CSE 2500-01: Homework 8

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Coded in LaTeX

Problems

A sequence $g_1, g_2, g_3, \ldots$ is defined by letting $g_1 = 3$, $g_2 = 5$, and $g_k = 3g_{k-1} - 2g_{k-2}$ for all integers $k \geq 3$.

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$ using strong mathematical induction.

Theorem 1: Strong Induction Proof 1

$$g_n = 2^n + 1 \textit{ for all integers } n \geq 1$$

Proof by Strong Mathematical Induction.
Let $P(n)$ be the expression:

$$g_n = 2^n + 1 \quad \text{for all integers} \quad n \geq 1$$

Base Cases:
We must show $P(1)$ and $P(2)$ are true.

$$g_1 = 2^1 + 1 = 3$$

$$g_2 = 2^2 + 1 = 5$$

Both adhere to our initial conditions, so we may proceed.
Inductive Step:
For an arbitrarily fixed integer $k \geq 2$, suppose $P(k)$ is true, that is:

$$g_i = s^i + 1 \quad \text{for all integers} \quad 1 \leq i \leq k.$$

We need to show $P(k+1)$ is true.
That is,

$$g_{k+1} = s^{k+1} + 1$$

From the definition of the sequence,

$$g_{k+1} = 3g_k - 2g_{k-1}$$

$$\begin{aligned} g_{k+1} &= 3(2^k + 1) - 2(2^{k-1} + 1) && \textit{(by inductive hypothesis)} \\ &= 3(2^k + 1) - 2 \cdot 2^{k-1} - 2 \\ &= 3 \cdot 2^k + 3 - 2^k - 2 \\ &= 2 \cdot 2^k + 1 \\ &= 2^{k+1} + 1 \end{aligned}$$

Since both the basis and inductive steps are true, $P(n)$ is true for all integers $n \geq 1$.
QED

Suppose $d_1, d_2, d_3, \ldots$ is a sequence defined by the formula $d_n = 3^n - 2^n$, for all integers $n \geq 0$.

Show that this sequence satisfies the recurrence relation for $k \geq 2$.

Theorem 2: Recurrence Relation

For every integer $n \geq 0$, $d_n = 3^n - 2^n$ satisfies $d_k = 5d_{k-1} - 6d_{k-2}$ for $k \geq 2$.
Then for each integer $k \geq 2$,

$$\begin{aligned} d_k &= 3^k - 2^k \\ d_{k-1} &= 3^{k-1} - 2^{k-1} \\ d_{k-2} &= 3^{k-2} - 2^{k-2} \end{aligned}$$

It follows that for each integer $k \geq 2$,

$$\begin{aligned} 5d_{k-1} - 6d_{k-2} &= 5(3^{k-1} - 2^{k-1}) - 6(3^{k-2} - 2^{k-2}) && \textit{(By Subsititution)} \\ &= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 2 \cdot 3 \cdot 3^{k-2} + 3 \cdot 2 \cdot 2^{k-2} \\ &= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 2 \cdot 3^{k-1} + 3 \cdot 2^{k-1} \\ &= 3 \cdot 3^{k-1} - 2 \cdot 2^{k-1} \\ &= 3^k - 2^k \\ &= d_k \end{aligned}$$

Thus $d_k = 5d_{k-1} - 6d_{k-2}$ for every integer $k \geq 2$.

1. Use iteration to guess an explicit formula for the recurrence relation $b_k = \frac{b_{k-1}}{1+b_{k-1}}$ for $k \geq 1$ with $b_0 = 1$.

2. Use mathematical induction to show that your guessed formula is correct for $k \geq 0$.

Theorem 3: Guessing a Formula

$b_k = \frac{b_{k-1}}{1+b_{k-1}}$ for $k \geq 1$ with $b_1 = 1$.

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= \frac{b_0}{1+b_0} = \frac{1}{1+1} = \frac{1}{2} \\
b_2 &= \frac{b_1}{1+b_1} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{2+1} = \frac{1}{3} \\
b_3 &= \frac{b_2}{1+b_2} = \frac{\frac{1}{3}}{1+\frac{1}{3}} = \frac{1}{3+1} = \frac{1}{4} \\
b_4 &= \frac{b_3}{1+b_3} = \frac{\frac{1}{4}}{1+\frac{1}{4}} = \frac{1}{4+1} = \frac{1}{5}
\end{aligned}
$$

Guess: $\frac{1}{n+1}$ for every integer $n \geq 0$.

Theorem 4: Checking a Formula

Proof by Mathematical Induction.

Let the sequence $b_0, b_1, b_2, \ldots$ be defined by:

$$b_0 = 1 \quad \text{and} \quad b_k = \frac{b_{k-1}}{1+b_{k-1}} \quad \text{for all integers } k \geq 1.$$

We claim that for every integer $k \geq 0$,

$$b_k = \frac{1}{k+1}.$$

Let $P(n)$ be the statement: $b_n = \frac{1}{n+1}$.
Base Case: We must show $P(0)$ holds.
Then,

$$b_0 = 1 = \frac{1}{0+1},$$

so $P(0)$ holds.

Inductive Step: Suppose $P(k)$ is true for some integer $k \geq 0$,

$$b_k = \frac{1}{k+1}.$$

We want to show that $P(k+1)$ is also true, that is,

$$b_{k+1} = \frac{1}{k+2}.$$

Using the recurrence relation:

$$b_{k+1} = \frac{b_k}{1+b_k}.$$

Substitute the inductive hypothesis:

$$b_{k+1} = \frac{\frac{1}{k+1}}{1+\frac{1}{k+1}} = \frac{\frac{1}{k+1}}{\frac{k+2}{k+1}} = \frac{1}{k+2}.$$

Therefore, $P(k+1)$ holds.

Thus, the formula $b_k = \frac{1}{k+1}$ holds for all integers $k \geq 0$. **QED**