

# CSE 2500-01: Homework 6

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Spring 2025

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Coded in L<sup>A</sup>T<sub>E</sub>X

## Problems

### Theorem 1

*For all integers  $k$  with  $k \geq 4$ ,  $2k^2 - 5k + 2$  is not prime.*

*Proof.* Suppose  $k$  is any integer such that  $k \geq 4$ .

By definition of prime, a number  $p \in \mathbb{Z}^+$  is prime if and only if  $p > 1$ , and for all positive integers  $x$  and  $y$  such that  $xy = p$ , either  $x = 1$  and  $y = p$ , or  $x = p$  and  $y = 1$ .

$$2k^2 - 5k + 2 = (2k - 1)(k - 2) \quad (\text{by factoring})$$

Since  $k \geq 4$

$$k - 2 \geq 2$$

$$2k \geq 8 \Rightarrow 2k - 1 \geq 7$$

Let  $a = 2k - 1$  and  $b = k - 2$ . Then  $a \cdot b = 2k^2 - 5k + 2$ , and both  $a, b \in \mathbb{Z}^+$  with  $a \geq 7$  and  $b \geq 2$ .

Hence,  $2k^2 - 5k + 2$  is a product of two positive integers greater than 1.

It follows by definition that  $2k^2 - 5k + 2$  is not prime.

**QED**

### Theorem 2

*If  $m$  and  $n$  are positive integers and  $mn$  is a perfect square, then  $m$  and  $n$  are both perfect squares.*

This is **False**.

Let  $m = n = 2 \rightarrow m \cdot n = 4$

By definition of perfect squares, 4 is a perfect square since  $4 = 2^2$  but neither  $m = 2$  nor  $n = 2$  is a perfect square.

**Theorem 3**

*Given two rational numbers  $r$  and  $s$  where  $r < s$ , there exists another rational number between  $r$  and  $s$ .*

*Proof.* Suppose  $r$  and  $s$  are any two distinct rational numbers.

By definition of rational numbers,  $r = \frac{a}{b}$  and  $s = \frac{c}{d}$  for some integers  $a, b, c$ , and  $d$  with  $b \neq 0$  and  $d \neq 0$

Then,

$$\begin{aligned} \frac{r+s}{2} &= \frac{\frac{a}{b} + \frac{c}{d}}{2} && \text{(By Substitution.)} \\ &= \frac{\frac{ad+bc}{bd}}{2} \\ &= \frac{ad+bc}{2bd} && \text{(By Algebra.)} \end{aligned}$$

Note that  $ad+cd$  and  $2bd$  are integers since  $a, b, c$ , and  $d$  are all integers and the product of integers and sums of integers is an integer. Also,  $2bd \neq 0$  by the zero product property.

It goes to say that by the definition of rational numbers,

$$\frac{r+s}{2} \tag{1}$$

is rational.

Suppose  $e$  and  $f$  are any real numbers such that  $e < f$ .

Dividing into cases and by the properties of inequalities,

**Case I**

$$\begin{aligned} (e+f) &< 2f \rightarrow \text{Dividing by Two,} \\ \frac{e+f}{2} &< f \end{aligned}$$

**Case II**

$$\begin{aligned} 2e &< (e+f) \rightarrow \text{Dividing by Two,} \\ e &< \frac{e+f}{2} \end{aligned}$$

Thus, by combination,

$$e < \frac{e+f}{2} < f \tag{2}$$

Let  $y = \frac{r+s}{2}$  where by (1)  $y$  is rational, and by (2),  $r < y < s$ .  
Hence there exists another rational number between  $r$  and  $s$ .

**QED**

## Problem 4

Is  $x$  necessarily rational? If so, prove it.

### Theorem 4

Let  $a, b, c, d \in \mathbb{Z}$  with  $a \neq c$ , and let  $x \in \mathbb{R}$  satisfy the equation

$$\frac{ax + b}{cx + d} = 1.$$

Then  $x \in \mathbb{Q}$ ; that is,  $x$  is rational.

*Proof.* Suppose  $a, b, c$ , and  $d$  are integers and  $a \neq c$ . Suppose  $x \in \mathbb{R}$  and  $\frac{ax+b}{cx+d} = 1$ .  
Then,

$$\frac{ax + b}{cx + d} = 1 \quad \text{(Starting Point)}$$

$$ax + b = cx + d \quad \text{(By Multiplication)}$$

$$x(a - c) = (d - b) \quad \text{(Separation of Variables)}$$

$$x = \frac{d - b}{a - c} \quad \text{(Algebra)}$$

Let  $t = \frac{d-b}{a-c}$ . Note that  $t$  is rational because it is the difference of two integers  $d$  and  $b$ .  
Also,  $a - c \neq 0$  since  $a \neq c$ .

It follows by the definition of rational numbers,  $t$  is quotient of integers with a non-zero denominator which makes  $x$  a rational number since  $x = t$ .

**QED**

## Problem 5

**(5 points)** Use the unique factorization theorem to write the following integers in standard factored form:

(a) 1176

(b) 5733

(c) 3675

## Problem 6

**(10 points)** Prove the following statement:

“The square of any integer has the form  $4k$  or  $4k + 1$  for some integer  $k$ .”

## Problem 7

(10 points) Show that the following statement is false:

“For all integers  $a$  and  $b$ , if  $3 \mid (a + b)$ , then  $3 \mid (a - b)$ .”

## Problem 8

(10 points) Prove that if  $n$  is any nonnegative integer whose decimal representation ends in 5, then  $5 \mid n$ .

## Problem 9

(10 points) Given any integer  $n > 3$ , could  $n$ ,  $n + 2$ , and  $n + 4$  all be prime? Prove or give a counterexample.

## Question 1

Prove the following properties. You should follow the procedures discussed and shown in the class.

### Theorem 5

*The difference of any even integer minus any odd integer is odd.*

Proof. Suppose:  $m$  is any even integer and  $n$  is any odd integer.

By definition of even and odd,  $m = 2r$  and  $n = 2s + 1$ , for some integers  $r$  and  $s$ .

Then

$$\begin{aligned} m - n &= (2r) - (2s + 1) && \text{(by substitution)} \\ &= 2r - 2s - 1 \\ &= 2(r - s - 1) + 1 && \text{(by algebra)} \end{aligned}$$

Let  $t = r - s - 1$

Note  $t$  is an integer since the difference of integers are integers.

Hence  $m - n = 2t + 1$ , where  $t$  is some integer.

It follows by definition of odd that  $m - n$  is odd.

**QED**