MA 372 : Stochastic Calculus for Finance

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Exercises 6

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1. Suppose that the price of a stock $\{S(t); t \geq 0\}$ follows geometric Brownian motion with drift 0.06 and volatility 0.2 so that it satisfies the stochastic differential equation
$$dS(t) = 0.06\, S(t)\, dW(t) + 0.2\, S(t)\, dt.$$
If the price of the stock at time zero is 40, determine the probability that the price of the stock at time $t = 5$ is less than 50.

2. Let $\mathcal{F}(t)$ be the filtration generated by Brownian motion $W(t)$. Find the martingale representation for the following martingales;
a) $M(t) = \mathbb{E}[W^2(T)|\mathcal{F}(t)]$
b) $M(t) = \mathbb{E}[W^3(T)|\mathcal{F}(t)]$
c) $M(t) = \mathbb{E}[\exp\{\sigma W(T)\}|\mathcal{F}(t)]$

3. Let $r(t)$ and $\sigma(t)$ be non-random functions. Suppose $S(t)$ satisfies the following:
$$S(t) = S(0)\exp\Big\{\int_0^t \sigma(s)d\tilde{W}(s) - \frac{1}{2}\int_0^t (r(s) - \frac{1}{2}\sigma^2(s))ds\Big\}$$
where $\tilde{W}(t)$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. The price of an European call at time t, given by the risk-neutral valuation formula is
$$c(0, S(0)) = \tilde{\mathbb{E}}\Big[\exp\Big\{-\int_0^T r(s)ds\Big\}\Big(S(T) - K\Big)^+\Big]$$
Let
$$\begin{aligned} BSM(T, x, K, R, b) &= xN\Big(\frac{1}{b\sqrt{T}}[\log(\frac{x}{K}) + (R + \frac{b^2}{2})T]\Big) \\ &\quad - \exp\{-RT\}KN\Big(\frac{1}{b\sqrt{T}}[\log(\frac{x}{K}) + (R - \frac{b^2}{2})T]\Big) \end{aligned}$$
Show that
$$c(0, S(0)) = BSM(T, S(0), K, \frac{1}{T}\int_0^T r(t)dt, \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt})$$

4. The BSM price of an European call at time zero, if the stock price is x is given by
$$\begin{aligned} c(0,x) &= \tilde{\mathbb{E}}\Big[\exp\Big\{-rT\Big\}\Big(S(T) - K\Big)^+\Big] \\ &= \tilde{\mathbb{E}}\Big[\exp\Big\{-rT\Big\}\Big(x\exp\Big\{\sigma\tilde{W}(T) + (r - \frac{1}{2}\sigma^2)T\Big\} - K\Big)^+\Big] \end{aligned}$$

Now if $h(s) = (s-K)^+$, then

$$\begin{aligned} \frac{dh}{ds} &= 0, \text{ if } s < K \\ &= 1, \text{ if } s > K \end{aligned}$$

Using this and the fact that $\mathbb{P}(S(T) = K) = 0$. Show that $c_x(0,x) = \hat{\mathbb{P}}(S(T) > K)$ where $\hat{\mathbb{P}}$ is a probability measure equivalent to $\tilde{\mathbb{P}}$. Show that $\hat{W}(t) = \tilde{W}(t) - \sigma t$ is a Brownian motion under $\hat{\mathbb{P}}$. Rewrite $S(T)$ in terms of $\hat{W}(T)$. Finally conclude that $\hat{\mathbb{P}}(S(T) > K) = N(d_+(\tau, x))$ where $N(x)$ is the normal distribution function and

$$d_+(T,x) = \frac{1}{\sigma\sqrt{T}}[\log(\frac{x}{K}) + (r + \frac{\sigma^2}{2})T].$$

5. Let $W(t),\ 0 \le t \le T$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t),\ 0 \le t \le T$ be a filtration for this Brownian motion. Consider a stock price process whose differential is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ \mu, \sigma \in \mathbb{R}, \sigma > 0.$$

(i) Write down the probability measure $\mathbb{Q}$ under which the discounted stock price $Y(t) = e^{-rt}S(t)$ is a martingale with respect to $\mathcal{F}(t)$.

(ii) Determine $d(e^{-rt}S(t))$ under the risk-neutral probability measure $\mathbb{Q}$.

6. Suppose the market has an arbitrage. So there ia a portfolio value process satisfying $X_1(0) = 0$ and $\mathbb{P}(X_1(T) \ge 0) = 1,\ \mathbb{P}(X_1(T) > 0) > 0$, for some positive time T.

a) Show that if $X_2(0)$ is positive, then there exists a portfolio value process $X_2(t)$ satisfying at $X_2(0)$ and satisfying

$$\mathbb{P}\Big(X_2(T) \ge \frac{X_2(0)}{D(T)}\Big) = 1 \text{ and } \mathbb{P}\Big(X_2(T) > \frac{X_2(0)}{D(T)}\Big) > 0$$

b) Suppose that the market has a portfolio process $X_2(t)$ such that $X_2(0)$ is positive and the above holds. Then show that the model has a portfolio value process $X_1(t)$ which is an arbitrage.

7. Consider a financial market consisting of a risk-free asset $B(t)$ and a stock $S(t)$, whose price at time $t,\ t > 0$ satisfy the following differentials:

$$\begin{aligned} dB(t) &= 2\ B(t)dt \\ dS(t) &= S(t)\Big(7\ dt + 2\ dW(t)\Big) \end{aligned}$$

(i) Is the above market arbitrage free?

(ii) Is the above market complete?

(iii) Find the risk-neutral probability measure $\mathbb{Q}$ for the above market.

(iv) Find $dS(t)$ in terms of $\tilde{W}(t)$, where $\tilde{W}(t)$ is a Brownian motion under $\mathbb{Q}$.

(v) Find the price of a European call whose payoff at time T is represented by $(S(T) - K)^+$ by using risk-neutral pricing formula.

8. Consider a market with m stocks, each satisfying the stochastic differential equation

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)dW_j(t)\,,$$

for $i = 1, 2, \ldots, m$ and where $W = (W_1, W_2, \ldots, W_d)$ is a $d-$dimensional Brownian motion. Set $\sigma_i(t) = \sqrt{\sum_{j=1}^{d}\sigma_{ij}^2(t)}$, which we assume is never zero. Define $B_i(t) = \sum_{j=1}^{d}\int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}dW_j(u)$, $i = 1, 2, \ldots, m$.

(a) Prove that each $B_i(t)$ is a continuous martingale

(b) Use Ito's product rule to show that $Cov(B_i(t), B_k(t)) = \mathbb{E}\left(\int_0^t \frac{\sum_{j=1}^{d}\sigma_{ij}(u)\sigma_{kj}(u)}{\sigma_i(u)\sigma_k(u)}du\right)$.