

MA 372 : Stochastic Calculus for Finance

July - November 2022

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Exercises 3

September 13, 2022

1. a) Prove that $\mathbb{E}[\exp\{iuW(t)\}] = \exp(-\frac{1}{2}u^2t)$
b) Deduce that $\mathbb{E}[W^4(t)] = 3t^2$ and more generally $\mathbb{E}[W^{2k}(t)] = \frac{(2k)!}{2^k k!} t^k$, $k \in \mathbb{N}$. Find $\mathbb{E}[W^6(t)]$.
c) Compute the moment generating function of $(W(t_1), W(t_2), \dots, W(t_m))$, i.e., find $\mathbb{E}[\exp\{u_1W(t_1) + u_2W(t_2) + \dots + u_mW(t_m)\}]$.
Hint: $u_1W(t_1) + u_2W(t_2) + \dots + u_mW(t_m) = u_m(W(t_m) - W(t_{m-1})) + (u_{m-1} + u_m)(W(t_{m-1}) - W(t_{m-2})) + \dots + (u_1 + u_2 + \dots + u_m)W(t_1)$
2. Let X_n be a symmetric random walk, that is

$$X_n = Y_1 + Y_2 + \dots + Y_n,$$

where Y_1, Y_2, \dots is a sequence of independent identical distributed random variables such that $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = 1/2$.

- a) Show that $X_n^2 - n$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.
- b) Show that $Z_n = (-1)^n \cos(\pi X_n)$ is a martingale with respect to the filtration \mathcal{F}_n .
- c) Show that $Z_n = (-1)^n \cos(\pi(X_n + 100))$ is a martingale with respect to the filtration \mathcal{F}_n .
- d) Let τ be a stopping time with respect to the filtration \mathcal{F}_n . Then the stopped process X^τ is defined for $t \geq 0$ and $\omega \in \Omega$ by

$$X_n^\tau(\omega) := X_{\{n \wedge \tau(\omega)\}}(\omega).$$

Show that X^τ is adapted to the filtration \mathcal{F}_n .

- e) Find $\mathbb{E}[(-1)^\tau]$, where τ the smallest n such that $|X_n| = 100$.

3. Let $W(t)$ be a Brownian motion. Check whether the process $X(t) = 2W(t) + 4t$ is a martingale with respect to Brownian filtration.
4. Assume that $\lim_{t \rightarrow \infty} \frac{W(t)}{t} = 0$, a.s. Then prove that

$$Y(t) = \begin{cases} tW(1/t) & \text{if } 0 < t < \infty \\ 0 & \text{if } t = 0 \end{cases}$$

is a Brownian motion if $W(t)$ is.

5. Let $c > 0$. Show that $X(t) = \frac{1}{c}W(c^2t); 0 \leq t < \infty$ is a Brownian motion if $W(t)$ is.

6. Show that for any fixed $T > 0$

$$X(t) = W(t+T) - W(T), \quad t \geq 0$$

is a Brownian motion if $W(t)$ is.

7. Let Y be a real valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[|Y|] < \infty.$$

Let $\mathcal{F}(t)$, $t \geq 0$, be any filtration. Define

$$M(t) = \mathbb{E}[Y|\mathcal{F}(t)], \quad t \geq 0.$$

Show that $M(t)$ is a martingale with respect to the filtration $\mathcal{F}(t)$, $t \geq 0$.

8. Show that $W^3(t) - 3tW(t)$ is a martingale with respect to the Brownian filtration.

9. Show that if $M(t)$ is a martingale with respect to the filtration $\mathcal{F}(t)$, $t \geq 0$, then

$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)]$$

for all $t \geq 0$. Give an example of a stochastic process $M(t)$ satisfying

$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)], \quad \forall t \geq 0$$

and which is not a martingale with respect to its own filtration, (i.e., $\mathcal{F}(t) := \sigma\{M(s) | s \leq t\}$).

10. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{(X_n, \mathcal{F}_n) : n \geq 0\}$ is a supermartingale and $\mathbb{E}(X_n) = c \in \mathbb{R}$, $\forall n$. Then show that $\{(X_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale.
11. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{(X_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale such that $|X_n| \leq M$ \mathbb{P} -almost everywhere on Ω for all $n \geq 0$. We define $Y_n := \sum_{k=1}^n \frac{1}{k}(X_k - X_{k-1})$ for all n . Show that $\{(Y_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale.
12. Let X_n be simple symmetric random walk, with $X_0 = 0$. Let $\tau = \inf\{n \geq 5 : X_{n+1} = X_n + 1\}$ be the first time after 4 which is just before the chain increases. Let $\rho = \tau + 1$
- Is τ a stopping time?
 - Is ρ a stopping time?