FREQUENCY OF PATTERNS IN CERTAIN GRAPHS AND IN PENROSE TILINGS

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FREQUENCY OF PATTERNS IN CERTAIN GRAPHS AND IN PENROSE TILINGS

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I - INTRODUCTION

Among the aperiodic tilings recently used in crystallography [1-8] some, as 2D-Penrose tilings, are invariant by an operation generalizing self similarity, namely the so called inflation (or deflation, depending on the mood !) procedure. It is then natural to study tilings and more generally graphs which are invariant by such an operation. Some of the ideas and techniques introduced by the author [9-11] to build an abstract setting for the study of Mandelbrot's squigs [12-15] are used again in the present work.
In it we prove theorems on the frequency of appearance of finite patterns in certain colored graphs, examples of which are the graphs dual to Penrose tilings.

Here is the main result, stated, for the sake of simplicity, for 2D-Penrose tilings. Let $\{\mathcal{R}_n\}_{n\geqslant 1}$ be a sequence of regular plane domains the areas of which tend to infinity while the ratio of the perimeter of $\mathcal{R}_n$ to its area tends to zero as n goes to infinity (this last condition means that $\mathcal{R}_n$ does not flatten too quickly). If α is a bounded pattern appearing in a certain Penrose tiling, let us denote $N_n(\alpha)$ the number of times the pattern α appears within the domain $\mathcal{R}_n$. Then the ratio $N_n(\alpha)/\text{area}(\mathcal{R}_n)$ has a non-zero limit.

This result is to be compared to Conway's weak periodicity as well as the property of almost periodicity brought into light be several authors [16-18]. But it is to be noticed that none of these properties implies any other of them although each one tells something about the correlations within such a tiling.

The use of graphs may seem complicated, but it allows a unified treatment of different situations, for instance decoration of Penrose tilings and aperiodic coloration of regular lattices.

For the reader who is not willing to go through all mathematical details, the particular case of word substitutions is recalled in section II and the outline of the proofs is given in section III for a particular realization of a 2D-Penrose tiling. The complete setting and proofs are given in sections IV-VII. Although the language of graph theory is used, no perequisites in this theory is needed.

The author acknowledges P. Assouad and F. Axel for first introducing him to Penrose tilings.

II - THE ONE DIMENSIONAL CASE : WORD SUBSTITUTIONS

Let A be a finite set which we call "alphabet". For each a in A, a word $\sigma(a)$, constructed over the alphabet A, is given. Such a data σ is called a substitution.

If $w = x_1 x_2 \dots x_\nu$ is a word over A, we denote $\sigma(w)$ the word $\sigma(x_1)\sigma(x_2)\dots\sigma(x_\nu)$ obtained by putting end to end the words $\sigma(x_1),\sigma(x_2),\dots,\sigma(x_\nu)$. These substitutions have been studied from different points of view by many authors [11, 19-23].

Let us take an example : $A = \{0,1\}$, $\sigma(0) = 011$, $\sigma(1) = 01$. We then have $\sigma^2(0) = 0110101$, $\sigma^3(0) = 01101010110101101$ and so on. One can remark that the word $\sigma^{n+1}(0)$ begins by $\sigma^n(0)$; Therefore there is an infinite sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$ of 0's and 1's which is the limit in a suitable sense of $\sigma^n(0)$. This infinite sequence is invariant under the application of σ.

Let x_n and y_n denote the numbers of 0's and 1's in the word $\sigma^n(0)$. As each 0 generates one 0 and two 1's and each 1 generates one 0 and one 1, one has the following recursion formula

$$x_{n+1} = x_n + y_n$$

$$y_{n+1} = 2x_n + y_n$$

or $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = M \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, where $M = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

(note that two different substitutions may have the same matrix M).

Thus $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

So, as a consequence of Perron-Frobenius theory (see the appendix), we have $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \sim \lambda^n \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, where λ is the largest eigenvalue of M and $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is a corresponding right eigenvector. So the relative frequency of 0's and 1's is α/β. Evidently such a result holds for general substitutions. It can be rephrased in the following form :

$$\lim_{n \to \infty} \frac{1}{|\sigma^n(0)|} \sum_{j=1}^{|\sigma^n(0)|} \varepsilon_j = \frac{\alpha}{\alpha+\beta} ,$$

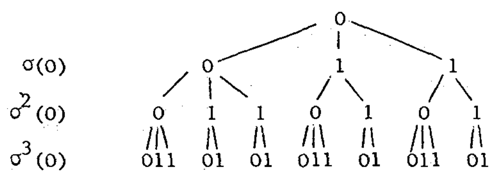
where $|\sigma^n(0)|$ is the length of the word $\sigma^n(0)$.

The next problem is to determine whether or not the sequence $\{\varepsilon_j\}$ has an autocorrelation sequence, i.e. to determine if $\frac{1}{|\sigma^n(0)|} \sum_{j=1}^{|\sigma_n(0)|-k} \varepsilon_j \varepsilon_{j+k}$ has a limit for each positive integer k. A way of counting the non-zero terms in this last sum is to count the number of occurences in $\sigma^n(0)$ of words of lenth $k+1$ which begin and end by a 1. So one is led to study the frequency of appearance of any word w in the sequence $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \dots$

Let us explain how the existence of a frequency for words of length 2 can be proved. The reader would then supply the proof for the general case (see also [11]. The tool is the change of alphabet described below.

If $w = x_1 x_2 \dots x_\nu$ is a word over A , let us write $\tilde{w} = (x_1x_2)(x_2x_3)(x_3x_4)\dots(x_{\nu-1}x_\nu)(x_\nu\varepsilon)$, where ε is a new symbol indicating ends of words. Then $\tilde{w}$ appears to be a word over a new alphabet the "letters" of which are words of length 2.

Let us consider the above example and draw the following diagram showing the "genealogy"



and perform the change of alphabet :

$$(0\varepsilon)$$

$(\sigma(0))^{\sim}$: $(01)\quad(11)\quad(1\varepsilon)$

$(\sigma^2(0))^{\sim}$: $(01)\ (11)\ (10)\quad(01)\ (10)\quad(01)\ (1\varepsilon)$

$(\sigma^3(0))^{\sim}$: $(01)(11)(10)\ \ (01)(10)\ \ (01)(10)\ \ (01)(11)(10)\ \ (01)(10)\ \ (01)(11)(10)\ \ (01)(1\varepsilon)$

It is then clear that $(\sigma^j(0))^{\sim} = (\tilde{\sigma})^j(0\varepsilon)$ where $\tilde{\sigma}$ is the following substitution acting on the alphabet $\tilde{A} = \{(01),(10),(11),(1\varepsilon)\}$:

$$\tilde{\sigma}(01) = (01)(11)(10)$$
$$\tilde{\sigma}(10) = (01)(10)$$
$$\tilde{\sigma}(11) = (01)(10)$$
$$\tilde{\sigma}(1\varepsilon) = (01)(1\varepsilon)\ .$$

Therefore the number of occurences of words of length 2 in $\sigma^n(0)$ is governed by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and their relative frequencies are the components of the normalized eigenvector corresponding to $\lambda = 1 + \sqrt{2}$.

Indeed this analysis is valid in general, not only on this example.

In fact we have only proved the following proposition : if w is a word, then the ratio

$$\frac{1}{|\sigma^n(0)|} \times \text{number of } w\text{'s in } \sigma^n(0)$$

tends to a limit as n goes to infinity. But a stronger result holds : if a_n and b_n are two sequences of positive integers such that b_n tends to infinity, then the ratio.

$$\frac{1}{b_n} \times \text{number of } w\text{'s in } \varepsilon_{a_n}\varepsilon_{a_n+1}\cdots\varepsilon_{a_n+b_n}$$

tends to a limit, as n goes to infinity, uniformly with respect tot a_n's.

The treatment of Penrose tilings and graphs follows the same lines although lots of technicalities indeed appear, due to the greater combinatorial complexity of tilings and graphs.

III - PENROSE TILINGS

There are several ways of constructing Penrose tilings [24-31]. We use Robinson's approach [26]. A similar description has been used by Dekking [29]. We consider the isoceles triangles of figure 1.

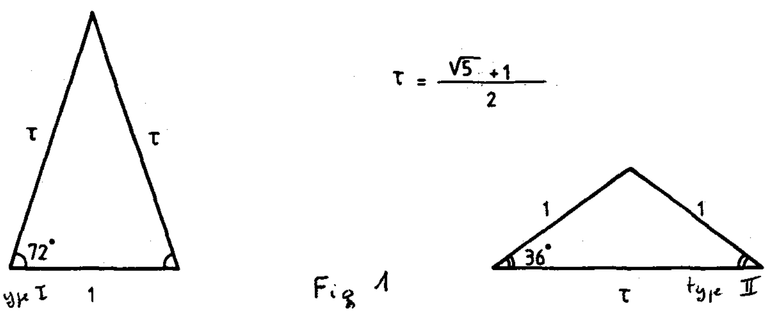


Fig 1

The tiling elements we consider are triangles equal to one of those, the vertices of which are colored with two colours so that the vertices of a triangle corresponding to equal angles have different colours. In the subsequent figures one of the colours, say the white, will not be represented, the other one, the black, will be indicated by small circles around the corresponding vertices. We observe that, due to different colorations and symmetries, there are eight different kinds of tiles as shown on figure 2.

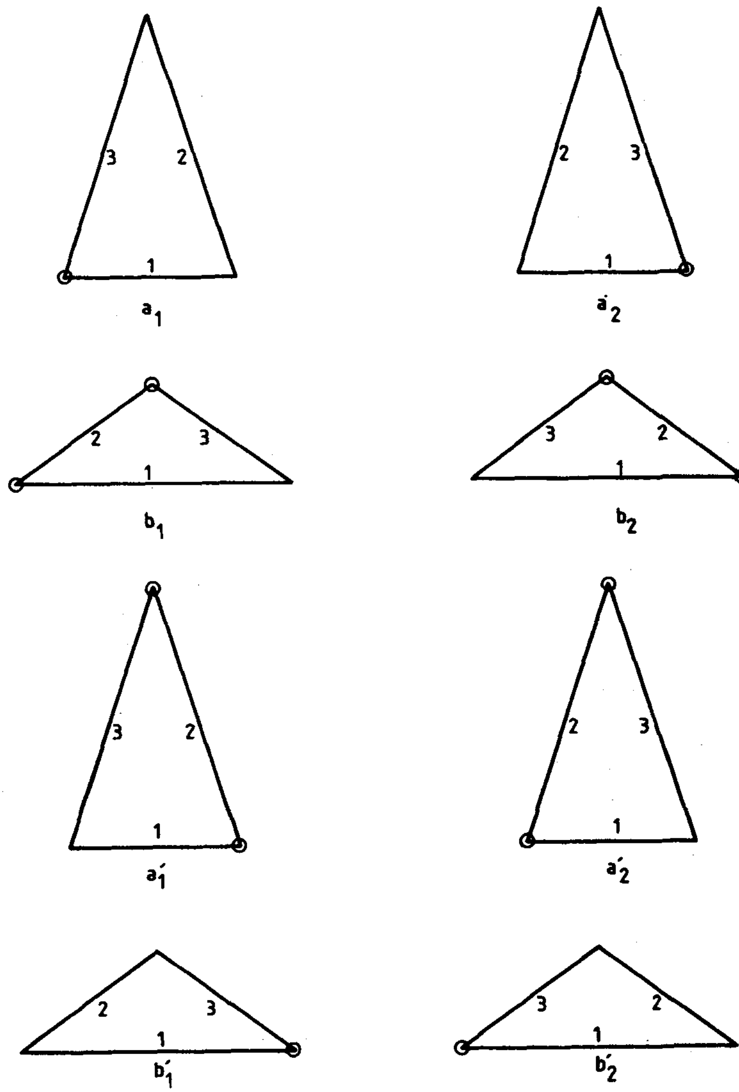


Figure 2

The eight different tiles with an enumeration of their sides.

The tilings we are considering obey the following rules :

1° They are composed either of tiles a_1, a_2, b_1 and b_2 or of tiles a_1', a_2', b_1' and b_2'.

2° The tiles are assembled so that the colours of their vertices match.

3° If two tiles are in contact along one side the vertices of which have the same colour, the larger angles of each tiles adjacent to this side also match.

Let us consider such a tiling of a plane domain. One obtains a finer tiling by replacing each tile according to the rule described by figure 3.

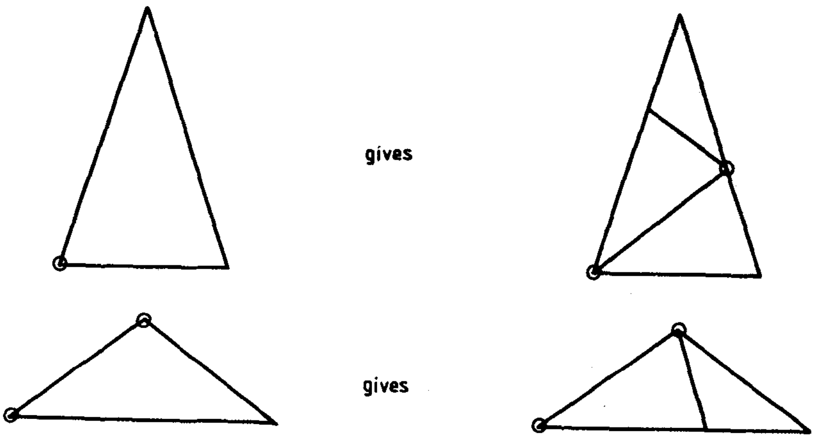


Figure 3
First step of the inflation procedure.

This figure only gives the substitution rules for two kinds of tiles, the other ones being deduced by permutation of colors or symmetry. If one expands the new tiling by the factor τ one gets a tiling of a plane domain by triangles of types I and II satisfying the above requirements on colors. The operation we have just described is commonly called inflation. It enables us, starting from a tesselation of a plane domain, to get a tesselation of a larger one. By iterating inflation and taking weak limits one gets tilings of the plane which are inflation invariant. These are the Penrose tilings.

Let us take an example. We consider OAB a triangle of type I with white vertices O and B. Let us apply the inflation procedure twice, using dilations centered at O. The result is shown on figure 4. The tile OAB appears in the new tiling, to within a symmetry, and would have been found exactly had we applied four times the inflation procedure. So the sequence of tilings obtained by applying $4n$ times the inflation procedure converges as n goes to infinity to a tiling of a sector of the plane of aperture $\pi/5$. The tesselation obtained by completing this one using symmetries and rotations of angle $2\pi/5$ is the Penrose tiling of the plane with pentagonal symmetry.

This operation of inflation is a kind of substitution : a tile of type a_1 gives one tile of type a_1', one of type a_2' and one of type b_2' and similarly for the other types. Therefore the numbers of each type of tiles after n applications of the inflation procedure are given by the n-th power of a certain matrix.

In order to be able to count not only each type of tiles but also the occurences of each finite pattern, we are going to describe tilings by the mean of their dual graphs.

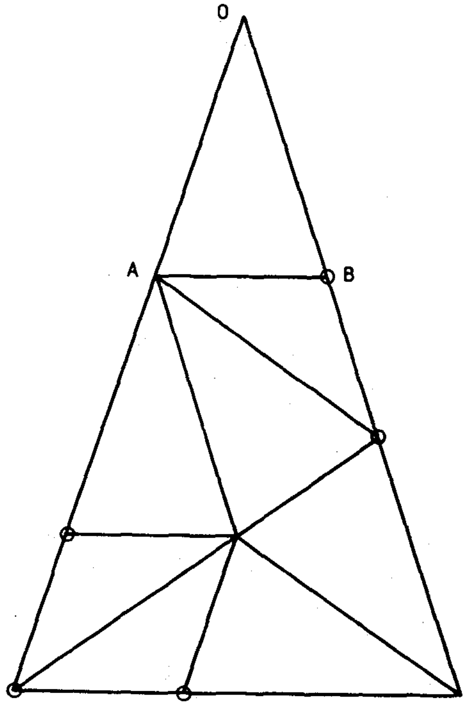


Figure 4
Result of two applications of the inflation procedure

Given a tiling (finite or not) one can consider the graph the vertices of which are the elements of the tiling, two tiles being linked by an edge of the graph if and only if they share one of their sides. One keep in memory the type of each tile by indicating it at the corresponding node of the graph and also the way tiles are connected by tagging each edge of the resulting graphs. The type of a tile will be called the colour of the corresponding vertex of the dual graph. Figure 5 shows a tesselation with indication of types and of the tagging of each side within each tile. Figure 6 shows the coding of this tesselation as a graph. Such a graph will be in the sequel called a coloured tagged graphs.

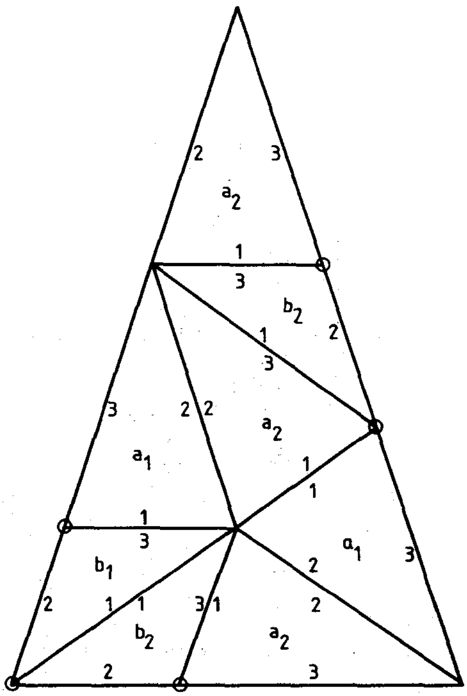


Figure 5 (on the left)

Same as figure 4, but with the indication of the "colours" of triangles and of the numeration of their sides.

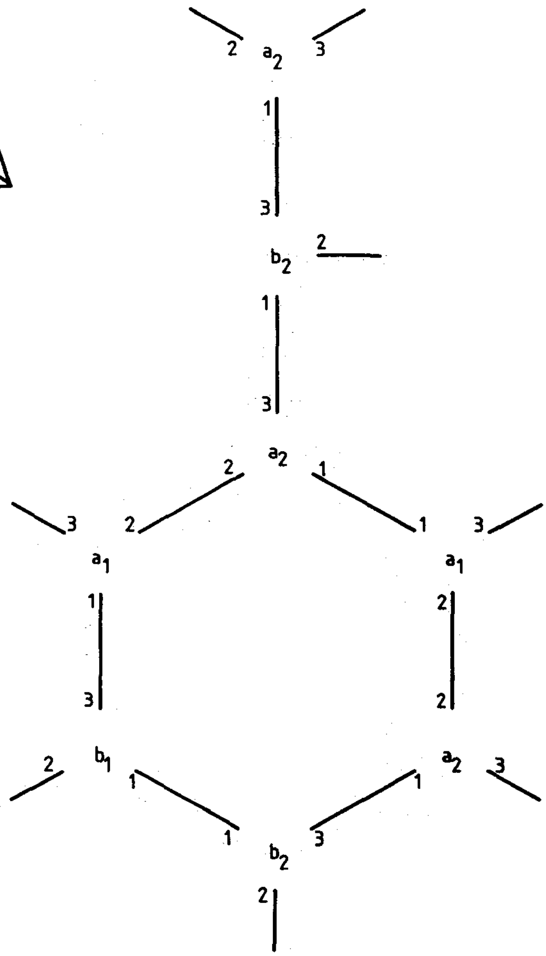


Figure 6 (on the right)

Dual to figure 5

Now, we have to describe in this setting the operation of inflation. Consider for instance the splitting of a tile of type a_2' as shown on figure 7.

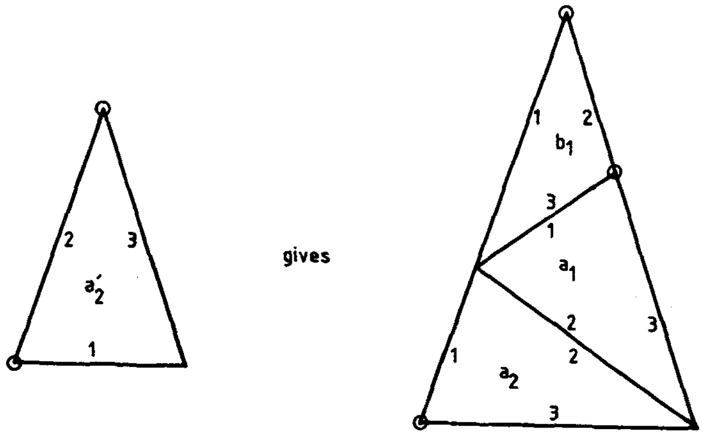


Figure 7 (inflation rule for tile a_2')

The coding of this situation is shown on figure 8. As this graph is to be linked to others, it has dangling bonds. These dangling bonds have been separated into three classes W^1 , W^2 and W^3, taking into account which edge of the original tile they come from. The arrows indicate an order of enumeration of these dangling bonds in each class. The resulting figure is an example of what is called an ion in the sequel.

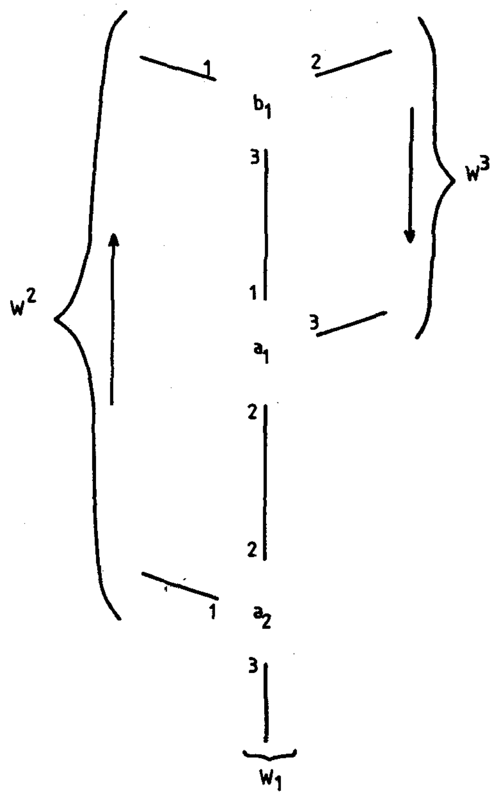


Figure 8

Figure 9 shows a two-tiles tesselation and the corresponding graph. Figure 10 shows the inflation process on this tesselation and the corresponding operation on graphs

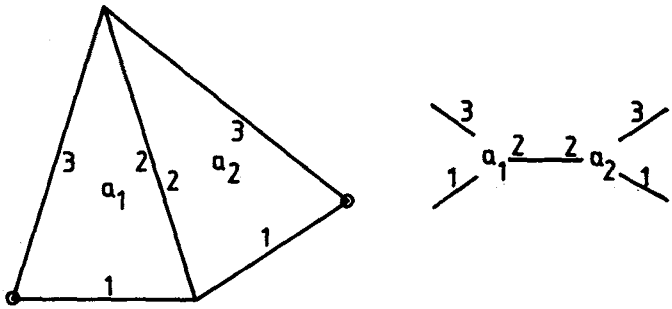


Figure 9

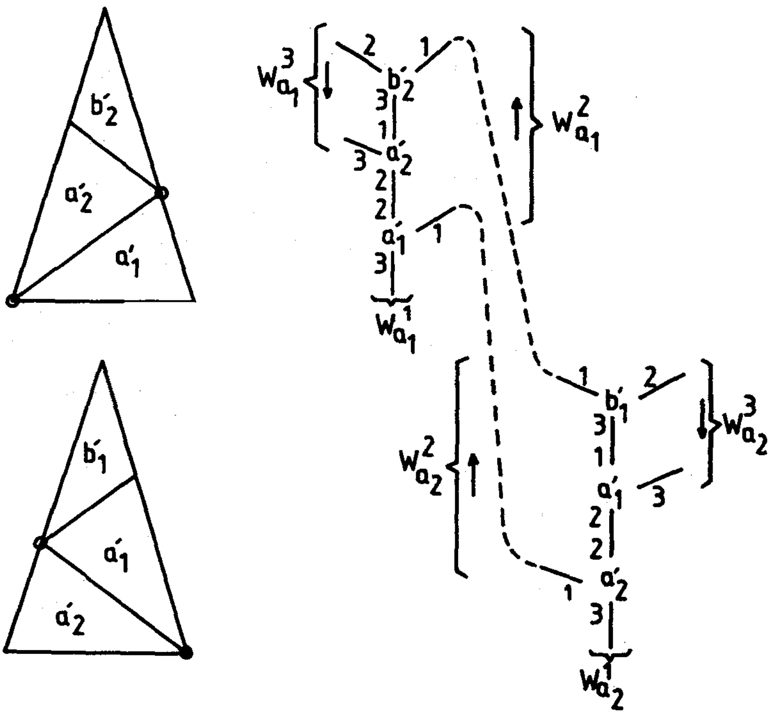


Figure 10

Binding of ions

This operation consists in replacing each node of the original graph by an ion and binding these ions as shown on figure 10 : each edge of the original graph directs the binding of corresponding dangling bonds. This operation is analogous to the word-substitutions previously considered. Figure 11 shows the result of this operation.

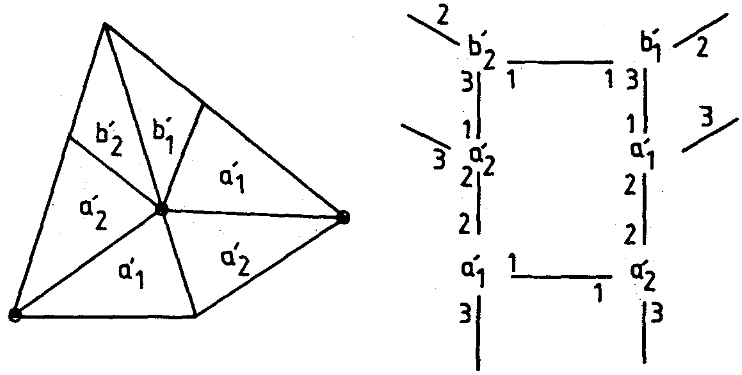


Figure 11

Up to now, we have just given an alternative description of tilings and inflation procedure, but this enables us to prove that patterns occur with a well determined frequency. As previously for words-substitutions, the proof relies upon changes of colours. Here is the sketch of the proof.

By successive applications of the inflation procedure we get a sequence $\{\underset{\sim}{G}_n\}_{n\geqslant 1}$ of colored tagged graphs. If we suitably define new colours, the sequence $\bar{G}_n$ obtained by changing colours in G_n can be generated by a new inflation procedure, which provides information on frequencies of new colours.

We use two changes of colours. This first one aims at studying the growth of the boundary $\partial\underset{\sim}{G}_n$ of G_n. It is shown that the number of elements of ∂G_n grows as $\tilde{\lambda}^n$ where λ is the Perron-Frobenius eigenvalue of a certain matrix. In the Penrose case one has $\tilde{\lambda} < \lambda$ (this can also be shown by using area-perimeter argument). In the general case we have to assume $\tilde{\lambda} < \lambda$. Then we perform a second change of colours : the new colour of a vertex describes its surroundings up to distance k. It provides us with a matrix which governs the numbers of patterns of size k in G_n. After that there is still some work to obtain the result for any limit point of the sequence $\{G_n\}$ and for any sequence $\{\mathcal{R}_n\}$ of domains.

IV - TAGGED GRAPHS, IONS, BINDINGS

1. Tagged graphs.

A finite non-empty set F is given throughout the paper. The graphs we are going to consider are non-directed and their vertices have orders less than or equal to $\# F$, the cardinality of F.

Definition. A tagged graph is a couple $Z = (V,E)$ where V is a set and E a subset of $V\times F\times V\times F$ so that

a) $(a,m,b,n)\in E$ implies $a \neq b$,

b) $(a,m,b,n)\in E$ implies $(b,n,a,m)\in E$,

c) if a and b are elements of V, then the set $\{(m,n)\in F\times F\ ;\ (a,m,b,n)\in E\}$ has one element at most,

d) if (a,m) is an element of $V\times F$, then the set $\{(b,n)\in V\times F\ ;\ (a,m,b,n)\in E\}$ has one element at most.

The element of V are the vertices of Z, those of E its edges.

As we shall have to consider several tagged graphs simultaneously, we shall be more specific if needed : if Z is a tagged graph, V_Z will denote the set of its vertices and E_Z that of its edges.

To each tagged graph Z we associate an ordinaty graph $Z^\natural$ in the following way. The graph $Z^\natural$ has the same vertices as Z. The set $E_Z^\natural$ of its edges is so defined : $(a,b) \in V_Z \times V_Z$ belongs to E_Z if and only if there exists (m,n) in $F \times F$ such that (a,m,b,n) belong to E_Z.

A tagged graph Z is said to be connected if $Z^\natural$ is. Then the distance within Z , denoted d_Z , of two of its vertices is, by definition, their geodesic distance along $Z^\natural$ (i.e. the minimum number of edges to go through to connext them).

If Z is a tagged graph we set

$$W_Z = \left\{(a,m) \in V_Z \times F \ ; \begin{array}{l} \text{there exists no } (b,n) \text{ in } V_Z \times F \\ \text{such that } (a,m,b,n) \in E_Z \end{array}\right\}$$

and

$$\partial Z = \{a \in V_Z \ ; \text{ there exists } m \text{ in } F \text{ such that } (a,m) \in W_Z\}.$$

Figure 6, if we forget the names of the nodes, shows a picture of a tagged graph.

Let us consider a tagged graph $Z = (V_Z, E_Z)$ and U a subset of V_Z. We are going to define a tagged graph which we call sub tagged graph of Z associated to U : its set of vertices is U and its set of edges is $\{(a,m,b,n) \ ; \ (a,b) \in U \times U\}$. If x is a vertex of Z and r a positive integer, $B_Z(x,r)$ stands for the sub tagged graph of Z associated to the ball $\{y \in V_Z \ ; \ d_Z(x,y) \leq r\}$. It will be called the ball of center x and radius r of Z.

Two tagged graphs Z_1 et Z_2 are isomorphic if there exists a one-to-one mapping φ from V_{Z_1} onto V_{Z_2} such that $(\varphi(a), m, \varphi(b), n)$ be in V_{Z_2} if and only if (a,m,b,n) is in V_{Z_1}.

2. Colorations, ions.

Let A be a finite set, which we call set of colours. An A-coloration of a tagged graph is a mapping g from V_Z to A. From now on, an A-colored tagged graph will be simply called an A-graph.

If Z is an A-graph and U a subset of V_Z , Z_U denotes the A-graph obtained by restricting the coloration of Z to the sub tagged graph of Z associated to U. $B_Z(x,r)$ will also denote the ball $B_Z(x,r)$ defined above endowed with the coloration of Z.

An isomorphism φ of the A-graph $Z = (V_Z, E_Z, g_Z)$ on the A-graph Z' is an isomorphism of the underlying tagged graphs which is compatible with the colorations : i.e. $g_{Z'} \circ \varphi = g_Z$.

<u>Definition</u>. <u>We shall call a finite A-graph</u> Z <u>with a partition</u> $\{W_Z^m\}_{m \in F}$ <u>of</u> W_Z <u>by non-empty sets and, for any</u> m <u>in</u> F , <u>a total ordering of</u> W_Z <u>an A-ion</u>.

Two A-ions $Z_j = (V_{Z_j}, E_{Z_j}, \{W_{Z_j}^m\}_{m \in F}, g_{Z_j})$, (for $j = 1,2$), are isomorphic if there exists a one-to-one mapping φ from V_{Z_1} onto V_{Z_2} such that φ be an isomorphism of the underlying tagged graphs and, for any m in F , an isomorphism of the ordered sets $W_{Z_1}^m$ and $W_{Z_2}^m$.

Figure 8 shows an ion.

3. Binding of ions.

We are given a tagged graph $Z = (V,E)$ and, for any x in V, an A-ion $Z_x = (V_{Z_x}, E_{Z_x}, \{W^m_{Z_x}\}_{m\in F}, g_{Z_x})$ in such a way that whenever (x,m,y,n) is in E the sets $W^m_{Z_x}$ and $W^n_{Z_y}$ have the same cardinality. Then we define an A-graph $Z' = (V',E',g')$ by the following three conditions.

a) V' is the disjoint union of the sets $\{V_{Z_x}\}_{x\in V}$,

b) g' extends any of the mappings g_{Z_x},

c) E' is the disjoint union of the sets $\{E_{Z_x}\}_{x\in E}$ and of the set $\tilde{E}$ so defined :

(α,j,β,k) is in $\tilde{E}$ if and only if there exists (x,m,y,n) in E such that (α,j) be in $W^m_{Z_x}$, (β,k) be in $W^n_{Z_y}$ and (α,j) and (β,k) match according to the orders on $W^m_{Z_x}$ and $W^n_{Z_y}$.

The A-graph Z' will be called the binding of the ions $\{Z_x\}_{x\in V}$ directed by the tagged graph Z.

The binding process is illustrated by figures 9, 10 and 11.

V - PENROSE TILINGS AS A-GRAPHS

Let F be the set $\{1,2,3\}$. The set A has eight elements a_1, a_2, b_1, b_2, a'_1, a'_2, b'_1, b'_2 which correspond to triangles as shown in figure 2. In figure 2 also appears a numbering of the edges.

Let us consider for instance the tesselation obtained by applying twice the inflation procedure to a triangle of type a_1 as shown in figure 4. Figure 5 then shows the types of triangles which appear and the numbering of their edges. The dual to figure 5 is figure 6. In other words it is an A-graph Z :

$$V_Z = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$$

$$g_Z(x_1) = a_2,\ g_Z(x_2) = b_2,\ g_Z(x_3) = a_2,\ g_Z(x_4) = a_1\ ,$$

$$g_Z(x_5) = b_1,\ g_Z(x_6) = b_2,\ g_Z(x_7) = a_2,\ g_Z(x_8) = a_1$$

$$\begin{aligned} E_Z = \{&(x_1,1,x_2,3),(x_2,3,x_1,1),(x_2,1,x_3,3),(x_3,3,x_2,1),\\ &(x_3,2,x_4,2),(x_4,2,x_3,2),(x_4,1,x_5,3),(x_5,3,x_4,1),\\ &(x_5,1,x_6,1),(x_6,1,x_5,1),(x_6,3,x_7,1),(x_7,1,x_6,3),\\ &(x_7,2,x_8,2),(x_8,2,x_7,2),(x_8,1,x_3,1),(x_3,1,x_8,1)\}\end{aligned}$$

$$W_Z = \{(x_1,2),(x_1,3),(x_2,2),(x_4,3),(x_5,2),(x_6,2),(x_7,3),(x_8,3)\}$$

$$\partial Z = \{x_1,x_2,x_4,x_5,x_6,x_7,x_8\}$$

VI - SUBSTITUTIONS

1. Definition of substitutions.

As previously two finite sets A and F are given. An A-substitution is a mapping σ, which associates an A-ion to each element of A, together with a set $\mathcal{G}$ of A-graphs, subjected to the following requirements :

a) $\mathcal{G}$ contains A, the elements of which are identified to A-graphs with a single vertex,

b) if $Z = (V,E,g)$ is in $\mathcal{G}$, then the binding of the family $\{\sigma(g(x))\}_{x \in V}$ directed by Z can be done and the resulting A-graph, denoted $\sigma(Z)$, is in $\mathcal{G}$.

In other words, one passes from Z to $\sigma(Z)$ by replacing each node of Z by a graph according to its color. The vertices of the ion by which a vertex x of Z has been replaced are called the descendents of the first generation of x.

This notion generalizes that of substitution operating on words. It has been introduced in [10] in order to give an abstract setting for Mandelbrot's squigs [12-15].

Several facts are to be noticed :

- If the A-graph Z is connected, so is $\sigma(Z)$.
- σ does not decrease distances. It means the following : let x and y be two elements of V, then if x' and y' are descendents of x and y respectively one has $d_{\sigma(Z)}(x',y') \geq d_Z(x,y)$.
- σ can be iterated : if $Z \in \mathcal{G}$, then we get a sequence $\sigma^n(Z)$ of A-graphs. We are mostly interested in the behaviour of $\sigma^n(a)$ for a in A.
- If $Z = (V,E)$ is an element of $\mathcal{G}$ and if U is a subset of V, then $\sigma^n(Z_U) = (\sigma^n(Z))_{U_n}$, where U_n is the n^{th}-generation offspring of the element of U.

2. Example.

The above construction of Penrose tilings can be rephrased in terms of substitution : in chapter V these tilings have been coded as A-graphs and the inflation procedure gives the rules of substitution. Let us for instance define the ion $\sigma(a_2')$. Figure 7 shows the inflation of a tile of type a_2' and figure 8 shows its codings as an ion :

$$V_{\sigma(a_2')} = \{(x_1,x_2,x_3)\}$$

$$g_{\sigma(a_2')}(x_1) = b_1,\ g_{\sigma(a_2')}(x_2) = a_1,\ g_{\sigma(a_2')}(x_3) = a_2$$

$$E_{\sigma(a_2')} = \{(x_1,3,x_2,1),(x_2,1,x_1,3),(x_2,2,x_3,2),(x_3,2,x_2,2)\}$$

$$W^1_{\sigma(a_2')} = \{(x_3,3)\}$$

$$W^2_{\sigma(a_2')} = \{(x_3,1),(x_1,1)\}$$

$$W^3_{\sigma(a_2')} = \{(x_1,2),(x_2,3)\}$$

(the order of enumeration of the elements of $W^m_{\sigma(a_2')}$ defines the total ordering of this set). This partition of $W_{\sigma(a_2')}$ and the corresponding orderings are obtained by analysing from which edge of the initial triangle the pending bonds come, and in which order they appear according to the orientation induced by the numbering of the edges of the initial triangle.

The other ions $\{\sigma(a)\}_{a \in A}$ are defined in the same way. If a is an element of A, the A-graph $\sigma^n(a)$ describes the Penrose tiling obtained after applying n times the inflation procedure to a triangle of type a.

3. Matrix of a substitution

If Z is an A-graph $L(Z)$ denotes the vector in $\mathbb{R}^A$ which describes the composition in colors of the vertices of Z : the component $L_a(Z)$ of $L(Z)$ corresponding to the element a in A is the number of vertices of Z the color of which is a.

A square matrix M indexed by $A \times A$ is associated to the substitution σ in the following way : the column of M corresponding to the element b of A

is $L(\sigma(b))$. For any Z in $\mathcal{G}$ the following relation holds : $L(\sigma(Z)) = ML(Z)$.

For instance the matrix of the Penrose substitution, described in the previous paragraph, is

$$M_P = \begin{bmatrix} 0 & M'_P \\ M'_P & 0 \end{bmatrix} \quad \text{where} \quad M'_P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

if the colors are so ordered : $a_1, a_2, b_1, b_2, a'_1, a'_2, b'_1, b'_2$.

4. First change of colours

Consider the set $\tilde{A} = A \times 2^F$, where 2^F stands for the set of subsets of F. If $Z = (V,E,g)$ is an A-graph, we define on it an A-coloration $\tilde{g}$: $\tilde{g}(x) = (g(x),B)$ where B is the set $B = \{m \in F$; there is no (y,n) in $V \times F$ such that (x,m,y,n) be in $E\}$. The A-graph so obtained will be denoted $\tilde{Z}$.

We aim at defining a substitution $\tilde{\sigma}$ acting on $\tilde{A}$ in such a way it be equivalent to σ. The set of A-graphs on which it will operate is $\tilde{\mathcal{G}} = \{\tilde{Z}\ ;\ Z \in \mathcal{G}\}$. Let us now define $\tilde{\sigma}(a,B)$ for a in A and B in 2^F.

- for $\sigma(a)$ and $\tilde{\sigma}(a,B)$ the underlying tagged graphs are the same,
- if x is a vertex of $\tilde{\sigma}(a,B)$ (therefore also of $\sigma(a)$), its

colour is $(g(x),B')$ where

$$B' = \left\{m \in F\ ;\ \{(y,n) \in V_{\sigma(a)} \times F\ ;\ (x,m,y,n) \in E\} = \emptyset\right\} \setminus \bigcup_{j \notin B} \left\{m \in F\ ;\ (x,m) \in W^j_{\sigma(a)}\right\}$$

- the ordered sets $W^j_{\sigma(a)}$ and $W^j_{\tilde{\sigma}(a,B)}$ are identical.

For any Z in $\mathcal{G}$ we have $\tilde{\sigma}(\tilde{Z}) = (\sigma(Z))^{\sim}$. The partition $A = (A \times \{\emptyset\}) \cup (\tilde{A} \setminus (A \times \{\emptyset\}))$ induces the following decomposition into blocks of the matrix $\tilde{M}$ of $\tilde{\sigma}$:

$$\tilde{M} = \begin{bmatrix} M & M' \\ 0 & M'' \end{bmatrix}$$

It is to be noticed that the matrix M of σ appears as one of these blocks.

As previously, a vector $\tilde{L}(H)$ in $\mathbb{R}^{\tilde{A}}$ in associated to any $\tilde{A}$-graph H. If Z is an A-graph, then the sum of the components of $\tilde{L}(\tilde{Z})$ which correspond to $\tilde{A} \setminus (A \times \{\emptyset\})$ is the number of elements of ∂Z.

5. Hypotheses

Terminology and a few facts about non-negative matrices can be found in the appendix.

The hypotheses we shall assume to hold from now on, unless otherwise specified, are the following :

a) the matrix M is primitive and its Perron-Frobenius eigenvalue λ is strictly greater than 1,

b) the largest eigenvalue $\tilde{\lambda}$ of M'' is strictly less than λ.

It results from hypothesis a) that, for any a in A, the vector $\lambda^{-n}L(\lambda^n(a))$, as n goes to infinity, tends to a vector any component of which is positive. Hypothesis b) implies that $\tilde{\lambda}^{-n}\ \mathrm{card}(\partial(\sigma^n(a)))$ has a polynomial growth as a function of n.

Example 1.

This example shows that hypotheses a) and b) are independent. Let $F = \{1,2,3\}$, $A = \{a\}$ and define the ion $\sigma(a) = (V,E,W^1,W^2,W^3)$ so :

$$V = \{u,v\} \qquad E = \{(u,1,v,1),(v,1,u,1)\}$$

$$W^1 = \{(v,2)\}\ ,\ W^2 = \{(u,2)\}\ ,\ W^3 = \{(u,3),(v,3)\}.$$

It is easily checked that $\lambda = \tilde{\lambda} = 2$.

Example 2 : the Penrose substitution

The matrix M_P is not primitive. But, if we consider σ^2, we see that when starting from a_1, it is a substitution in fact acting on $\{a_1, a_2, b_1, b_2\}$ as set of colors with matrix $(M_P')^2$, which is primitive.

It is easy to determine λ and $\tilde{\lambda}$ for the Penrose substitution. Starting from a triangle of type I and applying n times the inflation procedure we get a tesselation of T, a triangle of type I expanded by the factor τ^n. So, the number of tiles and the area of T have the same order of magnitude and so have its perimeter and the number of tiles touching its boundary. Therefore we have $\lambda = \tau^2$ and $\tilde{\lambda} = \tau$.

6. Second change of colours.

Let r be a nonnegative integer. Let us consider the set $\mathcal{A} = \{(Z,x)\ ;\ Z \in \mathcal{G}\ ,\ x \in V_Z\}$ and define an equivalence relation $\mathcal{R}_r$ on $\mathcal{A}$:

(Z,x) and (Z',x') are equivalent if there exists a mapping φ from $\{y \in V_Z\ ;\ d_Z(x,y) \leq r\}$ onto $\{y' \in V_{Z'}\ ;\ d_{Z'}(x',y') \leq r\}$ such that we have $\varphi(x) = x'$ and φ be an isomorphism of the A-graph $B_Z(x,r)$ onto $B_{Z'}(x',r)$.

The quotient space $\mathcal{A}/\mathcal{R}_r$ is denoted A_r. In other words, A_r is the set of classes of balls of radii r of elements of $\mathcal{G}$, modulo isomorphisms of A-graphs carrying centers onto centers.

Any A-graph Z in $\mathcal{G}$ gives an A_r-graph $Z^{(r)}$ in the following way :

- the underlying tagged graphs are the same for Z and $Z^{(r)}$,
- if x is a vertex, its colour in $Z^{(r)}$ is the class modulo $\mathcal{R}_r$ of $(B_Z(x,r),x)$.

In the same spirit as in paragraph 4 we are going to define an A_r-subtitution σ_r such that, for any Z in $\mathcal{G}$, we have $\sigma_r(Z^{(r)}) = (\sigma(Z))^{(r)}$. This substitution will act on the set $\mathcal{G}_r = \{Z^{(r)}\ ;\ Z \in \mathcal{G}\}$. Let α be an element of A_r. We now proceed to the definition of $\sigma^r(\alpha)$. Let us suppose that α is represented by $(B_Z(x,r),x)$ where $Z \in \mathcal{G}$ and $x \in V_Z$. The substitution procedure gives an isomorphism φ from the tagged graph lying under $\sigma(g_Z(x))$ onto the sub tagged graph of $\sigma(Z)$ the vertices of which are the descendents of x in $\sigma(Z)$. We now define a new coloration $g^{(r)}$ on $\sigma(g_Z(x))$: if y is a vertex of $\sigma(g_Z(x))$, then $g^{(r)}(y)$ is the class modulo $\mathcal{R}_r$ of $(B_{\sigma(Z)}(\varphi(y),r),\varphi(y))$. It results from the fact that σ does not decrease distances that the coloration g_r does not depend on the particular choice of the representant of α. Therefore, we can *define* $\sigma_r(\alpha)$ *as the ion* $\sigma(g_Z(x))$ the coloration of which has been replaced by $g^{(r)}$. It is easy to check that, as claimed above, for any Z in $\mathcal{G}$, we have $\sigma_r(Z^{(r)}) = (\sigma(Z))^{(r)}$ and thus $\sigma_r^n(Z^{(r)}) = (\sigma^n(Z))^{(r)}$.

Let us consider the following subset of $\mathcal{A}$: $\mathcal{A}_r = \{(Z,x)\ ;\ Z \in \mathcal{G}\ ,\ x \in V_Z\ ,\ d_Z(x,\partial Z) > r\}$ and denote A_r^1 the set of classes modulo $\mathcal{R}_r$ of elements of $\mathcal{A}_r$. We set $A_r^2 = A_r \setminus A_r^1$. Corresponding to the partition $A_r = A_r^1 \cup A_r^2$ the matrix M_r of σ_r decomposes into blocks :

$$M_r = \begin{bmatrix} M'_r & M'''_r \\ 0 & M''_r \end{bmatrix}$$

Proposition.

1° λ <u>is an eigenvalue of the matrix</u> M'.

2° <u>Any eigenvalue of</u> M' <u>has a modulus not larger than</u> λ .

3° <u>Any eigenvalue of</u> M'' <u>has a modulus not larger than</u> $\tilde{\lambda}$.

Proof.

These assertions result from the following facts :

- for any Z in $\mathcal{G}$, the number of vertices of $\sigma^n(Z)$ grows like λ^n,
- for any Z in $\mathcal{G}$, $\tilde{\lambda}^{-n}$ card$(\partial\sigma^n(Z))$ has a polynomial grows at most.

It is to be noticed that assertions 2 and 3 remain true without assuming hypotheses a) or b). It is, in fact, enough to conclude that the growth of λ^{-n} card$(V_{\sigma^n(Z)})$ be polynomial.

7. Precisions on the preceding paragraph.

Theorem. <u>Let</u> σ <u>be an A-substitution satisfying hypotheses a) and b). We suppose that</u> $\mathcal{G} = \{\sigma^n(a)\ ;\ n\geq 0,\ a\in A\}$. <u>Then, for any</u> r, <u>the matrix</u> M'_r <u>is primitive</u>.

Proof.

If α is an element of A^1_r , then we can choose an integer $n(\alpha)$, a in A and x in $V_{\sigma^n(a)}$ such that α be the class modulo $\mathcal{R}_r$ of $(\sigma^n(a),x)$ and such that $d_{\sigma^n(a)}(x, \partial\sigma^n(a)) > r$. Let us set $N = \sup\{n(\alpha)\ ;\ \alpha\in A^1_r\}$.

Let α be any element of A^1_r , then there exists $n(\alpha)$, a and x as above. Because M is primitive, there exists b in A such that $\sigma^{(N-n(\alpha))}(b)$ has a vertex the color of which is a. Due to the fact that σ does not decrease distances, there exists a vertex z of $\sigma^N(b)$ such that α be the classe modulo $\mathcal{R}_r$ of $(\sigma^N(z),z)$ and such that $d_{\sigma^N(b)}(z,\partial\sigma^N(b)) > r$.

Because M is primitive, there exists an integer N' such that, for any a in A, $\sigma^{N'}(a)$ has at least card(A^1_r) vertices of each color.

Let us consider now β in A^1_r. Then β is the class of (Z,x). As the preceding remarks show it, $(\sigma^{N+N'}(Z))^{(r)}$ contains vertices in the descent of x having any possible colour in A^1_r. Therefore the $(N+N')^{th}$ power of M'_r has all its entries strictly positive. This ends the proof.

It is to be noticed that hypothesis b) has not been used.

If α is in A^1_r and Z in $\mathcal{G}$, $L^r_\alpha(Z)$ denotes the number of vertices of $Z^{(r)}$ the colour of which is α . If Z is an ion or a graph $|Z|$ denotes the number of its vertices. The above theorem gives the following result.

Corollary. <u>For any</u> a <u>in</u> A, <u>the vector</u> $\lambda^{-n}L^r(\sigma^n(a))$ <u>converges to an eigenvector of</u> M'_r <u>associated to</u> λ . <u>The vector</u> $L^r(\sigma^n(a))/|\sigma^n(a)|$ <u>tends to</u>

<u>the eigenvector</u> ξ^r <u>the components of which are strictly positive and add up to</u> 1.

In the next paragraph we shall establish a strengthening of this corollary.

8. Frequency of appearance of patterns.

Let us remind the reader that by a sub-A-graph of Z we mean an A-graph of the form Z_U (see § IV-2) where U is a subset of V_Z. If we have such a sub-A-graph of $Z \in \mathcal{G}$, then $L^r(Z_U)$ is the vector in $\mathbb{R}^{A^1_r}$ the α^{th} component of which $L^r_\alpha(Z_U)$ is the number of vertices of $Z^{(r)}$ belonging to U and the color of which is α. In this framework, $\sigma^k(U)$ is the sub-A-graph of $\sigma^k(Z)$ the vertices of which are the k^{th} generation descendents of an element of U. The relation of inclusion between sub-A-graphs means the corresponding inclusion relation for the sets of vertices.

Theorem. <u>For any nonnegative integer</u> r, <u>there exists three positive numbers</u> $q<1$, h <u>and</u> C <u>such that, for any</u> a <u>in</u> A <u>and for any sequence</u> $\{R_n\}_{n\geq 0}$ <u>such that</u> R_n <u>be a sub-A-graph of</u> $\sigma^n(a)$, <u>we have, for any</u> $n\geq 0$ <u>and</u> α <u>in</u> A^1_r,

$$\left| \frac{|L^r_\alpha(R_n)|}{|R_n|} - \xi^r_\alpha \right| \leq C \sup \left| \left(\frac{|\partial R_n|}{|R_n|}\right)^h , q^n \right| .$$

Proof.

In what follows the letter C stands for a constant the value of which may depends on its occurence.

Let λ_1 be a number such that

- $\tilde{\lambda} < \lambda_1 < \lambda$
- λ_1 is strictly larger than the maximum of moduli of the eigenvalues distinct from λ of M and M'_r.

It results from the preceding paragraph that

$$\left| \frac{|L^r_\alpha(\sigma^n(a))|}{|\sigma^n(a)|} - \xi^r_\alpha \right| \leq C\left(\frac{\lambda_1}{\lambda}\right)^n .$$

C being independent of a, α and n.

Let n be an integer. Let j be an integer to be chosen later on . Let us consider two disjoint subsets E_1 and E_2 of vertices of $\sigma^j(a)$ such that

- $\sigma^{n-j}(E_1) \subset R_n \subset \sigma^{n-j}(E_1 \cup E_2)$
- E_1 is maximum for inclusion,
- E_2 is minimum for inclusion.

Then any of the sets $\{V_{\sigma^{n-j}(x)}\}_{x\in E_2}$, which are disjoint , intersets ∂R_n , so we have $|E_2| \leq |\partial R_n|$.

Now we are going to give an evaluation of the number of vertices of R_n the A^1_r-color of which is α : one has

$$|L^r_\alpha(R_n) - L^r_\alpha(\sigma^{n-j}(E_1))| \leq |\sigma^{n-j}(E_2)|$$

But if x is a vertex of $\sigma_r^j(a)$ the colour of which is $g_r(x)$ the A-graphs $\sigma_r^{n-j}(x)$ and $\sigma_r^{n-j}(g_r(x))$ differ only by their colorations : to be specific, the number of their vertices the colors of which differ is majorized by $C|\partial\sigma^{n-j}(x)|$. Thus

$$\left|\frac{L_\alpha^r(\sigma^{n-j}(x))}{|\sigma^{n-j}(x)|} - \xi_\alpha^r\right| \leqslant C\left(\frac{\lambda_1}{\lambda}\right)^n + \frac{|\partial\sigma^{n-j}(x)|}{|\sigma^{n-j}(x)|} \leqslant C\left(\frac{\lambda_1}{\lambda}\right)^{n-j} .$$

Therefore

$$\left|L_\alpha^r(R_n) - \xi_\alpha^r|\sigma^{n-j}(E_1)|\right| \leqslant |\sigma^{n-j}(E_2)| + C\left(\frac{\lambda_1}{\lambda}\right)^{n-j} |\sigma^{n-j}(E_1)|$$

and

$$\left|L_\alpha^r(R_n) - \xi_\alpha^r|R_n|\right| \leqslant C\left(\frac{\lambda_1}{\lambda}\right)^{n-j} |R_n| + \lambda^{n-j} |\partial R_n| .$$

By taking $j = \sup(0, [n - (\log(|R_n|/|\partial R_n|)/\log(\lambda^2/\lambda_1))])$, where [ ] denote the integral part, we get the result with $a = \lambda_1/\lambda$ and $h = (\log\lambda - \log\lambda_1)/(2\log\lambda - \log\lambda_1)$.

As we shall see later on (VII-3), this result on frequencies holds not only for patterns described by the various A_r-colorations but also for arbitrary finite patterns.

VII - GRAPHS INVARIANT BY A SUBSTITUTION

1. Limits of graphs.

A based A-graph is an A-graph one vertex of which, called base point, has been distinguished. Two based A-graphs are isomorphic if there is an isomorphism of these graphs which carries the base point of the first graph onto that of the second one.

Definition. <u>A sequence</u> $\{(Z_n,X_n)\}_{n\geqslant 0}$ <u>is said to converge to</u> (Z,x) <u>if, for any positive integer</u> r, <u>there exists an integer</u> N <u>such that, for any</u> $n\geqslant N$, <u>the based A-graphs</u> $(B_{Z_n}(x_n,r),x_n)$ <u>and</u> $(B_Z(x,r),x)$ <u>be isomorphic</u>.

2. Based substitutions.

Let σ be a substitution as defined in § VI-1. Let us distinguish a vertex x_a of the ion $\sigma(a)$ for each a in A. Then, if (Z,y) is a based A-graph with Z in $\mathcal{G}$, the A-graph $\sigma(Z)$ is based in a natural way : its base point is the vertex $x_{g(y)}$ of the ion $\sigma(g(y))$ which has replaced y.

An interesting case is the following : there exists ω in A such that $x_\omega = \omega$. Then the based A-graphs $(\sigma^n(\omega),\omega)$ have a limit $\sigma^\infty(\omega)$ which is an A-graph invariant by σ .

If such an ω does not exist it suffices to consider a suitable power of σ instead of σ to get such a color. It is what we have done with the Penrose substitution : we were led in chapter III to consider its fourth power.

3. Patterns in an invariant graph.

Let σ be a primitive substitution and Z a σ-invariant A-graph. Let us consider a sequence $\{R_n\}_{n\geqslant 0}$ of finite subsets of V_Z such that $|\partial R_n|/|R_n|$ converges to 0. Then, it results from the preceding theorems that $L^r(R_n)/|R_n|$ converges to ξ^r.

Up to now, we have proved the existence of frequency of certain patterns only, namely those which correspond to balls. We are going now to show that this is enough to get the result for any pattern. Let $\mathfrak{M}$ be a finite connected A-graph and x one of its vertices. Let ν be the cardinality of the set $\{y \; ; \; y$ is a vertex of $\mathfrak{M}$ and $(\mathfrak{M},x)$ and $(\mathfrak{M},y)$ are isomorphic as based A-graphs$\}$ and $r = \sup \{d_{\mathfrak{M}}(x,y) \; ; \; y$ vertex of $\mathfrak{M}\}$. Let us now consider an element α in A_r^1. It is represented by a based A-graph (T,t). Let us denote k_α the number of sub based A-graphs (T',t) of (T,t) which are isomorphic to $(\mathfrak{M},x)$. Then, if Z is an A-graph, the number of times $\mathfrak{M}$ appears in Z is equal to $\frac{1}{\nu} \sum_{\alpha \in A_r^1} k\, L_\alpha^r(Z)$, to within an error term majorized by $C|\partial Z|$ due to the boundary. Thus the preceding results are also válid for occurences of $\mathfrak{M}$.

We are now able to prove the property of Penrose tilings we had claimed. Let us consider a pattern (in other words, a bounded tesselation) appearing in a Penrose tiling. Let $\{R_n\}_{n \geqslant 0}$ be a sequence of real numbers converging to infinity and $\{z_n\}_{n \geqslant 0}$ a sequence of points in the plane. Let N_n be the number of appearances of this pattern in the disk of center z_n and radius R_n. Then N_n/R_n^2 has a limit : let us consider the tiles in the disk of center z_n and radius R_n ; by reasoning on areas, one can see that their number is minorized by CR_n^2, while the number of bording tiles is majorized by $C'R_n$, so we can apply the preceding result. As a consequence, if we denote N_n' and N_n'' the numbers of triangles of type I and of type II contained in the disk $D(z_n,R_n)$, the ratio N_n'/N_n'' converges to τ. Thus, setting $N_n = N_n' + N_n''$ and again reasoning on areas, we get the relation $N_n \sim \frac{4\pi\tau\sqrt{3-\tau}}{5} R_n^2$.

As an example, we are now going to determine the matrix M_1' for the Penrose substitution. The set A_1^1 has twenty elements, of which five are shown on figure 12 (more exactly this figure shows their duals).

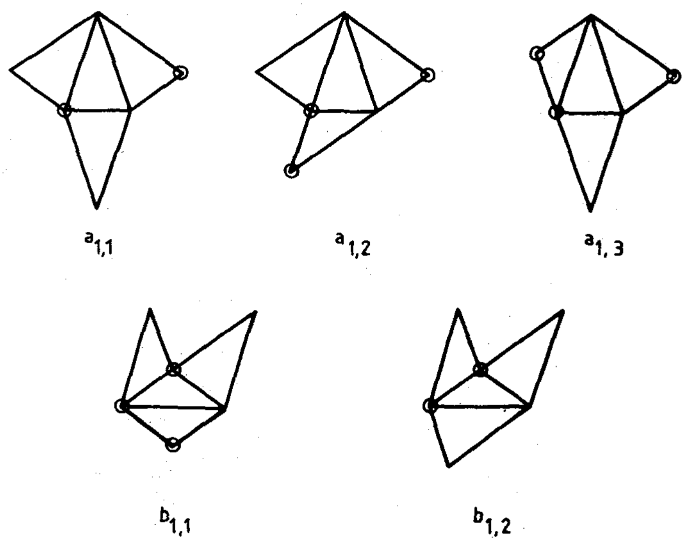


figure 12
five elements of A_1^1 for Penrose tilings.

Five more elements of A_1^1 are obtained by performing symmetries on the five first ones, they are denoted $a_{2,1}$, $a_{2,2}$, $a_{2,3}$, $b_{2,1}$ and $b_{2,2}$. By permuting the colorations of the vertices of these ten tiles, we get the ten other elements of A : they are named by dashing the element they come from by exchanging colours. If we use the following order of the elements of A_1^1 , $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{2,1}$, $a_{2,2}$, $a_{2,3}$ then b's , (a')'s and (b')'s, the matrix M' takes the form

$$M_1' = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix} \quad \text{where}$$

$$D = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Therefore, if we consider $4n^{th}$ powers of the inflation procedure, as in chapter III, when starting from a_1 the only patterns which occur are the ten first ones and their frequencies of appearances are the components of the right eigenvector of D corresponding to the eigenvalue τ^2 :
$\frac{1}{2}$ $(5\tau-8,\ 2-\tau,\ 5-3\tau,\ 5\tau-8,\ 2-\tau,\ 5-3\tau,\ 2\tau-3,\ 5-3\tau,\ 2\tau-3,\ 5-3\tau)$. Then, if γ_n denotes the number of times the pattern $a_{1,1}$, for instance, appears in $D(z_n,R_n)$, we have

$$\lim_{n\to\infty} \frac{\gamma_n}{\pi R_n^2} = \frac{2(5-3\tau)\sqrt{3-\tau}}{5} .$$

VIII - FINAL REMARKS

It results from the analysis in VI-7 that, as it is known for Penrose tilings, patterns appear in a relatively dense way. Let x be a vertex of a σ-invariant A-graph Z the A_r colour of which is the element α of A_r^1 . We are going to show that there exists R such that there is a vertex, different from x, of colour α in the ball $B_Z(x,R)$. Indeed, as $\sigma^{N+N'}(Z) = Z$, there is a vertex y of Z the offsprings of the $(N+N')^{th}$ generation of which contains $B_Z(x,R)$ as a sub-A-graph. If z is a vertex neighbour to y , then its $(N+N')^{th}$ generation offsprings contain a vertex the A_r-colour of which is α. But, as y and z are neighbours, any of their $(N+N')^{th}$ generation descents are at distance less than $C\lambda^{N+N'}$. It suffices then to take $R = C\lambda^{N+N'}$.

We could have defined a notion of A-graphs a bit more sophisticated that the one we used, allowing vertices to be linked to a variable number of other vertices, nevertheless without being in the boundary : we are given a mapping h from A to the set of positive integer. Then an A-graph would be a triple $Z = (V,E,g)$ where V is a set, g a mapping from V to A and E a subset of $\{(a,m,b,n)\ ;\ 1\leqslant m\leqslant h(g(a)),\ 1\leqslant n\leqslant h(g(b))\}$ satisfying the analogous of requirements a), b), c) and d) of IV-1.

One can also consider randomized such systems in the same spirit as in [9-11, 32].

APPENDIX. NONNEGATIVE MATRICES.

A square matrix is said to be nonnegative if any of its entries is. It is said primitive if there exists one of its power any entry of which be non-zero.

Let M be a primitive matrix. Then the Perron-Frobenius theorem asserts the following facts :

- M has a positive simple eigenvalue λ which is strictly larger than any other eigenvalue,
- the eigenspace associated with λ

is generated by a vector any component of which is strictly larger than 0.

The above λ is called the Perron-Frobenius eigenvalue of M. If a is a number in $]0,1[$ such that $a\lambda$ be strictly larger than the modulus of any other eigenvalue and if v and w are respectively right and left positive eigenvectors associated with λ the scalar product of which is 1, then it results from Perron-Frobenius theorem that, as n goes to infinity, we have

$$\lambda^{-n} M^n - w \otimes v = O(a^n)$$

Facts on nonnegative matrices can be found in [33].

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