CSE 2500-01: Homework 6

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Problems

Theorem 1

For all integers k with $k \geq 4$, $2k^2 - 5k + 2$ is not prime.

Proof. Suppose k is any integer such that $k \geq 4$.
By definition of prime, a number $p \in \mathbb{Z}^+$ is prime if and only if $p > 1$, and for all positive integers x and y such that $xy = p$, either $x = 1$ and $y = p$, or $x = p$ and $y = 1$.

$$2k^2 - 5k + 2 = (2k-1)(k-2) \quad \text{(by factoring)}$$

Since $k \geq 4$

$$\begin{aligned} &k - 2 \geq 2 \\ &2k \geq 8 \Rightarrow 2k - 1 \geq 7 \end{aligned}$$

Let $a = 2k - 1$ and $b = k - 2$. Then $a \cdot b = 2k^2 - 5k + 2$, and both $a, b \in \mathbb{Z}^+$ with $a \geq 7$ and $b \geq 2$.
Hence, $2k^2 - 5k + 2$ is a product of two positive integers greater than 1.
It follows by definition that $2k^2 - 5k + 2$ is not prime. **QED**

Theorem 2

If m and n are positive integers and mn is a perfect square, then m and n are both perfect squares.

This is **False**.
Let $m = n = 2 \rightarrow m \cdot n = 4$
By definition of perfect squares, 4 is a perfect square since $4 = 2^2$ but neither $m = 2$ nor $n = 2$ is a perfect square.

Theorem 3

Given two rational numbers r and s where $r < s$, there exists another rational number between r and s.

Proof. Suppose r and s are any two distinct rational numbers.
By definition of rational numbers, $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$
Then,

$$\begin{aligned}
\frac{r+s}{2} &= \frac{\frac{a}{b}+\frac{c}{d}}{2} && \text{(By Substitution.)} \\
&= \frac{\frac{ad+bc}{bd}}{2} \\
&= \frac{ad+bc}{2bd} && \text{(By Algebra.)}
\end{aligned}$$

Note that $ad+cd$ and $2bd$ are integers since a, b, c, and d are all integers and the product of integers and sums of integers is an integer. Also, $2bd \neq 0$ by the zero product property.

It goes to say that by the definition of rational numbers,

$$\frac{r+s}{2} \tag{1}$$

is rational.

Suppose e and f are any real numbers such that $e < f$.
Dividing into cases and by the properties of inequalities,
Case I

$$(e+f) < 2f \rightarrow \text{Dividing by Two,}$$
$$\frac{e+f}{2} < 2$$

Case II

$$2e < (e+f) \rightarrow \text{Dividing by Two,}$$
$$e < \frac{e+f}{2}$$

Thus, by combination,

$$e < \frac{e+f}{2} < f \tag{2}$$

Let $y = \frac{r+s}{2}$ where by (1) y is rational, and by (2), $r < y < s$.
Hence there exists another rational number between r and s. **QED**

Problem 4

Is x necessarily rational? If so, prove it.

Theorem 4

Let $a, b, c, d \in \mathbb{Z}$ with $a \neq c$, and let $x \in \mathbb{R}$ satisfy the equation

$$\frac{ax+b}{cx+d} = 1.$$

Then $x \in \mathbb{Q}$; that is, x is rational.

Proof. Suppose a, b, c, and d are integers and $a \neq c$. Suppose $x \in \mathbb{R}$ and $\frac{ax+b}{cx+d} = 1$
Then,

$$\begin{aligned} \frac{ax+b}{cx+d} &= 1 && \text{(Starting Point)} \\ ax+b &= cx+d && \text{(By Multiplication)} \\ x(a-c) &= (d-b) && \text{(Separation of Variables)} \\ x &= \frac{d-b}{a-c} && \text{(Algebra)} \end{aligned}$$

Let $t = \frac{d-b}{a-c}$. Note that t is rational because it is the difference of two integers d and b.
Also, $a - c \neq 0$ since $a \neq c$.
It follows by the definition of rational numbers, t is quotient of integers with a non-zero denominator which makes x a rational number since $x = t$. **QED**

Problem 5

(5 points) Use the unique factorization theorem to write the following integers in standard factored form:

(a) 1176

$$1176 = 2^3 \cdot 3 \cdot 7^2$$

(b) 5733

$$5733 = 3^2 \cdot 7^2 \cdot 13$$

(c) 3675

$$3675 = 3 \cdot 5^2 \cdot 7^2$$

Problem 6

Theorem 5

The square of any integer has the form $4k$ or $4k + 1$ for some integer k.

Proof. Suppose n is any integer.
By the Quotient-Remainder Theorem with a divisor equal to 2,
$n = 2q$ or $n = 2q + 1$ for some integer q.

Case I ($n = 2q$)

$$\begin{aligned} n^2 &= (2q)^2 && \text{(By Substitution)} \\ &= 4q^2 && \text{(By Algebra.)} \end{aligned}$$

Let $k = q^2$. Note that k is an integer because it is a product of integers.
It follows $n^2 = 4k$ for some integer k.

Case II ($n = 2q + 1$)

$$\begin{aligned} n^2 &= (2q + 1)^2 && \text{(By Substitution)} \\ &= (4q^2 + 4q + 1) \\ &= 4(q^2 + q) + 1 && \text{(By Algebra.)} \end{aligned}$$

Let $k = q^2 + q$. Note that k is an integer because it is a product of integers.
It follows $n^2 = 4k + 1$ for some integer k.

Hence, for both cases, there exists an integer k such that $n^2 = 4k$ or $n^2 = 4k+1$. **QED**

Problem 7

"For all integers a and b, if $3 \mid (a+b)$, then $3 \mid (a-b)$."

This is **False**. To prove this, I prove the inverse to be true.

Theorem 6

There exists integers a and b such that $3 \mid (a+b)$ and $3 \nmid (a-b)$.

Proof. Let $a = 2$ and $b = 1$.
Then,

$$a + b = 2 + 1 = 3 \qquad \text{(By Substitution)}$$

Thus, $3|(a+b)$ because $3 = 3 \cdot 1$.
Also,

$$a - b = 2 - 1 = 1 \qquad \text{(By Substitution)}$$

But, $3 \nmid 1$ because $\frac{1}{3}$ is not an integer.
Thus it is shown that $3 \nmid (a-b)$. **QED**

Problem 8

Theorem 7

If n is any nonnegative integer whose decimal representation ends in 5, then $5 \mid n$.

Proof. Suppose n is a nonnegative integer whose decimal representation ends in 5.
Let $n = 10m + 5$ for some integer, m.

$$\begin{aligned} n &= 10m + 5 \\ &= 5(2m+1) \qquad \text{(By Factoring.)} \end{aligned}$$

Note that the quantity $(2m+1)$ is an integer since it is the product and sum of integers.
Thus, by definition of divisibility, n is divisible by 5. **QED**

Problem 9

(10 points) Given any integer $n > 3$, could n, $n+2$, and $n+4$ all be prime? Prove or give a counterexample.
They cannot all be prime. To prove this, I will prove the inverse.

Theorem 8

There exists an integer $n > 3$ such that n, $n+2$, or $n+4$ is not prime.

Proof. Suppose n is any integer with $n > 3$.
Let $d = 3$. By the quotient remainder theorem,
$n = 3q$, or $n = 3q+1$ or $n = 3q+2$ for some integer q.
Note that since $n > 3$, either

$$[0]q > 1 \qquad\qquad \text{or} \qquad (3)$$
$$q = 1 \quad \text{and} \quad n = 4 = 3q+1 \qquad \text{or} \qquad (4)$$
$$q = 1 \quad \text{and} \quad n = 5 = 3q+1 \qquad (5)$$
$$(6)$$

QED

Question 1

Prove the following properties. You should follow the procedures discussed and shown in the class.

Theorem 9

The difference of any even integer minus any odd integer is odd.

Proof. Suppose: m is any even integer and n is any odd integer.
By definition of even and odd, $m = 2r$ and $n = 2s+1$, for some integers r and s.
Then

$$\begin{aligned} m - n &= (2r) - (2s+1) && \text{(by substitution)} \\ &= 2r - 2s - 1 \\ &= 2(r - s - 1) + 1 && \text{(by algebra)} \end{aligned}$$

Let $t = r - s - 1$
Note t is an integer since the difference of integers are integers.
Hence $m - n = 2t + 1$, where t is some integer.
It follows by definition of odd that $m - n$ is odd. **QED**