MA 372 : Stochastic Calculus for Finance

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Exercises 2

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1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $B \in \mathcal{F}$ an event with $\mathbb{P}(B) \neq 0$. We call
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$
the conditional probability of A given B. Prove that $A \longmapsto \mathbb{P}(A \mid B)$ is a probability measure on $\mathcal{F}$.

2. Let $H_1, H_2, \cdots$ be a partition of Ω such that $\mathbb{P}(H_n) \neq 0$ for any $n = 1, 2, \cdots$. Then for any event A
$$\mathbb{P}(A) = \sum_{n=1}^{\infty} \mathbb{P}(A \mid H_n)\mathbb{P}(H_n).$$

3. Set $\Omega = \{a, b, c, d\}, \mathcal{F} = 2^{\Omega}, \mathbb{P}(\{a\}) = 1/6, \mathbb{P}(\{b\}) = 1/3, \mathbb{P}(\{c\}) = 1/4, \mathbb{P}(\{d\}) = 1/4$. Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We next define two random variables, X and Y , by the formulas $X(a) = X(b) = 1, X(c) = X(d) = -1$ any $Y(a) = Y(c) = 1, Y(b) = Y(d) = -1$. We then define $Z = X + Y$.
(i) List the sets in $\sigma(X)$. (ii) Determine $E[Y|X]$.
(iii) Determine $E[Z|X]$. (iv) Compute $E[Z|X] - E[Y|X]$.

4. Let $\Omega = \{1, 2, 3, \cdots, 8\},\ \mathcal{F} = 2^{\Omega},\ \mathbb{P}(\{i\}) = 1/10$ for $i \leq 4$ and $\mathbb{P}(\{i\}) = 3/20$ for $i > 4$. Suppose $X = \mathbb{I}_{\{1,2,3,4\}} + 2\mathbb{I}_{\{5,6,7,8\}}$ and $Y = \mathbb{I}_{\{1,5\}} + 2\mathbb{I}_{\{2,3,4,6,7,8\}}$. Let $\mathcal{G}$ denote the σ-field generated by $\{\{1, 2\}, \{3, 4\}\}$ and let $\mathcal{H}$ denote the σ-field generated by $\{1, 2, 3, 4\}$. Show that
$$\mathbb{E}[\mathbb{E}[X \cdot Y|\mathcal{G}]|\mathcal{H}] = X \cdot \mathbb{E}[Y].$$
(Use two methods: direct calculation and applications of the three fundamental laws in conditional expectation.)

5. *Cauchy-Schwartz inequality.* Let X, Y be random variables with finite second moments. Show that
$$E(XY)^2 \leq EX^2EY^2.$$
(Hint: Use the fact that $E(tX + Y)^2 \geq 0$ for any $t \in \mathbb{R}$.)

6. Suppose that X and Y are jointly continuous random variables with join density $f_{X,Y}(x, y) = ce^{x+y}$ for $x, y \in (-\infty, 0]$ and $f_{X,Y}(x, y) = 0$ otherwise
a) what is the value of c ?
b)What is the probability that $X < Y$?
c)What are the marginal densities f_X and f_Y ?
d) Show that X and Y are independent.

7. Show that
   a) $Var(X + a) = Var(X)$ for any $a \in \mathbb{R}$.
   b) $Var(bX + a) = b^2 Var(X)$ for any $a, b \in \mathbb{R}$.
   c) $Var(X + Y) = Var(X) + Var(Y)$ if X and Y are independent.

8. Let (X, Y) be jointly normal, with the density function

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\Big[-\frac{1}{2(1-\rho^2)} \Big\{ \frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2} \Big\} \Big],$$

   where $\sigma_1 > 0, \sigma_2 > 0, |\rho| < 1$, and μ_1, μ_2 are real numbers. Define $W = Y - \frac{\rho\sigma_2}{\sigma_1} X$. show that X and W are independent. Calculate the joint density of (X, W).

9. Show that

$$E(\mathbb{I}_A | \mathbb{I}_B) = \begin{cases} \mathbb{P}(A \mid B) & \text{if } w \in B \\ \mathbb{P}(A \mid B^c) & \text{if } w \notin B \end{cases}$$

   for any B such that $1 \neq \mathbb{P}(B) \neq 0$.

10. Take $\Omega = [0, 1]$ with the σ-field of Borel sets and $\mathbb{P}$ the Lebesgue measure on $[0, 1]$. Compute $E[X|Y]$, where $X(x) = 2x$ and

$$Y(x) = \begin{cases} x & \text{if } 0 \leq x < 1/2 \\ 1/2 & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

11. Let X and Y have the joint distribution measure

$$\mu_{X,Y}(\{m, n\}) = \begin{cases} \frac{1}{2^{m+1}} & \text{if } m \geq n \\ 0 & \text{if } m < n \end{cases}$$

   for $m, n = 1, 2, 3, \cdots$ Compute the marginal distributions μ_X and μ_Y.

12. A die is rolled twice; X is the sum of the outcomes and Y is the outcomes of the first roll. Compute $E[X|Y]$.

13. Let X and Y be integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then $Y = Y_1 + Y_2$, where $Y_1 = E[Y|X]$ is $\sigma(X)$-measurable. Show that Y_2 and X are uncorrelated.

14. Let X and Y be two random variables defined on some probability space. Prove that

$$\mathbb{E}[X - \mathbb{E}(X|\mathcal{G})]^2 \leq \mathbb{E}[X - Y]^2$$

   for any $\mathcal{G}$ measurable function Y.

15. Let Ω be the unit square $[0, 1] \times [0, 1]$ with the Borel σ-field and $\mathbb{P}$ the Lebesgue measure on $[0, 1] \times [0, 1]$. Suppose that X and Y are random variables on Ω with join density $f_{X,Y}(x, y) = x + y$ for $x, y \in [0, 1]$ and $f_{X,Y}(x, y) = 0$ otherwise. Show that

$$\mathbb{E}[X|Y] = \frac{2 + 3Y}{3 + 6Y}.$$