CSE 2500-01: Homework 7

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Coded in LaTeX

Problems

$$\begin{array}{cc}
\textbf{LHS} & \textbf{RHS} \\
\sum_{i=0}^{0} r^i & \frac{r^{0+1}-1}{r-1} \\
\| & \| \\
r^0 & \frac{r-1}{r-1} \\
\| & \| \\
1 & 1 \\
\| & \| \\
1 & 1
\end{array}$$

Therefore, LHS = RHS

Theorem 1: Induction Proof 1

Prove the following statement using mathematical induction:

$$\forall n \in \mathbb{Z}, n \geq 1, \quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Proof by Mathematical Induction.

Let the property, $P(n)$ be the equation:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Base Case: Show that $P(1)$ is true.

$$\begin{array}{ccc}
\textbf{LHS} & & \textbf{RHS} \\
\frac{1}{1(1+1)} & & \frac{1}{1+1} \\
\| & & \| \\
\frac{1}{2} & = & \frac{1}{2}
\end{array}$$

Therefore, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true for some integer $k \geq 1$. That is, assume:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We need to show that $P(k+1)$ is true.

$$\begin{array}{ccc}
\textbf{LHS} & & \textbf{RHS} \\
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)((k+1)+1)} & & \frac{k+1}{(k+1)+1} \\
\| & & \| \\
\frac{k}{k+1}+\frac{1}{(k+1)((k+1)+1)} & & \frac{k+1}{k+2} \\
\| & & \Downarrow \\
\frac{k(k+1)+1}{(k+1)(k+2)} & & \\
\| & & \Downarrow \\
\frac{k^2+2k+1}{(k+1)(k+2)} & & \\
\| & & \Downarrow \\
\frac{(k+1)^2}{(k+1)(k+2)} & & \\
\| & & \Downarrow \\
\frac{k+1}{k+2} & = & \frac{k+1}{k+2}
\end{array}$$

Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

QED

Theorem 2: Induction Proof 2

Prove the following statement using mathematical induction:

$$\forall n \in \mathbb{Z}, n \geq 0, \quad \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

Proof by Mathematical Induction.

Let the property, $P(n)$ be the equation:

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$$

Base Case: Show that $P(0)$ is true.

$$\begin{array}{ccc}
\textbf{LHS} & & \textbf{RHS} \\
\sum_{i=1}^{0+1} i \cdot 2^i & & 0 \cdot 2^{0+2} + 2 \\
\| & & \| \\
1 \cdot 2^i & & 2 \\
\| & & \Downarrow \\
2 & = & 2
\end{array}$$

Inductive Step: Assume that $P(k)$ is true for some integer $k \geq 0$. That is, assume:

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$$

Then we need to show that $P(k+1)$ is true.

$$\begin{array}{ccc}
\textbf{LHS} & & \textbf{RHS} \\
\sum_{i=1}^{(k+1)+1} i \cdot 2^i & & (k+1) \cdot 2^{(k+1)+2} + 2 \\
\parallel & & \parallel \\
\sum_{i=1}^{(k+2)} i \cdot 2^i & & (k+1) \cdot 2^{(k+3)} + 2 \\
\parallel & & \Downarrow \\
\sum_{i=1}^{(k+1)} i \cdot 2^i + (k+2)2^{k+2} & & (k+1) \cdot 2^{(k+3)} + 2 \\
\parallel & & \Downarrow \\
k \cdot 2^{k+2} + 2 + (k+2)2^{k+2} & & \\
\parallel & & \Downarrow \\
2^{k+2}(k + (k+2)) + 2 & & \\
\parallel & & \Downarrow \\
2^{k+2}(2k+2) + 2 & & \\
\parallel & & \Downarrow \\
2^{k+3}(k+1) + 2 & & \\
\parallel & & \Downarrow \\
2^{k+3}(k+1) + 2 & = & 2^{k+3}(k+1) + 2
\end{array}$$

Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true. **QED**

Theorem 3: Induction Proof 3

Prove the following inequality using mathematical induction:

$$\forall n \in \mathbb{Z}, n \geq 2, \forall x \in \mathbb{R}, x \geq -1, \quad 1 + nx \leq (1+x)^n$$

Proof. Let the property, $P(n)$ be the inequality:

$$1 + nx \leq (1+x)^n$$

Base Case: Show that $P(2)$ is true.

$$\begin{array}{cc}
\textbf{LHS} & \textbf{RHS} \\
1 + 2x & (1+x)^2 \\
\parallel & \parallel \\
1 + 2x & 1 + 2x + x^2
\end{array}$$

$$1 + 2x \le 1 + 2x + x^2 \quad \text{is true for } x \ge -1$$

Inductive Step: Assume that $P(k)$ is true for some integer $k \ge 2$. That is, assume:

$$1 + kx \le (1 + x)^k$$

We must show that $P(k+1)$ is true.

LHS

$1 + (k+1)x$

$1 + kx + x$

$1 + x(k+1)$

RHS

$(1+x)^{k+1}$

$(1+x)(1+x)^k$

$\Downarrow$ (by inductive hypothesis)

$(1+x)(1+kx) \le (1+x)(1+x)^k$

$(kx^2 + kx + x + 1) \le (1+x)(1+x)^k$

$kx^2 + x(k+1) + 1 \le (1+x)(1+x)^k$

Since $1 + x(k+1) \le kx^2 + x(k+1) + 1$, it follows that:

$$1 + x(k+1) \le kx^2 + x(k+1) + 1 \le (1+x)(1+x)^k = (1+x)^{k+1}$$

Thus, we have shown that if $P(k)$ is true, then $P(k+1)$ is also true.

QED

Theorem 4: Recursively Defined Sequence

A sequence $c_0, c_1, c_2, \ldots$ is defined recursively as follows:

$$c_0 = 3, \quad c_k = (c_{k-1})^2 \text{ for all integers } k \ge 1.$$

Prove by induction that:

$$c_n = 3^{2^n} \quad \text{for all integers } n \ge 0.$$