anh(nh-h) + \frac{nh(nh-h)(2nh-h)}{6} \\
&= (b-a)a^2 + a(b-a)^2(1 - \frac{1}{n}) + \frac{1}{6}(b-a)^3(1 - \frac{1}{n})(2 - \frac{1}{n}).
\end{aligned}$$

Consider the sequence of partitions $\{P_n\}$ of $[a,b]$ with $\lim_{n\to\infty}\|P_n\| = \lim_{n\to\infty}\frac{b-a}{n} = 0$. Then $\overline{\int_a^b} f(x)dx = \lim_{n\to\infty} U(P_n, f) = (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}$ and

$$\begin{aligned}\underline{\int_a^b} f(x)dx &= \lim_{n\to\infty} L(P_n, f)\\ &= (b-a)a^2 + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3-a^3}{3}.\end{aligned}$$

As $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, $f$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = \frac{b^3-a^3}{3}$.

**Ex.** A function $f$ is defined on $[0,1]$ by

$$f(x) = \begin{cases} x \text{ if } x \in [0,1]\cap\mathbb{Q} \\ 0 \text{ if } x \in [0,1]\backslash\mathbb{Q}.\end{cases}$$

Find $\underline{\int_0^1} f(x)dx$ and $\overline{\int_0^1} f(x)dx$. Deduce that $f$ is not integrable on $[0,1]$.

**Ans:-** $f$ is bounded on $[0,1]$. Let us take the partition $P_n$ of $[0,1]$ defined by $P_n = (0, 1/n, 2/n, \cdots, n/n)$. Let $M_r = \sup_{x\in[\frac{r-1}{n},\frac{r}{n}]} f(x)$, $m_r = \inf_{x\in[\frac{r-1}{n},\frac{r}{n}]} f(x)$, for $r = 1, 2, \cdots, n$. Then $M_r = r/n$ and $m_r = 0$ for $r = 1, 2, \cdots n$.

$$\begin{aligned}U(P_n, f) &= M_1(\frac{1}{n} - 0) + M_2(\frac{2}{n} - \frac{1}{n}) + \cdots M_n(\frac{n}{n} - \frac{n-1}{n})\\ &= \frac{1}{n}[\frac{1}{n} + \frac{2}{n} + \cdots \frac{n}{n}]\\ &= \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}\end{aligned}$$

and

$$L(P_n, f) = m_1(\frac{1}{n} - 0) + m_2(\frac{2}{n} - \frac{1}{n}) + \cdots m_n(\frac{n}{n} - \frac{n-1}{n}) = 0.$$

Let us consider the sequence of partitions $\{P_n\}$ of $[0,1]$ with $\|P_n\| = \frac{1}{n}$ and $\lim_{n\to\infty}\|P_n\| = 0$. Then $\lim_{n\to\infty} U(P_n, f) = \overline{\int_0^1} f(x) = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}$ and $\lim_{n\to\infty} L(P_n, f) = \underline{\int_0^1} f(x)dx = 0$. Since $\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx$, $f$ is not Reimann integrable on $[0,1]$.

## 2 Lebesgue Integral

**Definition:-** A random variable $s : \Omega \to [0,\infty)$ is defined by $s(\omega) = \sum_{i=1}^{n} a_i\mathcal{X}_{A_i}(\omega)$, $\omega \in \Omega$, where $n$ is some positive integer, $a_1, a_2, \cdots, a_n$ are non-negative real-numbers, $A_i \in \mathcal{F}$ for every $i$; $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{n} A_i = \Omega$. Such a function $s$ is called a non-negative simple random variable. We say that $\sum_{i=1}^{n} a_i\mathcal{X}_{A_i}(\omega)$ is the standard representation of $s$ if $a_1, a_2, \cdots, a_n$ are all distinct. We denote by $\mathbb{L}_0^+$ the class of all non-negative simple random variables on $(\Omega, \mathcal{F})$.

**Examples**:

- If $s(\omega) \equiv c$ for some $c \in [0,\infty)$, then $s \in \mathbb{L}_0^+$.

- For $A \subset \Omega$, consider $\mathcal{X}_A : \Omega \to [0, \infty)$, the indicator function of the set $A$, i.e.,

$$\mathcal{X}(\omega) = \begin{cases} 0 \text{ if } \omega \notin A \\ 1 \text{ if } \omega \in A. \end{cases}$$

  Then $\mathcal{X}_A \in \mathbb{L}_0^+$ iff $A \in \mathcal{F}$.

- Let $A, B \in \mathcal{F}$. then $s = \mathcal{X}_A \mathcal{X}_B \in \mathbb{L}_0^+$ since $s = \mathcal{X}_{A \cap B}$.

- Let $A, B \in \mathcal{F}$. If $A \cap B = \emptyset$, then clearly, $\mathcal{X}_A + \mathcal{X}_B = \mathcal{X}_{A \cup B} \in \mathbb{L}_0^+$.

**Definition:-** For $s \in \mathbb{L}_0^+$ with a representation $s = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$, we define $\int_\Omega s(\omega) d\mathbb{P}(\omega)$, the integral of $s$ with respect to $\mathbb{P}$, by $\int_\Omega s(\omega) d\mathbb{P}(\omega) := \sum_{i=1}^{n} a_i \mathbb{P}(A_i)$.

We should check that $\int_\Omega s(\omega) d\mathbb{P}(\omega)$ is well defined i.e., if $s = \sum_{i=1}^n a_i \mathcal{X}_{A_i} = \sum_{j=1}^m b_i \mathcal{X}_{B_j}$ where $\{A_1, A_2, \cdots, A_n\}$ and $\{B_1, B_2, \cdots, B_m\}$ are partitions of $\Omega$ by elements of $\mathcal{F}$, then

$$\sum_{i=1}^{n} a_i \mathbb{P}(A_i) = \sum_{b=1}^{m} b_j \mathbb{P}(B_j).$$

For this, we note that we can write

$$s = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mathcal{X}_{A_i \cap B_j} = \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mathcal{X}_{A_i \cap B_i}.$$

Thus if $A_i \cap B_j \neq \emptyset$ then $a_i = b_j$. Hence using finite additivity of $\mathbb{P}$,

$$\begin{aligned} \sum_{i=1}^{n} a_i \mathbb{P}(A_i) &= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \mathbb{P}(A_i \cap B_j) \\ &= \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \mathbb{P}(A_i \cap B_j) = \sum_{j=1}^{m} b_j \mathbb{P}(B_j). \end{aligned}$$

Thus $\int_\Omega s(\omega) d\mathbb{P}(\omega)$ is independent of the representation of $s = \sum_{i=1}^n a_i \mathcal{X}_{A_i}$.

**Proposition 2.1.** *For $s, s_1, s_2 \in \mathbb{L}_0^+$ and $\alpha \in \mathbb{R}$ with $\alpha \geq 0$, the following hold:*

*(i)* $0 \leq \int_\Omega s \, d\mathbb{P} < +\infty$

*(ii)* $\alpha s \in \mathbb{L}_0^+$ *and* $\int_\Omega (\alpha s) \, d\mathbb{P} = \alpha \int_\Omega s \, d\mathbb{P}$

*(iii)* $s_1 + s_2 \in \mathbb{L}_0^+$ *and* $\int_\Omega (s_1 + s_2) \, d\mathbb{P} = \int_\Omega s_1 \, d\mathbb{P} + \int_\Omega s_2 \, d\mathbb{P}$.

*Proof.* Statements (i) and (ii) are obvious.

For (iii), let $s_1 = \sum_{i=1}^{n} a_i \mathcal{X}_{A_i}$ and $s_2 = \sum_{j=1}^m b_j \mathcal{X}_{B_j}$. Then we can write $s_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \mathcal{X}_{A_i \cap B_j}$ and $s_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \mathcal{X}_{A_i \cap B_j}$. Thus $s_1 + s_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) \mathcal{X}_{A_i \cap B_j}$. Hence $s_1 + s_2 \in \mathbb{L}_0^+$ and using these representations, it is clear that $\int_\Omega (s_1 + s_2) d\mathbb{P} = \int_\Omega s_1 d\mathbb{P} + \int_\Omega s_2 d\mathbb{P}$. □

**Excercise:-** Let $s_1, s_2 \in \mathbb{L}_0^+$. Then prove the followings

1. Let $s_1 \geq s_2$. Set $\phi = s_1 - s_2$. Show that $\phi \in \mathbb{L}_0^+$.

2. If $s_1 \geq s_2$, then $\int s_1 d\mathbb{P} \geq \int s_2 d\mathbb{P}$.

**Proposition 2.2.** *Let $X : \Omega \to \mathbb{R}$ a non-negative bounded random variable, then there exists a sequence $\{s_n\}_{n\geq 1}$ of random variables in $\mathbb{L}_0^+$ such that $\mathbb{P}\{\omega \in \Omega : \lim_{n\to\infty} s_n(\omega) = X(\omega)\} = 1$.*

*Proof.* Let $X$ be bounded by $M$. Then the sets $A_k^n = \{\omega : \dfrac{(k-1)M}{2^n} \leq X(\omega) < \dfrac{kM}{2^n}\}$, $1 \leq k \leq 2^n$. Then $\{A_k^n\}$ are disjoint, $A_k^n \in \mathcal{F}$ and have union $\cup_{k=1}^{2^n} A_k^n = \Omega$. We define function $s_n$ on $\Omega$ by

$$s_n(\omega) = \sum_{k=1}^{2^n} \frac{M(k-1)}{2^n} \mathcal{X}_{A_k^n}(\omega).$$

Clearly, $s_n \in \mathbb{L}_0^+$ and it is easy to check that for every $n$,

$$s_n(\omega) \leq s_{n+1}(\omega),\ \forall \omega \in \Omega.$$

If $\omega \in A_k$ for some $k$, $1 \leq k \leq 2^n$. Then

$$s_n(x) = \frac{(k-1)M}{2^n}$$

and $X(\omega) \in \left[\frac{(k-1)M}{2^n}, \frac{kM}{2^n}\right)$. Thus we have $s_n(\omega) \leq X(\omega)$ and $X(\omega) - s_n(\omega) \leq \frac{M}{2^n}$. In other words, $\lim_{n\to\infty} s_n(\omega) = X(\omega)$. $\square$

Consider the case n=1
Then $A_1^1 = [a, a_1] \cup (a_2, b]$
$A_2^1 = [a_1, a_2]$
$s_1 = 0.\mathcal{X}_{A_1^1} + \frac{M}{2}\mathcal{X}_{A_2^1}$.

Consider the case n=2
$A_1^2 = [a, a_2^{'}) \cup (a_2^{''}, b]$, $A_2^2 = [a_2^{'}, a_1) \cup (a_2, a_2^{''}]$, $A_3^2 = [a_1, c_2) \cup (c_2^{'}, a_2]$, $A_4^2 = [c_2, c_2^{'}]$.
$s_2 = 0.\mathcal{X}_{A_1^2} + \frac{M}{4}\mathcal{X}_{A_2^2} + \frac{M}{2}\mathcal{X}_{A_3^2} + \frac{3M}{4}\mathcal{X}_{A_4^2}$.

$$\begin{aligned}\int_\Omega s_1\ d\mathbb{P}(\omega) &= 0.\mathbb{P}(A_1^{'}) + \frac{M}{2}\mathbb{P}(A_2^{'})\\ &= \frac{M}{2}[a_2 - a_1].\end{aligned}$$

$$\int_\Omega s_2\ d\mathbb{P}(\omega) = \frac{M}{4}[(a_1 - a_2^{'}) + (a_2^{''} - a_2)] + \frac{M}{2}[(c_2 - a_1) + (a_2 - c_2^{'})] + \frac{3M}{4}(c_2^{'} - c_2).$$

Set $\mathbb{L}^+ = \{f : \Omega \to [0, \infty) : \exists$ an increasing sequence of random variables $\{s_n\}_{n\geq 1}$ in $\mathbb{L}_0^+$ such that $s_n(\omega)$ converges to $s(\omega)$ almost surely$\}$. For $f \in \mathbb{L}^+$, we define the integral of $f$ w.r.t. $\mathbb{P}$ by

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) := \lim_{n\to\infty} \int_\Omega s_n(\omega) d\mathbb{P}(\omega).$$

**Proposition 2.3.** *If $f \in \mathbb{L}^+$ and $s \in \mathbb{L}_0^+$ are such that $0 \leq s \leq f$, then $\int_\Omega s(\omega) d\mathbb{P}(\omega) \leq \int_\Omega f(\omega) d\mathbb{P}(\omega)$ and $\int_\Omega f(\omega) d\mathbb{P}(\omega) = \sup\{\int_\Omega s d\mathbb{P} | 0 \leq s \leq f, s \in \mathbb{L}_0^+\}$.*

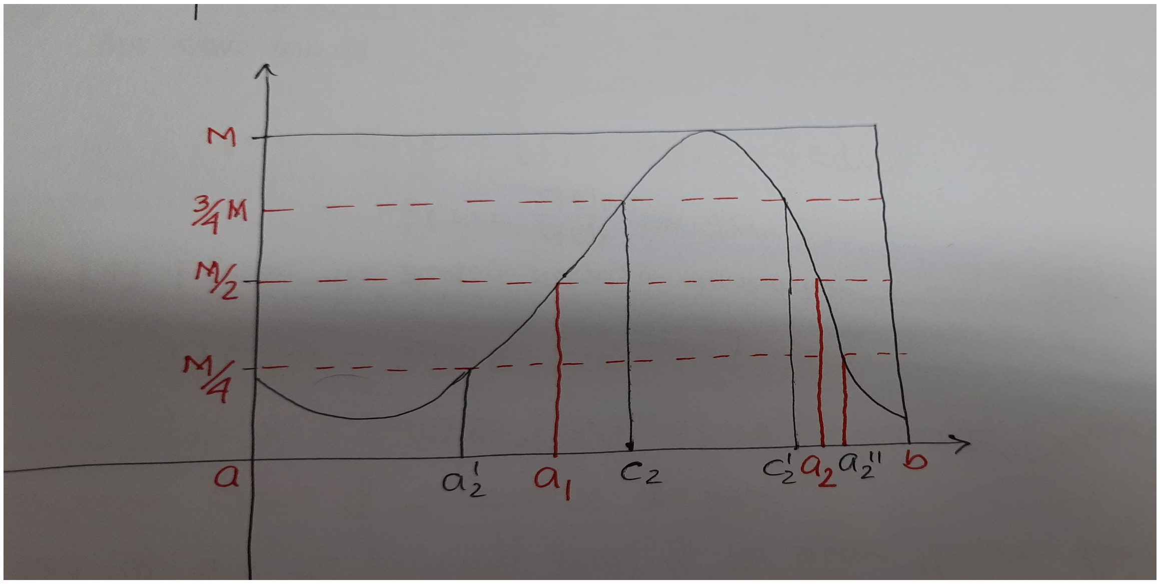


Figure 2:

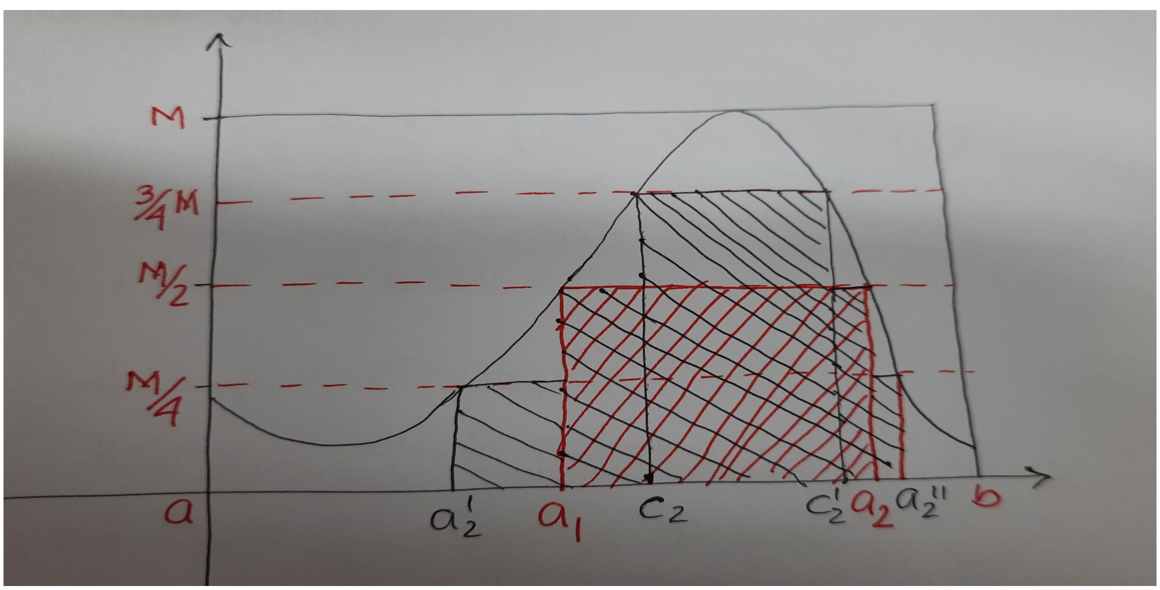


Figure 3:

Now for any random variable $X$, define

$$X^+(\omega) = \max\{X(\omega), 0\},\ X^- = \max\{-X(\omega), 0\}.$$

Then

$$X = X^+ - X^-.$$

We can define $\int_\Omega X^+ d\mathbb{P}(\omega)$ and $\int_\Omega X^- d\mathbb{P}(\omega)$ provided both of them are not infinite. Then we define

$$\int_\Omega X(\omega)d\mathbb{P}(\omega) = \int_\Omega X^+(\omega)d\mathbb{P}(\omega) - \int_\Omega X^-(\omega)d\mathbb{P}(\omega).$$

We say that $X$ is integrable if both $\int_\Omega X^+(\omega)d\mathbb{P}(\omega)$ and $\int_\Omega X^-(\omega)d\mathbb{P}(\omega)$ are finite. If both are infinite, then $\int_\Omega X(\omega)d\mathbb{P}(\omega)$ is not defined. If $\int_\Omega X^+(\omega)d\mathbb{P}(\omega)$ is finite and $\int_\Omega X^-(\omega)d\mathbb{P}(\omega) = \infty$ then $\int_\Omega X(\omega)d\mathbb{P}(\omega) = -\infty$. If $\int_\Omega X^-(\omega)d\mathbb{P}(\omega)$ is finite and $\int_\Omega X^+(\omega)d\mathbb{P}(\omega) = \infty$ then $\int_\Omega X(\omega)d\mathbb{P}(\omega) = \infty$.

# 3 Comparison of Riemann and Lebesgue integrals:-

Let $f$ be a bounded function defined on $\mathbb{R}$, and let $a < b$ be numbers.

1. The Riemann integral $\int_a^b f(x)dx$ is defined iff the set of points $x \in [a, b]$ where $f(x)$ is not continuous has Lebesgue measure zero.
2. If the Riemann integral $\int_a^b f(x)dx$ is defined, then $f$ is Borel measurable and so the Lebesgue integral $\int_a^b f(x)dx$ is also defined and the Riemann and Lebesgue integrals agree.

**Definition:-** Let $X$ be an integrable random variable. Then the expectation of $X$ is defined by $\mathbb{E}[X] := \int_\Omega X(\omega)d\mathbb{P}(\omega)$.

If $X \geq 0$, then $\mathbb{E}[X]$ is always defined [can be $+\infty$ as well].

**Examples:-**

1. Consider the infinite independent coin-toss space $(\Omega_\infty, \mathcal{F}_\infty, \mathbb{P}_\infty)$ with $p = \frac{1}{2}$. Let

$$Y_n(\omega) = \begin{cases} 1 \text{ if } \omega_n = H \\ 0 \text{ if } \omega_n = T. \end{cases}$$

$$\begin{aligned} \mathbb{E}[Y_n] &= 1 \cdot \mathbb{P}(Y_n = 1) + 0 \cdot \mathbb{P}(Y_n = 0) \\ &= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

2. Let $\Omega = [0, 1], \mathcal{F} = \mathcal{B}[0, 1]$ and let $\mathbb{P}$ be the Lebesgue measure on $[0, 1]$. Consider the random variable

$$X(\omega) = \begin{cases} 1 \text{ if } \omega \text{ is irrational} \\ 0 \text{ if } \omega \text{ is rational.} \end{cases}$$

$$
\begin{aligned}
\mathbb{E}[X] =& 1 \cdot \mathbb{P}(X=1) + 0 \cdot \mathbb{P}(X=0) \\
&= 1 \cdot \mathbb{P}(\omega \in [0,1]\backslash \mathbb{Q}) + 0 \cdot \mathbb{P}(\omega \in [0,1] \cap \mathbb{Q}) \\
&= 1 \cdot 1 + 0 \cdot 0 = 1.
\end{aligned}
$$

**Properties:-**

1. If $X$ takes only finitely many values $x_0, x_1, \cdots, x_n$, then
$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega) = \sum_{k=0}^{n} x_k \mathbb{P}(X = x_k).$$

2. The random variable $X$ is integrable iff $\int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty$.

3. If $X \leq Y$ and $X$ and $Y$ are integrable then
$$\int_\Omega X(\omega) d\mathbb{P}(\omega) \leq \int_\Omega Y(\omega) d\mathbb{P}(\omega).$$
**Note:** $|X| = X^+ + X^-$, $X^+ \leq |X|$, $X^- \leq |X|$.

4. If $\alpha$ and $\beta$ are real constant and $X$ and $Y$ are integrable or if $\alpha, \beta$ are non-negative constant and $X$ and $Y$ are non-negative. Then
$$\mathbb{E}[\alpha X + \beta Y] = \alpha \mathbb{E}[X] + \beta \mathbb{E}[Y].$$

**Two important convergence theorems:-**

**Definition:-** Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of random variables, all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $X$ be another random variable defined on the same space. We say that $X_1, X_2, \cdots$ converges to $X$ almost surely and write $\lim_{n\to\infty} X_n = X$ a.s. if $\mathbb{P}\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\} = 1$.

**Monotone convergence theorem:-** Let $X_1, X_2, \cdots$ be a sequence of random variables converging almost surely to another random variable $X$. If $0 \leq X_1 \leq X_2 \leq X_3 \cdots$ almost surely, then
$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

**Corollary 3.1.** *Suppose the non-negative random variable $X$ takes countable many values $x_0, x_1 \cdots$, then*
$$\mathbb{E}[X] = \sum_{k=0}^{\infty} x_k \mathbb{P}(X = x_k).$$

*Proof.* Let $A_k = \{X = x_k\}$. Then $X$ can be written as
$$X = \sum_{k=0}^{\infty} x_k \mathcal{X}_{A_k}.$$

Define
$$X_n = \sum_{k=0}^{n} x_k \mathcal{X}_{A_k}.$$
Then $0 \leq X_1 \leq X_2 \leq \cdots$ and $\lim_{n\to\infty} X_n = X$. Note that
$$\mathbb{E}[X_n] = \sum_{k=0}^{n} x_k \mathbb{P}(X = x_k).$$

Using Monotone convergence theorem, we obtain

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} \sum_{k=0}^{n} x_k \mathbb{P}(X = x_k) = \sum_{k=0}^{\infty} x_k \mathbb{P}(X = x_k).$$

$\square$

***Dominated Convergence Theorem:-*** *let $X_1, X_2, \cdots$ be a sequence of random variables converging almost surely to a random variable $X$. If there is another random variable $Y$ such that $\mathbb{E}[Y] < \infty$ and $|X_n| \leq Y$ almost surely for every $n$, then*

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}[X].$$

1. Consider the space $(\Omega, \mathcal{B}[0,1], \mathcal{L})$, where $\mathcal{L}$ is the Lebesgue measure. Define

$$X_n(\omega) = \begin{cases} n \text{ if } \omega \in [0, \frac{1}{n}) \\ 0 \text{ otherwise.} \end{cases}$$

Then $\lim_{n\to\infty} X_n(\omega) = 0$ a.s.

$$\lim_{n\to\infty} \int_0^1 X_n(\omega) d\mathbb{P}(\omega) = \lim_{n\to\infty} n \cdot \frac{1}{n} = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} X_n(\omega) d\mathbb{P}(\omega)$$

2. Consider a sequence of normal densities, each with mean zero and and the $n^{\text{th}}$ having variance $\frac{1}{n}$.

$$f_n(x) = \sqrt{\frac{n}{2\pi}} e^{\frac{-nx^2}{2}}.$$

If $x \neq 0$, then $\lim_{n\to\infty} f_n(x) = 0$ but

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} = \infty.$$

Therefore

$$f_n(x) \to 0 \text{ a.s.}$$

and

$$\int_{-\infty}^{\infty} f_n(x) dx = 1 \neq 0 = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(x) dx.$$