CSE 2500-01: Homework 6

Arturo Salinas-Aguayo
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Electrical and Computer Engineering Department

College of Engineering, University of Connecticut
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Problems

Theorem 1

For all integers k with $k \geq 4$, $2k^2 - 5k + 2$ is not prime.

Proof. Suppose k is any integer such that $k \geq 4$.
By definition of prime, a number $p \in \mathbb{Z}^+$ is prime if and only if $p > 1$, and for all positive integers x and y such that $xy = p$, either $x = 1$ and $y = p$, or $x = p$ and $y = 1$.

$$2k^2 - 5k + 2 = (2k-1)(k-2) \quad \text{(by factoring)}$$

Since $k \geq 4$

$$k - 2 \geq 2$$
$$2k \geq 8 \Rightarrow 2k - 1 \geq 7$$

Let $a = 2k - 1$ and $b = k - 2$. Then $a \cdot b = 2k^2 - 5k + 2$, and both $a, b \in \mathbb{Z}^+$ with $a \geq 7$ and $b \geq 2$.
Hence, $2k^2 - 5k + 2$ is a product of two positive integers greater than 1.
It follows by definition that $2k^2 - 5k + 2$ is not prime. **QED**

Theorem 2

If m and n are positive integers and mn is a perfect square, then m and n are both perfect squares.
This is **False**

Disproof by Counterexample. Let $m = n = 2 \to m \cdot n = 4$
By definition of perfect squares, 4 is a perfect square since $4 = 2^2$ but neither $m = 2$ nor $n = 2$ is a perfect square. **QED**

Theorem 3

Given two rational numbers r and s where $r < s$, there exists another rational number between r and s.

Proof. Suppose r and s are any two distinct rational numbers.
By definition of rational numbers, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$
Then,

$$\begin{aligned}
\frac{r+s}{2} &= \frac{\frac{a}{b}+\frac{c}{d}}{2} && \text{(By Substitution.)} \\
&= \frac{\frac{ad+bc}{bd}}{2} \\
&= \frac{ad+bc}{2bd} && \text{(By Algebra.)}
\end{aligned}$$

Note that $ad+cd$ and $2bd$ are integers since a, b, c, and d are all integers and the product of integers and sums of integers is an integer. Also, $2bd \neq 0$ by the zero product property.

It goes to say that by the definition of rational numbers,

$$\frac{r+s}{2} \tag{1}$$

is rational.

Suppose e and f are any real numbers such that $e < f$.
Dividing into cases and by the properties of inequalities,
Case I (adding a f to both sides.)

$$(e+f) < 2f \rightarrow \text{Dividing by Two,}$$
$$\frac{e+f}{2} < 2$$

Case II (adding an e to both sides.)

$$2e < (e+f) \rightarrow \text{Dividing by Two,}$$
$$e < \frac{e+f}{2}$$

Thus, by combination,

$$e < \frac{e+f}{2} < f \tag{2}$$

Let $y = \frac{r+s}{2}$ where by (1) y is rational, and by (2), $r < y < s$.
Hence there exists another rational number between r and s. **QED**

Theorem 4

Let $a, b, c, d \in \mathbb{Z}$ with $a \neq c$, and let $x \in \mathbb{R}$ satisfy the equation

$$\frac{ax+b}{cx+d} = 1.$$

Then $x \in \mathbb{Q}$; that is, x is rational.

Proof. Suppose a, b, c, and d are integers and $a \neq c$. Suppose $x \in \mathbb{R}$ and $\frac{ax+b}{cx+d} = 1$
Then,

$$\begin{aligned}
\frac{ax+b}{cx+d} &= 1 && \text{(Starting Point)} \\
ax+b &= cx+d && \text{(By Multiplication)} \\
x(a-c) &= (d-b) && \text{(Separation of Variables)} \\
x &= \frac{d-b}{a-c} && \text{(Algebra)}
\end{aligned}$$

Let $t = \frac{d-b}{a-c}$. Note that t is rational because it is the difference of two integers d and b.
Also, $a - c \neq 0$ since $a \neq c$.
It follows by the definition of rational numbers, t is quotient of integers with a non-zero denominator which makes x a rational number since $x = t$. **QED**

Theorem 5: Unique Factorization of Integers Theorem.

Given any integer $n > 1$, there exist a positive integer k, distinct prime numbers p_1, p_2, $\cdots$, p_k and positive integers e_1, e_2, $\cdots$, e_k such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$$

and any other expression for n as a product of prime numbers is identical to this except, perhaps, for the order in which the factors are written.

(a) 1176

$$1176 = 2^3 \cdot 3 \cdot 7^2$$

(b) 5733

$$5733 = 3^2 \cdot 7^2 \cdot 13$$

(c) 3675

$$3675 = 3 \cdot 5^2 \cdot 7^2$$

Theorem 6

The square of any integer has the form $4k$ or $4k+1$ for some integer k.

Proof. Suppose n is any integer.
By the Quotient-Remainder Theorem with a divisor equal to 2,
$n = 2q$ or $n = 2q + 1$ for some integer q.

Case I ($n = 2q$)

$$\begin{aligned} n^2 &= (2q)^2 && \text{(By Substitution)} \\ &= 4q^2 && \text{(By Algebra.)} \end{aligned}$$

Let $k = q^2$. Note that k is an integer because it is a product of integers.
It follows $n^2 = 4k$ for some integer k.

Case II ($n = 2q + 1$)

$$\begin{aligned} n^2 &= (2q+1)^2 && \text{(By Substitution)} \\ &= (4q^2 + 4q + 1) \\ &= 4(q^2 + q) + 1 && \text{(By Algebra.)} \end{aligned}$$

Let $k = q^2 + q$. Note that k is an integer because it is a product of integers.
It follows $n^2 = 4k + 1$ for some integer k.

Hence, for both cases, there exists an integer k such that $n^2 = 4k$ or $n^2 = 4k+1$. **QED**

Problem 7

"For all integers a and b, if $3 \mid (a+b)$, then $3 \mid (a-b)$."

This is **False**.

Theorem 7

There exists integers a and b such that $3 \mid (a+b)$ and $3 \nmid (a-b)$.

Disproof by Counterexample. Let $a = 2$ and $b = 1$.
Then,

$$a + b = 2 + 1 = 3 \qquad \text{(By Substitution)}$$

Thus, $3|(a+b)$ because $3 = 3 \cdot 1$.
Also,

$$a - b = 2 - 1 = 1 \qquad \text{(By Substitution)}$$

But, $3 \nmid 1$ because $\frac{1}{3}$ is not an integer.
Thus it is shown that $3 \nmid (a-b)$. **QED**

Theorem 8

If n is any nonnegative integer whose decimal representation ends in 5, then $5 \mid n$.

Proof. Suppose n is a nonnegative integer whose decimal representation ends in 5.
Let $n = 10m + 5$ for some integer, m.

$$\begin{aligned} n &= 10m + 5 \\ &= 5(2m+1) \qquad \text{(By Factoring.)} \end{aligned}$$

Note that the quantity $(2m+1)$ is an integer since it is the product and sum of integers.
Thus, by definition of divisibility, n is divisible by 5. **QED**

Problem 9

Given any integer $n > 3$, could n, $n+2$, and $n+4$ all be prime? Prove or give a counterexample.
They cannot all be prime

Theorem 9

There exists an integer $n > 3$ such that n, $n+2$, or $n+4$ is not prime.

Disproof by Counterexample. Suppose n is any integer with $n > 3$.
Let $d = 3$.
By the quotient-remainder theorem,
$n = 3q$, or $n = 3q+1$, or $n = 3q+2$ for some integer q.
Note that since $n > 3$, either

$$q > 1 \quad \text{or} \tag{1}$$
$$q = 1 \quad \text{and} \quad n = 4 = 3q+1 \quad \text{or} \tag{2}$$
$$q = 1 \quad \text{and} \quad n = 5 = 3q+2 \tag{3}$$

Case I ($q > 1$ and $n = 3q$)
n is not prime because it is a product of 3 and q, and both 3 and q are greater than 1.
Case II ($q \geq 1$ and $n = 3q+1$)

$$\begin{aligned} n+2 &= (3q+1)+2 && \text{(By substitution)} \\ &= 3q+3 \\ &= 3(q+1) && \text{(By algebra)} \end{aligned}$$

So $n+2$ is not prime because it is a product of 3 and $q+1$, and both 3 and $q+1$ are greater than 1.

Case III ($q \geq 1$ and $n = 3q+2$)

$$\begin{aligned} n+4 &= (3q+2)+4 && \text{(By substitution)} \\ &= 3q+6 \\ &= 3(q+2) && \text{(By algebra)} \end{aligned}$$

So $n+4$ is not prime because it is a product of 3 and $q+2$, and both 3 and $q+2$ are greater than 1.

Conclusion: In all three cases, at least one of n, $n+2$, or $n+4$ is not prime. **QED**