MA 372 : Stochastic Calculus for Finance

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Exercises 5

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1. Let $M(t), t \geq 0$ be a martingale relative to a filtration $\mathcal{F}_t$. Assume that f is a step stochastic process in $L^2_{step}([0,T] \times \Omega)$, i.e.,

$$f(t,\omega) = \sum_{i=1}^{n} \xi_{i-1} 1_{[t_{i-1},t_i)}(t)\,,$$

where $\{t_0, t_1, \ldots, t_n\}$ is a partition of $[0,T]$ and ξ_{i-1} is $\mathcal{F}_{t_{i-1}}$ measurable and $\mathbb{E}(\xi_{i-1}^2) < \infty$. For $t_k < t \leq t_{k+1}$, define

$$I_t(f) = \sum_{i=0}^{k-1} f(t_i)(M(t_{i+1}) - M(t_i)) + f(t_k)(M(t) - M(t_k)) := \int_0^t f(s)dM(s).$$

Show that the stochastic integral $\int_0^t f(s)dM(s)$ is a martingale relative to a the same filtration $\mathcal{F}_t$.

2. Check whether the process $X(t) = W_1(t)W_2(t)$, where $(W_1(t), W_2(t))$ is 2-dimensional Brownian motion, is a martingale with respect to Brownian filtration

3. Use Ito's-formula to write the stochastic process $Y(t) = e^{W(t)} + t^2$ on the standard form

$$dY(t) = b(t, Y(t))dt + \sigma(t, Y(t))dW(t).$$

4. For $c, \alpha_1, \cdots, \alpha_n$ constants, define $X(t) = e^{ct + \sum_i^n \alpha_i W_i(t)}$, where $(W_1(t), \cdots, W_n(t))$ is a n-dimensional Brownian motion. Prove that

$$dX(t) = (c + \frac{1}{2}\sum_i^n \alpha_i^2)X(t)dt + X(t)\sum_i^n \alpha_i dW_i(t)$$

5. Let $(W_1(t), W_2(t))$ be a 2-dimensional Brownian motion. Which one of the following is a Brownian motion?

   (i) $B_1(t) = \int_0^t \sin(s)dW_1(s) + \int_0^t \cos(s)dW_1(s)$

   (i) $B_2(t) = \int_0^t \sin(s)dW_1(s) + \int_0^t \cos(s)dW_2(s)$

   (i) $B_3(t) = \int_0^t \sin(W_1(s))dW_1(s) + \int_0^t \cos(W_1(s))dW_2(s)$

6. Suppose $W_1(t)$ and $W_2(t)$ are Brownian motions and

$$dW_1(t)dW_2(t) = \rho(t)dt,$$

where ρ is a stochastic process taking value strictly between 1 and -1. Define processes $B_1(t)$ and $B_2(t)$ such that

$$\begin{aligned} W_1(t) &= B_1(t),\ B_2(0) = 0, \text{ and} \\ W_2(t) &= \int_0^t \rho(s)dB_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dB_2(s). \end{aligned}$$

Show that (B_1, B_2) is a 2-dimensional Brownian motion.

7. Let $(W_1(t), W_2(t), W_3(t))$ be a 3-dimensional Brownian motion and
   $X(t) = \int_0^t \sin(W_3(s))dW_1(s) + \int_0^t \cos(W_3(s))dW_2(s)$
   $Y(t) = \int_0^t \cos(W_3(s))dW_1(s) + \int_0^t \sin(W_3(s))dW_2(s)$
   (i) Is $X(t)$ a Brownian motion?
   (ii) Is $Y(t)$ a Brownian motion?
   (iii) Is $(X(t), Y(t))$ a two-dimensional Brownian motion?

8. Consider the Black-Scholes Market. Define $\theta = \frac{\alpha - r}{\sigma}$ and

$$Y(t) = e^{-\theta W(t) - (r + \frac{1}{2}\theta^2)t}$$

   a) Show that $dY(t) = -\theta Y(t)dW(t) - rY(t)dt$.
   b) Let $X(t)$ denote the value of an investor's portfolio when he uses a portfolio process $\Delta(t)$. Hence we have

$$dX(t) = rX(t)dt + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)S(t)\sigma dW(t).$$

   Show that $Y(t)X(t)$ is a martingale.
   c) Let $T > 0$ be a fixed terminal time. Show that if an investor wants to begin with some initial capital $X(0)$ and invest in order to have portfolio value $V(T)$ at time T, where $V(T)$ is a given $\mathcal{F}(T)$-measurable random variable, then he must begin with initial capital

$$X(0) = \mathbb{E}[Y(T)V(T)].$$

   In other words, the present value at time zero of the random payment $V(T)$ at time T is $\mathbb{E}[Y(T)V(T)]$.

9. Let $W_1(t)$ and $W_2(t)$ be two independent Brownian motions and $-1 < \rho < 1$. Show that $W_3(t)$ defined by

$$W_3(t) = \rho W_1(t) + \sqrt{1 - \rho^2}W_2(t)$$

   is a Brownian motion. Use Ito's product rule to find $\mathbb{E}(W_1(t)W_3(t))$.

10. Let $W(t)$ be a Brownian motion and define

$$B(t) = \int_0^t sgn(W(s))dW(s)\,,$$

   where

$$sgn(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

   Show that $B(t)$ is a Brownian motion.