MA 372 : Stochastic Calculus for Finance

July - November 2022

Department of Mathematics, Indian Institute of Technology Guwahati

Exercises 4

October 28, 2022

1. We shall call $f(t), t \in [0,T]$ a simple process if there is a finite sequence of numbers $0 = t_0 < t_1 < \cdots < t_n = T$ and square integrable random variables $\eta_0, \eta_1, \cdots, \eta_{n-1}$ such that $f(t,w) = \sum_{j=0}^{n-1} \eta_j(w)\mathbb{I}_{[t_j,t_{j+1})}(t)$, where η_j is $\mathcal{F}_{t_j}$-measurable. The set of simple processes will be denoted by $M^2_{step}([0,T]\times\Omega)$

   a) Show that $M^2_{step}([0,T]\times\Omega)$ is a vector space, that is, $af+bg \in M^2_{step}([0,T]\times\Omega)$ for any $f,g \in M^2_{step}([0,T]\times\Omega)$ and $a,b \in \mathbb{R}$.

   b) Show that $I : M^2_{step}([0,T]\times\Omega) \to L^2$ is a linear map, i.e., for any $f,g \in M^2_{step}([0,T]\times\Omega)$ and $a,b \in \mathbb{R}$

   $$I(af+bg) = aI(f) + bI(g).$$

   c) For any $f,g \in M^2_{step}([0,T]\times\Omega)$

   $$E\Big[I(f)I(g)\Big] = E\Big[\int_0^T f(t)g(t)dt\Big]$$

2. Check whether the following processes $X(t)$ are martingale with respect to Brownian filtration

   a) $X(t) = W(t) + 4t$ b) $X(t) = W^2(t)$
   c) $X(t) = t^2 W(t) - 2\int_0^t sW(s)ds$

3. Use Ito's formula to prove that the following stochastic process are martingale with respect to Brownian filtration

   a) $X(t) = e^{\frac{t}{2}}\cos W(t)$ b) $X(t) = e^{\frac{t}{2}}\sin W(t)$
   c) $X(t) = e^{W(t)-\frac{t}{2}}$ d) $X(t) = (W(t)+t)e^{-W(t)-\frac{t}{2}}$

4. Define $\beta_k(t) = \mathbb{E}[W^k(t)]$; $k = 0,1,2,\cdots$; $t \geq 0$
   Use Ito's formula to prove that

   $$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds; \quad k \geq 2$$

   a) Deduce that $\mathbb{E}[W^4(t)] = 3t^2$ and find $\mathbb{E}[W^6(t)]$.
   b) Show that $\mathbb{E}[W^{2k+1}(t)] = 0$ and $\mathbb{E}[W^{2k}(t)] = \frac{(2k)!t^k}{2^k k!}$

5. For c, α constants, define

   $$X(t) = e^{ct+\alpha W(t)}.$$

   Prove that

   $$dX(t) = (c + \frac{1}{2}\alpha^2)X(t)dt + \alpha X(t)dW(t)$$

6. Let $h(t) = \sum_{j=0}^{2}(j+1)\mathbb{I}_{[j,j+1)}(t)$. Define

$$I(t) = \int_0^t h(s)dW(s), \quad 0 \le t \le 3.$$

Find the distribution function of the random variable $I(2)$. Find the variance of the random variable $I(3)$.

7. Let $\Pi = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$. For $\alpha \in [0,1]$, consider the sum

$$S_\alpha(\Pi) = \sum_{j=0}^{n-1}\Big[(1-\alpha)W(t_j) + \alpha W(t_{j+1})\Big](W(t_{j+1}) - W(t_j)).$$

Evaluate the limit $\lim\limits_{||\Pi||\to 0} S_\alpha(\Pi)$ (in L^2), where $\|\pi\| = \max\limits_{j=1,2,\cdots,n}(t_j - t_{j-1})$.

8. If $f(t,x) = e^{t/2}(\sin x + \cos x)$, then check whether the process $f(t, W(t))$ is a martingale with respect to Brownian filtration.

9. Let $\Pi = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$. For $\alpha \in [0,1]$, consider the sum

$$S_\alpha(\Pi) = \sum_{j=0}^{n-1}\Big[(W(t_{j+1}) - W(t_j))^2 - (t_{j+1} - t_j)\Big].$$

Evaluate the limit $\lim\limits_{||\Pi||\to 0} S_\alpha(\Pi)$ (in L^2), where $\|\pi\| = \max\limits_{j=1,2,\cdots,n}(t_j - t_{j-1})$.

10. If $f(t,x) = x^5 - 10tx^3 + 15t^2x$, then check whether the process $f(t, W(t))$ is a martingale with respect to Brownian filtration.

11. Suppose that $\{W(t); t \ge 0\}$ is a standard Brownian motion with $W(0) = 0$. Determine an expression for

$$\int_0^t \sin(W(s))dW(s)$$

that does not involve Ito integrals.

12. Suppose $f(t)$ is a deterministic function. Let $X(t) = X(0) + \int_0^t f(s)dW(s)$. Determine an expression for

$$\int_0^t f(s)X(s)dW(s)$$

that does not involve Ito integrals.

13. Suppose $f(t)$ is a deterministic function. Let $X(t) = \int_0^t f(t)[\sin(W(t) + \cos(W(t))]dW(t)$. Find the mean and variance of the random variable $X(2)$.

14. The solution to the BSM PDE with the specified terminal and boundary conditions is given by

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)) \quad 0 \le t < T,\ x > 0\,,$$

where $d_\pm(T-t,x) = \frac{1}{\sigma\sqrt{T-t}}[\log(x/K) + (r \pm \frac{\sigma^2}{2})(T-t)]$ and N is the CDF of $N(0,1)$. Note that $c(t,x)$ is not defined for $t = T$ and $x = 0$. But $c(t,x)$ is defined in such a way that $\lim_{t\to T} c(t,x) = (x-K)^+$ and $\lim_{x\downarrow 0} c(t,x) = 0$.

(a) Verify that $Ke^{-r(T-t)}N'(d_-) = xN'(d_+)$.

(b) Prove that $c_x(t,x) = N(d_+(T-t,x))$.

(c) Prove that $c_t(t,x) = -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x))$.

(d) Prove that $c_{xx}(t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x))$.

(e) Use the above formulas to show that $c(\cdot,\cdot)$ satisfies the BSM PDE.