CSE 2500-01: Homework 9

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Coded in LaTeX

Problems

1. (10 points) Let $A = \{n \in \mathbb{Z} \mid n = 5r, \exists r \in \mathbb{Z}\}$ and $B = \{m \in \mathbb{Z} \mid m = 20s, \exists s \in \mathbb{Z}\}$.

   (a) $A \subseteq B$
   $A \not\subseteq B$. For example $5 \in A$ because $5 = 5(1)$, but $5 \neq 20k$ for any integer k. Therefore, $5 \notin B$.

   (b) $B \subseteq A$
   Yes, every element of B is in A. Suppose n is any element of B, then $n = 20k$ for some integer k, and $n = 5(4k)$. So, since $4k$ is an integer, $n \in A$.

2. (10 points) Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{b, c, e\}$

   (a) Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?

      i. $A \cap (B \cup C) = \{a, b, c\} \cap \{b, c, d, e\} = \{b, c\}$
      ii. $(A \cap B) \cup C = \{b, c\} \cup \{b, c, e\} = \{b, c, e\}$
      iii. $(A \cap B) \cup (A \cap C) = \{b, c\} \cup \{b, c\} = \{b, c\}$
      iv. Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (b) Find $(A - B) - C$ and $A - (B - C)$. Are these two sets equal?

      i. $(A - B) - C = \{a\} - \{b, c, e\} = \{a\}$
      ii. $A - (B - C) = \{a, b, c\} - \{d\} = \{a, b, c\}$
      iii. No, these sets are not equal.

3. Let $S_i = \{x \in \mathbb{R} \mid 1 < x < 1 + \frac{1}{i}\} = (1, 1 + \frac{1}{i})$ for all positive integers i.

   (a) $\bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} (1, 1 + \frac{1}{i}) = (1, 2) = S_1$

   (b) $\bigcap_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} (1, 1 + \frac{1}{i}) = \emptyset$

4. (a) Is $\{\{5, 4\}, \{7, 2\}, \{1, 3, 4\}, \{6, 8\}\}$ a partition of $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$?
   No, because 4 is in two sets, $\{5, 4\}$ and $\{1, 3, 4\}$.

   (b) Is $\{\{1, 5\}, \{4, 7\}, \{2, 8, 6, 3\}\}$ a partition of $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$?
   Yes, every element in {1,2,3,4,5,6,7,8} is in exactly one set and no element is in more than one set of the partition.

5. Find $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.
   $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

6. Let $A_1 = \{1, 2, 3\}$, $A_2 = \{u, v\}$, and $A_3 = \{m, n\}$. Find the following sets:

   (a)
$$\begin{aligned}(A_1 \times A_2) \times A_3 = \{&((1, u), m),\ ((1, u), n),\ ((1, v), m),\ ((1, v), n),\\ &((2, u), m),\ ((2, u), n),\ ((2, v), m),\ ((2, v), n),\\ &((3, u), m),\ ((3, u), n),\ ((3, v), m),\ ((3, v), n)\}\end{aligned}$$

(b)

$$\begin{aligned} A_1 \times A_2 \times A_3 = \{&(1,u,m),\ (1,u,n),\ (1,v,m),\ (1,v,n),\\ &(2,u,m),\ (2,u,n),\ (2,v,m),\ (2,v,n),\\ &(3,u,m),\ (3,u,n),\ (3,v,m),\ (3,v,n)\} \end{aligned}$$

(c) How are these sets different?

- In part (a), each element is a **pair**, where the first component is itself a **pair**. The structure is $((x, y), z)$.
- In part (b), each element is a **triple** (x, y, z).
- Although the same elements are used (from A_1, A_2, and A_3), the organization and grouping differ.

7. (10 points) Use an elementary argument to prove the following statement using contrapositive. You may assume that all subsets are sets of a universal set U. It may be helpful to proceed by division into cases. For all sets A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$.

Theorem 1: Contrapositive Proof

Proof by Contrapositive. By the contrapositive:

Suppose $(A \cup B) \not\subseteq C$.

Then, there exists an element $x \in A \cup B$ such that $x \notin C$.

By definition of union, $x \in A$ or $x \in B$.

We proceed by cases:

- **Case 1:** $x \in A$. Since $x \notin C$, it follows that $A \not\subseteq C$.
- **Case 2:** $x \in B$. Since $x \notin C$, it follows that $B \not\subseteq C$.

Thus, if $(A \cup B) \not\subseteq C$, then either $A \not\subseteq C$ or $B \not\subseteq C$.

Therefore, by contrapositive, we conclude: If $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$. **QED**

8. (10 points) Use the element method for proving a set equals the empty set to prove: For all sets A, B, C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Theorem 2: Proving a Set Equals $\emptyset$

Proof. Let A, B, C be sets such that $A \subseteq B$ and $B \cap C = \emptyset$. Suppose $A \cap C \neq \emptyset$.
Then there is an element x such that $x \in A \cap C$.
By definition of intersection, this means $x \in A$ and $x \in C$.
Since $A \subseteq B$, it follows that $x \in B$. However, since $B \cap C = \emptyset$, we have $x \notin C$.
This is a contradiction.
Therefore, our assumption that $A \cap C \neq \emptyset$ must be false. Thus, we conclude that

$A \cap C = \emptyset$. **QED**

9. (10 points) Use the element method for proving a set equals the empty set to prove: For all sets A, B, C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

Theorem 3: Proving a Set Equals $\emptyset$

Let A, B, and C be any sets such that $B \cup C \subseteq A$.
Suppose $(C - A) \cap (B - A) \neq \emptyset$.
By definition of empty set, thgere is an element x such that $x \in (C-A) \cap (B-A_{=}\emptyset$.
By definition of intersection, $x \in C - A$ and $x \in B - A$.
So, by definition of set difference, $x \in C$ and $x \notin A$ and $x \in B$ and $x \notin A$.
Since $x \in C$ and $x \in B$, $x \in B \cap C$ by the definition of intersection.
But $B \cap C \subseteq A$, so $x \in A$.
This is a contradiction, since we have shown that $x \in A$ and $x \notin A$.
Hence the supposition is false, and we conclude that $(C - A) \cap (B - A) = \emptyset$.

10. (10 points) Construct an algebraic proof for the following (cite a set identity to justify each step): For all sets A, B, C,

$$(A - B) - (B - C) = A - B$$

(a) i. (1) $(A-B)-(B-C)$ (2) $= (A \cap B^c) \cap (B \cap C^c)^c$ by definition of set difference
 ii. (1) $= (A \cap B^c) \cap (B^c \cup C)$ by De Morgan's Law
 iii. (1) $= (A \cap B^c \cap B^c) \cup (A \cap B^c \cap C)$ by Distributive Law
 iv. (1) $= (A \cap B^c) \cup (A \cap B^c \cap C)$ by Idempotent Law
 v. (1) $= A \cap B^c$ by Absorption Law
 vi. (1) $= A - B$ by definition of set difference

11. (10 points) Simplify the expression, citing a set identity in each step:

$$((A \cap (B \cap C)) \cap (A - B)) \cap (B \cup C^c)$$

(a) i. (1) $((A \cap (B \cap C)) \cap (A-B)) \cap (B \cup C^c)$ (2) $= ((A \cap B \cap C) \cap (A \cap B^c)) \cap (B \cup C^c)$ by definition of set difference
 ii. (1) $= (A \cap (B \cap B^c) \cap C) \cap (B \cup C^c)$ by Associative and Commutative Laws
 iii. (1) $= (A \cap \emptyset \cap C) \cap (B \cup C^c)$ because $B \cap B^c = \emptyset$
 iv. (1) $= \emptyset \cap (B \cup C^c)$ because $A \cap \emptyset = \emptyset$
 v. (1) $= \emptyset$ by Identity Law ($\emptyset$ intersect anything is $\emptyset$)