# Invariant Extended Kalman Filter Notes

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January 3, 2023

## 1 Introduction

When the process model (excluding noise) is group affine, the error dynamics are state-independent, as stated in [1],

*Theorem 1: State-Independent Error Dynamics*

*If the function $f(\boldsymbol{\chi}(t), \mathbf{u}(t))$ is group affine and the error is either left- or right-invariant, then the error propagation will be state independent.*

Therefore, Jacobians are state-independent, avoiding some linearization errors. The choice of using a left- or right-invariant error depends on the left- or right-invariant form of the measurements, as defined next.

### 1.1 Notation

Hereafter we assume the following notation,

$\mathcal{M}$ a Lie group with origin $\mathcal{E} \in \mathcal{M}$.
$\boldsymbol{\chi}, \boldsymbol{\gamma} \in \mathcal{M}$ elements of the group.
$\mathfrak{m}$ the group tangent space.
$\boldsymbol{\tau}^\wedge \in \mathfrak{m}$ an element of the tangent space.
${}^{\mathbf{R}}X$ superscript marks the right-invariance.
${}^{\mathbf{L}}X$ superscript marks the left-invariance.

### 1.2 Invariant Filtering in Discrete Time

#### 1.2.1 System Model

Let us consider the discrete-time system model, noted $f(\cdot)$, as follows,

$$\begin{aligned} \boldsymbol{\chi}_k &= f(\hat{\boldsymbol{\chi}}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}) \ , \\ &\approx f(\hat{\boldsymbol{\chi}}_{k-1}, \mathbf{u}_{k-1}) \operatorname{Exp}(\mathbf{w}_{k-1}) \ . \end{aligned} \tag{1}$$

It must be group affine, meaning that after neglecting noise, it must satisfy,

$$\mathbf{F}(\boldsymbol{\chi}_1\boldsymbol{\chi}_2, \mathbf{u}, \mathbf{0}) = \mathbf{F}(\boldsymbol{\chi}_1, \mathbf{u}, \mathbf{0})\mathbf{F}(\mathcal{E}, \mathbf{u}, \mathbf{0})^{-1}\mathbf{F}(\boldsymbol{\chi}_2, \mathbf{u}, \mathbf{0}) \ . \tag{2}$$

After neglecting noise, the following three statements are equivalents,

- There exists a function $\mathbf{G}$ such that $\mathbf{F}(\boldsymbol{\chi}_2, \mathbf{u})^{-1}\mathbf{F}(\boldsymbol{\chi}_1, \mathbf{u}) = \mathbf{G}(\boldsymbol{\chi}_2^{-1}\boldsymbol{\chi}_1, \mathbf{u})$ ,
- There exists a function $\mathbf{G}$ such that $\mathbf{F}(\boldsymbol{\chi}_2, \mathbf{u})\mathbf{F}(\boldsymbol{\chi}_1, \mathbf{u})^{-1} = \mathbf{G}(\boldsymbol{\chi}_2\boldsymbol{\chi}_1^{-1}, \mathbf{u})$ ,
- Equation 2 is satisfied.

Moreover, for each $\mathbf{u} \in \mathbb{R}^d$, there exists a matrix $\mathbf{A} \in \mathbb{R}^{d\times d}$ such that,

$$\mathbf{G}(\mathrm{Exp}(\mathcal{E}), \mathbf{u}) = \mathrm{Exp}(\mathbf{A}\mathcal{E}) \ . \tag{3}$$

The proof of Eq. 3 may be found in [1].

#### 1.2.2 Measurement Model

Let us also consider the discrete-time measurement model, noted $h(\cdot)$, as follows,

$$\begin{aligned} \mathbf{y}_k &= h(\boldsymbol{\chi}_k, \mathbf{v}) \\ &= h(\boldsymbol{\chi}_k) + \mathbf{v} \ . \end{aligned} \tag{4}$$

#### 1.2.3 Prediction

$$\boldsymbol{\chi}_k = f(\boldsymbol{\chi}_{k-1}, \mathbf{u}_{k-1}, \mathbf{0}) \ . \tag{5}$$

$$\mathbf{P}_k = \mathbf{A}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{A}_{k-1}^{\top} + \mathbf{L}_{k-1}\hat{\mathbf{Q}}_{k-1}\mathbf{L}_{k-1}^{\top} \ . \tag{6}$$

#### 1.2.4 Update

### 1.3 Errors

#### 1.3.1 Left & Right errors

The invariant error definitions commonly encountered in the literature reads:

**Left-invariant**

$$\begin{aligned} {}^{\mathbf{L}}\delta\boldsymbol{\chi} &= \boldsymbol{\chi}^{-1}\hat{\boldsymbol{\chi}} \\ &= \hat{\boldsymbol{\chi}} \ominus^{\cdot} \boldsymbol{\chi} \ . \end{aligned} \tag{7}$$

**Right-invariant**

$$\begin{aligned} {}^{\mathbf{R}}\delta\boldsymbol{\chi} &= \hat{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1} \ , \\ &= \hat{\boldsymbol{\chi}} {}^{\cdot}\!\ominus \boldsymbol{\chi} \ . \end{aligned} \tag{8}$$

where the last expressions are the notation for the '*right*/*left*-minus' found in [3]. These errors are said to be *left*/*right*-invariant since,

$$\begin{aligned} {}^{\mathbf{L}}\delta\boldsymbol{\chi} &= \boldsymbol{\chi}^{-1}\hat{\boldsymbol{\chi}} \\ &= (\boldsymbol{\gamma}\boldsymbol{\chi})^{-1}(\boldsymbol{\gamma}\hat{\boldsymbol{\chi}}) \\ &= \boldsymbol{\chi}^{-1}\boldsymbol{\gamma}^{-1}\boldsymbol{\gamma}\hat{\boldsymbol{\chi}} \\ &= \boldsymbol{\chi}^{-1}\hat{\boldsymbol{\chi}} \ . \end{aligned} \tag{9}$$

$$\begin{aligned} {}^{\mathbf{R}}\delta\boldsymbol{\chi} &= \hat{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1} \\ &= (\hat{\boldsymbol{\chi}}\boldsymbol{\gamma})(\boldsymbol{\chi}\boldsymbol{\gamma})^{-1}, \\ &= \hat{\boldsymbol{\chi}}\boldsymbol{\gamma}\boldsymbol{\gamma}^{-1}\boldsymbol{\chi}^{-1}, \\ &= \hat{\boldsymbol{\chi}}\boldsymbol{\chi}^{-1} \ . \end{aligned} \tag{10}$$

These error models leads to the following updates,

$$
\begin{aligned}
{}^{\mathbf{L}}\delta\boldsymbol{\chi} &= \hat{\boldsymbol{\chi}}^{-1}\check{\boldsymbol{\chi}} \\
\hat{\boldsymbol{\chi}}\,{}^{\mathbf{L}}\delta\boldsymbol{\chi} &= \check{\boldsymbol{\chi}} \\
\hat{\boldsymbol{\chi}} &= \check{\boldsymbol{\chi}}\,{}^{\mathbf{L}}\delta\boldsymbol{\chi}^{-1} \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(\mathbf{Kz})^{-1} \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(-(\mathbf{Kz}))\ .
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
{}^{\mathbf{R}}\delta\boldsymbol{\chi} &= \check{\boldsymbol{\chi}}\hat{\boldsymbol{\chi}}^{-1} \\
{}^{\mathbf{R}}\delta\boldsymbol{\chi}\hat{\boldsymbol{\chi}} &= \check{\boldsymbol{\chi}} \\
\hat{\boldsymbol{\chi}} &= {}^{\mathbf{R}}\delta\boldsymbol{\chi}^{-1}\check{\boldsymbol{\chi}} \\
&= \mathrm{Exp}(\mathbf{Kz})^{-1}\check{\boldsymbol{\chi}} \\
&= \mathrm{Exp}(-\mathbf{Kz})\check{\boldsymbol{\chi}}.
\end{aligned} \tag{12}
$$

At this point, I wonder why the errors are $\check{\boldsymbol{\chi}} \ominus \boldsymbol{\chi}$ and not $\boldsymbol{\chi} \ominus \check{\boldsymbol{\chi}}$

#### 1.3.2 Mixed-Invariant

The choice for right- or left- invariant error results from the measurement models used. However, when both measurement types are available, the covariance must remains consistent.

We know from the Lie theory that one can use the adjoint matrix to linearly transform vectors of the (local) tangent space at $\boldsymbol{\chi}$ onto vectors of the (global) tangent space at the origin with,

$$
{}^{\mathcal{E}}\boldsymbol{\tau} = \mathbf{Ad}_{\boldsymbol{\chi}}\,{}^{\boldsymbol{\chi}}\boldsymbol{\tau}\ , \tag{13}
$$

and conversely,

$$
{}^{\boldsymbol{\chi}}\boldsymbol{\tau} = \mathbf{Ad}_{\boldsymbol{\chi}}^{-1}\,{}^{\mathcal{E}}\boldsymbol{\tau}\ . \tag{14}
$$

Thus, one can write,

$$
\begin{aligned}
\hat{\boldsymbol{\chi}} &= \check{\boldsymbol{\chi}}\,{}^{\mathbf{L}}\delta\boldsymbol{\chi}^{-1} \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(-\,{}^{\check{\boldsymbol{\chi}}}\boldsymbol{\tau}) \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(-\mathbf{Ad}_{\check{\boldsymbol{\chi}}}^{-1}\,{}^{\mathcal{E}}\boldsymbol{\tau}) \\
&= \check{\boldsymbol{\chi}}\check{\boldsymbol{\chi}}^{-1}\,\mathrm{Exp}(-\,{}^{\mathcal{E}}\boldsymbol{\tau})\check{\boldsymbol{\chi}} \\
&= \mathrm{Exp}(-\,{}^{\mathcal{E}}\boldsymbol{\tau})\check{\boldsymbol{\chi}} \\
&= \mathrm{Exp}(-\mathbf{Ad}_{\check{\boldsymbol{\chi}}}\,{}^{\check{\boldsymbol{\chi}}}\boldsymbol{\tau})\check{\boldsymbol{\chi}}\ .
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\hat{\boldsymbol{\chi}} &= {}^{\mathbf{R}}\delta\boldsymbol{\chi}^{-1}\check{\boldsymbol{\chi}} \\
&= \mathrm{Exp}(-\,{}^{\mathcal{E}}\boldsymbol{\tau})\check{\boldsymbol{\chi}} \\
&= \mathrm{Exp}(-\mathbf{Ad}_{\check{\boldsymbol{\chi}}}\,{}^{\check{\boldsymbol{\chi}}}\boldsymbol{\tau})\check{\boldsymbol{\chi}} \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(-\,{}^{\check{\boldsymbol{\chi}}}\boldsymbol{\tau})\check{\boldsymbol{\chi}}^{-1}\check{\boldsymbol{\chi}} \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(-\,{}^{\check{\boldsymbol{\chi}}}\boldsymbol{\tau}) \\
&= \check{\boldsymbol{\chi}}\,\mathrm{Exp}(-\mathbf{Ad}_{\check{\boldsymbol{\chi}}}^{-1}\,{}^{\mathcal{E}}\boldsymbol{\tau})\ .
\end{aligned} \tag{16}
$$

Therefore, in a right-invariant framework, when a left-invariant measurement is available, one can simply map the right-invariant covariance, ${}^{\mathbf{R}}\mathbf{P}$, to its left counter-part, ${}^{\mathbf{L}}\mathbf{P}$, using the matrix representation of the adjoint operator. The left-invariant framework can then be normally applied throughout the update step. At last, the covariance update must be mapped back to it's right-invariant form. Note that this 'trick' first appeared in [2].

$$
{}^{\mathbf{L}}\mathbf{P} = \mathbf{Ad}_{\boldsymbol{\chi}}^{-1}\,{}^{\mathbf{L}}\mathbf{P}\mathbf{Ad}_{\boldsymbol{\chi}}^{-\top}\ . \tag{17}
$$

$$
{}^{\mathbf{R}}\mathbf{P} = \mathbf{Ad}_{\boldsymbol{\chi}}\,{}^{\mathbf{L}}\mathbf{P}\mathbf{Ad}_{\boldsymbol{\chi}}^{\top}\ , \tag{18}
$$

We also recall the following property of the adjoint which may come handy for optimization purpose when implementing the above equations,

$$
\mathbf{Ad}_{\boldsymbol{\chi}}^{-1} = \mathbf{Ad}_{\boldsymbol{\chi}^{-1}}\ . \tag{19}
$$

### 1.4 System

From Eq. 4 we consider the following models for respectively, left and right-invariant measurement models,

$$^{\mathbf{L}}\mathbf{y} = \boldsymbol{\chi}\mathbf{b} + \mathbf{v} \ . \tag{20}$$

$$^{\mathbf{R}}\mathbf{y} = \boldsymbol{\chi}^{-1}\mathbf{b} + \mathbf{v} \ . \tag{21}$$

In order to have the invariant error appears, the innovation is rewritten as,

$$\begin{aligned} ^{\mathbf{L}}\mathbf{z} &= \check{\boldsymbol{\chi}}^{-1}(\mathbf{y} - \check{\mathbf{y}}) \ , \\ &= \check{\boldsymbol{\chi}}^{-1}(\boldsymbol{\chi}\mathbf{b} + \mathbf{v} - \check{\boldsymbol{\chi}}\mathbf{b}) \ , \\ &= {}^{\mathbf{L}}\delta\boldsymbol{\chi}^{-1}\mathbf{b} - \mathbf{b} + \check{\boldsymbol{\chi}}^{-1}\mathbf{v} \ . \end{aligned} \tag{22}$$

$$\begin{aligned} ^{\mathbf{R}}\mathbf{z} &= \check{\boldsymbol{\chi}}(\mathbf{y} - \check{\mathbf{y}}) \ , \\ &= \check{\boldsymbol{\chi}}(\boldsymbol{\chi}^{-1}\mathbf{b} + \mathbf{v} - \check{\boldsymbol{\chi}}^{-1}\mathbf{b}) \ , \\ &= {}^{\mathbf{R}}\delta\boldsymbol{\chi}\mathbf{b} - \mathbf{b} + \check{\boldsymbol{\chi}}\mathbf{v} \ . \end{aligned} \tag{23}$$

### 1.5 Metrics

#### 1.5.1 Root Mean Square Error

$$\boldsymbol{\tau}_k = \hat{\boldsymbol{\chi}}_k \ominus \boldsymbol{\chi}_k \ .$$

$$\mathbf{RMSE} = (\frac{1}{n}\sum_{k=1}^{n} \|\boldsymbol{\tau}_k\|^2)^{\frac{1}{2}} \ . \tag{24}$$

#### 1.5.2 Absolute Trajectory Error

$$\boldsymbol{\gamma}_k = \boldsymbol{\chi}_k^{-1}\hat{\boldsymbol{\chi}}_k \ .$$

$$\mathbf{ATE} = (\frac{1}{n}\sum_{k=1}^{n} \|trans(\boldsymbol{\gamma}_k)\|^2)^{\frac{1}{2}} \ . \tag{25}$$

#### 1.5.3 Absolute Orientation Error

$$\boldsymbol{\gamma}_k = \boldsymbol{\chi}_k^{-1}\hat{\boldsymbol{\chi}}_k \ .$$

$$\mathbf{AOE} = \frac{1}{n}\sum_{k=1}^{n} \angle(rot(\boldsymbol{\gamma}_k)) \ . \tag{26}$$

with,

$$\begin{aligned} &\angle(rot(\boldsymbol{\gamma}_k)) = |\theta|, && \text{for } \boldsymbol{\gamma}_k \in \mathrm{SE}(2) \ , \\ &\angle(rot(\boldsymbol{\gamma}_k)) = \text{angular\_distance}(\mathbf{q}, \mathbf{q}i), && \text{for } \boldsymbol{\gamma}_k \in \mathrm{SE}(3) \ . \end{aligned} \tag{27}$$

with $\mathbf{q}$ the rotational part of $\boldsymbol{\gamma}_k$ in quaternion form and $\mathbf{q}i$ the quaternion identity.

#### 1.5.4 Final Pose Error

$$\mathbf{FPE} = \|\hat{\boldsymbol{\chi}}_K \ominus \boldsymbol{\chi}_K\| \ . \tag{28}$$

#### 1.5.5 Relative Errors

The trajectory is divided into subsets $\mathbf{S}\{\boldsymbol{\gamma}\}_l = \{\boldsymbol{\gamma}_k, ..., \boldsymbol{\gamma}_{k+\Delta}\}$, where

$$\boldsymbol{\gamma}_{k+\Delta} = (\boldsymbol{\chi}_k^{-1}\boldsymbol{\chi}_{k+\Delta})^{-1}(\hat{\boldsymbol{\chi}}_k^{-1}\hat{\boldsymbol{\chi}}_{k+\Delta}) \ .$$

$$\mathbf{RRMSE} = \sum_{l=1}^{n} \mathbf{RMSE}(\ \mathbf{S}\{\boldsymbol{\gamma}\}_l\ ) \ . \tag{29}$$

$$\mathbf{RTE} = \sum_{l=1}^{n} \mathbf{ATE}(\ \mathbf{S}\{\boldsymbol{\gamma}\}_l\ ) \ . \tag{30}$$

$$\mathbf{ROE} = \sum_{l=1}^{n} \mathbf{AOE}(\ \mathbf{S}\{\boldsymbol{\gamma}\}_l\ ) \ . \tag{31}$$

# References


[1] A. Barrau and S. Bonnabel. “The Invariant Extended Kalman Filter as a Stable Observer”. In: *IEEE Transactions on Automatic Control* 62.4 (2017), pp. 1797–1812. DOI: `10.1109/TAC.2016.2594085`.

[2] R. Hartley et al. “Contact-aided invariant extended Kalman filtering for robot state estimation”. In: *The International Journal of Robotics Research* 39.4 (2020), pp. 402–430. DOI: `10.1177/0278364919894385`. eprint: `https://doi.org/10.1177/0278364919894385`. URL: `https://doi.org/10.1177/0278364919894385`.

[3] Joan Solà, Jeremie Deray, and Dinesh Atchuthan. *A micro Lie theory for state estimation in robotics*. Tech. rep. IRI-TR-18-01. Barcelona: Institut de Robòtica i Informàtica Industrial, 2018.