CSE 2500-01: Homework 9

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Spring 2025
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Coded in LaTeX

Problems

1. Let $A = \{n \in \mathbb{Z} \mid n = 5r, \exists r \in \mathbb{Z}\}$ and $B = \{m \in \mathbb{Z} \mid m = 20s, \exists s \in \mathbb{Z}\}$.

Theorem 1: Proofs of Set Inclusion

(a) $A \subseteq B$
No, $A \not\subseteq B$.

Disproof by Counterexample. $A \not\subseteq B$. For example $5 \in A$ because $5 = 5 \cdot 1$, but $5 \neq 20k$ for any integer k. Therefore, $5 \notin B$. **QED**

(b) $B \subseteq A$ Yes, every element of B is in A.

Proof. Suppose n is any element of B.
$n = 20k$ for some integer k, and $n = 5(4k)$.
So, since $4k$ is an integer, $n \in A$. **QED**

2. Let $A = \{a, b, c\}$, $B = \{b, c, d\}$, and $C = \{b, c, e\}$

Theorem 2: Union and Intersect Operations

(a) Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?

i. $A \cap (B \cup C) = \{a, b, c\} \cap \{b, c, d, e\} = \{b, c\}$
ii. $(A \cap B) \cup C = \{b, c\} \cup \{b, c, e\} = \{b, c, e\}$
iii. $(A \cap B) \cup (A \cap C) = \{b, c\} \cup \{b, c\} = \{b, c\}$
iv. Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) Find $(A - B) - C$ and $A - (B - C)$. Are these two sets equal?

i. $(A - B) - C = \{a\} - \{b, c, e\} = \{a\}$
ii. $A - (B - C) = \{a, b, c\} - \{d\} = \{a, b, c\}$
iii. No, these sets are not equal.

3. Let $S_i = \{x \in \mathbb{R} \mid 1 < x < 1 + \frac{1}{i}\} = (1, 1 + \frac{1}{i})$ for all positive integers i.

Theorem 3: Set Operations

(a) $\bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} (1, 1 + \frac{1}{i}) = (1, 2) = S_1$

(b) $\bigcap_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} (1, 1 + \frac{1}{i}) = \emptyset$

4. Identify if the following sets are partitions:

Theorem 4: Partitions

(a) Is $\{\{5, 4\}, \{7, 2\}, \{1, 3, 4\}, \{6, 8\}\}$ a partition of $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$?
No, because 4 is in two sets, $\{5, 4\}$ and $\{1, 3, 4\}$.

(b) Is $\{\{1, 5\}, \{4, 7\}, \{2, 8, 6, 3\}\}$ a partition of $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$?
Yes, every element in {1,2,3,4,5,6,7,8} is in exactly one set and no element is in more than one set of the partition.

5. Find $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.

Theorem 5: Power Sets

$$\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$$

6. Let $A_1 = \{1, 2, 3\}$, $A_2 = \{u, v\}$, and $A_3 = \{m, n\}$. Find the following sets:

Theorem 6: Set Products

(a) $(A_1 \times A_2) \times A_3$

$$\begin{aligned}(A_1 \times A_2) \times A_3 = \{&((1, u), m),\ ((1, u), n),\ ((1, v), m),\ ((1, v), n),\\ &((2, u), m),\ ((2, u), n),\ ((2, v), m),\ ((2, v), n),\\ &((3, u), m),\ ((3, u), n),\ ((3, v), m),\ ((3, v), n)\}\end{aligned}$$

(b) $A_1 \times A_2 \times A_3$

$$\begin{aligned}A_1 \times A_2 \times A_3 = \{&(1, u, m),\ (1, u, n),\ (1, v, m),\ (1, v, n),\\ &(2, u, m),\ (2, u, n),\ (2, v, m),\ (2, v, n),\\ &(3, u, m),\ (3, u, n),\ (3, v, m),\ (3, v, n)\}\end{aligned}$$

(c) How are these sets different?

- In part (a), each element is a pair, where the first component is itself a pair. The structure is $((x, y), z)$.j
- In part (b), each element is a triple (x, y, z).

7. Use an elementary argument to prove the following statement using contrapositive. You may assume that all subsets are sets of a universal set U. It may be helpful to proceed by division into cases. For all sets A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$.

Theorem 7: Contrapositive Proof

For all sets A, B, C, if $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$.

Proof by Contrapositive. Suppose $x \in A \cup B$.
By definition of union, $x \in A$ or $x \in B$.

- **Case 1:** $x \in A$. $x \in C$ because $A \subseteq C$.
- **Case 2:** $x \in B$. $x \in C$ because $B \subseteq C$.

Therefore, by contrapositive:
If $A \subseteq C$ and $B \subseteq C$, then $(A \cup B) \subseteq C$. **QED**

8. Use the element method for proving a set equals the empty set to prove: For all sets A, B, C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Theorem 8: Proving a Set Equals $\emptyset$

For all sets A, B, C, if $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$.

Proof by Contradiction. Let A, B, C be sets such that $A \subseteq B$ and $B \cap C = \emptyset$.
Suppose $A \cap C \neq \emptyset$. Then there is an element x such that $x \in A \cap C$.
By definition of intersection,

$$x \in A \quad \text{and} \quad x \in C$$

Since $A \subseteq B$, it follows that $x \in B$. However, since $B \cap C = \emptyset$, we have $x \notin C$.
This is a contradiction.
Therefore, our assumption that $A \cap C \neq \emptyset$ must be false. Thus, we conclude that $A \cap C = \emptyset$. **QED**

9. Use the element method for proving a set equals the empty set to prove: For all sets A, B, C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

Theorem 9: Proving a Set Equals $\emptyset$

For all sets A, B, C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

Proof by Contradiction. Let A, B, and C be any sets such that $B \cap C \subseteq A$.
Suppose $(C - A) \cap (B - A) \neq \emptyset$.
By definition of empty set, there is an element x such that

$$x \in (C - A) \cap (B - A) = \emptyset$$

By definition of intersection,

$$x \in C - A \quad \text{and} \quad x \in B - A$$

By definition of set difference, $x \in C$ and $x \notin A$ and $x \in B$ and $x \notin A$.
Since $x \in C$ and $x \in B$,

$$x \in B \cap C$$

by the definition of intersection.
But $B \cap C \subseteq A$, so $x \in A$.
This is a contradiction, since we have shown that $x \in A$ and $x \notin A$.
Hence the supposition is false, and we conclude that $(C-A)\cap(B-A) = \emptyset$. **QED**

10. Construct an algebraic proof for the following (cite a set identity to justify each step): For all sets A, B, C,
$$(A - B) - (B - C) = A - B$$

Theorem 10: Algebraic Proof

Proof by Algebraic Properties. Let A, B, and C be any sets.
$(A - B) - (B - C)$

$= (A \cap B^c) \cap (B \cap C^c)^c$	by definition of set difference
$= (A \cap B^c) \cap (B^c \cup C)$	by De Morgan's Law
$= (A \cap B^c) \cap (B^c \cap C)$	by double complement law
$= ((A \cap B^c) \cap B^c) \cap ((A \cap B^c) \cap C)$	by the distributive law
$= (A \cap (B^c \cap B^c)) \cup ((A \cap B^c) \cap C)$	by the associative law for $\cap$
$= (A \cap B^c) \cup ((A \cap B^c) \cap C)$	by Idempotent Law for $\cap$
$= A \cap B^c$	by Absorption Law
$= A - B$	by definition of set difference

QED

11. Simplify the expression, citing a set identity in each step:

$$((A \cap (B \cap C)) \cap (A - B)) \cap (B \cup C^c)$$

Theorem 11: Algebraic Proof

Proof by Set Identities. Let A, B, and C be any sets.

$$((A \cap (B \cap C)) \cap (A - B)) \cap (B \cup C^c)$$

$= ((A \cap (B \cup C)) \cap (A - B)) \cap (B \cup C^c)$	by the set difference law
$= ((A \cap B^c) \cap (A \cap (B \cup C))) \cap (B \cup C^c)$	by the commutative law for $\cap$
$= (((A \cap B^c) \cap A) \cap (B \cup C)) \cap (B \cup C^c)$	by the associative law for $\cap$
$= ((A \cap (A \cap B^c)) \cap (B \cup C)) \cup (B \cup C^c)$	by the commutative law for $\cap$
$= (((A \cap A) \cap B^c) \cap (B \cup C)) \cap (B \cup C^c)$	by the associative law for $\cap$
$= ((A \cap B^c) \cap (B \cup C)) \cap (B \cup C^c)$	by the idempotent law for $\cup$
$= (A \cap B^c) \cap (B \cup C)) \cap (B \cup C^c)$	by the associative law for $\cap$
$= (A \cap B^c) \cap (B \cup (C \cap C^c))$	by the distributive law
$= (A \cap B^c) \cap (B \cup \emptyset)$	by the complement law for $\cap$
$= (A \cap B^c) \cap B$	by the identity law for $\cup$
$= A \cap (B^c \cap B)$	by the commutative law for $\cap$
$= A \cap \emptyset$	by the complement law for $\cap$
$= \emptyset$	by the universal bound law for $\cap$

QED