(Q1) (a) (i)

$$
\begin{aligned}
\tilde{\mathbb{P}}(\Omega) &= \int_\Omega Z(\omega) d\mathbb{P}(\omega) \\
&= \int_\Omega \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X(\omega)} d\mathbb{P}(\omega) \\
&= \int_{-\infty}^{\infty} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \lambda e^{-\lambda x} dx \\
&= \int_0^{\infty} \tilde{\lambda} e^{-\tilde{\lambda}x} e^{\lambda x} e^{-\lambda x} dx \\
&= \tilde{\lambda} \frac{e^{-\tilde{\lambda}x}}{-\tilde{\lambda}} \bigg|_0^\infty = 0 + 1 = 1.
\end{aligned}
$$

(ii) $\tilde{\mathbb{P}}(A) \geq 0 \; \forall \in \mathcal{F}$, since $Z > 0$. Therefore $1 \geq \tilde{\mathbb{P}}(A) \geq 0$

(iii) Let $\{A_i\}_{i \geq 1}$ be a disjoint sequence of events from $\mathcal{F}$. Set $B_\infty = \cup_{i=1}^\infty A_i$ and $B_n = \cup_{i=1}^n A_i$. Then

$$
\mathbb{I}_{B_n} = \sum_{i=1}^n \mathbb{I}_{A_i} \text{ and } \mathbb{I}_{B_n} \uparrow \mathbb{I}_{B_\infty} = \sum_{i=1}^\infty \mathbb{I}_{A_i}.
$$

This implies $Z\mathbb{I}_{B_n} \uparrow Z\mathbb{I}_{B_\infty}$. By monotone convergence theorem

$$
\begin{aligned}
\tilde{\mathbb{P}}(B_\infty) &= \mathbb{E}[Z\mathbb{I}_{B_\infty}] \\
&= \lim_{n\to\infty} \mathbb{E}[Z\mathbb{I}_{B_n}] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^n Z\mathbb{I}_{A_i}\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^n \tilde{\mathbb{P}}(A_i) = \sum_{i=1}^\infty \tilde{\mathbb{P}}(A_i).
\end{aligned}
$$

From (i), (ii) and (iii), we conclude that $\tilde{\mathbb{P}}$ is a probability measure.

(b)

$$
\begin{aligned}
\tilde{\mathbb{P}}(X \leq a) &= \int_{\{X \leq a\}} Z(\omega) d\mathbb{P}(\omega) \\
&= \int_\Omega \mathbb{I}_{\{X \leq a\}} \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)X} d\mathbb{P}(\omega) \\
&= \int_0^a \frac{\tilde{\lambda}}{\lambda} e^{-(\tilde{\lambda}-\lambda)x} \lambda e^{-\lambda x} dx \\
&= \int_0^a \tilde{\lambda} e^{-\tilde{\lambda}x} dx \\
&= -e^{-\tilde{\lambda}x} \bigg|_0^a \\
&= 1 - e^{-\tilde{\lambda}a}.
\end{aligned}
$$

(Q2) (a) Let $x_1 \leq x_2$. Then

$$A \cap [0, x_1] \subseteq A \cap [0, x_2].$$

This implies $\mathbb{P}(A \cap [0, x_1]) \leq \mathbb{P}(A \cap [0, x_2])$ and so, $f(x_1) \leq f(x_2)$.

(b) Let $x_n \downarrow x$. Set $B_n = A \cap [0, x_n]$ and $B = A \cap [0, x]$. Then

$$B = \cap_{n \geq 1} B_n.$$

So,

$$f(x) = \mathbb{P}(A \cap [0, x]) = \lim_{n \to \infty} \mathbb{P}(B_n) = \lim_{n \to \infty} f(x_n).$$

(c) If $x_n \uparrow x$, then

$$A \cap [0, x) = \cup_{n \geq 1}(A \cap [0, x_n]) \text{ and } \mathbb{P}(\{x\}) = 0.$$

So,

$$\begin{aligned} f(x) =& \mathbb{P}(A \cap [0, x]) \\ &= \mathbb{P}(A \cap [0, x)) \\ &= \lim_{n \to \infty} \mathbb{P}(A \cap [0, x_n]) \\ &= \lim_{n \to \infty} f(x_n). \end{aligned}$$

Hence f is continuous. Note that $f(0) = 0$ and $f(1) = \frac{1}{2}$ and f is continuous on $[0, 1]$. Therefore by I.V.P, $\exists\, x_0 \in (0, 1)$ such that $f(x_0) = \frac{1}{4}$.

(Q3) Range of $X = \{3, 4, 5, 6\}$. Now,

$$X^{-1}(\{3\}) = \{\omega \in \Omega | X(\omega) = 3\} = \{1\} \notin \mathcal{F}.$$

So, X is not a random variable with respect to $\mathcal{F}$. Let

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega \in \{2\} \\ 0 & \text{if } \omega \in \{3, 4, 5\}. \end{cases}$$

Then $Y^{-1}(B) \in \mathcal{F}\ \forall B \in \mathcal{B}(\mathbb{R})$. Also note that Y is a non-constant random variable.