## Lie theory cheat sheet

$\text{Lie group } \mathcal{M}, \circ$		size	dim	$\mathcal{X} \in \mathcal{M}$	Constraint	$m{ au}^\wedge \in \mathfrak{m}$	$oldsymbol{ au} \in \mathbb{R}^m$
Vector n-D	$  \mathbb{R}^n, +$	$\mid n \mid$	$\mid n \mid$	$\mathbf{v} \in \mathbb{R}^n$	$  \mathbf{v} - \mathbf{v} = 0$	$oldsymbol{\mathbf{v}} \in \mathbb{R}^n$	$\mathbf{v} \in \mathbb{R}^n$
Unit Complex number	$  S^1, \cdot $	2	1	$\mathbf{z}\in\mathbb{C}$	$\mathbf{z}^*\mathbf{z} = 1$	$i\theta \in i\mathbb{R}$	$\theta \in \mathbb{R}$
2D Rotation	$\mid SO(2), \cdot \mid$	4	1	R		$\left[ egin{aligned} [ heta]_{ imes} &= egin{bmatrix} 0 & - heta \  heta & 0 \end{bmatrix} \in \mathfrak{so}(2) \end{aligned}  ight.$	$\theta \in \mathbb{R}$
2D Rigid Motion	$\mid$ SE(2), ·	9	3		$ \mid \ \mathbf{R}^{\top}\mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\theta]_{\times} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(2)$	$\left  \begin{array}{c} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{array} \right  \in \mathbb{R}^3 \ \left  \begin{array}{c} \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{array} \right $
Unit Quaternion	$  S^3, \cdot  $	4	3	$\mathbf{q}\in\mathbb{H}$	$\mathbf{q}^*\mathbf{q} = 1$	$oldsymbol{ heta}/2\in\mathbb{H}_p$	$oldsymbol{ heta} \in \mathbb{R}^3$
3D Rotation	$\mid SO(3), \cdot \mid$	9	3	R	$R^{\top} \mathbf{R} = \mathbf{I}$	$   [\boldsymbol{\theta}]_{\times} = \begin{bmatrix} 0 & -\theta_z & \theta_y \\ \theta_z & 0 & -\theta_x \\ -\theta_y & \theta_x & 0 \end{bmatrix} \in \mathfrak{so}(3) $	$oldsymbol{ heta} \in \mathbb{R}^3$
3D Rigid Motion	$\mid$ SE(3), ·	16	6	$\mathbf{M} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$	$R^{\top} \mathbf{R} = \mathbf{I}$	$\begin{bmatrix} [\boldsymbol{\theta}]_{\times} & \boldsymbol{\rho} \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3)$	$igg egin{array}{c} iggr  $

Operation	Inverse	Compose	Exp	Log	$\mid$ Right- $\oplus$	$ $ Right- $\ominus$
Right Jacobians	$\mathbf{J}_{\mathcal{X}}^{\mathcal{X}^{-1}} = -\mathbf{A}\mathbf{d}_{\mathcal{X}}$	$ \begin{vmatrix} \mathbf{J}_{\mathcal{X}}^{\mathcal{X} \circ \mathcal{Y}} = \mathbf{A} \mathbf{d}_{\mathcal{Y}^{-1}} \\ \mathbf{J}_{\mathcal{Y}}^{\mathcal{X} \circ \mathcal{Y}} = \mathbf{I} \end{vmatrix} $	$oxed{\mathbf{J}^{\mathrm{Exp}(oldsymbol{ au})}_{oldsymbol{ au}} = \mathbf{J}_r(oldsymbol{ au})}$	$\mathbf{J}^{\operatorname{Log}(\mathcal{X})}_{\mathcal{X}} = \mathbf{J}^{-1}_r(oldsymbol{ au})$	$egin{aligned} \mathbf{J}_{\mathcal{X}}^{\mathcal{X}\oplusoldsymbol{ au}} &= \mathbf{Ad}_{\mathrm{Exp}(oldsymbol{ au})^{-1}} \ \mathbf{J}_{oldsymbol{ au}}^{\mathcal{X}\oplusoldsymbol{ au}} &= \mathbf{J}_r(oldsymbol{ au}) \end{aligned}$	$ \begin{vmatrix} \mathbf{J}_{\mathcal{X}}^{\mathcal{Y}\ominus\mathcal{X}} = -\mathbf{J}_{l}^{-1}(\boldsymbol{\tau}) \\ \mathbf{J}_{\mathcal{Y}}^{\mathcal{Y}\ominus\mathcal{X}} = \mathbf{J}_{r}^{-1}(\boldsymbol{\tau}) \end{vmatrix} $

Note: In accordance to manif implementation, all Jacobians in this document are right Jacobians, whose definition reads:  $\frac{\delta f(X)}{\delta X} = \lim_{\varphi \to 0} \frac{f(X \oplus \varphi) \ominus f(X)}{\varphi}$ . However, notice that one can relate the left- and right- Jacobians with the Adjoint,  $\frac{\varepsilon_{\partial f(X)}}{\partial \mathcal{X}} \mathbf{Ad}_{\mathcal{X}} = \mathbf{Ad}_{f(\mathcal{X})} \frac{{}^{\chi}_{\partial f(\mathcal{X})}}{\partial \mathcal{X}}$ , see [1] Eq. (46).

<sup>[1]</sup> J. Solà, J. Deray, and D. Atchuthan, "A micro Lie theory for state estimation in robotics," Tech. Rep. IRI-TR-18-01, Institut de Robòtica i Informàtica Industrial, Barcelona, 2018. Available at arxiv.org/abs/1812.01537.

$\mathcal{M}, \circ$ Op	Identity	Inverse	Compose	Act	Exp	Log
$\mathbb{R}^n$ , +	$\mathbf{v} = [0]$	$-\mathbf{v}$	$\mathbf{v}_1 + \mathbf{v}_2$	$\mathbf{v} + \mathbf{p}$	v	v
$S^1, \cdot$	z = 1 + i  0	z*	$z_1$ $z_2$	z v	$z = \cos\theta + i\sin\theta$	$\theta = \arctan2(\operatorname{Im}(z), \operatorname{Re}(z))$
$SO(2), \cdot$	$\mathbf{R} = \mathbf{I}$	$\mathbf{R}^{-1} = \mathbf{R}^{\top}$	$\mathbf{R}_1 \; \mathbf{R}_2$	$\mathbf{R} \cdot \mathbf{v}$	$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	$\theta = \arctan2(r_{21}, r_{11})$
$\mathrm{SE}(2), \cdot$	$\mathbf{M} = egin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{I}$		$\mathbf{M}_1\mathbf{M}_2 = egin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1\mathbf{t}_2 \ 0 & 1 \end{bmatrix}$	$\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R} \mathbf{p}$	$\mathbf{M} = \begin{bmatrix} \operatorname{Exp}(\theta) & \mathbf{V}(\theta)  \boldsymbol{\rho} \\ 0 & 1 \end{bmatrix}  (1)$	$oldsymbol{ au} = egin{bmatrix} oldsymbol{ ho} \  heta \end{bmatrix} = egin{bmatrix} \mathbf{V}^{-1}( heta)  \mathbf{p} \  heta \end{bmatrix}  ^{(1)}$
$\mathrm{S}^3,\cdot$	$\mathbf{q} = 1 + i0 + j0 + k0$	$   \mathbf{q}^* = w - ix - jy - jz $	$\mathbf{q}_1 \; \mathbf{q}_2$	$\mathbf{q}  \mathbf{v}  \mathbf{q}^*$	$\mathbf{q} = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2}$	$oldsymbol{ heta} = 2 \mathbf{v} rac{\arctan 2(\ \mathbf{v}\ , w)}{\ \mathbf{v}\ }$
$SO(3), \cdot$	$\mathbf{R} = \mathbf{I}$	$\mathbf{R}^{-1} = \mathbf{R}^{\top}$	$\mathbf{R}_1 \; \mathbf{R}_2$	$\mathbf{R} \cdot \mathbf{v}$	$\mathbf{R} = \mathbf{I} + \sin \theta \left[ \mathbf{u} \right]_{\times} + (1 - \cos \theta) \left[ \mathbf{u} \right]_{\times}^{2}$	$oldsymbol{ heta} = rac{ heta(\mathbf{R} - \mathbf{R}^T)^\wedge}{2\sin heta}$
$\mathrm{SE}(3), \cdot$	$\mathbf{M} = egin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{I}$		$\mathbf{M}_1  \mathbf{M}_2 = egin{bmatrix} \mathbf{R}_1 \mathbf{R}_2 & \mathbf{t}_1 + \mathbf{R}_1 \mathbf{t}_2 \ 0 & 1 \end{bmatrix}$	$\mathbf{M} \cdot \mathbf{p} = \mathbf{t} + \mathbf{R}\mathbf{p}$	$\mathbf{M} = \begin{bmatrix} \operatorname{Exp}(\boldsymbol{\theta}) & \mathbf{V}(\boldsymbol{\theta})  \boldsymbol{\rho} \\ 0 & 1 \end{bmatrix}  (2)$	$oldsymbol{ au} = egin{bmatrix} oldsymbol{ ho} \ oldsymbol{ heta} \end{bmatrix} = egin{bmatrix} \mathbf{V}^{-1}(oldsymbol{ heta}) \ oldsymbol{ heta} \end{bmatrix} (2) \end{bmatrix}$

$\mathcal{M}, \circ$ Ad/Jac	$oxed{\mathrm{Ad}/\mathrm{Jac}}$ Ad $oxed{\mathrm{J}_r}$		${\color{red}\mathbf{J}_l}$	$\begin{array}{c c} \mathbf{J}_{\boldsymbol{\chi}}^{\boldsymbol{\chi}\cdot\mathbf{p}} & \mathbf{J}_{\mathbf{p}}^{\boldsymbol{\chi}\cdot\mathbf{p}} \\ & (\mathrm{Act}) \end{array}$	
$\mathbb{R}^n$ , +	$\mathbf{I} \in \mathbb{R}^{n \times n}$	I	I	I	I
$\mathrm{S}^1, \cdot$	1	1	1	$\mathbf{R}\left[1\right]_{\times}\mathbf{v}$	$\mathbf{R}$
$SO(2), \cdot$	1 1	1	1	$\mathbf{R}\left[1\right]_{\times}\mathbf{v}$	R
$\mathrm{SE}(2), \cdot$	$ \left  \begin{array}{cc} \mathbf{R} & -\begin{bmatrix} 1 \end{bmatrix}_{\times} \mathbf{t} \\ 0 & 1 \end{array} \right] $	$\begin{bmatrix} \sin \theta/\theta & (1-\cos \theta)/\theta & (\theta\rho_1-\rho_2+\rho_2\cos \theta-\rho_1\sin \theta)/\theta^2\\ (\cos \theta-1)/\theta & \sin \theta/\theta & (\rho_1+\theta\rho_2-\rho_1\cos \theta-\rho_2\sin \theta)/\theta^2\\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \sin \theta/\theta & (\cos \theta - 1)/\theta & (\theta \rho_1 + \rho_2 - \rho_2 \cos \theta - \rho_1 \sin \theta)/\theta^2 \\ (1 - \cos \theta)/\theta & \sin \theta/\theta & (-\rho_1 + \theta \rho_2 + \rho_1 \cos \theta - \rho_2 \sin \theta)/\theta^2 \\ 0 & 0 & 1 \end{bmatrix}$	$ \left  \begin{array}{cc} \left[ \mathbf{R} & \mathbf{R}\left[1\right]_{\times} \mathbf{p} \right] \end{array} \right  $	R
$S^3, \cdot$	$\mathbf{R}(\mathbf{q})$	$\mathbf{I} - rac{1-\cos heta}{ heta^2}\left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3}\left[oldsymbol{ heta} ight]_ imes$	$\mathbf{I} + rac{1-\cos heta}{ heta^2} \left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3} \left[oldsymbol{ heta} ight]_ imes$	$\left  \begin{array}{c} -\mathbf{R}(\mathbf{q}) \left[ \mathbf{v} \right]_{\times} \end{array} \right $	$\mathbf{R}(\mathbf{q})^{(3)}$
$SO(3), \cdot$	R	$\mathbf{I} - rac{1-\cos heta}{ heta^2}\left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3}\left[oldsymbol{ heta} ight]_ imes$	$\mathbf{I} + rac{1-\cos heta}{ heta^2} \left[oldsymbol{ heta} ight]_ imes + rac{ heta-\sin heta}{ heta^3} \left[oldsymbol{ heta} ight]_ imes$	$\left  -\mathbf{R} \left[ \mathbf{v} \right]_{\times} \right $	${f R}$
$SE(3), \cdot$	$\left  \begin{array}{cc} \left[ egin{matrix} \mathbf{R} & \left[ \mathbf{t}  ight]_{ imes} \mathbf{R} \\ 0 & \mathbf{R} \end{array}  ight] \  ight $	$\begin{bmatrix} \mathbf{J}_r(\boldsymbol{\theta}) & \mathbf{Q}(-\boldsymbol{\rho}, -\boldsymbol{\theta}) \\ 0 & \mathbf{J}_r(\boldsymbol{\theta}) \end{bmatrix}{}_{(4)}$	$\begin{bmatrix} \mathbf{J}_l(\boldsymbol{\theta}) & \mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta}) \\ 0 & \mathbf{J}_l(\boldsymbol{\theta}) \end{bmatrix}_{(4)}$	$\left  \begin{array}{cc} \left[ \mathbf{R} & -\mathbf{R} \left[ \mathbf{p} \right]_{\times} \right] \end{array} \right $	R

Some useful identities:

$$\mathcal{X} \oplus \boldsymbol{\tau} = \mathbf{A}\mathbf{d}_{\mathcal{X}} \ \boldsymbol{\tau} \oplus \mathcal{X} \ \mid \ \mathbf{A}\mathbf{d}_{\mathcal{X}}^{-1} = \mathbf{A}\mathbf{d}_{\mathcal{X}^{-1}} \ \mid \ \mathbf{A}\mathbf{d}_{\mathcal{X}\mathcal{Y}} = \mathbf{A}\mathbf{d}_{\mathcal{X}}\mathbf{A}\mathbf{d}_{\mathcal{Y}} \ \mid \ \mathbf{J}_l(\boldsymbol{\tau}) = \mathbf{A}\mathbf{d}_{\mathrm{Exp}(\boldsymbol{\tau})}\mathbf{J}_r(\boldsymbol{\tau}) \ \mid \ \mathbf{J}_l(\boldsymbol{\tau}) = \mathbf{J}_r(-\boldsymbol{\tau})$$

$$^{(1)}\mathbf{V}(\theta) = \frac{\sin\theta}{\theta}\mathbf{I} + \frac{1-\cos\theta}{\theta}\left[1\right]_{\times}$$

$$^{(2)}\mathbf{V}(oldsymbol{ heta}) = \mathbf{I} + rac{1-\cos heta}{ heta}\left[\mathbf{u}
ight]_{ imes} + rac{ heta-\sin heta}{ heta}\left[\mathbf{u}
ight]_{ imes}^2$$

$${}^{(\beta)}\mathbf{R}(\mathbf{q}) = \begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2(xy - wz) & 2(xz + wy) \\ 2(xy + wz) & w^2 - x^2 + y^2 - z^2 & 2(yz - wx) \\ 2(xz - wy) & 2(yz + wx) & w^2 - x^2 - y^2 + z^2 \end{bmatrix}$$

$${}^{(4)}\mathbf{Q}(\boldsymbol{\rho},\boldsymbol{\theta}) = 1/2\left[\boldsymbol{\rho}\right]_{\times} + \frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta}}{\boldsymbol{\theta}^3}(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\rho}\right]_{\times} + \left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times} + \left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right) - \frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4}(\left[\boldsymbol{\theta}\right]_{\times}^2\left[\boldsymbol{\rho}\right]_{\times} + \left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right) - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\rho}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta} - \sin\boldsymbol{\theta} - \frac{\boldsymbol{\theta}^3}{6}}{\boldsymbol{\theta}^5})(\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\left[\boldsymbol{\theta}\right]_{\times}\right] - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta}^2}{\boldsymbol{\theta}^4})(\boldsymbol{\theta}^2) - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4} - 3\frac{\boldsymbol{\theta}^2}{\boldsymbol{\theta}^4})(\boldsymbol{\theta}^2) - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4})(\boldsymbol{\theta}^2) - \frac{1}{2}(\frac{1 - \frac{\theta^2}{2} - \cos\boldsymbol{\theta}}{\boldsymbol{\theta}^4$$