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- At $[\frac{1}{2}, \frac{1}{2}]^T$ any $\mathbf{d} \in \mathbb{R}^2$ will be a feasible direction.

First order necessary conditions for a local minimum

- **Remark 1:** If $\mathbf{x}^*$ is an interior point of Ω then any $\mathbf{d} \in \mathbb{R}^n, \mathbf{d} \neq \mathbf{0}$ is a feasible direction at $\mathbf{x}^*$.

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- **Theorem 2 :** Let $f : \Omega \to \mathbb{R}$ be a continuously differentiable function. Let $\mathbf{x}^*$ be an interior point of Ω. If $\mathbf{x}^*$ is a local minimum point of f then
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- So $[6, 9]^T$ and $[0, 0]^T$ satisfies the **first order necessary conditions** for a local minimum.
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- **Exercise:** Will $[0, 0]^T$ be a local minimum point of f given in **Example 1**, if $\Omega = \mathbb{R}^2$?

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- Check that f has a global minimum at $x_1 = \frac{1}{2}, x_2 = 0$

Second order necessary conditions for a point to be a local minimum

- **Theorem 3:** Let $f : \Omega \to \mathbb{R}$ be a twice continuously differentiable function (that is all the second order partial derivatives of f (given by $\frac{\partial^2 f}{\partial x_j \partial x_i}$) exists and are continuous as functions from Ω to $\mathbb{R}$).

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- Hence the first order necessary conditions are **necessary** but **not sufficient** for a point to be a local minimum.

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- **Definition:** A real symmetric matrix A is said to be positive semidefinite (negative semidefinite) if $\mathbf{x}^T A\mathbf{x} \geq 0$ ($\mathbf{x}^T A\mathbf{x} \leq 0$) for all $\mathbf{x} \in \mathbb{R}^n$.

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- For example the $\mathbf{0}$ matrix has all n eigenvalues equal to 0, the identity matrix I_n has all n eigenvalues equal to 1, and for an **upper triangular matrix**, the diagonal entries are its eigenvalues.

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- **Remark:** Since maximizing f is same as minimizing $-f$, all the previous theorems have corresponding analogues for a maximization problem with some obvious changes.

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- The above result is **not** necessarily true if f is **not convex** as you have already seen in **Example 1**.