MA372

Stochastic Calculus for Finance

1. (a) If f is a simple process

then

$$f(t,\omega) = \sum_{j=0}^{n-1} \eta_j(\omega) \mathbb{I}_{[t_j, t_{j+1})}(t)$$

For $a \in \mathbb{R}$ $a\eta_j(\omega)$ is also a sq. integrable r.v.

Therefore,

$$af(t,\omega) = \sum_{j=0}^{n-1} \underbrace{\left[a\eta_j(\omega)\right]}_{\eta_j'(\omega)} \mathbb{I}[t_j, t_{j+1})$$

$\Rightarrow af(t,\omega)$ is also a simple process.

So we have two simple process $af(t,\omega)$ & $bg(t,\omega)$.

Say $af(t,\omega)$ has time intervals $0 = t_0 < t_1 < t_2 \ldots\ldots < t_n = T$
& $bf(t,\omega)$ has " " $0 = t_0' < t_1' < t_2' \ldots\ldots < t_m' = T$

Say $A_i = [t_i, t_{i+1})$ & $B_i = [t_i', t_{i+1}')$

Then

$$f(t,\omega) = \sum_{j=0}^{n-1} \eta_j(\omega) \mathbb{I}_{[t_j, t_{j+1})}$$

$$= \sum_{j=0}^{n-1} \eta_j(\omega) \sum_{i=0}^{m-1} \mathbb{I}_{A_j \cap B_i}$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \eta_j(\omega) \mathbb{I}_{A_j \cap B_i}$$

Similarly

$$g(t,\omega) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} m_i(\omega)\, \mathbb{I}_{B_i \cap A_j}(t)$$

Therefore $\quad af(t,\omega) + bg(t,\omega) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (a n_j(\omega) + b\, m_i(\omega))\, \mathbb{I}_{B_i \cap A_j}(t)$

Here $\{B_i \cap A_j\}$ are disjoint sets and $\cup \{B_i \cap A_j\} = [0, T)$

Therefore we conclude its a finite partition (maximum size is $m \times n$) and therefore $af(t,\omega) + bg(t,\omega)$ is also $M^2_{step}([0,T] \times \Omega)$

(b)

$$\cancel{I(f) = \sum_{i=0}^{n-1} n_i(\omega)\, \mathbb{I}_{[t_i, t_{i+1})}(t)\, (-\omega(t_i) + \omega(t_{i+1}))}$$

$$I(f) = \sum_{i=0}^{n-1} n_i(\omega)\, (\omega(t_{i+1}) - \omega(t_i)) \qquad \text{(By defn)}$$

By using above construction

$$I(f) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} n_i(\omega)\, M_{ij} \quad \text{—(I)} \qquad \& \qquad I(g) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} m_j(\omega)\, M_{ij} \quad \text{—(II)}$$

where $M_{ij} = \omega(t'_{ij+1}) - \omega(t'_{ij})$

where $t'_{ij} = \min\{0 \cup (A_i \cap B_j)\}$

& $t'_{ij+1} = \max\{\phi \cup (A_i \cap B_j)\}$

$$af + bg = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} (a\,n_i(\omega) + b\,m_j(\omega))\, \mathbb{I}^{(t)}_{B_j \cap A_i} \quad \text{(iii)}$$

Then $\mathbb{I}(af + bg) = aI(f) + bI(g)$ Using (i), (ii) & (iii)

(c)

$$I(f) = \sum_{i=0}^{n-1} n_i(\omega)\, M_i$$

$$I(f) \cdot I(g) = \left(\sum_{i=0}^{n-1} n_i(\omega)\, M_i\right)\left(\sum_{j=0}^{m-1} m_j(\omega)\, M_j'\right) = \sum_{i,j} n_i(\omega) m_j(\omega) M_i M_j'$$

$$f(t) \cdot g(t) = \left(\sum n_i(\omega)\, \mathbb{I}_{A_i}(t)\right)\left(\sum m_j(\omega)\, \mathbb{I}_{B_j}(t)\right)$$

$$= \sum_{i,j} n_i(\omega) m_j(\omega)\, \mathbb{I}_{A_i \cap B_j}(t)$$

$$M(i) = \omega(t_{i+1}) - \omega(t_i) = \int_{t_i}^{t_{i+1}} d(\omega(t))$$

$$\int f(t) g(t)\, dt = \cancel{\sum_{i,j} n_i(\omega) m_j(\omega) M_{ij}}$$

$$E[I f(t) I(g)] = \sum_{i,j} n_i(\omega) m_j(\omega) E[M_i M_j']$$

$$\& \quad E\left[\int_0^T f(t) g(t)\, dt\right] = \sum_{i,j} n_i(\omega) m_j(\omega) E[M_{ij}].$$

where $M_i = \omega(t_{i+1}) - \omega(t_i)$

& $M_j' = \omega(t_{j+1}') - \omega(t_j')$

& M_{ij} is defined previously.

If we prove $E[M_i M_j'] = E[M_{ij}]$ then we are done.

$$f \to \sum_{i=0}^{k-1} \xi_i' \, \mathbb{1}_{(d_i, d_{i+1})}.$$

$$g \to \sum_{i=0}^{k-1} \gamma_i' \, \mathbb{1}_{(d_i, d_{i+1})}.$$

$$I(f)\, I(g) = \sum_{i=0}^{k-1} \xi_i' \gamma_i' \left(w(d_{i+1}) - w(d_i)\right)^2$$

$$+ \sum_{\substack{i=0 \\ i \neq j}}^{k-1} \sum_{j=0}^{k-1} \xi_i' \gamma_j' \left(w(d_{i+1}) - w(d_i)\right) \left(w(d_{j+1}) - w(d_j)\right)$$

$$\cancel{I(g)} \cancel{\int_0^T f g \, dt = \sum_i \xi_i' \gamma_j' \; w(d_{i+1}) - w(d_i)}$$

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$$E\left[\int_0^T \left(\sum \xi_i' \gamma_j' \, \mathbb{1}_{(d_i, d_{i+1})}\right) dt\right].$$

$$= E\left[\sum_{i=0}^{m-1} \xi_i' \gamma_i' (d_{i+1} - d_i)\right]$$

$$f(t)\cdot g(t) = \sum_{i,j} m_i(\omega)\, m_j(\omega)\, \mathbb{I}_{A_i \cap B_j}$$

Define $\int \mathbb{I}_{A_i \cap B_j}\, dt = 0$ if $A_i \cap B_j = \phi$

Otherwise reimenan integration i.e $\int_{\mathbb{I}_{A_i \cap B_j}} 1\, dt = t'_{j+1} - t'_j$

Now,

$E[M_i M_j']$

Case ① $A_i \cap B_j = \phi$

then $E[M_i M_j'] = E[M_i]\, E[M_j] = 0$

Case ② $t_i \le t_j < t_{i+1} \le t_{j+1}$ WLOG.

$$E[(W(t_{i+1}) - W(t_i)) \cdot (W(t_{j+1}) - W(t_j))]$$

$$= E[W(t_{i+1}) \cdot W(t_{j+1})] + E[W(t_i) \cdot W(t_j)] - E[W(t_{i+1}) W(t_j)] - E[W(t_i) \cdot W(t_{j+1})]$$

$$= t_{i+1} + t_i - t_j - t_i$$

$$= t_{i+1} - t_j$$

$$= t'_{j+1} - t'_j$$

$\therefore$

$$\int \mathbb{I}_{A_i \cap B_j} dt = E[M_i M_j']$$

② a) $X(t) = W(t) + 4t$ → <u>Adapted Process</u>

$$E[X(t) \mid \mathcal{F}_s] = E[W(t) + 4t \mid \mathcal{F}_s] = W(s) + 4t \neq W(s) + 4s$$

~~and~~ therefore not a martingale.

b) $X(t) = W^2(t)$

$$E[W^2(t) \mid \mathcal{F}_s] = E[(W(t) - W(s) + W(s))^2 \mid \mathcal{F}_s]$$

$$= E[(W(t) - W(s))^2 + 2(W(t) - W(s))W(s) + W(s)^2 \mid \mathcal{F}_s]$$

$$= E[(W(t) - W(s))^2] + 0 + W(s)^2$$

$$\cancel{= E[W(t)^2 + W(s)^2 - 2W(t)W(s)] = E(W(s))}$$

$$\cancel{= E[E[W(t)^2 + W(s)^2 - 2W(t)W(s) \mid \mathcal{F}_s]]}$$

$$= t - s + W(s)^2 \neq W(s)^2 \qquad [Var(W(t)) = t]$$

Not a martingale.

c) $$t^2 W(t) - 0^2 W(0) = \int_0^T 2t\,W(t)\,dt + \int_0^T t^2\,dW(t) + \frac{1}{2}\int_0^T t^2 \cdot \underline{0}\; dW(t) \cdot dW(t)$$

$$X(t) = t^2 W(t) - 2\int_0^t s\,W(s)\,ds = \int_0^t s^2\,dW(s) \leftarrow \text{Ito's Integral}$$

<u>Martingale</u>

3. (a) $X(t) = e^{\frac{t}{2}} \cos W(t)$

$f(t,x) = e^{\frac{t}{2}} \cos x$

$f_t = \frac{1}{2} e^{\frac{t}{2}} \cos x$

$f_x = -e^{\frac{t}{2}} \sin x$

$f_{xx} = -e^{\frac{t}{2}} \cos x$

$$e^{\frac{t}{2}} \cos W(t) - \underbrace{e^{\frac{0}{2}} \cos W(0)}_{\text{constant}} = \cancel{\int_0^T f_t \, dt} + \int_0^T f_x \, dx + \cancel{\frac{1}{2} \int_0^T f_{xx} \, dt}$$

$$= \int -e^{\frac{t}{2}} \sin W(t) \, dW(t) \quad \text{Martingale}$$

(b) $X(t) = e^{\frac{t}{2}} \sin W(t)$

$$e^{\frac{t}{2}} \sin W(t) - e^{\frac{0}{2}} \sin W(0) = \cancel{\int_0^T \frac{1}{2} e^{\frac{t}{2}} \sin W(t) \, dt} + \int_0^T e^{\frac{t}{2}} \cos W(t) \, dW(t)$$

$$\cancel{- \frac{1}{2} \int_0^T e^{\frac{t}{2}} \sin W(t) \, dt}$$

$$= \int_0^T e^{\frac{t}{2}} \cos W(t) \, dW(t) \quad \text{Martingl}$$

$f(t,x) = x^2$

$\rightarrow f_t = 0$

$f_x = 2x$

$f_{xx} = 2$

$$W^2(T) - \cancel{W(0)} = \cancel{\int_0^T 2W(t) \, dW(t)} + T$$

$$\boxed{E[W^2(T)] = T}$$

(c) $x(t) = e^{W(t) - t/2}$

$f(t,x) = e^{x - t/2}$

$f_t = -\frac{1}{2} e^{x - t/2}$

$f_x = e^{x - t/2}$

$f_{xx} = e^{x - t/2}$

$$X(t) - X(0) = \cancel{\int_0^T f_t\, dt + \int_0^T f_x\, dx + \frac{1}{2}\int_0^T f_{xx}\, dt}$$

$$= \int_0^T f_x\, dx \rightarrow \text{Martingale}$$

(d) $X(t) = (W(t) + t)\, e^{-W(t) - t/2}$

$f(t,x) = (x+t) e^{-x - t/2}$

$$f_t = e^{-x-t/2} + (x+t) e^{-x-t/2} \left(-\frac{1}{2}\right) = \left(\frac{2-x-t}{2}\right) e^{-x-t/2}$$

$$f_x = e^{-x-t/2} - (x+t) e^{-x-t/2} = (1-x-t)\, e^{-x-t/2}$$

$$f_{xx} = -e^{-x-t/2} - e^{-x-t/2} + (x+t) e^{-x-t/2} = [-2 + (x+t)] e^{-x-t/2}$$

$$f_t + \frac{1}{2} f_{xx} = e^{-x-t/2}\left[1 - \left(\frac{x+t}{2}\right) - 1 + \left(\frac{x+t}{2}\right)\right] = 0$$

$$X(t) - \underbrace{X(0)}_{0} = \int_0^T f_x \, dx + \int_0^T f_t \, dt + \frac{1}{2}\int_0^T f_{xx} \, dt$$

$$= \int_0^T f_x \, dx + \int_0^T \underbrace{\left(f_t + \frac{1}{2} f_{xx}\right)}_{\to 0} dt$$

$$= \int_0^T f_x \, dx \leftarrow \underline{\text{Martingale}}$$

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④ $\beta_k(t) = E\left[W^k(t)\right]$

$f(t,x) = x^k$

<u>for $k \geq 2$.</u>

$f_t = 0$

$f_x = k x^{k-1}$ $\qquad f_{xx} = k(k-1) x^{k-2}$

$$W^k(t) - W^k(0) = \int_0^T 0 \cdot dt + \int_0^T k W(t)^{k-1} dW + \frac{1}{2}\int_0^T W(t)^{k-2} k(k-1) dt$$

$$= \frac{1}{2} k(k-1) \int \beta_{k-2} + \underbrace{\int_0^T k W(t)^{k-1} dW}_{\text{Martingale}}$$

$$E\left[W^k(t)\right] = \frac{1}{2} k(k-1) \int_0^T E\left(W^{k-2}(t)\right) dt + kE\left[\int_0^T W^{k-1} dW\right]$$

$$\beta_k(t) = \frac{1}{2} k(k-1) \int \beta_{k-2}(s) \, ds + k \, E[0] \to 0$$

(b) $E[W] = 0$ → Base case

$\Rightarrow \beta_{2k+1}(t) = \frac{1}{2}k(k-1)\int \beta_{2k-1}\, ds$ for $k \geq 1$

$= 0$ → Induction

$E[1] = 1$

$E[W^2] = \frac{1}{2} \cdot 2 \cdot (2-1) \int_0^t ds = t$ (Variance)

$E[W^{2k}(t)] = \frac{1}{2} 2k(2k-1) \int_0^T \frac{(2k-2)!\, t^{k-1}}{2^{k-1}(k-1)!}\, dt$ $\qquad \int t^n\, dt = \frac{t^{n+1}}{n+1}$

$= \frac{2k(2k-1)(2k-2)!\, k}{2^k\, k!} \frac{t^k}{k}$

$= \frac{(2k)!\, t^k}{2^k\, k!} \frac{\cancel{k}}{\cancel{k}}$

(a) Trivial

⑤. $X(t) = e^{ct + \alpha W(t)}$

<u>Use ito's formula</u>

$\hookrightarrow$ $f_t dt + f_x dx + \frac{1}{2} f_{xx} dx dx$

⑥ $I(t) = \int_0^t h(s)\, dW(s)$, $0 \le t \le 3$

$h(t) = \sum_{j=0}^{2} (j+1) \mathbb{I}_{[j, j+1)}(t)$

$I(2) = \int_0^2 h(s)\, dW(s) = \sum_0^2 (j+1)(W(t_{j+1}) - W(t_j))$

$= W(1) + 2W(2) - 2W(1)$

$= 2\{W(2) - W(1)\}$

$= 2W(2) - W(1)$

Linear combination of normal is <u>normal</u>.

$E[I(2)] = E[2W(2) - W(1)] = 0$

$Var[I(2)] = E[(2W(2) - W(1))^2]$

$= E[4W^2(2) + W^2(1) - 4W(1)W(2)]$

$= 4 \times 2 + 1 - 4 = 5$

Distribution $I(3) \sim N(0, \sqrt{5})$ → Variance is written here.

$$J(3) = 3W(3) - W(2) - W(1)$$

$$Var(I(3)) = E[(3W(3) - W(2) - W(1))^2]$$

$$= E[9W^2(3) + W^2(2) + W^2(1) - 6W(3)W(2) - 6W(3)W(1) + 2W(2)W(1)]$$

$$= 27 + 2 + 1 - 12 - 6 + 2$$

$$= \underline{\underline{14}}$$

(7) $$S_\alpha(\Pi) = \sum_{j=0}^{n-1} \left[(1-\alpha)W(t_j) + \alpha W(t_{j+1})\right](D_j)$$

where $D_j = W(t_{j+1}) - W(t_j)$

$$\lim_{\|\Pi\| \to 0} S_\alpha(\Pi) = \sum_{j=0}^{n-1} D_j W_j + \alpha(W_{j+1} - W_j) D_j$$

$$= \sum_{j=0}^{n-1} D_j W_j + \alpha \Sigma D_j^2$$

$$\lim_{\|\Pi\| \to 0} S_\alpha(\Pi) = \lim_{\|\Pi\| \to 0} \Sigma D_j W_j + \alpha T$$

By defn $$\int_0^T W(t)\,dW(t) + \alpha T$$

$$= \frac{1}{2} W^2(T) + (\alpha - 1)T$$

$$E[S_\alpha(\Pi)] =$$