Lab 05: First-Order Circuits

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Contents

1 Abstract

The course has so far introduced a few analog components such as resistors, LEDs, Operational Amplifiers, microphones, and speakers, however the use of other passive elements such as capacitors and inductors has been somewhat limited to filter out noise from the sound mixer design project. This week's experimental work creates the next leap by introducing the analysis of the capacitor and inductor which lead to first-order linear differential equations. By studying these and their use in circuits, although quite limited for DC applications, the ability to understand the response of these components allows for the further study of circuits involving both capacitive and inductive components.

2 Introduction

First-Order circuits are characterized by a first-order linear differential equation. This may seem a bit overwhelming at first, but a common theme in the experimental work is to build intuition with the analysis of these circuits and simplify the analysis greatly by being able to calculate the Thevenin or Norton equivalent circuit and fitting an algebraic equation to match the behavior of the inductive or capacitive component.

Up until now, all components analyzed have been static components for the most part. Node voltages stay the same once solved and branch currents also stay constant. The inclusion of the inductor and capacitor moves away from this and introduces a *time* based component to the analysis where depending on the value of the inductor or capacitor, a circuits response will vary in *time constants*.

The Node Voltage Method and the Mesh Current Method still apply to these circuits and can always be relied upon in a pinch in case the analyzer becomes befuddled or confused about how the branch response affects the rest of the circuit, however knowing the general form and behavior of these circuits allows for much of the more complicated math to be bypassed completely.

3 Theory

3.1 RC Circuits

A *Resistive Capacitive*, or RC circuit is simply one which contains a resistive element and a capacitive element. One is shown in Figure 1.

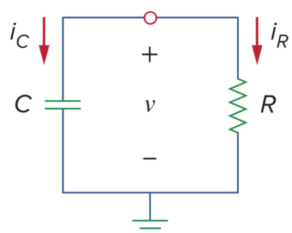


Figure 1: An RC Circuit

A capacitor is said to block DC voltage at steady state, that is, when there is no change or step response from the sources, the capacitor resembles an open circuit. However, when a sudden change or step response from the throwing of a switch or the application of a source, the capacitor contributes to the response of the circuit.

The natural response of the capacitor is such that its voltage (remember that voltage corresponds with a force in the physical world), cannot change suddenly, however the current flow of a capacitor will have discontinuities. Working from the circuit in Figure 1, KCL at the top node produces:

$$i_C + i_R = 0$$

Where $i_C = C\frac{dv}{dt}$ and $i_R = \frac{v}{R}$. Eventually, once the differential equation is solved by separation of variables, the following exponential equation is obtained:

$$v(t) = V_0 e^{-t/RC}$$

Notice that the power which the exponential is raised to depends on two factors. The product of RC and a new variable, t which stands for time. RC is the product of these two values in which resistance is in ohms and capacitance is in Farads. The product of these two values produces a unit of *seconds*.

Therefore, it is commonly notated as

$$\tau = RC$$

The dynamic nature of the circuit dies down after about 5 time constants where the circuit is, for all intensive purposes, considered to be at steady state again and the capacitor again appears like an open circuit at its leads.

When a stepped input is added as alluded to earlier, the equation becomes:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

Where $v(0)$ is the initial voltage felt at the "Open" of the capacitor, and $v(\infty)$ is the final voltage felt at the "Open" of the capacitor. τ is the Thevenin resistance of the final circuit and the value of the capacitor.

3.1.1 The Technique

The best method to analyze RC circuits is again by utilizing some intuition and following a simple procedure to obtain initial, transient, and final responses.

1. Find Thevenin Equivalence at initial values.

2. Find Thevenin Equivalence at final values.

3. Analyze and plug in constants to obtain transient response equations

That's it. No need to muddle about with differential equations or anything of the sort. A similar method exists for RL circuits.

3.2 RL Circuits

A *Resistive Inductive*, or RL circuit is one which contains a resistive element and an inductive element. One such configuration is shown in Figure 2.

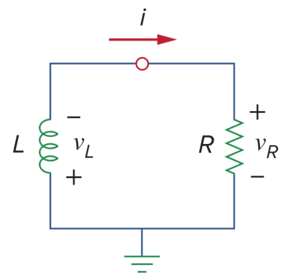


Figure 2: An RL Circuit

An inductor, in contrast to a capacitor, is said to block changes in current. At steady state (i.e., after all transients have died out), an inductor behaves like a short circuit, offering no resistance to current flow. However, during transients—like when a switch is thrown or a source is applied—the inductor resists changes in current, and this gives rise to the circuit's time-dependent behavior.

Recall that voltage across an inductor is defined as:

$$v_L = L\frac{di}{dt}$$

While the voltage across the resistor is:

$$v_R = iR$$

Applying KVL around the loop of Figure 2 gives:

$$v_L + v_R = 0 \Rightarrow L\frac{di}{dt} + iR = 0$$

Solving the differential equation yields an exponential decay of current:

$$i(t) = I_0 e^{-t/RC}$$

But since this is an RL circuit, the time constant is instead:

$$\tau = \frac{L}{R}$$

As before, this τ has units of seconds and represents the characteristic time it takes for the transient to decay significantly. After approximately 5τ, the inductor's effect is negligible, and the circuit is considered to have reached steady state.

When a stepped source is added, the current response becomes:

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

Where $i(0)$ is the initial current through the inductor (which cannot change suddenly), and $i(\infty)$ is the final current the inductor allows at steady-state. $\tau = \frac{L}{R}$, with L in Henries and R in Ohms.

3.2.1 The Technique

Just like the RC case, RL circuits are analyzed by identifying initial, transient, and final behavior:

1. Find the Norton equivalent circuit seen by the inductor initially.
2. Find the Norton equivalent circuit seen by the inductor at final conditions.
3. Analyze and plug in constants to obtain transient response equations.

3.3 The Shape of the Response

Knowing the fact that these two differential equations refer to an exponential equation is quite a strong observation. This allows for only two options graphically depending on the circuit. Either the voltage is building exponentially or decreasing exponentially in the same shape always—or the current is.

This exponential shape is always the same: it either rises from an initial value toward a final value asymptotically, or it decays from some initial value down toward zero. The only difference lies in what the initial and final values are, and whether it's voltage or current we're examining. Regardless, the time constant τ controls how "fast" or "slow" this curve progresses.

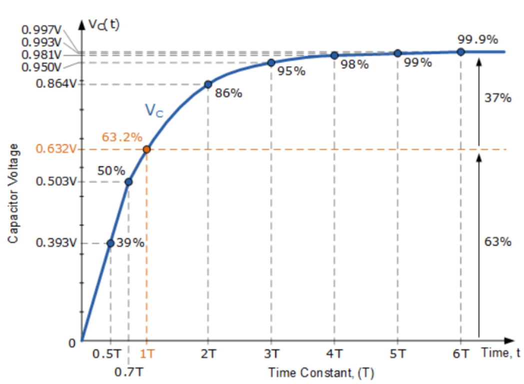
For a rising exponential (build-up), such as when a capacitor charges or current ramps up in an inductor:

$$y(t) = y(\infty)\left(1 - e^{-t/\tau}\right)$$

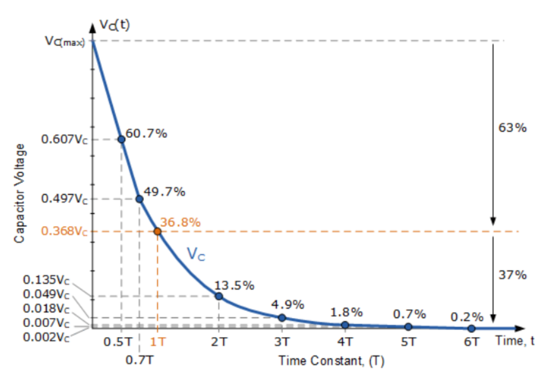
For a falling exponential (decay), such as when a capacitor discharges or an inductor's current dies out:

$$y(t) = y(0)e^{-t/\tau}$$

These expressions describe the same curve flipped around—either rising or falling depending on the physical setup. Figures 3(a) and 3(b) show both cases.



((a)) Rising Exponential Response



((b)) Falling Exponential Response

Figure 3: Exponential response shapes in first-order circuits.

After one time constant τ, the curve has reached about 63% of the way to its final value. After about 5τ, it's considered close enough to steady state that the transient is effectively over.

This universal shape simplifies analysis and allows for more of an intuition of how the circuit will behave.

4 Experimental Procedures

4.1 Circuit One: Charging (Growth)

The first experimental setup focused on observing the charging behavior of a capacitor in a high-impedance RC circuit. With no initial voltage across the capacitor and a DC supply of 10 V applied at $t = 0$, the capacitor begins charging through a 10 MΩ resistor.

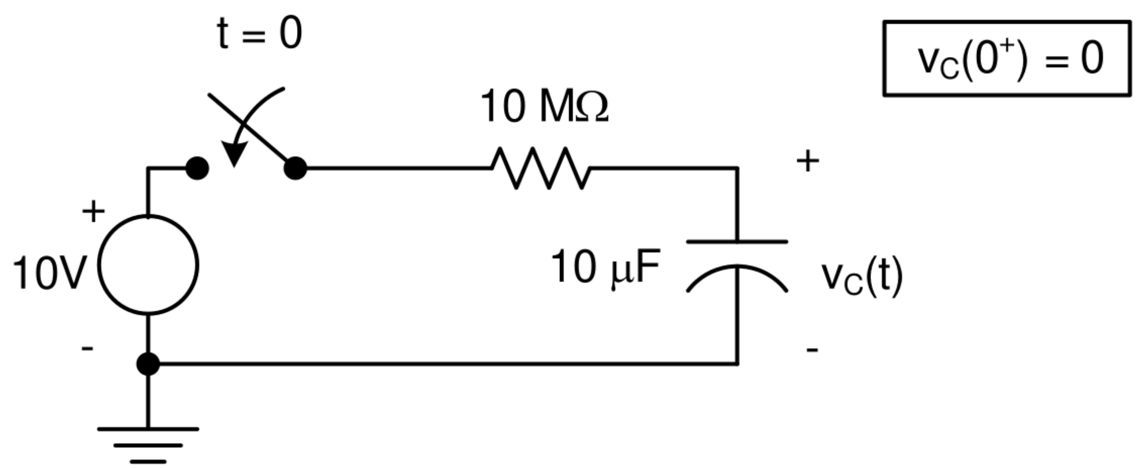


Figure 4: RC Circuit Undergoing Charging (Virgin Configuration)

The component values were:

$$V_0 = 0\,\text{V}, \quad R = 10\,\text{M}\Omega, \quad C = 10\,\mu\text{F}, \quad V_f = 10\,\text{V}$$

$$\tau = RC = 10\,\text{M}\Omega \cdot 10\,\mu\text{F} = 100\,\text{s}$$

This relatively large time constant results in a slow rise in voltage across the capacitor. The expected transient response is given by:

$$v_C(t) = 10 - 10e^{-t/100}$$

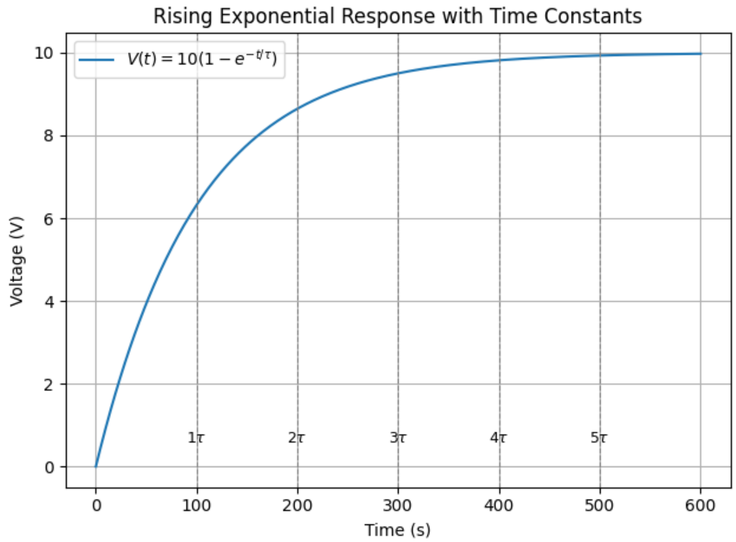


Figure 5: Expected Charging Curve for Circuit One

4.2 Circuit One: Discharging (Decay)

The second configuration reversed the behavior: starting with the capacitor fully charged to 10 V and then disconnected from the source, the voltage was allowed to decay through the same 10 MΩ resistor.

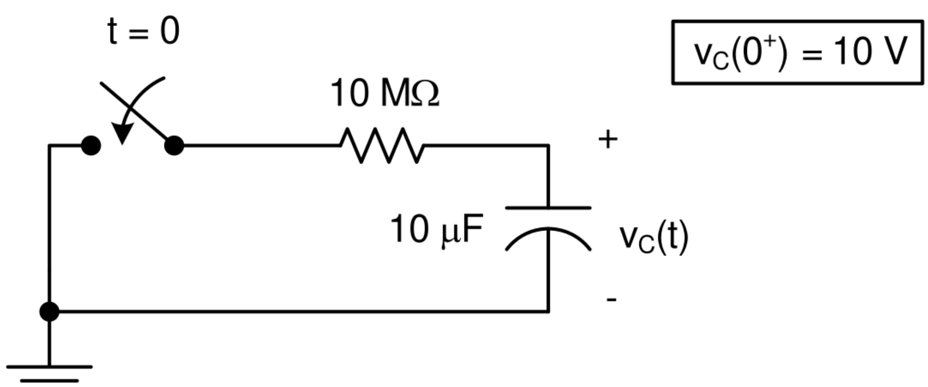


Figure 6: RC Circuit Undergoing Discharge (Excited Configuration)

The parameters remained the same:

$$V_0 = 10\,\text{V}, \quad R = 10\,\text{M}\Omega, \quad C = 10\,\mu\text{F}, \quad V_f = 0\,\text{V}$$

$$\tau = RC = 10\,\text{M}\Omega \cdot 10\,\mu\text{F} = 100\,\text{s}$$

The expected voltage decay follows the form:

$$v_C(t) = 10\,e^{-t/100}$$

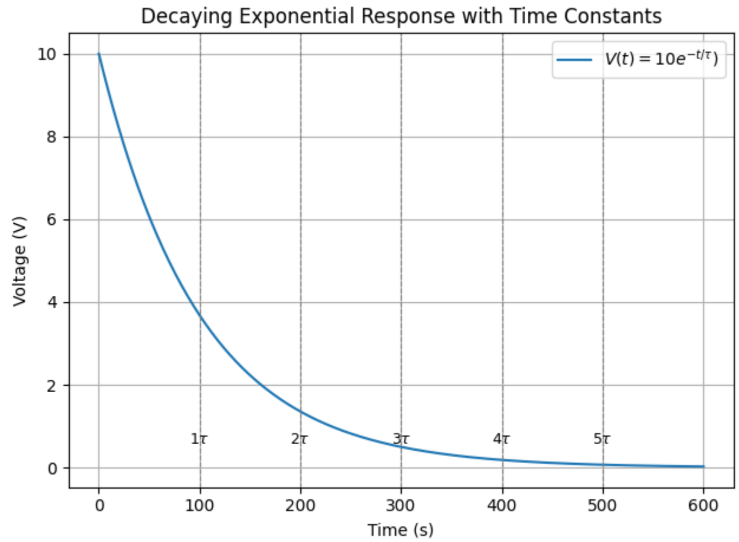


Figure 7: Expected Discharging Curve for Circuit One

4.3 Circuit Three: RC with Square Wave Input

Circuit Three introduced a periodic square wave input instead of a static DC source. The capacitor alternately charges and discharges in response to the high and low states of the wave, creating a periodic exponential waveform across its terminals.

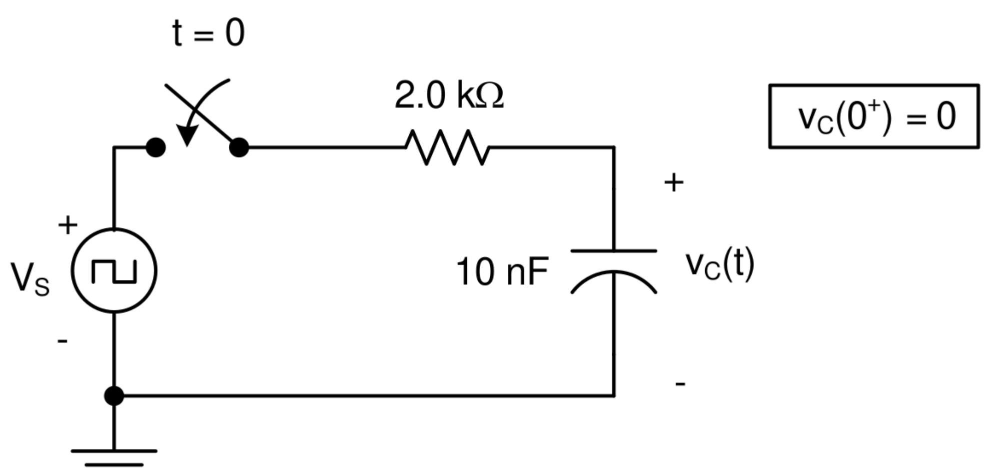


Figure 8: RC Circuit with Square Wave Input

$$R = 2.0\,\text{k}\Omega, \quad C = 10\,\text{nF}, \quad V_c(0^+) = 0\,\text{V}$$

$$\tau = RC = 2.0\,\text{k}\Omega \cdot 10\,\text{nF} = 20\,\mu\text{s}$$

A square wave of 5 V amplitude was applied, and the period was set to:

$$T = 10\tau = 200\,\mu\text{s}$$

The resulting exponential voltage response for the capacitor is:

$$v_C(t) = 5\left(1 - e^{-t/20\mu s}\right) \quad \text{(charging)}$$

$$v_C(t) = 5\,e^{-t/20\mu s} \quad \text{(discharging)}$$

4.4 Circuit Four: RC Circuit with Adjusted Components

This circuit is topologically identical to Circuit Three but with different R and C values, leading to a different time constant while retaining the same input waveform structure.

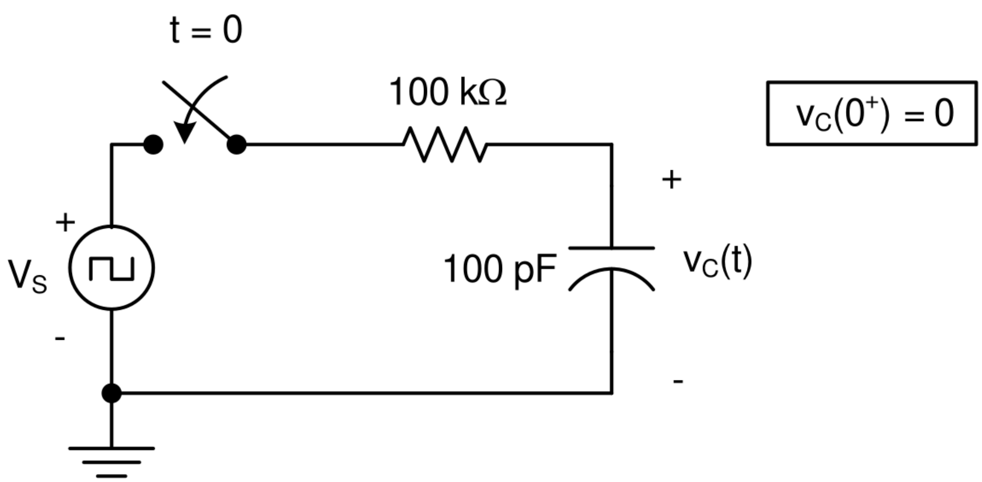


Figure 9: RC Circuit with Smaller Time Constant

$$R = 100\,\text{k}\Omega, \quad C = 100\,\text{pF}, \quad V_c(0^+) = 0\,\text{V}$$

$$\tau = RC = 100\,\text{k}\Omega \cdot 100\,\text{pF} = 10\,\mu\text{s}$$

The square wave had an amplitude of 5 V and a calculated period of:

$$T = 10\tau = 100\,\mu\text{s}$$

Voltage response equations:

$$v_C(t) = 5\left(1 - e^{-t/10\mu s}\right) \quad \text{(charging)}$$

$$v_C(t) = 5\,e^{-t/10\mu s} \quad \text{(discharging)}$$

4.5 Circuit Five: RL Response to Square Wave

Circuit Five replaces the capacitor with an inductor and analyzes the current behavior in response to a square wave input. Due to the high voltages generated at switching transitions, the amplitude was reduced to 1 V for safety and clarity.

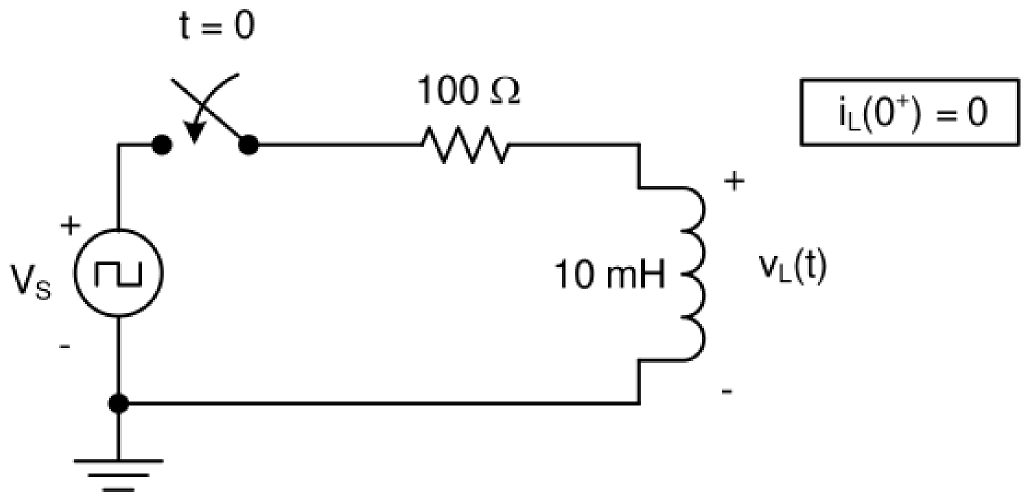


Figure 10: RL Circuit with Square Wave Input

$$L = 10\,\text{mH}, \quad R = 100\,\Omega, \quad I_L(0^+) = 0\,\text{A}$$
$$\tau = \frac{L}{R} = \frac{10\,\text{mH}}{100\,\Omega} = 100\,\mu\text{s}$$

The input waveform was a 1 V square wave with a period of:

$$T = 10\tau = 1.0\,\text{ms}$$

The current response is modeled by:

$$i_L(t) = \frac{V}{R}\left(1 - e^{-t/\tau}\right) = \frac{1}{100}\left(1 - e^{-t/100\mu s}\right) = 10\,\text{mA}\left(1 - e^{-t/100\mu s}\right) \quad \text{(ramp-up)}$$
$$i_L(t) = \frac{V}{R}e^{-t/\tau} = 10\,\text{mA} \cdot e^{-t/100\mu s} \quad \text{(decay)}$$

The inductor resists instantaneous current changes, leading to characteristic spikes in voltage at the moment the square wave toggles state.

5 Results and Discussion

5.1 Circuit One: Charging Transient

For the first circuit, a large time constant was calculated. This circuit (Figure 4) will take a long time to charge to final voltage, however it also serves as a reminder that measurement tools such as voltmeters contain their own resistance which will act as loads on any given circuit. In order to obtain clean data which represents the behavior of the capacitor, only momentary touches of the leads to the capacitor were allowed at the intervals shown in Table 1.

Time (s)	Voltage (V)
0	0
20	1.6
30	2.63
60	4.00
90	5.09
120	5.47
150	5.85
180	6.15
210	6.46
240	6.70
270	6.90
300	7.10
330	7.27
360	7.40
390	7.52

Table 1: Measured Data for Charging Capacitor

The behavior of the voltage charging event was exactly as expected well within tolerances. Due to the inherent resistance of the measurement tool, the final capacitor voltage was a little bit lower, however for experimental purposes, this still confirmed the time constant and behavioral characteristics.

Without the multimeter attached to the terminals, a voltage of 10 V equal to the supply voltage provided by the dedicated power supply was correctly identified later, after allowing the multimeter to remain disconnected during the charge period.

5.1.1 Time Constant Analysis.

From the measured data, the voltage rose toward a final value of approximately 7.52 V. The theoretical final value was 10 V, but the presence of the voltmeter's input resistance resulted in a reduced effective final voltage.

To determine the experimental time constant, we note that at $t = \tau$, the voltage should reach approximately 63% of its final value:

$$V_f \approx 7.52\,\mathrm{V}, \quad V_{63\%} = 0.63 \cdot 7.52 \approx 4.74\,\mathrm{V}$$

From the data:

$$V(60\,\mathrm{s}) = 4.00\,\mathrm{V}, \quad V(90\,\mathrm{s}) = 5.09\,\mathrm{V}$$

Using linear interpolation:

$$t_{63\%} \approx 60 + \frac{4.74 - 4.00}{5.09 - 4.00} \cdot (90 - 60) \approx 80.4\,\mathrm{s}$$

This gives an experimental time constant:

$$\tau_{\mathrm{exp}} \approx 80.4\,\mathrm{s}$$

Compared to the theoretical value:

$$\tau_{\mathrm{theory}} = RC = (10 \times 10^6)(0.01) = 100\,\mathrm{s}$$

The discrepancy is attributed to the voltmeter briefly loading the circuit during measurement. While the meter was not continuously connected, its momentary presence still impacted the effective resistance slightly.

5.1.2 Voltmeter Resistance Estimate

During the charging phase, the capacitor was expected to charge to the full supply voltage of 10 V; however, the final recorded voltage was only 7.52 V even after a long period. This drop can be attributed to the voltmeter acting as a parallel load with the 10 MΩ resistor, lowering the effective resistance and thus the final voltage seen at the capacitor.

Using the voltage divider relationship for the parallel case, the final voltage across the capacitor becomes:

$$V_{\mathrm{final}} = V_s \cdot \frac{R_m}{R + R_m}$$

Solving for R_m:

$$\begin{aligned}
\frac{R_m}{R + R_m} &= \frac{7.52}{10} = 0.752 \\
&\Rightarrow 0.752(R + R_m) = R_m \\
&\Rightarrow 0.752R + 0.752R_m = R_m \\
&\Rightarrow 0.752R = R_m - 0.752R_m \\
&\Rightarrow 0.752R = R_m(1 - 0.752) \\
&\Rightarrow R_m = \frac{0.752}{0.248}R \approx 3.032R
\end{aligned}$$

$$R_m \approx 3.032 \cdot 10\,\mathrm{M\Omega} = \boxed{30.3\,\mathrm{M\Omega}}$$

Therefore, the estimated input resistance of the voltmeter is approximately 30.3 MΩ. This value is consistent with typical digital multimeters configured for voltage measurements, and confirms the subtle but measurable influence of the instrument on the experiment.

5.2 Circuit One: Discharging

This section involved the same circuit, but in reverse. Starting with a charged capacitor, the voltage transient was recorded on the way down as it discharged to ground. In this case, it was vital to ground the input of the circuit to ensure an even response from the capacitor voltage value as expected.

Time (s)	Voltage (V)
0	8.36
30	8.10
60	7.89
90	7.51
120	6.19
150	6.06
180	5.96
210	5.85
240	5.61
270	5.32
300	4.20
330	3.18
360	2.45
390	1.90

Table 2: Measured Data for Discharging Capacitor

Again, utilizing the same experimental setup as in the previous section, the initial voltage was slightly lower due to the inherent resistance of the multimeter in parallel with the circuit. Educationally, this still acted as a resistive load and slightly changed the maximum voltage value.

The shape of the discharge curve still follows the expected exponential form with an initial voltage of approximately 8.36 V.

5.2.1 Time Constant Analysis.

For the discharging capacitor, the voltage should fall to 37% of its initial value at $t = \tau$:

$$V_0 \approx 8.36\,\mathrm{V}, \quad V_{37\%} = 0.37 \cdot 8.36 \approx 3.09\,\mathrm{V}$$

From the data:

$$V(330\,\mathrm{s}) = 3.18\,\mathrm{V}, \quad V(360\,\mathrm{s}) = 2.45\,\mathrm{V}$$

Interpolating:

$$t_{63\%} \approx 330 + \frac{3.09 - 3.18}{2.45 - 3.18} \cdot (360 - 330) \approx 333.7\,\mathrm{s}$$

Therefore:

$$\tau_{\mathrm{exp}} \approx 333.7\,\mathrm{s}$$

This is significantly higher than the theoretical 100 s value. The reason is that the voltmeter remained connected for the entire duration of the discharge, effectively increasing the circuit's Thevenin resistance and extending the time constant.

The result is consistent with expectations when the influence of the meter is taken into account. It highlights the importance of considering instrument loading effects in high-impedance RC circuits.

5.3 Circuit Three: Response to a Square Wave

Utilizing the time constant that was calculated for this circuit of $20\mu s$ and increasing it by a factor of 10, the input frequency of $5KHz$ was obtained, which allowed the capacitor to charge up periodically to a value close to its max voltage. Recall that for all practical purposes, a capacitor becomes fully charged after 5 time constants. Refer to Figure 11 for a graphical representation of the effect from Scopy. The orange signal is the input from the signal generator while the purple is the signal off the top leg of the capacitor.

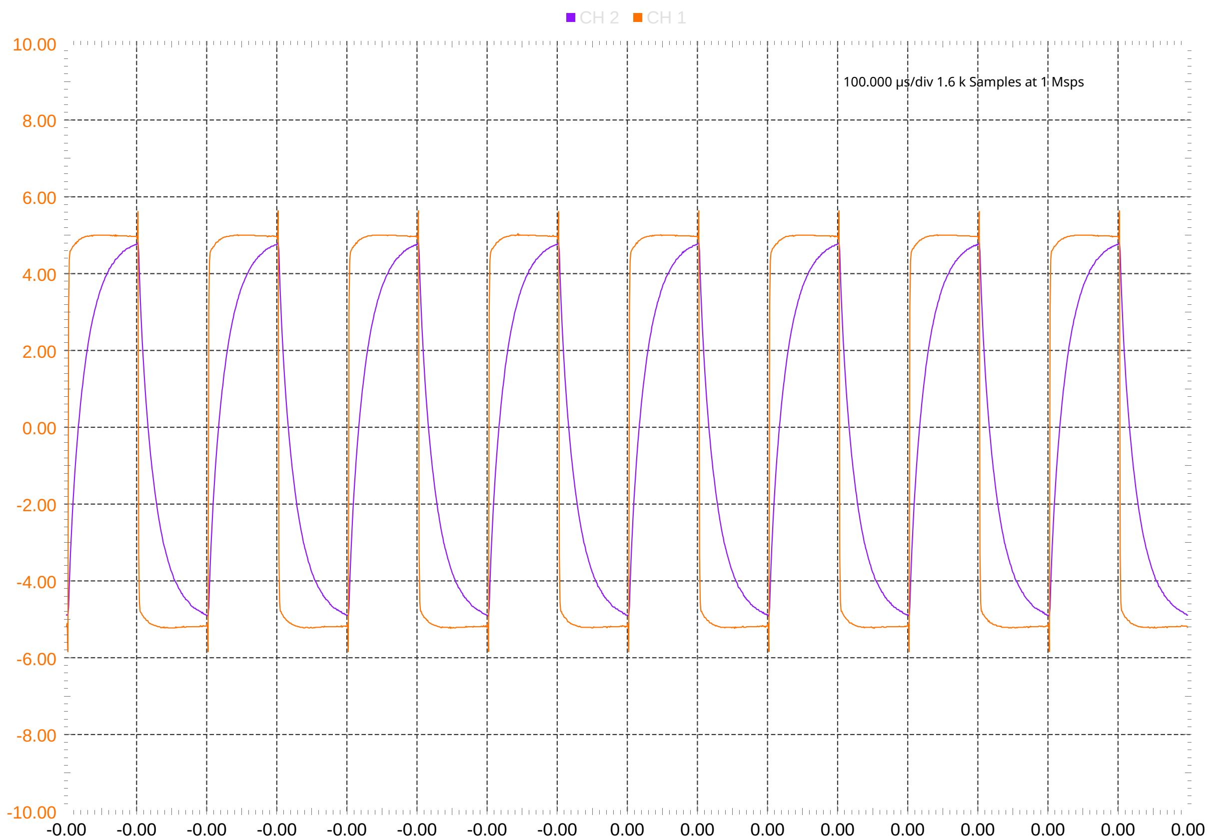


Figure 11: Oscilloscope display for Circuit Three

Notice how the purple waveform gets nearly to the top level of the input waveform. This is the expected outcome.

Comparing this to the SPICE analysis in Figure 12,

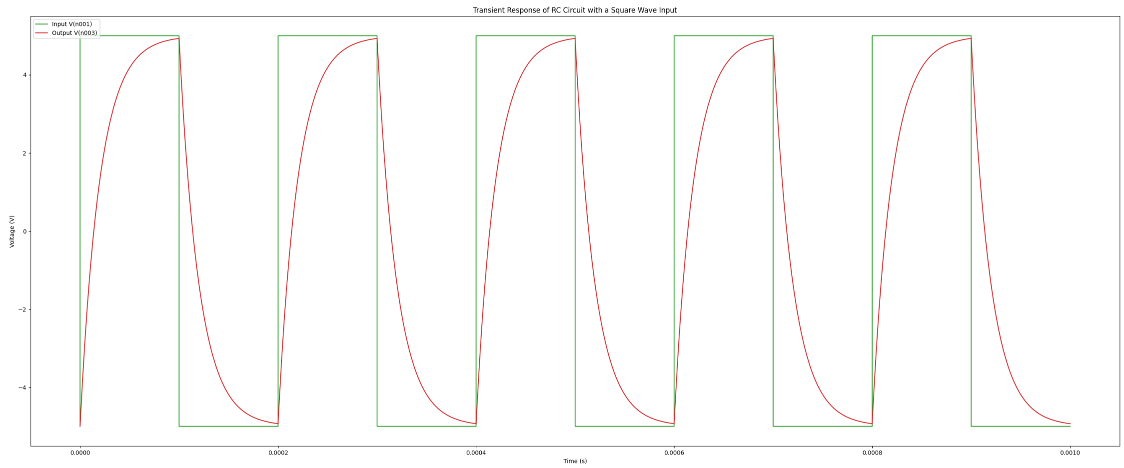


Figure 12: LTSpice Simulated Values

5.3.1 Analysis

To determine the experimental time constant, we observe the waveform in Figure 11. The purple trace, representing the voltage across the capacitor, follows an exponential charging and discharging curve in response to the square wave input.

The signal generator was set to a square wave of frequency $f = 5\,\text{kHz}$, corresponding to a period:

$$T = \frac{1}{f} = \frac{1}{5\,000} = 200\,\mu\text{s}$$

Since the square wave toggles every half period, each high or low duration is:

$$T_{\text{high}} = T_{\text{low}} = \frac{T}{2} = 100\,\mu\text{s}$$

Our calculated time constant was $\tau = 20\,\mu\text{s}$, meaning:

$$\frac{T_{\text{high}}}{\tau} = \frac{100\,\mu\text{s}}{20\,\mu\text{s}} = 5$$

This matches the rule of thumb that a capacitor reaches over 99% of its final value after 5 time constants. The purple waveform nearly reaches the full amplitude of the orange square wave before discharging again, consistent with this theoretical behavior.

Comparing this to the LTSpice simulation in Figure 12, we observe excellent agreement. Both the simulated and measured waveforms exhibit the same exponential rise and fall within the $100\mu s$ half-period.

Therefore, the experimental time constant agrees well with both the calculated value and the simulated output, with only minimal deviation due to real-world component tolerances and scope sampling resolution.

5.4 Circuit Four: Square Wave Response

Circuit four had a frequency calculated for its input square wave of 10 kHz due to the change in the sizes of both the resistive and capacitive portions of the circuit. This higher frequency results in a shorter available charge/discharge window per cycle.

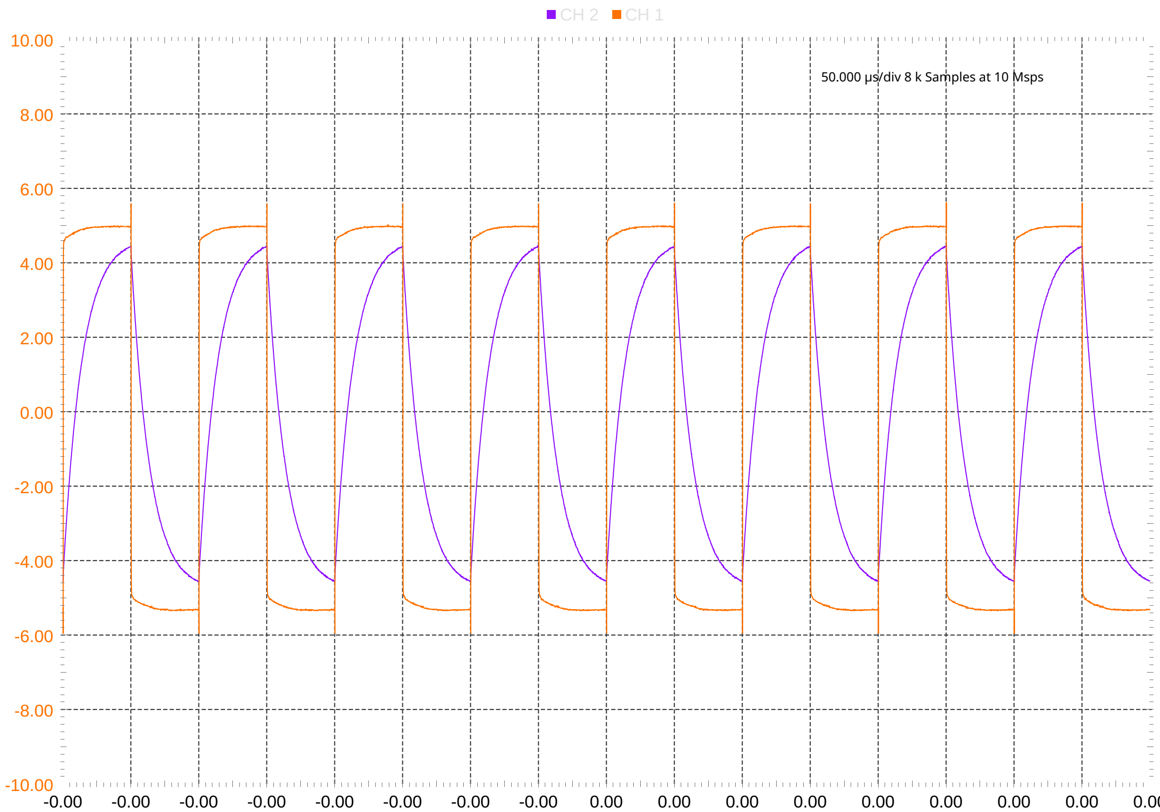


Figure 13: Oscilloscope display for Circuit Four

5.4.1 Analysis

The signal generator was set to a 10 kHz square wave, giving a full period of:

$$T = \frac{1}{f} = \frac{1}{10\,000} = 100\,\mu\text{s}$$

With the square wave toggling every half-cycle:

$$T_{\text{high}} = T_{\text{low}} = 50\,\mu\text{s}$$

The time scale in Figure 13 confirms that the capacitor does not fully charge or discharge before the square wave switches direction, which is expected. The waveform (purple) rises and falls exponentially, but does not reach the full $5V$ input level due to the shorter available time. This matches the calculated design for a smaller time constant relative to Circuit Three.

From the image, the capacitor reaches approximately 63% of its full excursion within one-fifth of the half-period, suggesting:

$$\tau_{\text{exp}} \approx \frac{50\,\mu\text{s}}{5} = 10\,\mu\text{s}$$

This agrees with the calculated value used to determine the input frequency. The exponential curves and timing visually confirm this result.

5.4.2 SPICE Simulation

Comparing this to the LTSpice simulation in Figure 14, the agreement is excellent. The red curve (capacitor voltage) in simulation matches the real purple waveform in both shape and amplitude behavior, undershooting the full square wave level due to insufficient time per cycle to reach steady-state.

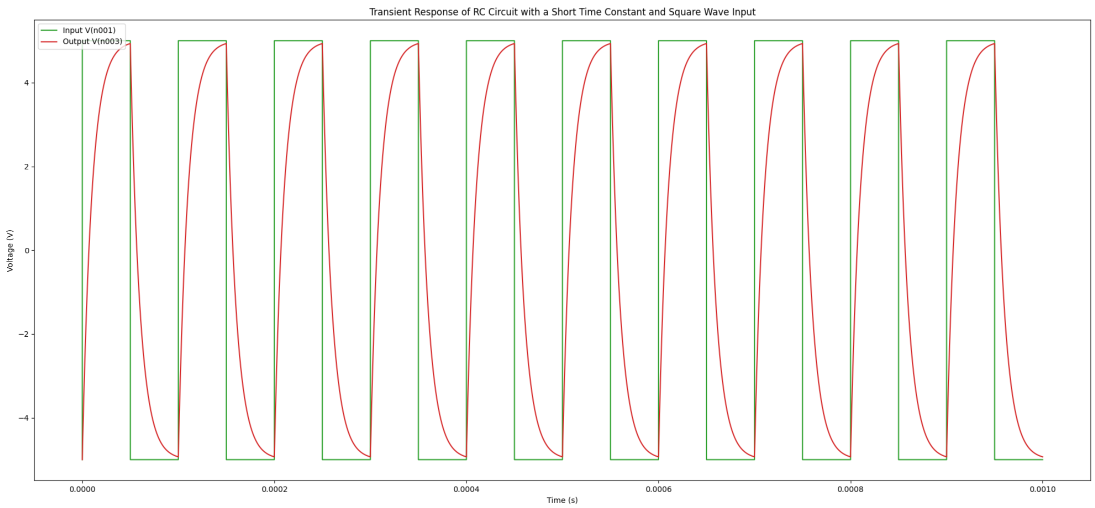


Figure 14: LTSpice Simulated Values

The experimental data strongly agrees with both the theoretical time constant and the LTSpice simulation. The waveform exhibits behavior exactly predicted by the time constant formula, and the result confirms the exponential nature of the charging and discharging phases even in rapid time-domain switching scenarios.

5.5 Circuit Five: RL Circuit Response

Utilizing a smaller amplitude of 1 V and a frequency of 1 kHz, the following waveform was output from the implemented RL circuit. This circuit replaces the capacitive element with an inductive one, introducing a fundamentally different dynamic governed by the inductor's opposition to changes in current.

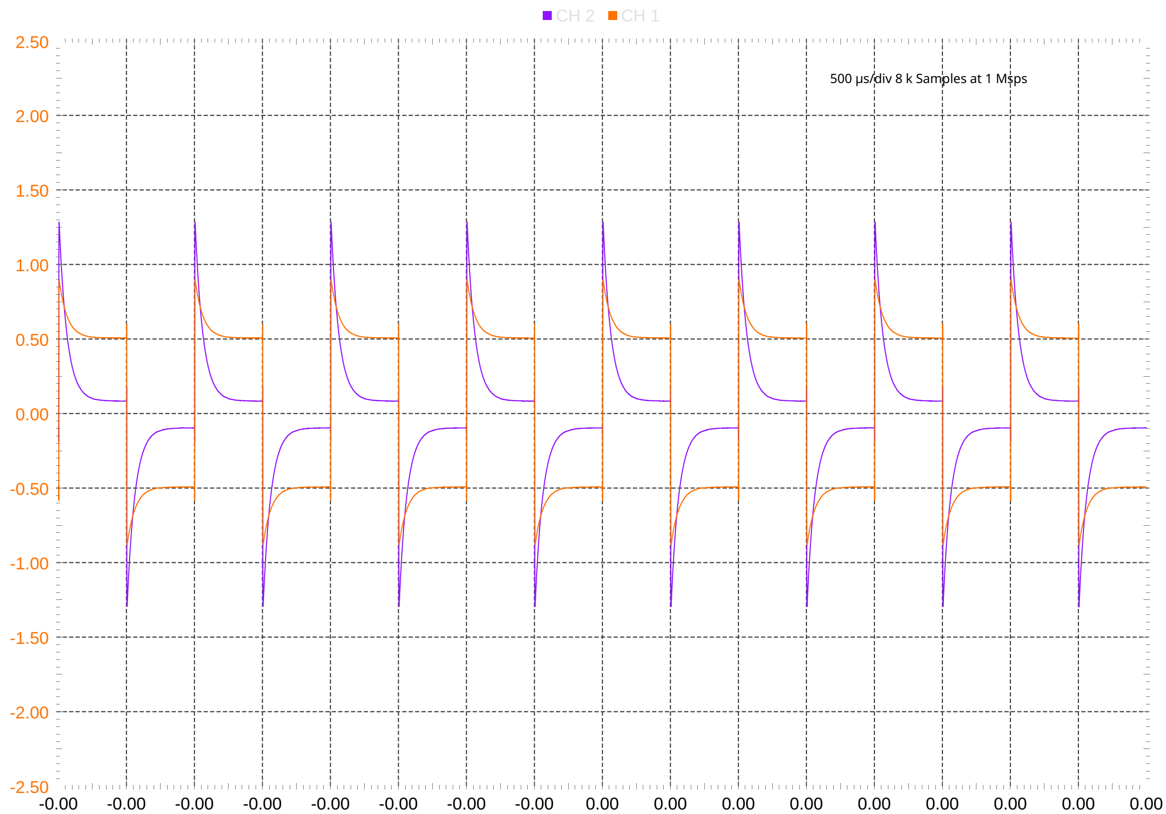


Figure 15: Oscilloscope display for Circuit Five

5.5.1 Analysis

In contrast to RC circuits, the voltage waveform here displays characteristic spikes at each edge of the square wave. These voltage discontinuities occur because an inductor resists sudden changes in current, not voltage. At the exact moment of a step input (rising or falling), the current through the inductor cannot jump instantaneously, so the voltage must spike to force the change in current.

The waveform captured in Figure 15 shows these expected traits clearly. The orange trace is the square wave input (channel 1), while the purple trace (channel 2) shows the voltage across the inductor. As the square wave toggles, the inductor generates a back-emf that causes a voltage spike opposite the direction of the step, then decays exponentially as the current ramps up or down.

Given the input frequency of 1 kHz, the period is:

$$T = \frac{1}{f} = 1\,\text{ms} \quad \Rightarrow \quad T_{\text{high}} = T_{\text{low}} = 0.5\,\text{ms}$$

The observed time it takes for the purple trace to decay to near zero aligns with an exponential curve over one or two time constants. This supports the theoretical result that:

$$\tau = \frac{L}{R}$$

where τ is small enough that the system settles during each half-period.

5.5.2 SPICE Simulation

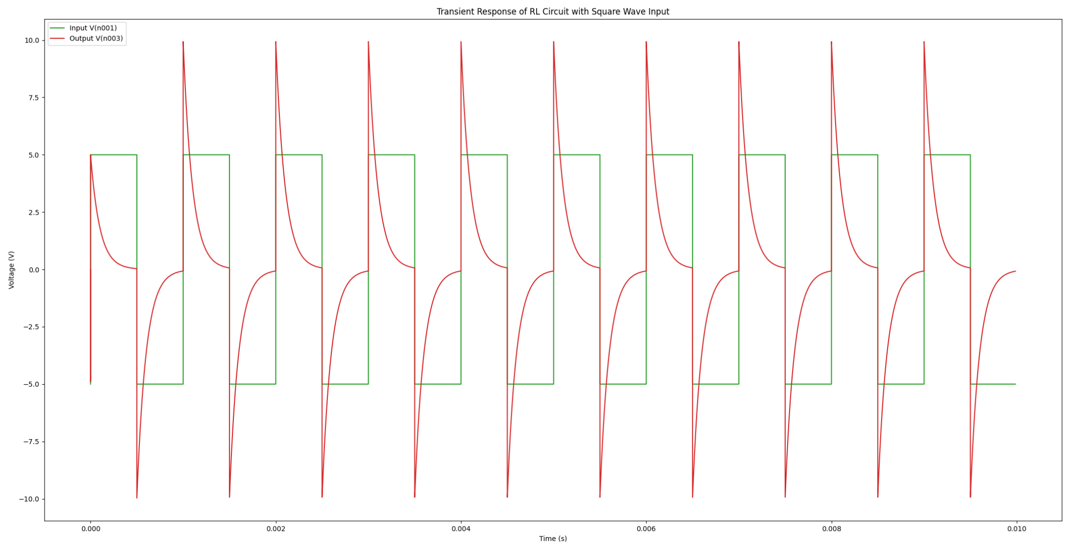


Figure 16: SPICE Simulated Values

Figure 16 shows the simulated waveform for the same RL configuration. Here, the red trace (inductor voltage) spikes at each transition and then decays exponentially, matching both the measured waveform and theoretical expectations. The simulation shows these transitions more cleanly because it doesn't account for practical limitations like inductor non-idealities or noise.

The experimental RL waveform exhibits behavior expected of an inductor-dominant circuit. The voltage across the inductor spikes when the square wave toggles and then decays, in accordance with $\tau = \frac{L}{R}$. Both the oscilloscope data and the LTSpice simulation agree, confirming the transient behavior of inductors under periodic forcing functions.

6 Conclusion

This lab served to introduce and explore the concept of first-order circuits—those containing either capacitors or inductors alongside resistors—and demonstrated their distinctive exponential behaviors through both long and short time constant scenarios.

From the slow voltage ramp in the high-impedance RC circuit to the rapid switching transients in the square wave experiments, the exponential form governed each response consistently. The experimental work reinforced the theory: capacitors resist changes in voltage and settle over time as they charge or discharge, while inductors oppose changes in current, generating voltage spikes as they adjust to new current levels.

SPICE simulations matched the measured results across all configurations, confirming both the validity of the exponential time constant model and the effect of real-world tolerances. Discrepancies in the RC measurements—such as reduced final voltages—highlighted the importance of accounting for the input resistance of test equipment when working with high-impedance circuits.

Ultimately, by understanding the role of τ in shaping transient behavior, the analysis of first-order circuits becomes intuitive and efficient—requiring only a few well-chosen constants and a solid grasp of exponential dynamics to fully characterize a circuit's response over time.