CSE 2500-01: Homework 7

Arturo Salinas-Aguayo
Spring 2025
Electrical and Computer Engineering Department

College of Engineering, University of Connecticut
Coded in LaTeX

Problems

Theorem 1: Induction Proof 1

Prove the following statement using mathematical induction:

$$\forall n \in \mathbb{Z}, n \geq 1, \quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Proof by Mathematical Induction.
Let the property, $P(n)$ be the equation:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Base Case: Show that $P(1)$ is true.

LHS		**RHS**
$\frac{1}{1(1+1)}$		$\frac{1}{1+1}$
$\parallel$		$\parallel$
$\frac{1}{2}$	$=$	$\frac{1}{2}$

Therefore, $P(1)$ is true.
Inductive Step: Assume that $P(k)$ is true for some integer $k \geq 1$. That is, assume:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We need to show that $P(k+1)$ is true.

LHS

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k+1)((k+1)+1)}$$
$$\parallel$$
$$\frac{k}{k+1} + \frac{1}{(k+1)((k+1)+1)}$$
$$\parallel$$
$$\frac{k(k+1)+1}{(k+1)(k+2)}$$
$$\parallel$$
$$\frac{k^2+2k+1}{(k+1)(k+2)}$$
$$\parallel$$
$$\frac{(k+1)^2}{(k+1)(k+2)}$$
$$\parallel$$
$$\frac{k+1}{k+2}$$

RHS

$$\frac{k+1}{(k+1)+1}$$
$$\parallel$$
$$\frac{k+1}{k+2}$$

The LHS is equal to the RHS.
Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

QED

Theorem 2: Induction Proof 2

Prove the following statement using mathematical induction:

$$\forall n \in \mathbb{Z}, n \geq 0, \quad \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

Proof by Mathematical Induction.
Let the property, $P(n)$ be the equation:

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

Base Case: Show that $P(0)$ is true.

$$\begin{array}{ccc} \textbf{LHS} & & \textbf{RHS} \\ \sum_{i=1}^{0+1} i \cdot 2^i & & 0 \cdot 2^{0+2} + 2 \\ \| & & \| \\ 1 \cdot 2^i & & 2 \\ \| & & \Downarrow \\ 2 & = & 2 \end{array}$$

Inductive Step: Assume that $P(k)$ is true for some integer $k \geq 0$. That is, assume:

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$$

Then we need to show that $P(k+1)$ is true.

LHS	**RHS**
$\sum_{i=1}^{(k+1)+1} i \cdot 2^i$	$(k+1) \cdot 2^{(k+1)+2} + 2$
$\|$	$\|$
$\sum_{i=1}^{k+2} i \cdot 2^i$	$(k+1) \cdot 2^{k+3} + 2$
$\|$	$\|$
$\sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2}$	$2^{k+3}(k+1) + 2$
$\|$	
$k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2}$	
$\|$	
$2^{k+2}(k + (k+2)) + 2$	
$\|$	
$2^{k+2}(2k+2) + 2$	
$\|$	
$2^{k+3}(k+1) + 2$	
$\|$	
$2^{k+3}(k+1) + 2$	

The LHS is equal to the RHS.
Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true. **QED**

Theorem 3: Induction Proof 3

Prove the following inequality using mathematical induction:

$$\forall n \in \mathbb{Z}, n \geq 2, \forall x \in \mathbb{R}, x \geq -1, \quad 1 + nx \leq (1+x)^n$$

Proof by Mathematical Induction.
Let the property, $P(n)$ be the inequality:

$$1 + nx \leq (1+x)^n$$

Base Case: Show that $P(2)$ is true.

LHS	**RHS**
$1 + 2x$	$(1+x)^2$
$\parallel$	$\parallel$
$1 + 2x$	$1 + 2x + x^2$

$$1 + 2x \leq 1 + 2x + x^2 \quad \text{is true for } x \geq -1$$

Inductive Step: Assume that $P(k)$ is true for some integer $k \geq 2$. That is, assume:

$$1 + kx \leq (1+x)^k$$

We must show that $P(k+1)$ is true.

LHS

$$\begin{gathered} 1 + (k+1)x \\ \parallel \\ 1 + kx + x \\ \parallel \\ 1 + x(k+1) \end{gathered}$$

RHS

$$\begin{gathered} (1+x)^{k+1} \\ \parallel \\ (1+x)(1+x)^k \\ \Downarrow \text{(by inductive hypothesis)} \\ (1+x)(1+kx) \leq (1+x)(1+x)^k \\ \parallel \\ (kx^2 + kx + x + 1) \leq (1+x)(1+x)^k \\ \parallel \\ kx^2 + x(k+1) + 1 \leq (1+x)(1+x)^k \end{gathered}$$

Since $1 + x(k+1) \leq kx^2 + x(k+1) + 1$, it follows that:

$$1 + x(k+1) \leq kx^2 + x(k+1) + 1 \leq (1+x)(1+x)^k = (1+x)^{k+1}$$

Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

QED

Theorem 4: Recursively Defined Sequence

A sequence $c_0, c_1, c_2, \ldots$ is defined recursively as follows:

$$c_0 = 3, \quad c_k = (c_{k-1})^2 \text{ for all integers } k \geq 1.$$

Prove by induction that:

$$c_n = 3^{2^n} \quad \text{for all integers } n \geq 0.$$

Proof by Mathematical Induction.
Let the property $P(n)$ be the equation:

$$c_n = 3^{2^n} \text{ for all integers } n \geq 0.$$

Base Case: Show that $P(0)$ is true.

$$c_0 = 3 \quad \text{and} \quad 3^{2^0} = 3^1 = 3$$

Inductive Step: Assume that $P(k)$ is true for some integer $k \geq 0$. That is, assume:

$$c_k = 3^{2^k}$$

We must show that $P(k+1)$ is true.
By Substitution:

LHS	**RHS**
c_{k+1} $\|\|$ $(c_k)^2$ $\Downarrow$ (by inductive hypothesis) $(3^{2^k})^2$ $\|\|$ $3^{2^{k+1}}$	$3^{2^{k+1}}$

The LHS is equal to the RHS.
Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

QED