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- Since $\sum_i a_i = \sum_j d_j$
  if $\mathbf{x} = (x_{ij})_{mn \times 1}$ satisfies any $(m + n - 1)$ equations then it automatically satisfies all the $(m + n)$ equations.

- The constraints are of the form
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$$A_{(m+n)\times mn} = \begin{bmatrix} \underset{n}{\overbrace{111..11}} & \mathbf{0_n} & \mathbf{0_n} & \dots & . & \mathbf{0_n} \\ \mathbf{0_n} & \underset{n}{\overbrace{111..11}} & \mathbf{0_n} & \dots & . & \mathbf{0_n} \\ \mathbf{0_n} & \mathbf{0_n} & \underset{n}{\overbrace{111..11}} & \mathbf{0_n} & . & \mathbf{0_n} \\ . & . & . & . & \dots & . \\ \mathbf{0_n} & . & . & \dots & \mathbf{0_n} & \underset{n}{\overbrace{111..11}} \\ \overbrace{100...0} & \overbrace{100...0} & . & . & \dots & \overbrace{100...0} \\ \overbrace{010...0} & \overbrace{010...0} & . & . & \dots & \overbrace{010...0} \\ . & . & . & \dots & \dots & . \\ \overbrace{000..01} & \overbrace{000..01} & . & . & \dots & \overbrace{000..01} \end{bmatrix}$$

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- Such matrices (such as B) are called triangular matrices, and because of this special structure of B it is easy to solve system of equations of the form $\mathbf{B}\mathbf{x}_B = \mathbf{b}$ (which will give a basic solution of the transportation problem).

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- Similarly the n **columns** correspond to the n **demand constraints** and the sum of the values of the variables in **column** j is given by d_j.

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- **Remark 2:** Note that a basic set corresponds to a basic solution of the transportation problem, where the variables corresponding to the basic cells are basic variables and the rest are nonbasic variables.

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- **Remark 3:** Let $\mathcal{B}$ be a basic set of cells. If we consider the **submatrix** of $A_{(m+n-1)\times mn}$ obtained by taking the **columns** corresponding to the variables associated with the basic set $\mathcal{B}$, then the submatrix will be a basis matrix, a square nonsingular matrix of dimension $m+n-1$.

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  If row i contains a single nonzero entry at (i, j) th position, then the submatrix obtained from B after **deleting** the i th row and the j th column from B again has the same property.

- If $\mathcal{B}$ is a **basic** set of cells and if the **row or column** having a single basic cell is struck off from the transportation array, then in the **reduced** (or remaining) array there will again be a **row** or **column** with a **single** basic cell.

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- **Example 1:** Consider the transportation problem with a_i and d_j as given below:
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	$j = 1$	2	3	4	5	6	a_i
$i = 1$							7
2							17
3							5
4							24
d_j	15	10	9	3	8	8	

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- Since the minimum of a_2 and d_3 is $d_3 = 9$, we take $x_{23} = 9$. Delete the third column and change a_2 from 17 to $a_2' = 17 - 9 = 8$.

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- In the reduced array choose a basic cell say $(2,4)$. Take $x_{24} = 3$ since $3 = min\{d_4 = 3, a_2' = 8\}$. Proceeding in this way we get the following BFS.

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2			[9]	[3]	[5]		17
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  Consider the following examples.

	1	2	3	4
1	$\circ$			$\circ$
2	$\circ$	$\circ$		
3		$\circ$	$\circ$	
4			$\circ$	$\circ$

	1	2	3	4
1	○			○
2	○	○		
3		○		
4				

	1	2	3	4
1	○			○
2	○	○		
3		○		
4				

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3			○	○
4			○	○

-

	1	2	3	4
1	$\circ$			$\circ$
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3		$\circ$		
4				

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	1	2	3	4
1	$\circ$	$\circ$		
2	$\circ$	$\circ$		
3			$\circ$	$\circ$
4			$\circ$	$\circ$

- In the second and third example, the marked cells do not form a θ loop of the 4×4 transportation array, since it violates properties 2 and 3, respectively.

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1	○			○
2	○	○		
3		○		
4				

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- The first one however is a θ loop.

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  Corollary 6: So from the previous theorems we can conclude that a subset of cells $\triangle$ of the transportation array is **linearly independent** if and only if it contains **no** θ loop.

- **Theorem 7:** If $\mathcal{B}$ is a collection of $m + n - 1$ basic cells of the transportation array and $(p, q) \notin \mathcal{B}$, then $\mathcal{B} \cup \{(p, q)\}$ contains one and only one θ-loop and this loop includes the cell (p, q).

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- We have $m + n - 1$ equations and $m + n - 1$ unknowns, which can be easily solved.

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- **Example:** Consider the following transportation problem (P) with c_{ij}'s, a_i's (40,30,30) and d_j's (30,50,20) as given below:

	2	5	1	40
	1	4	5	30
	1	5	3	30
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- Check whether the initial basic feasible solution $\mathbf{x}_0$ with basic cells
$\mathcal{B} = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 2)\}$, is optimal for (P) (by taking $v_2 = 0$, where v_2 is the dual variable corresponding to the second demand constraint).
Also find the optimal solution.

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.

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- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by
  $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.
- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by
  $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.
- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- On solving we get, $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$).

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.
- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- On solving we get, $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$).
- Check that

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by
  $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.
- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- On solving we get, $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$).
- Check that
  $c_{13} - u_1 - v_3 = 1 - 5 - 1 = -5, c_{21} - u_2 - v_1 = 1 - 4 - (-3) = 0,$
  $c_{31} - u_3 - v_1 = 1 - 5 - (-3) = -1,$
  $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3$

- The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by
  $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables.
- The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ for easier calculations.
- The other u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- On solving we get, $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$).
- Check that
  $c_{13} - u_1 - v_3 = 1 - 5 - 1 = -5, c_{21} - u_2 - v_1 = 1 - 4 - (-3) = 0,$
  $c_{31} - u_3 - v_1 = 1 - 5 - (-3) = -1,$
  $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3$, which is indicated in the

-

	0	0	-5	40
	0	0	0	30
	-1	0	-3	30
	30	50	20	

	0	0	-5	40
	0	0	0	30
	-1	0	-3	30
	30	50	20	

- Since all the $c_{ij} - u_i - v_j$ values are not non negative, the above table is not optimal.

	0	0	-5	40
	0	0	0	30
	-1	0	-3	30
	30	50	20	

- Since all the $c_{ij} - u_i - v_j$ values are not non negative, the above table is not optimal.
- The most negative value of $c_{ij} - u_i - v_j$ is in cell $(1, 3)$, so this will be the entering variable of the new BFS.

	0	0	-5	40
	0	0	0	30
	-1	0	-3	30
	30	50	20	

- Since all the $c_{ij} - u_i - v_j$ values are not non negative, the above table is not optimal.
- The most negative value of $c_{ij} - u_i - v_j$ is in cell $(1,3)$, so this will be the entering variable of the new BFS.
- The unique θ- loop in $\mathcal{B} \cup (1,3)$ is given by $\{(1,2),(2,2),(2,3),(1,3)\}$.

0	0	-5	40
0	0	0	30
-1	0	-3	30
30	50	20	

- Since all the $c_{ij} - u_i - v_j$ values are not non negative, the above table is not optimal.
- The most negative value of $c_{ij} - u_i - v_j$ is in cell $(1,3)$, so this will be the entering variable of the new BFS.
- The unique θ- loop in $\mathcal{B} \cup (1,3)$ is given by $\{(1,2),(2,2),(2,3),(1,3)\}$.
- Since $(1,3)$ is the entering variable, so if we give $+\theta$ allocation to cell $(1,3)$ (or value of $x_{13} = +\theta$) then $x_{12} = 10 - \theta$, $x_{22} = 10 + \theta$, $x_{23} = 20 - \theta$.

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1,1),(1,3),(2,2),(2,3),(3,2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$
- If we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below,

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1,1), (1,3), (2,2), (2,3), (3,2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$
- If we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below,
- $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5.$
  we get

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1,1),(1,3),(2,2),(2,3),(3,2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$
- If we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below,
- $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5.$
  we get
- $v_1 = 2, v_2 = 0, v_3 = 1, u_2 = 4, u_3 = 5.$

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1,1),(1,3),(2,2),(2,3),(3,2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$
- If we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below,
- $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5.$
  we get
- $v_1 = 2, v_2 = 0, v_3 = 1, u_2 = 4, u_3 = 5.$
- $c_{21} - u_2 - v_1 = 1 - 4 - 2 = -5, c_{12} - u_1 - v_2 = 5 - 0 - 0 = 5,$
  $c_{31} - u_3 - v_1 = 1 - 5 - 2 = -6,$
  $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3.$

- $x_{13} = 10$ is in the basis of the new BFS and x_{12} leaves the basis.
- New $\mathcal{B} = \{(1,1),(1,3),(2,2),(2,3),(3,2)\}$ and the values of the basic variables are given by:
  $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30.$
- If we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below,
- $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
  we get
- $v_1 = 2, v_2 = 0, v_3 = 1, u_2 = 4, u_3 = 5.$
- $c_{21} - u_2 - v_1 = 1 - 4 - 2 = -5, c_{12} - u_1 - v_2 = 5 - 0 - 0 = 5,$
  $c_{31} - u_3 - v_1 = 1 - 5 - 2 = -6,$
  $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3.$
  The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with
  $\mathcal{B} = \{(1,1),(1,3),(2,3),(2,2),(3,2)\}$

-

	0	5	0	40
	-5	0	0	30
	-6	0	-3	30
	30	50	20	

-

	0	5	0	40
	-5	0	0	30
	-6	0	-3	30
	30	50	20	

- The entering variable for the new BFS is x_{31}.

	0	5	0	40
	-5	0	0	30
	-6	0	-3	30
	30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3,1)$ which is given by $\{(3,1),(3,2),(2,2),(2,3),(1,3),(1,1)\}$.

-

0	5	0	40
-5	0	0	30
-6	0	-3	30
30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3,1)$ which is given by $\{(3,1),(3,2),(2,2),(2,3),(1,3),(1,1)\}$.
- $(3,1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3,1)$ (or value of $x_{31} = +\theta$) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$.

	0	5	0	40
	-5	0	0	30
	-6	0	-3	30
	30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3,1)$ which is given by $\{(3,1),(3,2),(2,2),(2,3),(1,3),(1,1)\}$.
- $(3,1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3,1)$ (or value of $x_{31} = +\theta$) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$.
- So $\theta = 10$.

0	5	0	40
-5	0	0	30
-6	0	-3	30
30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3, 1)$ which is given by $\{(3, 1), (3, 2), (2, 2), (2, 3), (1, 3), (1, 1)\}$.
- $(3, 1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3, 1)$ (or value of $x_{31} = +\theta$) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$.
- So $\theta = 10$.
- The entering variable for the new BFS is $x_{31} = 10$ and x_{23} is the leaving variable.

0	5	0	40
-5	0	0	30
-6	0	-3	30
30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3,1)$ which is given by $\{(3,1),(3,2),(2,2),(2,3),(1,3),(1,1)\}$.
- $(3,1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3,1)$ (or value of $x_{31} = +\theta$) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$.
- So $\theta = 10$.
- The entering variable for the new BFS is $x_{31} = 10$ and x_{23} is the leaving variable.
- The values of the basic variables in the new BFS is given by $x_{11} = 20, x_{13} = 20, x_{22} = 30, x_{31} = 10, x_{32} = 20$.

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0	5	0	40
-5	0	0	30
-6	0	-3	30
30	50	20	

- The entering variable for the new BFS is x_{31}.
- The unique θ- loop in $\mathcal{B} \cup (3, 1)$ which is given by $\{(3, 1), (3, 2), (2, 2), (2, 3), (1, 3), (1, 1)\}$.
- $(3, 1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3, 1)$ (or value of $x_{31} = +\theta$) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$.
- So $\theta = 10$.
- The entering variable for the new BFS is $x_{31} = 10$ and x_{23} is the leaving variable.
- The values of the basic variables in the new BFS is given by $x_{11} = 20, x_{13} = 20, x_{22} = 30, x_{31} = 10, x_{32} = 20$.
- The basic set of cells is given by $\mathcal{B} = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 2)\}$.

- We take $u_1 = 0$, then by solving the 5 equations given below:

$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
$c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.

- We take $u_1 = 0$, then by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- Check that $v_1 = 2, v_2 = 6, v_3 = 1, u_2 = -2, u_3 = -1$.

- We take $u_1 = 0$, then by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- Check that $v_1 = 2, v_2 = 6, v_3 = 1, u_2 = -2, u_3 = -1$.
- Check that $c_{23} - u_2 - v_3 = 5 - (-2) - 1 = 6$, $c_{21} - u_2 - v_1 = 1 - (-2) - 2 = 1$, $c_{12} - u_1 - v_2 = 5 - 0 - 6 = -1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3$.

- We take $u_1 = 0$, then by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
  $c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
- Check that $v_1 = 2, v_2 = 6, v_3 = 1, u_2 = -2, u_3 = -1$.
- Check that $c_{23} - u_2 - v_3 = 5 - (-2) - 1 = 6$, $c_{21} - u_2 - v_1 = 1 - (-2) - 2 = 1$, $c_{12} - u_1 - v_2 = 5 - 0 - 6 = -1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3$.
- The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with $\mathcal{B} = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$.

0	-1	0	40
1	0	6	30
0	0	3	30
30	50	20	

- The entering variable is x_{12}.

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.
- If $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$.

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.
- If $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$.
- Take $\theta = 20$.

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.
- If $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$.
- Take $\theta = 20$.
  Any one of x_{11} or x_{32} can be the leaving variable.

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.
- If $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$.
- Take $\theta = 20$.
  Any one of x_{11} or x_{32} can be the leaving variable.
- Let x_{32} leave the basis.

- The entering variable is x_{12}.
- The θ-loop is $\{(3,1),(3,2),(1,2),(1,1)\}$.
- If $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$.
- Take $\theta = 20$.
  Any one of x_{11} or x_{32} can be the leaving variable.
- Let x_{32} leave the basis.
- If we take $u_1 = 0$, then by solving the 5 equations given below:
  $c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
  $c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
  $c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
  $c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
  $c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$.
  we get

- $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1.$

- $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1.$
- $c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5, c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0$, $c_{32} - u_3 - v_2 = 5 - (-1) - 5 = 1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3.$

- $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1.$
- $c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5, c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0$, $c_{32} - u_3 - v_2 = 5 - (-1) - 5 = 1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3.$
- Since $c_{ij} - u_i - v_j \geq 0$ for all i, j,
  the above BFS is optimal and the optimal value is given by:

- $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1.$
- $c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5, c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0$, $c_{32} - u_3 - v_2 = 5 - (-1) - 5 = 1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3.$
- Since $c_{ij} - u_i - v_j \geq 0$ for all i, j,
  the above BFS is optimal and the optimal value is given by:
  $c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{22}x_{22} + c_{31}x_{31} = 2 \times 0 + 5 \times 20 + 1 \times 20 + 4 \times 30 + 1 \times 30 = 270.$