JuMP

The JuMP core developers and contributors

May 9, 2022

Contents

Part I

Introduction

Chapter 1

Introduction

Welcome to the documentation for JuMP!

1.1 What is JuMP?

JuMP is a domain-specific modeling language for mathematical optimization embedded in Julia. It currently supports a number of open-source and commercial solvers for a variety of problem classes, including linear, mixed-integer, second-order conic, semidefinite, and nonlinear programming.

> **Tip**
>
> If you aren't sure if you should use JuMP, read Should I use JuMP?.

1.2 Resources for getting started

There are few ways to get started with JuMP:

- Read the Installation Guide.
- Read the introductory tutorials Getting started with Julia and Getting started with JuMP.
- Browse some of our modeling tutorials, including classics such as The diet problem, or the Maximum likelihood estimation problem using nonlinear programming.

> **Tip**
>
> Need help? Join the community forum to search for answers to commonly asked questions.
>
> Before asking a question, make sure to read the post make it easier to help you, which contains a number of tips on how to ask a good question.

1.3 How the documentation is structured

Having a high-level overview of how this documentation is structured will help you know where to look for certain things.

- **Tutorials** contain worked examples of solving problems with JuMP. Start here if you are new to JuMP, or you have a particular problem class you want to model.

- The **Manual** contains short code-snippets that explain how to achieve specific tasks in JuMP. Look here if you want to know how to achieve a particular task, such as how to Delete a variable or how to Modify an objective coefficient.
- The **API Reference** contains a complete list of the functions you can use in JuMP. Look here if you want to know how to use a particular function.
- The **Background information** section contains background reading material to provide context to JuMP. Look here if you want an understanding of what JuMP is and why we created it, rather than how to use it.
- The **Developer docs** section contains information for people contributing to JuMP development or writing JuMP extensions. Don't worry about this section if you are using JuMP to formulate and solve problems as a user.
- The **MathOptInterface** section is a self-contained copy of the documentation for MathOptInterface. Look here for functions and constants beginning with `MOI.`, as well as for general information on how MathOptInterface works.

1.4 Citing JuMP

If you find JuMP useful in your work, we kindly request that you cite the following paper (pdf):

```
@article{DunningHuchetteLubin2017,
author = {Iain Dunning and Joey Huchette and Miles Lubin},
title = {JuMP: A Modeling Language for Mathematical Optimization},
journal = {SIAM Review},
volume = {59},
number = {2},
pages = {295-320},
year = {2017},
doi = {10.1137/15M1020575},
}
```

For an earlier work where we presented a prototype implementation of JuMP, see here:

```
@article{LubinDunningIJOC,
author = {Miles Lubin and Iain Dunning},
title = {Computing in Operations Research Using Julia},
journal = {INFORMS Journal on Computing},
volume = {27},
number = {2},
pages = {238-248},
year = {2015},
doi = {10.1287/ijoc.2014.0623},
}
```

A preprint of this paper is freely available.

1.5 NumFOCUS

JuMP is a Sponsored Project of NumFOCUS, a 501(c)(3) nonprofit charity in the United States. NumFOCUS provides JuMP with fiscal, legal, and administrative support to help ensure the health and sustainability of the project. Visit numfocus.org for more information.

You can support JuMP by donating.

Figure 1.1: NumFOCUS logo

Donations to JuMP are managed by NumFOCUS. For donors in the United States, your gift is tax-deductible to the extent provided by law. As with any donation, you should consult with your tax adviser about your particular tax situation.

JuMP's largest expense is the annual JuMP-dev workshop. Donations will help us provide travel support for JuMP-dev attendees and take advantage of other opportunities that arise to support JuMP development.

Chapter 2

Should I use JuMP?

JuMP is an algebraic modeling language for mathematical optimization written in the Julia language.

This page explains when you should consider using JuMP, and importantly, when you should not use JuMP.

2.1 When should I use JuMP?

You should use JuMP if you have a constrained optimization problem for which you can formulate:

- a set of decision variables
- a scalar objective function
- a set of constraints.

Key reasons to use JuMP include:

- User friendliness
  - JuMP has syntax that mimics natural mathematical expressions. (See the section on algebraic modeling languages.)
- Speed
  - Benchmarking has shown that JuMP can create problems at similar speeds to special-purpose modeling languages such as AMPL.
  - JuMP communicates with most solvers in memory, avoiding the need to write intermediary files.
- Solver independence
  - JuMP uses a generic solver-independent interface provided by the MathOptInterface package, making it easy to change between a number of open-source and commercial optimization software packages ("solvers"). The Supported solvers section contains a table of the currently supported solvers.
- Access to advanced algorithmic techniques
  - JuMP supports efficient in-memory re-solves of linear programs, which previously required using solver-specific or low-level C++ libraries.

  - JuMP provides access to solver-independent and solver-dependent Callbacks.

- Ease of embedding

  - JuMP itself is written purely in Julia. Solvers are the only binary dependencies.
  - Automated install of many solver dependencies.
    * JuMP provides automatic installation of many open-source solvers. This is different to modeling languages in Python which require you to download and install a solver yourself.
  - Being embedded in a general-purpose programming language makes it easy to solve optimization problems as part of a larger workflow (e.g., inside a simulation, behind a web server, or as a subproblem in a decomposition algorithm).
    * As a trade-off, JuMP's syntax is constrained by the syntax available in Julia.
  - JuMP is MPL licensed, meaning that it can be embedded in commercial software that complies with the terms of the license.

2.2 When should I not use JuMP?

JuMP supports a broad range of optimization classes. However, there are still some that it doesn't support, or that are better supported by other software packages.

You want to optimize a complicated Julia function

Packages in Julia compose well. It's common for people to pick two unrelated packages and use them in conjunction to create novel behavior. JuMP isn't one of those packages.

If you want to optimize an ordinary differential equation from DifferentialEquations.jl or tune a neural network from Flux.jl, consider using other packages such as:

- Optim.jl
- GalacticOptim.jl
- Nonconvex.jl

Black-box, derivative free, or unconstrained optimization

JuMP does support nonlinear programs with constraints and objectives containing user-defined functions. However, the functions must be automatically differentiable, or need to provide explicit derivatives. (See User-defined Functions for more information.)

If your function is a black-box that is non-differentiable (e.g., the output of a simulation written in C++), JuMP is not the right tool for the job. This also applies if you want to use a derivative free method.

Even if your problem is differentiable, if it is unconstrained there is limited benefit (and downsides in the form of more overhead) to using JuMP over tools which are only concerned with function minimization.

Alternatives to consider are:

- Optim.jl
- GalacticOptim.jl
- NLopt.jl

Optimal control problems

JuMP supports formulating optimal control problems as large nonlinear programs (see, for example, Optimal control for a Space Shuttle reentry trajectory). However, the nonlinear interface has a number of limitations (for example, the need to write out the dynamics in algebraic form) that mean JuMP might not be the right tool for the job.

Alternatives to consider are:

- CasADi, CasADi.jl
- InfiniteOpt.jl
- pyomo.DAE

Multiobjective programs

If your problem has more than one objective, JuMP is not the right tool for the job. However, we're working on fixing this!.

Alternatives to consider are:

- vOptGeneric.jl

Disciplined convex programming

JuMP does not support disciplined convex programming (DCP).

Alternatives to consider are:

- Convex.jl

  Note

  `Convex.jl` is also built on MathOptInterface, and shares the same set of underlying solvers. However, you input problems differently, and Convex.jl checks that the problem is DCP.

Stochastic programming

JuMP requires deterministic input data.

If you have stochastic input data, consider using a JuMP extension such as:

- InfiniteOpt.jl
- StochasticPrograms.jl
- SDDP.jl

Chapter 3

Installation Guide

This guide explains how to install Julia and JuMP. If you have installation troubles, read the Common installation issues section below.

3.1 Install Julia

JuMP is a package for Julia. To use JuMP, first download and install Julia.

> **Tip**
>
> If you are new to Julia, read our Getting started with Julia tutorial.

Which version should I pick?

You can install the "Current stable release" or the "Long-term support (LTS) release".

- The "Current stable release" is the latest release of Julia. It has access to newer features, and is likely faster.
- The "Long-term support release" is an older version of Julia that has continued to receive bug and security fixes. However, it may not have the latest features or performance improvements.

For most users, you should install the "Current stable release", and whenever Julia releases a new version of the current stable release, you should update your version of Julia. Note that any code you write on one version of the current stable release will continue to work on all subsequent releases.

For users in restricted software environments (e.g., your enterprise IT controls what software you can install), you may be better off installing the long-term support release because you will not have to update Julia as frequently.

3.2 Install JuMP

From Julia, JuMP is installed using the built-in package manager:

```julia
import Pkg
Pkg.add("JuMP")
```

Tip

We recommend you create a Pkg environment for each project you use JuMP for, instead of adding lots of packages to the global environment. The Pkg manager documentation has more information on this topic.

When we release a new version of JuMP, you can update with:

```
import Pkg
Pkg.update("JuMP")
```

3.3 Install a solver

JuMP depends on solvers to solve optimization problems. Therefore, you will need to install one before you can solve problems with JuMP.

Install a solver using the Julia package manager, replacing `"Clp"` by the Julia package name as appropriate.

```
import Pkg
Pkg.add("Clp")
```

Once installed, you can use Clp as a solver with JuMP as follows, using `set_optimizer_attributes` to set solver-specific options:

```
using JuMP
using Clp
model = Model(Clp.Optimizer)
set_optimizer_attributes(model, "LogLevel" => 1, "PrimalTolerance" => 1e-7)
```

Note

Most packages follow the `ModuleName.Optimizer` naming convention, but exceptions may exist. See the README of the Julia package's GitHub repository for more details on how to use a particular solver, including any solver-specific options.

3.4 Supported solvers

Most solvers are not written in Julia, and some require commercial licenses to use, so installation is often more complex.

- If a solver has `Manual` in the `Installation` column, the solver requires a manual installation step, such as downloading and installing a binary, or obtaining a commercial license. Consult the README of the relevant Julia package for more information.
- If the solver has `Manualᴹ` in the `Installation` column, the solver requires an installation of MATLAB.
- If the `Installation` column is missing an entry, installing the Julia package will download and install any relevant solver binaries automatically, and you shouldn't need to do anything other than `Pkg.add`.

Solvers with a missing entry in the `Julia Package` column are written in Julia. The link in the `Solver` column is the corresponding Julia package.

Where:

Solver	Julia Package	Installation	License	Supports
Alpine.jl			Triad NS	(MI)NLP
Artelys Knitro	KNITRO.jl	Manual	Comm.	(MI)LP, (MI)SOCP, (MI)NLP
BARON	BARON.jl	Manual	Comm.	(MI)NLP
Bonmin	AmplNLWriter.jl		EPL	(MI)NLP
Cbc	Cbc.jl		EPL	(MI)LP
CDCS	CDCS.jl	Manual[M]	GPL	LP, SOCP, SDP
CDD	CDDLib.jl		GPL	LP
Clp	Clp.jl		EPL	LP
COSMO.jl			Apache	LP, QP, SOCP, SDP
Couenne	AmplNLWriter.jl		EPL	(MI)NLP
CPLEX	CPLEX.jl	Manual	Comm.	(MI)LP, (MI)SOCP
CSDP	CSDP.jl		EPL	LP, SDP
EAGO.jl			MIT	NLP
ECOS	ECOS.jl		GPL	LP, SOCP
FICO Xpress	Xpress.jl	Manual	Comm.	(MI)LP, (MI)SOCP
GLPK	GLPK.jl		GPL	(MI)LP
Gurobi	Gurobi.jl	Manual	Comm.	(MI)LP, (MI)SOCP
HiGHS	HiGHS.jl		MIT	(MI)LP
Hypatia.jl			MIT	LP, SOCP, SDP
Ipopt	Ipopt.jl		EPL	LP, QP, NLP
Juniper.jl			MIT	(MI)SOCP, (MI)NLP
MadNLP.jl			MIT	LP, QP, NLP
MOSEK	MosekTools.jl	Manual	Comm.	(MI)LP, (MI)SOCP, SDP
NLopt	NLopt.jl		GPL	LP, QP, NLP
OSQP	OSQP.jl		Apache	LP, QP
PATH	PATHSolver.jl		MIT	MCP
Pavito.jl			MPL-2	(MI)NLP
Penbmi	Penopt.jl		Comm.	Bilinear SDP
ProxSDP.jl			MIT	LP, SOCP, SDP
RAPOSa	AmplNLWriter.jl	Manual	RAPOSa	(MI)NLP
SCIP	SCIP.jl		ZIB	(MI)LP, (MI)NLP
SCS	SCS.jl		MIT	LP, SOCP, SDP
SDPA	SDPA.jl, SDPAFamily.jl		GPL	LP, SDP
SDPNAL	SDPNAL.jl	Manual[M]	CC BY-SA	LP, SDP
SDPT3	SDPT3.jl	Manual[M]	GPL	LP, SOCP, SDP
SeDuMi	SeDuMi.jl	Manual[M]	GPL	LP, SOCP, SDP
Tulip.jl			MPL-2	LP

- LP = Linear programming
- QP = Quadratic programming
- SOCP = Second-order conic programming (including problems with convex quadratic constraints or objective)
- MCP = Mixed-complementarity programming
- NLP = Nonlinear programming
- SDP = Semidefinite programming

- (MI)XXX = Mixed-integer equivalent of problem type XXX

Note

Developed a solver or solver wrapper? This table is open for new contributions! Start by making a pull request to edit the installation.md file.

Note

Developing a solver or solver wrapper? See Models and the MathOptInterface docs for more details on how JuMP interacts with solvers. Please get in touch via the Developer Chatroom with any questions about connecting new solvers with JuMP.

3.5 AMPL-based solvers

Use AmplNLWriter to access solvers that support the nl format.

Some solvers, such as Bonmin and Couenne can be installed via the Julia package manager. Others need to be manually installed.

Consult the AMPL documentation for a complete list of supported solvers.

3.6 GAMS-based solvers

Use GAMS.jl to access solvers available through GAMS. Such solvers include: AlphaECP, Antigone, BARON, CONOPT, Couenne, LocalSolver, PATHNLP, SHOT, SNOPT, SoPlex. See a complete list here.

Note

GAMS.jl requires an installation of the commercial software GAMS for which a free community license exists.

3.7 NEOS-based solvers

Use NEOSServer.jl to access solvers available through the NEOS Server.

3.8 Common installation issues

Tip

When in doubt, run `import Pkg; Pkg.update()` to see if updating your packages fixes the issue. Remember you will need to exit Julia and start a new session for the changes to take effect.

Check the version of your packages

Each package is versioned with a three-part number of the form `vX.Y.Z`. You can check which versions you have installed with `import Pkg; Pkg.status()`.

This should almost always be the most-recent release. You can check the releases of a package by going to the relevant GitHub page, and navigating to the "releases" page. For example, the list of JuMP releases is available at: https://github.com/jump-dev/JuMP.jl/releases.

If you post on the community forum, please include the output of `Pkg.status()`!

Unsatisfiable requirements detected

Did you get an error like `Unsatisfiable requirements detected for package JuMP`? The Pkg documentation has a section on how to understand and manage these conflicts.

Installing new packages can make JuMP downgrade to an earlier version

Another common complaint is that after adding a new package, code that previously worked no longer works.

This usually happens because the new package is not compatible with the latest version of JuMP. Therefore, the package manager rolls-back JuMP to an earlier version! Here's an example.

First, we add JuMP:

```
(jump_example) pkg> add JuMP
  Resolving package versions...
Updating `~/jump_example/Project.toml`
  [4076af6c] + JuMP v0.21.5
Updating `~/jump_example/Manifest.toml`
  ... lines omitted ...
```

The `+ JuMP v0.21.5` line indicates that JuMP has been added at version `0.21.5`. However, watch what happens when we add JuMPeR:

```
(jump_example) pkg> add JuMPeR
  Resolving package versions...
Updating `~/jump_example/Project.toml`
  [4076af6c] ↓ JuMP v0.21.5 ⇒ v0.18.6
  [707a9f91] + JuMPeR v0.6.0
Updating `~/jump_example/Manifest.toml`
  ... lines omitted ...
```

JuMPeR gets added at version `0.6.0` (`+ JuMPeR v0.6.0`), but JuMP gets downgraded from `0.21.5` to `0.18.6` (`↓ JuMP v0.21.5 ⇒ v0.18.6`)! The reason for this is that JuMPeR doesn't support a version of JuMP newer than `0.18.6`.

> **Tip**
>
> Pay careful attention to the output of the package manager when adding new packages, especially when you see a package being downgraded!

Part II

Tutorials

Chapter 4

Getting started

4.1 Introduction

The purpose of these "Getting started" tutorials is to teach new users the basics of Julia and JuMP.

How these tutorials are structured

Having a high-level overview of how this part of the documentation is structured will help you know where to look for certain things.

- The "Getting started with ..." tutorials are basic introductions to different aspects of JuMP and Julia. If you are new to JuMP and Julia, start by reading them in the following order:
  - Getting started with Julia
  - Getting started with JuMP
  - Getting started with sets and indexing
  - Getting started with data and plotting
- Julia has a reputation for being "fast." Unfortunately, it is also easy to write slow Julia code. Performance tips contains a number of important tips on how to improve the performance of models you write in JuMP.
- Design patterns for larger models is a more advanced tutorial that is aimed at users writing large JuMP models. It's in the "Getting started" section to give you an early preview of how JuMP makes it easy to structure larger models. If you are new to JuMP you may want to skip or briefly skim this tutorial, and come back to it once you have written a few JuMP models.

4.2 Getting started with Julia

Because JuMP is embedded in Julia, knowing some basic Julia is important before you start learning JuMP.

> **Tip**
>
> This tutorial is designed to provide a minimalist crash course in the basics of Julia. You can find resources that provide a more comprehensive introduction to Julia here.

Where to get help

- Read the documentation
  - JuMP https://jump.dev/JuMP.jl/stable/
  - Julia https://docs.julialang.org/en/v1/
- Ask (or browse) the Julia community forum: https://discourse.julialang.org
  - If the question is JuMP-related, ask in the Optimization (Mathematical) section, or tag your question with "jump"

To access the built-in help at the REPL, type ?, followed by the name of the function to lookup:

```
help?> help
search: help schedule Channel hasfield check_belongs_to_model @threadcall AbstractChannel
↪  searchsortedlast

 Welcome to Julia 1.6.2. The full manual is available at

 https://docs.julialang.org

 as well as many great tutorials and learning resources:

 https://julialang.org/learning/

 For help on a specific function or macro, type ? followed by its name, e.g. ?cos, or ?@time, and
 ↪  press enter. Type ; to enter shell mode, ] to enter package mode.
```

Installing Julia

To install Julia, download the latest stable release, then follow the platform specific install instructions.

> **Tip**
>
> Unless you know otherwise, you probably want the 64-bit version.

Next, you need an IDE to develop in. VS Code is a popular choice, so follow these install instructions.

Numbers and arithmetic

Since we want to solve optimization problems, we're going to be using a lot of math. Luckily, Julia is great for math, with all the usual operators:

```
@show 1 + 1
@show 1 - 2
@show 2 * 2
@show 4 / 5
@show 3^2
```

```
1 + 1 = 2
1 - 2 = -1
2 * 2 = 4
4 / 5 = 0.8
3 ^ 2 = 9
```

Info

The `@` in front of something indicates that it is a macro, which is a special type of function. In this case, `@show` prints the expression as typed (e.g., `1 - 2`), as well as the evaluation of the expression (`-1`).

Did you notice how Julia didn't print `.0` after some of the numbers? Julia is a dynamic language, which means you never have to explicitly declare the type of a variable. However, in the background, Julia is giving each variable a type. Check the type of something using the `typeof` function:

```
@show typeof(1)
@show typeof(1.0)
```

```
typeof(1) = Int64
typeof(1.0) = Float64
```

Here `1` is an `Int64`, which is an integer with 64 bits of precision, and `1.0` is a `Float64`, which is a floating point number with 64-bits of precision.

Tip

If you aren't familiar with floating point numbers, make sure to read the Floating point numbers section.

We create complex numbers using `im`:

```
x = 2 + 1im
@show real(x)
@show imag(x)
@show typeof(x)
@show x * (1 - 2im)
```

```
real(x) = 2
imag(x) = 1
typeof(x) = Complex{Int64}
x * (1 - 2im) = 4 - 3im
```

Info

The curly brackets surround what we call the parameters of a type. You can read `Complex{Int64}` as "a complex number, where the real and imaginary parts are represented by `Int64`." If we call `typeof(1.0 + 2.0im)` it will be `Complex{Float64}`, which a complex number with the parts represented by `Float64`.

There are also some cool things like an irrational representation of π.

```
π
```

```
π
 = 3.1415926535897...
```

Tip

To make π (and most other Greek letters), type `\pi` and then press `[TAB]`.

```
typeof(π)
```

```
Irrationalπ{:}
```

However, if we do math with irrational numbers, they get converted to `Float64`:

```
typeof(2π / 3)
```

```
Float64
```

Floating point numbers

Warning

If you aren't familiar with floating point numbers, make sure to read this section carefully.

A `Float64` is a floating point approximation of a real number using 64-bits of information.

Because it is an approximation, things we know hold true in mathematics don't hold true in a computer! For example:

```
0.1 * 3 == 0.3
```

```
false
```

```
sin(2π / 3) == √3 / 2
```

```
false
```

Tip

Get √ by typing `\sqrt` then press `[TAB]`.

Let's see what the differences are:

```
0.1 * 3 - 0.3
```

```
5.551115123125783e-17
```

```
sin(2π / 3) - √3 / 2
```

```
1.1102230246251565e-16
```

They are small, but not zero!

One way of explaining this difference is to consider how we would write `1 / 3` and `2 / 3` using only four digits after the decimal point. We would write `1 / 3` as `0.3333`, and `2 / 3` as `0.6667`. So, despite the fact that `2 * (1 / 3) == 2 / 3`, `2 * 0.3333 == 0.6666 != 0.6667`.

Let's try that again using ≈ (`\approx + [TAB]`) instead of `==`:

```
0.1 * 3 ≈ 0.3
```

```
true
```

```
sin(2π / 3) ≈ √3 / 2
```

```
true
```

≈ is a clever way of calling the `isapprox` function:

```
isapprox(sin(2π / 3), √3 / 2; atol = 1e-8)
```

```
true
```

> **Warning**
>
> Floating point is the reason solvers use tolerances when they solve optimization models. A common mistake you're likely to make is checking whether a binary variable is 0 using `value(z) == 0`. Always remember to use something like `isapprox` when comparing floating point numbers.

Note that `isapprox` will always return `false` if one of the number being compared is `0` and `atol` is zero (its default value).

```
1e-300 ≈ 0.0
```

```
false
```

so always set a nonzero value of `atol` if one of the arguments can be zero.

```
isapprox(1e-9, 0.0, atol = 1e-8)
```

```
true
```

> **Tip**
>
> Gurobi has a good series of articles on the implications of floating point in optimization if you want to read more.

If you aren't careful, floating point arithmetic can throw up all manner of issues. For example:

```
1 + 1e-16 == 1
```

```
true
```

It even turns out that floating point numbers aren't associative!

```
(1 + 1e-16) - 1e-16 == 1 + (1e-16 - 1e-16)
```

```
false
```

It's important to note that this issue isn't Julia-specific. It happens in every programming language (try it out in Python).

Vectors, matrices and arrays

Similar to Matlab, Julia has native support for vectors, matrices and tensors; all of which are represented by arrays of different dimensions. Vectors are constructed by comma-separated elements surrounded by square brackets:

```
b = [5, 6]
```

```
2-element Vector{Int64}:
 5
 6
```

Matrices can by constructed with spaces separating the columns, and semicolons separating the rows:

```
A = [1.0 2.0; 3.0 4.0]
```

```
2×2 Matrix{Float64}:
 1.0  2.0
 3.0  4.0
```

We can do linear algebra:

```
x = A \ b
```

```
2-element Vector{Float64}:
 -3.9999999999999987
  4.499999999999999
```

> **Info**
>
> Here is floating point at work again! `x` is approximately `[-4, 4.5]`.

```
A * x
```

```
2-element Vector{Float64}:
 5.0
 6.0
```

```
A * x ≈ b
```

```
true
```

Note that when multiplying vectors and matrices, dimensions matter. For example, you can't multiply a vector by a vector:

```
    b * b
```

```
MethodError: no method matching *(::Vector{Int64}, ::Vector{Int64})
Closest candidates are:
  *(::Any, ::Any, !Matched::Any, !Matched::Any...) at operators.jl:560
  *(!Matched::StridedMatrix{T}, ::StridedVector{S}) where {T<:Union{Float32, Float64, ComplexF32,
     ComplexF64}, S<:Real} at /buildworker/worker/package_linux64/build/usr/share/julia/stdlib/v1.6/
     LinearAlgebra/src/matmul.jl:44
  *(::StridedVecOrMat{T} where T, !Matched::LinearAlgebra.Adjoint{var"#s814", var"#s813"} where {var
     "#s814", var"#s813"<:LinearAlgebra.LQPackedQ}) at /buildworker/worker/package_linux64/build/usr/
     share/julia/stdlib/v1.6/LinearAlgebra/src/lq.jl:254
  ...
```

But multiplying transposes works:

```
b' * b
```

```
61
```

```
b * b'
```

```
2×2 Matrix{Int64}:
 25  30
 30  36
```

Other common types

Strings

Double quotes are used for strings:

```
typeof("This is Julia")
```

```
String
```

Unicode is fine in strings:

```
typeof("π is about 3.1415")
```

```
String
```

Use println to print a string:

```
println("Hello, World!")
```

```
Hello, World!
```

Use `$()` to interpolate values into a string:

```
x = 123
println("The value of x is: $(x)")
```

```
The value of x is: 123
```

Symbols

Julia `Symbols` are a data structure from the compiler that represent Julia identifiers (i.e., variable names).

```
println("The value of x is: $(eval(:x))")
```

```
The value of x is: 123
```

> **Tip**
>
> We used `eval` here to demonstrate how Julia links `Symbols` to variables. However, avoid calling `eval` in your code. It is usually a sign that your code is doing something that could be more easily achieved a different way. The Community Forum is a good place to ask for advice on alternative approaches.

```
typeof(:x)
```

```
Symbol
```

You can think of a `Symbol` as a `String` that takes up less memory, and that can't be modified.

Convert between `String` and `Symbol` using their constructors:

```
String(:abc)
```

```
"abc"
```

```
Symbol("abc")
```

```
:abc
```

> **Tip**
>
> `Symbols` are often (ab)used to stand in for a `String` or an `Enum`, when one of the former is likely a better choice. The JuMP Style guide recommends reserving `Symbols` for identifiers. See @enum vs. Symbol for more.

Tuples

Julia makes extensive use of a simple data structure called Tuples. Tuples are immutable collections of values. For example:

```
t = ("hello", 1.2, :foo)
```

```
("hello", 1.2, :foo)
```

```
typeof(t)
```

```
Tuple{String, Float64, Symbol}
```

Tuples can be accessed by index, similar to arrays:

```
t[2]
```

```
1.2
```

And they can be "unpacked" like so:

```
a, b, c = t
b
```

```
1.2
```

The values can also be given names, which is a convenient way of making light-weight data structures.

```
t = (word = "hello", num = 1.2, sym = :foo)
```

```
(word = "hello", num = 1.2, sym = :foo)
```

Values can be accessed using dot syntax:

```
t.word
```

```
"hello"
```

Dictionaries

Similar to Python, Julia has native support for dictionaries. Dictionaries provide a very generic way of mapping keys to values. For example, a map of integers to strings:

```
d1 = Dict(1 => "A", 2 => "B", 4 => "D")
```

```
Dict{Int64, String} with 3 entries:
  4 => "D"
  2 => "B"
  1 => "A"
```

> **Info**
>
> Type-stuff again: `Dict{Int64,String}` is a dictionary with `Int64` keys and `String` values.

Looking up a values uses the bracket syntax:

```
d1[2]
```

```
"B"
```

Dictionaries support non-integer keys and can mix data types:

```
Dict("A" => 1, "B" => 2.5, "D" => 2 - 3im)
```

```
Dict{String, Number} with 3 entries:
  "B" => 2.5
  "A" => 1
  "D" => 2-3im
```

> **Info**
>
> Julia types form a hierarchy. Here the value type of the dictionary is `Number`, which is a generalization of `Int64`, `Float64`, and `Complex{Int}`. Leaf nodes in this hierarchy are called "concrete" types, and all others are called "Abstract". In general, having variables with abstract types like `Number` can lead to slower code, so you should try to make sure every element in a dictionary or vector is the same type. For example, in this case we could represent every element as a `Complex{Float64}`:

```
Dict("A" => 1.0 + 0.0im, "B" => 2.5 + 0.0im, "D" => 2.0 - 3.0im)
```

```
Dict{String, ComplexF64} with 3 entries:
  "B" => 2.5+0.0im
  "A" => 1.0+0.0im
  "D" => 2.0-3.0im
```

Dictionaries can be nested:

```
d2 = Dict("A" => 1, "B" => 2, "D" => Dict(:foo => 3, :bar => 4))
```

```
Dict{String, Any} with 3 entries:
  "B" => 2
  "A" => 1
  "D" => Dict(:bar=>4, :foo=>3)
```

```
d2["B"]
```

```
2
```

```
d2["D"][:foo]
```

```
3
```

Structs

You can define custom datastructures with `struct`:

```
struct MyStruct
    x::Int
    y::String
    z::Dict{Int,Int}
end

a = MyStruct(1, "a", Dict(2 => 3))
```

```
Main.MyStruct(1, "a", Dict(2 => 3))
```

```
a.x
```

```
1
```

By default, these are not mutable

```
    a.x = 2
```

```
setfield! immutable struct of type MyStruct cannot be changed
```

However, you can declare a `mutable struct` which is mutable:

```
mutable struct MyStructMutable
    x::Int
    y::String
    z::Dict{Int,Int}
end

a = MyStructMutable(1, "a", Dict(2 => 3))
a.x
```

```
1
```

```
a.x = 2

a
```

```
Main.MyStructMutable(2, "a", Dict(2 => 3))
```

Loops

Julia has native support for for-each style loops with the syntax `for <value> in <collection> end`:

```
for i in 1:5
    println(i)
end
```

```
1
2
3
4
5
```

> **Info**
>
> Ranges are constructed as `start:stop`, or `start:step:stop`.

```
for i in 1.2:1.1:5.6
    println(i)
end
```

```
1.2
2.3
3.4
4.5
5.6
```

This for-each loop also works with dictionaries:

```
for (key, value) in Dict("A" => 1, "B" => 2.5, "D" => 2 - 3im)
    println("$(key): $(value)")
end
```

```
B: 2.5
A: 1
D: 2 - 3im
```

Note that in contrast to vector languages like Matlab and R, loops do not result in a significant performance degradation in Julia.

Control flow

Julia control flow is similar to Matlab, using the keywords `if-elseif-else-end`, and the logical operators `||` and && for **or** and **and** respectively:

```
for i in 0:5:15
    if i < 5
        println("$(i) is less than 5")
    elseif i < 10
        println("$(i) is less than 10")
    else
        if i == 10
            println("the value is 10")
        else
            println("$(i) is bigger than 10")
        end
    end
end
```

```
0 is less than 5
5 is less than 10
the value is 10
15 is bigger than 10
```

Comprehensions

Similar to languages like Haskell and Python, Julia supports the use of simple loops in the construction of arrays and dictionaries, called comprehensions.

A list of increasing integers:

```
[i for i in 1:5]
```

```
5-element Vector{Int64}:
 1
 2
 3
 4
 5
```

Matrices can be built by including multiple indices:

```
[i * j for i in 1:5, j in 5:10]
```

```
5×6 Matrix{Int64}:
  5   6   7   8   9  10
 10  12  14  16  18  20
 15  18  21  24  27  30
 20  24  28  32  36  40
 25  30  35  40  45  50
```

Conditional statements can be used to filter out some values:

```
[i for i in 1:10 if i % 2 == 1]
```

```
5-element Vector{Int64}:
 1
 3
 5
 7
 9
```

A similar syntax can be used for building dictionaries:

```
Dict("$(i)" => i for i in 1:10 if i % 2 == 1)
```

```
Dict{String, Int64} with 5 entries:
  "1" => 1
  "5" => 5
  "7" => 7
  "9" => 9
  "3" => 3
```

Functions

A simple function is defined as follows:

```
function print_hello()
    return println("hello")
end
print_hello()
```

```
hello
```

Arguments can be added to a function:

```
function print_it(x)
    return println(x)
end
print_it("hello")
print_it(1.234)
print_it(:my_id)
```

```
hello
1.234
my_id
```

Optional keyword arguments are also possible:

```
function print_it(x; prefix = "value:")
    return println("$(prefix) $(x)")
end
print_it(1.234)
print_it(1.234, prefix = "val:")
```

```
value: 1.234
val: 1.234
```

The keyword `return` is used to specify the return values of a function:

```
function mult(x; y = 2.0)
    return x * y
end

mult(4.0)
```

```
8.0
```

```
mult(4.0, y = 5.0)
```

```
20.0
```

Anonymous functions

The syntax `input -> output` creates an anonymous function. These are most useful when passed to other functions. For example:

```
f = x -> x^2
f(2)
```

```
4
```

```
map(x -> x^2, 1:4)
```

```
4-element Vector{Int64}:
  1
  4
  9
 16
```

Type parameters

We can constrain the inputs to a function using type parameters, which are `::` followed by the type of the input we want. For example:

```
function foo(x::Int)
    return x^2
end

function foo(x::Float64)
    return exp(x)
end

function foo(x::Number)
    return x + 1
```

```
end

@show foo(2)
@show foo(2.0)
@show foo(1 + 1im)
```

```
foo(2) = 4
foo(2.0) = 7.38905609893065
foo(1 + 1im) = 2 + 1im
```

But what happens if we call `foo` with something we haven't defined it for?

```
    foo([1, 2, 3])
```

```
MethodError: no method matching foo(::Vector{Int64})
Closest candidates are:
  foo(!Matched::Int64) at getting_started_with_julia.md:637
  foo(!Matched::Float64) at getting_started_with_julia.md:641
  foo(!Matched::Number) at getting_started_with_julia.md:645
```

We get a dreaded `MethodError`! A `MethodError` means that you passed a function something that didn't match the type that it was expecting. In this case, the error message says that it doesn't know how to handle an `Vector{Int64}`, but it does know how to handle `Float64`, `Int64`, and `Number`.

> **Tip**
>
> Read the "Closest candidates" part of the error message carefully to get a hint as to what was expected.

Broadcasting

In the example above, we didn't define what to do if `f` was passed a `Vector`. Luckily, Julia provides a convenient syntax for mapping `f` element-wise over arrays! Just add a `.` between the name of the function and the opening `(`. This works for any function, including functions with multiple arguments. For example:

```
f.([1, 2, 3])
```

```
3-element Vector{Int64}:
 1
 4
 9
```

> **Tip**
>
> Get a `MethodError` when calling a function that takes a `Vector`, `Matrix`, or `Array`? Try broadcasting it!

Mutable vs immutable objects

Some types in Julia are mutable, which means you can change the values inside them. A good example is an array. You can modify the contents of an array without having to make a new array.

In contrast, types like `Float64` are immutable. You can't modify the contents of a `Float64`.

This is something to be aware of when passing types into functions. For example:

```
function mutability_example(mutable_type::Vector{Int}, immutable_type::Int)
    mutable_type[1] += 1
    immutable_type += 1
    return
end

mutable_type = [1, 2, 3]
immutable_type = 1

mutability_example(mutable_type, immutable_type)

println("mutable_type: $(mutable_type)")
println("immutable_type: $(immutable_type)")
```

```
mutable_type: [2, 2, 3]
immutable_type: 1
```

Because `Vector{Int}` is a mutable type, modifying the variable inside the function changed the value outside of the function. In contrast, the change to `immutable_type` didn't modify the value outside the function.

You can check mutability with the `isimmutable` function:

```
isimmutable([1, 2, 3])
```

```
false
```

```
isimmutable(1)
```

```
true
```

The package manager

Installing packages

No matter how wonderful Julia's base language is, at some point you will want to use an extension package. Some of these are built-in, for example random number generation is available in the `Random` package in the standard library. These packages are loaded with the commands `using` and `import`.

```
using Random  # The equivalent of Python's `from Random import *`
import Random  # The equivalent of Python's `import Random`

Random.seed!(33)

[rand() for i in 1:10]
```

```
10-element Vector{Float64}:
 0.8245577112736127
 0.2928364052074266
 0.8765793121770682
 0.41615145984974955
 0.7113242552761618
 0.7762718106176869
 0.407423649552187
 0.15761624576044575
 0.8889767003637221
 0.017829104289712516
```

The Package Manager is used to install packages that are not part of Julia's standard library.

For example the following can be used to install JuMP,

```julia
using Pkg
Pkg.add("JuMP")
```

For a complete list of registered Julia packages see the package listing at JuliaHub.

From time to you may wish to use a Julia package that is not registered. In this case a git repository URL can be used to install the package.

```julia
using Pkg
Pkg.add("https://github.com/user-name/MyPackage.jl.git")
```

Package environments

By default, `Pkg.add` will add packages to Julia's global environment. However, Julia also has built-in support for virtual environments.

Activate a virtual environment with:

```julia
import Pkg; Pkg.activate("/path/to/environment")
```

You can see what packages are installed in the current environment with `Pkg.status()`.

> **Tip**
>
> We strongly recommend you create a Pkg environment for each project that you create in Julia, and add only the packages that you need, instead of adding lots of packages to the global environment. The Pkg manager documentation has more information on this topic.

> **Tip**
>
> This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

4.3 Getting started with JuMP

This tutorial is aimed at providing a quick introduction to writing and solving optimization models with JuMP.

If you're new to Julia, start by reading Getting started with Julia.

What is JuMP?

JuMP ("Julia for Mathematical Programming") is an open-source modeling language that is embedded in Julia. It allows users to formulate various classes of optimization problems (linear, mixed-integer, quadratic, conic quadratic, semidefinite, and nonlinear) with easy-to-read code. These problems can then be solved using state-of-the-art open-source and commercial solvers.

JuMP also makes advanced optimization techniques easily accessible from a high-level language.

What is a solver?

A solver is a software package that incorporates algorithms for finding solutions to one or more classes of problem.

For example, HiGHS is a solver for linear programming (LP) and mixed integer programming (MIP) problems. It incorporates algorithms such as the simplex method and the interior-point method.

The Supported-solvers table lists the open-source and commercial solvers that JuMP currently supports.

What is MathOptInterface?

Each solver has its own concepts and data structures for representing optimization models and obtaining results.

MathOptInterface (MOI) is an abstraction layer that JuMP uses to convert from the problem written in JuMP to the solver-specific data structures for each solver.

MOI can be used directly, or through a higher-level modeling interface like JuMP.

Because JuMP is built on top of MOI, you'll often see the `MathOptInterface.` prefix displayed when JuMP types are printed. However, you'll only need to understand and interact with MOI to accomplish advanced tasks such as creating solver-independent callbacks.

Installation

JuMP is a package for Julia. From Julia, JuMP is installed by using the built-in package manager.

```julia
import Pkg
Pkg.add("JuMP")
```

You also need to include a Julia package which provides an appropriate solver. One such solver is `HiGHS.Optimizer`, which is provided by the HiGHS.jl package.

```julia
import Pkg
Pkg.add("HiGHS")
```

See Installation Guide for a list of other solvers you can use.

An example

Let's solve the following linear programming problem using JuMP and HiGHS. We will first look at the complete code to solve the problem and then go through it step by step.

Here's the problem:

$$
\begin{aligned}
\min \quad & 12x + 20y \\
\text{s.t.} \quad & 6x + 8y \geq 100 \\
& 7x + 12y \geq 120 \\
& x \geq 0 \\
& y \in [0, 3]
\end{aligned}
$$

And here's the code to solve this problem:

```
using JuMP
using HiGHS
model = Model(HiGHS.Optimizer)
@variable(model, x >= 0)
@variable(model, 0 <= y <= 3)
@objective(model, Min, 12x + 20y)
@constraint(model, c1, 6x + 8y >= 100)
@constraint(model, c2, 7x + 12y >= 120)
print(model)
optimize!(model)
@show termination_status(model)
@show primal_status(model)
@show dual_status(model)
@show objective_value(model)
@show value(x)
@show value(y)
@show shadow_price(c1)
@show shadow_price(c2)
```

```
Min 12 x + 20 y
Subject to
 c1 : 6 x + 8 y ≥ 100.0
 c2 : 7 x + 12 y ≥ 120.0
 x ≥ 0.0
 y ≥ 0.0
 y ≤ 3.0
Presolving model
2 rows, 2 cols, 4 nonzeros
2 rows, 2 cols, 4 nonzeros
Presolve : Reductions: rows 2(-0); columns 2(-0); elements 4(-0)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 2(220) 0s
          2     2.0500000000e+02 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 2
Objective value     :  2.0500000000e+02
HiGHS run time      :          0.00
termination_status(model) = MathOptInterface.OPTIMAL
primal_status(model) = MathOptInterface.FEASIBLE_POINT
dual_status(model) = MathOptInterface.FEASIBLE_POINT
objective_value(model) = 205.0
```

```
value(x) = 15.0
value(y) = 1.25
shadow_price(c1) = -0.25
shadow_price(c2) = -1.5
```

Step-by-step

Once JuMP is installed, to use JuMP in your programs write:

```julia
using JuMP
```

We also need to include a Julia package which provides an appropriate solver. We want to use `HiGHS.Optimizer` here which is provided by the `HiGHS.jl` package.

```julia
using HiGHS
```

JuMP builds problems incrementally in a `Model` object. Create a model by passing an optimizer to the `Model` function:

```julia
model = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Variables are modeled using `@variable`:

```julia
@variable(model, x >= 0)
```

$$x$$

They can have lower and upper bounds.

```julia
@variable(model, 0 <= y <= 30)
```

$$y$$

The objective is set using `@objective`:

```julia
@objective(model, Min, 12x + 20y)
```

$$12x + 20y$$

Constraints are modeled using `@constraint`. Here, c1 and c2 are the names of our constraint.

```
@constraint(model, c1, 6x + 8y >= 100)
```

c1 : $6x + 8y \geq 100.0$

```
@constraint(model, c2, 7x + 12y >= 120)
```

c2 : $7x + 12y \geq 120.0$

Call `print` to display the model:

```
print(model)
```

```
Min 12 x + 20 y
Subject to
 c1 : 6 x + 8 y ≥ 100.0
 c2 : 7 x + 12 y ≥ 120.0
 x ≥ 0.0
 y ≥ 0.0
 y ≤ 30.0
```

To solve the optimization problem, call the `optimize!` function.

```
optimize!(model)
```

```
Presolving model
2 rows, 2 cols, 4 nonzeros
2 rows, 2 cols, 4 nonzeros
Presolve : Reductions: rows 2(-0); columns 2(-0); elements 4(-0)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 2(220) 0s
          2     2.0500000000e+02 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 2
Objective value     :  2.0500000000e+02
HiGHS run time      :          0.00
```

Info

The `!` after optimize is part of the name. It's nothing special. Julia has a convention that functions which mutate their arguments should end in `!`. A common example is `push!`.

Now let's see what information we can query about the solution.

`termination_status` tells us why the solver stopped:

```
termination_status(model)
```

```
OPTIMAL::TerminationStatusCode = 1
```

In this case, the solver found an optimal solution.

Check `primal_status` to see if the solver found a primal feasible point:

```
primal_status(model)
```

```
FEASIBLE_POINT::ResultStatusCode = 1
```

and `dual_status` to see if the solver found a dual feasible point:

```
dual_status(model)
```

```
FEASIBLE_POINT::ResultStatusCode = 1
```

Now we know that our solver found an optimal solution, and that it has a primal and a dual solution to query.

Query the objective value using `objective_value`:

```
objective_value(model)
```

```
205.0
```

the primal solution using `value`:

```
value(x)
```

```
15.0
```

```
value(y)
```

```
1.25
```

and the dual solution using `shadow_price`:

```
shadow_price(c1)
```

```
-0.25
```

```
shadow_price(c2)
```

```
-1.5
```

That's it for our simple model. In the rest of this tutorial, we expand on some of the basic JuMP operations.

Model basics

Create a model by passing an optimizer:

```
model = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Alternatively, call `set_optimizer` at any point before calling `optimize!`:

```
model = Model()
set_optimizer(model, HiGHS.Optimizer)
```

For some solvers, you can also use `direct_model`, which offers a more efficient connection to the underlying solver:

```
model = direct_model(HiGHS.Optimizer())
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: DIRECT
Solver name: HiGHS
```

> **Warning**
>
> Some solvers do not support `direct_model`!

Solver Options

Pass options to solvers with `optimizer_with_attributes`:

```
model =
    Model(optimizer_with_attributes(HiGHS.Optimizer, "output_flag" => false))
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

> **Note**
>
> These options are solver-specific. To find out the various options available, see the GitHub README of the individual solver packages. The link to each solver's GitHub page is in the Supported solvers table.

You can also pass options with `set_optimizer_attribute`

```
model = Model(HiGHS.Optimizer)
set_optimizer_attribute(model, "output_flag", false)
```

Solution basics

We saw above how to use `termination_status` and `primal_status` to understand the solution returned by the solver.

However, only query solution attributes like `value` and `objective_value` if there is an available solution. Here's a recommended way to check:

```
function solve_infeasible()
    model = Model(HiGHS.Optimizer)
    @variable(model, 0 <= x <= 1)
    @variable(model, 0 <= y <= 1)
    @constraint(model, x + y >= 3)
    @objective(model, Max, x + 2y)
    optimize!(model)
    if termination_status(model) != OPTIMAL
        @warn("The model was not solved correctly.")
        return nothing
    end
    return value(x), value(y)
end

solve_infeasible()
```

```
Presolving model
Problem status detected on presolve: Infeasible
Model   status      : Infeasible
Objective value     :  0.0000000000e+00
HiGHS run time      :          0.00
ERROR:   No invertible representation for getDualRay┌
 Warning: The model was not solved correctly.└
 @ Main getting_started_with_JuMP.md:302
```

Variable basics

Let's create a new empty model to explain some of the variable syntax:

```
model = Model()
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
```

Variable bounds

All of the variables we have created till now have had a bound. We can also create a free variable.

```
@variable(model, free_x)
```

$$free\_x$$

While creating a variable, instead of using the <= and >= syntax, we can also use the `lower_bound` and `upper_bound` keyword arguments.

```
@variable(model, keyword_x, lower_bound = 1, upper_bound = 2)
```

$$keyword\_x$$

We can query whether a variable has a bound using the `has_lower_bound` and `has_upper_bound` functions. The values of the bound can be obtained using the `lower_bound` and `upper_bound` functions.

```
has_upper_bound(keyword_x)
```

```
true
```

```
upper_bound(keyword_x)
```

```
2.0
```

Note querying the value of a bound that does not exist will result in an error.

```
    lower_bound(free_x)
```

```
Variable free_x does not have a lower bound.
```

Containers

We have already seen how to add a single variable to a model using the `@variable` macro. Now let's look at ways to add multiple variables to a model.

JuMP provides data structures for adding collections of variables to a model. These data structures are referred to as containers and are of three types: `Arrays`, `DenseAxisArrays`, and `SparseAxisArrays`.

Arrays JuMP arrays are created when you have integer indices that start at 1:

```
@variable(model, a[1:2, 1:2])
```

```
2×2 Matrix{VariableRef}:
 a[1,1]  a[1,2]
 a[2,1]  a[2,2]
```

Create an n-dimensional variable $x \in R^n$ with bounds $l \leq x \leq u$ ($l, u \in R^n$) as follows:

```
n = 10
l = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
u = [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]

@variable(model, l[i] <= x[i = 1:n] <= u[i])
```

```
10-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]
 x[4]
 x[5]
 x[6]
 x[7]
 x[8]
 x[9]
 x[10]
```

We can also create variable bounds that depend upon the indices:

```
@variable(model, y[i = 1:2, j = 1:2] >= 2i + j)
```

```
2×2 Matrix{VariableRef}:
 y[1,1]  y[1,2]
 y[2,1]  y[2,2]
```

DenseAxisArrays `DenseAxisArrays` are used when the indices are not one-based integer ranges. The syntax is similar except with an arbitrary vector as an index as opposed to a one-based range:

```
@variable(model, z[i = 2:3, j = 1:2:3] >= 0)
```

```
2-dimensional DenseAxisArray{VariableRef,2,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, 1:2:3
And data, a 2×2 Matrix{VariableRef}:
 z[2,1]  z[2,3]
 z[3,1]  z[3,3]
```

Indices do not have to be integers. They can be any Julia type:

```
@variable(model, w[1:5, ["red", "blue"]] <= 1)
```

```
2-dimensional DenseAxisArray{VariableRef,2,...} with index sets:
    Dimension 1, Base.OneTo(5)
    Dimension 2, ["red", "blue"]
And data, a 5×2 Matrix{VariableRef}:
 w[1,red]  w[1,blue]
 w[2,red]  w[2,blue]
 w[3,red]  w[3,blue]
 w[4,red]  w[4,blue]
 w[5,red]  w[5,blue]
```

SparseAxisArrays `SparseAxisArrays` are created when the indices do not form a rectangular set. For example, this applies when indices have a dependence upon previous indices (called triangular indexing):

```
@variable(model, u[i = 1:2, j = i:3])
```

```
JuMP.Containers.SparseAxisArray{VariableRef, 2, Tuple{Int64, Int64}} with 5 entries:
  [1, 1]  =  u[1,1]
  [1, 2]  =  u[1,2]
  [1, 3]  =  u[1,3]
  [2, 2]  =  u[2,2]
  [2, 3]  =  u[2,3]
```

We can also conditionally create variables by adding a comparison check that depends upon the named indices and is separated from the indices by a semi-colon `;`:

```
@variable(model, v[i = 1:9; mod(i, 3) == 0])
```

```
JuMP.Containers.SparseAxisArray{VariableRef, 1, Tuple{Int64}} with 3 entries:
  [3]  =  v[3]
  [6]  =  v[6]
  [9]  =  v[9]
```

Integrality

JuMP can create binary and integer variables. Binary variables are constrained to the set $\{0, 1\}$, and integer variables are constrained to the set $\mathbb{Z}$.

Integer variables Create an integer variable by passing `Int`:

```
@variable(model, integer_x, Int)
```

$$integer\_x$$

or setting the `integer` keyword to `true`:

```
@variable(model, integer_z, integer = true)
```

$$integer\_z$$

Binary variables Create a binary variable by passing `Bin`:

```
@variable(model, binary_x, Bin)
```

$$binary\_x$$

or setting the `binary` keyword to `true`:

```
@variable(model, binary_z, binary = true)
```

$$binary\_z$$

Constraint basics

We'll need a need a new model to explain some of the constraint basics:

```
model = Model()
@variable(model, x)
@variable(model, y)
@variable(model, z[1:10]);
```

Containers

Just as we had containers for variables, JuMP also provides `Arrays`, `DenseAxisArrays`, and `SparseAxisArrays` for storing collections of constraints. Examples for each container type are given below.

Arrays

```
@constraint(model, [i = 1:3], i * x <= i + 1)
```

```
3-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 x ≤ 2.0
 2 x ≤ 3.0
 3 x ≤ 4.0
```

DenseAxisArrays

```
@constraint(model, [i = 1:2, j = 2:3], i * x <= j + 1)
```

```
2-dimensional DenseAxisArray{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape},2,...} with
     index sets:
    Dimension 1, Base.OneTo(2)
    Dimension 2, 2:3
And data, a 2×2 Matrix{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 x ≤ 3.0    x ≤ 4.0
 2 x ≤ 3.0  2 x ≤ 4.0
```

SparseAxisArrays

```
@constraint(model, [i = 1:2, j = 1:2; i != j], i * x <= j + 1)
```

```
JuMP.Containers.SparseAxisArray{ConstraintRef{Model, MathOptInterface.ConstraintIndex{
    MathOptInterface.ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape
    }, 2, Tuple{Int64, Int64}} with 2 entries:
  [1, 2]  =  x ≤ 3.0
  [2, 1]  =  2 x ≤ 2.0
```

Constraints in a loop

We can add constraints using regular Julia loops:

```
for i in 1:3
    @constraint(model, 6x + 4y >= 5i)
end
```

or use for each loops inside the `@constraint` macro:

```
@constraint(model, [i in 1:3], 6x + 4y >= 5i)
```

```
3-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
    ScalarAffineFunction{Float64}, MathOptInterface.GreaterThan{Float64}}, ScalarShape}}:
 6 x + 4 y ≥ 5.0
 6 x + 4 y ≥ 10.0
 6 x + 4 y ≥ 15.0
```

We can also create constraints such as $\sum_{i=1}^{10} z_i \leq 1$:

```
@constraint(model, sum(z[i] for i in 1:10) <= 1)
```

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7 + z_8 + z_9 + z_{10} \leq 1.0$$

Objective functions

Set an objective function with `@objective`:

```
model = Model(HiGHS.Optimizer)
@variable(model, x >= 0)
@variable(model, y >= 0)
@objective(model, Min, 2x + y)
```

$$2x + y$$

Create a maximization objective using Max:

```
@objective(model, Max, 2x + y)
```

$$2x + y$$

Tip

Calling `@objective` multiple times will over-write the previous objective. This can be useful when you want to solve the same problem with different objectives.

Vectorized syntax

We can also add constraints and an objective to JuMP using vectorized linear algebra. We'll illustrate this by solving an LP in standard form i.e.

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

```
vector_model = Model(HiGHS.Optimizer)

A = [
    1 1 9 5
    3 5 0 8
    2 0 6 13
]

b = [7; 3; 5]

c = [1; 3; 5; 2]

@variable(vector_model, x[1:4] >= 0)
@constraint(vector_model, A * x .== b)
@objective(vector_model, Min, c' * x)

optimize!(vector_model)
```

```
Presolving model
3 rows, 4 cols, 10 nonzeros
3 rows, 4 cols, 10 nonzeros
Presolve : Reductions: rows 3(-0); columns 4(-0); elements 10(-0)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 3(13.5) 0s
          4     4.9230769231e+00 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 4
Objective value     :  4.9230769231e+00
HiGHS run time      :          0.00
```

```
objective_value(vector_model)
```

```
4.923076923076923
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

4.4 Getting started with sets and indexing

Most introductory courses to linear programming will teach you to identify sets over which the decision variables and constraints are indexed. Therefore, it is common to write variables such as x_i for all $i \in I$.

A common stumbling block for new users to JuMP is that JuMP does not provide specialized syntax for constructing and manipulating these sets.

We made this decision because Julia already provides a wealth of data structures for working with sets.

In contrast, because tools like AMPL are stand-alone software packages, they had to define their own syntax for set construction and manipulation. Indeed, the AMPL Book has two entire chapters devoted to sets and indexing (V: Simple Sets and Indexing, and VI: Compound Sets and Indexing).

The purpose of this tutorial is to demonstrate a variety of ways in which you can construct and manipulate sets for optimization models.

If you haven't already, you should first read Getting started with JuMP.

```julia
using JuMP
```

Unordered sets

Unordered sets are useful to describe non-numeric indices, such as the names of cities or types of products.

The most common way to construct a set is by creating a vector:

```julia
animals = ["dog", "cat", "chicken", "cow", "pig"]
model = Model()
@variable(model, x[animals])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, ["dog", "cat", "chicken", "cow", "pig"]
And data, a 5-element Vector{VariableRef}:
 x[dog]
 x[cat]
 x[chicken]
 x[cow]
 x[pig]
```

We can also use things like the keys of a dictionary:

```julia
weight_of_animals = Dict(
    "dog" => 20.0,
    "cat" => 5.0,
    "chicken" => 2.0,
    "cow" => 720.0,
```

```
    "pig" => 150.0,
)
animal_keys = keys(weight_of_animals)
model = Model()
@variable(model, x[animal_keys])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, ["cow", "chicken", "cat", "pig", "dog"]
And data, a 5-element Vector{VariableRef}:
 x[cow]
 x[chicken]
 x[cat]
 x[pig]
 x[dog]
```

A third option is to use Julia's `Set` object.

```
animal_set = Set()
for animal in keys(weight_of_animals)
    push!(animal_set, animal)
end
animal_set
```

```
Set{Any} with 5 elements:
  "cow"
  "chicken"
  "cat"
  "pig"
  "dog"
```

The nice thing about `Set`s is that they automatically remove duplicates:

```
push!(animal_set, "dog")
animal_set
```

```
Set{Any} with 5 elements:
  "cow"
  "chicken"
  "cat"
  "pig"
  "dog"
```

Note how dog does not appear twice.

```
model = Model()
@variable(model, x[animal_set])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, ["cow", "chicken", "cat", "pig", "dog"]
And data, a 5-element Vector{VariableRef}:
 x[cow]
 x[chicken]
 x[cat]
 x[pig]
 x[dog]
```

Sets of numbers

Sets of numbers are useful to decribe sets that are ordered, such as years or elements in a vector. The easiest way to create sets of numbers is to use Julia's range syntax.

These can start at 1:

```
model = Model()
@variable(model, x[1:4])
```

```
4-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]
 x[4]
```

but they don't have to:

```
model = Model()
@variable(model, x[2012:2021])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, 2012:2021
And data, a 10-element Vector{VariableRef}:
 x[2012]
 x[2013]
 x[2014]
 x[2015]
 x[2016]
 x[2017]
 x[2018]
 x[2019]
 x[2020]
 x[2021]
```

Ranges also have a `start:step:stop` syntax. So the Olympic years are:

```
model = Model()
@variable(model, x[1896:4:2020])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, 1896:4:2020
And data, a 32-element Vector{VariableRef}:
 x[1896]
 x[1900]
 x[1904]
 x[1908]
 x[1912]
 x[1916]
 x[1920]
 x[1924]
 x[1928]
 x[1932]
```

```
 x[1988]
 x[1992]
 x[1996]
 x[2000]
 x[2004]
 x[2008]
 x[2012]
 x[2016]
 x[2020]
```

Sets of other things

An important observation is that you can have any Julia type as the element of a set. It doesn't have to be a `String` or a `Number`. For example, you can have tuples:

```
sources = ["A", "B", "C"]
sinks = ["D", "E"]
S = [(source, sink) for source in sources, sink in sinks]
model = Model()
@variable(model, x[S])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, [("A", "D"), ("B", "D"), ("C", "D"), ("A", "E"), ("B", "E"), ("C", "E")]
And data, a 6-element Vector{VariableRef}:
 x[("A", "D")]
 x[("B", "D")]
 x[("C", "D")]
 x[("A", "E")]
 x[("B", "E")]
 x[("C", "E")]
```

```
x[("A", "D")]
```

$$x_{("A","D")}$$

For multi-dimensional sets, you can use JuMP's syntax for constructing Containers:

```
model = Model()
@variable(model, x[sources, sinks])
```

```
2-dimensional DenseAxisArray{VariableRef,2,...} with index sets:
    Dimension 1, ["A", "B", "C"]
    Dimension 2, ["D", "E"]
And data, a 3×2 Matrix{VariableRef}:
 x[A,D]  x[A,E]
 x[B,D]  x[B,E]
 x[C,D]  x[C,E]
```

```
x["A", "D"]
```

$$x_{A,D}$$

Info

Note how we indexed `x["A", "D"]` instead of `x[("A", "D")]` as above.

Set operations

Julia has built-in support for set operations such as `union`, `intersect`, and `setdiff`.

Therefore, to create a set of all years in which the summer Olympics were held, we can use:

```
baseline = 1896:4:2020
cancelled = [1916, 1940, 1944, 2020]
off_year = [2021]
olympic_years = union(setdiff(baseline, cancelled), off_year)
```

```
29-element Vector{Int64}:
 1896
 1900
 1904
 1908
 1912
 1920
 1924
 1928
 1932
 1936
 
 1988
 1992
 1996
 2000
 2004
 2008
 2012
 2016
 2021
```

You can also find the number of elements (i.e., the cardinality) in a set using `length`:

```
length(olympic_years)
```

```
29
```

Set membership operations

To compute membership of sets, use the `in` function.

```
2000 in olympic_years
```

```
true
```

```
2001 in olympic_years
```

```
false
```

Indexing expressions

Use Julia's generator syntax to compute new sets, such as the list of Olympic years that are divisible by 3:

```
olympic_3_years = [year for year in olympic_years if mod(year, 3) == 0]
model = Model()
@variable(model, x[olympic_3_years])
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, [1896, 1908, 1920, 1932, 1956, 1968, 1980, 1992, 2004, 2016]
And data, a 10-element Vector{VariableRef}:
 x[1896]
 x[1908]
 x[1920]
 x[1932]
 x[1956]
 x[1968]
 x[1980]
 x[1992]
 x[2004]
 x[2016]
```

Alternatively, use JuMP's syntax for constructing Containers:

```
model = Model()
@variable(model, x[year in olympic_years; mod(year, 3) == 0])
```

```
JuMP.Containers.SparseAxisArray{VariableRef, 1, Tuple{Int64}} with 10 entries:
  [1896]  =  x[1896]
  [1908]  =  x[1908]
  [1920]  =  x[1920]
  [1932]  =  x[1932]
  [1956]  =  x[1956]
  [1968]  =  x[1968]
  [1980]  =  x[1980]
  [1992]  =  x[1992]

  [2004]  =  x[2004]
  [2016]  =  x[2016]
```

Compound sets

Consider a transportation problem in which we need to ship goods between cities. We have been provided a list of cities:

```
cities = ["Auckland", "Wellington", "Christchurch", "Dunedin"]
```

```
4-element Vector{String}:
 "Auckland"
 "Wellington"
 "Christchurch"
 "Dunedin"
```

and a distance matrix which records the shipping distance between pairs of cities. If we can't ship between two cities, the distance is `0`.

```
distances = [0 643 1071 1426; 0 0 436 790; 0 0 0 360; 1426 0 0 0]
```

```
4×4 Matrix{Int64}:
    0  643  1071  1426
    0    0   436   790
    0    0     0   360
 1426    0     0     0
```

Let's have a look at ways we could write a model with an objective function to minimize the total shipping cost. For simplicity, we'll ignore all constraints.

Fix unused variables

One approach is to fix all variables that we can't use to zero. Most solvers are smart-enough to remove these during a presolve phase, so it has a very small impact on performance:

```
N = length(cities)
model = Model()
@variable(model, x[1:N, 1:N] >= 0)
for i in 1:N, j in 1:N
    if distances[i, j] == 0
        fix(x[i, j], 0.0; force = true)
    end
end
@objective(model, Min, sum(distances[i, j] * x[i, j] for i in 1:N, j in 1:N))
```

$$643x_{1,2} + 1071x_{1,3} + 1426x_{1,4} + 436x_{2,3} + 790x_{2,4} + 360x_{3,4} + 1426x_{4,1}$$

Filtered summation

Another approach is to define filters whenever we want to sum over our decision variables:

```
N = length(cities)
model = Model()
@variable(model, x[1:N, 1:N] >= 0)
@objective(
    model,
    Min,
    sum(
        distances[i, j] * x[i, j] for i in 1:N, j in 1:N if distances[i, j] > 0
    ),
)
```

$$643x_{1,2} + 1071x_{1,3} + 1426x_{1,4} + 436x_{2,3} + 790x_{2,4} + 360x_{3,4} + 1426x_{4,1}$$

Filtered indexing

We could also use JuMP's support for Containers:

```
N = length(cities)
model = Model()
@variable(model, x[i = 1:N, j = 1:N; distances[i, j] > 0])
@objective(model, Min, sum(distances[i...] * x[i] for i in eachindex(x)))
```

$$790x_{2,4} + 643x_{1,2} + 1071x_{1,3} + 1426x_{4,1} + 360x_{3,4} + 1426x_{1,4} + 436x_{2,3}$$

> **Note**
>
> The `i...` is called a "splat". It converts a tuple like `(1, 2)` into two indices like `distances[1, 2]`.

Converting to a different data structure

Another approach, and one that is often the most readable, is to convert the data you have into something that is easier to work with. Originally, we had a vector of strings and a matrix of distances. What we really need is something that maps usable origin-destination pairs to distances. A dictionary is an obvious choice:

```
routes = Dict(
    (a, b) => distances[i, j] for
    (i, a) in enumerate(cities), (j, b) in enumerate(cities) if
    distances[i, j] > 0
)
```

```
Dict{Tuple{String, String}, Int64} with 7 entries:
  ("Auckland", "Wellington")     => 643
  ("Wellington", "Christchurch") => 436
  ("Wellington", "Dunedin")      => 790
  ("Christchurch", "Dunedin")    => 360
  ("Auckland", "Dunedin")        => 1426
  ("Dunedin", "Auckland")        => 1426
  ("Auckland", "Christchurch")   => 1071
```

Then, we can create our model like so:

```
model = Model()
@variable(model, x[keys(routes)])
@objective(model, Min, sum(v * x[k] for (k, v) in routes))
```

$$643x_{("Auckland","Wellington")}+436x_{("Wellington","Christchurch")}+790x_{("Wellington","Dunedin")}+360x_{("Christchurch","Dunedin}$$

This has a number of benefits over the other approaches, including a compacter algebraic model and variables that are named in a more meaningful way.

> **Tip**
>
> If you're struggling to formulate a problem using the available syntax in JuMP, it's probably a sign that you should convert your data into a different form.

Next steps

The purpose of this tutorial was to show how JuMP does not have specialized syntax for set creation and manipulation. Instead, you should use the tools provided by Julia itself.

This is both an opportunity and a challenge, because you are free to pick the syntax and data structures that best suit your problem, but for new users it can be daunting to decide which structure to use.

Read through some of the other JuMP tutorials to get inspiration and ideas for how you can use Julia's syntax and data structures to your advantage.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

4.5 Getting started with data and plotting

In this tutorial we will learn how to read tabular data into Julia, and some of the basics of plotting.

If you're new to Julia, start by reading Getting started with Julia and Getting started with JuMP first.

Note

There are multiple ways to read the same kind of data into Julia. This tutorial focuses on DataFrames.jl because it provides the ecosystem to work with most of the required file types in a straightforward manner.

Before we get started, we need this constant to point to where the data files are.

```
const DATA_DIR = joinpath(@__DIR__, "data")
```

```
"/home/runner/work/JuMP.jl/JuMP.jl/docs/latex_build/tutorials/getting_started/data"
```

Where to get help

Read the documentation

- Plots.jl: http://docs.juliaplots.org/latest/
- CSV.jl: http://csv.juliadata.org/stable
- DataFrames.jl: https://dataframes.juliadata.org/stable/

Preliminaries

To get started, we need to install some packages.

DataFrames.jl

The `DataFrames` package provides a set of tools for working with tabular data. It is available through the Julia package manager.

```julia
using Pkg
Pkg.add("DataFrames")
```

```julia
import DataFrames
```

!!! info What is a DataFrame? A DataFrame is a data structure like a table or spreadsheet. You can use it for storing and exploring a set of related data values. Think of it as a smarter array for holding tabular data.

Plots.jl

The `Plots` package provides a set of tools for plotting. It is available through the Julia package manager.

```julia
using Pkg
Pkg.add("Plots")
```

```julia
import Plots
```

CSV .jl

CSV and other delimited text files can be read by the CSV.jl package.

```julia
Pkg.add("CSV")
```

```julia
import CSV
```

DataFrame basics

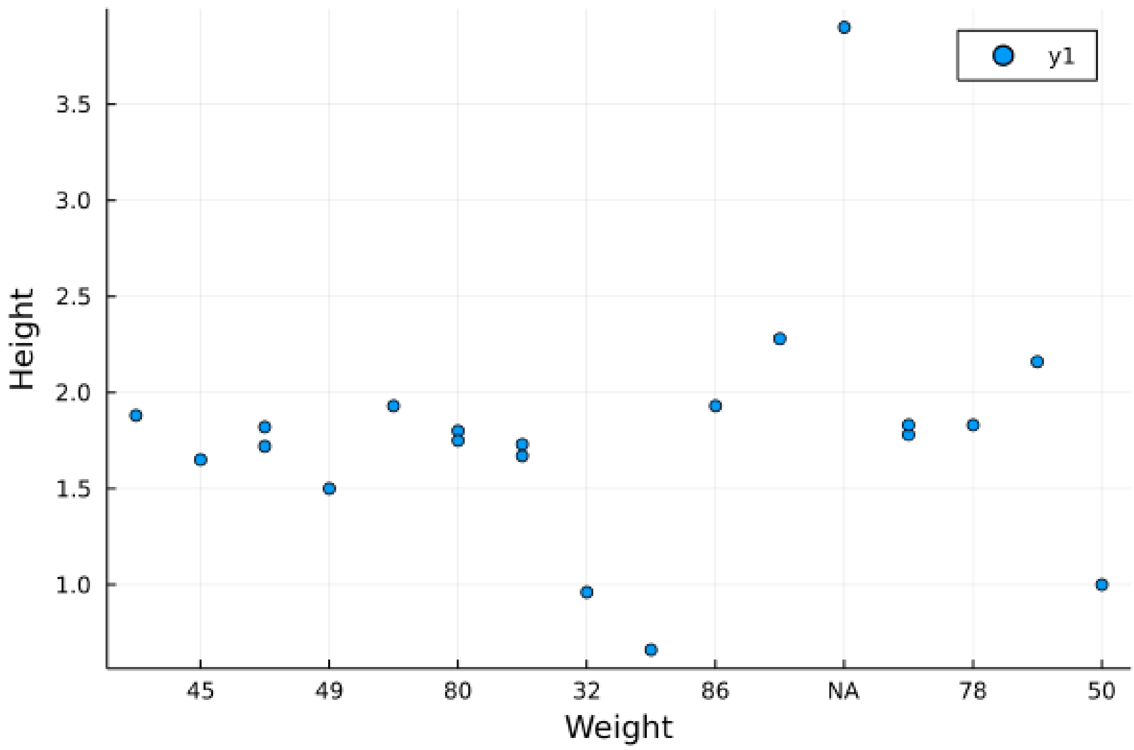
To read a CSV file into a DataFrame, we use the `CSV.read` function.

```julia
csv_df = CSV.read(joinpath(DATA_DIR, "StarWars.csv"), DataFrames.DataFrame)
```

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	D
	String31	String7	Float64	String7	String15	String7	String15	String15	String15	Stri
1	Anakin Skywalker	male	1.88	84	blue	blond	fair	Tatooine	41.9BBY	4/
2	Padme Amidala	female	1.65	45	brown	brown	light	Naboo	46BBY	19
3	Luke Skywalker	male	1.72	77	blue	blond	fair	Tatooine	19BBY	unk
4	Leia Skywalker	female	1.5	49	brown	brown	light	Alderaan	19BBY	unk
5	Qui-Gon Jinn	male	1.93	88.5	blue	brown	light	unk_planet	92BBY	32
6	Obi-Wan Kenobi	male	1.82	77	bluegray	auburn	fair	Stewjon	57BBY	0
7	Han Solo	male	1.8	80	brown	brown	light	Corellia	29BBY	unk
8	Sheev Palpatine	male	1.73	75	blue	red	pale	Naboo	82BBY	10
9	R2-D2	male	0.96	32	NA	NA	NA	Naboo	33BBY	unk
10	C-3PO	male	1.67	75	NA	NA	NA	Tatooine	112BBY	3/
11	Yoda	male	0.66	17	brown	brown	green	unk_planet	896BBY	4/
12	Darth Maul	male	1.75	80	yellow	none	red	Dathomir	54BBY	unk
13	Dooku	male	1.93	86	brown	brown	light	Serenno	102BBY	19
14	Chewbacca	male	2.28	112	blue	brown	NA	Kashyyyk	200BBY	25
15	Jabba	male	3.9	NA	yellow	none	tan-green	Tatooine	unk_born	4/
16	Lando Calrissian	male	1.78	79	brown	blank	dark	Socorro	31BBY	unk
17	Boba Fett	male	1.83	78	brown	black	brown	Kamino	31.5BBY	unk
18	Jango Fett	male	1.83	79	brown	black	brown	ConcordDawn	66BBY	22
19	Grievous	male	2.16	159	gold	black	orange	Kalee	unk_born	19
20	Chief Chirpa	male	1.0	50	black	gray	brown	Endor	unk_born	4/

Let's try plotting some of this data

```
Plots.scatter(
    csv_df.Weight,
    csv_df.Height,
    xlabel = "Weight",
    ylabel = "Height",
)
```



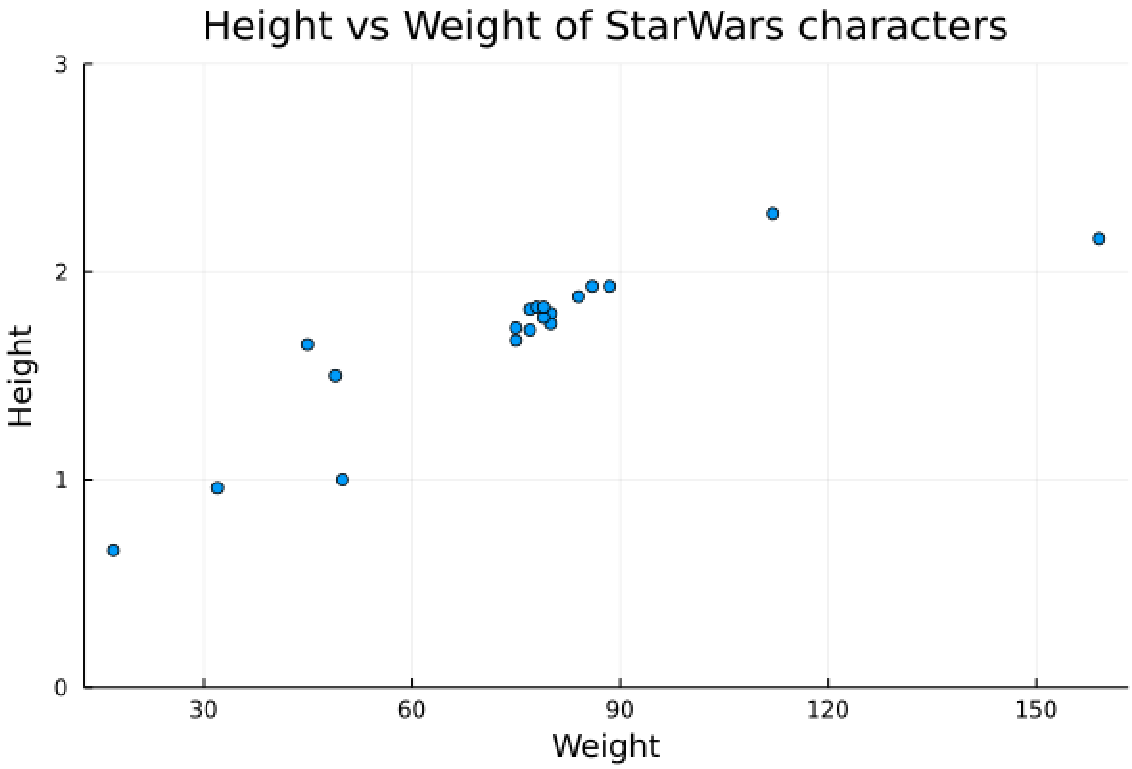
That doesn't look right. What happened? If you look at the dataframe above, it read `Weight` in as a `String` column because there are "NA" fields. Let's correct that, by telling CSV to consider "NA" as `missing`.

```
csv_df = CSV.read(
    joinpath(DATA_DIR, "StarWars.csv"),
    DataFrames.DataFrame,
    missingstring = "NA",
)
```

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	
	String31	String7	Float64	Float64?	String15	String7	String15	String15	String15	St
1	Anakin Skywalker	male	1.88	84.0	blue	blond	fair	Tatooine	41.9BBY	4
2	Padme Amidala	female	1.65	45.0	brown	brown	light	Naboo	46BBY	1
3	Luke Skywalker	male	1.72	77.0	blue	blond	fair	Tatooine	19BBY	ur
4	Leia Skywalker	female	1.5	49.0	brown	brown	light	Alderaan	19BBY	ur
5	Qui-Gon Jinn	male	1.93	88.5	blue	brown	light	unk_planet	92BBY	3
6	Obi-Wan Kenobi	male	1.82	77.0	bluegray	auburn	fair	Stewjon	57BBY	(
7	Han Solo	male	1.8	80.0	brown	brown	light	Corellia	29BBY	ur
8	Sheev Palpatine	male	1.73	75.0	blue	red	pale	Naboo	82BBY	1
9	R2-D2	male	0.96	32.0	missing	missing	missing	Naboo	33BBY	ur
10	C-3PO	male	1.67	75.0	missing	missing	missing	Tatooine	112BBY	[illegible]
11	Yoda	male	0.66	17.0	brown	brown	green	unk_planet	896BBY	4
12	Darth Maul	male	1.75	80.0	yellow	none	red	Dathomir	54BBY	ur
13	Dooku	male	1.93	86.0	brown	brown	light	Serenno	102BBY	1
14	Chewbacca	male	2.28	112.0	blue	brown	missing	Kashyyyk	200BBY	2
15	Jabba	male	3.9	missing	yellow	none	tan-green	Tatooine	unk_born	4
16	Lando Calrissian	male	1.78	79.0	brown	blank	dark	Socorro	31BBY	ur
17	Boba Fett	male	1.83	78.0	brown	black	brown	Kamino	31.5BBY	ur
18	Jango Fett	male	1.83	79.0	brown	black	brown	ConcordDawn	66BBY	2
19	Grievous	male	2.16	159.0	gold	black	orange	Kalee	unk_born	1
20	Chief Chirpa	male	1.0	50.0	black	gray	brown	Endor	unk_born	4

Then let's re-plot our data

```
Plots.scatter(
    csv_df.Weight,
    csv_df.Height,
    title = "Height vs Weight of StarWars characters",
    xlabel = "Weight",
    ylabel = "Height",
    label = false,
    ylims = (0, 3),
)
```



Better!

Tip

Read the CSV documentation for other parsing options.

DataFrames.jl supports manipulation using functions similar to pandas. For example, split the dataframe into groups based on eye-color:

```
by_eyecolor = DataFrames.groupby(csv_df, :Eyecolor)
```

GroupedDataFrame with 7 groups based on key: Eyecolor

First Group (5 rows): Eyecolor = "blue"

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	Died
	String31	String7	Float64	Float64?	String15	String7	String15	String15	String15	String1
1	Anakin Skywalker	male	1.88	84.0	blue	blond	fair	Tatooine	41.9BBY	4ABY
2	Luke Skywalker	male	1.72	77.0	blue	blond	fair	Tatooine	19BBY	unk_die
3	Qui-Gon Jinn	male	1.93	88.5	blue	brown	light	unk_planet	92BBY	32BBY
4	Sheev Palpatine	male	1.73	75.0	blue	red	pale	Naboo	82BBY	10ABY
5	Chewbacca	male	2.28	112.0	blue	brown	missing	Kashyyyk	200BBY	25ABY

. . .

Last Group (1 row): Eyecolor = "black"

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	Died	
	String31	String7	Float64	Float64?	String15	String7	String15	String15	String15	String15	St
1	Chief Chirpa	male	1.0	50.0	black	gray	brown	Endor	unk_born	4ABY	n

Then recombine into a single dataframe based on a function operating over the split dataframes:

```
eyecolor_count = DataFrames.combine(by_eyecolor) do df
    return DataFrames.nrow(df)
end
```

	Eyecolor	x1
	String15	Int64
1	blue	5
2	brown	8
3	bluegray	1
4	missing	2
5	yellow	2
6	gold	1
7	black	1

We can rename columns:

```
DataFrames.rename!(eyecolor_count, :x1 => :count)
```

	Eyecolor	count
	String15	Int64
1	blue	5
2	brown	8
3	bluegray	1
4	missing	2
5	yellow	2
6	gold	1
7	black	1

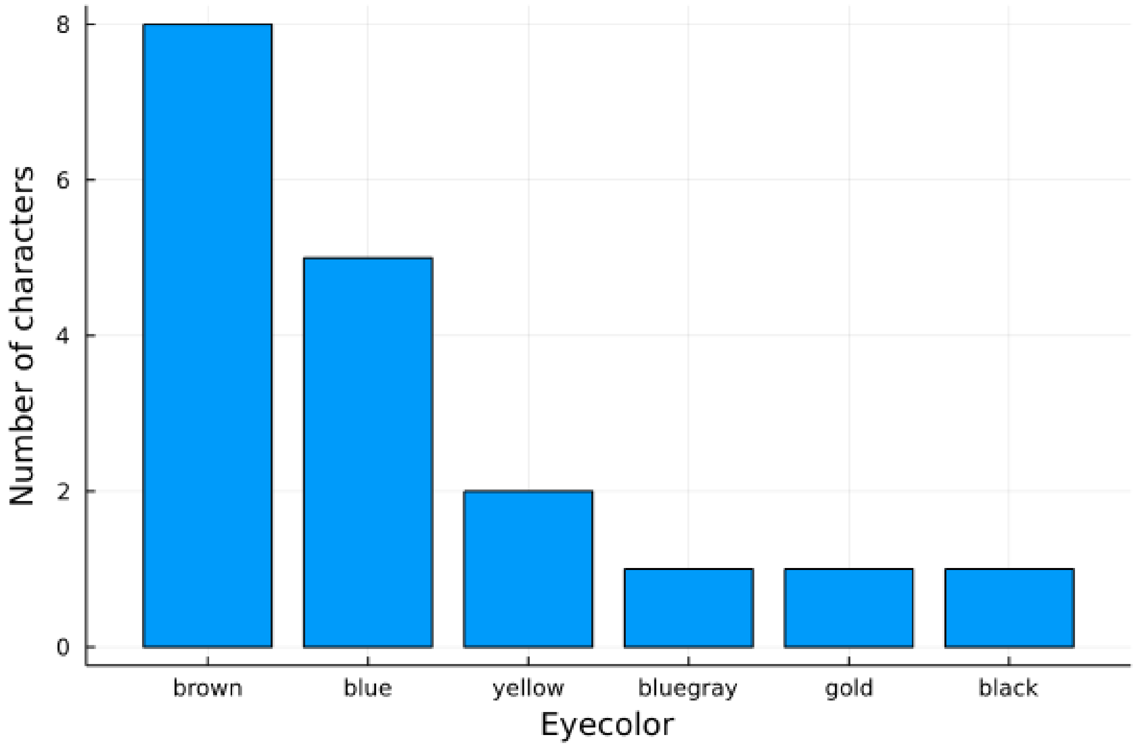
Drop some missing rows:

```
DataFrames.dropmissing!(eyecolor_count, :Eyecolor)
```

	Eyecolor	count
	String15	Int64
1	blue	5
2	brown	8
3	bluegray	1
4	yellow	2
5	gold	1
6	black	1

Then we can visualize the data:

```
sort!(eyecolor_count, :count, rev = true)
Plots.bar(
    eyecolor_count.Eyecolor,
    eyecolor_count.count,
    xlabel = "Eyecolor",
    ylabel = "Number of characters",
    label = false,
)
```



Other Delimited Files

We can also use the `CSV.jl` package to read any other delimited text file format.

By default, CSV.File will try to detect a file's delimiter from the first 10 lines of the file.

Candidate delimiters include `','`, `'\t'`, `' '`, `'|'`, `';'`, and `':'`. If it can't auto-detect the delimiter, it will assume `','`.

Let's take the example of space separated data.

```
ss_df = CSV.read(joinpath(DATA_DIR, "Cereal.txt"), DataFrames.DataFrame)
```

	Name	Cups	Calories	Carbs	Fat	Fiber	Potassium	Protein	Sodium	Sugars
	String31	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
1	CapnCrunch	0.75	120	12.0	2	0.0	35	1	220	12
2	CocoaPuffs	1.0	110	12.0	1	0.0	55	1	180	13
3	Trix	1.0	110	13.0	1	0.0	25	1	140	12
4	AppleJacks	1.0	110	11.0	0	1.0	30	2	125	14
5	CornChex	1.0	110	22.0	0	0.0	25	2	280	3
6	CornFlakes	1.0	100	21.0	0	1.0	35	2	290	2
7	Nut&Honey	0.67	120	15.0	1	0.0	40	2	190	9
8	Smacks	0.75	110	9.0	1	1.0	40	2	70	15
9	MultiGrain	1.0	100	15.0	1	2.0	90	2	220	6
10	CracklinOat	0.5	110	10.0	3	4.0	160	3	140	7
11	GrapeNuts	0.25	110	17.0	0	3.0	90	3	179	3
12	HoneyNutCheerios	0.75	110	11.5	1	1.5	90	3	250	10
13	NutriGrain	0.67	140	21.0	2	3.0	130	3	220	7
14	Product19	1.0	100	20.0	0	1.0	45	3	320	3
15	TotalRaisinBran	1.0	140	15.0	1	4.0	230	3	190	14
16	WheatChex	0.67	100	17.0	1	3.0	115	3	230	3
17	Oatmeal	0.5	130	13.5	2	1.5	120	3	170	10
18	Life	0.67	100	12.0	2	2.0	95	4	150	6
19	Maypo	1.0	100	16.0	1	0.0	95	4	0	3
20	QuakerOats	0.5	100	14.0	1	2.0	110	4	135	6
21	Muesli	1.0	150	16.0	3	3.0	170	4	150	11
22	Cheerios	1.25	110	17.0	2	2.0	105	6	290	1
23	SpecialK	1.0	110	16.0	0	1.0	55	6	230	3

We can also specify the delimiter by passing the `delim` argument.

```
delim_df = CSV.read(
    joinpath(DATA_DIR, "Soccer.txt"),
    DataFrames.DataFrame,
    delim = "::",
)
```

	Team	Played	Wins	Draws	Losses	Goals_for	Goals_against
	String31	Int64	Int64	Int64	Int64	String15	String15
1	Barcelona	38	30	4	4	110 goals	21 goals
2	Real Madrid	38	30	2	6	118 goals	38 goals
3	Atletico Madrid	38	23	9	6	67 goals	29 goals
4	Valencia	38	22	11	5	70 goals	32 goals
5	Seville	38	23	7	8	71 goals	45 goals
6	Villarreal	38	16	12	10	48 goals	37 goals
7	Athletic Bilbao	38	15	10	13	42 goals	41 goals
8	Celta Vigo	38	13	12	13	47 goals	44 goals
9	Malaga	38	14	8	16	42 goals	48 goals
10	Espanyol	38	13	10	15	47 goals	51 goals
11	Rayo Vallecano	38	15	4	19	46 goals	68 goals
12	Real Sociedad	38	11	13	14	44 goals	51 goals
13	Elche	38	11	8	19	35 goals	62 goals
14	Levante	38	9	10	19	34 goals	67 goals
15	Getafe	38	10	7	21	33 goals	64 goals
16	Deportivo La Coruna	38	7	14	17	35 goals	60 goals
17	Granada	38	7	14	17	29 goals	64 goals
18	Eibar	38	9	8	21	34 goals	55 goals
19	Almeria	38	8	8	22	35 goals	64 goals
20	Cordoba	38	3	11	24	22 goals	68 goals

Working with DataFrames

Now that we have read the required data into a DataFrame, let us look at some basic operations we can perform on it.

Querying Basic Information

The `size` function gets us the dimensions of the DataFrame:

```
DataFrames.size(ss_df)
```

```
(23, 10)
```

We can also use the `nrow` and `ncol` functions to get the number of rows and columns respectively:

```
DataFrames.nrow(ss_df), DataFrames.ncol(ss_df)
```

```
(23, 10)
```

The `describe` function gives basic summary statistics of data in a DataFrame:

```
DataFrames.describe(ss_df)
```

	variable	mean	min	median	max	nmissing	eltype
	Symbol	Union...	Any	Union...	Any	Int64	DataType
1	Name		AppleJacks		WheatChex	0	String31
2	Cups	0.823043	0.25	1.0	1.25	0	Float64
3	Calories	113.043	100	110.0	150	0	Int64
4	Carbs	15.0435	9.0	15.0	22.0	0	Float64
5	Fat	1.13043	0	1.0	3	0	Int64
6	Fiber	1.56522	0.0	1.5	4.0	0	Float64
7	Potassium	86.3043	25	90.0	230	0	Int64
8	Protein	2.91304	1	3.0	6	0	Int64
9	Sodium	189.957	0	190.0	320	0	Int64
10	Sugars	7.52174	1	7.0	15	0	Int64

Names of every column can be obtained by the `names` function:

```
DataFrames.names(ss_df)
```

```
10-element Vector{String}:
 "Name"
 "Cups"
 "Calories"
 "Carbs"
 "Fat"
 "Fiber"
 "Potassium"
 "Protein"
 "Sodium"
 "Sugars"
```

Corresponding data types are obtained using the broadcasted `eltype` function:

```
eltype.(ss_df)
```

	Name	Cups	Calories	Carbs	Fat	Fiber	Potassium	Protein	Sodium	Sugars
	DataType	DataType	DataType	DataType	DataType	DataType	DataType	DataType	DataType	DataType
1	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
2	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
3	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
4	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
5	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
6	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
7	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
8	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
9	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
10	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
11	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
12	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
13	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
14	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
15	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
16	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
17	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
18	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
19	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
20	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
21	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
22	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64
23	Char	Float64	Int64	Float64	Int64	Float64	Int64	Int64	Int64	Int64

Accessing the Data

Similar to regular arrays, we use numerical indexing to access elements of a DataFrame:

```
csv_df[1, 1]
```

```
"Anakin Skywalker"
```

The following are different ways to access a column:

```
csv_df[!, 1]
```

```
20-element Vector{InlineStrings.String31}:
 "Anakin Skywalker"
 "Padme Amidala"
 "Luke Skywalker"
 "Leia Skywalker"
 "Qui-Gon Jinn"
 "Obi-Wan Kenobi"
 "Han Solo"
 "Sheev Palpatine"
 "R2-D2"
 "C-3PO"
 "Yoda"
 "Darth Maul"
 "Dooku"
```

```
 "Chewbacca"
 "Jabba"
 "Lando Calrissian"
 "Boba Fett"
 "Jango Fett"
 "Grievous"
 "Chief Chirpa"
```

```
csv_df[!, :Name]
```

```
20-element Vector{InlineStrings.String31}:
 "Anakin Skywalker"
 "Padme Amidala"
 "Luke Skywalker"
 "Leia Skywalker"
 "Qui-Gon Jinn"
 "Obi-Wan Kenobi"
 "Han Solo"
 "Sheev Palpatine"
 "R2-D2"
 "C-3PO"
 "Yoda"
 "Darth Maul"
 "Dooku"
 "Chewbacca"
 "Jabba"
 "Lando Calrissian"
 "Boba Fett"
 "Jango Fett"
 "Grievous"
 "Chief Chirpa"
```

```
csv_df.Name
```

```
20-element Vector{InlineStrings.String31}:
 "Anakin Skywalker"
 "Padme Amidala"
 "Luke Skywalker"
 "Leia Skywalker"
 "Qui-Gon Jinn"
 "Obi-Wan Kenobi"
 "Han Solo"
 "Sheev Palpatine"
 "R2-D2"
 "C-3PO"
 "Yoda"
 "Darth Maul"
 "Dooku"
 "Chewbacca"
 "Jabba"
 "Lando Calrissian"
 "Boba Fett"
 "Jango Fett"
 "Grievous"
 "Chief Chirpa"
```

```
csv_df[:, 1] # Note that this creates a copy.
```

```
20-element Vector{InlineStrings.String31}:
 "Anakin Skywalker"
 "Padme Amidala"
 "Luke Skywalker"
 "Leia Skywalker"
 "Qui-Gon Jinn"
 "Obi-Wan Kenobi"
 "Han Solo"
 "Sheev Palpatine"
 "R2-D2"
 "C-3PO"
 "Yoda"
 "Darth Maul"
 "Dooku"
 "Chewbacca"
 "Jabba"
 "Lando Calrissian"
 "Boba Fett"
 "Jango Fett"
 "Grievous"
 "Chief Chirpa"
```

The following are different ways to access a row:

```
csv_df[1:1, :]
```

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	Died
	String31	String7	Float64	Float64?	String15	String7	String15	String15	String15	String15
1	Anakin Skywalker	male	1.88	84.0	blue	blond	fair	Tatooine	41.9BBY	4ABY

```
csv_df[1, :] # This produces a DataFrameRow.
```

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	Died
	String31	String7	Float64	Float64?	String15	String7	String15	String15	String15	String15
1	Anakin Skywalker	male	1.88	84.0	blue	blond	fair	Tatooine	41.9BBY	4ABY

We can change the values just as we normally assign values.

Assign a range to scalar:

```
csv_df[1:3, :Height] .= 1.83
```

```
3-element view(::Vector{Float64}, 1:3) with eltype Float64:
 1.83
 1.83
 1.83
```

Assign a vector:

```
csv_df[4:6, :Height] = [1.8, 1.6, 1.8]
```

```
3-element Vector{Float64}:
 1.8
 1.6
 1.8
```

```
csv_df
```

	Name	Gender	Height	Weight	Eyecolor	Haircolor	Skincolor	Homeland	Born	
	String31	String7	Float64	Float64?	String15	String7	String15	String15	String15	St
1	Anakin Skywalker	male	1.83	84.0	blue	blond	fair	Tatooine	41.9BBY	4
2	Padme Amidala	female	1.83	45.0	brown	brown	light	Naboo	46BBY	1
3	Luke Skywalker	male	1.83	77.0	blue	blond	fair	Tatooine	19BBY	ur
4	Leia Skywalker	female	1.8	49.0	brown	brown	light	Alderaan	19BBY	ur
5	Qui-Gon Jinn	male	1.6	88.5	blue	brown	light	unk_planet	92BBY	3
6	Obi-Wan Kenobi	male	1.8	77.0	bluegray	auburn	fair	Stewjon	57BBY	[illegible]
7	Han Solo	male	1.8	80.0	brown	brown	light	Corellia	29BBY	ur
8	Sheev Palpatine	male	1.73	75.0	blue	red	pale	Naboo	82BBY	1
9	R2-D2	male	0.96	32.0	missing	missing	missing	Naboo	33BBY	ur
10	C-3PO	male	1.67	75.0	missing	missing	missing	Tatooine	112BBY	[illegible]
11	Yoda	male	0.66	17.0	brown	brown	green	unk_planet	896BBY	4
12	Darth Maul	male	1.75	80.0	yellow	none	red	Dathomir	54BBY	ur
13	Dooku	male	1.93	86.0	brown	brown	light	Serenno	102BBY	1
14	Chewbacca	male	2.28	112.0	blue	brown	missing	Kashyyyk	200BBY	2
15	Jabba	male	3.9	missing	yellow	none	tan-green	Tatooine	unk_born	4
16	Lando Calrissian	male	1.78	79.0	brown	blank	dark	Socorro	31BBY	ur
17	Boba Fett	male	1.83	78.0	brown	black	brown	Kamino	31.5BBY	ur
18	Jango Fett	male	1.83	79.0	brown	black	brown	ConcordDawn	66BBY	2
19	Grievous	male	2.16	159.0	gold	black	orange	Kalee	unk_born	1
20	Chief Chirpa	male	1.0	50.0	black	gray	brown	Endor	unk_born	4

Tip

There are a lot more things which can be done with a DataFrame. Read the docs for more information.

For information on dplyr-type syntax:

- Read the DataFrames.jl documentation
- Check out DataFramesMeta.jl

Example: the passport problem

Let's now apply what we have learned to solve a real problem.

Data manipulation

The Passport Index Dataset lists travel visa requirements for 199 countries, in `.csv` format. Our task is to find the minimum number of passports required to visit all countries.

```
passport_data = CSV.read(
    joinpath(DATA_DIR, "passport-index-matrix.csv"),
    DataFrames.DataFrame,
)
```

	Passport	Afghanistan	Albania	Algeria	Andorra	Angola	Antigua and Barbuda	Arg
	String	Int64	Int64	Int64	Int64	Int64	Int64	I
1	Afghanistan	-1	0	0	0	0	0	
2	Albania	0	-1	0	3	0	3	
3	Algeria	0	0	-1	0	1	0	
4	Andorra	0	3	0	-1	0	3	
5	Angola	0	0	0	0	-1	0	
6	Antigua and Barbuda	0	3	0	3	0	-1	
7	Argentina	0	3	0	3	1	3	
8	Armenia	0	3	0	0	0	3	
9	Australia	0	3	0	3	1	3	
10	Austria	0	3	0	3	1	3	
11	Azerbaijan	0	3	0	0	0	3	
12	Bahamas	0	3	0	3	0	3	
13	Bahrain	0	3	0	0	0	0	
14	Bangladesh	0	0	0	0	0	0	
15	Barbados	0	3	0	3	0	3	
16	Belarus	0	3	0	0	0	3	
17	Belgium	0	3	0	3	1	3	
18	Belize	0	0	0	0	0	3	
19	Benin	0	0	0	0	0	0	
20	Bhutan	0	0	0	0	0	0	
21	Bolivia	0	0	0	0	0	0	
22	Bosnia and Herzegovina	0	3	0	3	0	0	
23	Botswana	0	0	0	0	3	3	
24	Brazil	0	3	0	3	1	3	
25	Brunei	0	3	0	3	0	3	
26	Bulgaria	0	3	0	3	1	3	
27	Burkina Faso	0	0	0	0	0	0	
28	Burundi	0	0	0	0	0	0	
29	Cambodia	0	0	0	0	0	0	
30	Cameroon	0	0	0	0	0	0	
31	Canada	0	3	0	3	1	3	
32	Cape Verde	0	0	0	0	1	0	
33	Central African Republic	0	0	0	0	0	0	
34	Chad	0	0	0	0	0	0	
35	Chile	0	3	0	3	1	3	
36	China	0	3	0	0	1	3	
37	Colombia	0	3	0	3	0	3	
38	Comoros	0	0	0	0	0	0	
39	Congo	0	0	0	0	0	0	
40	DR Congo	0	0	0	0	0	0	
41	Costa Rica	0	3	0	3	0	0	
42	Ivory Coast	0	0	0	0	0	0	
43	Croatia	0	3	0	3	1	3	
44	Cuba	0	0	0	0	1	3	
45	Cyprus	0	3	0	3	1	3	
46	Czech Republic	0	3	0	3	1	3	
47	Denmark	0	3	0	3	1	3	
48	Djibouti	0	0	0	0	0	0	
49	Dominica	0	0	0	3	0	3	
50	Dominican Republic	0	0	0	0	0	0	
51	Ecuador	0	0	0	0	0	0	
52	Egypt	0	0	0	0	0	0	
53	El Salvador	0	3	0	3	0	0	
54	Equatorial Guinea	0	0	0	0	0	0	
55	Eritrea	0	0	0	0	0	0	

In this dataset, the first column represents a passport (=from) and each remaining column represents a foreign country (=to).

The values in each cell are as follows:

- 3 = visa-free travel
- 2 = eTA is required
- 1 = visa can be obtained on arrival
- 0 = visa is required
- -1 is for all instances where passport and destination are the same

Our task is to find out the minimum number of passports needed to visit every country without requiring a visa.

The values we are interested in are -1 and 3. Let's modify the dataframe so that the -1 and 3 are 1 (true), and all others are `0` (false):

```
function modifier(x)
    if x == -1 || x == 3
        return 1
    else
        return 0
    end
end

for country in passport_data.Passport
    passport_data[!, country] = modifier.(passport_data[!, country])
end

passport_data
```

	Passport	Afghanistan	Albania	Algeria	Andorra	Angola	Antigua and Barbuda	Arg
	String	Int64	Int64	Int64	Int64	Int64	Int64	I
1	Afghanistan	1	0	0	0	0	0	
2	Albania	0	1	0	1	0	1	
3	Algeria	0	0	1	0	0	0	
4	Andorra	0	1	0	1	0	1	
5	Angola	0	0	0	0	1	0	
6	Antigua and Barbuda	0	1	0	1	0	1	
7	Argentina	0	1	0	1	0	1	
8	Armenia	0	1	0	0	0	1	
9	Australia	0	1	0	1	0	1	
10	Austria	0	1	0	1	0	1	
11	Azerbaijan	0	1	0	0	0	1	
12	Bahamas	0	1	0	1	0	1	
13	Bahrain	0	1	0	0	0	0	
14	Bangladesh	0	0	0	0	0	0	
15	Barbados	0	1	0	1	0	1	
16	Belarus	0	1	0	0	0	1	
17	Belgium	0	1	0	1	0	1	
18	Belize	0	0	0	0	0	1	
19	Benin	0	0	0	0	0	0	
20	Bhutan	0	0	0	0	0	0	
21	Bolivia	0	0	0	0	0	0	
22	Bosnia and Herzegovina	0	1	0	1	0	0	
23	Botswana	0	0	0	0	1	1	
24	Brazil	0	1	0	1	0	1	
25	Brunei	0	1	0	1	0	1	
26	Bulgaria	0	1	0	1	0	1	
27	Burkina Faso	0	0	0	0	0	0	
28	Burundi	0	0	0	0	0	0	
29	Cambodia	0	0	0	0	0	0	
30	Cameroon	0	0	0	0	0	0	
31	Canada	0	1	0	1	0	1	
32	Cape Verde	0	0	0	0	0	0	
33	Central African Republic	0	0	0	0	0	0	
34	Chad	0	0	0	0	0	0	
35	Chile	0	1	0	1	0	1	
36	China	0	1	0	0	0	1	
37	Colombia	0	1	0	1	0	1	
38	Comoros	0	0	0	0	0	0	
39	Congo	0	0	0	0	0	0	
40	DR Congo	0	0	0	0	0	0	
41	Costa Rica	0	1	0	1	0	0	
42	Ivory Coast	0	0	0	0	0	0	
43	Croatia	0	1	0	1	0	1	
44	Cuba	0	0	0	0	0	1	
45	Cyprus	0	1	0	1	0	1	
46	Czech Republic	0	1	0	1	0	1	
47	Denmark	0	1	0	1	0	1	
48	Djibouti	0	0	0	0	0	0	
49	Dominica	0	0	0	1	0	1	
50	Dominican Republic	0	0	0	0	0	0	
51	Ecuador	0	0	0	0	0	0	
52	Egypt	0	0	0	0	0	0	
53	El Salvador	0	1	0	1	0	0	
54	Equatorial Guinea	0	0	0	0	0	0	
55	Eritrea	0	0	0	0	0	0	

The values in the cells now represent:

- 1 = no visa required for travel
- 0 = visa required for travel

JuMP Modeling

To model the problem as a mixed-integer linear program, we need a binary decision variable x_c for each country c. x_c is 1 if we select passport c and 0 otherwise. Our objective is to minimize the sum $\sum x_c$ over all countries.

Since we wish to visit all the countries, for every country, we must own at least one passport that lets us travel to that country visa free. For one destination, this can be mathematically represented as $\sum_{c \in C} a_{c,d} \cdot x_d \geq 1$, where a is the `passport_data` dataframe.

Thus, we can represent this problem using the following model:

$$\begin{aligned} \min \quad & \sum_{c \in C} x_c \\ \text{s.t.} \quad & \sum_{c \in C} a_{c,d} x_c \geq 1 \quad \forall d \in C \\ & x_c \in \{0,1\} \quad \forall c \in C. \end{aligned}$$

We'll now solve the problem using JuMP:

```
using JuMP
import HiGHS
```

First, create the set of countries:

```
C = passport_data.Passport
```

```
199-element Vector{String}:
 "Afghanistan"
 "Albania"
 "Algeria"
 "Andorra"
 "Angola"
 "Antigua and Barbuda"
 "Argentina"
 "Armenia"
 "Australia"
 "Austria"
 ⋮
 "Uruguay"
 "Uzbekistan"
 "Vanuatu"
 "Vatican"
 "Venezuela"
 "Viet Nam"
 "Yemen"
 "Zambia"
 "Zimbabwe"
```

Then, create the model and initialize the decision variables:

```
model = Model(HiGHS.Optimizer)
@variable(model, x[C], Bin)
@objective(model, Min, sum(x))
@constraint(model, [d in C], passport_data[!, d]' * x >= 1)
model
```

```
A JuMP Model
Minimization problem with:
Variables: 199
Objective function type: AffExpr
`AffExpr`-in-`MathOptInterface.GreaterThan{Float64}`: 199 constraints
`VariableRef`-in-`MathOptInterface.ZeroOne`: 199 constraints
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
Names registered in the model: x
```

Now optimize!

```
optimize!(model)
```

```
Presolving model
161 rows, 190 cols, 8925 nonzeros
97 rows, 168 cols, 2714 nonzeros
30 rows, 46 cols, 185 nonzeros
Objective function is integral with scale 1

Solving MIP model with:
   30 rows
   46 cols (38 binary, 8 integer, 0 implied int., 0 continuous)
   185 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) Found 1 generators and 2 full orbitope(s) acting on 4 columns

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

         0       0         0   0.00%   15              inf                  inf        0      0
      0         0     0.0s
 T       0       0         0   0.00%   15              23                34.78%        0      0
      0        27     0.0s

Solving report
  Status            Optimal
  Primal bound      23
  Dual bound        23
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    23 (objective)
                    0 (bound viol.)
```

```
                    0 (int. viol.)
                    0 (row viol.)
  Timing            0.01 (total)
                    0.01 (presolve)
                    0.00 (postsolve)
  Nodes             1
  LP iterations     27 (total)
                    0 (strong br.)
                    0 (separation)
                    0 (heuristics)
```

We can use the `solution_summary` function to get an overview of the solution:

```
solution_summary(model)
```

```
* Solver : HiGHS

* Status
  Termination status : OPTIMAL
  Primal status      : FEASIBLE_POINT
  Dual status        : NO_SOLUTION
  Message from the solver:
  "kHighsModelStatusOptimal"

* Candidate solution
  Objective value      : 2.30000e+01
  Objective bound      : 2.30000e+01

* Work counters
  Solve time (sec)   : 6.90818e-03
  Simplex iterations : -1
  Barrier iterations : -1
```

Solution

Let's have a look at the solution in more detail:

```
println("Minimum number of passports needed: ", objective_value(model))
```

```
Minimum number of passports needed: 23.0
```

```
println("Optimal passports:")
for c in C
    if value(x[c]) > 0.5
        println(" * ", c)
    end
end
```

```
Optimal passports:
 * Afghanistan
 * Comoros
 * Djibouti
```

```
* Gabon
* Georgia
* Hong Kong
* India
* Jordan
* Madagascar
* Malaysia
* Maldives
* New Zealand
* Niger
* North Korea
* Papua New Guinea
* Somalia
* Sri Lanka
* Sweden
* Turkey
* Uganda
* United Arab Emirates
* United States
* Zimbabwe
```

Interesting! We need some passports, like Australia and the United States, which have widespread access to a large number of countries. However, we also need passports like North Korea which only have visa-free access to a very limited number of countries.

> **Note**
>
> We use `value(x[c]) > 0.5` rather than `value(x[c]) == 1` to avoid excluding solutions like `x[c] = 0.99999` that are "1" to some tolerance.

> **Tip**
>
> This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

4.6 Performance tips

By now you should have read the other "getting started" tutorials. You're almost ready to write your own models, but before you do so there are some important things to be aware of.

Read the Julia performance tips

The first thing to do is read the Performance tips section of the Julia manual. The most important rule is to avoid global variables! This is particularly important if you're learning JuMP after using a language like MATLAB.

The "time-to-first-solve" issue

Similar to the infamous time-to-first-plot plotting problem, JuMP suffers from time-to-first-solve latency. This latency occurs because the first time you call JuMP code in each session, Julia needs to compile a lot of code specific to your problem. This issue is actively being worked on, but there are a few things you can do to improve things.

Don't call JuMP from the command line

In other languages, you might be used to a workflow like:

```
$ julia my_script.jl
```

This doesn't work for JuMP, because we have to pay the compilation latency every time you run the script. Instead, use one of the suggested workflows from the Julia documentation.

Disable bridges if none are being used

At present, the majority of the latency problems are caused by JuMP's bridging mechanism. If you only use constraints that are natively supported by the solver, you can disable bridges by passing `add_bridges = false` to `Model`.

```julia
model = Model(HiGHS.Optimizer; add_bridges = false)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Use PackageCompiler

As an example of compilation latency, consider the following linear program with two variables and two constraints:

```julia
using JuMP, HiGHS
model = Model(HiGHS.Optimizer)
@variable(model, x >= 0)
@variable(model, 0 <= y <= 3)
@objective(model, Min, 12x + 20y)
@constraint(model, c1, 6x + 8y >= 100)
@constraint(model, c2, 7x + 12y >= 120)
optimize!(model)
open("model.log", "w") do io
    print(io, solution_summary(model; verbose = true))
    return
end
```

```
Presolving model
2 rows, 2 cols, 4 nonzeros
2 rows, 2 cols, 4 nonzeros
Presolve : Reductions: rows 2(-0); columns 2(-0); elements 4(-0)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 2(220) 0s
          2     2.0500000000e+02 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 2
Objective value     :  2.0500000000e+02
HiGHS run time      :          0.00
```

Saving the problem in `model.jl` and calling from the command line results in:

```
$ time julia model.jl
15.78s user 0.48s system 100% cpu 16.173 total
```

Clearly, 16 seconds is a large overhead to pay for solving this trivial model. However, the compilation latency is independent on the problem size, and so 16 seconds of additional overhead may be tolerable for larger models that take minutes or hours to solve.

In cases where the compilation latency is intolerable, JuMP is compatible with the PackageCompiler.jl package, which makes it easy to generate a custom sysimage (a binary extension to Julia that caches compiled code) that dramatically reduces the compilation latency. A custom image for our problem can be created as follows:

```julia
using PackageCompiler, Libdl
PackageCompiler.create_sysimage(
    ["JuMP", "HiGHS"],
    sysimage_path = "customimage." * Libdl.dlext,
    precompile_execution_file = "model.jl",
)
```

When Julia is run with the custom image, the run time is now 0.7 seconds instead of 16:

```
$ time julia --sysimage customimage model.jl
0.68s user 0.22s system 153% cpu 0.587 total
```

Other performance tweaks, such as disabling bridges or using direct mode can reduce this time futher.

> **Note**
>
> `create_sysimage` only needs to be run once, and the same sysimage can be used–to a slight detriment of performance–even if we modify `model.jl` or run a different file.

Use macros to build expressions

What

Use JuMP's macros (or `add_to_expression!`) to build expressions. Avoid constructing expressions outside the macros.

Why

Constructing an expression outside the macro results in intermediate copies of the expression. For example,

```julia
x[1] + x[2] + x[3]
```

is equivalent to

```julia
a = x[1]
b = a + x[2]
c = b + x[3]
```

Since we only care about `c`, the `a` and `b` expressions are not needed and constructing them slows the program down!

JuMP's macros rewrite the expressions to operate in-place and avoid these extra copies. Because they allocate less memory, they are faster, particularly for large expressions.

Example

```
model = Model()
@variable(model, x[1:3])
```

```
3-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]
```

Here's what happens if we construct the expression outside the macro:

```
@allocated x[1] + x[2] + x[3]
```

```
1344
```

> **Info**
>
> The `@allocated` measures how many bytes were allocated during the evaluation of an expression. Fewer is better.

If we use the `@expression` macro, we get many fewer allocations:

```
@allocated @expression(model, x[1] + x[2] + x[3])
```

```
800
```

> **Tip**
>
> This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

4.7 Design patterns for larger models

JuMP makes it easy to build and solve optimization models. However, once you start to construct larger models, and especially ones that interact with external data sources or have customizable sets of variables and constraints based on client choices, you may find that your scripts become unwieldy. This tutorial demonstrates a variety of ways in which you can structure larger JuMP models to improve their readability and maintainability.

> **Tip**
>
> This tutorial is more advanced than the other "Getting started" tutorials. It's in the "Getting started" section to give you an early preview of how JuMP makes it easy to structure larger models. However, if you are new to JuMP you may want to briefly skim the tutorial, and come back to it once you have written a few JuMP models.

Overview

This tutorial uses explanation-by-example. We're going to start with a simple knapsack model, and then expand it to add various features and structure.

A simple script

Your first prototype of a JuMP model is probably a script that uses a small set of hard-coded data.

```
using JuMP, HiGHS
profit = [5, 3, 2, 7, 4]
weight = [2, 8, 4, 2, 5]
capacity = 10
N = 5
model = Model(HiGHS.Optimizer)
@variable(model, x[1:N], Bin)
@objective(model, Max, sum(profit[i] * x[i] for i in 1:N))
@constraint(model, sum(weight[i] * x[i] for i in 1:N) <= capacity)
optimize!(model)
value.(x)
```

```
5-element Vector{Float64}:
  1.0
  0.0
 -0.0
  1.0
  1.0
```

The benefits of this approach are:

- it is quick to code
- it is quick to make changes.

The downsides include:

- all variables are global (read Performance tips)
- it is easy to introduce errors, e.g., having `profit` and `weight` be vectors of different lengths, or not match `N`
- the solution, `x[i]`, is hard to interpret without knowing the order in which we provided the data.

Wrap the model in a function

A good next step is to wrap your model in a function. This is useful for a few reasons:

- it removes global variables
- it encapsulates the JuMP model and forces you to clarify your inputs and outputs
- we can add some error checking.

```
function solve_knapsack_1(profit::Vector, weight::Vector, capacity::Real)
    if length(profit) != length(weight)
        throw(DimensionMismatch("profit and weight are different sizes"))
    end
    N = length(weight)
    model = Model(HiGHS.Optimizer)
```

```
    @variable(model, x[1:N], Bin)
    @objective(model, Max, sum(profit[i] * x[i] for i in 1:N))
    @constraint(model, sum(weight[i] * x[i] for i in 1:N) <= capacity)
    optimize!(model)
    return value.(x)
end

solve_knapsack_1([5, 3, 2, 7, 4], [2, 8, 4, 2, 5], 10)
```

```
5-element Vector{Float64}:
  1.0
  0.0
 -0.0
  1.0
  1.0
```

Create better data structures

Although we can check for errors like mis-matched vector lengths, if you start to develop models with a lot of data, keeping track of vectors and lengths and indices is fragile and a common source of bugs. A good solution is to use Julia's type system to create an abstraction over your data.

For example, we can create a `struct` that represents a single object, with a constructor that lets us validate assumptions on the input data:

```
struct KnapsackObject
    profit::Float64
    weight::Float64
    function KnapsackObject(profit::Float64, weight::Float64)
        if weight < 0
            throw(DomainError("Weight of object cannot be negative"))
        end
        return new(profit, weight)
    end
end
```

as well as a `struct` that holds a dictionary of objects and the knapsack's capacity:

```
struct KnapsackData
    objects::Dict{String,KnapsackObject}
    capacity::Float64
end
```

Here's what our data might look like now:

```
objects = Dict(
    "apple" => KnapsackObject(5.0, 2.0),
    "banana" => KnapsackObject(3.0, 8.0),
    "cherry" => KnapsackObject(2.0, 4.0),
    "date" => KnapsackObject(7.0, 2.0),
    "eggplant" => KnapsackObject(4.0, 5.0),
)
data = KnapsackData(objects, 10.0)
```

```
Main.KnapsackData(Dict{String, Main.KnapsackObject}("cherry" => Main.KnapsackObject(2.0, 4.0), "
     banana" => Main.KnapsackObject(3.0, 8.0), "date" => Main.KnapsackObject(7.0, 2.0), "eggplant" =>
      Main.KnapsackObject(4.0, 5.0), "apple" => Main.KnapsackObject(5.0, 2.0)), 10.0)
```

If you want, you can add custom printing to make it easier to visualize:

```
function Base.show(io::IO, data::KnapsackData)
    println(io, "A knapsack with capacity $(data.capacity) and possible items:")
    for (k, v) in data.objects
        println(
            io,
            "  $(rpad(k, 8)) : profit = $(v.profit), weight = $(v.weight)",
        )
    end
    return
end

data
```

```
A knapsack with capacity 10.0 and possible items:
  cherry   : profit = 2.0, weight = 4.0
  banana   : profit = 3.0, weight = 8.0
  date     : profit = 7.0, weight = 2.0
  eggplant : profit = 4.0, weight = 5.0
  apple    : profit = 5.0, weight = 2.0
```

Then, we can re-write our `solve_knapsack` function to take our `KnapsackData` as input:

```
function solve_knapsack_2(data::KnapsackData)
    model = Model(HiGHS.Optimizer)
    @variable(model, x[keys(data.objects)], Bin)
    @objective(model, Max, sum(v.profit * x[k] for (k, v) in data.objects))
    @constraint(
        model,
        sum(v.weight * x[k] for (k, v) in data.objects) <= data.capacity,
    )
    optimize!(model)
    return value.(x)
end

solve_knapsack_2(data)
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 -0.0
  0.0
  1.0
  1.0
  1.0
```

Read in data from files

Having a data structure is a good step. But it is still annoying that we have to hard-code the data into Julia. A good next step is to separate the data into an external file format; JSON is a common choice.

The JuMP repository has a file we're going to use for this tutorial. To run this tutorial locally, download the file and then update `data_filename` as appropriate.

To build this version of the JuMP documentation, we needed to set the filename:

```
data_filename = joinpath(@__DIR__, "data", "knapsack.json");
```

`knapsack.json` has the following contents:

```
println(read(data_filename, String))
```

```
{
    "objects": {
        "apple": {"profit": 5.0, "weight": 2.0},
        "banana": {"profit": 3.0, "weight": 8.0},
        "cherry": {"profit": 2.0, "weight": 4.0},
        "date": {"profit": 7.0, "weight": 2.0},
        "eggplant": {"profit": 4.0, "weight": 5.0}
    },
    "capacity": 10.0
}
```

Now let's write a function that reads this file and builds a `KnapsackData` object:

```
import JSON

function read_data(filename)
    d = JSON.parsefile(filename)
    return KnapsackData(
        Dict(
            k => KnapsackObject(v["profit"], v["weight"]) for
            (k, v) in d["objects"]
        ),
        d["capacity"],
    )
end

data = read_data(data_filename)
```

```
A knapsack with capacity 10.0 and possible items:
  cherry   : profit = 2.0, weight = 4.0
  banana   : profit = 3.0, weight = 8.0
  date     : profit = 7.0, weight = 2.0
  eggplant : profit = 4.0, weight = 5.0
  apple    : profit = 5.0, weight = 2.0
```

Add options via if-else

At this point, we have data in a file format which we can load and solve a single problem. For many users, this might be sufficient. However, at some point you may be asked to add features like "but what if I want to take more than one of a particular item?"

If this is the first time that you've been asked to add a feature, adding options via `if-else` statements is a good approach. For example, we might write:

```
function solve_knapsack_3(data::KnapsackData; binary_knapsack::Bool)
    model = Model(HiGHS.Optimizer)
    if binary_knapsack
        @variable(model, x[keys(data.objects)], Bin)
    else
        @variable(model, x[keys(data.objects)] >= 0, Int)
    end
    @objective(model, Max, sum(v.profit * x[k] for (k, v) in data.objects))
    @constraint(
        model,
        sum(v.weight * x[k] for (k, v) in data.objects) <= data.capacity,
    )
    optimize!(model)
    return value.(x)
end
```

```
solve_knapsack_3 (generic function with 1 method)
```

Now we can solve the binary knapsack:

```
solve_knapsack_3(data; binary_knapsack = true)
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 -0.0
  0.0
  1.0
  1.0
  1.0
```

And an integer knapsack where we can take more than one copy of each item:

```
solve_knapsack_3(data; binary_knapsack = false)
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 0.0
 0.0
 5.0
 0.0
 0.0
```

Add configuation options via dispatch

If you get repeated requests to add different options, you'll quickly find yourself in a mess of different flags and `if-else` statements. It's hard to write, hard to read, and hard to ensure you haven't introduced any bugs. A good solution is to use Julia's type dispatch to control the configuration of the model. The easiest way to explain this is by example.

First, start by defining a new abstract type, as well as new subtypes for each of our options. These types are going to control the configuration of the knapsack model.

```
abstract type AbstractConfiguration end

struct BinaryKnapsackConfig <: AbstractConfiguration end

struct IntegerKnapsackConfig <: AbstractConfiguration end
```

Then, we rewrite our `solve_knapsack` function to take a `config` argument, and we introduce an `add_knapsack_variables` function to abstract the creation of our variables.

```
function solve_knapsack_4(data::KnapsackData, config::AbstractConfiguration)
    model = Model(HiGHS.Optimizer)
    x = add_knapsack_variables(model, data, config)
    @objective(model, Max, sum(v.profit * x[k] for (k, v) in data.objects))
    @constraint(
        model,
        sum(v.weight * x[k] for (k, v) in data.objects) <= data.capacity,
    )
    optimize!(model)
    return value.(x)
end
```

```
solve_knapsack_4 (generic function with 1 method)
```

For the binary knapsack problem, `add_knapsack_variables` looks like this:

```
function add_knapsack_variables(
    model::Model,
    data::KnapsackData,
    ::BinaryKnapsackConfig,
)
    return @variable(model, x[keys(data.objects)], Bin)
end
```

```
add_knapsack_variables (generic function with 1 method)
```

For the integer knapsack problem, `add_knapsack_variables` looks like this:

```
function add_knapsack_variables(
    model::Model,
    data::KnapsackData,
    ::IntegerKnapsackConfig,
)
    return @variable(model, x[keys(data.objects)] >= 0, Int)
end
```

```
add_knapsack_variables (generic function with 2 methods)
```

Now we can solve the binary knapsack:

```
solve_knapsack_4(data, BinaryKnapsackConfig())
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 -0.0
  0.0
  1.0
  1.0
  1.0
```

and the integer knapsack problem:

```
solve_knapsack_4(data, IntegerKnapsackConfig())
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 0.0
 0.0
 5.0
 0.0
 0.0
```

The main benefit of the dispatch approach is that you can quickly add new options without needing to modify the existing code. For example:

```julia
struct UpperBoundedKnapsackConfig <: AbstractConfiguration
    limit::Int
end

function add_knapsack_variables(
    model::Model,
    data::KnapsackData,
    config::UpperBoundedKnapsackConfig,
)
    return @variable(model, 0 <= x[keys(data.objects)] <= config.limit, Int)
end

solve_knapsack_4(data, UpperBoundedKnapsackConfig(3))
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 0.0
 0.0
 3.0
 0.0
 2.0
```

Generalize constraints and objectives

It's easy to extend the dispatch approach to constraints and objectives as well. The key points to notice in the next two functions are that:

- we can access registered variables via `model[:x]`
- we can define generic functions which accept any `AbstractConfiguration` as a configuration argument. That means we can implement a single method and have it apply to multiple configuration types.

```julia
function add_knapsack_constraints(
    model::Model,
    data::KnapsackData,
    ::AbstractConfiguration,
)
    x = model[:x]
    @constraint(
        model,
        capacity_constraint,
        sum(v.weight * x[k] for (k, v) in data.objects) <= data.capacity,
    )
    return
end

function add_knapsack_objective(
    model::Model,
    data::KnapsackData,
    ::AbstractConfiguration,
)
    x = model[:x]
    @objective(model, Max, sum(v.profit * x[k] for (k, v) in data.objects))
    return
end

function solve_knapsack_5(data::KnapsackData, config::AbstractConfiguration)
    model = Model(HiGHS.Optimizer)
    add_knapsack_variables(model, data, config)
    add_knapsack_constraints(model, data, config)
    add_knapsack_objective(model, data, config)
    optimize!(model)
    return value.(model[:x])
end

solve_knapsack_5(data, BinaryKnapsackConfig())
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 -0.0
  0.0
  1.0
  1.0
  1.0
```

Remove solver dependence, add error checks

Compared to where we started, our knapsack model is now significantly different. We've wrapped it in a function, defined some data types, and introduced configuration options to control the variables and constraints that get added. There are a few other steps we can do to further improve things:

- remove the dependence on `HiGHS`
- add checks that we found an optimal solution
- add a helper function to avoid the need to explicitly construct the data.

```
function solve_knapsack_6(
    optimizer,
    data::KnapsackData,
    config::AbstractConfiguration,
)
    model = Model(optimizer)
    add_knapsack_variables(model, data, config)
    add_knapsack_constraints(model, data, config)
    add_knapsack_objective(model, data, config)
    optimize!(model)
    if termination_status(model) != OPTIMAL
        @warn("Model not solved to optimality")
        return nothing
    end
    return value.(model[:x])
end

function solve_knapsack_6(
    optimizer,
    data::String,
    config::AbstractConfiguration,
)
    return solve_knapsack_6(optimizer, read_data(data), config)
end

solution =
    solve_knapsack_6(HiGHS.Optimizer, data_filename, BinaryKnapsackConfig())
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 -0.0
  0.0
  1.0
  1.0
  1.0
```

Create a module

Now we're ready to expose our model to the wider world. That might be as part of a larger Julia project that we're contributing to, or as a stand-alone script that we can run on-demand. In either case, it's good practice to wrap everything in a module. This further encapsulates our code into a single namespace, and we can add documentation in the form of docstrings.

Some good rules to follow when creating a module are:

- use `import` in a module instead of `using` to make it clear which functions are from which packages
- use _ to start function and type names that are considered private
- add docstrings to all public variables and functions.

```julia
module KnapsackModel

import JuMP
import JSON

struct _KnapsackObject
    profit::Float64
    weight::Float64
    function _KnapsackObject(profit::Float64, weight::Float64)
        if weight < 0
            throw(DomainError("Weight of object cannot be negative"))
        end
        return new(profit, weight)
    end
end

struct _KnapsackData
    objects::Dict{String,_KnapsackObject}
    capacity::Float64
end

function _read_data(filename)
    d = JSON.parsefile(filename)
    return _KnapsackData(
        Dict(
            k => _KnapsackObject(v["profit"], v["weight"]) for
            (k, v) in d["objects"]
        ),
        d["capacity"],
    )
end

abstract type _AbstractConfiguration end

"""
    BinaryKnapsackConfig()

Create a binary knapsack problem where each object can be taken 0 or 1 times.
"""
struct BinaryKnapsackConfig <: _AbstractConfiguration end

"""
    IntegerKnapsackConfig()

Create an integer knapsack problem where each object can be taken any number of
times.
"""
struct IntegerKnapsackConfig <: _AbstractConfiguration end
```

```julia
function _add_knapsack_variables(
    model::JuMP.Model,
    data::_KnapsackData,
    ::BinaryKnapsackConfig,
)
    return JuMP.@variable(model, x[keys(data.objects)], Bin)
end

function _add_knapsack_variables(
    model::JuMP.Model,
    data::_KnapsackData,
    ::IntegerKnapsackConfig,
)
    return JuMP.@variable(model, x[keys(data.objects)] >= 0, Int)
end

function _add_knapsack_constraints(
    model::JuMP.Model,
    data::_KnapsackData,
    ::_AbstractConfiguration,
)
    x = model[:x]
    JuMP.@constraint(
        model,
        capacity_constraint,
        sum(v.weight * x[k] for (k, v) in data.objects) <= data.capacity,
    )
    return
end

function _add_knapsack_objective(
    model::JuMP.Model,
    data::_KnapsackData,
    ::_AbstractConfiguration,
)
    x = model[:x]
    JuMP.@objective(model, Max, sum(v.profit * x[k] for (k, v) in data.objects))
    return
end

function _solve_knapsack(
    optimizer,
    data::_KnapsackData,
    config::_AbstractConfiguration,
)
    model = JuMP.Model(optimizer)
    _add_knapsack_variables(model, data, config)
    _add_knapsack_constraints(model, data, config)
    _add_knapsack_objective(model, data, config)
    JuMP.optimize!(model)
    if JuMP.termination_status(model) != JuMP.OPTIMAL
        @warn("Model not solved to optimality")
        return nothing
    end
    return JuMP.value.(model[:x])
```

```julia
end

"""
    solve_knapsack(
        optimizer,
        data_filename::String,
        config::_AbstractConfiguration,
    )

Solve the knapsack problem and return the optimal primal solution

# Arguments

 * `optimizer` : an object that can be passed to `JuMP.Model` to construct a new
   JuMP model.
 * `data_filename` : the filename of a JSON file containing the data for the
   problem.
 * `config` : an object to control the type of knapsack model constructed.
   Valid options are:
    * `BinaryKnapsackConfig()`
    * `IntegerKnapsackConfig()`

# Returns

 * If an optimal solution exists: a `JuMP.DenseAxisArray` that maps the `String`
   name of each object to the number of objects to pack into the knapsack.
 * Otherwise, `nothing`, indicating that the problem does not have an optimal
   solution.

# Examples

```julia
solution = solve_knapsack(
    HiGHS.Optimizer,
    "path/to/data.json",
    BinaryKnapsackConfig(),
)
```

```julia
solution = solve_knapsack(
    MOI.OptimizerWithAttributes(HiGHS.Optimizer, "output_flag" => false),
    "path/to/data.json",
    IntegerKnapsackConfig(),
)
```
"""
function solve_knapsack(
    optimizer,
    data_filename::String,
    config::_AbstractConfiguration,
)
    return _solve_knapsack(optimizer, _read_data(data_filename), config)
end

end
```

```
Main.KnapsackModel
```

Finally, you can call your model:

```
import .KnapsackModel

KnapsackModel.solve_knapsack(
    HiGHS.Optimizer,
    joinpath(@__DIR__, "data", "knapsack.json"),
    KnapsackModel.BinaryKnapsackConfig(),
)
```

```
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, ["cherry", "banana", "date", "eggplant", "apple"]
And data, a 5-element Vector{Float64}:
 -0.0
  0.0
  1.0
  1.0
  1.0
```

Note

The `.` in `.KnapsackModel` denotes that it is a submodule and not a separate package that we installed with `Pkg.add`. If you put the `KnapsackModel` in a separate file, load it with:

```
include("path/to/KnapsackModel.jl")
import .KnapsackModel
```

Add tests

As a final step, you should add tests for your model. This often means testing on a small problem for which you can work out the optimal solution by hand. The Julia standard library `Test` has good unit-testing functionality.

```
import .KnapsackModel
using Test

@testset "KnapsackModel" begin
    @testset "feasible_binary_knapsack" begin
        x = KnapsackModel.solve_knapsack(
            HiGHS.Optimizer,
            joinpath(@__DIR__, "data", "knapsack.json"),
            KnapsackModel.BinaryKnapsackConfig(),
        )
        @test isapprox(x["apple"], 1, atol = 1e-5)
        @test isapprox(x["banana"], 0, atol = 1e-5)
        @test isapprox(x["cherry"], 0, atol = 1e-5)
        @test isapprox(x["date"], 1, atol = 1e-5)
        @test isapprox(x["eggplant"], 1, atol = 1e-5)
    end
    @testset "feasible_integer_knapsack" begin
        x = KnapsackModel.solve_knapsack(
            HiGHS.Optimizer,
            joinpath(@__DIR__, "data", "knapsack.json"),
```

```
            KnapsackModel.IntegerKnapsackConfig(),
        )
        @test isapprox(x["apple"], 0, atol = 1e-5)
        @test isapprox(x["banana"], 0, atol = 1e-5)
        @test isapprox(x["cherry"], 0, atol = 1e-5)
        @test isapprox(x["date"], 5, atol = 1e-5)
        @test isapprox(x["eggplant"], 0, atol = 1e-5)
    end
    @testset "infeasible_binary_knapsack" begin
        x = KnapsackModel.solve_knapsack(
            HiGHS.Optimizer,
            # This file contains data that makes the problem infeasible.
            joinpath(@__DIR__, "data", "knapsack_infeasible.json"),
            KnapsackModel.BinaryKnapsackConfig(),
        )
        @test x === nothing
    end
end
```

```
Test.DefaultTestSet("KnapsackModel", Any[Test.DefaultTestSet("feasible_binary_knapsack", Any[], 5,
    false, false), Test.DefaultTestSet("feasible_integer_knapsack", Any[], 5, false, false), Test.
    DefaultTestSet("infeasible_binary_knapsack", Any[], 1, false, false)], 0, false, false)
```

Tip

Place these tests in a separate file `test_knapsack_model.jl` so that you can run the tests by adding `include("test_knapsack_model.jl")` to any file where needed.

Next steps

We've only briefly scratched the surface of ways to create and structure large JuMP models, so consider this tutorial a starting point, rather than a comprehensive list of all the possible ways to structure JuMP models. If you are embarking on a large project that uses JuMP, a good next step is to look at ways people have written large JuMP projects "in the wild".

Here are some good examples (all co-incidentally related to energy):

- AnyMOD.jl
  - JuMP-dev 2021 talk
  - source code
- PowerModels.jl
  - JuMP-dev 2021 talk
  - source code
- PowerSimulations.jl
  - JuliaCon 2021 talk
  - source code
- UnitCommitment.jl

  - JuMP-dev 2021 talk
  - source code

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

Chapter 5

Linear programs

5.1 Introduction

Linear programs (LPs) are a fundamental class of optimization problems of the form:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} c_i x_i \tag{5.1}$$

$$\text{s.t.} l_j \le \sum_{i=1}^{n} a_{ij} x_i \le u_j \qquad j = 1 \ldots m \tag{5.2}$$

$$l_i \le x_i \le u_i \qquad i = 1 \ldots n. \tag{5.3}$$

The most important thing to note is that all terms are of the form `coefficient * variable`, and that there are no nonlinear terms or multiplications between variables.

Mixed-integer linear programs (MILPs) are extensions of linear programs in which some (or all) of the decision variables take discrete values.

How to choose a solver

Almost all solvers support linear programs; look for "LP" in the list of Supported solvers. However, fewer solvers support mixed-integer linear programs. Solvers supporting discrete variables start with "(MI)" in the list of Supported solvers.

How these tutorials are structured

Having a high-level overview of how this part of the documentation is structured will help you know where to look for certain things.

- The following tutorials are worked examples that present a problem in words, then formulate it in mathematics, and then solve it in JuMP. This usually involves some sort of visualization of the solution. Start here if you are new to JuMP.
  - The diet problem
  - The cannery problem
  - The facility location problem

  - Financial modeling problems
  - Network flow problems
  - N-Queens
  - Sudoku

- The Tips and tricks tutorial contains a number of helpful reformulations and tricks you can use when modeling linear programs. Look here if you are stuck trying to formulate a problem as a linear program.
- The Sensitivity analysis of a linear program tutorial explains how to create sensitivity reports like those produced by the Excel Solver.
- The Callbacks tutorial explains how to write a variety of solver-independent callbacks. Look here if you want to write a callback.
- The remaining tutorials are less verbose and styled in the form of short code examples. These tutorials have less explanation, but may contain useful code snippets, particularly if they are similar to a problem you are trying to solve.

5.2 Tips and tricks

Originally Contributed by: Arpit Bhatia

> **Tip**
>
> A good source of tips is the Mosek Modeling Cookbook.

This tutorial collates some tips and tricks you can use when formulating mixed-integer programs. It uses the following packages:

```
using JuMP
```

Boolean operators

Binary variables can be used to construct logical operators. Here are some example.

Or

$$x_3 = x_1 \lor x_2$$

```
model = Model()
@variable(model, x[1:3], Bin)
@constraints(model, begin
    x[1] <= x[3]
    x[2] <= x[3]
    x[3] <= x[1] + x[2]
end)
```

```
(x[1] - x[3] ≤ 0.0, x[2] - x[3] ≤ 0.0, -x[1] - x[2] + x[3] ≤ 0.0)
```

And

$$x_3 = x_1 \land x_2$$

```
model = Model()
@variable(model, x[1:3], Bin)
@constraints(model, begin
    x[3] <= x[1]
    x[3] <= x[2]
    x[3] >= x[1] + x[2] - 1
end)
```

```
(-x[1] + x[3] ≤ 0.0, -x[2] + x[3] ≤ 0.0, -x[1] - x[2] + x[3] ≥ -1.0)
```

Not

$$x_1 \neg x_2$$

```
model = Model()
@variable(model, x[1:2], Bin)
@constraint(model, x[1] == 1 - x[2])
```

$$x_1 + x_2 = 1.0$$

Implies

$$x_1 \implies x_2$$

```
model = Model()
@variable(model, x[1:2], Bin)
@constraint(model, x[1] <= x[2])
```

$$x_1 - x_2 \leq 0.0$$

Disjunctions

Problem

Suppose that we have two constraints $a^\top x \leq b$ and $c^\top x \leq d$, and we want at least one to hold.

Trick

Introduce a "big-M" multiplied by a binary variable to relax one of the constraints.

Example Either $x_1 \leq 1$ or $x_2 \leq 2$.

```
model = Model()
@variable(model, x[1:2])
@variable(model, y, Bin)
M = 100
@constraint(model, x[1] <= 1 + M * y)
@constraint(model, x[2] <= 2 + M * (1 - y))
```

$$x_2 + 100y \leq 102.0$$

Warning

If `M` is too small, the solution may be suboptimal. If `M` is too big, the solver may encounter numerical issues. Try to use domain knowledge to choose an `M` that is just right. Gurobi has a good documentation section on this topic.

Indicator constraints

Problem

Suppose we want to model that a certain linear inequality must be satisfied when some other event occurs, i.e., for a binary variable z, we want to model the implication:

$$z = 1 \implies a^T x \leq b$$

Trick 1

Some solvers have native support for indicator constraints.

Example $x_1 + x_2 \leq 1$ if $z = 1$.

```
model = Model()
@variable(model, x[1:2])
@variable(model, z, Bin)
@constraint(model, z => {sum(x) <= 1})
```

$$z => x_1 + x_2 \leq 1.0$$

Example $x_1 + x_2 \leq 1$ if $z = 0$.

```
model = Model()
@variable(model, x[1:2])
@variable(model, z, Bin)
@constraint(model, !z => {sum(x) <= 1})
```

$$!z => x_1 + x_2 \leq 1.0$$

Trick 2

If the solver doesn't support indicator constraints, you an use the big-M trick.

Example $x_1 + x_2 \leq 1$ if $z = 1$.

```
model = Model()
@variable(model, x[1:2])
@variable(model, z, Bin)
M = 100
@constraint(model, sum(x) <= 1 + M * (1 - z))
```

$$x_1 + x_2 + 100z \leq 101.0$$

Example $x_1 + x_2 \leq 1$ if $z = 0$.

```
model = Model()
@variable(model, x[1:2])
@variable(model, z, Bin)
M = 100
@constraint(model, sum(x) <= 1 + M * z)
```

$$x_1 + x_2 - 100z \leq 1.0$$

Semi-continuous variables

Info

This section uses sets from MathOptInterface. By default, JuMP exports the `MOI` symbol as an alias for the MathOptInterface.jl package. We recommend making this more explicit in your code by adding the following lines:

```
import MathOptInterface
const MOI = MathOptInterface
```

A semi-continuous variable is a continuous variable between bounds $[l, u]$ that also can assume the value zero. ie. $x \in \{0\} \cup [l, u]$.

Example $x \in \{0\} \cup [1, 2]$

```
model = Model()
@variable(model, x in MOI.Semicontinuous(1.0, 2.0))
```

$$x$$

Semi-integer variables

A semi-integer variable is a variable which assumes integer values between bounds $[l, u]$ and can also assume the value zero: $x \in \{0\} \cup [l, u] \cap \mathbb{Z}$.

```
model = Model()
@variable(model, x in MOI.Semiinteger(5.0, 10.0))
```

$$x$$

Special Ordered Sets of Type I

A Special Ordered Set of Type I is a set of variables, at most one of which can take a non-zero value, all others being at 0.

They most frequently apply where a set of variables are actually binary variables. In other words, we have to choose at most one from a set of possibilities.

```
model = Model()
@variable(model, x[1:3], Bin)
@constraint(model, x in SOS1())
```

$$[x_1, x_2, x_3] \in \text{MathOptInterface.SOS1\{Float64\}([1.0, 2.0, 3.0])}$$

You can optionally pass SOS1 a weight vector like

```
@constraint(model, x in SOS1([0.2, 0.5, 0.3]))
```

$$[x_1, x_2, x_3] \in \text{MathOptInterface.SOS1\{Float64\}([0.2, 0.5, 0.3])}$$

If the decision variables are related and have a physical ordering, then the weight vector, although not used directly in the constraint, can help the solver make a better decision in the solution process.

Special Ordered Sets of Type II

A Special Ordered Set of type 2 is a set of non-negative variables, of which at most two can be non-zero, and if two are non-zero these must be consecutive in their ordering.

```
model = Model()
@variable(model, x[1:3])
@constraint(model, x in SOS2([3.0, 1.0, 2.0]))
```

$$[x_1, x_2, x_3] \in \text{MathOptInterface.SOS2\{Float64\}([3.0, 1.0, 2.0])}$$

The ordering provided by the weight vector is more important in this case as the variables need to be consecutive according to the ordering. For example, in the above constraint, the possible pairs are:

- Consecutive
  - (`x[1]` and `x[3]`) as they correspond to 3 and 2 resp. and thus can be non-zero
  - (`x[2]` and `x[3]`) as they correspond to 1 and 2 resp. and thus can be non-zero
- Non-consecutive
  - (`x[1]` and `x[2]`) as they correspond to 3 and 1 resp. and thus cannot be non-zero

Piecewise linear approximations

SOSII constraints are most often used to form piecewise linear approximations of a function.

Given a set of points for x:

```
x̂ = -1:0.5:2
```

```
-1.0:0.5:2.0
```

and a set of corresponding points for y:

```
ŷ = x̂ .^ 2
```

```
7-element Vector{Float64}:
 1.0
 0.25
 0.0
 0.25
 1.0
 2.25
 4.0
```

the piecewise linear approximation is constructed by representing x and y as convex combinations of x̂ and ŷ.

```
N = length(x̂)
model = Model()
@variable(model, -1 <= x <= 2)
@variable(model, y)
@variable(model, 0 <= λ[1:N] <= 1)
@objective(model, Max, y)
@constraints(model, begin
    x == sum(x̂[i] * λ[i] for i in 1:N)
    y == sum(ŷ[i] * λ[i] for i in 1:N)
    sum(λ) == 1
    λ in SOS2()
end)
```

```
(x + λ[1] + 0.5 λ[2] - 0.5 λ[4] - λ[5] - 1.5 λ[6] - 2 λ[7] = 0.0, y - λ[1] - 0.25 λ[2] - 0.25 λ[4] -
     λ[5] - 2.25 λ[6] - 4 λ[7] = 0.0, λ[1] + λ[2] + λ[3] + λ[4] + λ[5] + λ[6] + λ[7] = 1.0, λ[[1],
    λ[2], λ[3], λ[4], λ[5], λ[6], λ[7]] ∈ MathOptInterface.SOS2{Float64}([1.0, 2.0, 3.0, 4.0, 5.0,
    6.0, 7.0]))
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.3 The diet problem

This tutorial solves the classic "diet problem", also known as the Stigler diet.

Required packages

```
using JuMP
import DataFrames
import HiGHS
```

Formulation

Suppose we wish to cook a nutritionally balanced meal by choosing the quantity of each food f to eat from a set of foods F in our kitchen.

Each food f has a cost, c_f, as well as a macronutrient profile $a_{m,f}$ for each macronutrient $m \in M$.

Because we care about a nutritionally balanced meal, we set some minimum and maximum limits for each nutrient, which we denote l_m and u_m respectively.

Furthermore, because we are optimizers, we seek the minimum cost solution.

With a little effort, we can formulate our dinner problem as the following linear program:

$$
\begin{aligned}
\min & \sum_{f \in F} c_f x_f \\
\text{s.t. } & l_m \leq \sum_{f \in F} a_{m,f} x_f \leq u_m, & \forall m \in M \\
& x_f \geq 0, & \forall f \in F
\end{aligned}
$$

In the rest of this tutorial, we will create and solve this problem in JuMP, and learn what we should cook for dinner.

Data

First, we need some data for the problem:

```
foods = DataFrames.DataFrame(
    [
        "hamburger" 2.49 410 24 26 730
        "chicken" 2.89 420 32 10 1190
        "hot dog" 1.50 560 20 32 1800
        "fries" 1.89 380 4 19 270
        "macaroni" 2.09 320 12 10 930
        "pizza" 1.99 320 15 12 820
        "salad" 2.49 320 31 12 1230
        "milk" 0.89 100 8 2.5 125
        "ice cream" 1.59 330 8 10 180
    ],
    ["name", "cost", "calories", "protein", "fat", "sodium"],
)
```

	name	cost	calories	protein	fat	sodium
	Any	Any	Any	Any	Any	Any
1	hamburger	2.49	410	24	26	730
2	chicken	2.89	420	32	10	1190
3	hot dog	1.5	560	20	32	1800
4	fries	1.89	380	4	19	270
5	macaroni	2.09	320	12	10	930
6	pizza	1.99	320	15	12	820
7	salad	2.49	320	31	12	1230
8	milk	0.89	100	8	2.5	125
9	ice cream	1.59	330	8	10	180

Here, F is `foods.name` and c_f is `foods.cost`. (We're also playing a bit loose the term "macronutrient" by including calories and sodium.)

Tip

Although we hard-coded the data here, you could also read it in from a file. See Getting started with data and plotting for details.

We also need our minimum and maximum limits:

```
limits = DataFrames.DataFrame(
    [
        "calories" 1800 2200
        "protein" 91 Inf
        "fat" 0 65
        "sodium" 0 1779
    ],
    ["name", "min", "max"],
)
```

	name	min	max
	Any	Any	Any
1	calories	1800	2200
2	protein	91	Inf
3	fat	0	65
4	sodium	0	1779

JuMP formulation

Now we're ready to convert our mathematical formulation into a JuMP model.

First, create a new JuMP model. Since we have a linear program, we'll use HiGHS as our optimizer:

```
model = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Next, we create a set of decision variables x, indexed over the foods in the `data` DataFrame. Each x has a lower bound of `0`.

```
@variable(model, x[foods.name] >= 0)
```

```
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, Any["hamburger", "chicken", "hot dog", "fries", "macaroni", "pizza", "salad", "milk
     ", "ice cream"]
And data, a 9-element Vector{VariableRef}:
 x[hamburger]
 x[chicken]
 x[hot dog]
 x[fries]
 x[macaroni]
 x[pizza]
 x[salad]
 x[milk]
 x[ice cream]
```

Our objective is to minimize the total cost of purchasing food. We can write that as a sum over the rows in `data`.

```
@objective(
    model,
    Min,
    sum(food["cost"] * x[food["name"]] for food in eachrow(foods)),
)
```

$$2.49x_{hamburger}+2.89x_{chicken}+1.5x_{hotdog}+1.89x_{fries}+2.09x_{macaroni}+1.99x_{pizza}+2.49x_{salad}+0.89x_{milk}+1.59x_{icecream}$$

For the next component, we need to add a constraint that our total intake of each component is within the limits contained in the `limits` DataFrame. To make this more readable, we introduce a JuMP `@expression`

```
for limit in eachrow(limits)
    intake = @expression(
        model,
        sum(food[limit["name"]] * x[food["name"]] for food in eachrow(foods)),
    )
    @constraint(model, limit.min <= intake <= limit.max)
end
```

What does our model look like?

```
print(model)
```

```
Min 2.49 x[hamburger] + 2.89 x[chicken] + 1.5 x[hot dog] + 1.89 x[fries] + 2.09 x[macaroni] + 1.99 x
     [pizza] + 2.49 x[salad] + 0.89 x[milk] + 1.59 x[ice cream]
Subject to
 410 x[hamburger] + 420 x[chicken] + 560 x[hot dog] + 380 x[fries] + 320 x[macaroni] + 320 x[pizza]
     + 320 x[salad] + 100 x[milk] + 330 x[ice cream] ∈ [1800.0, 2200.0]
```

```
24 x[hamburger] + 32 x[chicken] + 20 x[hot dog] + 4 x[fries] + 12 x[macaroni] + 15 x[pizza] + 31 x[
    salad] + 8 x[milk] + 8 x[ice cream] ∈ [91.0, Inf]
26 x[hamburger] + 10 x[chicken] + 32 x[hot dog] + 19 x[fries] + 10 x[macaroni] + 12 x[pizza] + 12 x
    [salad] + 2.5 x[milk] + 10 x[ice cream] ∈ [0.0, 65.0]
730 x[hamburger] + 1190 x[chicken] + 1800 x[hot dog] + 270 x[fries] + 930 x[macaroni] + 820 x[pizza
    ] + 1230 x[salad] + 125 x[milk] + 180 x[ice cream] ∈ [0.0, 1779.0]
x[hamburger] ≥ 0.0
x[chicken] ≥ 0.0
x[hot dog] ≥ 0.0
x[fries] ≥ 0.0
x[macaroni] ≥ 0.0
x[pizza] ≥ 0.0
x[salad] ≥ 0.0
x[milk] ≥ 0.0
x[ice cream] ≥ 0.0
```

Solution

```
optimize!(model)


solution_summary(model)
```

```
* Solver : HiGHS

* Status
  Termination status : OPTIMAL
  Primal status      : FEASIBLE_POINT
  Dual status        : FEASIBLE_POINT
  Message from the solver:
  "kHighsModelStatusOptimal"

* Candidate solution
  Objective value      : 1.18289e+01
  Objective bound      : 0.00000e+00
  Dual objective value : 1.18289e+01

* Work counters
  Solve time (sec)   : 3.91006e-04
  Simplex iterations : 6
  Barrier iterations : 0
```

Success! We found an optimal solution. Let's see what the optimal solution is:

```
for food in foods.name
    println(food, " = ", value(x[food]))
end
```

```
hamburger = 0.6045138888888888
chicken = 0.0
hot dog = 0.0
fries = 0.0
macaroni = 0.0
pizza = 0.0
```

```
salad = 0.0
milk = 6.970138888888889
ice cream = 2.5913194444444447
```

That's a lot of milk and ice cream! And sadly, we only get `0.6` of a hamburger.

Problem modification

JuMP makes it easy to take an existing model and modify it by adding extra constraints. Let's see what happens if we add a constraint that we can buy at most 6 units of milk or ice cream combined.

```
@constraint(model, x["milk"] + x["ice cream"] <= 6)
```

$$x_{milk} + x_{icecream} \leq 6.0$$

```
optimize!(model)


solution_summary(model)
```

```
* Solver : HiGHS

* Status
  Termination status : INFEASIBLE
  Primal status      : NO_SOLUTION
  Dual status        : INFEASIBILITY_CERTIFICATE
  Message from the solver:
  "kHighsModelStatusInfeasible"

* Candidate solution
  Objective value      : 1.18289e+01
  Objective bound      : 0.00000e+00
  Dual objective value : 3.56146e+00

* Work counters
  Solve time (sec)   : 6.01768e-04
  Simplex iterations : 0
  Barrier iterations : 0
```

Uh oh! There exists no feasible solution to our problem. Looks like we're stuck eating ice cream for dinner.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.4 The cannery problem

Original author: Louis Luangkesorn, January 30, 2015.

This tutorial solves the cannery problem from Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, NJ, 1963. This class of problem is known as a transshipment problem.

The purpose of this tutorial is to demonstrate how to use JSON data in the formulation of a JuMP model.

Required packages

```julia
using JuMP
import HiGHS
import JSON
```

Formulation

The cannery problem assumes we are optimizing the shipment of cans from production plants $p \in P$ to markets $m \in M$.

Each production plant p has a capacity, c_p, and each market m has a demand d_m. The distance from plant to market is $d_{p,m}$.

With a little effort, we can formulate our problem as the following linear program:

$$
\begin{aligned}
\min & \sum_{p \in P} \sum_{m \in M} d_{p,m} x_{p,m} & \\
\text{s.t.} & \sum_{m \in M} x_{p,m} \leq c_p, & \forall p \in P \\
& \sum_{p \in P} x_{p,m} \geq d_m, & \forall m \in M \\
& x_{p,m} \geq 0, & \forall p \in P, m \in M
\end{aligned}
$$

Data

A key feature of the tutorial is to demonstrate how to load data from JSON.

For simplicity, we've hard-coded it below. But if the data was available as a `.json` file, we could use `data = JSON.parsefile(filename)` to read in the data.

```julia
data = JSON.parse("""
{
    "plants": {
        "Seattle": {"capacity": 350},
        "San-Diego": {"capacity": 600}
    },
    "markets": {
        "New-York": {"demand": 300},
        "Chicago": {"demand": 300},
        "Topeka": {"demand": 300}
    },
    "distances": {
        "Seattle => New-York": 2.5,
        "Seattle => Chicago": 1.7,
        "Seattle => Topeka": 1.8,
        "San-Diego => New-York": 2.5,
        "San-Diego => Chicago": 1.8,
        "San-Diego => Topeka": 1.4
    }
}
""")
```

```
Dict{String, Any} with 3 entries:
  "plants"    => Dict{String, Any}("Seattle"=>Dict{String, Any}("capacity…"=>350
  "distances" => Dict{String, Any}("San-Diego => New-York"=>2.5, "Seattle => …To
  "markets"   => Dict{String, Any}("Chicago"=>Dict{String, Any}("demand"=>300)…,
```

Create the set of plants:

```
P = keys(data["plants"])
```

```
KeySet for a Dict{String, Any} with 2 entries. Keys:
  "Seattle"
  "San-Diego"
```

Create the set of markets:

```
M = keys(data["markets"])
```

```
KeySet for a Dict{String, Any} with 3 entries. Keys:
  "Chicago"
  "Topeka"
  "New-York"
```

We also need a function to compute the distance from plant to market:

```
distance(p::String, m::String) = data["distances"]["$(p) => $(m)"]
```

```
distance (generic function with 1 method)
```

JuMP formulation

Now we're ready to convert our mathematical formulation into a JuMP model.

First, create a new JuMP model. Since we have a linear program, we'll use HiGHS as our optimizer:

```
model = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Our decision variables are indexed over the set of plants and markets:

```
@variable(model, x[P, M] >= 0)
```

```
2-dimensional DenseAxisArray{VariableRef,2,...} with index sets:
    Dimension 1, ["Seattle", "San-Diego"]
    Dimension 2, ["Chicago", "Topeka", "New-York"]
And data, a 2×3 Matrix{VariableRef}:
 x[Seattle,Chicago]    x[Seattle,Topeka]    x[Seattle,New-York]
 x[San-Diego,Chicago]  x[San-Diego,Topeka]  x[San-Diego,New-York]
```

We need a constraint that each plant can ship no more than its capacity:

```
@constraint(model, [p in P], sum(x[p, :]) <= data["plants"][p]["capacity"])
```

```
1-dimensional DenseAxisArray{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape},1,...} with
     index sets:
    Dimension 1, ["Seattle", "San-Diego"]
And data, a 2-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 x[Seattle,Chicago] + x[Seattle,Topeka] + x[Seattle,New-York] ≤ 350.0
 x[San-Diego,Chicago] + x[San-Diego,Topeka] + x[San-Diego,New-York] ≤ 600.0
```

and each market must receive at least its demand:

```
@constraint(model, [m in M], sum(x[:, m]) >= data["markets"][m]["demand"])
```

```
1-dimensional DenseAxisArray{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.GreaterThan{Float64}}, ScalarShape},1,...} with
     index sets:
    Dimension 1, ["Chicago", "Topeka", "New-York"]
And data, a 3-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.GreaterThan{Float64}}, ScalarShape}}:
 x[Seattle,Chicago] + x[San-Diego,Chicago] ≥ 300.0
 x[Seattle,Topeka] + x[San-Diego,Topeka] ≥ 300.0
 x[Seattle,New-York] + x[San-Diego,New-York] ≥ 300.0
```

Finally, our objective is to minimize the transportation distance:

```
@objective(model, Min, sum(distance(p, m) * x[p, m] for p in P, m in M))
```

$$1.7x_{Seattle,Chicago}+1.8x_{Seattle,Topeka}+2.5x_{Seattle,New-York}+1.8x_{San-Diego,Chicago}+1.4x_{San-Diego,Topeka}+2.5x_{San-D}$$

Solution

```
optimize!(model)

solution_summary(model)
```

```
* Solver : HiGHS

* Status
  Termination status : OPTIMAL
  Primal status      : FEASIBLE_POINT
  Dual status        : FEASIBLE_POINT
  Message from the solver:
  "kHighsModelStatusOptimal"

* Candidate solution
  Objective value      : 1.68000e+03
  Objective bound      : 0.00000e+00
```

```
  Dual objective value : 1.68000e+03

* Work counters
  Solve time (sec)   : 3.65973e-04
  Simplex iterations : 3
  Barrier iterations : 0
```

What's the optimal shipment?

```
println("RESULTS:")
for p in P, m in M
    println(p, " => ", m, ": ", value(x[p, m]))
end
```

```
RESULTS:
Seattle => Chicago: 300.0
Seattle => Topeka: 0.0
Seattle => New-York: 0.0
San-Diego => Chicago: 0.0
San-Diego => Topeka: 300.0
San-Diego => New-York: 300.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.5 The facility location problem

It was originally contributed by Mathieu Tanneau (@mtanneau) and Alexis Montoison (@amontoison).

```
using JuMP
import HiGHS
import LinearAlgebra
import Plots
import Random
```

Uncapacitated facility location

Problem description

We are given

- A set $M = \{1, \dots, m\}$ of clients
- A set $N = \{1, \dots, n\}$ of sites where a facility can be built

Decision variables Decision variables are split into two categories:

- Binary variable y_j indicates whether facility j is built or not
- Binary variable $x_{i,j}$ indicates whether client i is assigned to facility j

Objective The objective is to minimize the total cost of serving all clients. This costs breaks down into two components:

- Fixed cost of building a facility.

In this example, this cost is $f_j = 1, \ \forall j$.

- Cost of serving clients from the assigned facility.

In this example, the cost $c_{i,j}$ of serving client i from facility j is the Euclidean distance between the two.

Constraints

- Each customer must be served by exactly one facility
- A facility cannot serve any client unless it is open

MILP formulation

The problem can be formulated as the following MILP:

$$
\begin{aligned}
\min_{x,y} \quad & \sum_{i,j} c_{i,j} x_{i,j} + \sum_{j} f_j y_j \\
s.t. \sum_{j} & x_{i,j} = 1, && \forall i \in M \\
& x_{i,j} \leq y_j, && \forall i \in M, j \in N \\
& x_{i,j}, y_j \in \{0, 1\}, && \forall i \in M, j \in N
\end{aligned}
$$

where the first set of constraints ensures that each client is served exactly once, and the second set of constraints ensures that no client is served from an unopened facility.

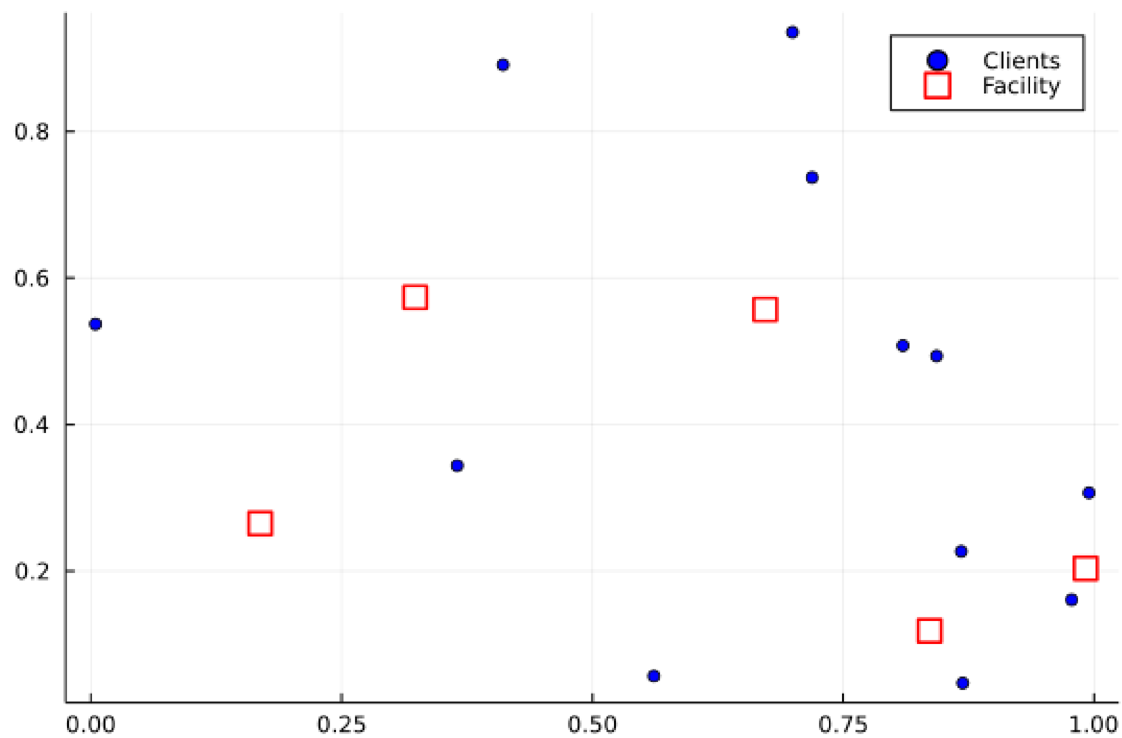
Problem data

```
Random.seed!(314)

# number of clients
m = 12
# number of facility locations
n = 5

# Clients' locations
Xc = rand(m)
Yc = rand(m)

# Facilities' potential locations
Xf = rand(n)
Yf = rand(n)

# Fixed costs
f = ones(n);
```

```
# Distance
c = zeros(m, n)
for i in 1:m
    for j in 1:n
        c[i, j] = LinearAlgebra.norm([Xc[i] - Xf[j], Yc[i] - Yf[j]], 2)
    end
end
```

Display the data

```
Plots.scatter(
    Xc,
    Yc,
    label = "Clients",
    markershape = :circle,
    markercolor = :blue,
)
Plots.scatter!(
    Xf,
    Yf,
    label = "Facility",
    markershape = :square,
    markercolor = :white,
    markersize = 6,
    markerstrokecolor = :red,
    markerstrokewidth = 2,
)
```



JuMP implementation

Create a JuMP model

```
ufl = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Variables

```
@variable(ufl, y[1:n], Bin);
@variable(ufl, x[1:m, 1:n], Bin);
```

Each client is served exactly once

```
@constraint(ufl, client_service[i in 1:m], sum(x[i, j] for j in 1:n) == 1);
```

A facility must be open to serve a client

```
@constraint(ufl, open_facility[i in 1:m, j in 1:n], x[i, j] <= y[j]);
```

Objective

```
@objective(ufl, Min, f'y + sum(c .* x));
```

Solve the uncapacitated facility location problem with HiGHS

```
optimize!(ufl)
```

```
Presolving model
72 rows, 65 cols, 180 nonzeros
72 rows, 65 cols, 180 nonzeros

Solving MIP model with:
   72 rows
   65 cols (65 binary, 0 integer, 0 implied int., 0 continuous)
   180 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) No symmetry present

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

```

```
         0       0         0   0.00%   0                inf                inf        0      0
      0         0     0.0s
 T       0       0         0   0.00%   0                5.22641797       100.00%        0      0
      0        32     0.0s

Solving report
  Status            Optimal
  Primal bound      5.22641797047
  Dual bound        5.22641797047
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    5.22641797047 (objective)
                    0 (bound viol.)
                    0 (int. viol.)
                    0 (row viol.)
  Timing            0.00 (total)
                    0.00 (presolve)
                    0.00 (postsolve)
  Nodes             1
  LP iterations     32 (total)
                    0 (strong br.)
                    0 (separation)
                    0 (heuristics)
```

```
println("Optimal value: ", objective_value(ufl))
```

```
Optimal value: 5.226417970467934
```

Visualizing the solution

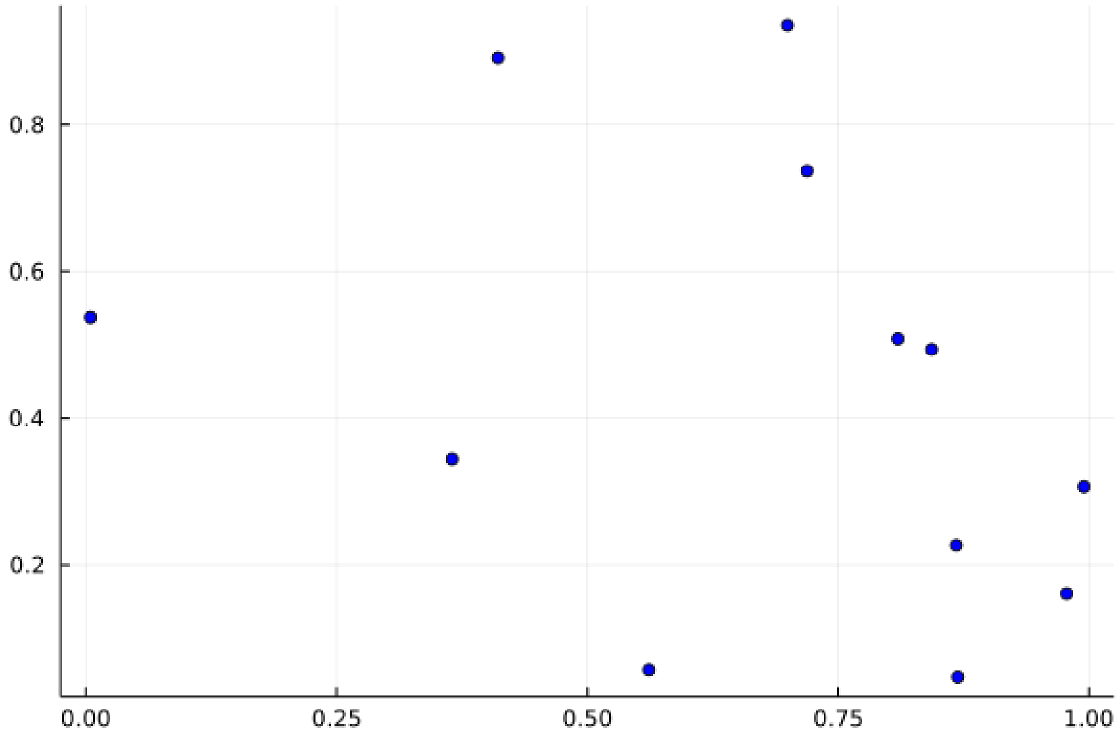
The threshold 1e-5 ensure that edges between clients and facilities are drawn when x[i, j] ≈ 1.

```
x_ = value.(x) .> 1 - 1e-5
y_ = value.(y) .> 1 - 1e-5
```

```
5-element BitVector:
 0
 0
 1
 1
 0
```

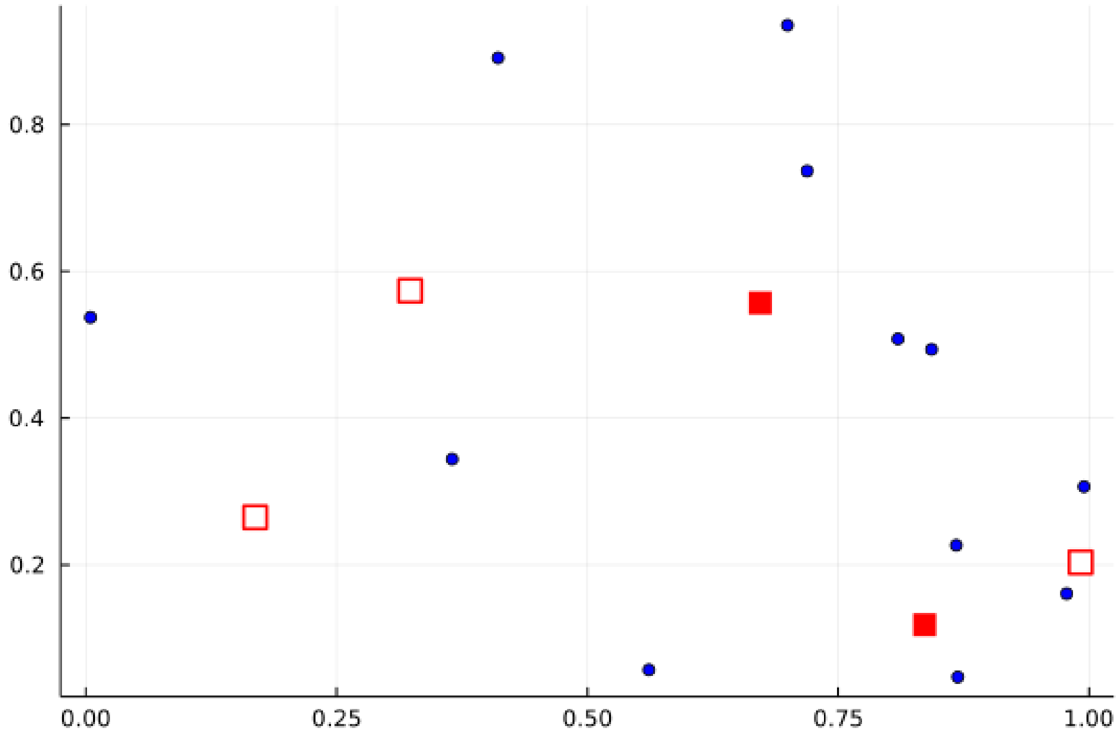
Display clients

```
p = Plots.scatter(
    Xc,
    Yc,
    markershape = :circle,
    markercolor = :blue,
    label = nothing,
)
```



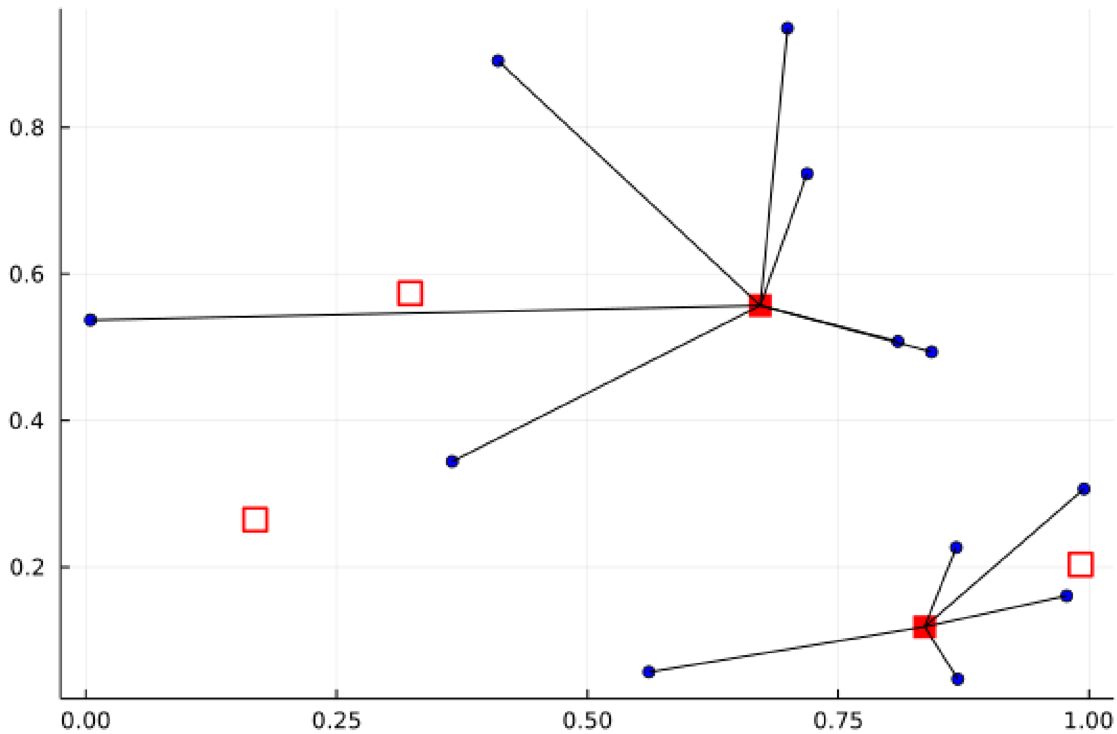
Show open facility

```
mc = [(y_[j] ? :red : :white) for j in 1:n]
Plots.scatter!(
    Xf,
    Yf,
    markershape = :square,
    markercolor = mc,
    markersize = 6,
    markerstrokecolor = :red,
    markerstrokewidth = 2,
    label = nothing,
)
```



Show client-facility assignment

```
for i in 1:m
    for j in 1:n
        if x_[i, j] == 1
            Plots.plot!(
                [Xc[i], Xf[j]],
                [Yc[i], Yf[j]],
                color = :black,
                label = nothing,
            )
        end
    end
end

p
```



Capacitated facility location

Problem formulation

The capacitated variant introduces a capacity constraint on each facility, i.e., clients have a certain level of demand to be served, while each facility only has finite capacity which cannot be exceeded.

Specifically,

- The demand of client i is denoted by $a_i \geq 0$
- The capacity of facility j is denoted by $q_j \geq 0$

The capacity constraints then write

$$\sum_i a_i x_{i,j} \leq q_j y_j \quad \forall j \in N$$

Note that, if y_j is set to 0, the capacity constraint above automatically forces $x_{i,j}$ to 0.

Thus, the capacitated facility location can be formulated as follows

$$
\begin{aligned}
\min_{x,y} \quad & \sum_{i,j} c_{i,j} x_{i,j} + \sum_j f_j y_j \\
s.t. & \sum_j x_{i,j} = 1, && \forall i \in M \\
& \sum_i a_i x_{i,j} \le q_j y_j, && \forall j \in N \\
& x_{i,j}, y_j \in \{0, 1\}, && \forall i \in M, j \in N
\end{aligned}
$$

For simplicity, we will assume that there is enough capacity to serve the demand, i.e., there exists at least one feasible solution.

Demands

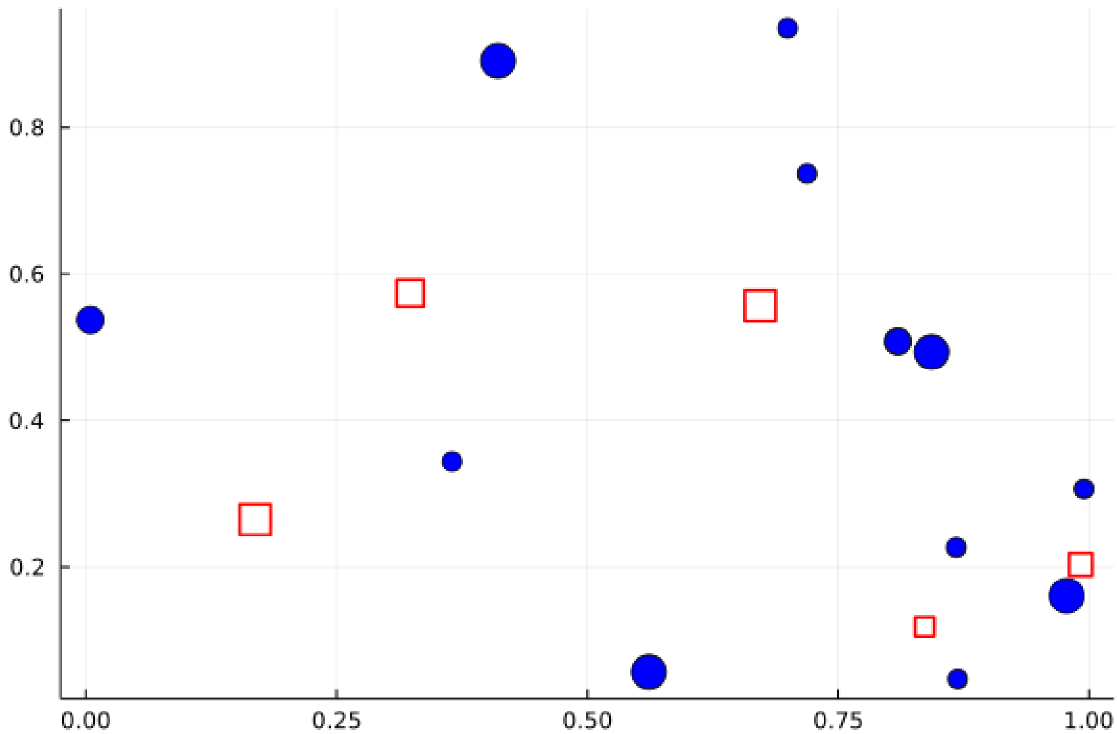
```
a = rand(1:3, m);
```

Capacities

```
q = rand(5:10, n);
```

Display the data

```
Plots.scatter(
    Xc,
    Yc,
    label = nothing,
    markershape = :circle,
    markercolor = :blue,
    markersize = 2 .* (2 .+ a),
)

Plots.scatter!(
    Xf,
    Yf,
    label = nothing,
    markershape = :rect,
    markercolor = :white,
    markersize = q,
    markerstrokecolor = :red,
    markerstrokewidth = 2,
)
```



JuMP implementation

Create a JuMP model

```
cfl = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Variables

```
@variable(cfl, y[1:n], Bin);
@variable(cfl, x[1:m, 1:n], Bin);
```

Each client is served exactly once

```
@constraint(cfl, client_service[i in 1:m], sum(x[i, :]) == 1);
```

Capacity constraint

```
@constraint(cfl, capacity, x'a .<= (q .* y));
```

Objective

```
@objective(cfl, Min, f'y + sum(c .* x));
```

Solve the problem

```
optimize!(cfl)
```

```
Presolving model
17 rows, 65 cols, 125 nonzeros
17 rows, 65 cols, 125 nonzeros

Solving MIP model with:
   17 rows
   65 cols (65 binary, 0 integer, 0 implied int., 0 continuous)
   125 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) No symmetry present

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

         0       0         0   0.00%   0               inf                  inf        0      0
      0          0     0.0s
 S       0       0         0   0.00%   0               7.54947525       100.00%        0      0
      0          0     0.0s
         0       0         0   0.00%   5.636380012     7.54947525        25.34%        0      0
      3         21     0.0s
 S       0       0         0   0.00%   5.636380012     7.373597157       23.56%       12      1
      3         21     0.0s
 C       0       0         0   0.00%   6.078302661     7.204177775       15.63%       96     14
      6         37     0.0s
 L       0       0         0   0.00%   6.173715063     6.173715063        0.00%      276     23
      6         50     0.0s

Solving report
  Status            Optimal
  Primal bound      6.17371506253
  Dual bound        6.17371506253
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    6.17371506253 (objective)
                    0 (bound viol.)
                    0 (int. viol.)
                    0 (row viol.)
  Timing            0.02 (total)
                    0.00 (presolve)
                    0.00 (postsolve)
  Nodes             1
  LP iterations     69 (total)
                    0 (strong br.)
                    29 (separation)
                    19 (heuristics)
```

```
println("Optimal value: ", objective_value(cfl))
```

```
Optimal value: 6.17371506253207
```

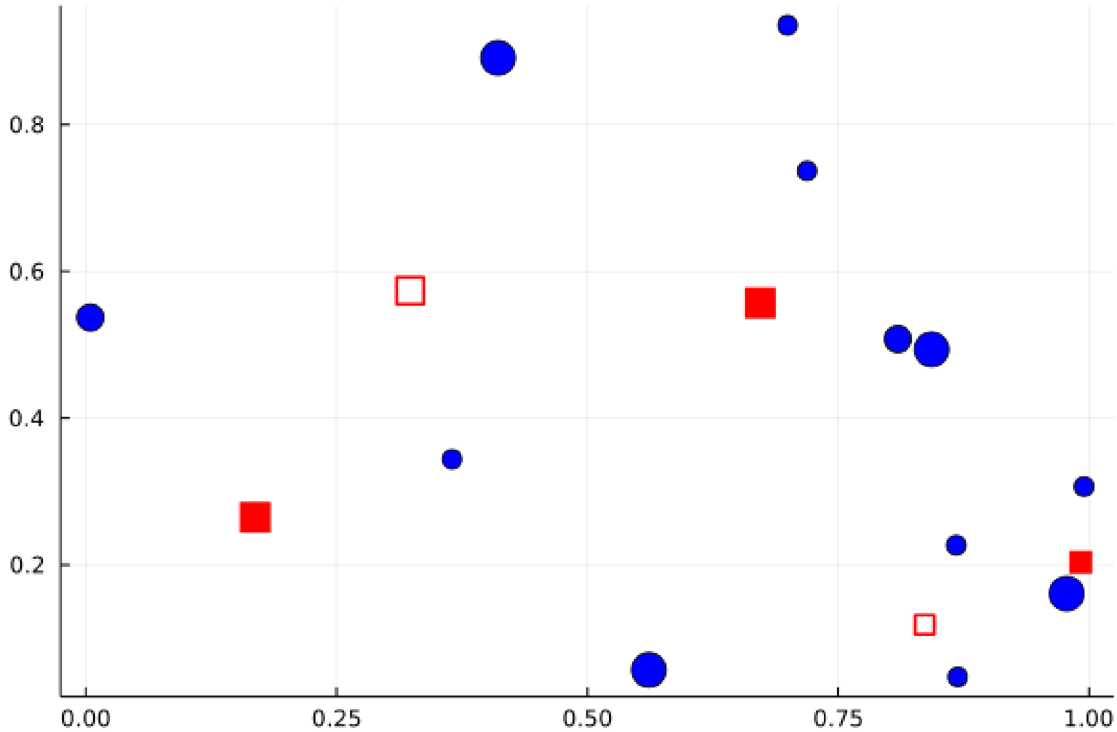
Visualizing the solution

The threshold 1e-5 ensure that edges between clients and facilities are drawn when x[i, j] ≈ 1.

```
x_ = value.(x) .> 1 - 1e-5;
y_ = value.(y) .> 1 - 1e-5;
```

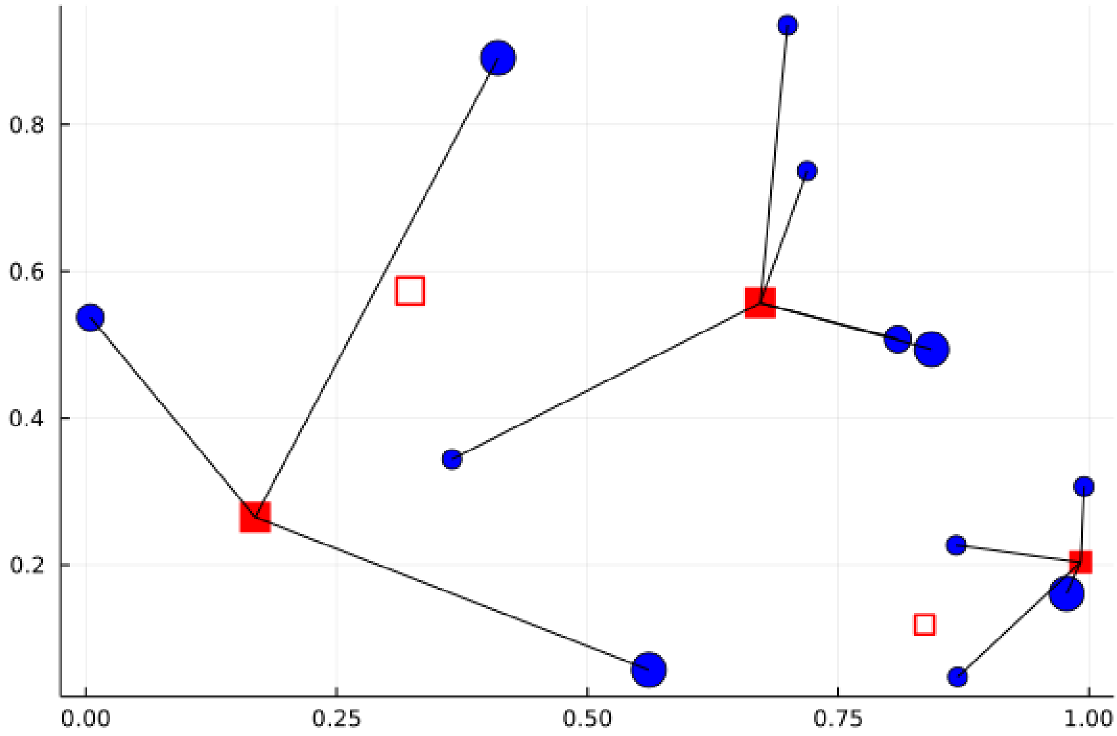
Display the solution

```
p = Plots.scatter(
    Xc,
    Yc,
    label = nothing,
    markershape = :circle,
    markercolor = :blue,
    markersize = 2 .* (2 .+ a),
)

mc = [(y_[j] ? :red : :white) for j in 1:n]
Plots.scatter!(
    Xf,
    Yf,
    label = nothing,
    markershape = :rect,
    markercolor = mc,
    markersize = q,
    markerstrokecolor = :red,
    markerstrokewidth = 2,
)
```



Show client-facility assignment

```
for i in 1:m
    for j in 1:n
        if x_[i, j] == 1
            Plots.plot!(
                [Xc[i], Xf[j]],
                [Yc[i], Yf[j]],
                color = :black,
                label = nothing,
            )
            break
        end
    end
end
p
```



Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.6 The factory schedule example

This is a Julia translation of part 5 from "Introduction to Linear Programming with Python" available at https://github.com/benalexke
to-linear-programming

For 2 factories (A, B), minimize the cost of production over the course of 12 months while meeting monthly demand. Factory B has a planned outage during month 5.

It was originally contributed by @Crghilardi.

```
using JuMP
import HiGHS
import Test

function example_factory_schedule()
    # Sets in the problem:
    months, factories = 1:12, [:A, :B]
    # This function takes a matrix and converts it to a JuMP container so we can
    # refer to elements such as `d_max_cap[1, :A]`.
    containerize(A::Matrix) = Containers.DenseAxisArray(A, months, factories)
    # Maximum production capacity in (month, factory) [units/month]:
    d_max_cap = containerize(
```

```
    [
        100000 50000
        110000 55000
        120000 60000
        145000 100000
        160000 0
        140000 70000
        155000 60000
        200000 100000
        210000 100000
        197000 100000
        80000 120000
        150000 150000
    ],
)
# Minimum production capacity in (month, factory) [units/month]:
d_min_cap = containerize(
    [
        20000 20000
        20000 20000
        20000 20000
        20000 20000
        20000 0
        20000 20000
        20000 20000
        20000 20000
        20000 20000
        20000 20000
        20000 20000
        20000 20000
    ],
)
# Variable cost of production in (month, factory) [$/unit]:
d_var_cost = containerize([
    10 5
    11 4
    12 3
    9 5
    8 0
    8 6
    5 4
    7 6
    9 8
    10 11
    8 10
    8 12
])
# Fixed cost of production in (month, factory) # [$/month]:
d_fixed_cost = containerize(
    [
        500 600
        500 600
        500 600
        500 600
        500 0
        500 600
```

```
            500 600
            500 600
            500 600
            500 600
            500 600
            500 600
        ],
    )
    # Demand in each month [units/month]:
    d_demand = [
        120_000,
        100_000,
        130_000,
        130_000,
        140_000,
        130_000,
        150_000,
        170_000,
        200_000,
        190_000,
        140_000,
        100_000,
    ]
    # The model!
    model = Model(HiGHS.Optimizer)
    # Decision variables
    @variables(model, begin
        status[m in months, f in factories], Bin
        production[m in months, f in factories], Int
    end)
    # The production cannot be less than minimum capacity.
    @constraint(
        model,
        [m in months, f in factories],
        production[m, f] >= d_min_cap[m, f] * status[m, f],
    )
    # The production cannot be more that maximum capacity.
    @constraint(
        model,
        [m in months, f in factories],
        production[m, f] <= d_max_cap[m, f] * status[m, f],
    )
    # The production must equal demand in a given month.
    @constraint(model, [m in months], sum(production[m, :]) == d_demand[m])
    # Factory B is shut down during month 5, so production and status are both
    # zero.
    fix(status[5, :B], 0.0)
    fix(production[5, :B], 0.0)
    # The objective is to minimize the cost of production across all time
    ##periods.
    @objective(
        model,
        Min,
        sum(
            d_fixed_cost[m, f] * status[m, f] +
            d_var_cost[m, f] * production[m, f] for m in months, f in factories
```

```
        )
    )
    # Optimize the problem
    optimize!(model)
    # Check the solution!
    Test.@testset "Check the solution against known optimal" begin
        Test.@test termination_status(model) == OPTIMAL
        Test.@test objective_value(model) == 12_906_400.0
        Test.@test value.(production)[1, :A] == 70_000
        Test.@test value.(status)[1, :A] == 1
        Test.@test value.(status)[5, :B] == 0
        Test.@test value.(production)[5, :B] == 0
    end
    println("The production schedule is:")
    println(value.(production))
    return
end

example_factory_schedule()
```

```
Presolving model
44 rows, 30 cols, 82 nonzeros
4 rows, 10 cols, 8 nonzeros
4 rows, 3 cols, 8 nonzeros
Objective function is integral with scale 0.25

Solving MIP model with:
   4 rows
   3 cols (2 binary, 1 integer, 0 implied int., 0 continuous)
   8 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) No symmetry present

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

         0       0         0   0.00%   12906400        inf                  inf        0      0
      0         0     0.0s

Solving report
  Status            Optimal
  Primal bound      12906400
  Dual bound        12906400
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    12906400 (objective)
                    0 (bound viol.)
                    0 (int. viol.)
                    0 (row viol.)
  Timing            0.00 (total)
                    0.00 (presolve)
                    0.00 (postsolve)
  Nodes             1
```

```
  LP iterations     1 (total)
                    0 (strong br.)
                    0 (separation)
                    0 (heuristics)
Test Summary:                           | Pass  Total
Check the solution against known optimal |    6      6
The production schedule is:
2-dimensional DenseAxisArray{Float64,2,...} with index sets:
    Dimension 1, 1:12
    Dimension 2, [:A, :B]
And data, a 12×2 Matrix{Float64}:
  70000.0   50000.0
  45000.0   55000.0
  70000.0   60000.0
  30000.0  100000.0
 140000.0       0.0
  60000.0   70000.0
  90000.0   60000.0
  70000.0  100000.0
 100000.0  100000.0
 190000.0       0.0
  80000.0   60000.0
 100000.0      -0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.7 Financial modeling problems

Originally Contributed by: Arpit Bhatia

Optimization models play an increasingly important role in financial decisions. Many computational finance problems can be solved efficiently using modern optimization techniques.

In this tutorial we will discuss two such examples taken from the book Optimization Methods in Finance.

This tutorial uses the following packages

```julia
using JuMP
import HiGHS
```

Short-term financing

Corporations routinely face the problem of financing short term cash commitments such as the following:

Month	Jan	Feb	Mar	Apr	May	Jun
Net Cash Flow	-150	-100	200	-200	50	300

Net cash flow requirements are given in thousands of dollars. The company has the following sources of funds:

- A line of credit of up to \$100K at an interest rate of 1% per month,

- In any one of the first three months, it can issue 90-day commercial paper bearing a total interest of 2% for the 3-month period,
- Excess funds can be invested at an interest rate of 0.3% per month.

Our task is to find out the most economical way to use these 3 sources such that we end up with the most amount of money at the end of June.

We model this problem in the following manner:

We will use the following decision variables:

- the amount u_i drawn from the line of credit in month i
- the amount v_i of commercial paper issued in month i
- the excess funds w_i in month i

Here we have three types of constraints:

1. for every month, cash inflow = cash outflow for each month
2. upper bounds on u_i
3. nonnegativity of the decision variables u_i, v_i and w_i.

Our objective will be to simply maximize the company's wealth in June, which say we represent with the variable m.

```
financing = Model(HiGHS.Optimizer)

@variables(financing, begin
    0 <= u[1:5] <= 100
    0 <= v[1:3]
    0 <= w[1:5]
    m
end)

@objective(financing, Max, m)

@constraints(
    financing,
    begin
        u[1] + v[1] - w[1] == 150 # January
        u[2] + v[2] - w[2] - 1.01u[1] + 1.003w[1] == 100 # February
        u[3] + v[3] - w[3] - 1.01u[2] + 1.003w[2] == -200 # March
        u[4] - w[4] - 1.02v[1] - 1.01u[3] + 1.003w[3] == 200 # April
        u[5] - w[5] - 1.02v[2] - 1.01u[4] + 1.003w[4] == -50 # May
        -m - 1.02v[3] - 1.01u[5] + 1.003w[5] == -300 # June
    end
)

optimize!(financing)

objective_value(financing)
```

```
92.49694915254233
```

Combinatorial auctions

In many auctions, the value that a bidder has for a set of items may not be the sum of the values that he has for individual items.

Examples are equity trading, electricity markets, pollution right auctions and auctions for airport landing slots.

To take this into account, combinatorial auctions allow the bidders to submit bids on combinations of items.

Let $M = \{1, 2, \ldots, m\}$ be the set of items that the auctioneer has to sell. A bid is a pair $B_j = (S_j, p_j)$ where $S_j \subseteq M$ is a nonempty set of items and p_j is the price offer for this set.

Suppose that the auctioneer has received n bids $B_1, B_2, \ldots, B_n$. The goal of this problem is to help an auctioneer determine the winners in order to maximize his revenue.

We model this problem by taking a decision variable y_j for every bid. We add a constraint that each item i is sold at most once. This gives us the following model:

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} p_j y_j \\ \text{s.t.} \quad & \sum_{j : i \in S_j} y_j \leq 1 \quad \forall i = \{1, 2 \ldots m\} \\ & y_j \in \{0, 1\} \quad \forall j \in \{1, 2 \ldots n\} \end{aligned}$$

```julia
bid_values = [6 3 12 12 8 16]
bid_items = [[1], [2], [3 4], [1 3], [2 4], [1 3 4]]

auction = Model(HiGHS.Optimizer)
@variable(auction, y[1:6], Bin)
@objective(auction, Max, sum(y' .* bid_values))
for i in 1:6
    @constraint(auction, sum(y[j] for j in 1:6 if i in bid_items[j]) <= 1)
end

optimize!(auction)

objective_value(auction)
```

```
21.0
```

```julia
value.(y)
```

```
6-element Vector{Float64}:
  1.0
  1.0
  1.0
 -0.0
 -0.0
  0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.8 Geographical clustering

Originally Contributed by: Matthew Helm (with help from Mathieu Tanneau on Julia Discourse)

The goal of this exercise is to cluster n cities into k groups, minimizing the total pairwise distance between cities and ensuring that the variance in the total populations of each group is relatively small.

This tutorial uses the following packages:

```
using JuMP
import DataFrames
import HiGHS
import LinearAlgebra
```

For this example, we'll use the 20 most populous cities in the United States.

```
cities = DataFrames.DataFrame(
    city = [
        "New York, NY",
        "Los Angeles, CA",
        "Chicago, IL",
        "Houston, TX",
        "Philadelphia, PA",
        "Phoenix, AZ",
        "San Antonio, TX",
        "San Diego, CA",
        "Dallas, TX",
        "San Jose, CA",
        "Austin, TX",
        "Indianapolis, IN",
        "Jacksonville, FL",
        "San Francisco, CA",
        "Columbus, OH",
        "Charlotte, NC",
        "Fort Worth, TX",
        "Detroit, MI",
        "El Paso, TX",
        "Memphis, TN",
    ],
    population = [
        8.405,
        3.884,
        2.718,
        2.195,
        1.553,
        1.513,
        1.409,
        1.355,
        1.257,
        0.998,
        0.885,
        0.843,
        0.842,
        0.837,
        0.822,
        0.792,
```

```
        0.792,
        0.688,
        0.674,
        0.653,
    ],
    lat = [
        40.7127,
        34.0522,
        41.8781,
        29.7604,
        39.9525,
        33.4483,
        29.4241,
        32.7157,
        32.7766,
        37.3382,
        30.2671,
        39.7684,
        30.3321,
        37.7749,
        39.9611,
        35.2270,
        32.7554,
        42.3314,
        31.7775,
        35.1495,
    ],
    lon = [
        -74.0059,
        -118.2436,
        -87.6297,
        -95.3698,
        -75.1652,
        -112.0740,
        -98.4936,
        -117.1610,
        -96.7969,
        -121.8863,
        -97.7430,
        -86.1580,
        -81.6556,
        -122.4194,
        -82.9987,
        -80.8431,
        -97.3307,
        -83.0457,
        -106.4424,
        -90.0489,
    ],
)
```

	city	population	lat	lon
	String	Float64	Float64	Float64
1	New York, NY	8.405	40.7127	-74.0059
2	Los Angeles, CA	3.884	34.0522	-118.244
3	Chicago, IL	2.718	41.8781	-87.6297
4	Houston, TX	2.195	29.7604	-95.3698
5	Philadelphia, PA	1.553	39.9525	-75.1652
6	Phoenix, AZ	1.513	33.4483	-112.074
7	San Antonio, TX	1.409	29.4241	-98.4936
8	San Diego, CA	1.355	32.7157	-117.161
9	Dallas, TX	1.257	32.7766	-96.7969
10	San Jose, CA	0.998	37.3382	-121.886
11	Austin, TX	0.885	30.2671	-97.743
12	Indianapolis, IN	0.843	39.7684	-86.158
13	Jacksonville, FL	0.842	30.3321	-81.6556
14	San Francisco, CA	0.837	37.7749	-122.419
15	Columbus, OH	0.822	39.9611	-82.9987
16	Charlotte, NC	0.792	35.227	-80.8431
17	Fort Worth, TX	0.792	32.7554	-97.3307
18	Detroit, MI	0.688	42.3314	-83.0457
19	El Paso, TX	0.674	31.7775	-106.442
20	Memphis, TN	0.653	35.1495	-90.0489

Model Specifics

We will cluster these 20 cities into 3 different groups and we will assume that the ideal or target population P for a group is simply the total population of the 20 cities divided by 3:

```
n = size(cities, 1)
k = 3
P = sum(cities.population) / k
```

```
11.038333333333334
```

Obtaining the distances between each city

Let's compute the pairwise Haversine distance between each of the cities in our data set and store the result in a variable we'll call `dm`:

```
"""
    haversine(lat1, long1, lat2, long2, r = 6372.8)

Compute the haversine distance between two points on a sphere of radius `r`,
where the points are given by the latitude/longitude pairs `lat1/long1` and
`lat2/long2` (in degrees).
"""
function haversine(lat1, long1, lat2, long2, r = 6372.8)
    lat1, long1 = deg2rad(lat1), deg2rad(long1)
    lat2, long2 = deg2rad(lat2), deg2rad(long2)
    hav(a, b) = sin((b - a) / 2)^2
    inner_term = hav(lat1, lat2) + cos(lat1) * cos(lat2) * hav(long1, long2)
    d = 2 * r * asin(sqrt(inner_term))
    # Round distance to nearest kilometer.
```

```
    return round(Int, d)
end
```

```
Main.haversine
```

Our distance matrix is symmetric so we'll convert it to a `LowerTriangular` matrix so that we can better interpret the objective value of our model:

```
dm = LinearAlgebra.LowerTriangular([
    haversine(cities.lat[i], cities.lon[i], cities.lat[j], cities.lon[j])
    for i in 1:n, j in 1:n
])
```

```
20×20 LinearAlgebra.LowerTriangular{Int64, Matrix{Int64}}:
    0     ⋅     ⋅     ⋅     ⋅     ⋅     ⋅  …     ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 3937     0     ⋅     ⋅     ⋅     ⋅     ⋅        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 1145  2805     0     ⋅     ⋅     ⋅     ⋅        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 2282  2207  1516     0     ⋅     ⋅     ⋅        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
  130  3845  1068  2157     0     ⋅     ⋅        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 3445   574  2337  1633  3345     0     ⋅  …     ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 2546  1934  1695   304  2423  1363     0        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 3908   179  2787  2094  3812   481  1813        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 2206  1993  1295   362  2089  1424   406        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 4103   492  2958  2588  4023   989  2336        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 2432  1972  1577   235  2310  1398   118  …     ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 1036  2907   265  1394   938  2409  1609        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 1345  3450  1391  1321  1221  2883  1626        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
 4130   559  2986  2644  4052  1051  2394        ⋅     ⋅     ⋅     ⋅     ⋅  ⋅
  767  3177   444  1598   668  2679  1834        0     ⋅     ⋅     ⋅     ⋅  ⋅
  855  3405   946  1490   725  2863  1777  …   560     0     ⋅     ⋅     ⋅  ⋅
 2251  1944  1327   382  2134  1375   387     1511  1543     0     ⋅     ⋅  ⋅
  774  3186   382  1780   711  2716  1994      264   813  1646     0     ⋅  ⋅
 3054  1130  2010  1081  2945   559   804     2292  2398   864  2374     0  ⋅
 1534  2576   777   780  1415  2028  1017      820   837   722  1003  1565  0
```

Build the model

Now that we have the basics taken care of, we can set up our model, create decision variables, add constraints, and then solve.

First, we'll set up a model that leverages the Cbc solver. Next, we'll set up a binary variable $x_{i,k}$ that takes the value 1 if city i is in group k and 0 otherwise. Each city must be in a group, so we'll add the constraint $\sum_k x_{i,k} = 1$ for every i.

```
model = Model(HiGHS.Optimizer)
set_silent(model)
```

```
@variable(model, x[1:n, 1:k], Bin)
```

```
20×3 Matrix{VariableRef}:
 x[1,1]   x[1,2]   x[1,3]
 x[2,1]   x[2,2]   x[2,3]
 x[3,1]   x[3,2]   x[3,3]
```

```
 x[4,1]   x[4,2]   x[4,3]
 x[5,1]   x[5,2]   x[5,3]
 x[6,1]   x[6,2]   x[6,3]
 x[7,1]   x[7,2]   x[7,3]
 x[8,1]   x[8,2]   x[8,3]
 x[9,1]   x[9,2]   x[9,3]
 x[10,1]  x[10,2]  x[10,3]
 x[11,1]  x[11,2]  x[11,3]
 x[12,1]  x[12,2]  x[12,3]
 x[13,1]  x[13,2]  x[13,3]
 x[14,1]  x[14,2]  x[14,3]
 x[15,1]  x[15,2]  x[15,3]
 x[16,1]  x[16,2]  x[16,3]
 x[17,1]  x[17,2]  x[17,3]
 x[18,1]  x[18,2]  x[18,3]
 x[19,1]  x[19,2]  x[19,3]
 x[20,1]  x[20,2]  x[20,3]
```

```
@constraint(model, [i = 1:n], sum(x[i, :]) == 1)
```

```
20-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.EqualTo{Float64}}, ScalarShape}}:
 x[1,1] + x[1,2] + x[1,3] = 1.0
 x[2,1] + x[2,2] + x[2,3] = 1.0
 x[3,1] + x[3,2] + x[3,3] = 1.0
 x[4,1] + x[4,2] + x[4,3] = 1.0
 x[5,1] + x[5,2] + x[5,3] = 1.0
 x[6,1] + x[6,2] + x[6,3] = 1.0
 x[7,1] + x[7,2] + x[7,3] = 1.0
 x[8,1] + x[8,2] + x[8,3] = 1.0
 x[9,1] + x[9,2] + x[9,3] = 1.0
 x[10,1] + x[10,2] + x[10,3] = 1.0
 x[11,1] + x[11,2] + x[11,3] = 1.0
 x[12,1] + x[12,2] + x[12,3] = 1.0
 x[13,1] + x[13,2] + x[13,3] = 1.0
 x[14,1] + x[14,2] + x[14,3] = 1.0
 x[15,1] + x[15,2] + x[15,3] = 1.0
 x[16,1] + x[16,2] + x[16,3] = 1.0
 x[17,1] + x[17,2] + x[17,3] = 1.0
 x[18,1] + x[18,2] + x[18,3] = 1.0
 x[19,1] + x[19,2] + x[19,3] = 1.0
 x[20,1] + x[20,2] + x[20,3] = 1.0
```

To reduce symmetry, we fix the first city to belong to the first group.

```
fix(x[1, 1], 1; force = true)
```

The total population of a group k is $Q_k = \sum_i x_{i,k} q_i$ where q_i is simply the ith value from the `population` column in our `cities` DataFrame. Let's add constraints so that $\alpha \leq (Q_k - P) \leq \beta$. We'll set α equal to -3 million and β equal to 3. By adjusting these thresholds you'll find that there is a tradeoff between having relatively even populations between groups and having geographically close cities within each group. In other words, the larger the absolute values of α and β, the closer together the cities in a group will be but the variance between the group populations will be higher.

```
@variable(model, -3 <= population_diff[1:k] <= 3)
@constraint(model, population_diff .== x' * cities.population .- P)
```

```
3-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
    ScalarAffineFunction{Float64}, MathOptInterface.EqualTo{Float64}}, ScalarShape}}:
 -8.405 x[1,1] - 3.884 x[2,1] - 2.718 x[3,1] - 2.195 x[4,1] - 1.553 x[5,1] - 1.513 x[6,1] - 1.409 x
    [7,1] - 1.355 x[8,1] - 1.257 x[9,1] - 0.998 x[10,1] - 0.885 x[11,1] - 0.843 x[12,1] - 0.842 x
    [13,1] - 0.837 x[14,1] - 0.822 x[15,1] - 0.792 x[16,1] - 0.792 x[17,1] - 0.688 x[18,1] - 0.674 x
    [19,1] - 0.653 x[20,1] + population_diff[1] = -11.038333333333334
 -8.405 x[1,2] - 3.884 x[2,2] - 2.718 x[3,2] - 2.195 x[4,2] - 1.553 x[5,2] - 1.513 x[6,2] - 1.409 x
    [7,2] - 1.355 x[8,2] - 1.257 x[9,2] - 0.998 x[10,2] - 0.885 x[11,2] - 0.843 x[12,2] - 0.842 x
    [13,2] - 0.837 x[14,2] - 0.822 x[15,2] - 0.792 x[16,2] - 0.792 x[17,2] - 0.688 x[18,2] - 0.674 x
    [19,2] - 0.653 x[20,2] + population_diff[2] = -11.038333333333334
 -8.405 x[1,3] - 3.884 x[2,3] - 2.718 x[3,3] - 2.195 x[4,3] - 1.553 x[5,3] - 1.513 x[6,3] - 1.409 x
    [7,3] - 1.355 x[8,3] - 1.257 x[9,3] - 0.998 x[10,3] - 0.885 x[11,3] - 0.843 x[12,3] - 0.842 x
    [13,3] - 0.837 x[14,3] - 0.822 x[15,3] - 0.792 x[16,3] - 0.792 x[17,3] - 0.688 x[18,3] - 0.674 x
    [19,3] - 0.653 x[20,3] + population_diff[3] = -11.038333333333334
```

Now we need to add one last binary variable $z_{i,j}$ to our model that we'll use to compute the total distance between the cities in our groups, defined as $\sum_{i,j} d_{i,j} z_{i,j}$. Variable $z_{i,j}$ will equal 1 if cities i and j are in the same group, and 0 if they are not in the same group.

To ensure that $z_{i,j} = 1$ if and only if cities i and j are in the same group, we add the constraints $z_{i,j} \geq x_{i,k} + x_{j,k} - 1$ for every pair i, j and every k:

```
@variable(model, z[i = 1:n, j = 1:i], Bin)
```

```
JuMP.Containers.SparseAxisArray{VariableRef, 2, Tuple{Int64, Int64}} with 210 entries:
  [12, 10]  =  z[12,10]
  [12, 2 ]  =  z[12,2]
  [12, 3 ]  =  z[12,3]
  [16, 12]  =  z[16,12]
  [16, 14]  =  z[16,14]
  [16, 16]  =  z[16,16]
  [17, 12]  =  z[17,12]
  [18, 14]  =  z[18,14]

  [18, 16]  =  z[18,16]
  [18, 18]  =  z[18,18]
  [19, 12]  =  z[19,12]
  [19, 14]  =  z[19,14]
  [19, 16]  =  z[19,16]
  [19, 19]  =  z[19,19]
  [20, 15]  =  z[20,15]
```

```
for k in 1:k, i in 1:n, j in 1:i
    @constraint(model, z[i, j] >= x[i, k] + x[j, k] - 1)
end
```

We can now add an objective to our model which will simply be to minimize the dot product of z and our distance matrix, dm.

```
@objective(model, Min, sum(dm[i, j] * z[i, j] for i in 1:n, j in 1:i))
```

$$3937z_{2,1}+1145z_{3,1}+2805z_{3,2}+2282z_{4,1}+2207z_{4,2}+1516z_{4,3}+130z_{5,1}+3845z_{5,2}+1068z_{5,3}+2157z_{5,4}+3445z_{6,1}+574z_{6,2}+$$

We can then call `optimize!` and review the results.

```
optimize!(model)
```

Reviewing the Results

Now that we have results, we can add a column to our `cities` DataFrame for the group and then loop through our x variable to assign each city to its group. Once we have that, we can look at the total population for each group and also look at the cities in each group to verify that they are grouped by geographic proximity.

```
cities.group = zeros(n)

for i in 1:n, j in 1:k
    if round(Int, value(x[i, j])) == 1
        cities.group[i] = j
    end
end

for group in DataFrames.groupby(cities, :group)
    @show group
    println("")
    @show sum(group.population)
    println("")
end
```

```
group = 7×5 SubDataFrame
 Row │ city              population  lat      lon       group
     │ String            Float64     Float64  Float64   Float64
─────┼───────────────────────────────────────────────────────────

   1 │ New York, NY           8.405  40.7127  -74.0059      1.0
   2 │ Philadelphia, PA       1.553  39.9525  -75.1652      1.0
   3 │ Indianapolis, IN       0.843  39.7684  -86.158       1.0
   4 │ Jacksonville, FL       0.842  30.3321  -81.6556      1.0
   5 │ Columbus, OH           0.822  39.9611  -82.9987      1.0
   6 │ Charlotte, NC          0.792  35.227   -80.8431      1.0
   7 │ Detroit, MI            0.688  42.3314  -83.0457      1.0

sum(group.population) = 13.944999999999999

group = 6×5 SubDataFrame
 Row │ city               population  lat      lon       group
     │ String             Float64     Float64  Float64   Float64
─────┼────────────────────────────────────────────────────────────

   1 │ Los Angeles, CA         3.884  34.0522  -118.244      2.0
   2 │ Phoenix, AZ             1.513  33.4483  -112.074      2.0
   3 │ San Diego, CA           1.355  32.7157  -117.161      2.0
   4 │ San Jose, CA            0.998  37.3382  -121.886      2.0
   5 │ San Francisco, CA       0.837  37.7749  -122.419      2.0
```

```
   6 │ El Paso, TX             0.674  31.7775  -106.442      2.0

sum(group.population) = 9.261000000000001

group = 7×5 SubDataFrame
 Row │ city             population  lat      lon       group
     │ String           Float64     Float64  Float64   Float64
─────┼──────────────────────────────────────────────────────────

   1 │ Chicago, IL           2.718  41.8781  -87.6297      3.0
   2 │ Houston, TX           2.195  29.7604  -95.3698      3.0
   3 │ San Antonio, TX       1.409  29.4241  -98.4936      3.0
   4 │ Dallas, TX            1.257  32.7766  -96.7969      3.0
   5 │ Austin, TX            0.885  30.2671  -97.743       3.0
   6 │ Fort Worth, TX        0.792  32.7554  -97.3307      3.0
   7 │ Memphis, TN           0.653  35.1495  -90.0489      3.0

sum(group.population) = 9.909
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.9 The knapsack problem

Formulate and solve a simple knapsack problem:

```
max sum(p_j x_j)
 st sum(w_j x_j) <= C
    x binary
```

```julia
using JuMP
import HiGHS
import Test

function example_knapsack(; verbose = true)
    profit = [5, 3, 2, 7, 4]
    weight = [2, 8, 4, 2, 5]
    capacity = 10
    model = Model(HiGHS.Optimizer)
    @variable(model, x[1:5], Bin)
    # Objective: maximize profit
    @objective(model, Max, profit' * x)
    # Constraint: can carry all
    @constraint(model, weight' * x <= capacity)
    # Solve problem using MIP solver
    optimize!(model)
    if verbose
        println("Objective is: ", objective_value(model))
        println("Solution is:")
        for i in 1:5
            print("x[$i] = ", value(x[i]))
            println(", p[$i]/w[$i] = ", profit[i] / weight[i])
```

```julia
        end
    end
    Test.@test termination_status(model) == OPTIMAL
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) == 16.0
    return
end

example_knapsack()
```

```
Presolving model
1 rows, 5 cols, 5 nonzeros
1 rows, 4 cols, 4 nonzeros
Objective function is integral with scale 1

Solving MIP model with:
   1 rows
   4 cols (4 binary, 0 integer, 0 implied int., 0 continuous)
   4 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) No symmetry present

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

         0       0         0   0.00%   18              -inf                 inf        0      0
      0         0     0.0s

Solving report
  Status            Optimal
  Primal bound      16
  Dual bound        16
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    16 (objective)
                    0 (bound viol.)
                    0 (int. viol.)
                    0 (row viol.)
  Timing            0.00 (total)
                    0.00 (presolve)
                    0.00 (postsolve)
  Nodes             1
  LP iterations     1 (total)
                    0 (strong br.)
                    0 (separation)
                    0 (heuristics)
Objective is: 16.0
Solution is:
x[1] = 1.0, p[1]/w[1] = 2.5
x[2] = 0.0, p[2]/w[2] = 0.375
x[3] = -0.0, p[3]/w[3] = 0.5
x[4] = 1.0, p[4]/w[4] = 3.5
x[5] = 1.0, p[5]/w[5] = 0.8
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.10 The multi-commodity flow problem

JuMP implementation of the multicommodity transportation model AMPL: A Modeling Language for Mathematical Programming, 2nd ed by Robert Fourer, David Gay, and Brian W. Kernighan 4-1.

Originally contributed by Louis Luangkesorn, February 26, 2015.

```
using JuMP
import HiGHS
import Test

function example_multi(; verbose = true)
    orig = ["GARY", "CLEV", "PITT"]
    dest = ["FRA", "DET", "LAN", "WIN", "STL", "FRE", "LAF"]
    prod = ["bands", "coils", "plate"]
    numorig = length(orig)
    numdest = length(dest)
    numprod = length(prod)
    # supply(prod, orig) amounts available at origins
    supply = [
        400 700 800
        800 1600 1800
        200 300 300
    ]
    # demand(prod, dest) amounts required at destinations
    demand = [
        300 300 100 75 650 225 250
        500 750 400 250 950 850 500
        100 100 0 50 200 100 250
    ]
    # limit(orig, dest) of total units from any origin to destination
    limit = [625.0 for j in 1:numorig, i in 1:numdest]
    # cost(dest, orig, prod) Shipment cost per unit
    cost = reshape(
        [
            [
                [30, 10, 8, 10, 11, 71, 6]
                [22, 7, 10, 7, 21, 82, 13]
                [19, 11, 12, 10, 25, 83, 15]
            ]
            [
                [39, 14, 11, 14, 16, 82, 8]
                [27, 9, 12, 9, 26, 95, 17]
                [24, 14, 17, 13, 28, 99, 20]
            ]
            [
                [41, 15, 12, 16, 17, 86, 8]
                [29, 9, 13, 9, 28, 99, 18]
                [26, 14, 17, 13, 31, 104, 20]
            ]
```

```
        ],
        7,
        3,
        3,
    )
    # DECLARE MODEL
    multi = Model(HiGHS.Optimizer)
    # VARIABLES
    @variable(multi, trans[1:numorig, 1:numdest, 1:numprod] >= 0)
    # OBJECTIVE
    @objective(
        multi,
        Max,
        sum(
            cost[j, i, p] * trans[i, j, p] for i in 1:numorig, j in 1:numdest,
            p in 1:numprod
        )
    )
    # CONSTRAINTS
    # Supply constraint
    @constraint(
        multi,
        supply_con[i in 1:numorig, p in 1:numprod],
        sum(trans[i, j, p] for j in 1:numdest) == supply[p, i]
    )
    # Demand constraint
    @constraint(
        multi,
        demand_con[j in 1:numdest, p in 1:numprod],
        sum(trans[i, j, p] for i in 1:numorig) == demand[p, j]
    )
    # Total shipment constraint
    @constraint(
        multi,
        total_con[i in 1:numorig, j in 1:numdest],
        sum(trans[i, j, p] for p in 1:numprod) - limit[i, j] <= 0
    )
    optimize!(multi)
    Test.@test termination_status(multi) == OPTIMAL
    Test.@test primal_status(multi) == FEASIBLE_POINT
    Test.@test objective_value(multi) == 225_700.0
    if verbose
        println("RESULTS:")
        for i in 1:length(orig)
            for j in 1:length(dest)
                for p in 1:length(prod)
                    print(
                        " $(prod[p]) $(orig[i]) $(dest[j]) = $(value(trans[i, j, p]))\t",
                    )
                end
                println()
            end
        end
    end
    return
end
```

```
example_multi()
```

```
Presolving model
44 rows, 60 cols, 165 nonzeros
41 rows, 60 cols, 149 nonzeros
Presolve : Reductions: rows 41(-10); columns 60(-3); elements 149(-40)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -1.6599957764e+03 Ph1: 41(149); Du: 60(1660) 0s
         60    -2.2570000000e+05 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 60
Objective value     :  2.2570000000e+05
HiGHS run time      :          0.00
RESULTS:
 bands GARY FRA = 25.0   coils GARY FRA = 500.0  plate GARY FRA = 100.0
 bands GARY DET = 125.0  coils GARY DET = 0.0    plate GARY DET = 50.0
 bands GARY LAN = 0.0    coils GARY LAN = 0.0    plate GARY LAN = 0.0
 bands GARY WIN = 0.0    coils GARY WIN = 0.0    plate GARY WIN = 50.0
 bands GARY STL = 250.0  coils GARY STL = 300.0  plate GARY STL = 0.0
 bands GARY FRE = 0.0    coils GARY FRE = 0.0    plate GARY FRE = 0.0
 bands GARY LAF = 0.0    coils GARY LAF = 0.0    plate GARY LAF = 0.0
 bands CLEV FRA = 275.0  coils CLEV FRA = 0.0    plate CLEV FRA = 0.0
 bands CLEV DET = 0.0    coils CLEV DET = 300.0  plate CLEV DET = 50.0
 bands CLEV LAN = 100.0  coils CLEV LAN = 0.0    plate CLEV LAN = -0.0
 bands CLEV WIN = 0.0    coils CLEV WIN = 0.0    plate CLEV WIN = 0.0
 bands CLEV STL = 0.0    coils CLEV STL = 625.0  plate CLEV STL = 0.0
 bands CLEV FRE = 225.0  coils CLEV FRE = 400.0  plate CLEV FRE = 0.0
 bands CLEV LAF = 100.0  coils CLEV LAF = 275.0  plate CLEV LAF = 250.0
 bands PITT FRA = 0.0    coils PITT FRA = 0.0    plate PITT FRA = 0.0
 bands PITT DET = 175.0  coils PITT DET = 450.0  plate PITT DET = -0.0
 bands PITT LAN = 0.0    coils PITT LAN = 400.0  plate PITT LAN = 0.0
 bands PITT WIN = 75.0   coils PITT WIN = 250.0  plate PITT WIN = 0.0
 bands PITT STL = 400.0  coils PITT STL = 25.0   plate PITT STL = 200.0
 bands PITT FRE = 0.0    coils PITT FRE = 450.0  plate PITT FRE = 100.0
 bands PITT LAF = 150.0  coils PITT LAF = 225.0  plate PITT LAF = 0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.11 N-Queens

Originally Contributed by: Matthew Helm (with help from @mtanneau on Julia Discourse)

The N-Queens problem involves placing N queens on an N x N chessboard such that none of the queens attacks another. In chess, a queen can move vertically, horizontally, and diagonally so there cannot be more than one queen on any given row, column, or diagonal.

Note that none of the queens above are able to attack any other as a result of their careful placement.

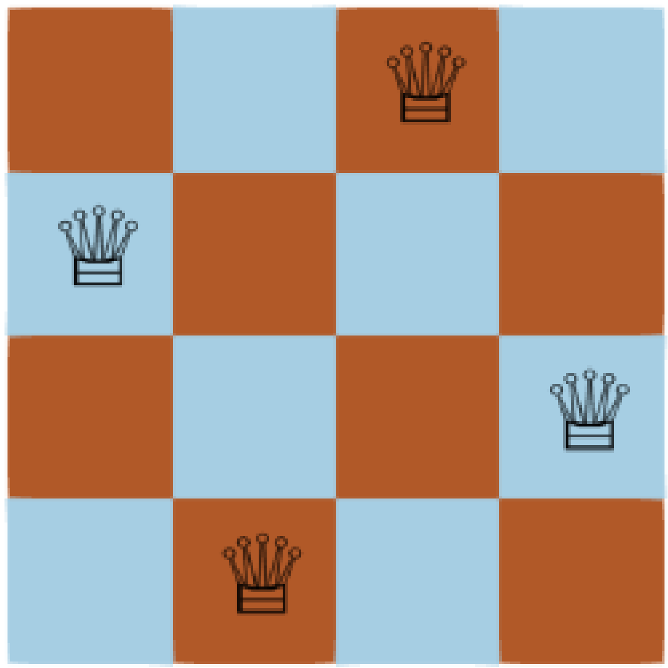


Figure 5.1: Four Queens

```
using JuMP
import HiGHS
import LinearAlgebra
```

N-Queens

```
N = 8

model = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Next, let's create an N x N chessboard of binary values. 0 will represent an empty space on the board and 1 will represent a space occupied by one of our queens:

```
@variable(model, x[1:N, 1:N], Bin)
```

```
8×8 Matrix{VariableRef}:
 x[1,1]  x[1,2]  x[1,3]  x[1,4]  x[1,5]  x[1,6]  x[1,7]  x[1,8]
 x[2,1]  x[2,2]  x[2,3]  x[2,4]  x[2,5]  x[2,6]  x[2,7]  x[2,8]
 x[3,1]  x[3,2]  x[3,3]  x[3,4]  x[3,5]  x[3,6]  x[3,7]  x[3,8]
 x[4,1]  x[4,2]  x[4,3]  x[4,4]  x[4,5]  x[4,6]  x[4,7]  x[4,8]
```

```
 x[5,1]  x[5,2]  x[5,3]  x[5,4]  x[5,5]  x[5,6]  x[5,7]  x[5,8]
 x[6,1]  x[6,2]  x[6,3]  x[6,4]  x[6,5]  x[6,6]  x[6,7]  x[6,8]
 x[7,1]  x[7,2]  x[7,3]  x[7,4]  x[7,5]  x[7,6]  x[7,7]  x[7,8]
 x[8,1]  x[8,2]  x[8,3]  x[8,4]  x[8,5]  x[8,6]  x[8,7]  x[8,8]
```

Now we can add our constraints:

There must be exactly one queen in a given row/column

```
for i in 1:N
    @constraint(model, sum(x[i, :]) == 1)
    @constraint(model, sum(x[:, i]) == 1)
end
```

There can only be one queen on any given diagonal

```
for i in -(N - 1):(N-1)
    @constraint(model, sum(LinearAlgebra.diag(x, i)) <= 1)
    @constraint(model, sum(LinearAlgebra.diag(reverse(x, dims = 1), i)) <= 1)
end
```

That's it! We are ready to put our model to work and see if it is able to find a feasible solution:

```
optimize!(model)
```

```
Presolving model
42 rows, 64 cols, 252 nonzeros
42 rows, 64 cols, 270 nonzeros
Objective function is integral with scale 1

Solving MIP model with:
   42 rows
   64 cols (64 binary, 0 integer, 0 implied int., 0 continuous)
   270 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) Found 1 full orbitope(s) acting on 64 columns

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

         0       0         0   0.00%   0               inf                  inf        0      0
      0         0     0.0s
         0       0         0   0.00%   0               inf                  inf        0      0
      4        29     0.0s

Solving report
  Status            Optimal
  Primal bound      0
  Dual bound        0
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
```

```
                   0 (objective)
                   0 (bound viol.)
                   8.82206753991e-15 (int. viol.)
                   0 (row viol.)
  Timing           0.03 (total)
                   0.00 (presolve)
                   0.00 (postsolve)
  Nodes            1
  LP iterations    142 (total)
                   0 (strong br.)
                   90 (separation)
                   23 (heuristics)
```

We can now review the solution that our model found:

```
solution = round.(Int, value.(x))
```

```
8×8 Matrix{Int64}:
 0  1  0  0  0  0  0  0
 0  0  0  0  1  0  0  0
 0  0  0  0  0  0  1  0
 1  0  0  0  0  0  0  0
 0  0  1  0  0  0  0  0
 0  0  0  0  0  0  0  1
 0  0  0  0  0  1  0  0
 0  0  0  1  0  0  0  0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.12 Sensitivity analysis of a linear program

This tutorial explains how to use the `lp_sensitivity_report` function to create sensitivity reports like those that are produced by the Excel Solver. This is most often used in introductory classes to linear programming.

In brief, sensitivity analysis of a linear program is about asking two questions:

1. Given an optimal solution, how much can I change the objective coefficients before a different solution becomes optimal?
2. Given an optimal solution, how much can I change the right-hand side of a linear constraint before a different solution becomes optimal?

JuMP provides a function, `lp_sensitivity_report`, to help us compute these values, but this tutorial extends that to create more informative tables in the form of a `DataFrame`.

Setup

This tutorial uses the following packages:

```julia
using JuMP
import HiGHS
import DataFrames
```

as well as this small linear program:

```julia
model = Model(HiGHS.Optimizer)
@variable(model, x >= 0)
@variable(model, 0 <= y <= 3)
@variable(model, z <= 1)
@objective(model, Min, 12x + 20y - z)
@constraint(model, c1, 6x + 8y >= 100)
@constraint(model, c2, 7x + 12y >= 120)
@constraint(model, c3, x + y <= 20)
optimize!(model)
solution_summary(model; verbose = true)
```

```
* Solver : HiGHS

* Status
  Termination status : OPTIMAL
  Primal status      : FEASIBLE_POINT
  Dual status        : FEASIBLE_POINT
  Result count       : 1
  Has duals          : true
  Message from the solver:
  "kHighsModelStatusOptimal"

* Candidate solution
  Objective value      : 2.04000e+02
  Objective bound      : 0.00000e+00
  Dual objective value : 2.04000e+02
  Primal solution :
    x : 1.50000e+01
    y : 1.25000e+00
    z : 1.00000e+00
  Dual solution :
    c1 : 2.50000e-01
    c2 : 1.50000e+00
    c3 : 0.00000e+00

* Work counters
  Solve time (sec)   : 3.20196e-04
  Simplex iterations : 2
  Barrier iterations : 0
```

Can you identify:

- The objective coefficient of each variable?
- The right-hand side of each constraint?
- The optimal primal and dual solutions?

Sensitivity reports

Now let's call `lp_sensitivity_report`:

```
report = lp_sensitivity_report(model)
```

```
SensitivityReport(Dict{ConstraintRef, Tuple{Float64, Float64}}(c3 : x + y ≤ 20.0 => (-3.75, Inf), c1
     : 6 x + 8 y ≥ 100.0 => (-4.0, 2.857142857142857), y ≥ 0.0 => (-Inf, 1.25), y ≤ 3.0 => (-1.75,
    Inf), z ≤ 1.0 => (-Inf, Inf), x ≥ 0.0 => (-Inf, 15.0), c2 : 7 x + 12 y ≥ 120.0 =>
    (-3.3333333333333335, 4.666666666666667)), Dict{VariableRef, Tuple{Float64, Float64}}(x =>
    (-0.3333333333333333, 3.0), z => (-Inf, 1.0), y => (-4.0, 0.5714285714285714)))
```

It returns a `SensitivityReport` object, which maps:

- Every variable reference to a tuple `(d_lo, d_hi)::Tuple{Float64,Float64}`, explaining how much the objective coefficient of the corresponding variable can change by, such that the original basis remains optimal.
- Every constraint reference to a tuple `(d_lo, d_hi)::Tuple{Float64,Float64}`, explaining how much the right-hand side of the corresponding constraint can change by, such that the basis remains optimal.

Both tuples are relative, rather than absolute. So, given an objective coefficient of `1.0` and a tuple `(-0.5, 0.5)`, the objective coefficient can range between `1.0 - 0.5` an `1.0 + 0.5`.

For example:

```
report[x]
```

```
(-0.3333333333333333, 3.0)
```

indicates that the objective coefficient on `x`, that is, 12, can decrease by `-0.333` or increase by `3.0` and the primal solution `(15, 1.25)` will remain optimal. In addition:

```
report[c1]
```

```
(-4.0, 2.857142857142857)
```

means that the right-hand side of the `c1` constraint (100), can decrease by 4 units, or increase by 2.85 units, aand the primal solution `(15, 1.25)` will remain optimal.

Variable sensitivity

By themselves, the tuples aren't informative. Let's put them in context by collating a range of other information about a variable:

```
function variable_report(xi)
    return (
        name = name(xi),
        lower_bound = has_lower_bound(xi) ? lower_bound(xi) : -Inf,
        value = value(xi),
        upper_bound = has_upper_bound(xi) ? upper_bound(xi) : Inf,
        reduced_cost = reduced_cost(xi),
```

```
        obj_coefficient = coefficient(objective_function(model), xi),
        allowed_decrease = report[xi][1],
        allowed_increase = report[xi][2],
    )
end
```

```
variable_report (generic function with 1 method)
```

Calling our function on x:

```
x_report = variable_report(x)
```

```
(name = "x", lower_bound = 0.0, value = 15.0, upper_bound = Inf, reduced_cost = 0.0, obj_coefficient
     = 12.0, allowed_decrease = -0.3333333333333333, allowed_increase = 3.0)
```

That's a bit hard to read, so let's call this on every variable in the model and put things into a DataFrame:

```
variable_df =
    DataFrames.DataFrame(variable_report(xi) for xi in all_variables(model))
```

	name	lower_bound	value	upper_bound	reduced_cost	obj_coefficient	allowed_decrease	allowed_increase
	String	Float64	Float64	Float64	Float64	Float64	Float64	Float64
1	x	0.0	15.0	Inf	0.0	12.0	-0.333333	3.0
2	y	0.0	1.25	3.0	0.0	20.0	-4.0	0.571429
3	z	-Inf	1.0	1.0	-1.0	-1.0	-Inf	1.0

Great! That looks just like the reports in Excel.

Constraint sensitivity

We can do something similar with constraints:

```
function constraint_report(ci)
    return (
        name = name(ci),
        value = value(ci),
        rhs = normalized_rhs(ci),
        slack = normalized_rhs(ci) - value(ci),
        shadow_price = shadow_price(ci),
        allowed_decrease = report[ci][1],
        allowed_increase = report[ci][2],
    )
end

c1_report = constraint_report(c1)
```

```
(name = "c1", value = 100.0, rhs = 100.0, slack = 0.0, shadow_price = -0.25, allowed_decrease =
    -4.0, allowed_increase = 2.857142857142857)
```

That's a bit hard to read, so let's call this on every variable in the model and put things into a DataFrame:

```
constraint_df = DataFrames.DataFrame(
    constraint_report(ci) for (F, S) in list_of_constraint_types(model) for
    ci in all_constraints(model, F, S) if F == AffExpr
)
```

	name	value	rhs	slack	shadow_price	allowed_decrease	allowed_increase
	String	Float64	Float64	Float64	Float64	Float64	Float64
1	c1	100.0	100.0	0.0	-0.25	-4.0	2.85714
2	c2	120.0	120.0	0.0	-1.5	-3.33333	4.66667
3	c3	16.25	20.0	3.75	0.0	-3.75	Inf

Analysis questions

Now we can use these dataframes to ask questions of the solution.

For example, we can find basic variables by looking for variables with a reduced cost of 0:

```
basic = filter(row -> iszero(row.reduced_cost), variable_df)
```

	name	lower_bound	value	upper_bound	reduced_cost	obj_coefficient	allowed_decrease	allowed_increase
	String	Float64	Float64	Float64	Float64	Float64	Float64	Float64
1	x	0.0	15.0	Inf	0.0	12.0	-0.333333	3.0
2	y	0.0	1.25	3.0	0.0	20.0	-4.0	0.571429

and non-basic variables by looking for non-zero reduced costs:

```
non_basic = filter(row -> !iszero(row.reduced_cost), variable_df)
```

	name	lower_bound	value	upper_bound	reduced_cost	obj_coefficient	allowed_decrease	allowed_increase
	String	Float64	Float64	Float64	Float64	Float64	Float64	Float64
1	z	-Inf	1.0	1.0	-1.0	-1.0	-Inf	1.0

we can also find constraints that are binding by looking for zero slacks:

```
binding = filter(row -> iszero(row.slack), constraint_df)
```

	name	value	rhs	slack	shadow_price	allowed_decrease	allowed_increase
	String	Float64	Float64	Float64	Float64	Float64	Float64
1	c1	100.0	100.0	0.0	-0.25	-4.0	2.85714
2	c2	120.0	120.0	0.0	-1.5	-3.33333	4.66667

or non-zero shadow prices:

```
binding2 = filter(row -> !iszero(row.shadow_price), constraint_df)
```

	name	value	rhs	slack	shadow_price	allowed_decrease	allowed_increase
	String	Float64	Float64	Float64	Float64	Float64	Float64
1	c1	100.0	100.0	0.0	-0.25	-4.0	2.85714
2	c2	120.0	120.0	0.0	-1.5	-3.33333	4.66667

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.13 Network flow problems

Originally Contributed by: Arpit Bhatia

In graph theory, a flow network (also known as a transportation network) is a directed graph where each edge has a capacity and each edge receives a flow. The amount of flow on an edge cannot exceed the capacity of the edge.

Often in operations research, a directed graph is called a network, the vertices are called nodes and the edges are called arcs.

A flow must satisfy the restriction that the amount of flow into a node equals the amount of flow out of it, unless it is a source, which has only outgoing flow, or sink, which has only incoming flow.

A network can be used to model traffic in a computer network, circulation with demands, fluids in pipes, currents in an electrical circuit, or anything similar in which something travels through a network of nodes.

```
using JuMP
import HiGHS
import LinearAlgebra
```

The shortest path problem

Suppose that each arc (i, j) of a graph is assigned a scalar cost $a_{i,j}$, and suppose that we define the cost of a forward path to be the sum of the costs of its arcs.

Given a pair of nodes, the shortest path problem is to find a forward path that connects these nodes and has minimum cost.

$$
\begin{aligned}
\min && \sum_{\forall e(i,j) \in E} a_{i,j} \times x_{i,j} \\
s.t. && b(i) = \sum_j x_{ij} - \sum_k x_{ki} = \begin{cases} 1 & \text{if } i \text{ is the starting node,} \\ -1 & \text{if } i \text{ is the ending node,} \\ 0 & \text{otherwise.} \end{cases} \\
&& x_e \in \{0, 1\} \quad \forall e \in E
\end{aligned}
$$

```
G = [
    0 100 30 0 0
    0 0 20 0 0
    0 0 0 10 60
    0 15 0 0 50
    0 0 0 0 0
]

n = size(G)[1]

shortest_path = Model(HiGHS.Optimizer)

@variable(shortest_path, x[1:n, 1:n], Bin)
```

```
5×5 Matrix{VariableRef}:
 x[1,1]  x[1,2]  x[1,3]  x[1,4]  x[1,5]
 x[2,1]  x[2,2]  x[2,3]  x[2,4]  x[2,5]
```

```
 x[3,1]  x[3,2]  x[3,3]  x[3,4]  x[3,5]
 x[4,1]  x[4,2]  x[4,3]  x[4,4]  x[4,5]
 x[5,1]  x[5,2]  x[5,3]  x[5,4]  x[5,5]
```

Arcs with zero cost are not a part of the path as they do no exist

```
@constraint(shortest_path, [i = 1:n, j = 1:n; G[i, j] == 0], x[i, j] == 0)
```

```
JuMP.Containers.SparseAxisArray{ConstraintRef{Model, MathOptInterface.ConstraintIndex{
    MathOptInterface.ScalarAffineFunction{Float64}, MathOptInterface.EqualTo{Float64}}, ScalarShape
    }, 2, Tuple{Int64, Int64}} with 18 entries:
  [1, 1]  =  x[1,1] = 0.0
  [1, 4]  =  x[1,4] = 0.0
  [2, 2]  =  x[2,2] = 0.0
  [2, 4]  =  x[2,4] = 0.0
  [2, 5]  =  x[2,5] = 0.0
  [3, 1]  =  x[3,1] = 0.0
  [3, 2]  =  x[3,2] = 0.0
  [3, 3]  =  x[3,3] = 0.0

  [4, 1]  =  x[4,1] = 0.0
  [4, 3]  =  x[4,3] = 0.0
  [4, 4]  =  x[4,4] = 0.0
  [5, 2]  =  x[5,2] = 0.0
  [5, 3]  =  x[5,3] = 0.0
  [5, 4]  =  x[5,4] = 0.0
  [5, 5]  =  x[5,5] = 0.0
```

Flow conservation constraint

```
@constraint(
    shortest_path,
    [i = 1:n; i != 1 && i != 2],
    sum(x[i, :]) == sum(x[:, i])
)
```

```
JuMP.Containers.SparseAxisArray{ConstraintRef{Model, MathOptInterface.ConstraintIndex{
    MathOptInterface.ScalarAffineFunction{Float64}, MathOptInterface.EqualTo{Float64}}, ScalarShape
    }, 1, Tuple{Int64}} with 3 entries:
  [3]  =  x[3,1] + x[3,2] - x[1,3] - x[2,3] - x[4,3] - x[5,3] + x[3,4] + x[3,5] = 0.0
  [4]  =  x[4,1] + x[4,2] + x[4,3] - x[1,4] - x[2,4] - x[3,4] - x[5,4] + x[4,5] = 0.0
  [5]  =  x[5,1] + x[5,2] + x[5,3] + x[5,4] - x[1,5] - x[2,5] - x[3,5] - x[4,5] = 0.0
```

Flow coming out of source = 1

```
@constraint(shortest_path, sum(x[1, :]) - sum(x[:, 1]) == 1)
```

$$-x_{2,1} - x_{3,1} - x_{4,1} - x_{5,1} + x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} = 1.0$$

Flowing coming out of destination = -1 i.e. Flow entering destination = 1

```
@constraint(shortest_path, sum(x[2, :]) - sum(x[:, 2]) == -1)
@objective(shortest_path, Min, LinearAlgebra.dot(G, x))

optimize!(shortest_path)
objective_value(shortest_path)
```

```
55.0
```

```
value.(x)
```

```
5×5 Matrix{Float64}:
 0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0
 0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0
```

The assignment problem

Suppose that there are n persons and n objects that we have to match on a one-to-one basis. There is a benefit or value $a_{i,j}$ for matching person i with object j, and we want to assign persons to objects so as to maximize the total benefit.

There is also a restriction that person i can be assigned to object j only if (i, j) belongs to a given set of pairs A.

Mathematically, we want to find a set of person-object pairs $(1, j_1), ..., (n, j_n)$ from A such that the objects $j_1, ..., j_n$ are all distinct, and the total benefit $\sum_{i=1}^{y} a_{ij_i}$ is maximized.

$$
\begin{aligned}
\max \quad & \sum_{(i,j)\in A} a_{i,j} \times y_{i,j} \\
s.t. \quad & \sum_{\{j|(i,j)\in A\}} y_{i,j} = 1 \qquad \forall i = \{1, 2....n\} \\
& \sum_{\{i|(i,j)\in A\}} y_{i,j} = 1 \qquad \forall j = \{1, 2....n\} \\
& y_{i,j} \in \{0, 1\} \quad \forall (i, j) \in \{1, 2...k\}
\end{aligned}
$$

```
G = [
    6 4 5 0
    0 3 6 0
    5 0 4 3
    7 5 5 5
]

n = size(G)[1]

assignment = Model(HiGHS.Optimizer)
@variable(assignment, y[1:n, 1:n], Bin)
```

```
4×4 Matrix{VariableRef}:
 y[1,1]  y[1,2]  y[1,3]  y[1,4]
 y[2,1]  y[2,2]  y[2,3]  y[2,4]
 y[3,1]  y[3,2]  y[3,3]  y[3,4]
 y[4,1]  y[4,2]  y[4,3]  y[4,4]
```

One person can only be assigned to one object

```
@constraint(assignment, [i = 1:n], sum(y[:, i]) == 1)
```

```
4-element Vector{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
    ScalarAffineFunction{Float64}, MathOptInterface.EqualTo{Float64}}, ScalarShape}}:
 y[1,1] + y[2,1] + y[3,1] + y[4,1] = 1.0
 y[1,2] + y[2,2] + y[3,2] + y[4,2] = 1.0
 y[1,3] + y[2,3] + y[3,3] + y[4,3] = 1.0
 y[1,4] + y[2,4] + y[3,4] + y[4,4] = 1.0
```

One object can only be assigned to one person

```
@constraint(assignment, [j = 1:n], sum(y[j, :]) == 1)
@objective(assignment, Max, LinearAlgebra.dot(G, y))

optimize!(assignment)
objective_value(assignment)
```

```
20.0
```

```
value.(y)
```

```
4×4 Matrix{Float64}:
 -0.0  1.0  -0.0   0.0
  0.0  0.0   1.0   0.0
  1.0  0.0   0.0  -0.0
 -0.0  0.0   0.0   1.0
```

The max-flow problem

In the max-flow problem, we have a graph with two special nodes: the $source$, denoted by s, and the $sink$, denoted by t.

The objective is to move as much flow as possible from s into t while observing the capacity constraints.

$$
\begin{aligned}
\max \quad & & \sum_{v:(s,v)\in E} f(s,v) & \\
s.t. \quad & \sum_{u:(u,v)\in E} f(u,v) = & \sum_{w:(v,w)\in E} f(v,w) & \quad \forall v \in V - \{s,t\} \\
& & f(u,v) \leq c(u,v) & \quad \forall (u,v) \in E \\
& & f(u,v) \geq 0 & \quad \forall (u,v) \in E
\end{aligned}
$$

```
G = [
    0 3 2 2 0 0 0 0
    0 0 0 0 5 1 0 0
    0 0 0 0 1 3 1 0
    0 0 0 0 0 1 0 0
    0 0 0 0 0 0 0 4
    0 0 0 0 0 0 0 2
    0 0 0 0 0 0 0 4
    0 0 0 0 0 0 0 0
]

n = size(G)[1]

max_flow = Model(HiGHS.Optimizer)

@variable(max_flow, f[1:n, 1:n] >= 0)
```

```
8×8 Matrix{VariableRef}:
 f[1,1]  f[1,2]  f[1,3]  f[1,4]  f[1,5]  f[1,6]  f[1,7]  f[1,8]
 f[2,1]  f[2,2]  f[2,3]  f[2,4]  f[2,5]  f[2,6]  f[2,7]  f[2,8]
 f[3,1]  f[3,2]  f[3,3]  f[3,4]  f[3,5]  f[3,6]  f[3,7]  f[3,8]
 f[4,1]  f[4,2]  f[4,3]  f[4,4]  f[4,5]  f[4,6]  f[4,7]  f[4,8]
 f[5,1]  f[5,2]  f[5,3]  f[5,4]  f[5,5]  f[5,6]  f[5,7]  f[5,8]
 f[6,1]  f[6,2]  f[6,3]  f[6,4]  f[6,5]  f[6,6]  f[6,7]  f[6,8]
 f[7,1]  f[7,2]  f[7,3]  f[7,4]  f[7,5]  f[7,6]  f[7,7]  f[7,8]
 f[8,1]  f[8,2]  f[8,3]  f[8,4]  f[8,5]  f[8,6]  f[8,7]  f[8,8]
```

Capacity constraints

```
@constraint(max_flow, [i = 1:n, j = 1:n], f[i, j] <= G[i, j])
```

```
8×8 Matrix{ConstraintRef{Model, MathOptInterface.ConstraintIndex{MathOptInterface.
     ScalarAffineFunction{Float64}, MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 f[1,1] ≤ 0.0  f[1,2] ≤ 3.0  f[1,3] ≤ 2.0  …  f[1,7] ≤ 0.0  f[1,8] ≤ 0.0
 f[2,1] ≤ 0.0  f[2,2] ≤ 0.0  f[2,3] ≤ 0.0     f[2,7] ≤ 0.0  f[2,8] ≤ 0.0
 f[3,1] ≤ 0.0  f[3,2] ≤ 0.0  f[3,3] ≤ 0.0     f[3,7] ≤ 1.0  f[3,8] ≤ 0.0
 f[4,1] ≤ 0.0  f[4,2] ≤ 0.0  f[4,3] ≤ 0.0     f[4,7] ≤ 0.0  f[4,8] ≤ 0.0
 f[5,1] ≤ 0.0  f[5,2] ≤ 0.0  f[5,3] ≤ 0.0     f[5,7] ≤ 0.0  f[5,8] ≤ 4.0
 f[6,1] ≤ 0.0  f[6,2] ≤ 0.0  f[6,3] ≤ 0.0  …  f[6,7] ≤ 0.0  f[6,8] ≤ 2.0
 f[7,1] ≤ 0.0  f[7,2] ≤ 0.0  f[7,3] ≤ 0.0     f[7,7] ≤ 0.0  f[7,8] ≤ 4.0
 f[8,1] ≤ 0.0  f[8,2] ≤ 0.0  f[8,3] ≤ 0.0     f[8,7] ≤ 0.0  f[8,8] ≤ 0.0
```

Flow conservation constraints

```
@constraint(max_flow, [i = 1:n; i != 1 && i != 8], sum(f[i, :]) == sum(f[:, i]))
@objective(max_flow, Max, sum(f[1, :]))

optimize!(max_flow)
objective_value(max_flow)
```

```
6.0
```

```
value.(f)
```

```
8×8 Matrix{Float64}:
 -0.0   3.0   2.0  1.0  -0.0  -0.0  -0.0  -0.0
  0.0   0.0   0.0  0.0   2.0   1.0   0.0   0.0
  0.0   0.0   0.0  0.0   1.0  -0.0   1.0   0.0
 -0.0  -0.0  -0.0  0.0  -0.0   1.0  -0.0  -0.0
  0.0   0.0   0.0  0.0   0.0   0.0   0.0   3.0
  0.0   0.0   0.0  0.0   0.0   0.0   0.0   2.0
  0.0   0.0   0.0  0.0   0.0   0.0   0.0   1.0
  0.0   0.0   0.0  0.0   0.0   0.0   0.0   0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.14 The workforce scheduling problem

This model determines a set of workforce levels that will most economically meet demands and inventory requirements over time. The formulation is motivated by the experiences of a large producer in the United States. The data are for three products and 13 periods.

Problem taken from the Appendix C of the expanded version of Fourer, Gay, and Kernighan, A Modeling Language for Mathematical Programming

Originally contributed by Louis Luangkesorn, February 26, 2015.

```julia
using JuMP
import HiGHS
import Test

function example_prod(; verbose = true)
    # PRODUCTION SETS AND PARAMETERS
    prd = ["18REG" "24REG" "24PRO"]
    # Members of the product group
    numprd = length(prd)
    pt = [1.194, 1.509, 1.509]
    # Crew-hours to produce 1000 units
    pc = [2304, 2920, 2910]
    # Nominal production cost per 1000, used
    # to compute inventory and shortage costs
    #
    # TIME PERIOD SETS AND PARAMETERS
    firstperiod = 1
    # Index of first production period to be modeled
    lastperiod = 13
    # Index of last production period to be modeled
    numperiods = firstperiod:lastperiod
    # 'planning horizon' := first..last;
    # EMPLOYMENT PARAMETERS
    # Workers per crew
    cs = 18
    # Regular-time hours per shift
    sl = 8
    # Wage per hour for regular-time labor
    rtr = 16.00
```

```
# Wage per hour for overtime labor
otr = 43.85
# Crews employed at start of first period
iw = 8
# Regular working days in a production period
dpp = [19.5, 19, 20, 19, 19.5, 19, 19, 20, 19, 20, 20, 18, 18]
# Maximum crew-hours of overtime in a period
ol = [96, 96, 96, 96, 96, 96, 96, 96, 96, 96, 96, 96, 96]
# Lower limit on average employment in a period
cmin = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
# Upper limit on average employment in a period
cmax = [8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8]
# Penalty cost of hiring a crew
hc = [
    7500,
    7500,
    7500,
    7500,
    15000,
    15000,
    15000,
    15000,
    15000,
    15000,
    7500,
    7500,
    7500,
]
# Penalty cost of laying off a crew
lc = [
    7500,
    7500,
    7500,
    7500,
    15000,
    15000,
    15000,
    15000,
    15000,
    15000,
    7500,
    7500,
    7500,
]
# DEMAND PARAMETERS
d18REG = [
    63.8,
    76,
    88.4,
    913.8,
    115,
    133.8,
    79.6,
    111,
    121.6,
    470,
```

```
        78.4,
        99.4,
        140.4,
        63.8,
    ]
    d24REG = [
        1212,
        306.2,
        319,
        208.4,
        298,
        328.2,
        959.6,
        257.6,
        335.6,
        118,
        284.8,
        970,
        343.8,
        1212,
    ]
    d24PRO = [0, 0, 0, 0, 0, 0, 0, 0, 0, 1102, 0, 0, 0, 0]
    # Requirements (in 1000s) to be met from current production and inventory
    dem = Array[d18REG, d24REG, d24PRO]
    # true if product will be the subject of a special promotion in the period
    pro = Array[
        [0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
        [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 1],
        [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
    ]
    # INVENTORY AND SHORTAGE PARAMETERS
    # Proportion of non-promoted demand that must be in inventory the previous
    # period
    rir = 0.75
    # Proportion of promoted demand that must be in inventory the previous
    # period
    pir = 0.80
    # Upper limit on number of periods that any product may sit in inventory
    life = 2
    # Inventory cost per 1000 units is cri times nominal production cost
    cri = [0.015, 0.015, 0.015]
    # Shortage cost per 1000 units is crs times nominal production cost
    crs = [1.1, 1.1, 1.1]
    # Inventory at start of first period; age unknown
    iinv = [82, 792.2, 0]
    # Initial inventory still available for allocation at end of period t
    iil = [
        [
            max(0, iinv[p] - sum(dem[p][v] for v in firstperiod:t)) for
            t in numperiods
        ] for p in 1:numprd
    ]
    # Lower limit on inventory at end of period t
    function checkpro(
        product,
        timeperiod,
```

```
        production,
        promotionalrate,
        regularrate,
    )
        if production[product][timeperiod+1] == 1
            return promotionalrate
        else
            return regularrate
        end
    end
    minv = [
        [dem[p][t+1] * checkpro(p, t, pro, pir, rir) for t in numperiods]
        for p in 1:numprd
    ]
    # DEFINE MODEL
    prod = Model(HiGHS.Optimizer)
    # VARIABLES
    # Average number of crews employed in each period
    @variable(prod, Crews[0:lastperiod] >= 0)
    # Crews hired from previous to current period
    @variable(prod, Hire[numperiods] >= 0)
    # Crews laid off from previous to current period
    @variable(prod, Layoff[numperiods] >= 0)
    # Production using regular-time labor, in 1000s
    @variable(prod, Rprd[1:numprd, numperiods] >= 0)
    # Production using overtime labor, in 1000s
    @variable(prod, Oprd[1:numprd, numperiods] >= 0)
    # a numperiods old -- produced in period (t+1)-a --
    # and still in storage at the end of period t
    @variable(prod, Inv[1:numprd, numperiods, 1:life] >= 0)
    # Accumulated unsatisfied demand at the end of period t
    @variable(prod, Short[1:numprd, numperiods] >= 0)
    # CONSTRAINTS
    # Hours needed to accomplish all regular-time production in a period must
    # not exceed hours available on all shifts
    @constraint(
        prod,
        [t = numperiods],
        sum(pt[p] * Rprd[p, t] for p in 1:numprd) <= sl * dpp[t] * Crews[t]
    )
    # Hours needed to accomplish all overtime production in a period must not
    # exceed the specified overtime limit
    @constraint(
        prod,
        [t = numperiods],
        sum(pt[p] * Oprd[p, t] for p in 1:numprd) <= ol[t]
    )
    # Use given initial workforce
    @constraint(prod, Crews[firstperiod-1] == iw)
    # Workforce changes by hiring or layoffs
    @constraint(
        prod,
        [t in numperiods],
        Crews[t] == Crews[t-1] + Hire[t] - Layoff[t]
    )
    # Workforce must remain within specified bounds
```

```
@constraint(prod, [t in numperiods], cmin[t] <= Crews[t])
@constraint(prod, [t in numperiods], Crews[t] <= cmax[t])
# 'first demand requirement
@constraint(
    prod,
    [p in 1:numprd],
    Rprd[p, firstperiod] + Oprd[p, firstperiod] + Short[p, firstperiod] -
    Inv[p, firstperiod, 1] == max(0, dem[p][firstperiod] - iinv[p])
)
# Production plus increase in shortage plus decrease in inventory must
# equal demand
for t in (firstperiod+1):lastperiod
    @constraint(
        prod,
        [p in 1:numprd],
        Rprd[p, t] + Oprd[p, t] + Short[p, t] - Short[p, t-1] +
        sum(Inv[p, t-1, a] - Inv[p, t, a] for a in 1:life) ==
        max(0, dem[p][t] - iil[p][t-1])
    )
end
# Inventory in storage at end of period t must meet specified minimum
@constraint(
    prod,
    [p in 1:numprd, t in numperiods],
    sum(Inv[p, t, a] + iil[p][t] for a in 1:life) >= minv[p][t]
)
# In the vth period (starting from first) no inventory may be more than v
# numperiods old (initial inventories are handled separately)
@constraint(
    prod,
    [p in 1:numprd, v in 1:(life-1), a in (v+1):life],
    Inv[p, firstperiod+v-1, a] == 0
)
# New inventory cannot exceed production in the most recent period
@constraint(
    prod,
    [p in 1:numprd, t in numperiods],
    Inv[p, t, 1] <= Rprd[p, t] + Oprd[p, t]
)
# Inventory left from period (t+1)-p can only decrease as time goes on
secondperiod = firstperiod + 1
@constraint(
    prod,
    [p in 1:numprd, t in 2:lastperiod, a in 2:life],
    Inv[p, t, a] <= Inv[p, t-1, a-1]
)
# OBJECTIVE
# Full regular wages for all crews employed, plus penalties for hiring and
# layoffs, plus wages for any overtime worked, plus inventory and shortage
# costs. (All other production costs are assumed to depend on initial
# inventory and on demands, and so are not included explicitly.)
@objective(
    prod,
    Min,
    sum(
        rtr * sl * dpp[t] * cs * Crews[t] +
```

```
            hc[t] * Hire[t] +
            lc[t] * Layoff[t] +
            sum(
                otr * cs * pt[p] * Oprd[p, t] +
                sum(cri[p] * pc[p] * Inv[p, t, a] for a in 1:life) +
                crs[p] * pc[p] * Short[p, t] for p in 1:numprd
            ) for t in numperiods
        )
    )
    # Obtain solution
    optimize!(prod)
    Test.@test termination_status(prod) == OPTIMAL
    Test.@test primal_status(prod) == FEASIBLE_POINT
    Test.@test objective_value(prod) ≈ 4_426_822.89 atol = 1e-2
    if verbose
        println("RESULTS:")
        println("Crews")
        for t in 0:length(Crews.data)-1
            print(" $(value(Crews[t])) ")
        end
        println()
        println("Hire")
        for t in 1:length(Hire.data)
            print(" $(value(Hire[t])) ")
        end
        println()
        println("Layoff")
        for t in 1:length(Layoff.data)
            print(" $(value(Layoff[t])) ")
        end
        println()
    end
    return
end

example_prod()
```

```
Presolving model
177 rows, 230 cols, 675 nonzeros
166 rows, 218 cols, 695 nonzeros
Presolve : Reductions: rows 166(-56); columns 218(-17); elements 695(-45)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0    -2.6189817551e+02 Ph1: 9(8.2635); Du: 3(261.898) 0s
        111     4.4268228908e+06 Pr: 0(0); Du: 0(1.42109e-13) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 111
Objective value     :  4.4268228908e+06
HiGHS run time      :          0.00
RESULTS:
Crews
 8.0  6.439849038461538  5.947339134615384  5.947339134615384  5.947339134615384  5.947339134615384
     6.162946095647774  6.340057203947365  7.805664375000001  7.805664375000001  7.805664375000001
    7.805664375000001  8.0  8.0
```

```
Hire
 0.0  0.0  0.0  0.0  0.0  0.21560696103238985  0.17711110829959065  1.4656071710526364  0.0  0.0
     0.0  0.194335624999999  0.0
Layoff
 1.5601509615384623  0.49250990384615356  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.15 The SteelT3 problem

The steelT3 model from AMPL: A Modeling Language for Mathematical Programming, 2nd ed by Robert Fourer, David Gay, and Brian W. Kernighan.

Originally contributed by Louis Luangkesorn, April 3, 2015.

```julia
using JuMP
import HiGHS
import Test

function example_steelT3(; verbose = true)
    T = 4
    prod = ["bands", "coils"]
    area = Dict(
        "bands" => ("east", "north"),
        "coils" => ("east", "west", "export"),
    )
    avail = [40, 40, 32, 40]
    rate = Dict("bands" => 200, "coils" => 140)
    inv0 = Dict("bands" => 10, "coils" => 0)
    prodcost = Dict("bands" => 10, "coils" => 11)
    invcost = Dict("bands" => 2.5, "coils" => 3)
    revenue = Dict(
        "bands" => Dict(
            "east" => [25.0, 26.0, 27.0, 27.0],
            "north" => [26.5, 27.5, 28.0, 28.5],
        ),
        "coils" => Dict(
            "east" => [30, 35, 37, 39],
            "west" => [29, 32, 33, 35],
            "export" => [25, 25, 25, 28],
        ),
    )
    market = Dict(
        "bands" => Dict(
            "east" => [2000, 2000, 1500, 2000],
            "north" => [4000, 4000, 2500, 4500],
        ),
        "coils" => Dict(
            "east" => [1000, 800, 1000, 1100],
            "west" => [2000, 1200, 2000, 2300],
            "export" => [1000, 500, 500, 800],
```

```
        ),
    )
    # Model
    model = Model(HiGHS.Optimizer)
    # Decision Variables
    @variables(
        model,
        begin
            make[p in prod, t in 1:T] >= 0
            inventory[p in prod, t in 0:T] >= 0
            0 <= sell[p in prod, a in area[p], t in 1:T] <= market[p][a][t]
        end
    )
    @constraints(
        model,
        begin
            [p = prod, a = area[p], t = 1:T], sell[p, a, t] <= market[p][a][t]
            # Total of hours used by all products may not exceed hours available,
            # in each week
            [t in 1:T], sum(1 / rate[p] * make[p, t] for p in prod) <= avail[t]
            # Initial inventory must equal given value
            [p in prod], inventory[p, 0] == inv0[p]
            # Tons produced and taken from inventory must equal tons sold and put
            # into inventory.
            [p in prod, t in 1:T],
            make[p, t] + inventory[p, t-1] ==
            sum(sell[p, a, t] for a in area[p]) + inventory[p, t]
        end
    )
    # Maximize total profit: total revenue less costs for all products in all
    # weeks.
    @objective(
        model,
        Max,
        sum(
            revenue[p][a][t] * sell[p, a, t] - prodcost[p] * make[p, t] -
            invcost[p] * inventory[p, t] for p in prod, a in area[p], t in 1:T
        )
    )
    optimize!(model)
    Test.@test termination_status(model) == OPTIMAL
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) == 172850.0
    if verbose
        println("RESULTS:")
        for p in prod
            println("make $(p)")
            for t in 1:T
                print(value(make[p, t]), "\t")
            end
            println()
            println("Inventory $(p)")
            for t in 1:T
                print(value(inventory[p, t]), "\t")
            end
            println()
```

```
            for a in area[p]
                println("sell $(p) $(a)")
                for t in 1:T
                    print(value(sell[p, a, t]), "\t")
                end
                println()
            end
        end
    end
    return
end

example_steelT3()
```

```
Presolving model
12 rows, 36 cols, 50 nonzeros
11 rows, 32 cols, 45 nonzeros
Presolve : Reductions: rows 11(-23); columns 32(-6); elements 45(-29)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Ph1: 0(0) 0s
         13    -1.7285000000e+05 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 13
Objective value     :  1.7285000000e+05
HiGHS run time      :          0.00
RESULTS:
make bands
5990.0  6000.0  4000.0  6500.0
Inventory bands
0.0     0.0     0.0     0.0
sell bands east
2000.0  2000.0  1500.0  2000.0
sell bands north
4000.0  4000.0  2500.0  4500.0
make coils
-0.0    800.0   1000.0  1050.0
Inventory coils
0.0     0.0     0.0     0.0
sell coils east
0.0     800.0   1000.0  1050.0
sell coils west
0.0     0.0     0.0     0.0
sell coils export
0.0     0.0     0.0     0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Figure 5.2: Partially solved Sudoku

5.16 Sudoku

Originally Contributed by: Iain Dunning

Sudoku is a popular number puzzle. The goal is to place the digits 1,...,9 on a nine-by-nine grid, with some of the digits already filled in. Your solution must satisfy the following rules:

- The numbers 1 to 9 must appear in each 3x3 square
- The numbers 1 to 9 must appear in each row
- The numbers 1 to 9 must appear in each column

Here is a partially solved Sudoku problem:

Solving a Sudoku isn't an optimization problem with an objective; its actually a feasibility problem: we wish to find a feasible solution that satisfies these rules. You can think of it as an optimization problem with an objective of 0.

We can model this problem using 0-1 integer programming: a problem where all the decision variables are binary. We'll use JuMP to create the model, and then we can solve it with any integer programming solver.

```
using JuMP
using HiGHS
```

We will define a binary variable (a variable that is either 0 or 1) for each possible number in each possible cell. The meaning of each variable is as follows: `x[i,j,k] = 1 if and only if cell (i,j) has number k`, where `i` is the row and `j` is the column.

Create a model

```
sudoku = Model(HiGHS.Optimizer)
```

```
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Create our variables

```
@variable(sudoku, x[i = 1:9, j = 1:9, k = 1:9], Bin)
```

```
9×9×9 Array{VariableRef, 3}:
[:, :, 1] =
 x[1,1,1]  x[1,2,1]  x[1,3,1]  x[1,4,1]  …  x[1,7,1]  x[1,8,1]  x[1,9,1]
 x[2,1,1]  x[2,2,1]  x[2,3,1]  x[2,4,1]     x[2,7,1]  x[2,8,1]  x[2,9,1]
 x[3,1,1]  x[3,2,1]  x[3,3,1]  x[3,4,1]     x[3,7,1]  x[3,8,1]  x[3,9,1]
 x[4,1,1]  x[4,2,1]  x[4,3,1]  x[4,4,1]     x[4,7,1]  x[4,8,1]  x[4,9,1]
 x[5,1,1]  x[5,2,1]  x[5,3,1]  x[5,4,1]     x[5,7,1]  x[5,8,1]  x[5,9,1]
 x[6,1,1]  x[6,2,1]  x[6,3,1]  x[6,4,1]  …  x[6,7,1]  x[6,8,1]  x[6,9,1]
 x[7,1,1]  x[7,2,1]  x[7,3,1]  x[7,4,1]     x[7,7,1]  x[7,8,1]  x[7,9,1]
 x[8,1,1]  x[8,2,1]  x[8,3,1]  x[8,4,1]     x[8,7,1]  x[8,8,1]  x[8,9,1]
 x[9,1,1]  x[9,2,1]  x[9,3,1]  x[9,4,1]     x[9,7,1]  x[9,8,1]  x[9,9,1]

[:, :, 2] =
 x[1,1,2]  x[1,2,2]  x[1,3,2]  x[1,4,2]  …  x[1,7,2]  x[1,8,2]  x[1,9,2]
 x[2,1,2]  x[2,2,2]  x[2,3,2]  x[2,4,2]     x[2,7,2]  x[2,8,2]  x[2,9,2]
 x[3,1,2]  x[3,2,2]  x[3,3,2]  x[3,4,2]     x[3,7,2]  x[3,8,2]  x[3,9,2]
 x[4,1,2]  x[4,2,2]  x[4,3,2]  x[4,4,2]     x[4,7,2]  x[4,8,2]  x[4,9,2]
 x[5,1,2]  x[5,2,2]  x[5,3,2]  x[5,4,2]     x[5,7,2]  x[5,8,2]  x[5,9,2]
 x[6,1,2]  x[6,2,2]  x[6,3,2]  x[6,4,2]  …  x[6,7,2]  x[6,8,2]  x[6,9,2]
 x[7,1,2]  x[7,2,2]  x[7,3,2]  x[7,4,2]     x[7,7,2]  x[7,8,2]  x[7,9,2]
 x[8,1,2]  x[8,2,2]  x[8,3,2]  x[8,4,2]     x[8,7,2]  x[8,8,2]  x[8,9,2]
 x[9,1,2]  x[9,2,2]  x[9,3,2]  x[9,4,2]     x[9,7,2]  x[9,8,2]  x[9,9,2]

[:, :, 3] =
 x[1,1,3]  x[1,2,3]  x[1,3,3]  x[1,4,3]  …  x[1,7,3]  x[1,8,3]  x[1,9,3]
 x[2,1,3]  x[2,2,3]  x[2,3,3]  x[2,4,3]     x[2,7,3]  x[2,8,3]  x[2,9,3]
 x[3,1,3]  x[3,2,3]  x[3,3,3]  x[3,4,3]     x[3,7,3]  x[3,8,3]  x[3,9,3]
 x[4,1,3]  x[4,2,3]  x[4,3,3]  x[4,4,3]     x[4,7,3]  x[4,8,3]  x[4,9,3]
 x[5,1,3]  x[5,2,3]  x[5,3,3]  x[5,4,3]     x[5,7,3]  x[5,8,3]  x[5,9,3]
```

```
 x[6,1,3]  x[6,2,3]  x[6,3,3]  x[6,4,3]  …  x[6,7,3]  x[6,8,3]  x[6,9,3]
 x[7,1,3]  x[7,2,3]  x[7,3,3]  x[7,4,3]     x[7,7,3]  x[7,8,3]  x[7,9,3]
 x[8,1,3]  x[8,2,3]  x[8,3,3]  x[8,4,3]     x[8,7,3]  x[8,8,3]  x[8,9,3]
 x[9,1,3]  x[9,2,3]  x[9,3,3]  x[9,4,3]     x[9,7,3]  x[9,8,3]  x[9,9,3]

[:, :, 4] =
 x[1,1,4]  x[1,2,4]  x[1,3,4]  x[1,4,4]  …  x[1,7,4]  x[1,8,4]  x[1,9,4]
 x[2,1,4]  x[2,2,4]  x[2,3,4]  x[2,4,4]     x[2,7,4]  x[2,8,4]  x[2,9,4]
 x[3,1,4]  x[3,2,4]  x[3,3,4]  x[3,4,4]     x[3,7,4]  x[3,8,4]  x[3,9,4]
 x[4,1,4]  x[4,2,4]  x[4,3,4]  x[4,4,4]     x[4,7,4]  x[4,8,4]  x[4,9,4]
 x[5,1,4]  x[5,2,4]  x[5,3,4]  x[5,4,4]     x[5,7,4]  x[5,8,4]  x[5,9,4]
 x[6,1,4]  x[6,2,4]  x[6,3,4]  x[6,4,4]  …  x[6,7,4]  x[6,8,4]  x[6,9,4]
 x[7,1,4]  x[7,2,4]  x[7,3,4]  x[7,4,4]     x[7,7,4]  x[7,8,4]  x[7,9,4]
 x[8,1,4]  x[8,2,4]  x[8,3,4]  x[8,4,4]     x[8,7,4]  x[8,8,4]  x[8,9,4]
 x[9,1,4]  x[9,2,4]  x[9,3,4]  x[9,4,4]     x[9,7,4]  x[9,8,4]  x[9,9,4]

[:, :, 5] =
 x[1,1,5]  x[1,2,5]  x[1,3,5]  x[1,4,5]  …  x[1,7,5]  x[1,8,5]  x[1,9,5]
 x[2,1,5]  x[2,2,5]  x[2,3,5]  x[2,4,5]     x[2,7,5]  x[2,8,5]  x[2,9,5]
 x[3,1,5]  x[3,2,5]  x[3,3,5]  x[3,4,5]     x[3,7,5]  x[3,8,5]  x[3,9,5]
 x[4,1,5]  x[4,2,5]  x[4,3,5]  x[4,4,5]     x[4,7,5]  x[4,8,5]  x[4,9,5]
 x[5,1,5]  x[5,2,5]  x[5,3,5]  x[5,4,5]     x[5,7,5]  x[5,8,5]  x[5,9,5]
 x[6,1,5]  x[6,2,5]  x[6,3,5]  x[6,4,5]  …  x[6,7,5]  x[6,8,5]  x[6,9,5]
 x[7,1,5]  x[7,2,5]  x[7,3,5]  x[7,4,5]     x[7,7,5]  x[7,8,5]  x[7,9,5]
 x[8,1,5]  x[8,2,5]  x[8,3,5]  x[8,4,5]     x[8,7,5]  x[8,8,5]  x[8,9,5]
 x[9,1,5]  x[9,2,5]  x[9,3,5]  x[9,4,5]     x[9,7,5]  x[9,8,5]  x[9,9,5]

[:, :, 6] =
 x[1,1,6]  x[1,2,6]  x[1,3,6]  x[1,4,6]  …  x[1,7,6]  x[1,8,6]  x[1,9,6]
 x[2,1,6]  x[2,2,6]  x[2,3,6]  x[2,4,6]     x[2,7,6]  x[2,8,6]  x[2,9,6]
 x[3,1,6]  x[3,2,6]  x[3,3,6]  x[3,4,6]     x[3,7,6]  x[3,8,6]  x[3,9,6]
 x[4,1,6]  x[4,2,6]  x[4,3,6]  x[4,4,6]     x[4,7,6]  x[4,8,6]  x[4,9,6]
 x[5,1,6]  x[5,2,6]  x[5,3,6]  x[5,4,6]     x[5,7,6]  x[5,8,6]  x[5,9,6]
 x[6,1,6]  x[6,2,6]  x[6,3,6]  x[6,4,6]  …  x[6,7,6]  x[6,8,6]  x[6,9,6]
 x[7,1,6]  x[7,2,6]  x[7,3,6]  x[7,4,6]     x[7,7,6]  x[7,8,6]  x[7,9,6]
 x[8,1,6]  x[8,2,6]  x[8,3,6]  x[8,4,6]     x[8,7,6]  x[8,8,6]  x[8,9,6]
 x[9,1,6]  x[9,2,6]  x[9,3,6]  x[9,4,6]     x[9,7,6]  x[9,8,6]  x[9,9,6]

[:, :, 7] =
 x[1,1,7]  x[1,2,7]  x[1,3,7]  x[1,4,7]  …  x[1,7,7]  x[1,8,7]  x[1,9,7]
 x[2,1,7]  x[2,2,7]  x[2,3,7]  x[2,4,7]     x[2,7,7]  x[2,8,7]  x[2,9,7]
 x[3,1,7]  x[3,2,7]  x[3,3,7]  x[3,4,7]     x[3,7,7]  x[3,8,7]  x[3,9,7]
 x[4,1,7]  x[4,2,7]  x[4,3,7]  x[4,4,7]     x[4,7,7]  x[4,8,7]  x[4,9,7]
 x[5,1,7]  x[5,2,7]  x[5,3,7]  x[5,4,7]     x[5,7,7]  x[5,8,7]  x[5,9,7]
 x[6,1,7]  x[6,2,7]  x[6,3,7]  x[6,4,7]  …  x[6,7,7]  x[6,8,7]  x[6,9,7]
 x[7,1,7]  x[7,2,7]  x[7,3,7]  x[7,4,7]     x[7,7,7]  x[7,8,7]  x[7,9,7]
 x[8,1,7]  x[8,2,7]  x[8,3,7]  x[8,4,7]     x[8,7,7]  x[8,8,7]  x[8,9,7]
 x[9,1,7]  x[9,2,7]  x[9,3,7]  x[9,4,7]     x[9,7,7]  x[9,8,7]  x[9,9,7]

[:, :, 8] =
 x[1,1,8]  x[1,2,8]  x[1,3,8]  x[1,4,8]  …  x[1,7,8]  x[1,8,8]  x[1,9,8]
 x[2,1,8]  x[2,2,8]  x[2,3,8]  x[2,4,8]     x[2,7,8]  x[2,8,8]  x[2,9,8]
 x[3,1,8]  x[3,2,8]  x[3,3,8]  x[3,4,8]     x[3,7,8]  x[3,8,8]  x[3,9,8]
 x[4,1,8]  x[4,2,8]  x[4,3,8]  x[4,4,8]     x[4,7,8]  x[4,8,8]  x[4,9,8]
 x[5,1,8]  x[5,2,8]  x[5,3,8]  x[5,4,8]     x[5,7,8]  x[5,8,8]  x[5,9,8]
 x[6,1,8]  x[6,2,8]  x[6,3,8]  x[6,4,8]  …  x[6,7,8]  x[6,8,8]  x[6,9,8]
```

```
 x[7,1,8]  x[7,2,8]  x[7,3,8]  x[7,4,8]     x[7,7,8]  x[7,8,8]  x[7,9,8]
 x[8,1,8]  x[8,2,8]  x[8,3,8]  x[8,4,8]     x[8,7,8]  x[8,8,8]  x[8,9,8]
 x[9,1,8]  x[9,2,8]  x[9,3,8]  x[9,4,8]     x[9,7,8]  x[9,8,8]  x[9,9,8]

[:, :, 9] =
 x[1,1,9]  x[1,2,9]  x[1,3,9]  x[1,4,9]  …  x[1,7,9]  x[1,8,9]  x[1,9,9]
 x[2,1,9]  x[2,2,9]  x[2,3,9]  x[2,4,9]     x[2,7,9]  x[2,8,9]  x[2,9,9]
 x[3,1,9]  x[3,2,9]  x[3,3,9]  x[3,4,9]     x[3,7,9]  x[3,8,9]  x[3,9,9]
 x[4,1,9]  x[4,2,9]  x[4,3,9]  x[4,4,9]     x[4,7,9]  x[4,8,9]  x[4,9,9]
 x[5,1,9]  x[5,2,9]  x[5,3,9]  x[5,4,9]     x[5,7,9]  x[5,8,9]  x[5,9,9]
 x[6,1,9]  x[6,2,9]  x[6,3,9]  x[6,4,9]  …  x[6,7,9]  x[6,8,9]  x[6,9,9]
 x[7,1,9]  x[7,2,9]  x[7,3,9]  x[7,4,9]     x[7,7,9]  x[7,8,9]  x[7,9,9]
 x[8,1,9]  x[8,2,9]  x[8,3,9]  x[8,4,9]     x[8,7,9]  x[8,8,9]  x[8,9,9]
 x[9,1,9]  x[9,2,9]  x[9,3,9]  x[9,4,9]     x[9,7,9]  x[9,8,9]  x[9,9,9]
```

Now we can begin to add our constraints. We'll actually start with something obvious to us as humans, but what we need to enforce: that there can be only one number per cell.

```julia
for i in 1:9  ## For each row
    for j in 1:9  ## and each column
        # Sum across all the possible digits. One and only one of the digits
        # can be in this cell, so the sum must be equal to one.
        @constraint(sudoku, sum(x[i, j, k] for k in 1:9) == 1)
    end
end
```

Next we'll add the constraints for the rows and the columns. These constraints are all very similar, so much so that we can actually add them at the same time.

```julia
for ind in 1:9  ## Each row, OR each column
    for k in 1:9  ## Each digit
        # Sum across columns (j) - row constraint
        @constraint(sudoku, sum(x[ind, j, k] for j in 1:9) == 1)
        # Sum across rows (i) - column constraint
        @constraint(sudoku, sum(x[i, ind, k] for i in 1:9) == 1)
    end
end
```

Finally, we have the to enforce the constraint that each digit appears once in each of the nine 3x3 sub-grids. Our strategy will be to index over the top-left corners of each 3x3 square with `for` loops, then sum over the squares.

```julia
for i in 1:3:7
    for j in 1:3:7
        for k in 1:9
            # i is the top left row, j is the top left column.
            # We'll sum from i to i+2, e.g. i=4, r=4, 5, 6.
            @constraint(
                sudoku,
                sum(x[r, c, k] for r in i:(i+2), c in j:(j+2)) == 1
            )
        end
    end
end
```

The final step is to add the initial solution as a set of constraints. We'll solve the problem that is in the picture at the start of the tutorial. We'll put a `0` if there is no digit in that location.

The given digits

```
init_sol = [
    5 3 0 0 7 0 0 0 0
    6 0 0 1 9 5 0 0 0
    0 9 8 0 0 0 0 6 0
    8 0 0 0 6 0 0 0 3
    4 0 0 8 0 3 0 0 1
    7 0 0 0 2 0 0 0 6
    0 6 0 0 0 0 2 8 0
    0 0 0 4 1 9 0 0 5
    0 0 0 0 8 0 0 7 9
]
for i in 1:9
    for j in 1:9
        # If the space isn't empty
        if init_sol[i, j] != 0
            # Then the corresponding variable for that digit and location must
            # be 1.
            fix(x[i, j, init_sol[i, j]], 1; force = true)
        end
    end
end
```

solve problem

```
optimize!(sudoku)
```

```
Presolving model
151 rows, 115 cols, 466 nonzeros
0 rows, 0 cols, 0 nonzeros
Presolve: Optimal

Solving report
  Status            Optimal
  Primal bound      0
  Dual bound        0
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    0 (objective)
                    0 (bound viol.)
                    0 (int. viol.)
                    0 (row viol.)
  Timing            0.00 (total)
                    0.00 (presolve)
                    0.00 (postsolve)
  Nodes             0
  LP iterations     0 (total)
                    0 (strong br.)
                    0 (separation)
                    0 (heuristics)
```

Extract the values of x

```
x_val = value.(x)
```

```
9×9×9 Array{Float64, 3}:
[:, :, 1] =
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0

[:, :, 2] =
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0

[:, :, 3] =
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

[:, :, 4] =
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0

[:, :, 5] =
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
```

```
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0

[:, :, 6] =
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0

[:, :, 7] =
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0

[:, :, 8] =
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0

[:, :, 9] =
 0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
 0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
 0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
 0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
 1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
 0.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
```

Create a matrix to store the solution

```
sol = zeros(Int, 9, 9)  # 9x9 matrix of integers
for i in 1:9
    for j in 1:9
        for k in 1:9
            # Integer programs are solved as a series of linear programs so the
            # values might not be precisely 0 and 1. We can round them to
```

```
            # the nearest integer to make it easier.
            if round(Int, x_val[i, j, k]) == 1
                sol[i, j] = k
            end
        end
    end
end
```

Display the solution

```
sol
```

```
9×9 Matrix{Int64}:
 5  3  4  6  7  8  9  1  2
 6  7  2  1  9  5  3  4  8
 1  9  8  3  4  2  5  6  7
 8  5  9  7  6  1  4  2  3
 4  2  6  8  5  3  7  9  1
 7  1  3  9  2  4  8  5  6
 9  6  1  5  3  7  2  8  4
 2  8  7  4  1  9  6  3  5
 3  4  5  2  8  6  1  7  9
```

Which is the correct solution:

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.17 The transportation problem

Allocation of passenger cars to trains to minimize cars required or car-miles run. Based on:

Fourer, D.M. Gay and Brian W. Kernighan, A Modeling Language for Mathematical Programming, https://ampl.com/REFS/amplmod.ps.gz Appendix D.

Originally contributed by Louis Luangkesorn, January 30, 2015.

```
using JuMP
import HiGHS
import Test

function example_transp()
    ORIG = ["GARY", "CLEV", "PITT"]
    DEST = ["FRA", "DET", "LAN", "WIN", "STL", "FRE", "LAF"]
    supply = [1_400, 2_600, 2_900]
    demand = [900, 1_200, 600, 400, 1_700, 1_100, 1_000]
    Test.@test sum(supply) == sum(demand)
    cost = [
        39 14 11 14 16 82 8
        27 9 12 9 26 95 17
        24 14 17 13 28 99 20
    ]
```

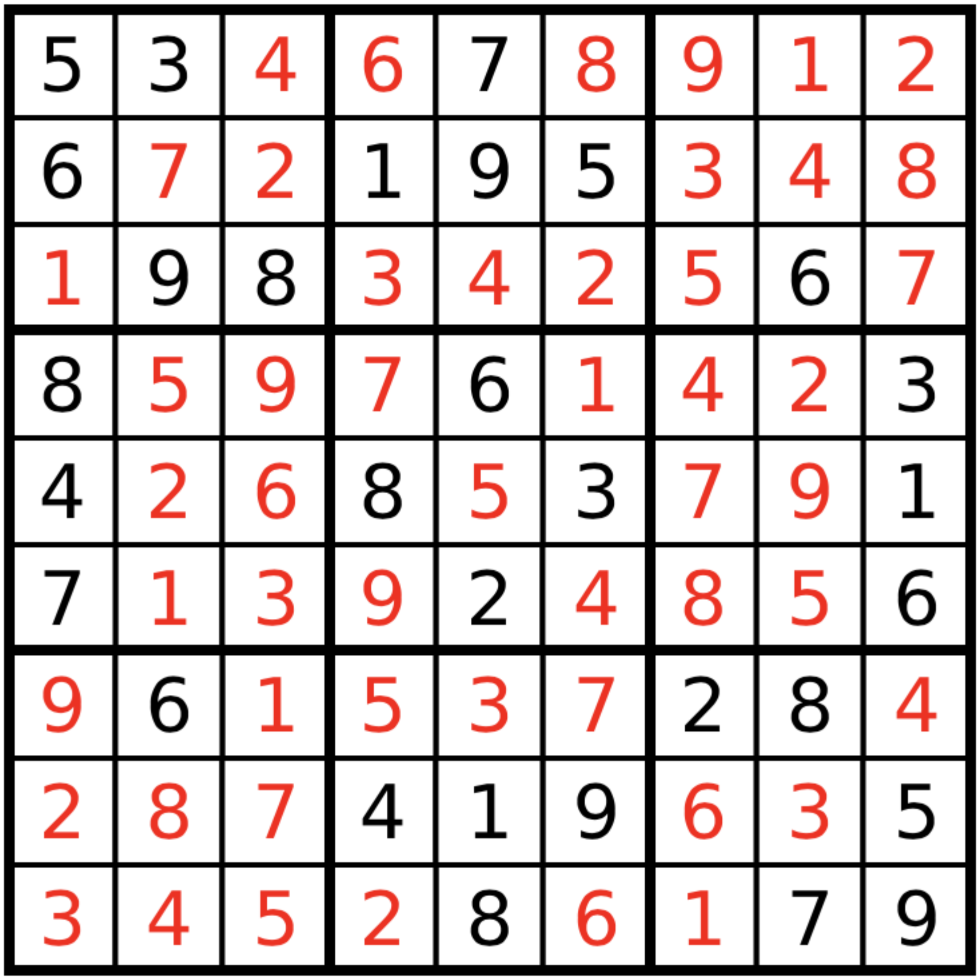


Figure 5.3: Solved Sudoku

```
model = Model(HiGHS.Optimizer)
@variable(model, trans[1:length(ORIG), 1:length(DEST)] >= 0)
@objective(
    model,
    Min,
    sum(
        cost[i, j] * trans[i, j] for i in 1:length(ORIG),
        j in 1:length(DEST)
    )
)
@constraints(
    model,
    begin
        [i in 1:length(ORIG)], sum(trans[i, :]) == supply[i]
        [j in 1:length(DEST)], sum(trans[:, j]) == demand[j]
    end
)
optimize!(model)
```

```
    Test.@test termination_status(model) == OPTIMAL
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) == 196200.0
    println("The optimal solution is:")
    println(value.(trans))
    return
end

example_transp()
```

```
Presolving model
10 rows, 21 cols, 42 nonzeros
9 rows, 21 cols, 39 nonzeros
Presolve : Reductions: rows 9(-1); columns 21(-0); elements 39(-3)
Solving the presolved LP
Using EKK dual simplex solver - serial
  Iteration        Objective     Infeasibilities num(sum)
          0     0.0000000000e+00 Pr: 9(12900) 0s
         10     1.9620000000e+05 Pr: 0(0) 0s
Solving the original LP from the solution after postsolve
Model   status      : Optimal
Simplex   iterations: 10
Objective value     :  1.9620000000e+05
HiGHS run time      :          0.00
The optimal solution is:
[0.0 0.0 0.0 0.0 300.0 1100.0 0.0; 0.0 1200.0 600.0 400.0 0.0 0.0 400.0; 900.0 0.0 0.0 0.0 1400.0
     0.0 600.0]
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.18 The urban planning problem

An "urban planning" problem based on an example from puzzlor.

```
using JuMP
import HiGHS
import Test

function example_urban_plan()
    model = Model(HiGHS.Optimizer)
    # x is indexed by row and column
    @variable(model, 0 <= x[1:5, 1:5] <= 1, Int)
    # y is indexed by R or C, the points, and an index in 1:5. Note how JuMP
    # allows indexing on arbitrary sets.
    rowcol = ["R", "C"]
    points = [5, 4, 3, -3, -4, -5]
    @variable(model, 0 <= y[rowcol, points, 1:5] <= 1, Int)
    # Objective - combine the positive and negative parts
    @objective(
        model,
        Max,
```

```julia
        sum(
            3 * (y["R", 3, i] + y["C", 3, i]) +
            1 * (y["R", 4, i] + y["C", 4, i]) +
            1 * (y["R", 5, i] + y["C", 5, i]) -
            3 * (y["R", -3, i] + y["C", -3, i]) -
            1 * (y["R", -4, i] + y["C", -4, i]) -
            1 * (y["R", -5, i] + y["C", -5, i]) for i in 1:5
        )
    )
    # Constrain the number of residential lots
    @constraint(model, sum(x) == 12)
    # Add the constraints that link the auxiliary y variables to the x variables
    for i in 1:5
        @constraints(model, begin
            # Rows
            y["R", 5, i] <= 1 / 5 * sum(x[i, :]) # sum = 5
            y["R", 4, i] <= 1 / 4 * sum(x[i, :]) # sum = 4
            y["R", 3, i] <= 1 / 3 * sum(x[i, :]) # sum = 3
            y["R", -3, i] >= 1 - 1 / 3 * sum(x[i, :]) # sum = 2
            y["R", -4, i] >= 1 - 1 / 2 * sum(x[i, :]) # sum = 1
            y["R", -5, i] >= 1 - 1 / 1 * sum(x[i, :]) # sum = 0
            # Columns
            y["C", 5, i] <= 1 / 5 * sum(x[:, i]) # sum = 5
            y["C", 4, i] <= 1 / 4 * sum(x[:, i]) # sum = 4
            y["C", 3, i] <= 1 / 3 * sum(x[:, i]) # sum = 3
            y["C", -3, i] >= 1 - 1 / 3 * sum(x[:, i]) # sum = 2
            y["C", -4, i] >= 1 - 1 / 2 * sum(x[:, i]) # sum = 1
            y["C", -5, i] >= 1 - 1 / 1 * sum(x[:, i]) # sum = 0
        end)
    end
    # Solve it
    optimize!(model)
    Test.@test termination_status(model) == OPTIMAL
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) ≈ 14.0
    return
end

example_urban_plan()
```

```
Presolving model
61 rows, 85 cols, 385 nonzeros
61 rows, 85 cols, 385 nonzeros
Objective function is integral with scale 1

Solving MIP model with:
   61 rows
   85 cols (85 binary, 0 integer, 0 implied int., 0 continuous)
   385 nonzeros

( 0.0s) Starting symmetry detection
( 0.0s) Found 5 generators

        Nodes      |    B&B Tree     |            Objective Bounds              |  Dynamic
     Constraints |       Work
```

```
     Proc. InQueue |  Leaves   Expl. | BestBound       BestSol              Gap |   Cuts   InLp
     Confl. | LpIters     Time

         0       0         0   0.00%   50              -inf                 inf        0      0
      0         0     0.0s
 S       0       0         0   0.00%   50              -6               933.33%        0      0
      0         0     0.0s
         0       0         0   0.00%   28.8            -6               580.00%        0      0
      4        63     0.0s
 H       0       0         0   0.00%   14              14                 0.00%     3645     99
     34       642     0.1s

Solving report
  Status            Optimal
  Primal bound      14
  Dual bound        14
  Gap               0% (tolerance: 0.01%)
  Solution status   feasible
                    14 (objective)
                    0 (bound viol.)
                    2.22044604925e-16 (int. viol.)
                    0 (row viol.)
  Timing            0.13 (total)
                    0.00 (presolve)
                    0.00 (postsolve)
  Nodes             1
  LP iterations     642 (total)
                    0 (strong br.)
                    579 (separation)
                    0 (heuristics)
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

5.19 Callbacks

The purpose of the tutorial is to demonstrate the various solver-independent and solver-dependent callbacks that are supported by JuMP.

The tutorial uses the following packages:

```julia
using JuMP
import GLPK
import Random
```

Info

This tutorial uses the MathOptInterface API. By default, JuMP exports the `MOI` symbol as an alias for the MathOptInterface.jl package. We recommend making this more explicit in your code by adding the following lines:

```julia
import MathOptInterface
const MOI = MathOptInterface
```

Lazy constraints

An example using a lazy constraint callback.

```julia
function example_lazy_constraint()
    model = Model(GLPK.Optimizer)
    @variable(model, 0 <= x <= 2.5, Int)
    @variable(model, 0 <= y <= 2.5, Int)
    @objective(model, Max, y)
    lazy_called = false
    function my_callback_function(cb_data)
        lazy_called = true
        x_val = callback_value(cb_data, x)
        y_val = callback_value(cb_data, y)
        println("Called from (x, y) = ($x_val, $y_val)")
        status = callback_node_status(cb_data, model)
        if status == MOI.CALLBACK_NODE_STATUS_FRACTIONAL
            println(" - Solution is integer infeasible!")
        elseif status == MOI.CALLBACK_NODE_STATUS_INTEGER
            println(" - Solution is integer feasible!")
        else
            @assert status == MOI.CALLBACK_NODE_STATUS_UNKNOWN
            println(" - I don't know if the solution is integer feasible :(")
        end
        if y_val - x_val > 1 + 1e-6
            con = @build_constraint(y - x <= 1)
            println("Adding $(con)")
            MOI.submit(model, MOI.LazyConstraint(cb_data), con)
        elseif y_val + x_val > 3 + 1e-6
            con = @build_constraint(y + x <= 3)
            println("Adding $(con)")
            MOI.submit(model, MOI.LazyConstraint(cb_data), con)
        end
    end
    MOI.set(model, MOI.LazyConstraintCallback(), my_callback_function)
    optimize!(model)
    println("Optimal solution (x, y) = ($(value(x)), $(value(y)))")
    return
end

example_lazy_constraint()
```

```
Called from (x, y) = (0.0, 2.0)
 - Solution is integer feasible!
Adding ScalarConstraint{AffExpr, MathOptInterface.LessThan{Float64}}(y - x, MathOptInterface.
     LessThan{Float64}(1.0))
Called from (x, y) = (1.0, 2.0)
 - Solution is integer feasible!
Optimal solution (x, y) = (1.0, 2.0)
```

User-cuts

An example using a user-cut callback.

```julia
function example_user_cut_constraint()
    Random.seed!(1)
```

```
    N = 30
    item_weights, item_values = rand(N), rand(N)
    model = Model(GLPK.Optimizer)
    @variable(model, x[1:N], Bin)
    @constraint(model, sum(item_weights[i] * x[i] for i in 1:N) <= 10)
    @objective(model, Max, sum(item_values[i] * x[i] for i in 1:N))
    callback_called = false
    function my_callback_function(cb_data)
        callback_called = true
        x_vals = callback_value.(Ref(cb_data), x)
        accumulated = sum(item_weights[i] for i = 1:N if x_vals[i] > 1e-4)
        println("Called with accumulated = $(accumulated)")
        n_terms = sum(1 for i = 1:N if x_vals[i] > 1e-4)
        if accumulated > 10
            con = @build_constraint(
                sum(x[i] for i = 1:N if x_vals[i] > 0.5) <= n_terms - 1
            )
            println("Adding $(con)")
            MOI.submit(model, MOI.UserCut(cb_data), con)
        end
    end
    MOI.set(model, MOI.UserCutCallback(), my_callback_function)
    optimize!(model)
    @show callback_called
    return
end

example_user_cut_constraint()
```

```
Called with accumulated = 10.245300779183612
Adding ScalarConstraint{AffExpr, MathOptInterface.LessThan{Float64}}(x[1] + x[2] + x[4] + x[6] + x
     [7] + x[9] + x[10] + x[11] + x[12] + x[13] + x[14] + x[15] + x[16] + x[17] + x[18] + x[19] + x
     [20] + x[21] + x[22] + x[23] + x[24] + x[26] + x[29] + x[30], MathOptInterface.LessThan{Float64
     }(23.0))
Called with accumulated = 10.276817515233951
Adding ScalarConstraint{AffExpr, MathOptInterface.LessThan{Float64}}(x[1] + x[2] + x[4] + x[6] + x
     [7] + x[9] + x[10] + x[11] + x[12] + x[13] + x[14] + x[16] + x[17] + x[18] + x[19] + x[20] + x
     [21] + x[22] + x[23] + x[24] + x[26] + x[29] + x[30], MathOptInterface.LessThan{Float64}(23.0))
Called with accumulated = 10.812296027897915
Adding ScalarConstraint{AffExpr, MathOptInterface.LessThan{Float64}}(x[1] + x[2] + x[4] + x[6] + x
     [7] + x[9] + x[10] + x[11] + x[12] + x[13] + x[14] + x[16] + x[17] + x[18] + x[19] + x[20] + x
     [21] + x[22] + x[23] + x[24] + x[26] + x[29] + x[30], MathOptInterface.LessThan{Float64}(23.0))
callback_called = true
```

Heuristic solutions

An example using a heuristic solution callback.

```
function example_heuristic_solution()
    Random.seed!(1)
    N = 30
    item_weights, item_values = rand(N), rand(N)
    model = Model(GLPK.Optimizer)
    @variable(model, x[1:N], Bin)
    @constraint(model, sum(item_weights[i] * x[i] for i in 1:N) <= 10)
```

```
    @objective(model, Max, sum(item_values[i] * x[i] for i in 1:N))
    callback_called = false
    function my_callback_function(cb_data)
        callback_called = true
        x_vals = callback_value.(Ref(cb_data), x)
        ret =
            MOI.submit(model, MOI.HeuristicSolution(cb_data), x, floor.(x_vals))
        println("Heuristic solution status = $(ret)")
    end
    MOI.set(model, MOI.HeuristicCallback(), my_callback_function)
    optimize!(model)
    return
end

example_heuristic_solution()
```

```
Heuristic solution status = HEURISTIC_SOLUTION_ACCEPTED
Heuristic solution status = HEURISTIC_SOLUTION_ACCEPTED
Heuristic solution status = HEURISTIC_SOLUTION_REJECTED
Heuristic solution status = HEURISTIC_SOLUTION_REJECTED
```

GLPK solver-dependent callback

An example using GLPK's solver-dependent callback.

```
function example_solver_dependent_callback()
    model = Model(GLPK.Optimizer)
    @variable(model, 0 <= x <= 2.5, Int)
    @variable(model, 0 <= y <= 2.5, Int)
    @objective(model, Max, y)
    lazy_called = false
    function my_callback_function(cb_data)
        lazy_called = true
        reason = GLPK.glp_ios_reason(cb_data.tree)
        println("Called from reason = $(reason)")
        if reason != GLPK.GLP_IROWGEN
            return
        end
        x_val = callback_value(cb_data, x)
        y_val = callback_value(cb_data, y)
        if y_val - x_val > 1 + 1e-6
            con = @build_constraint(y - x <= 1)
            println("Adding $(con)")
            MOI.submit(model, MOI.LazyConstraint(cb_data), con)
        elseif y_val + x_val > 3 + 1e-6
            con = @build_constraint(y - x <= 1)
            println("Adding $(con)")
            MOI.submit(model, MOI.LazyConstraint(cb_data), con)
        end
    end
    MOI.set(model, GLPK.CallbackFunction(), my_callback_function)
    optimize!(model)
    return
end
```

```
example_solver_dependent_callback()
```

```
Called from reason = 6
Called from reason = 7
Called from reason = 1
Adding ScalarConstraint{AffExpr, MathOptInterface.LessThan{Float64}}(y - x, MathOptInterface.
     LessThan{Float64}(1.0))
Called from reason = 7
Called from reason = 1
Called from reason = 2
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

Chapter 6

Nonlinear programs

6.1 Introduction

Nonlinear programs (NLPs) are a class of optimization problems in which some of the constraints or the objective function are nonlinear:

$$\min_{x \in \mathbb{R}^n} f_0(x) \tag{6.1}$$

$$\text{s.t.} l_j \leq f_j(x) \leq u_j \qquad j = 1 \ldots m \tag{6.2}$$

$$l_i \leq x_i \leq u_i \qquad i = 1 \ldots n. \tag{6.3}$$

Mixed-integer nonlinear linear programs (MINLPs) are extensions of nonlinear programs in which some (or all) of the decision variables take discrete values.

How to choose a solver

JuMP supports a range of nonlinear solvers; look for "NLP" in the list of Supported solvers. However, very few solvers support mixed-integer nonlinear linear programs. Solvers supporting discrete variables start with "(MI)" in the list of Supported solvers.

If the only nonlinearities in your model are quadratic terms (that is, multiplication between two decision variables), you can also use second-order cone solvers, which are indicated by "SOCP." In most cases, these solvers are restricted to convex quadratic problems and will error if you pass a nonconvex quadratic function; however, Gurobi has the ability to solve nonconvex quadratic terms.

How these tutorials are structured

Having a high-level overview of how this part of the documentation is structured will help you know where to look for certain things.

- The following tutorials are worked examples that present a problem in words, then formulate it in mathematics, and then solve it in JuMP. This usually involves some sort of visualization of the solution. Start here if you are new to JuMP.
  - Rocket Control
  - Optimal control for a Space Shuttle reentry trajectory
  - Quadratic portfolio optimization

- The Tips and tricks tutorial contains a number of helpful reformulations and tricks you can use when modeling nonlinear programs. Look here if you are stuck trying to formulate a problem as a nonlinear program.
- The Computing Hessians is an advanced tutorial which explains how to compute the Hessian of the Lagrangian of a nonlinear program. This is useful only in particular cases.
- The remaining tutorials are less verbose and styled in the form of short code examples. These tutorials have less explanation, but may contain useful code snippets, particularly if they are similar to a problem you are trying to solve.

6.2 Tips and tricks

This example collates some tips and tricks you can use when formulating nonlinear programs. It uses the following packages:

```
using JuMP
import Ipopt
import Test
```

User-defined functions with vector outputs

A common situation is to have a user-defined function like the following that returns multiple outputs (we define `function_calls` to keep track of how many times we call this method):

```
function_calls = 0
function foo(x, y)
    global function_calls += 1
    common_term = x^2 + y^2
    term_1 = sqrt(1 + common_term)
    term_2 = common_term
    return term_1, term_2
end
```

```
foo (generic function with 1 method)
```

For example, the first term might be used in the objective, and the second term might be used in a constraint, and often they share work that is expensive to evaluate.

This is a problem for JuMP, because it requires user-defined functions to return a single number. One option is to define two separate functions, the first returning the first argument, and the second returning the second argument.

```
foo_1(x, y) = foo(x, y)[1]
foo_2(x, y) = foo(x, y)[2]
```

```
foo_2 (generic function with 1 method)
```

However, if the common term is expensive to compute, this approach is wasteful because it will evaluate the expensive term twice. Let's have a look at how many times we evaluate `x^2 + y^2` during a solve:

```
model = Model(Ipopt.Optimizer)
set_silent(model)
@variable(model, x[1:2] >= 0, start = 0.1)
register(model, :foo_1, 2, foo_1; autodiff = true)
register(model, :foo_2, 2, foo_2; autodiff = true)
@NLobjective(model, Max, foo_1(x[1], x[2]))
@NLconstraint(model, foo_2(x[1], x[2]) <= 2)
function_calls = 0
optimize!(model)
Test.@test objective_value(model) ≈ √3 atol = 1e-4
Test.@test value.(x) ≈ [1.0, 1.0] atol = 1e-4
println("Naive approach: function calls = $(function_calls)")
```

```
Naive approach: function calls = 40
```

An alternative approach is to use memoization, which uses a cache to store the result of function evaluations. We can write a memoization function as follows:

```
"""
    memoize(foo::Function, n_outputs::Int)

Take a function `foo` and return a vector of length `n_outputs`, where each
element is a function that returns the `i`'th output of `foo`.

To avoid duplication of work, cache the most-recent evaluations of `foo`.
Because `foo_i` is auto-differentiated with ForwardDiff, our cache needs to
work when `x` is a `Float64` and a `ForwardDiff.Dual`.
"""
function memoize(foo::Function, n_outputs::Int)
    last_x, last_f = nothing, nothing
    last_dx, last_dfdx = nothing, nothing
    function foo_i(i, x::T...) where {T<:Real}
        if T == Float64
            if x != last_x
                last_x, last_f = x, foo(x...)
            end
            return last_f[i]::T
        else
            if x != last_dx
                last_dx, last_dfdx = x, foo(x...)
            end
            return last_dfdx[i]::T
        end
    end
    return [(x...) -> foo_i(i, x...) for i in 1:n_outputs]
end
```

```
Main.memoize
```

Let's see how it works. First, construct the memoized versions of foo:

```
memoized_foo = memoize(foo, 2)
```

```
2-element Vector{Main.var"#4#7"{Int64, Main.var"#foo_i#5"{typeof(Main.foo)}}}:
 #4 (generic function with 1 method)
 #4 (generic function with 1 method)
```

Now try evaluating the first element of `memoized_foo`.

```
function_calls = 0
memoized_foo[1](1.0, 1.0)
println("function_calls = ", function_calls)
```

```
function_calls = 1
```

As expected, this evaluated the function once. However, if we call the function again, we hit the cache instead of needing to re-compute `foo` and `function_calls` is still 1!

```
memoized_foo[1](1.0, 1.0)
println("function_calls = ", function_calls)
```

```
function_calls = 1
```

Now let's see how this works during a real solve:

```
model = Model(Ipopt.Optimizer)
set_silent(model)
@variable(model, x[1:2] >= 0, start = 0.1)
register(model, :foo_1, 2, memoized_foo[1]; autodiff = true)
register(model, :foo_2, 2, memoized_foo[2]; autodiff = true)
@NLobjective(model, Max, foo_1(x[1], x[2]))
@NLconstraint(model, foo_2(x[1], x[2]) <= 2)
function_calls = 0
optimize!(model)
Test.@test objective_value(model) ≈ √3 atol = 1e-4
Test.@test value.(x) ≈ [1.0, 1.0] atol = 1e-4
println("Memoized approach: function_calls = $(function_calls)")
```

```
Memoized approach: function_calls = 20
```

Compared to the naive approach, the memoized approach requires half as many function evaluations!

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.3 Quadratic portfolio optimization

Originally Contributed by: Arpit Bhatia

Optimization models play an increasingly important role in financial decisions. Many computational finance problems can be solved efficiently using modern optimization techniques.

This tutorial solves the famous Markowitz Portfolio Optimization problem with data from lecture notes from a course taught at Georgia Tech by Shabir Ahmed.

This tutorial uses the following packages

```
using JuMP
import Ipopt
import Statistics
```

Suppose we are considering investing 1000 dollars in three non-dividend paying stocks, IBM (IBM), Walmart (WMT), and Southern Electric (SEHI), for a one-month period.

We will use the initial money to buy shares of the three stocks at the current market prices, hold these for one month, and sell the shares off at the prevailing market prices at the end of the month.

As a rational investor, we hope to make some profit out of this endeavor, i.e., the return on our investment should be positive.

Suppose we bought a stock at p dollars per share in the beginning of the month, and sold it off at s dollars per share at the end of the month. Then the one-month return on a share of the stock is $\frac{s-p}{p}$.

Since the stock prices are quite uncertain, so is the end-of-month return on our investment. Our goal is to invest in such a way that the expected end-of-month return is at least $50 or 5%. Furthermore, we want to make sure that the "risk" of not achieving our desired return is minimum.

Note that we are solving the problem under the following assumptions:

1. We can trade any continuum of shares.
2. No short-selling is allowed.
3. There are no transaction costs.

We model this problem by taking decision variables $x_i, i = 1, 2, 3,$ denoting the dollars invested in each of the 3 stocks.

Let us denote by $\tilde{r}_i$ the random variable corresponding to the monthly return (increase in the stock price) per dollar for stock i.

Then, the return (or profit) on x_i dollars invested in stock i is $\tilde{r}_i x_i$, and the total (random) return on our investment is $\sum_{i=1}^{3} \tilde{r}_i x_i$. The expected return on our investment is then $\mathbb{E}\left[\sum_{i=1}^{3} \tilde{r}_i x_i\right] = \sum_{i=1}^{3} \overline{r}_i x_i,$ where $\overline{r}_i$ is the expected value of the $\tilde{r}_i$.

Now we need to quantify the notion of "risk" in our investment.

Markowitz, in his Nobel prize winning work, showed that a rational investor's notion of minimizing risk can be closely approximated by minimizing the variance of the return of the investment portfolio. This variance is given by:

$$\mathrm{Var}\left[\sum_{i=1}^{3} \tilde{r}_i x_i\right] = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \sigma_{ij}$$

where σ_{ij} is the covariance of the return of stock i with stock j.

Note that the right hand side of the equation is the most reduced form of the expression and we have not shown the intermediate steps involved in getting to this form. We can also write this equation as:

$$\mathrm{Var}\left[\sum_{i=1}^{3} \tilde{r}_i x_i\right] = x^T Q x$$

Where Q is the covariance matrix for the random vector $\tilde{r}$.

Finally, we can write the model as:

$$
\begin{aligned}
\min x^T Q x \\
\text{s.t.} \sum_{i=1}^{3} x_i \leq 1000.00 \\
\overline{r}^T x \geq 50.00 \\
x \geq 0
\end{aligned}
$$

After that long discussion, let's now use JuMP to solve the portfolio optimization problem for the data given below.

Month	IBM	WMT	SEHI
November-00	93.043	51.826	1.063
December-00	84.585	52.823	0.938
January-01	111.453	56.477	1.000
February-01	99.525	49.805	0.938
March-01	95.819	50.287	1.438
April-01	114.708	51.521	1.700
May-01	111.515	51.531	2.540
June-01	113.211	48.664	2.390
July-01	104.942	55.744	3.120
August-01	99.827	47.916	2.980
September-01	91.607	49.438	1.900
October-01	107.937	51.336	1.750
November-01	115.590	55.081	1.800

```
stock_data = [
    93.043 51.826 1.063
    84.585 52.823 0.938
    111.453 56.477 1.000
    99.525 49.805 0.938
    95.819 50.287 1.438
    114.708 51.521 1.700
    111.515 51.531 2.540
    113.211 48.664 2.390
    104.942 55.744 3.120
    99.827 47.916 2.980
    91.607 49.438 1.900
    107.937 51.336 1.750
    115.590 55.081 1.800
]
```

```
13×3 Matrix{Float64}:
  93.043  51.826  1.063
  84.585  52.823  0.938
 111.453  56.477  1.0
  99.525  49.805  0.938
  95.819  50.287  1.438
```

```
 114.708  51.521  1.7
 111.515  51.531  2.54
 113.211  48.664  2.39
 104.942  55.744  3.12
  99.827  47.916  2.98
  91.607  49.438  1.9
 107.937  51.336  1.75
 115.59   55.081  1.8
```

Calculating stock returns

```
stock_returns = Array{Float64}(undef, 12, 3)
for i in 1:12
    stock_returns[i, :] =
        (stock_data[i+1, :] .- stock_data[i, :]) ./ stock_data[i, :]
end
stock_returns
```

```
12×3 Matrix{Float64}:
 -0.0909042   0.0192374    -0.117592
  0.317645    0.0691744     0.0660981
 -0.107023   -0.118137     -0.062
 -0.0372369   0.00967774    0.533049
  0.197132    0.0245391     0.182197
 -0.0278359   0.000194096   0.494118
  0.0152087  -0.0556364    -0.0590551
 -0.0730406   0.145487      0.305439
 -0.0487412  -0.140428     -0.0448718
 -0.0823425   0.0317639    -0.362416
  0.178261    0.0383915    -0.0789474
  0.0709025   0.0729508     0.0285714
```

Calculating the expected value of monthly return:

```
r = Statistics.mean(stock_returns, dims = 1)
```

```
1×3 Matrix{Float64}:
 0.0260022  0.00810132  0.0737159
```

Calculating the covariance matrix Q

```
Q = Statistics.cov(stock_returns)
```

```
3×3 Matrix{Float64}:
 0.018641    0.00359853  0.00130976
 0.00359853  0.00643694  0.00488727
 0.00130976  0.00488727  0.0686828
```

JuMP Model

```
portfolio = Model(Ipopt.Optimizer)
set_silent(portfolio)
@variable(portfolio, x[1:3] >= 0)
```

```
@objective(portfolio, Min, x' * Q * x)
@constraint(portfolio, sum(x) <= 1000)
@constraint(portfolio, sum(r[i] * x[i] for i in 1:3) >= 50)
optimize!(portfolio)

objective_value(portfolio)
```

```
22634.417849884143
```

```
value.(x)
```

```
3-element Vector{Float64}:
 497.0455298498641
  -9.670578501817138e-9
 502.95448015948085
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.4 Quadratically constrained programs

A simple quadratically constrained program based on an example from Gurobi.

```
using JuMP
import Ipopt
import Test

function example_qcp(; verbose = true)
    model = Model(Ipopt.Optimizer)
    set_silent(model)
    @variable(model, x)
    @variable(model, y >= 0)
    @variable(model, z >= 0)
    @objective(model, Max, x)
    @constraint(model, x + y + z == 1)
    @constraint(model, x * x + y * y - z * z <= 0)
    @constraint(model, x * x - y * z <= 0)
    optimize!(model)
    if verbose
        print(model)
        println("Objective value: ", objective_value(model))
        println("x = ", value(x))
        println("y = ", value(y))
    end
    Test.@test termination_status(model) == LOCALLY_SOLVED
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) ≈ 0.32699 atol = 1e-5
    Test.@test value(x) ≈ 0.32699 atol = 1e-5
    Test.@test value(y) ≈ 0.25707 atol = 1e-5
    return
```

```
end

example_qcp()
```

```
Max x
Subject to
 x + y + z = 1.0
 x² + y² - z² ≤ 0.0
 x² - z*y ≤ 0.0
 y ≥ 0.0
 z ≥ 0.0
Objective value: 0.3269928349138724
x = 0.3269928349138724
y = 0.2570658388068964
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.5 Optimal control for a Space Shuttle reentry trajectory

Originally Contributed by: Henrique Ferrolho

This tutorial demonstrates how to compute a reentry trajectory for the Space Shuttle, by formulating and solving a nonlinear programming problem. The problem was drawn from Chapter 6 of "Practical Methods for Optimal Control and Estimation Using Nonlinear Programming", by John T. Betts.

Tip

This tutorial is a more-complicated version of the Rocket Control example. If you are new to solving nonlinear programs in JuMP, you may want to start there instead.

The motion of the vehicle is defined by the following set of DAEs:

$$
\begin{aligned}
\dot{h} &= v \sin\gamma, \\
\dot{\phi} &= \frac{v}{r}\cos\gamma\sin\psi/\cos\theta, \\
\dot{\theta} &= \frac{v}{r}\cos\gamma\cos\psi, \\
\dot{v} &= -\frac{D}{m} - g\sin\gamma, \\
\dot{\gamma} &= \frac{L}{mv}\cos(\beta) + \cos\gamma\left(\frac{v}{r} - \frac{g}{v}\right), \\
\dot{\psi} &= \frac{1}{mv\cos\gamma}L\sin(\beta) + \frac{v}{r\cos\theta}\cos\gamma\sin\psi\sin\theta, \\
q &\le q_U,
\end{aligned}
$$

where the aerodynamic heating on the vehicle wing leading edge is $q = q_a q_r$ and the dynamic variables are

h	altitude (ft),	γ	flight path angle (rad),
ϕ	longitude (rad),	ψ	azimuth (rad),
θ	latitude (rad),	α	angle of attack (rad),
v	velocity (ft/sec),	β	bank angle (rad).

The aerodynamic and atmospheric forces on the vehicle are specified by the following quantities (English units):

$$\begin{aligned}
D &= \frac{1}{2}c_D S\rho v^2, & a_0 &= -0.20704,\\
L &= \frac{1}{2}c_L S\rho v^2, & a_1 &= 0.029244,\\
g &= \mu/r^2, & \mu &= 0.14076539 \times 10^{17},\\
r &= R_e + h, & b_0 &= 0.07854,\\
\rho &= \rho_0 \exp[-h/h_r], & b_1 &= -0.61592 \times 10^{-2},\\
\rho_0 &= 0.002378, & b_2 &= 0.621408 \times 10^{-3},\\
h_r &= 23800, & q_r &= 17700\sqrt{\rho}(0.0001v)^{3.07},\\
c_L &= a_0 + a_1\hat{\alpha}, & q_a &= c_0 + c_1\hat{\alpha} + c_2\hat{\alpha}^2 + c_3\hat{\alpha}^3,\\
c_D &= b_0 + b_1\hat{\alpha} + b_2\hat{\alpha}^2, & c_0 &= 1.0672181,\\
\hat{\alpha} &= 180\alpha/\pi, & c_1 &= -0.19213774 \times 10^{-1},\\
R_e &= 20902900, & c_2 &= 0.21286289 \times 10^{-3},\\
S &= 2690, & c_3 &= -0.10117249 \times 10^{-5}.
\end{aligned}$$

The reentry trajectory begins at an altitude where the aerodynamic forces are quite small with a weight of $w = 203000$ (lb) and mass $m = w/g_0$ (slug), where $g_0 = 32.174$ (ft/sec^2). The initial conditions are as follows:

$$\begin{aligned}
h &= 260000 \text{ ft}, & v &= 25600 \text{ ft/sec},\\
\phi &= 0 \text{ deg}, & \gamma &= -1 \text{ deg},\\
\theta &= 0 \text{ deg}, & \psi &= 90 \text{ deg}.
\end{aligned}$$

The final point on the reentry trajectory occurs at the unknown (free) time t_F, at the so-called terminal area energy management (TAEM) interface, which is defined by the conditions

$$h = 80000 \text{ ft}, \qquad v = 2500 \text{ ft/sec}, \qquad \gamma = -5 \text{ deg}.$$

As explained in the book, our goal is to maximize the final cross-range, which is equivalent to maximizing the final latitude of the vehicle, i.e., $J = \theta(t_F)$.

Approach

We will use a discretized model of time, with a fixed number of discretized points, n. The decision variables at each point are going to be the state of the vehicle and the controls commanded to it. In addition, we will also

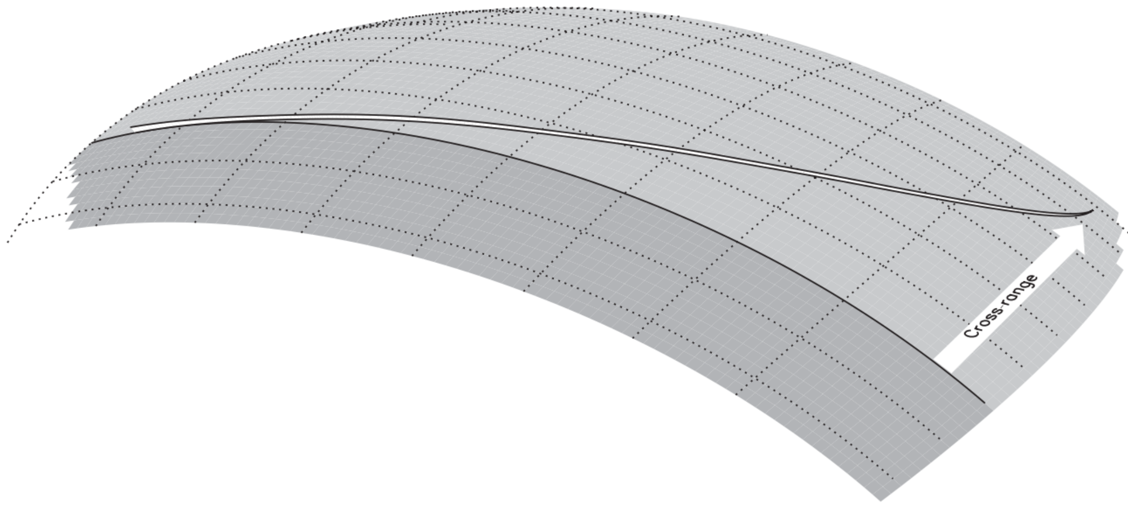


Figure 6.1: Max cross-range shuttle reentry

make each time step size Δt a decision variable; that way, we can either fix the time step size easily, or allow the solver to fine-tune the duration between each adjacent pair of points. Finally, in order to approximate the derivatives of the problem dynamics, we will use either rectangular or trapezoidal integration.

Warning

Do not try to actually land a Space Shuttle using this notebook! There's no mesh refinement going on, which can lead to unrealistic trajectories having position and velocity errors with orders of magnitude 10^4 ft and 10^2 ft/sec, respectively.

```
using JuMP
import Interpolations
import Ipopt

# Global variables
const w = 203000.0  # weight (lb)
const g₀ = 32.174    # acceleration (ft/sec^2)
const m = w / g₀    # mass (slug)

# Aerodynamic and atmospheric forces on the vehicle
const ρ₀ = 0.002378
const hᵣ = 23800.0
const Rₑ = 20902900.0
const μ = 0.14076539e17
const S = 2690.0
const a₀ = -0.20704
const a₁ = 0.029244
const b₀ = 0.07854
const b₁ = -0.61592e-2
const b₂ = 0.621408e-3
const c₀ = 1.0672181
const c₁ = -0.19213774e-1
const c₂ = 0.21286289e-3
```

```julia
const c₃ = -0.10117249e-5

# Initial conditions
const h_s = 2.6           # altitude (ft) / 1e5
const ϕ_s = deg2rad(0)    # longitude (rad)
const θ_s = deg2rad(0)    # latitude (rad)
const v_s = 2.56          # velocity (ft/sec) / 1e4
const γ_s = deg2rad(-1)   # flight path angle (rad)
const ψ_s = deg2rad(90)   # azimuth (rad)
const α_s = deg2rad(0)    # angle of attack (rad)
const β_s = deg2rad(0)    # bank angle (rad)
const t_s = 1.00          # time step (sec)

# Final conditions, the so-called Terminal Area Energy Management (TAEM)
const h_t = 0.8           # altitude (ft) / 1e5
const v_t = 0.25          # velocity (ft/sec) / 1e4
const γ_t = deg2rad(-5)   # flight path angle (rad)

# Number of mesh points (knots) to be used
const n = 503

# Integration scheme to be used for the dynamics
const integration_rule = "rectangular"
```

Choose a good linear solver

Picking a good linear solver is **extremely important** to maximize the performance of nonlinear solvers. For the best results, it is advised to experiment different linear solvers.

For example, the linear solver MA27 is outdated and can be quite slow. MA57 is a much better alternative, especially for highly-sparse problems (such as trajectory optimization problems).

```julia
# Uncomment the lines below to pass user options to the solver
user_options = (
# "mu_strategy" => "monotone",
# "linear_solver" => "ma27",
)

# Create JuMP model, using Ipopt as the solver
model = Model(optimizer_with_attributes(Ipopt.Optimizer, user_options...))

@variables(model, begin
    0 ≤ scaled_h[1:n]                # altitude (ft) / 1e5
    ϕ[1:n]                # longitude (rad)
    deg2rad(-89) ≤ θ[1:n] ≤ deg2rad(89)  # latitude (rad)
    1e-4 ≤ scaled_v[1:n]                # velocity (ft/sec) / 1e4
    deg2rad(-89) ≤ γ[1:n] ≤ deg2rad(89)  # flight path angle (rad)
    ψ[1:n]                # azimuth (rad)
    deg2rad(-90) ≤ α[1:n] ≤ deg2rad(90)  # angle of attack (rad)
    deg2rad(-89) ≤ β[1:n] ≤ deg2rad(1)  # bank angle (rad)
    #        3.5 ≤       Δt[1:n] ≤ 4.5          # time step (sec)
    Δt[1:n] == 4.0         # time step (sec)
end)
```

```
(VariableRef[scaled_h[1], scaled_h[2], scaled_h[3], scaled_h[4], scaled_h[5], scaled_h[6], scaled_h
    [7], scaled_h[8], scaled_h[9], scaled_h[10]  …  scaled_h[494], scaled_h[495], scaled_h[496],
    scaled_h[497], scaled_h[498], scaled_h[499], scaled_h[500], scaled_h[501], scaled_h[502],
    scaled_h[503]], VariableRefϕ[[1], ϕ[2], ϕ[3], ϕ[4], ϕ[5], ϕ[6], ϕ[7], ϕ[8], ϕ[9], ϕ[10]  …
    ϕ[494], ϕ[495], ϕ[496], ϕ[497], ϕ[498], ϕ[499], ϕ[500], ϕ[501], ϕ[502], ϕ[503]], VariableRef
    θ[[1], θ[2], θ[3], θ[4], θ[5], θ[6], θ[7], θ[8], θ[9], θ[10]  …  θ[494], θ[495], θ[496], θ[497],
     θ[498], θ[499], θ[500], θ[501], θ[502], θ[503]], VariableRef[scaled_v[1], scaled_v[2], scaled_v
    [3], scaled_v[4], scaled_v[5], scaled_v[6], scaled_v[7], scaled_v[8], scaled_v[9], scaled_v[10]
     …  scaled_v[494], scaled_v[495], scaled_v[496], scaled_v[497], scaled_v[498], scaled_v[499],
    scaled_v[500], scaled_v[501], scaled_v[502], scaled_v[503]], VariableRefγ[[1], γ[2], γ[3], γ[4],
     γ[5], γ[6], γ[7], γ[8], γ[9], γ[10]  …  γ[494], γ[495], γ[496], γ[497], γ[498], γ[499], γ[500],
     γ[501], γ[502], γ[503]], VariableRefψ[[1], ψ[2], ψ[3], ψ[4], ψ[5], ψ[6], ψ[7], ψ[8], ψ[9],
    ψ[10]  …  ψ[494], ψ[495], ψ[496], ψ[497], ψ[498], ψ[499], ψ[500], ψ[501], ψ[502], ψ[503]],
    VariableRefα[[1], α[2], α[3], α[4], α[5], α[6], α[7], α[8], α[9], α[10]  …  α[494], α[495],
    α[496], α[497], α[498], α[499], α[500], α[501], α[502], α[503]], VariableRefβ[[1], β[2], β[3],
    β[4], β[5], β[6], β[7], β[8], β[9], β[10]  …  β[494], β[495], β[496], β[497], β[498], β[499],
    β[500], β[501], β[502], β[503]], VariableRefΔ[t[1], Δt[2], Δt[3], Δt[4], Δt[5], Δt[6], Δt[7], Δt
    [8], Δt[9], Δt[10]  …  Δt[494], Δt[495], Δt[496], Δt[497], Δt[498], Δt[499], Δt[500], Δt[501],
    Δt[502], Δt[503]])
```

Info

Above you can find two alternatives for the `Δt` variables.

The first one, `3.5 ≤ Δt[1:n] ≤ 4.5` (currently commented), allows some wiggle room for the solver to adjust the time step size between pairs of mesh points. This is neat because it allows the solver to figure out which parts of the flight require more dense discretization than others. (Remember, the number of discretized points is fixed, and this example does not implement mesh refinement.) However, this makes the problem more complex to solve, and therefore leads to a longer computation time.

The second line, `Δt[1:n] == 4.0`, fixes the duration of every time step to exactly 4.0 seconds. This allows the problem to be solved faster. However, to do this we need to know beforehand that the close-to-optimal total duration of the flight is ~2009 seconds. Therefore, if we split the total duration in slices of 4.0 seconds, we know that we require `n = 503` knots to discretize the whole trajectory.

```julia
# Fix initial conditions
fix(scaled_h[1], h_s; force = true)
fix(ϕ[1], ϕ_s; force = true)
fix(θ[1], θ_s; force = true)
fix(scaled_v[1], v_s; force = true)
fix(γ[1], γ_s; force = true)
fix(ψ[1], ψ_s; force = true)

# Fix final conditions
fix(scaled_h[n], h_t; force = true)
fix(scaled_v[n], v_t; force = true)
fix(γ[n], γ_t; force = true)

# Initial guess: linear interpolation between boundary conditions
x_s = [h_s, ϕ_s, θ_s, v_s, γ_s, ψ_s, α_s, β_s, t_s]
x_t = [h_t, ϕ_s, θ_s, v_t, γ_t, ψ_s, α_s, β_s, t_s]
interp_linear = Interpolations.LinearInterpolation([1, n], [x_s, x_t])
initial_guess = mapreduce(transpose, vcat, interp_linear.(1:n))
set_start_value.(all_variables(model), vec(initial_guess))
```

```julia
# Functions to restore `h` and `v` to their true scale
@NLexpression(model, h[j = 1:n], scaled_h[j] * 1e5)
@NLexpression(model, v[j = 1:n], scaled_v[j] * 1e4)

# Helper functions
@NLexpression(model, c_L[j = 1:n], a₀ + a₁ * rad2deg(α[j]))
@NLexpression(
    model,
    c_D[j = 1:n],
    b₀ + b₁ * rad2deg(α[j]) + b₂ * rad2deg(α[j])^2
)
@NLexpression(model, ρ[j = 1:n], ρ₀ * exp(-h[j] / hᵣ))
@NLexpression(model, D[j = 1:n], 0.5 * c_D[j] * S * ρ[j] * v[j]^2)
@NLexpression(model, L[j = 1:n], 0.5 * c_L[j] * S * ρ[j] * v[j]^2)
@NLexpression(model, r[j = 1:n], Rₑ + h[j])
@NLexpression(model, g[j = 1:n], μ / r[j]^2)

# Motion of the vehicle as a differential-algebraic system of equations (DAEs)
@NLexpression(model, δh[j = 1:n], v[j] * sin(γ[j]))
@NLexpression(
    model,
    δϕ[j = 1:n],
    (v[j] / r[j]) * cos(γ[j]) * sin(ψ[j]) / cos(θ[j])
)
@NLexpression(model, δθ[j = 1:n], (v[j] / r[j]) * cos(γ[j]) * cos(ψ[j]))
@NLexpression(model, δv[j = 1:n], -(D[j] / m) - g[j] * sin(γ[j]))
@NLexpression(
    model,
    δγ[j = 1:n],
    (L[j] / (m * v[j])) * cos(β[j]) +
    cos(γ[j]) * ((v[j] / r[j]) - (g[j] / v[j]))
)
@NLexpression(
    model,
    δψ[j = 1:n],
    (1 / (m * v[j] * cos(γ[j]))) * L[j] * sin(β[j]) +
    (v[j] / (r[j] * cos(θ[j]))) * cos(γ[j]) * sin(ψ[j]) * sin(θ[j])
)

# System dynamics
for j in 2:n
    i = j - 1  # index of previous knot

    if integration_rule == "rectangular"
        # Rectangular integration
        @NLconstraint(model, h[j] == h[i] + Δt[i] * δh[i])
        @NLconstraint(model, ϕ[j] == ϕ[i] + Δt[i] * δϕ[i])
        @NLconstraint(model, θ[j] == θ[i] + Δt[i] * δθ[i])
        @NLconstraint(model, v[j] == v[i] + Δt[i] * δv[i])
        @NLconstraint(model, γ[j] == γ[i] + Δt[i] * δγ[i])
        @NLconstraint(model, ψ[j] == ψ[i] + Δt[i] * δψ[i])
    elseif integration_rule == "trapezoidal"
        # Trapezoidal integration
        @NLconstraint(model, h[j] == h[i] + 0.5 * Δt[i] * (δh[j] + δh[i]))
        @NLconstraint(model, ϕ[j] == ϕ[i] + 0.5 * Δt[i] * (δϕ[j] + δϕ[i]))
```

```
        @NLconstraint(model, θ[j] == θ[i] + 0.5 * Δt[i] * (δθ[j] + δθ[i]))
        @NLconstraint(model, v[j] == v[i] + 0.5 * Δt[i] * (δv[j] + δv[i]))
        @NLconstraint(model, γ[j] == γ[i] + 0.5 * Δt[i] * (δγ[j] + δγ[i]))
        @NLconstraint(model, ψ[j] == ψ[i] + 0.5 * Δt[i] * (δψ[j] + δψ[i]))
    else
        @error "Unexpected integration rule '$(integration_rule)'"
    end
end

# Objective: Maximize cross-range
@objective(model, Max, θ[n])

set_silent(model)  # Hide solver's verbose output
optimize!(model)  # Solve for the control and state
@assert termination_status(model) == LOCALLY_SOLVED

# Show final cross-range of the solution
println(
    "Final latitude θ = ",
    round(objective_value(model) |> rad2deg, digits = 2),
    "°",
)
```

```
Final latitude θ = 34.18°
```

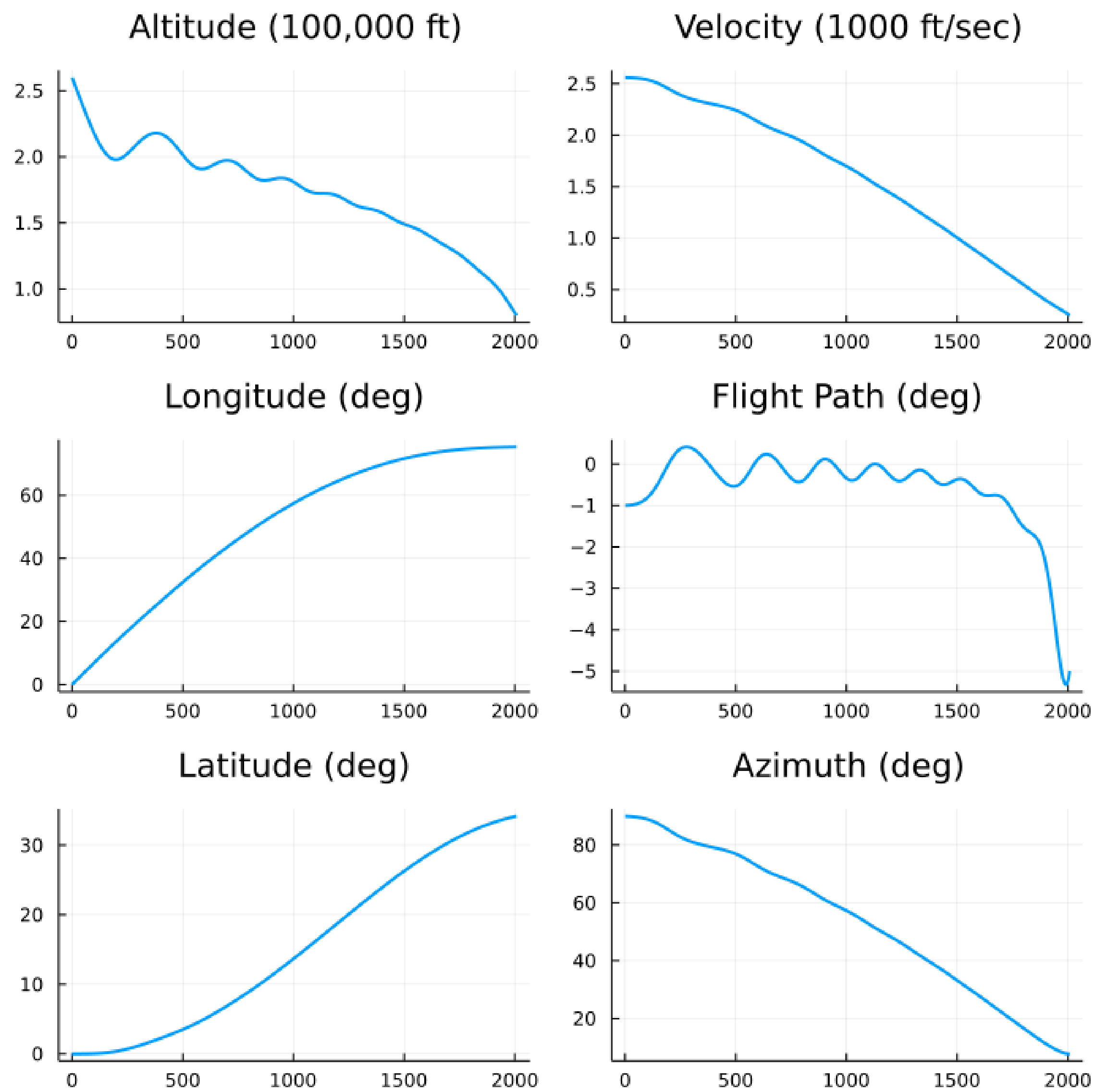
Plotting the results

```
using Plots
```

```
ts = cumsum([0; value.(Δt)])[1:end-1]
```

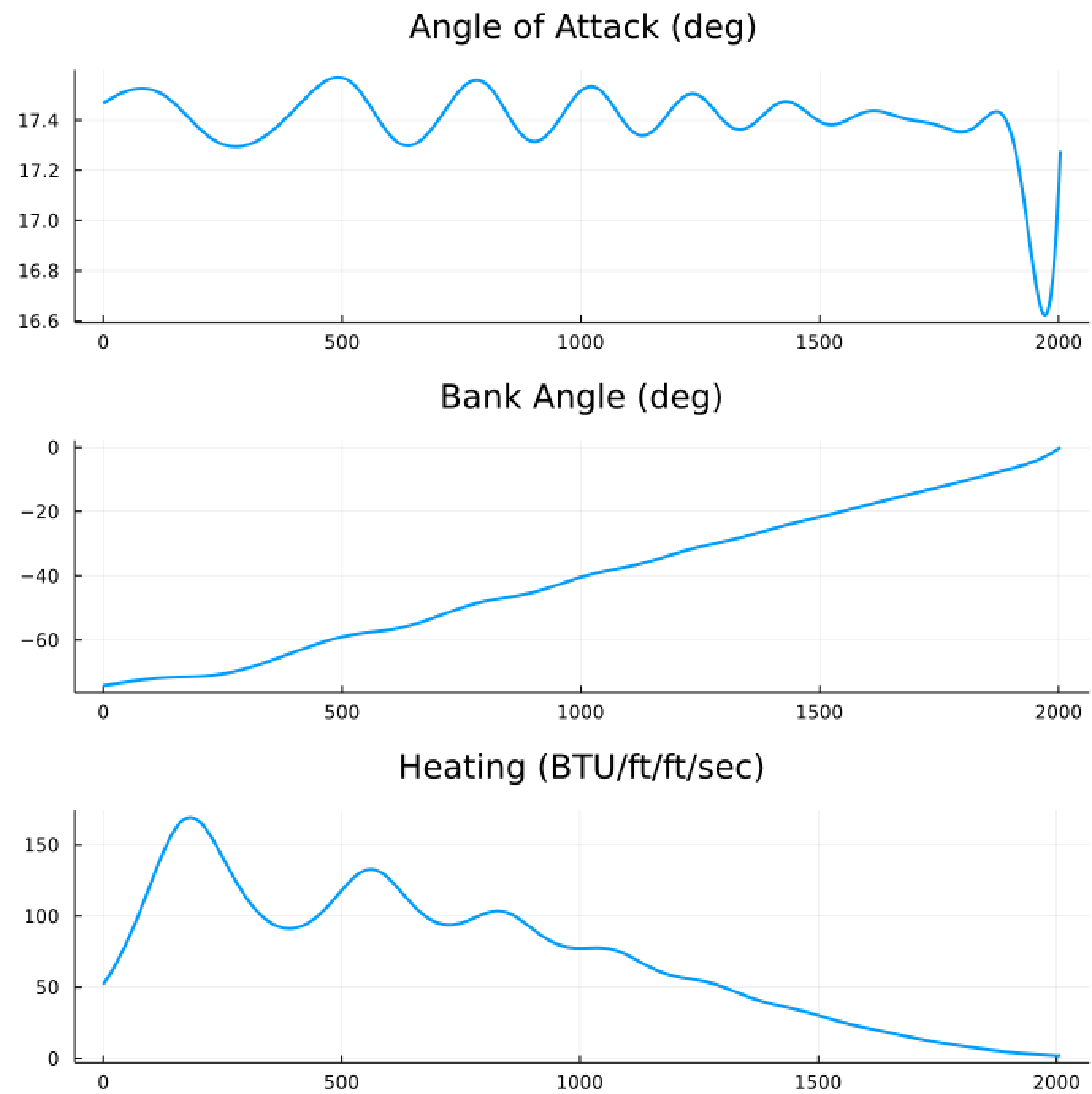
```
plt_altitude = plot(
    ts,
    value.(scaled_h),
    legend = nothing,
    title = "Altitude (100,000 ft)",
)
plt_longitude =
    plot(ts, rad2deg.(value.(ϕ)), legend = nothing, title = "Longitude (deg)")
plt_latitude =
    plot(ts, rad2deg.(value.(θ)), legend = nothing, title = "Latitude (deg)")
plt_velocity = plot(
    ts,
    value.(scaled_v),
    legend = nothing,
    title = "Velocity (1000 ft/sec)",
)
plt_flight_path =
    plot(ts, rad2deg.(value.(γ)), legend = nothing, title = "Flight Path (deg)")
plt_azimuth =
    plot(ts, rad2deg.(value.(ψ)), legend = nothing, title = "Azimuth (deg)")

plt = plot(
    plt_altitude,
    plt_velocity,
```

```
    plt_longitude,
    plt_flight_path,
    plt_latitude,
    plt_azimuth,
    layout = grid(3, 2),
    linewidth = 2,
    size = (700, 700),
)
```

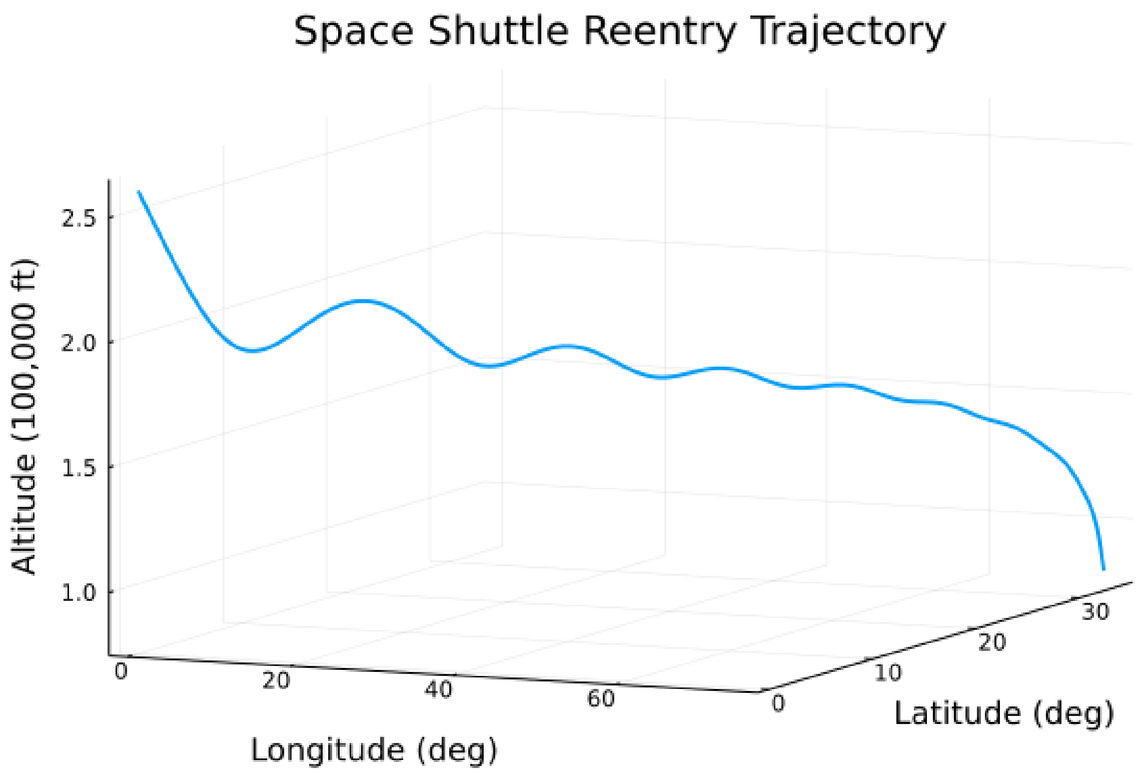



```
function q(h, v, a)
    ρ(h) = ρ₀ * exp(-h / hᵣ)
    qᵣ(h, v) = 17700 * √ρ(h) * (0.0001 * v)^3.07
    qₐ(a) = c₀ + c₁ * rad2deg(a) + c₂ * rad2deg(a)^2 + c₃ * rad2deg(a)^3
    # Aerodynamic heating on the vehicle wing leading edge
```

```
    return qₐ(a) * qᵣ(h, v)
end

plt_attack_angle = plot(
    ts[1:end-1],
    rad2deg.(value.(α)[1:end-1]),
    legend = nothing,
    title = "Angle of Attack (deg)",
)
plt_bank_angle = plot(
    ts[1:end-1],
    rad2deg.(value.(β)[1:end-1]),
    legend = nothing,
    title = "Bank Angle (deg)",
)
plt_heating = plot(
    ts,
    q.(value.(scaled_h) * 1e5, value.(scaled_v) * 1e4, value.(α)),
    legend = nothing,
    title = "Heating (BTU/ft/ft/sec)",
)

plt = plot(
    plt_attack_angle,
    plt_bank_angle,
    plt_heating,
    layout = grid(3, 1),
    linewidth = 2,
    size = (700, 700),
)
```




```
plt = plot(
    rad2deg.(value.(ϕ)),
    rad2deg.(value.(θ)),
    value.(scaled_h),
    linewidth = 2,
    legend = nothing,
    title = "Space Shuttle Reentry Trajectory",
    xlabel = "Longitude (deg)",
    ylabel = "Latitude (deg)",
    zlabel = "Altitude (100,000 ft)",
)
```



Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.6 Rocket Control

Originally Contributed by: Iain Dunning

This tutorial shows how to solve a nonlinear rocketry control problem. The problem was drawn from the COPS3 benchmark.

Our goal is to maximize the final altitude of a vertically launched rocket.

We can control the thrust of the rocket, and must take account of the rocket mass, fuel consumption rate, gravity, and aerodynamic drag.

Let us consider the basic description of the model (for the full description, including parameters for the rocket, see the COPS3 PDF)

Overview

We will use a discretized model of time, with a fixed number of time steps, n.

We will make the time step size Δt, and thus the final time $t_f = n \cdot \Delta t$, a variable in the problem. To approximate the derivatives in the problem we will use the trapezoidal rule.

State and Control

We will have three state variables:

- Velocity, v
- Altitude, h
- Mass of rocket and remaining fuel, m

and a single control variable, thrust T.

Our goal is thus to maximize $h(t_f)$.

Each of these corresponds to a JuMP variable indexed by the time step.

Dynamics

We have three equations that control the dynamics of the rocket:

Rate of ascent: $h' = v$ Acceleration: $v' = \frac{T-D(h,v)}{m} - g(h)$ Rate of mass loss: $m' = -\frac{T}{c}$

where drag $D(h, v)$ is a function of altitude and velocity, and gravity $g(h)$ is a function of altitude.

These forces are defined as

$$D(h, v) = D_c v^2 exp\left(-h_c \left(\frac{h - h(0)}{h(0)}\right)\right)$$

and $g(h) = g_0 \left(\frac{h(0)}{h}\right)^2$

The three rate equations correspond to JuMP constraints, and for convenience we will represent the forces with nonlinear expressions.

```
using JuMP
import Ipopt
import Plots
```

Create JuMP model, using Ipopt as the solver

```
rocket = Model(Ipopt.Optimizer)
set_silent(rocket)
```

Constants

Note that all parameters in the model have been normalized to be dimensionless. See the COPS3 paper for more info.

```
h_0 = 1    # Initial height
v_0 = 0    # Initial velocity
m_0 = 1    # Initial mass
g_0 = 1    # Gravity at the surface
```

```
T_c = 3.5  # Used for thrust
h_c = 500  # Used for drag
v_c = 620  # Used for drag
m_c = 0.6  # Fraction of initial mass left at end

c = 0.5 * sqrt(g_0 * h_0)  # Thrust-to-fuel mass
m_f = m_c * m_0              # Final mass
D_c = 0.5 * v_c * m_0 / g_0  # Drag scaling
T_max = T_c * g_0 * m_0        # Maximum thrust

n = 800    # Time steps
```

```
800
```

Decision variables

```
@variables(rocket, begin
    Δt ≥ 0, (start = 1 / n) # Time step
    # State variables
    v[1:n] ≥ 0            # Velocity
    h[1:n] ≥ h_0          # Height
    m_f ≤ m[1:n] ≤ m_0    # Mass
    # Control variables
    0 ≤ T[1:n] ≤ T_max    # Thrust
end)
```

```
Δ
(t, VariableRef[v[1], v[2], v[3], v[4], v[5], v[6], v[7], v[8], v[9], v[10]  …  v[791], v[792], v
    [793], v[794], v[795], v[796], v[797], v[798], v[799], v[800]], VariableRef[h[1], h[2], h[3], h
    [4], h[5], h[6], h[7], h[8], h[9], h[10]  …  h[791], h[792], h[793], h[794], h[795], h[796], h
    [797], h[798], h[799], h[800]], VariableRef[m[1], m[2], m[3], m[4], m[5], m[6], m[7], m[8], m
    [9], m[10]  …  m[791], m[792], m[793], m[794], m[795], m[796], m[797], m[798], m[799], m[800]],
    VariableRef[T[1], T[2], T[3], T[4], T[5], T[6], T[7], T[8], T[9], T[10]  …  T[791], T[792], T
    [793], T[794], T[795], T[796], T[797], T[798], T[799], T[800]])
```

Objective

The objective is to maximize altitude at end of time of flight.

```
@objective(rocket, Max, h[n])
```

$$h_{800}$$

Initial conditions

```
fix(v[1], v_0; force = true)
fix(h[1], h_0; force = true)
fix(m[1], m_0; force = true)
fix(m[n], m_f; force = true)
```

Forces

```
@NLexpressions(
    rocket,
    begin
        # Drag(h,v) = Dc v^2 exp( -hc * (h - h0) / h0 )
        drag[j = 1:n], D_c * (v[j]^2) * exp(-h_c * (h[j] - h_0) / h_0)
        # Grav(h)   = go * (h0 / h)^2
        grav[j = 1:n], g_0 * (h_0 / h[j])^2
        # Time of flight
        t_f, Δt * n
    end
)
```

```
(NonlinearExpression[subexpression[1]: 310.0 * v[1] ^ 2.0 * exp((-500.0 * (h[1] - 1.0)) / 1.0),
    subexpression[2]: 310.0 * v[2] ^ 2.0 * exp((-500.0 * (h[2] - 1.0)) / 1.0), subexpression[3]:
    310.0 * v[3] ^ 2.0 * exp((-500.0 * (h[3] - 1.0)) / 1.0), subexpression[4]: 310.0 * v[4] ^ 2.0 *
    exp((-500.0 * (h[4] - 1.0)) / 1.0), subexpression[5]: 310.0 * v[5] ^ 2.0 * exp((-500.0 * (h[5] -
     1.0)) / 1.0), subexpression[6]: 310.0 * v[6] ^ 2.0 * exp((-500.0 * (h[6] - 1.0)) / 1.0),
    subexpression[7]: 310.0 * v[7] ^ 2.0 * exp((-500.0 * (h[7] - 1.0)) / 1.0), subexpression[8]:
    310.0 * v[8] ^ 2.0 * exp((-500.0 * (h[8] - 1.0)) / 1.0), subexpression[9]: 310.0 * v[9] ^ 2.0 *
    exp((-500.0 * (h[9] - 1.0)) / 1.0), subexpression[10]: 310.0 * v[10] ^ 2.0 * exp((-500.0 * (h
    [10] - 1.0)) / 1.0)  …  subexpression[791]: 310.0 * v[791] ^ 2.0 * exp((-500.0 * (h[791] - 1.0))
     / 1.0), subexpression[792]: 310.0 * v[792] ^ 2.0 * exp((-500.0 * (h[792] - 1.0)) / 1.0),
    subexpression[793]: 310.0 * v[793] ^ 2.0 * exp((-500.0 * (h[793] - 1.0)) / 1.0), subexpression
    [794]: 310.0 * v[794] ^ 2.0 * exp((-500.0 * (h[794] - 1.0)) / 1.0), subexpression[795]: 310.0 *
    v[795] ^ 2.0 * exp((-500.0 * (h[795] - 1.0)) / 1.0), subexpression[796]: 310.0 * v[796] ^ 2.0 *
    exp((-500.0 * (h[796] - 1.0)) / 1.0), subexpression[797]: 310.0 * v[797] ^ 2.0 * exp((-500.0 * (
    h[797] - 1.0)) / 1.0), subexpression[798]: 310.0 * v[798] ^ 2.0 * exp((-500.0 * (h[798] - 1.0))
    / 1.0), subexpression[799]: 310.0 * v[799] ^ 2.0 * exp((-500.0 * (h[799] - 1.0)) / 1.0),
    subexpression[800]: 310.0 * v[800] ^ 2.0 * exp((-500.0 * (h[800] - 1.0)) / 1.0)],
    NonlinearExpression[subexpression[801]: 1.0 * (1.0 / h[1]) ^ 2.0, subexpression[802]: 1.0 * (1.0
     / h[2]) ^ 2.0, subexpression[803]: 1.0 * (1.0 / h[3]) ^ 2.0, subexpression[804]: 1.0 * (1.0 / h
    [4]) ^ 2.0, subexpression[805]: 1.0 * (1.0 / h[5]) ^ 2.0, subexpression[806]: 1.0 * (1.0 / h[6])
     ^ 2.0, subexpression[807]: 1.0 * (1.0 / h[7]) ^ 2.0, subexpression[808]: 1.0 * (1.0 / h[8]) ^
    2.0, subexpression[809]: 1.0 * (1.0 / h[9]) ^ 2.0, subexpression[810]: 1.0 * (1.0 / h[10]) ^ 2.0
      …  subexpression[1591]: 1.0 * (1.0 / h[791]) ^ 2.0, subexpression[1592]: 1.0 * (1.0 / h[792])
    ^ 2.0, subexpression[1593]: 1.0 * (1.0 / h[793]) ^ 2.0, subexpression[1594]: 1.0 * (1.0 / h
    [794]) ^ 2.0, subexpression[1595]: 1.0 * (1.0 / h[795]) ^ 2.0, subexpression[1596]: 1.0 * (1.0 /
     h[796]) ^ 2.0, subexpression[1597]: 1.0 * (1.0 / h[797]) ^ 2.0, subexpression[1598]: 1.0 * (1.0
     / h[798]) ^ 2.0, subexpression[1599]: 1.0 * (1.0 / h[799]) ^ 2.0, subexpression[1600]: 1.0 *
    (1.0 / h[800]) ^ 2.0], subexpression[1601]: Δt * 800.0)
```

Dynamics

```
for j in 2:n
    # h' = v
    # Rectangular integration
    # @NLconstraint(rocket, h[j] == h[j - 1] + Δt * v[j - 1])
    # Trapezoidal integration
    @NLconstraint(rocket, h[j] == h[j-1] + 0.5 * Δt * (v[j] + v[j-1]))
    # v' = (T-D(h,v))/m - g(h)
    # Rectangular integration
    # @NLconstraint(
    #     rocket,
    #     v[j] == v[j - 1] + Δt *((T[j - 1] - drag[j - 1]) / m[j - 1] - grav[j - 1])
```

```
    # )
    # Trapezoidal integration
    @NLconstraint(
        rocket,
        v[j] ==
        v[j-1] +
        0.5 *
        Δt *
        (
            (T[j] - drag[j] - m[j] * grav[j]) / m[j] +
            (T[j-1] - drag[j-1] - m[j-1] * grav[j-1]) / m[j-1]
        )
    )
    # m' = -T/c
    # Rectangular integration
    # @NLconstraint(rocket, m[j] == m[j - 1] - Δt * T[j - 1] / c)
    # Trapezoidal integration
    @NLconstraint(rocket, m[j] == m[j-1] - 0.5 * Δt * (T[j] + T[j-1]) / c)
end
```

Solve for the control and state

```
println("Solving...")
optimize!(rocket)
solution_summary(rocket)
```

```
* Solver : Ipopt

* Status
  Termination status : LOCALLY_SOLVED
  Primal status      : FEASIBLE_POINT
  Dual status        : FEASIBLE_POINT
  Message from the solver:
  "Solve_Succeeded"

* Candidate solution
  Objective value      : 1.01283e+00

* Work counters
  Solve time (sec)   : 1.30007e+00
```

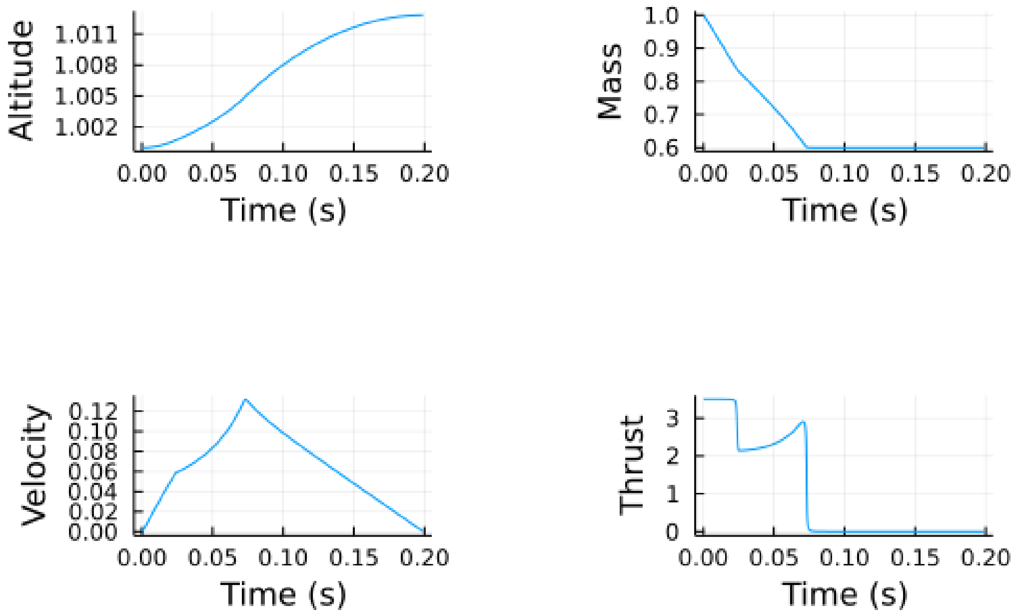
Display results

```
println("Max height: ", objective_value(rocket))
```

```
Max height: 1.0128340648308019
```

```
function my_plot(y, ylabel)
    return Plots.plot(
        (1:n) * value.(Δt),
        value.(y)[:];
        xlabel = "Time (s)",
        ylabel = ylabel,
```

```
    )
end

Plots.plot(
    my_plot(h, "Altitude"),
    my_plot(m, "Mass"),
    my_plot(v, "Velocity"),
    my_plot(T, "Thrust");
    layout = (2, 2),
    legend = false,
    margin = 1Plots.cm,
)
```



Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.7 The Rosenbrock function

A nonlinear example of the classical Rosenbrock function.

```
using JuMP
import Ipopt
import Test
```

```
function example_rosenbrock()
    model = Model(Ipopt.Optimizer)
    set_silent(model)
    @variable(model, x)
    @variable(model, y)
    @NLobjective(model, Min, (1 - x)^2 + 100 * (y - x^2)^2)
    optimize!(model)

    Test.@test termination_status(model) == LOCALLY_SOLVED
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) ≈ 0.0 atol = 1e-10
    Test.@test value(x) ≈ 1.0
    Test.@test value(y) ≈ 1.0
    return
end

example_rosenbrock()
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.8 Maximum likelihood estimation

Use nonlinear optimization to compute the maximum likelihood estimate (MLE) of the parameters of a normal distribution, a.k.a., the sample mean and variance.

```
using JuMP
import Ipopt
import Random
import Statistics
import Test

function example_mle(; verbose = true)
    n = 1_000
    Random.seed!(1234)
    data = randn(n)
    model = Model(Ipopt.Optimizer)
    set_silent(model)
    @variable(model, μ, start = 0.0)
    @variable(model, σ >= 0.0, start = 1.0)
    @NLobjective(
        model,
        Max,
        n / 2 * log(1 / (2 * π * σ^2)) -
        sum((data[i] - μ)^2 for i in 1:n) / (2 * σ^2)
    )
    optimize!(model)
    if verbose
        println("μ             = ", value(μ))
        println("mean(data)    = ", Statistics.mean(data))
        println("σ^2           = ", value(σ)^2)
```

```
        println("var(data)     = ", Statistics.var(data))
        println("MLE objective = ", objective_value(model))
    end
    Test.@test value(μ) ≈ Statistics.mean(data) atol = 1e-3
    Test.@test value(σ)^2 ≈ Statistics.var(data) atol = 1e-2
    # You can even do constrained MLE!
    @NLconstraint(model, μ == σ^2)
    optimize!(model)
    Test.@test value(μ) ≈ value(σ)^2
    if verbose
        println()
        println("With constraint μ == σ^2:")
        println("μ                         = ", value(μ))
        println("σ^2                       = ", value(σ)^2)
        println("Constrained MLE objective = ", objective_value(model))
    end
    return
end

example_mle()
```

```
μ
             = -0.022943864572027958
mean(data)    = -0.022943864572027916σ
^2           = 1.0096978289461431
var(data)     = 1.0107085374810936
MLE objective = -1423.76408661786

With constraint μ == σ^2:μ
                         = 0.6225971004178991σ
^2                       = 0.6225971004178991
Constrained MLE objective = -1827.5516590930729
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.9 The clnlbeam problem

Based on an AMPL model by Hande Y. Benson

Copyright (C) 2001 Princeton University All Rights Reserved

Permission to use, copy, modify, and distribute this software and its documentation for any purpose and without fee is hereby granted, provided that the above copyright notice appear in all copies and that the copyright notice and this permission notice appear in all supporting documentation.

Source: H. Maurer and H.D. Mittelman, "The non-linear beam via optimal control with bound state variables", Optimal Control Applications and Methods 12, pp. 19-31, 1991.

```
using JuMP
import Ipopt

function example_clnlbeam()
```

```
    N = 1000
    h = 1 / N
    alpha = 350
    model = Model(Ipopt.Optimizer)
    @variables(model, begin
        -1 <= t[1:(N+1)] <= 1
        -0.05 <= x[1:(N+1)] <= 0.05
        u[1:(N+1)]
    end)
    @NLobjective(
        model,
        Min,
        sum(
            0.5 * h * (u[i+1]^2 + u[i]^2) +
            0.5 * alpha * h * (cos(t[i+1]) + cos(t[i])) for i in 1:N
        ),
    )
    @NLconstraint(
        model,
        [i = 1:N],
        x[i+1] - x[i] - 0.5 * h * (sin(t[i+1]) + sin(t[i])) == 0,
    )
    @constraint(
        model,
        [i = 1:N],
        t[i+1] - t[i] - 0.5 * h * u[i+1] - 0.5 * h * u[i] == 0,
    )
    optimize!(model)
    println("""
    termination_status = $(termination_status(model))
    primal_status      = $(primal_status(model))
    objective_value    = $(objective_value(model))
    """)
    return
end

example_clnlbeam()
```

```
This is Ipopt version 3.14.4, running with linear solver MUMPS 5.4.1.

Number of nonzeros in equality constraint Jacobian...:     8000
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:     4002

Total number of variables............................:     3003
                     variables with only lower bounds:        0
                variables with lower and upper bounds:     2002
                     variables with only upper bounds:        0
Total number of equality constraints.................:     2000
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.5000000e+02 0.00e+00 0.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
```

```
   1  3.5000000e+02 0.00e+00 0.00e+00  -1.7 0.00e+00    -  1.00e+00 1.00e+00   0
   2  3.5000000e+02 0.00e+00 0.00e+00  -3.8 0.00e+00  -2.0 1.00e+00 1.00e+00T  0
   3  3.5000000e+02 0.00e+00 0.00e+00  -5.7 0.00e+00   0.2 1.00e+00 1.00e+00T  0
   4  3.5000000e+02 0.00e+00 0.00e+00  -8.6 0.00e+00  -0.2 1.00e+00 1.00e+00T  0

Number of Iterations....: 4

                                   (scaled)                 (unscaled)
Objective...............:   3.5000000000000318e+02    3.5000000000000318e+02
Dual infeasibility......:   0.0000000000000000e+00    0.0000000000000000e+00
Constraint violation....:   0.0000000000000000e+00    0.0000000000000000e+00
Variable bound violation:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   2.5059035596802450e-09    2.5059035596802450e-09
Overall NLP error.......:   2.5059035596802450e-09    2.5059035596802450e-09


Number of objective function evaluations             = 5
Number of objective gradient evaluations             = 5
Number of equality constraint evaluations            = 5
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 5
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 4
Total seconds in IPOPT                               = 0.108

EXIT: Optimal Solution Found.
termination_status = LOCALLY_SOLVED
primal_status      = FEASIBLE_POINT
objective_value    = 350.0000000000032
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

6.10 Computing Hessians

The purpose of this tutorial is to demonstrate how to compute the Hessian of the Lagrangian of a nonlinear program.

Warning

This is an advanced tutorial that interacts with the low-level nonlinear interface of MathOptInterface.

By default, JuMP exports the `MOI` symbol as an alias for the MathOptInterface.jl package. We recommend making this more explicit in your code by adding the following lines:

```
import MathOptInterface
const MOI = MathOptInterface
```

Given a nonlinear program:

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{6.4}$$

$$\text{s.t.} \quad l \leq g_i(x) \leq u \tag{6.5}$$

the Hessian of the Lagrangian is computed as:

$$H(x, \sigma, \mu) = \sigma \nabla^2 f(x) + \sum_{i=1}^{m} \mu_i \nabla^2 g_i(x)$$

where x is a primal point, σ is a scalar (typically 1), and μ is a vector of weights corresponding to the Lagrangian dual of the constraints.

This tutorial uses the following packages:

```
using JuMP
import Ipopt
import LinearAlgebra
import Random
import SparseArrays
```

The basic model

To demonstrate how to interact with the lower-level nonlinear interface, we need an example model. The exact model isn't important; we use the model from The Rosenbrock function tutorial, with some additional constraints to demonstrate various features of the lower-level interface.

```
model = Model(Ipopt.Optimizer)
set_silent(model)
@variable(model, x[i = 1:2], start = -i)
@constraint(model, g_1, x[1]^2 <= 1)
@NLconstraint(model, g_2, (x[1] + x[2])^2 <= 2)
@NLobjective(model, Min, (1 - x[1])^2 + 100 * (x[2] - x[1]^2)^2)
optimize!(model)
```

The analytic solution

With a little work, it is possible to analytically derive the correct hessian:

```
function analytic_hessian(x, σ, μ)
    g_1_H = [2.0 0.0; 0.0 0.0]
    g_2_H = [2.0 2.0; 2.0 2.0]
    f_H = zeros(2, 2)
    f_H[1, 1] = 2.0 + 1200.0 * x[1]^2 - 400.0 * x[2]
    f_H[1, 2] = f_H[2, 1] = -400.0 * x[1]
    f_H[2, 2] = 200.0
    return σ * f_H + μ' * [g_1_H, g_2_H]
end
```

```
analytic_hessian (generic function with 1 method)
```

Here are various points:

```
analytic_hessian([1, 1], 0, [0, 0])
```

```
2×2 Matrix{Float64}:
 0.0  0.0
 0.0  0.0
```

```
analytic_hessian([1, 1], 0, [1, 0])
```

```
2×2 Matrix{Float64}:
 2.0  0.0
 0.0  0.0
```

```
analytic_hessian([1, 1], 0, [0, 1])
```

```
2×2 Matrix{Float64}:
 2.0  2.0
 2.0  2.0
```

```
analytic_hessian([1, 1], 1, [0, 0])
```

```
2×2 Matrix{Float64}:
  802.0  -400.0
 -400.0   200.0
```

Initializing the NLPEvaluator

JuMP stores all information relating to the nonlinear portions of the model in a `NLPEvaluator` struct:

```
d = NLPEvaluator(model)
```

```
An NLPEvaluator with available features:
  * :Grad
  * :Jac
  * :JacVec
  * :ExprGraph
  * :Hess
  * :HessVec
```

Before computing anything with the NLPEvaluator, we need to initialize it. Use `MOI.features_available` to see what we can query:

```
MOI.features_available(d)
```

```
6-element Vector{Symbol}:
 :Grad
 :Jac
 :JacVec
 :ExprGraph
 :Hess
 :HessVec
```

Consult the MOI documentation for specifics. But to obtain the Hessian matrix, we need to initialize `:Hess`:

```
MOI.initialize(d, [:Hess])
```

MOI represents the Hessian as a sparse matrix. Get the sparsity pattern as follows:

```
hessian_sparsity = MOI.hessian_lagrangian_structure(d)
```

```
6-element Vector{Tuple{Int64, Int64}}:
 (1, 1)
 (2, 2)
 (2, 1)
 (1, 1)
 (2, 2)
 (2, 1)
```

The sparsity pattern has a few properties of interest:

- Each element `(i, j)` indicates a structural non-zero in row `i` and column `j`
- The list may contain duplicates, in which case we should add the values together
- The list does not need to be sorted
- The list may contain any mix of lower- or upper-triangular indices

This format matches Julia's sparse-triplet form of a SparseArray, so we can convert from the sparse Hessian representation to a Julia SparseArray as follows:

```
I = [i for (i, _) in hessian_sparsity]
J = [j for (_, j) in hessian_sparsity]
V = zeros(length(hessian_sparsity))
n = num_variables(model)
H = SparseArrays.sparse(I, J, V, n, n)
```

```
2×2 SparseArrays.SparseMatrixCSC{Float64, Int64} with 3 stored entries:
 0.0   ⋅
 0.0  0.0
```

Of course, knowing where the zeros are isn't very interesting. We really want to compute the value of the Hessian matrix at a point.

```
num_g = num_nonlinear_constraints(model)
MOI.eval_hessian_lagrangian(d, V, ones(n), 1.0, ones(num_g))
H = SparseArrays.sparse(I, J, V, n, n)
```

```
2×2 SparseArrays.SparseMatrixCSC{Float64, Int64} with 3 stored entries:
  804.0     ⋅
 -398.0  202.0
```

In practice, we often want to compute the value of the hessian at the optimal solution.

First, we compute the primal solution. To do so, we need a vector of the variables in the order that they were passed to the solver:

```
x = all_variables(model)
```

```
2-element Vector{VariableRef}:
 x[1]
 x[2]
```

Here `x[1]` is the variable that corresponds to column 1, and so on. Here's the optimal primal solution:

```
x_optimal = value.(x)
```

```
2-element Vector{Float64}:
 0.7903587551555908
 0.6238546250475736
```

Next, we need the optimal dual solution associated with the nonlinear constraints:

```
y_optimal = dual.(all_nonlinear_constraints(model))
```

```
1-element Vector{Float64}:
 -0.057440893636909414
```

Now we can compute the Hessian at the optimal primal-dual point:

```
MOI.eval_hessian_lagrangian(d, V, x_optimal, 1.0, y_optimal)
H = SparseArrays.sparse(I, J, V, n, n)
```

```
2×2 SparseArrays.SparseMatrixCSC{Float64, Int64} with 3 stored entries:
  501.944     ⋅
 -316.258  199.885
```

However, this Hessian isn't quite right because it isn't symmetric. We can fix this by filling in the appropriate off-diagonal terms:

```
function fill_off_diagonal(H)
    ret = H + H'
    row_vals = SparseArrays.rowvals(ret)
    non_zeros = SparseArrays.nonzeros(ret)
    for col in 1:size(ret, 2)
        for i in SparseArrays.nzrange(ret, col)
            if col == row_vals[i]
                non_zeros[i] /= 2
            end
        end
```

```
    end
    return ret
end

fill_off_diagonal(H)
```

```
2×2 SparseArrays.SparseMatrixCSC{Float64, Int64} with 4 stored entries:
  501.944  -316.258
 -316.258   199.885
```

Moreover, this Hessian only accounts for the objective and constraints entered using `@NLobjective` and `@NLconstraint`. If we want to take quadratic objectives and constraints written using `@objective` or `@constraint` into account, we'll need to handle them separately.

> **Tip**
>
> If you don't want to do this, you can replace calls to `@objective` and `@constraint` with `@NLobjective` and `@NLconstraint`.

Hessians from QuadExpr functions

To compute the hessian from a quadratic expression, let's see how JuMP represents a quadratic constraint:

```
f = constraint_object(g_1).func
```

$$x_1^2$$

`f` is a quadratic expression of the form:

```
f(x) = Σᵢⱼq * ᵢx * ⱼx + Σᵢa ᵢx + c
```

So $\nabla^2 f(x)$ is the matrix formed by $[q_{ij}]_{ij}$ if `i != j` and $2[q_{ij}]_{ij}$ if `i = j`.

```
variables_to_column = Dict(x[i] => i for i in 1:n)

function add_to_hessian(H, f::QuadExpr, μ)
    for (vars, coef) in f.terms
        i = variables_to_column[vars.a]
        j = variables_to_column[vars.b]
        H[i, j] += μ * coef
    end
    return
end
```

```
add_to_hessian (generic function with 1 method)
```

If the function `f` is not a `QuadExpr`, do nothing because it is an `AffExpr` or a `VariableRef`. In both cases, the second derivative is zero.

```
add_to_hessian(H, f::Any, μ) = nothing
```

```
add_to_hessian (generic function with 2 methods)
```

Then we iterate over all constraints in the model and add their Hessian components:

```
for (F, S) in list_of_constraint_types(model)
    for cref in all_constraints(model, F, S)
        f = constraint_object(cref).func
        add_to_hessian(H, f, dual(cref))
    end
end

H
```

```
2×2 SparseArrays.SparseMatrixCSC{Float64, Int64} with 3 stored entries:
  501.944     ⋅
 -316.258  199.885
```

Finally, we need to take into account the objective function:

```
add_to_hessian(H, objective_function(model), 1.0)

fill_off_diagonal(H)
```

```
2×2 SparseArrays.SparseMatrixCSC{Float64, Int64} with 4 stored entries:
  501.944  -316.258
 -316.258   199.885
```

Putting everything together:

```
function compute_optimal_hessian(model)
    d = NLPEvaluator(model)
    MOI.initialize(d, [:Hess])
    hessian_sparsity = MOI.hessian_lagrangian_structure(d)
    I = [i for (i, _) in hessian_sparsity]
    J = [j for (_, j) in hessian_sparsity]
    V = zeros(length(hessian_sparsity))
    x = all_variables(model)
    x_optimal = value.(x)
    y_optimal = dual.(all_nonlinear_constraints(model))
    MOI.eval_hessian_lagrangian(d, V, x_optimal, 1.0, y_optimal)
    n = num_variables(model)
    H = SparseArrays.sparse(I, J, V, n, n)
    vmap = Dict(x[i] => i for i in 1:n)
    add_to_hessian(H, f::Any, μ) = nothing
    function add_to_hessian(H, f::QuadExpr, μ)
        for (vars, coef) in f.terms
            if vars.a != vars.b
                H[vmap[vars.a], vmap[vars.b]] += μ * coef
            else
                H[vmap[vars.a], vmap[vars.b]] += 2 * μ * coef
            end
        end
    end
    for (F, S) in list_of_constraint_types(model)
        for cref in all_constraints(model, F, S)
            add_to_hessian(H, constraint_object(cref).func, dual(cref))
```

```
        end
    end
    add_to_hessian(H, objective_function(model), 1.0)
    return Matrix(fill_off_diagonal(H))
end

H_star = compute_optimal_hessian(model)
```

```
2×2 Matrix{Float64}:
  501.944  -316.258
 -316.258   199.885
```

If we compare our solution against the analytical solution:

```
analytic_hessian(value.(x), 1.0, dual.([g_1, g_2]))
```

```
2×2 Matrix{Float64}:
  501.944  -316.258
 -316.258   199.885
```

If we look at the eigenvalues of the Hessian:

```
LinearAlgebra.eigvals(H_star)
```

```
2-element Vector{Float64}:
   0.44439959255495864
 701.3843408744132
```

we see that they are all positive. Therefore, the Hessian is positive definite, and so the solution found by Ipopt is a local minimizer.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

Chapter 7

Conic programs

7.1 Introduction

Conic programs are a class of convex nonlinear optimization problems which use cones to represent the nonlinearities. They have the form:

$$\min_{x \in \mathbb{R}^n} \quad f_0(x) \tag{7.1}$$

$$\text{s.t.} \quad f_j(x) \in \mathcal{S}_j \quad j = 1 \ldots m \tag{7.2}$$

Mixed-integer conic programs (MICPs) are extensions of conic programs in which some (or all) of the decision variables take discrete values.

How to choose a solver

JuMP supports a range of conic solvers, although support differs on what types of cones each solver supports. In the list of Supported solvers, "SOCP" denotes solvers supporting second-order cones and "SDP" denotes solvers supporting semidefinite cones. In addition, solvers such as SCS and Mosek have support for the exponential cone. Moreover, due to the bridging system in MathOptInterface, many of these solvers support a much wider range of exotic cones than they natively support. Solvers supporting discrete variables start with "(MI)" in the list of Supported solvers.

How these tutorials are structured

Having a high-level overview of how this part of the documentation is structured will help you know where to look for certain things.

- The following tutorials are worked examples that present a problem in words, then formulate it in mathematics, and then solve it in JuMP. This usually involves some sort of visualization of the solution. Start here if you are new to JuMP.
  - Experiment design
  - Logistic regression
- The Tips and tricks tutorial contains a number of helpful reformulations and tricks you can use when modeling conic programs. Look here if you are stuck trying to formulate a problem as a conic program.

- The remaining tutorials are less verbose and styled in the form of short code examples. These tutorials have less explanation, but may contain useful code snippets, particularly if they are similar to a problem you are trying to solve.

7.2 Primal and dual warm-starts

Some conic solvers have the ability to set warm-starts for the primal and dual solution. This can improve performance, particularly if you are repeatedly solving a sequence of related problems.

In this tutorial, we demonstrate how to write a function that sets the primal and dual starts as the optimal solution stored in a model. It is intended to be a starting point for which you can modify if you want to do something similar in your own code.

This tutorial uses the following packages:

```
using JuMP
import SCS
```

The main component of this tutorial is the following function. The most important observation is that we cache all of the solution values first, and then we modify the model second. (Alternating between querying a value and modifying the model is not allowed in JuMP.)

```
function set_optimal_start_values(model::Model)
    # Store a mapping of the variable primal solution
    variable_primal = Dict(x => value(x) for x in all_variables(model))
    # In the following, we loop through every constraint and store a mapping
    # from the constraint index to a tuple containing the primal and dual
    # solutions.
    constraint_solution = Dict()
    for (F, S) in list_of_constraint_types(model)
        # We add a try-catch here because some constraint types might not
        # support getting the primal or dual solution.
        try
            for ci in all_constraints(model, F, S)
                constraint_solution[ci] = (value(ci), dual(ci))
            end
        catch
            @info("Something went wrong getting $F-in-$S. Skipping")
        end
    end
    # Now we can loop through our cached solutions and set the starting values.
    for (x, primal_start) in variable_primal
        set_start_value(x, primal_start)
    end
    for (ci, (primal_start, dual_start)) in constraint_solution
        set_start_value(ci, primal_start)
        set_dual_start_value(ci, dual_start)
    end
    return
end
```

```
set_optimal_start_values (generic function with 1 method)
```

To test our function, we use the following linear program:

```
model = Model(SCS.Optimizer)
@variable(model, x[1:3] >= 0)
@constraint(model, sum(x) <= 1)
@objective(model, Max, sum(i * x[i] for i in 1:3))
optimize!(model)
```

By looking at the log (not shown in Documenter due to a bug), we can see that SCS took 100 iterations to find the optimal solution. Now we set the optimal solution as our starting point:

```
set_optimal_start_values(model)
```

and we re-optimize:

```
optimize!(model)
```

Now the optimization terminates after 0 iterations because our starting point is already optimal.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.3 Tips and Tricks

Originally Contributed by: Arpit Bhatia

This tutorial is aimed at providing a simplistic introduction to conic programming using JuMP.

It uses the following packages:

```
using JuMP
import SCS
import LinearAlgebra
```

Info

This tutorial uses sets from MathOptInterface. By default, JuMP exports the `MOI` symbol as an alias for the MathOptInterface.jl package. We recommend making this more explicit in your code by adding the following lines:

```
import MathOptInterface
const MOI = MathOptInterface
```

Tip

A good resource for learning more about functions which can be modeled using cones is the MOSEK Modeling Cookbook.

What is a cone?

A subset C of a vector space V is a cone if $\forall x \in C$ and positive scalars $\lambda > 0$, the product $\lambda x \in C$.

A cone C is a convex cone if $\lambda x + (1 - \lambda) y \in C$, for any $\lambda \in [0, 1]$, and any $x, y \in C$.

What is a conic program?

Conic programming problems are convex optimization problems in which a convex function is minimized over the intersection of an affine subspace and a convex cone. An example of a conic-form minimization problems, in the primal form is:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & a_0^T x + b_0 \\
\text{s.t.} \quad & A_i x + b_i \in \mathcal{C}_i \quad i = 1 \dots m
\end{aligned}
$$

The corresponding dual problem is:

$$
\begin{aligned}
\max_{y_1, \dots, y_m} \quad & -\sum_{i=1}^m b_i^T y_i + b_0 \\
\text{s.t.} \quad & a_0 - \sum_{i=1}^m A_i^T y_i = 0 \\
& y_i \in \mathcal{C}_i^* \quad i = 1 \dots m
\end{aligned}
$$

where each $\mathcal{C}_i$ is a closed convex cone and $\mathcal{C}_i^*$ is its dual cone.

Second-Order Cone

The Second-Order Cone (or Lorentz Cone) of dimension n is of the form:

$$
Q^n = \{(t, x) \in \mathbb{R}^n : t \ge ||x||_2\}
$$

Example

Minimize the L2 norm of a vector x.

```
model = Model()
@variable(model, x[1:3])
@variable(model, norm_x)
@constraint(model, [norm_x; x] in SecondOrderCone())
@objective(model, Min, norm_x)
```

$$
norm\_x
$$

Rotated Second-Order Cone

A Second-Order Cone rotated by $\pi/4$ in the (x_1, x_2) plane is called a Rotated Second-Order Cone. It is of the form:

$$
Q_r^n = \{(t, u, x) \in \mathbb{R}^n : 2tu \ge ||x||_2^2, t, u \ge 0\}
$$

Example

Given a set of predictors x, and observations y, find the parameter θ that minimizes the sum of squares loss between y_i and θx_i.

```
x = [1.0, 2.0, 3.0, 4.0]
y = [0.45, 1.04, 1.51, 1.97]
model = Model()
@variable(model, θ)
@variable(model, loss)
@constraint(model, [loss; 0.5; θ .* x .- y] in RotatedSecondOrderCone())
@objective(model, Min, loss)
```

$$loss$$

Exponential Cone

An Exponential Cone is a set of the form:

$$K_{exp} = \{(x, y, z) \in \mathbb{R}^3 : y \exp(x/y) \le z, y > 0\}$$

```
model = Model()
@variable(model, x[1:3] >= 0)
@constraint(model, x in MOI.ExponentialCone())
@objective(model, Min, x[3])
```

$$x_3$$

Example: Entropy Maximization

The entropy maximization problem consists of maximizing the entropy function, $H(x) = -x \log x$ subject to linear inequality constraints.

$$\begin{aligned} \max \quad & -\sum_{i=1}^{n} x_i \log x_i \\ \text{s.t.} \quad & \mathbf{1}'x = 1 \\ & Ax \le b \end{aligned}$$

We can model this problem using an exponential cone by using the following transformation:

$$t \le -x \log x \iff t \le x \log(1/x) \iff (t, x, 1) \in K_{exp}$$

Thus, our problem becomes,

$$\begin{aligned}
\max \quad & 1^T t \\
\text{s.t.} \quad & Ax \leq b \\
& 1^T x = 1 \\
& (t_i, x_i, 1) \in K_{exp} \quad \forall i = 1 \dots n
\end{aligned}$$

```
n = 15
m = 10
A = randn(m, n)
b = rand(m, 1)

model = Model(SCS.Optimizer)
set_silent(model)
@variable(model, t[1:n])
@variable(model, x[1:n])
@objective(model, Max, sum(t))
@constraint(model, sum(x) == 1)
@constraint(model, A * x .<= b)
@constraint(model, con[i = 1:n], [t[i], x[i], 1] in MOI.ExponentialCone())
optimize!(model)
```

```
objective_value(model)
```

```
2.708336083753836
```

Positive Semidefinite Cone

The set of positive semidefinite matrices (PSD) of dimension n form a cone in $\mathbb{R}^n$. We write this set mathematically as:

$$\mathcal{S}_+^n = \{X \in \mathcal{S}^n \mid z^T X z \geq 0, \, \forall z \in \mathbb{R}^n\}.$$

A PSD cone is represented in JuMP using the MOI sets `PositiveSemidefiniteConeTriangle` (for upper triangle of a PSD matrix) and `PositiveSemidefiniteConeSquare` (for a complete PSD matrix). However, it is preferable to use the PSDCone shortcut as illustrated below.

Example: largest eigenvalue of a symmetric matrix Suppose A has eigenvalues $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$. Then the matrix $tI - A$ has eigenvalues $t - \lambda_1, t - \lambda_2, \ldots, t - \lambda_n$. Note that $tI - A$ is PSD exactly when all these eigenvalues are non-negative, and this happens for values $t \geq \lambda_1$. Thus, we can model the problem of finding the largest eigenvalue of a symmetric matrix as:

$$\begin{aligned}
\lambda_1 &= \min t \\
\text{s.t. } tI - A &\succeq 0
\end{aligned}$$

```
A = [3 2 4; 2 0 2; 4 2 3]
I = Matrix{Float64}(LinearAlgebra.I, 3, 3)
model = Model(SCS.Optimizer)
```

```
set_silent(model)
@variable(model, t)
@objective(model, Min, t)
@constraint(model, t .* I - A in PSDCone())

optimize!(model)
```

```
objective_value(model)
```

```
8.000003377698672
```

Other Cones and Functions

For other cones supported by JuMP, check out the MathOptInterface Manual.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.4 Logistic regression

Originally Contributed by: François Pacaud

This tutorial shows how to solve a logistic regression problem with JuMP. Logistic regression is a well known method in machine learning, useful when we want to classify binary variables with the help of a given set of features. To this goal, we find the optimal combination of features maximizing the (log)-likelihood onto a training set. From a modern optimization glance, the resulting problem is convex and differentiable. On a modern optimization glance, it is even conic representable.

Formulating the logistic regression problem

Suppose we have a set of training data-point $i = 1, \cdots, n$, where for each i we have a vector of features $x_i \in \mathbb{R}^p$ and a categorical observation $y_i \in \{-1, 1\}$.

The log-likelihood is given by

$$l(\theta) = \sum_{i=1}^{n} \log(\frac{1}{1 + \exp(-y_i \theta^\top x_i)})$$

and the optimal θ minimizes the logistic loss function:

$$\min_{\theta} \sum_{i=1}^{n} \log(1 + \exp(-y_i \theta^\top x_i)).$$

Most of the time, instead of solving directly the previous optimization problem, we prefer to add a regularization term:

$$\min_{\theta} \sum_{i=1}^{n} \log(1 + \exp(-y_i \theta^\top x_i)) + \lambda \|\theta\|$$

with $\lambda \in \mathbb{R}_+$ a penalty and $\|.\|$ a norm function. By adding such a regularization term, we avoid overfitting on the training set and usually achieve a greater score in cross-validation.

Reformulation as a conic optimization problem

By introducing auxiliary variables $t_1, \cdots, t_n$ and r, the optimization problem is equivalent to

$$\begin{aligned} \min_{t,r,\theta} \; & \sum_{i=1}^{n} t_i + \lambda r \\ \text{subject to} \quad & t_i \geq \log(1 + \exp(-y_i \theta^\top x_i)) \\ & r \geq \|\theta\| \end{aligned}$$

Now, the trick is to reformulate the constraints $t_i \geq \log(1 + \exp(-y_i \theta^\top x_i))$ with the help of the exponential cone

$$K_{exp} = \{(x, y, z) \in \mathbb{R}^3 : \; y \exp(x/y) \leq z\}.$$

Indeed, by passing to the exponential, we see that for all $i = 1, \cdots, n$, the constraint $t_i \geq \log(1+\exp(-y_i \theta^\top x_i))$ is equivalent to

$$\exp(-t_i) + \exp(u_i - t_i) \leq 1$$

with $u_i = -y_i \theta^\top x_i$. Then, by adding two auxiliary variables z_{i1} and z_{i2} such that $z_{i1} \geq \exp(u_i - t_i)$ and $z_{i2} \geq \exp(-t_i)$, we get the equivalent formulation

$$\begin{cases} (u_i - t_i, 1, z_{i1}) \in K_{exp} \\ (-t_i, 1, z_{i2}) \in K_{exp} \\ z_{i1} + z_{i2} \leq 1 \end{cases}$$

In this setting, the conic version of the logistic regression problems writes out

$$\begin{aligned} \min_{t,z,r,\theta} \; & \sum_{i=1}^{n} t_i + \lambda r \\ \text{subject to} \quad & (u_i - t_i, 1, z_{i1}) \in K_{exp} \\ & (-t_i, 1, z_{i2}) \in K_{exp} \\ & z_{i1} + z_{i2} \leq 1 \\ & u_i = -y_i x_i^\top \theta \\ & r \geq \|\theta\| \end{aligned}$$

and thus encompasses $3n + p + 1$ variables and $3n + 1$ constraints ($u_i = -y_i \theta^\top x_i$ is only a virtual constraint used to clarify the notation). Thus, if $n \gg 1$, we get a large number of variables and constraints.

Fitting logistic regression with a conic solver

It is now time to pass to the implementation. We choose SCS as a conic solver.

```
using JuMP
import Random
import SCS
```

Info

This tutorial uses sets from MathOptInterface. By default, JuMP exports the `MOI` symbol as an alias for the MathOptInterface.jl package. We recommend making this more explicit in your code by adding the following lines:

```
import MathOptInterface
const MOI = MathOptInterface
```

```
Random.seed!(2713);
```

We start by implementing a function to generate a fake dataset, and where we could tune the correlation between the feature variables. The function is a direct transcription of the one used in this blog post.

```
function generate_dataset(n_samples = 100, n_features = 10; shift = 0.0)
    X = randn(n_samples, n_features)
    w = randn(n_features)
    y = sign.(X * w)
    X .+= 0.8 * randn(n_samples, n_features) # add noise
    X .+= shift # shift the points in the feature space
    X = hcat(X, ones(n_samples, 1))
    return X, y
end
```

```
generate_dataset (generic function with 3 methods)
```

We write a `softplus` function to formulate each constraint $t \geq \log(1 + \exp(u))$ with two exponential cones.

```
function softplus(model, t, u)
    z = @variable(model, [1:2], lower_bound = 0.0)
    @constraint(model, sum(z) <= 1.0)
    @constraint(model, [u - t, 1, z[1]] in MOI.ExponentialCone())
    @constraint(model, [-t, 1, z[2]] in MOI.ExponentialCone())
end
```

```
softplus (generic function with 1 method)
```

ℓ_2 regularized logistic regression

Then, with the help of the `softplus` function, we could write our optimization model. In the ℓ_2 regularization case, the constraint $r \geq \|\theta\|_2$ rewrites as a second order cone constraint.

```
function build_logit_model(X, y, λ)
    n, p = size(X)
    model = Model()
    @variable(model, θ[1:p])
    @variable(model, t[1:n])
    for i in 1:n
        u = -(X[i, :]' * θ) * y[i]
        softplus(model, t[i], u)
    end
    # Add 2 regularization
    @variable(model, 0.0 <= reg)
    @constraint(model, [reg; θ] in SecondOrderCone())
    # Define objective
    @objective(model, Min, sum(t) + λ * reg)
    return model
end
```

```
build_logit_model (generic function with 1 method)
```

We generate the dataset.

> **Warning**
>
> Be careful here, for large n and p SCS could fail to converge!

```
n, p = 200, 10
X, y = generate_dataset(n, p, shift = 10.0);

# We could now solve the logistic regression problem
λ = 10.0
model = build_logit_model(X, y, λ)
set_optimizer(model, SCS.Optimizer)
set_silent(model)
JuMP.optimize!(model)
```

```
θ♯ = JuMP.value.(model[:θ])
```

```
11-element Vector{Float64}:
  0.0015739335606945706
  0.6238309106055949
 -0.36068047422372984
  0.16711691158172787
  0.24900921079262353
 -0.4997291009406383
 -0.46482747131916013
  0.42189519037124157
 -0.1497536879547582
  0.02757377777527947
 -0.12513816667486083
```

It appears that the speed of convergence is not that impacted by the correlation of the dataset, nor by the penalty λ.

ℓ_1 regularized logistic regression

We now formulate the logistic problem with a ℓ_1 regularization term. The ℓ_1 regularization ensures sparsity in the optimal solution of the resulting optimization problem. Luckily, the ℓ_1 norm is implemented as a set in `MathOptInterface`. Thus, we could formulate the sparse logistic regression problem with the help of a `MOI.NormOneCone` set.

```
function build_sparse_logit_model(X, y, λ)
    n, p = size(X)
    model = Model()
    @variable(model, θ[1:p])
    @variable(model, t[1:n])
    for i in 1:n
        u = -(X[i, :]' * θ) * y[i]
        softplus(model, t[i], u)
    end
    # Add 1 regularization
    @variable(model, 0.0 <= reg)
    @constraint(model, [reg; θ] in MOI.NormOneCone(p + 1))
    # Define objective
    @objective(model, Min, sum(t) + λ * reg)
    return model
end

# Auxiliary function to count non-null components:
count_nonzero(v::Vector; tol = 1e-6) = sum(abs.(v) .>= tol)

# We solve the sparse logistic regression problem on the same dataset as before.
λ = 10.0
sparse_model = build_sparse_logit_model(X, y, λ)
set_optimizer(sparse_model, SCS.Optimizer)
set_silent(sparse_model)
JuMP.optimize!(sparse_model)
```

```
θ♯ = JuMP.value.(sparse_model[:θ])
println(
    "Number of non-zero components: ",
    count_nonzero(θ♯),
    " (out of ",
    p,
    " features)",
)
```

```
Number of non-zero components: 7 (out of 10 features)
```

Extensions

A direct extension would be to consider the sparse logistic regression with hard thresholding, which, on contrary to the soft version using a ℓ_1 regularization, adds an explicit cardinality constraint in its formulation:

$$\begin{aligned} \min_{\theta} \quad & \sum_{i=1}^{n} \log(1 + \exp(-y_i \theta^\top x_i)) + \lambda \|\theta\|_2^2 \\ \text{subject to} \quad & \|\theta\|_0 <= k \end{aligned}$$

where k is the maximum number of non-zero components in the vector θ, and $\|.\|_0$ is the ℓ_0 pseudo-norm:

$$\|x\|_0 = \#\{i : x_i \neq 0\}$$

The cardinality constraint $\|\theta\|_0 \leq k$ could be reformulated with binary variables. Thus the hard sparse regression problem could be solved by any solver supporting mixed integer conic problems.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.5 K-means clustering via SDP

From "Approximating K-means-type clustering via semidefinite programming" By Jiming Peng and Yu Wei.

Given a set of points $a_1, \ldots, a_m$ in R_n, allocate them to k clusters.

```
using JuMP
import LinearAlgebra
import SCS

function example_cluster(; verbose = true)
    # Data points
    n = 2
    m = 6
    a = Any[
        [2.0, 2.0],
        [2.5, 2.1],
        [7.0, 7.0],
        [2.2, 2.3],
        [6.8, 7.0],
        [7.2, 7.5],
    ]
    k = 2
    # Weight matrix
    W = zeros(m, m)
    for i in 1:m
        for j in i+1:m
            W[i, j] = W[j, i] = exp(-LinearAlgebra.norm(a[i] - a[j]) / 1.0)
        end
    end
    model = Model(SCS.Optimizer)
    set_silent(model)
    # Z >= 0, PSD
    @variable(model, Z[1:m, 1:m], PSD)
    @constraint(model, Z .>= 0)
    # min Tr(W(I-Z))
    I = Matrix(1.0 * LinearAlgebra.I, m, m)
    @objective(model, Min, LinearAlgebra.tr(W * (I - Z)))
    # Z e = e
    @constraint(model, Z * ones(m) .== ones(m))
    # Tr(Z) = k
```

```
    @constraint(model, LinearAlgebra.tr(Z) == k)
    optimize!(model)
    Z_val = value.(Z)
    # A simple rounding scheme
    which_cluster = zeros(Int, m)
    num_clusters = 0
    for i in 1:m
        if Z_val[i, i] <= 1e-3
            continue
        elseif which_cluster[i] == 0
            num_clusters += 1
            which_cluster[i] = num_clusters
            for j in i+1:m
                if LinearAlgebra.norm(Z_val[i, j] - Z_val[i, i]) <= 1e-3
                    which_cluster[j] = num_clusters
                end
            end
        end
    end
    if verbose
        # Print results
        for cluster in 1:k
            println("Cluster $cluster")
            for i in 1:m
                if which_cluster[i] == cluster
                    println(a[i])
                end
            end
        end
    end
    return
end

example_cluster()
```

```
Cluster 1
[2.0, 2.0]
[2.5, 2.1]
[2.2, 2.3]
Cluster 2
[7.0, 7.0]
[6.8, 7.0]
[7.2, 7.5]
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.6 The correlation problem

Given three random variables A, B, C and given bounds on two of the three correlation coefficients:

```
-0.2 <= ρ_AB <= -0.1
0.4 <= ρ_BC <=  0.5
```

We can use the following property of the correlations to determine bounds on ρ_AC by solving a SDP:

```
|  1    ρ_AB  ρ_AC |
| ρ_AB   1    ρ_BC |  ⩾ 0
| ρ_AC  ρ_BC   1   |
```

```
using JuMP
import SCS

function example_corr_sdp()
    model = Model(SCS.Optimizer)
    set_silent(model)
    @variable(model, X[1:3, 1:3], PSD)
    # Diagonal is 1s
    @constraint(model, X[1, 1] == 1)
    @constraint(model, X[2, 2] == 1)
    @constraint(model, X[3, 3] == 1)
    # Bounds on the known correlations
    @constraint(model, X[1, 2] >= -0.2)
    @constraint(model, X[1, 2] <= -0.1)
    @constraint(model, X[2, 3] >= 0.4)
    @constraint(model, X[2, 3] <= 0.5)
    # Find upper bound
    @objective(model, Max, X[1, 3])
    optimize!(model)
    println("An upper bound for X[1, 3] is $(value(X[1, 3]))")
    # Find lower bound
    @objective(model, Min, X[1, 3])
    optimize!(model)
    println("A lower bound for X[1, 3] is $(value(X[1, 3]))")
    return
end

example_corr_sdp()
```

```
An upper bound for X[1, 3] is 0.8719220303159311
A lower bound for X[1, 3] is -0.9779989594171764
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.7 Experiment design

Originally Contributed by: Arpit Bhatia, Chris Coey

This tutorial covers experiment design examples (D-optimal, A-optimal, and E-optimal) from section 7.5 of the book Convex Optimization by Boyd and Vandenberghe.

The tutorial uses the following packages

```
using JuMP
import SCS
import LinearAlgebra
import Random
```

Info

This tutorial uses sets from MathOptInterface. By default, JuMP exports the `MOI` symbol as an alias for the MathOptInterface.jl package. We recommend making this more explicit in your code by adding the following lines:

```
import MathOptInterface
const MOI = MathOptInterface
```

We set a seed so the random numbers are repeatable:

```
Random.seed!(1234)
```

```
MersenneTwister(1234)
```

The relaxed experiment design problem

The basic experiment design problem is as follows.

Given the menu of possible choices for experiments, $v_1, \ldots, v_p$, and the total number m of experiments to be carried out, choose the numbers of each type of experiment, i.e., $m_1, \ldots, m_p$ to make the error covariance E small (in some sense).

The variables $m_1, \ldots, m_p$ must, of course, be integers and sum to m the given total number of experiments. This leads to the optimization problem:

$$
\begin{aligned}
\min \left(\mathrm{w.r.t.} \mathbf{S}_+^n\right) E = \left(\sum_{j=1}^{p} m_j v_j v_j^T\right)^{-1} \\
\text{subject to} m_i \geq 0 \\
\sum_{i=1}^{p} m_i = m \\
m_i \in \mathbb{Z}, \quad i = 1, \ldots, p
\end{aligned}
$$

The basic experiment design problem can be a hard combinatorial problem when m, the total number of experiments, is comparable to n, since in this case the m_i are all small integers.

In the case when m is large compared to n, however, a good approximate solution can be found by ignoring, or relaxing, the constraint that the m_i are integers.

Let $\lambda_i = m_i/m$, which is the fraction of the total number of experiments for which $a_j = v_i$, or the relative frequency of experiment i. We can express the error covariance in terms of λ_i as:

$$
E = \frac{1}{m}\left(\sum_{i=1}^{p} \lambda_i v_i v_i^T\right)^{-1}
$$

The vector $\lambda \in \mathbf{R}^p$ satisfies $\lambda \succeq 0, \mathbf{1}^T \lambda = 1$, and also, each λ_i is an integer multiple of $1/m$. By ignoring this last constraint, we arrive at the problem:

$$
\begin{aligned}
\min \left(\text{w.r.t.} \mathbf{S}_+^n\right) & E = (1/m) \left(\sum_{i=1}^p \lambda_i v_i v_i^T\right)^{-1} \\
\text{subject to:} \quad & \lambda \succeq 0 \\
& \mathbf{1}^T \lambda = 1
\end{aligned}
$$

Several scalarizations have been proposed for the experiment design problem, which is a vector optimization problem over the positive semidefinite cone.

```
q = 4 # dimension of estimate space
p = 8 # number of experimental vectors
nmax = 3 # upper bound on lambda
n = 12

V = randn(q, p)

eye = Matrix{Float64}(LinearAlgebra.I, q, q);
```

A-optimal design

In A-optimal experiment design, we minimize tr E, the trace of the covariance matrix. This objective is simply the mean of the norm of the error squared:

$$
\mathbf{E}\|e\|_2^2 = \mathbf{E} \operatorname{tr}\left(ee^T\right) = \operatorname{tr} E
$$

The A-optimal experiment design problem in SDP form is

$$
\begin{aligned}
& \min \mathbf{1}^T u \\
\text{subject to} \quad & \begin{bmatrix} \sum_{i=1}^p \lambda_i v_i v_i^T e_k \\ e_k^T u_k \end{bmatrix} \succeq 0, \quad k = 1, \ldots, n \\
& \lambda \succeq 0 \\
& \mathbf{1}^T \lambda = 1
\end{aligned}
$$

```
aOpt = Model(SCS.Optimizer)
set_silent(aOpt)
@variable(aOpt, np[1:p], lower_bound = 0, upper_bound = nmax)
@variable(aOpt, u[1:q], lower_bound = 0)
@constraint(aOpt, sum(np) <= n)
for i in 1:q
    matrix = [
        V*LinearAlgebra.diagm(0 => np ./ n)*V' eye[:, i]
        eye[i, :]' u[i]
    ]
    @constraint(aOpt, matrix >= 0, PSDCone())
end
```

```
@objective(aOpt, Min, sum(u))
optimize!(aOpt)
objective_value(aOpt)
```

```
5.0412636484980755
```

```
value.(np)
```

```
8-element Vector{Float64}:
 1.747945718805369
 1.1153130074864266
 1.288808299639784e-7
 1.661952499684036
 2.9999994121581453
 0.8414258334554227
 1.3825595386035296
 2.25080346403116
```

E-optimal design

In E -optimal design, we minimize the norm of the error covariance matrix, i.e. the maximum eigenvalue of E.

Since the diameter (twice the longest semi-axis) of the confidence ellipsoid $\mathcal{E}$ is proportional to $\|E\|_2^{1/2}$, minimizing $\|E\|_2$ can be interpreted geometrically as minimizing the diameter of the confidence ellipsoid.

E-optimal design can also be interpreted as minimizing the maximum variance of $q^T e$, over all q with $\|q\|_2 = 1$. The E-optimal experiment design problem in SDP form is:

$$
\begin{aligned}
&\min t \\
\text{subject to } &\sum_{i=1}^{p} \lambda_i v_i v_i^T \succeq tI \\
&\lambda \succeq 0 \\
&\mathbf{1}^T \lambda = 1
\end{aligned}
$$

```
eOpt = Model(SCS.Optimizer)
set_silent(eOpt)
@variable(eOpt, 0 <= np[1:p] <= nmax)
@variable(eOpt, t)
@constraint(
    eOpt,
    V * LinearAlgebra.diagm(0 => np ./ n) * V' - (t .* eye) >= 0,
    PSDCone(),
)
@constraint(eOpt, sum(np) <= n)
@objective(eOpt, Max, t)
optimize!(eOpt)
objective_value(eOpt)
```

```
0.44896306556366
```

```
value.(np)
```

```
8-element Vector{Float64}:
  2.9999591973576862
  0.6746121070433458
 -4.077828668073517e-5
  1.0453736634429782
  2.999976477144745
  1.7873172744476913
  0.3017495901336785
  2.1911100940002943
```

D-optimal design

The most widely used scalarization is called D -optimal design, in which we minimize the determinant of the error covariance matrix E. This corresponds to designing the experiment to minimize the volume of the resulting confidence ellipsoid (for a fixed confidence level). Ignoring the constant factor $1/m$ in E, and taking the logarithm of the objective, we can pose this problem as convex optimization problem:

$$\min \log\det \left(\sum_{i=1}^{p} \lambda_i v_i v_i^T\right)^{-1}$$

subject to$\lambda \succeq 0$

$$\mathbf{1}^T\lambda = 1$$

```
dOpt = Model(SCS.Optimizer)
set_silent(dOpt)
@variable(dOpt, np[1:p], lower_bound = 0, upper_bound = nmax)
@variable(dOpt, t)
@objective(dOpt, Max, t)
@constraint(dOpt, sum(np) <= n)
E = V * LinearAlgebra.diagm(0 => np ./ n) * V'
@constraint(
    dOpt,
    [t, 1, (E[i, j] for i in 1:q for j in 1:i)...] in MOI.LogDetConeTriangle(q)
)
optimize!(dOpt)
objective_value(dOpt)
```

```
0.19012932811233788
```

```
value.(np)
```

```
8-element Vector{Float64}:
 8.947286889821718e-5
 2.5660639516185197
 0.00014367738919698677
 0.26257069021947865
 2.9421090046954608
 2.39212443758168
 2.8368058541508545
 0.9999018965107976
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.8 SDP relaxations: max-cut

Solves a semidefinite programming relaxation of the MAXCUT graph problem:

```
max   0.25 * •LX
s.t.  diag(X) == e
      X ≽ 0
```

Where L is the weighted graph Laplacian. Uses this relaxation to generate a solution to the original MAXCUT problem using the method from the paper:

Goemans, M. X., & Williamson, D. P. (1995). Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. Journal of the ACM (JACM), 42(6), 1115-1145.

```
using JuMP
import LinearAlgebra
import Random
import SCS
import Test

"""
    svd_cholesky(X::AbstractMatrix, rtol)

Return the matrix `U` of the Cholesky decomposition of `X` as `U' * U`.
Note that we do not use the `LinearAlgebra.cholesky` function as it it
requires the matrix to be positive definite while `X` may be only
positive *semi*definite.
We use the convention `U' * U` instead of `U * U'` to be consistent with
`LinearAlgebra.cholesky`.
"""
function svd_cholesky(X::AbstractMatrix)
    F = LinearAlgebra.svd(X)
    # We now have `X ≈ `F.U * D² * F.U'` where:
    D = LinearAlgebra.Diagonal(sqrt.(F.S))
    # So `X ≈ U' * U` where `U` is:
    return (F.U * D)'
end

function solve_max_cut_sdp(num_vertex, weights)
    # Calculate the (weighted) Lapacian of the graph: L = D - W.
    laplacian = LinearAlgebra.diagm(0 => weights * ones(num_vertex)) - weights
    # Solve the SDP relaxation
    model = Model(SCS.Optimizer)
    set_silent(model)
    # Start with X as the identity matrix to avoid numerical issues.
    @variable(
        model,
        X[i = 1:num_vertex, j = 1:num_vertex],
        PSD,
        start = (i == j ? 1.0 : 0.0),
```

```
    )
    @objective(model, Max, 1 / 4 * LinearAlgebra.dot(laplacian, X))
    @constraint(model, LinearAlgebra.diag(X) .== 1)
    optimize!(model)
    @assert termination_status(model) == MOI.OPTIMAL
    opt_X = value(X)
    V = svd_cholesky(opt_X)
    # Generate random vector on unit sphere.
    Random.seed!(num_vertex)
    r = rand(size(V, 1))
    r /= LinearAlgebra.norm(r)
    # Iterate over vertices, and assign each vertex to a side of cut.
    cut = ones(num_vertex)
    for i in 1:num_vertex
        if LinearAlgebra.dot(r, V[:, i]) <= 0
            cut[i] = -1
        end
    end
    println("Solution:")
    print(" (S, S′) = ({")
    print(join(findall(cut .== -1), ", "))
    print("}, {")
    print(join(findall(cut .== 1), ", "))
    println("})")
    #  (S, S′)  = ({1}, {2, 3, 4})
    return cut, 0.25 * sum(laplacian .* (cut * cut'))
end

function example_max_cut_sdp()
    println()
    println("Example 1:")
    #   [1] --- 5 --- [2]
    #
    # Solution:
    #  (S, S′)  = ({1}, {2})
    cut, cutval = solve_max_cut_sdp(2, [0.0 5.0; 5.0 0.0])
    Test.@test cut[1] != cut[2]

    println()
    println("Example 2:")
    #   [1] --- 5 --- [2]
    #    |  \          |
    #    |    \        |
    #    7      6      1
    #    |        \    |
    #    |          \  |
    #   [3] --- 1 --- [4]
    #
    # Solution:
    #  (S, S′)  = ({1}, {2, 3, 4})
    W = [
        0.0 5.0 7.0 6.0
        5.0 0.0 0.0 1.0
        7.0 0.0 0.0 1.0
        6.0 1.0 1.0 0.0
    ]
```

```
    cut, cutval = solve_max_cut_sdp(4, W)
    Test.@test cut[1] != cut[2]
    Test.@test cut[2] == cut[3] == cut[4]

    println()
    println("Example 3:")
    #   [1] --- 1 --- [2]
    #    |             |
    #    |             |
    #    5             9
    #    |             |
    #    |             |
    #   [3] --- 2 --- [4]
    #
    # Solution:
    #  (S, S′)  = ({1, 4}, {2, 3})
    W = [
        0.0 1.0 5.0 0.0
        1.0 0.0 0.0 9.0
        5.0 0.0 0.0 2.0
        0.0 9.0 2.0 0.0
    ]
    cut, cutval = solve_max_cut_sdp(4, W)
    Test.@test cut[1] == cut[4]
    Test.@test cut[2] == cut[3]
    Test.@test cut[1] != cut[2]
    return
end

example_max_cut_sdp()
```

```

Example 1:
Solution:
 (S, ′S) = ({1}, {2})

Example 2:
Solution:
 (S, ′S) = ({2, 3, 4}, {1})

Example 3:
Solution:
 (S, ′S) = ({2, 3}, {1, 4})
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.9 The minimum distortion problem

This example arises from computational geometry, in particular the problem of embedding a general finite metric space into a euclidean space.

It is known that the 4-point metric space defined by the star graph:

```
x
 \\
  x - x
 /
x
```

where distances are computed by length of the shortest path between vertices, cannot be exactly embedded into a euclidean space of any dimension.

Here we will formulate and solve an SDP to compute the best possible embedding, that is, the embedding f() that minimizes the distortion c such that

```
(1 / c) * D(a, b) ≤ ||f(a) - f(b)|| ≤ D(a, b)
```

for all points (a, b), where D(a, b) is the distance in the metric space.

Any embedding can be characterized by its Gram matrix Q, which is PSD, and

```
||f(a) - f(b)||^2 = Q[a, a] + Q[b, b] - 2 * Q[a, b]
```

We can therefore constrain

```
D[i, j]^2 ≤ Q[i, i] + Q[j, j] - 2 * Q[i, j] ≤ c^2 * D[i, j]^2
```

and minimize c^2, which gives us the SDP formulation below.

For more detail, see "Lectures on discrete geometry" by J. Matoušek, Springer, 2002, pp. 378-379.

```
using JuMP
import SCS
import Test

function example_min_distortion()
    model = Model(SCS.Optimizer)
    set_silent(model)
    D = [
        0.0 1.0 1.0 1.0
        1.0 0.0 2.0 2.0
        1.0 2.0 0.0 2.0
        1.0 2.0 2.0 0.0
    ]
    @variable(model, c² >= 1.0)
    @variable(model, Q[1:4, 1:4], PSD)
    for i in 1:4
        for j in (i+1):4
            @constraint(model, D[i, j]^2 <= Q[i, i] + Q[j, j] - 2 * Q[i, j])
            @constraint(
                model,
                Q[i, i] + Q[j, j] - 2 * Q[i, j] <= c² * D[i, j]^2
            )
        end
    end
    @objective(model, Min, c²)
    optimize!(model)
    Test.@test termination_status(model) == OPTIMAL
    Test.@test primal_status(model) == FEASIBLE_POINT
    Test.@test objective_value(model) ≈ 4 / 3 atol = 1e-4
    return
```

```
end

example_min_distortion()
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.10 Minimal ellipses

This example comes from section 8.4.1 of the book Convex Optimization by Boyd and Vandenberghe (2004).

Given a set of m ellipses of the form

$$E(A, b, c) = \{x : x'Ax + 2b'x + c \leq 0\},$$

we find the ellipse of smallest area that encloses the given ellipses.

It is convenient to parameterize the minimal enclosing ellipse as

$$\{x : ||Px + q|| \leq 1\}.$$

Then the optimal P and q are given by the convex semidefinite program

$$\begin{array}{lll} \text{maximize} & \log(\det(P)) & \\ \text{subject to} & \tau_i \geq 0, & i = 1, \ldots, m, \\ & \begin{bmatrix} P^2 - \tau_i A_i & Pq - \tau_i b_i & 0 \\ (Pq - \tau_i b_i)^T & -1 - \tau_i c_i & (Pq)^T \\ 0 & (Pq) & -P^2 \end{bmatrix} \preceq 0 \text{ (PSD)} & i = 1, \ldots, m \end{array}$$

with helper variables τ.

The program can be solved by using a variable representing P^2 (`Psqr` in the Julia code), a vector of variables $\tilde{q}$ (`q_tilde`) in place of Pq and the variables τ (`tau[i]`).

This tutorial uses the following packages:

```
using JuMP
using SCS
using Plots
using Test
```

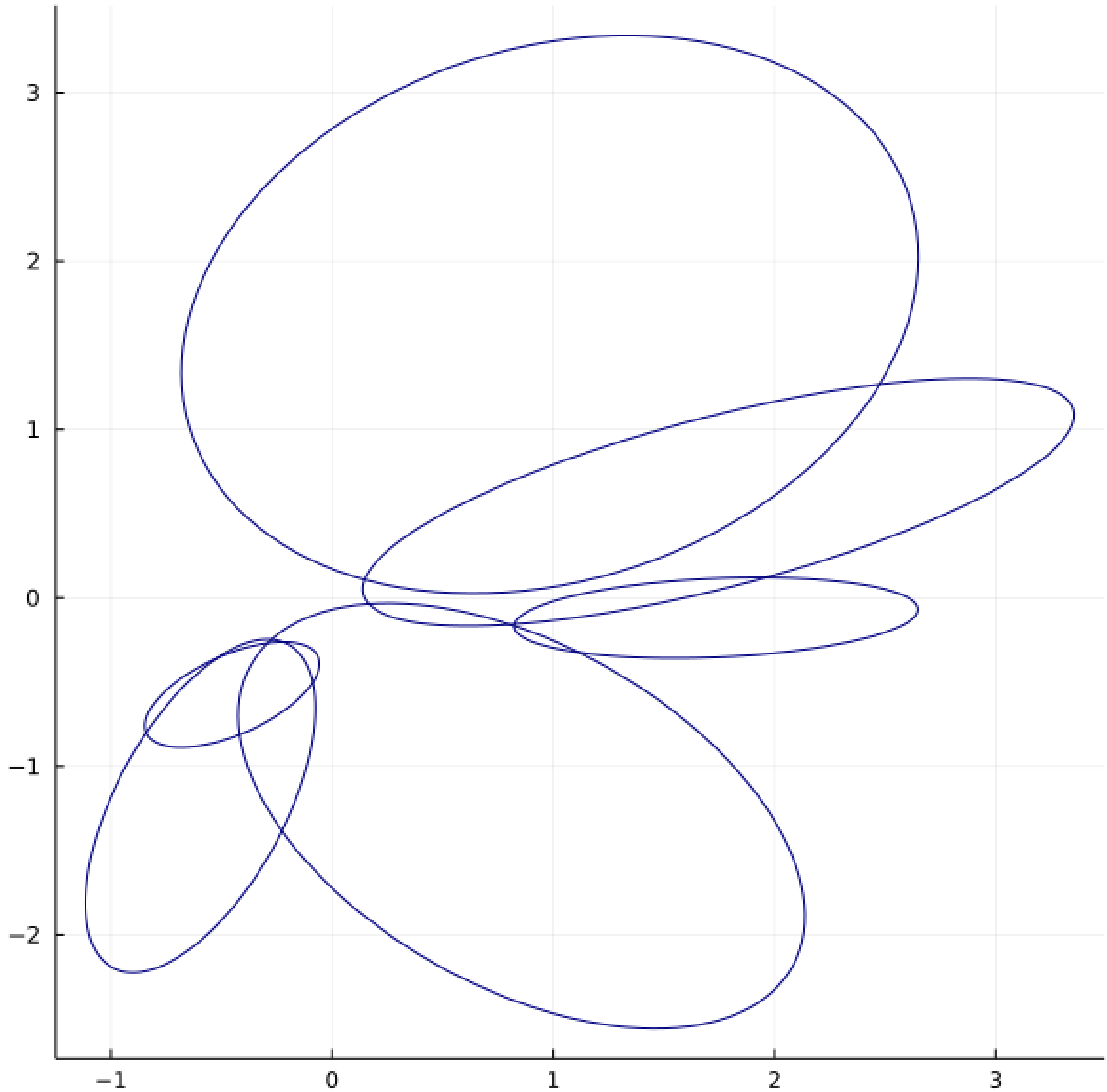
Set-up

First, define the m input ellipses (here $m = 6$), parameterized as $x^T A_i x + 2b_i^T x + c \leq 0$:

```
As = [
    [1.2576 -0.3873; -0.3873 0.3467],
    [1.4125 -2.1777; -2.1777 6.7775],
    [1.7018 0.8141; 0.8141 1.7538],
    [0.9742 -0.7202; -0.7202 1.5444],
    [0.6798 -0.1424; -0.1424 0.6871],
    [0.1796 -0.1423; -0.1423 2.6181],
];
bs = [
    [0.2722, 0.1969],
    [-1.228, -0.0521],
    [-0.4049, 1.5713],
    [0.0265, 0.5623],
    [-0.4301, -1.0157],
    [-0.3286, 0.557],
];
cs = [0.1831, 0.3295, 0.2077, 0.2362, 0.3284, 0.4931];
```

We visualise the ellipses using the Plots package:

```
pl = plot(size = (600, 600))
thetas = range(0, 2pi + 0.05, step = 0.05)
for (A, b, c) in zip(As, bs, cs)
    sqrtA = sqrt(A)
    b_tilde = sqrtA \ b
    alpha = b' * (A \ b) - c
    rhs = hcat(
        sqrt(alpha) * cos.(thetas) .- b_tilde[1],
        sqrt(alpha) * sin.(thetas) .- b_tilde[2],
    )
    ellipse = sqrtA \ rhs'
    plot!(pl, ellipse[1, :], ellipse[2, :], label = nothing, c = :navy)
end
plot(pl)
```



Build the model

Now let's build the initial model, using the change-of-variables Psqr = P^2 and q_tilde = Pq:

```
model = Model(SCS.Optimizer)
m = length(As)
n, _ = size(first(As))
@variable(model, tau[1:m] ≥ 0)
@variable(model, Psqr[1:n, 1:n], PSD)
@variable(model, q_tilde[1:n])
@variable(model, logdetP);
```

Next, create the PSD constraints and objective:

```
for (A, b, c, t) in zip(As, bs, cs, tau)
    if !(isreal(A) && transpose(A) == A)
        @error "Input matrices need to be real, symmetric matrices."
    end
    @constraint(
        model,
        -[
            #! format: off
            Psqr-t*A           q_tilde-t*b zeros(n, n)
            (q_tilde - t * b)' -1-t*c      q_tilde'
            zeros(n, n)        q_tilde     -Psqr
            #! format: on
        ] in PSDCone()
    )
end

@constraint(
    model,
    [logdetP; [Psqr[i, j] for i in 1:n for j in i:n]] in MOI.RootDetConeTriangle(n)
);
@objective(model, Max, logdetP);
```

Note that here the root-determinant cone is used for constructing the objective function. While the more consistent choice for the mathematical formulation is to use `MOI.LogDetConeTriangle(n)` instead, `MOI.RootDetConeTriangle(n)` will produce equivalent optimal solutions and is found to be more efficient for the SCS solver for this example.

Now, solve the program:

```
optimize!(model)
@test termination_status(model) == OPTIMAL;
@test primal_status(model) == FEASIBLE_POINT;
```

Results

After solving the model to optimality we can recover the original solution parameterization as

```
P = sqrt(value.(Psqr))
q = P \ value.(q_tilde)
```

```
2-element Vector{Float64}:
 -0.39617763941130074
 -0.021368863910555466
```

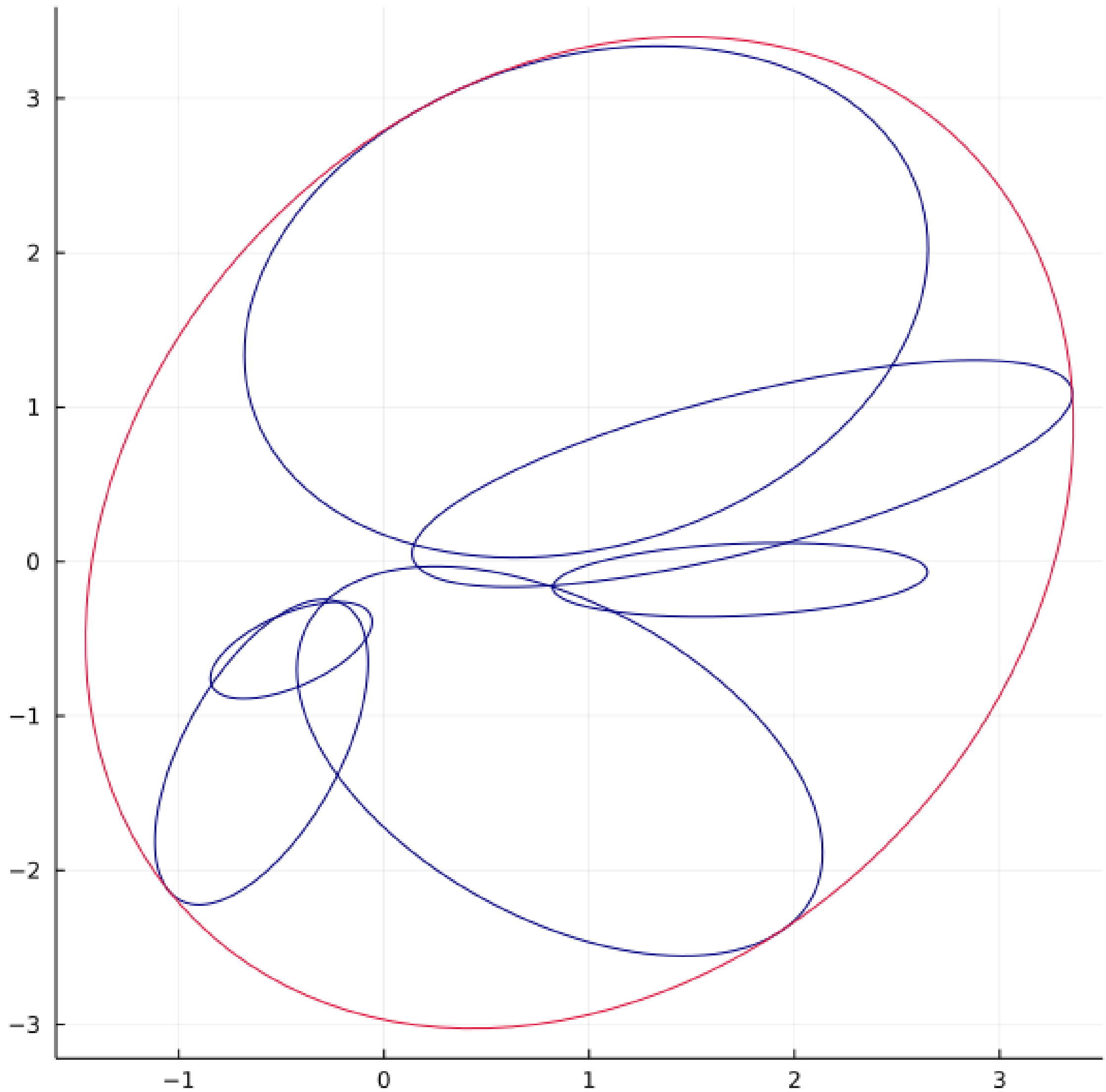
We can test that we get the expected results to within approximation tolerance.

```
@test isapprox(P, [0.4237 -0.0396; -0.0396 0.3163], atol = 1e-2);
@test isapprox(q, [-0.3960, -0.0214], atol = 1e-2);
```

Finally, overlaying the solution in the plot we see the minimal area enclosing ellipsoid.

```
plot!(
    pl,
    [tuple(P \ ([cos(theta), sin(theta)] - q)...) for theta in thetas],
```

```
    c = :crimson,
    label = nothing,
)
plot(pl)
```



Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

7.11 Robust uncertainty sets

Computes the Value at Risk for a data-driven uncertainty set; see "Data-Driven Robust Optimization" (Bertsimas 2013), section 6.1 for details. Closed-form expressions for the optimal value are available.

```
using JuMP
import SCS
import LinearAlgebra
import Test

function example_robust_uncertainty()
    R = 1
    d = 3
     = 0.05
    ε = 0.05
    N = ceil((2 + 2 * log(2 / ))^2) + 1
    c = randn(d)
    μhat = rand(d)
    M = rand(d, d)
    Σhat = 1 / (d - 1) * (M - ones(d) * μhat')' * (M - ones(d) * μhat')
    Γ1(, N) = R / sqrt(N) * (2 + sqrt(2 * log(1 / )))
    Γ2(, N) = 2 * R^2 / sqrt(N) * (2 + sqrt(2 * log(2 / )))
    model = Model(SCS.Optimizer)
    set_silent(model)
    @variable(model, Σ[1:d, 1:d], PSD)
    @variable(model, u[1:d])
    @variable(model, μ[1:d])
    @constraint(model, [Γ1( / 2, N); μ - μhat] in SecondOrderCone())
    @constraint(model, [Γ2( / 2, N); vec(Σ - Σhat)] in SecondOrderCone())
    @constraint(model, [((1-ε)/ε) (u - μ)'; (u-μ) Σ] in PSDCone())
    @objective(model, Max, LinearAlgebra.dot(c, u))
    optimize!(model)
    I = Matrix(1.0 * LinearAlgebra.I, d, d)
    exact =
        LinearAlgebra.dot(μhat, c) +
        Γ1( / 2, N) * LinearAlgebra.norm(c) +
        sqrt((1 - ε) / ε) *
        sqrt(LinearAlgebra.dot(c, (Σhat + Γ2( / 2, N) * I) * c))
    Test.@test objective_value(model) ≈ exact atol = 1e-2
    return
end

example_robust_uncertainty()
```

Tip

This tutorial was generated using Literate.jl. View the source .jl file on GitHub.

Chapter 8

Algorithms

8.1 Benders decomposition

Originally Contributed by: Shuvomoy Das Gupta

This tutorial describes how to implement Benders decomposition in JuMP. It uses the following packages:

```
using JuMP
import GLPK
import Printf
```

Theory

Benders decomposition is a useful algorithm for solving convex optimization problems with a large number of variables. It works best when a larger problem can be decomposed into two (or more) smaller problems that are invidually much easier to solve. This tutorial demonstrates Benders decomposition on the following mixed-integer linear program:

$$\begin{aligned} \text{min } & c_1^\top x + c_2^\top y \\ \text{subject to } & A_1 x + A_2 y \le b \\ & x \ge 0 \\ & y \ge 0 \\ & x \in \mathbb{Z}^n \end{aligned}$$

where $b \in \mathbb{R}^m$, $A_1 \in \mathbb{R}^{m\times n}$, $A_2 \in \mathbb{R}^{m\times p}$ and $\mathbb{Z}$ is the set of integers.

Any mixed integer programming problem can be written in the form above.

If there are relatively few integer variables, and many more continuous variables, then it may be beneficial to decompose the problem into a small problem containing only integer variables and a linear program containing only continuous variables. Hopefully, the linear program will be much easier to solve in isolation than in the full mixed-integer linear program.

For example, if we knew a feasible solution for x, we could obtain a solution for y by solving:

$$\begin{aligned} V_2(x) = & \text{min} & c_2^\top y & \\ & \text{subject to} & A_2 y \le b - A_1 x & \quad [\pi] \\ & & y \ge 0, & \end{aligned}$$

where π is the dual variable associated with the constraints. Because this is a linear program, it is easy to solve.

Replacing the $c_2^\top y$ component of the objective in our original problem with V_2 yields:

$$\begin{aligned}\text{min } & c_1^\top x + V_2(x)\\ \text{subject to } & x \geq 0\\ & x \in \mathbb{Z}^n\end{aligned}$$

This problem looks a lot simpler to solve, but we need to do something else with V_2 first.

Because x is a constant that appears on the right-hand side of the constraints, V_2 is a convex function with respect to x, and the dual variable π can be multiplied by $-A_1$ to obtain a subgradient of $V_2(x)$ with respect to x. Therefore, if we have a candidate solution x_k, then we can solve $V_2(x_k)$ and obtain a feasible dual vector π_k. Using these values, we can construct a first-order Taylor-series approximation of V_2 about the point x_k:

$$V_2(x) \geq V_2(x_k) + -\pi_k^\top A_1(x - x_k).$$

By convexity, we know that this inequality holds for all x, and we call these inequalities cuts.

Benders decomposition is an iterative technique that replaces $V_2(x)$ with a new decision variable θ, and approximates it from below using cuts:

$$\begin{aligned}V_1^K = \text{min} \quad & & c_1^\top x + \theta & \\ \text{subject to} \quad & & x \geq 0 & \\ & & x \in \mathbb{Z}^n & \\ & & \theta \geq M & \\ & & \theta \geq V_2(x_k) + \pi_k^\top (x - x_k) & \quad \forall k = 1, \ldots, K.\end{aligned}$$

This integer program is called the first-stage subproblem.

To generate cuts, we solve V_1^K to obtain a candidate first-stage solution x_k, then we use that solution to solve $V_2(x_k)$. Then, using the optimal objective value and dual solution from V_2, we add a new cut to form V_1^{K+1} and repeat.

Bounds

Due to convexity, we know that $V_2(x) \geq \theta$ for all x. Therefore, the optimal objective value of V_1^K provides a valid lower bound on the objective value of the full problem. In addition, if we take a feasible solution for x from the first-stage problem, then $c_1^\top x + V_2(x)$ is a valid upper bound on the objective value of the full problem.

Benders decomposition uses the lower and upper bounds to determine when it has found the global optimal solution.

Input data

As an example for this tutorial, we use the input data is from page 139 of Garfinkel, R. & Nemhauser, G. L. Integer programming. (Wiley, 1972).

```
c_1 = [1, 4]
c_2 = [2, 3]
dim_x = length(c_1)
dim_y = length(c_2)
b = [-2; -3]
A_1 = [1 -3; -1 -3]
A_2 = [1 -2; -1 -1]
M = -1000;
```

Iterative method

Warning

This is a basic implementation for pedagogical purposes. We haven't discussed Benders feasibility cuts, or any of the computational tricks that are required to build a performative implementation for large-scale problems.

We start by formulating the first-stage subproblem:

```
model = Model(GLPK.Optimizer)
@variable(model, x[1:dim_x] >= 0, Int)
@variable(model, θ >= M)
@objective(model, Min, c_1' * x + θ)
print(model)
```

```
Min x[1] + 4 x[2] + θ
Subject to
 x[1] ≥ 0.0
 x[2] ≥ 0.0
 θ ≥ -1000.0
 x[1] integer
 x[2] integer
```

FOr the next step, we need a function that takes a first-stage candidate solution x and returns the optimal solution from the second-stage subproblem:

```
function solve_subproblem(x)
    model = Model(GLPK.Optimizer)
    @variable(model, y[1:dim_y] >= 0)
    con = @constraint(model, A_2 * y .<= b - A_1 * x)
    @objective(model, Min, c_2' * y)
    optimize!(model)
    @assert termination_status(model) == OPTIMAL
    return (obj = objective_value(model), y = value.(y), π = dual.(con))
end
```

```
solve_subproblem (generic function with 1 method)
```

Note that `solve_subproblem` returns a `NamedTuple` of the objective value, the optimal primal solution for y, and the optimal dual solution for π.

We're almost ready for our optimization loop, but first, here's a helpful function for logging:

```
function print_iteration(k, args...)
    f(x) = Printf.@sprintf("%12.4e", x)
    println(lpad(k, 9), " ", join(f.(args), " "))
    return
end
```

```
print_iteration (generic function with 1 method)
```

We also need to put a limit on the number of iterations before termination:

```
MAXIMUM_ITERATIONS = 100
```

```
100
```

And a way to check if the lower and upper bounds are close-enough to terminate:

```
ABSOLUTE_OPTIMALITY_GAP = 1e-6
```

```
1.0e-6
```

Now we're ready to iterate Benders decomposition:

```
println("Iteration  Lower Bound  Upper Bound          Gap")
for k in 1:MAXIMUM_ITERATIONS
    optimize!(model)
    lower_bound = objective_value(model)
    x_k = value.(x)
    ret = solve_subproblem(x_k)
    upper_bound = c_1' * x_k + ret.obj
    gap = (upper_bound - lower_bound) / upper_bound
    print_iteration(k, lower_bound, upper_bound, gap)
    if gap < ABSOLUTE_OPTIMALITY_GAP
        println("Terminating with the optimal solution")
        break
    end
    cut = @constraint(model, θ >= ret.obj + -ret.π' * A_1 * (x .- x_k))
    @info "Adding the cut $(cut)"
end
```

```
Iteration  Lower Bound  Upper Bound          Gap
        1  -1.0000e+03   7.6667e+00   1.3143e+02
[ Info: Adding the cut 2 x[1] + 8 x[2] + θ ≥ 7.666666666666666
        2  -4.9600e+02   1.2630e+03   1.3927e+00
[ Info: Adding the cut -1.5 x[1] + 4.5 x[2] + θ ≥ 3.0
        3  -1.0800e+02   8.8800e+02   1.1216e+00
[ Info: Adding the cut θ ≥ 0.0
        4   4.0000e+00   4.0000e+00   0.0000e+00
Terminating with the optimal solution
```

Finally, we can obtain the optimal solution

```
optimize!(model)
x_optimal = value.(x)
```

```
2-element Vector{Float64}:
 0.0
 1.0
```

```
optimal_ret = solve_subproblem(x_optimal)
y_optimal = optimal_ret.y
```

```
2-element Vector{Float64}:
 0.0
 0.0
```

Callback method

The Iterative method section implemented Benders decomposition using a loop. In each iteration, we re-solved the first-stage subproblem to generate a candidate solution. However, modern MILP solvers such as CPLEX, Gurobi, and GLPK provide lazy constraint callbacks which allow us to add new cuts while the solver is running. This can be more efficient than an iterative method because we can avoid repeating work such as solving the root node of the first-stage MILP at each iteration.

> **Tip**
>
> For more information on callbacks, read the page Solver-independent callbacks.

As before, we construct the same first-stage subproblem:

```
lazy_model = Model(GLPK.Optimizer)
@variable(lazy_model, x[1:dim_x] >= 0, Int)
@variable(lazy_model, θ >= M)
@objective(lazy_model, Min, θ)
print(lazy_model)
```

```
Min θ
Subject to
 x[1] ≥ 0.0
 x[2] ≥ 0.0
 θ ≥ -1000.0
 x[1] integer
 x[2] integer
```

What differs is that we write a callback function instead of a loop:

```
k = 0

"""
    my_callback(cb_data)

A callback that implements Benders decomposition. Note how similar it is to the
inner loop of the iterative method.
"""
function my_callback(cb_data)
    global k += 1
    x_k = callback_value.(cb_data, x)
```

```
    θ_k = callback_value(cb_data, θ)
    lower_bound = c_1' * x_k + θ_k
    ret = solve_subproblem(x_k)
    upper_bound = c_1' * x_k + c_2' * ret.y
    gap = (upper_bound - lower_bound) / upper_bound
    print_iteration(k, lower_bound, upper_bound, gap)
    if gap < ABSOLUTE_OPTIMALITY_GAP
        println("Terminating with the optimal solution")
        return
    end
    cut = @build_constraint(θ >= ret.obj + -ret.π' * A_1 * (x .- x_k))
    MOI.submit(model, MOI.LazyConstraint(cb_data), cut)
    return
end

MOI.set(lazy_model, MOI.LazyConstraintCallback(), my_callback)
```

Now when we optimize!, our callback is run:

```
optimize!(lazy_model)
```

```
        1  -1.0000e+03   7.6667e+00   1.3143e+02
        2  -4.9617e+02   5.0383e+02   1.9848e+00
        3   3.8333e+00   4.0833e+00   6.1224e-02
        4   4.0000e+00   4.0000e+00   0.0000e+00
Terminating with the optimal solution
```

Note how this problem also takes 4 iterations to converge, but the sequence of bounds is different compared to the iterative method.

Finally, we can obtain the optimal solution:

```
x_optimal = value.(x)
```

```
2-element Vector{Float64}:
 0.0
 1.0
```

```
optimal_ret = solve_subproblem(x_optimal)
y_optimal = optimal_ret.y
```

```
2-element Vector{Float64}:
 0.0
 0.0
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

8.2 Column generation

This tutorial describes how to implement the Cutting stock problem in JuMP using column generation. It uses the following packages:

```
using JuMP
import GLPK
import SparseArrays
```

Mathematical formulation

The cutting stock problem is about cutting large rolls of paper into smaller pieces. There is a demand different sizes of pieces to meet, and all large rolls have the same width. The goal is to meet the demand while maximizing the total profit.

We denote the set of possible sized pieces that a roll can be cut into by $i \in 1, \ldots, I$. Each piece i has a width, w_i, and a demand, d_i. The width of the large roll is W.

Here's the data that we are going to use in this tutorial:

```
struct Piece
    w::Float64
    d::Int
end

struct Data
    pieces::Vector{Piece}
    W::Float64
end

function Base.show(io::IO, d::Data)
    println(io, "Data for the cutting stock problem:")
    println(io, "  W = $(d.W)")
    println(io, "with pieces:")
    println(io, "   i   w_i d_i")
    println(io, "  ------------")
    for (i, p) in enumerate(d.pieces)
        println(io, lpad(i, 4), " ", lpad(p.w, 5), " ", lpad(p.d, 3))
    end
    return
end

function get_data()
    data = [
        75.0 38
        75.0 44
        75.0 30
        75.0 41
        75.0 36
        53.8 33
        53.0 36
        51.0 41
        50.2 35
        32.2 37
        30.8 44
        29.8 49
```

```
        20.1 37
        16.2 36
        14.5 42
        11.0 33
        8.6 47
        8.2 35
        6.6 49
        5.1 42
    ]
    return Data([Piece(data[i, 1], data[i, 2]) for i in 1:size(data, 1)], 100.0)
end

data = get_data()
```

```
Data for the cutting stock problem:
  W = 100.0
with pieces:
   i   w_i d_i
  ------------
   1  75.0  38
   2  75.0  44
   3  75.0  30
   4  75.0  41
   5  75.0  36
   6  53.8  33
   7  53.0  36
   8  51.0  41
   9  50.2  35
  10  32.2  37
  11  30.8  44
  12  29.8  49
  13  20.1  37
  14  16.2  36
  15  14.5  42
  16  11.0  33
  17   8.6  47
  18   8.2  35
  19   6.6  49
  20   5.1  42
```

To formulate the cutting stock problem as a mixed-integer linear program, we assume that there is a set of large rolls $j = 1, \ldots, J$ to use. Then, we introduce two classes of decision variables:

- $x_{ij} \geq 0, \text{integer}, \forall i = 1, \ldots, I, j = 1, \ldots, J$
- $y_j \in \{0, 1\} \forall j = 1, \ldots, J.$

y_j is a binary variable that indicates if we use roll j, and x_{ij} counts how many pieces of size i that we cut from roll j.

Our mixed-integer linear program is therefore:

$$\min \sum_{j=1}^{J} y_j \tag{8.1}$$

$$\text{s.t.} \sum_{i=1}^{N} w_i x_{ij} \leq W y_j \qquad \forall j = 1, \ldots, J \tag{8.2}$$

$$\sum_{j=1}^{J} x_{ij} \geq d_i \qquad \forall i = 1, \ldots, I \tag{8.3}$$

$$x_{ij} \geq 0 \qquad \forall i = 1, \ldots, N, j = 1, \ldots, J \tag{8.4}$$

$$x_{ij} \in \mathbb{Z} \qquad \forall i = 1, \ldots, I, j = 1, \ldots, J \tag{8.5}$$

$$y_j \in \{0, 1\} \qquad \forall j = 1, \ldots, J \tag{8.6}$$

$$\tag{8.7}$$

The objective is to minimze the number of rolls that we use, and the two constraints ensure that we respect the total width of each large roll and that we satisfy demand exactly.

```
I = length(data.pieces)
J = 1000  # Some large number
model = Model(GLPK.Optimizer)
@variable(model, x[1:I, 1:J] >= 0, Int)
@variable(model, y[1:J], Bin)
@constraint(
    model,
    [j in 1:J],
    sum(data.pieces[i].w * x[i, j] for i in 1:I) <= data.W * y[j],
)
@constraint(model, [i in 1:I], sum(x[i, j] for j in 1:J) >= data.pieces[i].d)
@objective(model, Min, sum(y[j] for j in 1:J))
model
```

```
A JuMP Model
Minimization problem with:
Variables: 21000
Objective function type: AffExpr
`AffExpr`-in-`MathOptInterface.GreaterThan{Float64}`: 20 constraints
`AffExpr`-in-`MathOptInterface.LessThan{Float64}`: 1000 constraints
`VariableRef`-in-`MathOptInterface.GreaterThan{Float64}`: 20000 constraints
`VariableRef`-in-`MathOptInterface.Integer`: 20000 constraints
`VariableRef`-in-`MathOptInterface.ZeroOne`: 1000 constraints
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: GLPK
Names registered in the model: x, y
```

Unfortunately, we won't attempt to solve this formulation because it takes a very long time to solve. (Try it and see.)

```
# optimize!(model)
```

However, there is a formulation that solves much faster, and that is to use a column generation scheme.

Column generation theory

The key insight for column generation is to recognize that the x variables above encode cutting patterns. For example, if we look only at the roll $j = 1$, then feasible solutions are:

- $x_{1,1} = 1$, $x_{13,1} = 1$ and all the rest 0, which is 1 roll of piece #1 and 1 roll of piece #13
- $x_{1,20} = 19$ and all the rest 0, which is 19 rolls of piece #20.

Cutting patterns like $x_{1,1} = 1$ and $x_{2,1} = 1$ are infeasible because the combined length is greater than W.

Since there are a finite number of ways that we could cut a roll into a valid cutting pattern, we can create a set of all possible cutting patterns $p = 1, \ldots, P$, with data $a_{i,p}$ indicating how many pieces of size i we cut in pattern p. Then, we can formulate our mixed-integer linear program as:

$$\min \sum_{p=1}^{P} x_p \tag{8.8}$$

$$\text{s.t.} \sum_{p=1}^{P} a_{ip} x_p \geq d_i \qquad \forall i = 1, \ldots, I \tag{8.9}$$

$$x_p \geq 0 \qquad \forall p = 1, \ldots, P \tag{8.10}$$

$$x_p \in \mathbb{Z} \qquad \forall p = 1, \ldots, P \tag{8.11}$$

Unfortunately, there will be a very large number of these patterns, so it is often intractable to enumerate all columns $p = 1, \ldots, P$.

Column generation is an iterative algorithm that starts with a small set of initial patterns, and then cleverly chooses new columns to add to the main MILP so that we find the optimal solution without having to enumerate every column.

Choosing new columns

For now we assume that we have our mixed-integer linear program with a subset of the columns. If we have all of the columns that appear in an optimal solution then we are done. Otherwise, how do we choose a new column that leads to an improved solution?

Column generation chooses a new column by relaxing the integrality constraint on x and looking at the dual variable π_i associated with demand constraint i.

Using the economic interpretation of the dual variable, we can say that a one unit increase in demand for piece i will cost an extra π_i rolls. Alternatively, we can say that a one unit increase in the left-hand side (for example, due to a new cutting pattern) will save us π_i rolls. Therefore, we want a new column that maximizes the savings associated with the dual variables, while respecting the total width of the roll:

$$\max \sum_{i=1}^{I} \pi_i y_i \tag{8.12}$$

$$\text{s.t.} \sum_{i=1}^{I} w_i y_i \leq W \tag{8.13}$$

$$y_i \geq 0 \qquad \forall i = 1, \ldots, I \tag{8.14}$$

$$y_i \in \mathbb{Z} \qquad \forall i = 1, \ldots, I \tag{8.15}$$

$$\tag{8.16}$$

If this problem, called the pricing problem, has an objective value greater than 1, then we estimate than adding y as the coefficients of a new column will decrease the objective by more than the cost of an extra roll.

Here is code to solve the pricing problem:

```
function solve_pricing(data::Data, π::Vector{Float64})
    I = length(π)
    model = Model(GLPK.Optimizer)
    set_silent(model)
    @variable(model, y[1:I] >= 0, Int)
    @constraint(model, sum(data.pieces[i].w * y[i] for i in 1:I) <= data.W)
    @objective(model, Max, sum(π[i] * y[i] for i in 1:I))
    optimize!(model)
    if objective_value(model) > 1
        return round.(Int, value.(y))
    end
    return nothing
end
```

```
solve_pricing (generic function with 1 method)
```

Choosing the initial set of patterns

For the initial set of patterns, we create a trivial cutting pattern which cuts as many pieces of size i as will fit, or the amount demanded, whichever is smaller.

```
patterns = Vector{Int}[]
for i in 1:I
    pattern = zeros(Int, I)
    pattern[i] = floor(Int, min(data.W / data.pieces[i].w, data.pieces[i].d))
    push!(patterns, pattern)
end
P = length(patterns)
```

```
20
```

We can visualize the patterns by looking at the sparse matrix of the non-zeros:

```
SparseArrays.sparse(hcat(patterns...))
```

```
20×20 SparseArrays.SparseMatrixCSC{Int64, Int64} with 20 stored entries:
```

Solving the problem

First, we create our initial linear program:

```
model = Model(GLPK.Optimizer)
set_silent(model)
@variable(model, x[1:P] >= 0)
@objective(model, Min, sum(x))
@constraint(model, demand[i = 1:I], patterns[i]' * x == data.pieces[i].d)
model
```

```
A JuMP Model
Minimization problem with:
Variables: 20
Objective function type: AffExpr
`AffExpr`-in-`MathOptInterface.EqualTo{Float64}`: 20 constraints
`VariableRef`-in-`MathOptInterface.GreaterThan{Float64}`: 20 constraints
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: GLPK
Names registered in the model: demand, x
```

Then, we run the iterative column generation scheme:

```
while true
    # Solve the linear relaxation
    optimize!(model)
    # Obtain a new dual vector
    π = dual.(demand)
    # Solve the pricing problem
    new_pattern = solve_pricing(data, π)
    # Stop iterating if there is no new pattern
    if new_pattern === nothing
        break
    end
    push!(patterns, new_pattern)
    # Create a new column
    push!(x, @variable(model, lower_bound = 0))
    # Update the objective coefficients
    set_objective_coefficient(model, x[end], 1.0)
    # Update the non-zeros in the coefficient matrix
    for i in 1:I
        if new_pattern[i] > 0
            set_normalized_coefficient(demand[i], x[end], new_pattern[i])
        end
    end
end
```

Let's have a look at the patterns now:

```
SparseArrays.sparse(hcat(patterns...))
```

```
20×41 SparseArrays.SparseMatrixCSC{Int64, Int64} with 88 stored entries:
```

Nice! We found over 20 new patterns.

Here's pattern 21:

```
for i in 1:I
    if patterns[21][i] > 0
        println(patterns[21][i], " unit(s) of piece $i")
    end
end
```

```
1 unit(s) of piece 8
2 unit(s) of piece 13
1 unit(s) of piece 17
```

Looking at the solution

Since we only solved a linear relaxation, some of our columns have fractional solutions. We can create a integer feasible solution by rounding up the orders:

```
for p in 1:length(x)
    v = ceil(Int, value(x[p]))
    if v > 0
        println(lpad(v, 2), " roll(s) of pattern $p")
    end
end
```

```
31 roll(s) of pattern 1
44 roll(s) of pattern 2
30 roll(s) of pattern 3
41 roll(s) of pattern 4
15 roll(s) of pattern 5
15 roll(s) of pattern 21
23 roll(s) of pattern 22
16 roll(s) of pattern 25
 1 roll(s) of pattern 26
26 roll(s) of pattern 28
11 roll(s) of pattern 31
10 roll(s) of pattern 32
 3 roll(s) of pattern 33
19 roll(s) of pattern 34
11 roll(s) of pattern 36
 4 roll(s) of pattern 37
 8 roll(s) of pattern 38
15 roll(s) of pattern 39
12 roll(s) of pattern 40
 8 roll(s) of pattern 41
```

This requires 343 rolls:

```
sum(ceil.(Int, value.(x)))
```

```
343
```

Alternatively, we can re-introduce the integrality constraints and resolve the problem:

```
set_integer.(x)
optimize!(model)
for p in 1:length(x)
    v = round(Int, value(x[p]))
    if v > 0
        println(lpad(v, 2), " roll(s) of pattern $p, each roll of which makes:")
        for i in 1:I
            if patterns[p][i] > 0
                println("  ", patterns[p][i], " unit(s) of piece $i")
            end
        end
    end
end
```

```
33 roll(s) of pattern 1, each roll of which makes:
  1 unit(s) of piece 1
44 roll(s) of pattern 2, each roll of which makes:
  1 unit(s) of piece 2
30 roll(s) of pattern 3, each roll of which makes:
  1 unit(s) of piece 3
41 roll(s) of pattern 4, each roll of which makes:
  1 unit(s) of piece 4
 2 roll(s) of pattern 5, each roll of which makes:
  1 unit(s) of piece 5
 3 roll(s) of pattern 7, each roll of which makes:
  1 unit(s) of piece 7
 1 roll(s) of pattern 9, each roll of which makes:
  1 unit(s) of piece 9
11 roll(s) of pattern 21, each roll of which makes:
  1 unit(s) of piece 8
  2 unit(s) of piece 13
  1 unit(s) of piece 17
22 roll(s) of pattern 22, each roll of which makes:
  1 unit(s) of piece 9
  1 unit(s) of piece 12
  1 unit(s) of piece 15
  1 unit(s) of piece 20
 1 roll(s) of pattern 24, each roll of which makes:
  1 unit(s) of piece 8
  1 unit(s) of piece 12
  1 unit(s) of piece 17
  2 unit(s) of piece 20
17 roll(s) of pattern 25, each roll of which makes:
  1 unit(s) of piece 8
  1 unit(s) of piece 12
  1 unit(s) of piece 16
  1 unit(s) of piece 18
 2 roll(s) of pattern 26, each roll of which makes:
  1 unit(s) of piece 9
  1 unit(s) of piece 11
  1 unit(s) of piece 17
  2 unit(s) of piece 20
24 roll(s) of pattern 28, each roll of which makes:
  1 unit(s) of piece 7
```

```
  1 unit(s) of piece 11
  1 unit(s) of piece 14
12 roll(s) of pattern 31, each roll of which makes:
  1 unit(s) of piece 8
  1 unit(s) of piece 10
  1 unit(s) of piece 14
 9 roll(s) of pattern 32, each roll of which makes:
  1 unit(s) of piece 9
  1 unit(s) of piece 10
  2 unit(s) of piece 18
 1 roll(s) of pattern 33, each roll of which makes:
  1 unit(s) of piece 9
  1 unit(s) of piece 10
  1 unit(s) of piece 16
  1 unit(s) of piece 19
18 roll(s) of pattern 34, each roll of which makes:
  1 unit(s) of piece 6
  1 unit(s) of piece 11
  2 unit(s) of piece 19
 9 roll(s) of pattern 36, each roll of which makes:
  1 unit(s) of piece 7
  1 unit(s) of piece 12
  2 unit(s) of piece 17
 4 roll(s) of pattern 37, each roll of which makes:
  1 unit(s) of piece 5
  3 unit(s) of piece 19
  1 unit(s) of piece 20
 5 roll(s) of pattern 38, each roll of which makes:
  1 unit(s) of piece 1
  1 unit(s) of piece 15
  2 unit(s) of piece 20
15 roll(s) of pattern 39, each roll of which makes:
  1 unit(s) of piece 6
  1 unit(s) of piece 10
  1 unit(s) of piece 16
15 roll(s) of pattern 40, each roll of which makes:
  1 unit(s) of piece 5
  1 unit(s) of piece 15
  1 unit(s) of piece 17
15 roll(s) of pattern 41, each roll of which makes:
  1 unit(s) of piece 5
  1 unit(s) of piece 13
```

This now requires 334 rolls:

```
total_rolls = sum(ceil.(Int, value.(x)))
```

```
334
```

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

8.3 Traveling Salesperson Problem

Originally Contributed by: Daniel Schermer

This tutorial describes how to implement the Traveling Salesperson Problem in JuMP using solver-independent lazy constraints that dynamically separate subtours. To be more precise, we use lazy constraints to cut off infeasible subtours only when necessary and not before needed.

It uses the following packages:

```
using JuMP
import GLPK
import Random
import Plots
```

Mathematical Formulation

Assume that we are given a complete graph $\mathcal{G}(V, E)$ where V is the set of vertices (or cities) and E is the set of edges (or roads). For each pair of vertices $i, j \in V, i \neq j$ the edge $(i, j) \in E$ is associated with a weight (or distance) $d_{ij} \in \mathbb{R}^+$.

For this tutorial, we assume the problem to be symmetric, that is, $d_{ij} = d_{ji} \,\forall i, j \in V$.

In the Traveling Salesperson Problem, we are tasked with finding a tour with minimal length that visits every vertex exactly once and then returns to the point of origin, that is, a hamiltonian cycle with minimal weight.

To model the problem, we introduce a binary variable, $x_{ij} \in \{0, 1\} \;\forall i, j \in V$, that indicates if edge (i, j) is part of the tour or not. Using these variables, the Traveling Salesperson Problem can be modeled as the following integer linear program.

Objective Function

The objective is to minimize the length of the tour (due to the assumed symmetry, the second sum only contains $i < j$):

$$\text{min} \sum_{i \in V} \sum_{j \in V, i < j} d_{ij} x_{ij}.$$

Note that it is also possible to use the following objective function instead:

$$\text{min} \sum_{i \in V} \sum_{j \in V} \frac{d_{ij} x_{ij}}{2}.$$

Constraints

There are four classes of constraints in our formulation.

First, due to the presumed symmetry, the following constraints must hold:

$$x_{ij} = x_{ji} \quad \forall i, j \in V.$$

Second, for each vertex i, exactly two edges must be selected that connect it to other vertices j in the graph G:

$$\sum_{j \in V} x_{ij} = 2 \quad \forall i \in V.$$

Third, we do not permit loops to occur:

$$x_{ii} = 0 \quad \forall i \in V.$$

The fourth constraint is more complicated. A major difficulty of the Traveling Salesperson Problem arises from the fact that we need to prevent subtours, that is, several distinct Hamiltonian cycles existing on subgraphs of G.

Note that the previous constraints do not guarantee that the solution will be free of subtours. To this end, by S we label a subset of vertices. Then, for each proper subset $S \subset V$, the following constraints guarantee that no subtour may occur:

$$\sum_{i \in S} \sum_{j \in S, i < j} x_{ij} \leq |S| - 1 \quad \forall S \subset V.$$

Problematically, we require exponentially many of these constraints as $|V|$ increases. Therefore, we will add these constraints only when necessary.

Implementation

There are two ways we can eliminate subtours in JuMP, both of which will be shown in what follows:

- iteratively solving a new model that incorporates previously identified subtours,
- or adding violated subtours as lazy constraints.

Data

The vertices are assumed to be randomly distributed in the Euclidean space; thus, the weight (distance) of each edge is defined as follows.

```
function generate_distance_matrix(n; random_seed = 1)
    rng = Random.MersenneTwister(random_seed)
    X = 100 * rand(rng, n)
    Y = 100 * rand(rng, n)
    d = [sqrt((X[i] - X[j])^2 + (Y[i] - Y[j])^2) for i in 1:n, j in 1:n]
    return X, Y, d
end

n = 40
X, Y, d = generate_distance_matrix(n)
```

```
([23.603334566204694, 34.651701419196044, 31.27069683360675, 0.790928339056074, 48.86128300795012,
    21.096820215853597, 95.1916339835734, 99.99046588986135, 25.166218303197184, 98.66663668987997
    …  46.33502459235987, 18.582130997265377, 11.198087695816717, 97.6311881619359,
    5.161462067432709, 53.80295812064833, 45.56920516275036, 27.93951106725605, 17.824610354168602,
    54.89828719625274], [37.097066286146884, 89.41659192657593, 64.80537482231894,
    41.70393538841062, 14.456554241360564, 62.24031828206811, 87.23344353741976, 52.49746566167794,
    24.159060827129643, 88.48369255734127  …  66.12321555087209, 19.45678064479248,
    39.3193497656424, 99.07406554003964, 55.03342139580574, 58.07816346526631, 76.83586278313636,
    51.952465724186084, 51.486297110544356, 99.81360570779374], [0.0 53.473350122820904 …
    15.506244459460921 70.09092934998034; 53.473350122820904 0.0 … 41.49527995497558
    22.760099542720535; … ; 15.506244459460921 41.49527995497558 … 0.0 60.9096566304971;
    70.09092934998034 22.760099542720535 … 60.9096566304971 0.0])
```

For the JuMP model, we first initialize the model object. Then, we create the binary decision variables and add the objective function and constraints. By defining the `x` matrix as `Symmetric`, we do not need to add explicit constraints that `x[i, j] == x[j, i]`.

```
function build_tsp_model(d, n)
    model = Model(GLPK.Optimizer)
    @variable(model, x[1:n, 1:n], Bin, Symmetric)
    @objective(model, Min, sum(d .* x) / 2)
    @constraint(model, [i in 1:n], sum(x[i, :]) == 2)
    @constraint(model, [i in 1:n], x[i, i] == 0)
    return model
end
```

```
build_tsp_model (generic function with 1 method)
```

To search for violated constraints, based on the edges that are currently in the solution (that is, those that have value $x_{ij} = 1$), we identify the shortest cycle through the function `subtour()`. Whenever a subtour has been identified, a constraint corresponding to the form above can be added to the model.

```
function subtour(edges::Vector{Tuple{Int,Int}}, n)
    shortest_subtour, unvisited = collect(1:n), Set(collect(1:n))
    while !isempty(unvisited)
        this_cycle, neighbors = Int[], unvisited
        while !isempty(neighbors)
            current = pop!(neighbors)
            push!(this_cycle, current)
            if length(this_cycle) > 1
                pop!(unvisited, current)
            end
            neighbors =
                [j for (i, j) in edges if i == current && j in unvisited]
        end
        if length(this_cycle) < length(shortest_subtour)
            shortest_subtour = this_cycle
        end
    end
    return shortest_subtour
end
```

```
subtour (generic function with 1 method)
```

Let us declare a helper function `selected_edges()` that will be repeatedly used in what follows.

```
function selected_edges(x::Matrix{Float64}, n)
    return Tuple{Int,Int}[(i, j) for i in 1:n, j in 1:n if x[i, j] > 0.5]
end
```

```
selected_edges (generic function with 1 method)
```

Other helper functions for computing subtours:

```
subtour(x::Matrix{Float64}) = subtour(selected_edges(x, size(x, 1)), size(x, 1))
subtour(x::AbstractMatrix{VariableRef}) = subtour(value.(x))
```

```
subtour (generic function with 3 methods)
```

Iterative method

An iterative way of eliminating subtours is the following.

However, it is reasonable to assume that this is not the most efficient way: Whenever a new subtour elimination constraint is added to the model, the optimization has to start from the very beginning.

That way, the solver will repeatedly discard useful information encountered during previous solves (e.g., all cuts, the incumbent solution, or lower bounds).

> **Info**
>
> Note that, in principle, it would also be feasible to add all subtours (instead of just the shortest one) to the model. However, preventing just the shortest cycle is often sufficient for breaking other subtours and will keep the model size smaller.

```
iterative_model = build_tsp_model(d, n)
optimize!(iterative_model)
time_iterated = solve_time(iterative_model)
cycle = subtour(iterative_model[:x])
while 1 < length(cycle) < n
    println("Found cycle of length $(length(cycle))")
    S = [(i, j) for (i, j) in Iterators.product(cycle, cycle) if i < j]
    @constraint(
        iterative_model,
        sum(iterative_model[:x][i, j] for (i, j) in S) <= length(cycle) - 1,
    )
    optimize!(iterative_model)
    global time_iterated += solve_time(iterative_model)
    global cycle = subtour(iterative_model[:x])
end

objective_value(iterative_model)
```

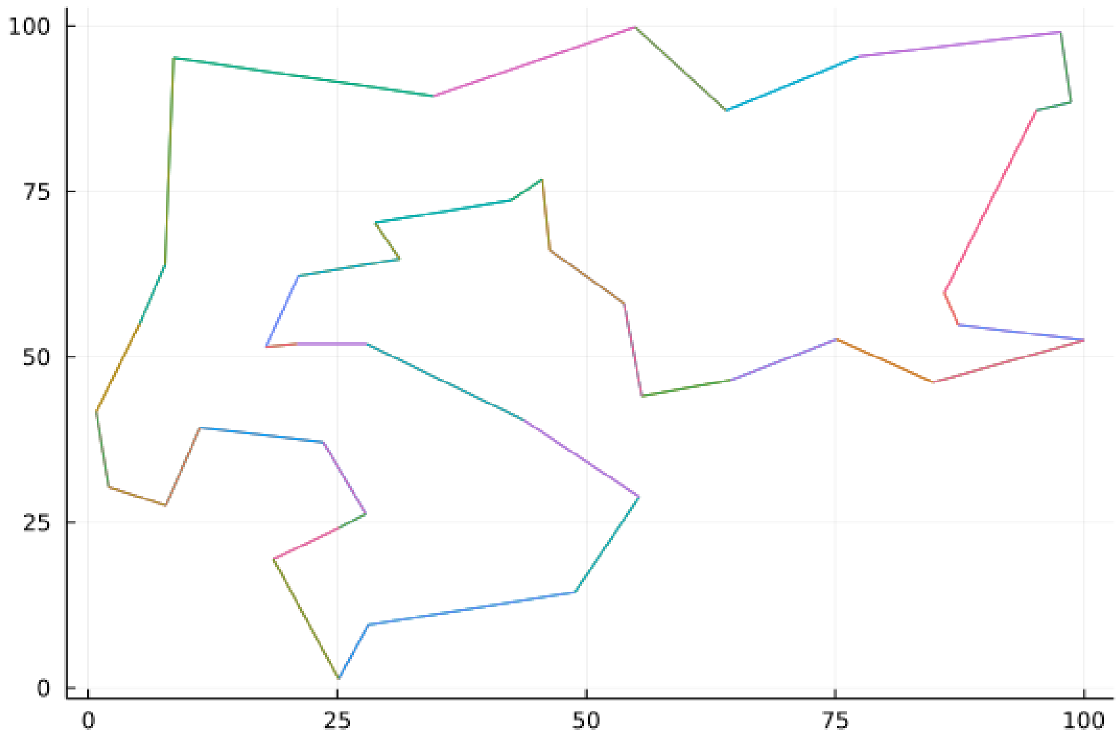
```
525.7039004442727
```

```
time_iterated
```

```
0.03724956512451172
```

As a quick sanity check, we visualize the optimal tour to verify that no subtour is present:

```
function plot_tour(X, Y, x)
    plt = Plots.plot()
    for (i, j) in selected_edges(x, size(x, 1))
        Plots.plot!([X[i], X[j]], [Y[i], Y[j]], legend = false)
    end
    return plt
end

plot_tour(X, Y, value.(iterative_model[:x]))
```



Lazy constraint method

A more sophisticated approach makes use of lazy constraints. To be more precise, we do this through the `subtour_elimination_callback()` below, which is only run whenever we encounter a new integer-feasible solution.

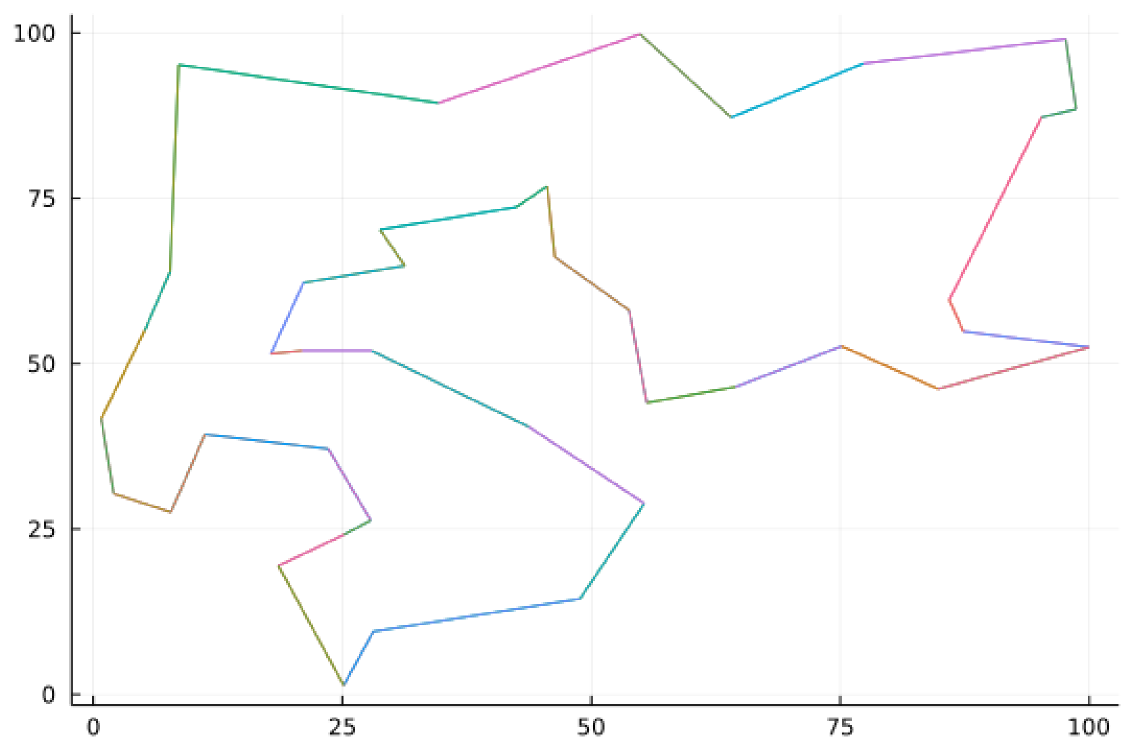
```
lazy_model = build_tsp_model(d, n)
function subtour_elimination_callback(cb_data)
    status = callback_node_status(cb_data, lazy_model)
    if status != MOI.CALLBACK_NODE_STATUS_INTEGER
        return  # Only run at integer solutions
    end
    cycle = subtour(callback_value.(cb_data, lazy_model[:x]))
```

```
    if !(1 < length(cycle) < n)
        return  # Only add a constraint if there is a cycle
    end
    println("Found cycle of length $(length(cycle))")
    S = [(i, j) for (i, j) in Iterators.product(cycle, cycle) if i < j]
    con = @build_constraint(
        sum(lazy_model[:x][i, j] for (i, j) in S) <= length(cycle) - 1,
    )
    MOI.submit(lazy_model, MOI.LazyConstraint(cb_data), con)
    return
end
MOI.set(lazy_model, MOI.LazyConstraintCallback(), subtour_elimination_callback)
optimize!(lazy_model)
objective_value(lazy_model)
```

```
525.7039004442727
```

This finds the same optimal tour:

```
plot_tour(X, Y, value.(lazy_model[:x]))
```



Surprisingly, for this particular model with GLPK, the solution time is worse than the iterative method:

```
time_lazy = solve_time(lazy_model)
```

```
0.15094304084777832
```

In most other cases and solvers, however, the lazy time should be faster than the iterative method.

Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

Chapter 9

Applications

9.1 Power Systems

Originally Contributed by: Yury Dvorkin and Miles Lubin

This tutorial demonstrates how to formulate basic power systems engineering models in JuMP.

We will consider basic "economic dispatch" and "unit commitment" models without taking into account transmission constraints.

For this tutorial, we use the following packages:

```
using JuMP
import DataFrames
import HiGHS
import Plots
import StatsPlots
```

Economic dispatch

Economic dispatch (ED) is an optimization problem that minimizes the cost of supplying energy demand subject to operational constraints on power system assets. In its simplest modification, ED is an LP problem solved for an aggregated load and wind forecast and for a single infinitesimal moment.

Mathematically, the ED problem can be written as follows:

$$\min \sum_{i \in I} c^g_i \cdot g_i + c^w \cdot w,$$

where c_i and g_i are the incremental cost (\$/MWh) and power output (MW) of the i^{th} generator, respectively, and c^w and w are the incremental cost (\$/MWh) and wind power injection (MW), respectively.

Subject to the constraints:

- Minimum ($g^{\min}$) and maximum ($g^{\max}$) limits on power outputs of generators: $g^{\min}_i \leq g_i \leq g^{\max}_i$.
- Constraint on the wind power injection: $0 \leq w \leq w^f$, where w and w^f are the wind power injection and wind power forecast, respectively.
- Power balance constraint: $\sum_{i \in I} g_i + w = d^f$, where d^f is the demand forecast.

Further reading on ED models can be found in A. J. Wood, B. F. Wollenberg, and G. B. Sheblé, "Power Generation, Operation and Control", Wiley, 2013.

Define some input data about the test system.

We define some thermal generators:

```
function ThermalGenerator(
    min::Float64,
    max::Float64,
    fixed_cost::Float64,
    variable_cost::Float64,
)
    return (
        min = min,
        max = max,
        fixed_cost = fixed_cost,
        variable_cost = variable_cost,
    )
end

generators = [
    ThermalGenerator(0.0, 1000.0, 1000.0, 50.0),
    ThermalGenerator(300.0, 1000.0, 0.0, 100.0),
]
```

```
2-element Vector{NamedTuple{(:min, :max, :fixed_cost, :variable_cost), NTuple{4, Float64}}}:
 (min = 0.0, max = 1000.0, fixed_cost = 1000.0, variable_cost = 50.0)
 (min = 300.0, max = 1000.0, fixed_cost = 0.0, variable_cost = 100.0)
```

A wind generator

```
WindGenerator(variable_cost::Float64) = (variable_cost = variable_cost,)

wind_generator = WindGenerator(50.0)
```

```
(variable_cost = 50.0,)
```

And a scenario

```
function Scenario(demand::Float64, wind::Float64)
    return (demand = demand, wind = wind)
end

scenario = Scenario(1500.0, 200.0)
```

```
(demand = 1500.0, wind = 200.0)
```

Create a function `solve_ed`, which solves the economic dispatch problem for a given set of input parameters.

```
function solve_ed(generators::Vector, wind, scenario)
    # Define the economic dispatch (ED) model
    ed = Model(HiGHS.Optimizer)
    set_silent(ed)
```

```
    # Define decision variables
    # power output of generators
    N = length(generators)
    @variable(ed, generators[i].min <= g[i = 1:N] <= generators[i].max)
    # wind power injection
    @variable(ed, 0 <= w <= scenario.wind)
    # Define the objective function
    @objective(
        ed,
        Min,
        sum(generators[i].variable_cost * g[i] for i in 1:N) +
        wind.variable_cost * w,
    )
    # Define the power balance constraint
    @constraint(ed, sum(g[i] for i in 1:N) + w == scenario.demand)
    # Solve statement
    optimize!(ed)
    # return the optimal value of the objective function and its minimizers
    return (
        g = value.(g),
        w = value(w),
        wind_spill = scenario.wind - value(w),
        total_cost = objective_value(ed),
    )
end
```

```
solve_ed (generic function with 1 method)
```

Solve the economic dispatch problem

```
solution = solve_ed(generators, wind_generator, scenario);

println("Dispatch of Generators: ", solution.g, " MW")
println("Dispatch of Wind: ", solution.w, " MW")
println("Wind spillage: ", solution.wind_spill, " MW")
println("Total cost: \$", solution.total_cost)
```

```
Dispatch of Generators: [1000.0, 300.0] MW
Dispatch of Wind: 200.0 MW
Wind spillage: 0.0 MW
Total cost: $90000.0
```

Economic dispatch with adjustable incremental costs

In the following exercise we adjust the incremental cost of generator G1 and observe its impact on the total cost.

```
function scale_generator_cost(g, scale)
    return ThermalGenerator(g.min, g.max, g.fixed_cost, scale * g.variable_cost)
end

start = time()
c_g_scale_df = DataFrames.DataFrame(
    # Scale factor
```

```
    scale = Float64[],
    # Dispatch of Generator 1 [MW]
    dispatch_G1 = Float64[],
    # Dispatch of Generator 2 [MW]
    dispatch_G2 = Float64[],
    # Dispatch of Wind [MW]
    dispatch_wind = Float64[],
    # Spillage of Wind [MW]
    spillage_wind = Float64[],
    # Total cost [$]
    total_cost = Float64[],
)
for c_g1_scale in 0.5:0.1:3.0
    # Update the incremental cost of the first generator at every iteration.
    new_generators = scale_generator_cost.(generators, [c_g1_scale, 1.0])
    # Solve the ed problem with the updated incremental cost
    sol = solve_ed(new_generators, wind_generator, scenario)
    push!(
        c_g_scale_df,
        (c_g1_scale, sol.g[1], sol.g[2], sol.w, sol.wind_spill, sol.total_cost),
    )
end
print(string("elapsed time: ", time() - start, " seconds"))
```

```
elapsed time: 0.15087413787841797 seconds
```

```
c_g_scale_df
```

	scale	dispatch_G1	dispatch_G2	dispatch_wind	spillage_wind	total_cost
	Float64	Float64	Float64	Float64	Float64	Float64
1	0.5	1000.0	300.0	200.0	0.0	65000.0
2	0.6	1000.0	300.0	200.0	0.0	70000.0
3	0.7	1000.0	300.0	200.0	0.0	75000.0
4	0.8	1000.0	300.0	200.0	0.0	80000.0
5	0.9	1000.0	300.0	200.0	0.0	85000.0
6	1.0	1000.0	300.0	200.0	0.0	90000.0
7	1.1	1000.0	300.0	200.0	0.0	95000.0
8	1.2	1000.0	300.0	200.0	0.0	100000.0
9	1.3	1000.0	300.0	200.0	0.0	105000.0
10	1.4	1000.0	300.0	200.0	0.0	110000.0
11	1.5	1000.0	300.0	200.0	0.0	115000.0
12	1.6	1000.0	300.0	200.0	0.0	120000.0
13	1.7	1000.0	300.0	200.0	0.0	125000.0
14	1.8	1000.0	300.0	200.0	0.0	130000.0
15	1.9	1000.0	300.0	200.0	0.0	135000.0
16	2.0	300.0	1000.0	200.0	0.0	140000.0
17	2.1	300.0	1000.0	200.0	0.0	141500.0
18	2.2	300.0	1000.0	200.0	0.0	143000.0
19	2.3	300.0	1000.0	200.0	0.0	144500.0
20	2.4	300.0	1000.0	200.0	0.0	146000.0
21	2.5	300.0	1000.0	200.0	0.0	147500.0
22	2.6	300.0	1000.0	200.0	0.0	149000.0
23	2.7	300.0	1000.0	200.0	0.0	150500.0
24	2.8	300.0	1000.0	200.0	0.0	152000.0
25	2.9	300.0	1000.0	200.0	0.0	153500.0
26	3.0	300.0	1000.0	200.0	0.0	155000.0

Modifying the JuMP model in-place

Note that in the previous exercise we entirely rebuilt the optimization model at every iteration of the internal loop, which incurs an additional computational burden. This burden can be alleviated if instead of re-building the entire model, we modify a specific constraint(s) or the objective function, as it shown in the example below.

Compare the computing time in case of the above and below models.

```julia
function solve_ed_inplace(
    generators::Vector,
    wind,
    scenario,
    scale::AbstractVector{Float64},
)
    obj_out = Float64[]
    w_out = Float64[]
    g1_out = Float64[]
    g2_out = Float64[]
    # This function only works for two generators
    @assert length(generators) == 2
    ed = Model(HiGHS.Optimizer)
    set_silent(ed)
    N = length(generators)
    @variable(ed, generators[i].min <= g[i = 1:N] <= generators[i].max)
    @variable(ed, 0 <= w <= scenario.wind)
```

```
    @objective(
        ed,
        Min,
        sum(generators[i].variable_cost * g[i] for i in 1:N) +
        wind.variable_cost * w,
    )
    @constraint(ed, sum(g[i] for i in 1:N) + w == scenario.demand)
    for c_g1_scale in scale
        @objective(
            ed,
            Min,
            c_g1_scale * generators[1].variable_cost * g[1] +
            generators[2].variable_cost * g[2] +
            wind.variable_cost * w,
        )
        optimize!(ed)
        push!(obj_out, objective_value(ed))
        push!(w_out, value(w))
        push!(g1_out, value(g[1]))
        push!(g2_out, value(g[2]))
    end
    df = DataFrames.DataFrame(
        scale = scale,
        dispatch_G1 = g1_out,
        dispatch_G2 = g2_out,
        dispatch_wind = w_out,
        spillage_wind = scenario.wind .- w_out,
        total_cost = obj_out,
    )
    return df
end

start = time()
inplace_df = solve_ed_inplace(generators, wind_generator, scenario, 0.5:0.1:3.0)
print(string("elapsed time: ", time() - start, " seconds"))
```

```
elapsed time: 0.22209811210632324 seconds
```

Adjusting specific constraints or the objective function is faster than re-building the entire model.

```
inplace_df
```

	scale	dispatch_G1	dispatch_G2	dispatch_wind	spillage_wind	total_cost
	Float64	Float64	Float64	Float64	Float64	Float64
1	0.5	1000.0	300.0	200.0	0.0	65000.0
2	0.6	1000.0	300.0	200.0	0.0	70000.0
3	0.7	1000.0	300.0	200.0	0.0	75000.0
4	0.8	1000.0	300.0	200.0	0.0	80000.0
5	0.9	1000.0	300.0	200.0	0.0	85000.0
6	1.0	1000.0	300.0	200.0	0.0	90000.0
7	1.1	1000.0	300.0	200.0	0.0	95000.0
8	1.2	1000.0	300.0	200.0	0.0	100000.0
9	1.3	1000.0	300.0	200.0	0.0	105000.0
10	1.4	1000.0	300.0	200.0	0.0	110000.0
11	1.5	1000.0	300.0	200.0	0.0	115000.0
12	1.6	1000.0	300.0	200.0	0.0	120000.0
13	1.7	1000.0	300.0	200.0	0.0	125000.0
14	1.8	1000.0	300.0	200.0	0.0	130000.0
15	1.9	1000.0	300.0	200.0	0.0	135000.0
16	2.0	1000.0	300.0	200.0	0.0	140000.0
17	2.1	300.0	1000.0	200.0	0.0	141500.0
18	2.2	300.0	1000.0	200.0	0.0	143000.0
19	2.3	300.0	1000.0	200.0	0.0	144500.0
20	2.4	300.0	1000.0	200.0	0.0	146000.0
21	2.5	300.0	1000.0	200.0	0.0	147500.0
22	2.6	300.0	1000.0	200.0	0.0	149000.0
23	2.7	300.0	1000.0	200.0	0.0	150500.0
24	2.8	300.0	1000.0	200.0	0.0	152000.0
25	2.9	300.0	1000.0	200.0	0.0	153500.0
26	3.0	300.0	1000.0	200.0	0.0	155000.0

Inefficient usage of wind generators

The economic dispatch problem does not perform commitment decisions and, thus, assumes that all generators must be dispatched at least at their minimum power output limit. This approach is not cost efficient and may lead to absurd decisions. For example, if $d = \sum_{i \in I} g_i^{\min}$, the wind power injection must be zero, i.e. all available wind generation is spilled, to meet the minimum power output constraints on generators.

In the following example, we adjust the total demand and observed how it affects wind spillage.

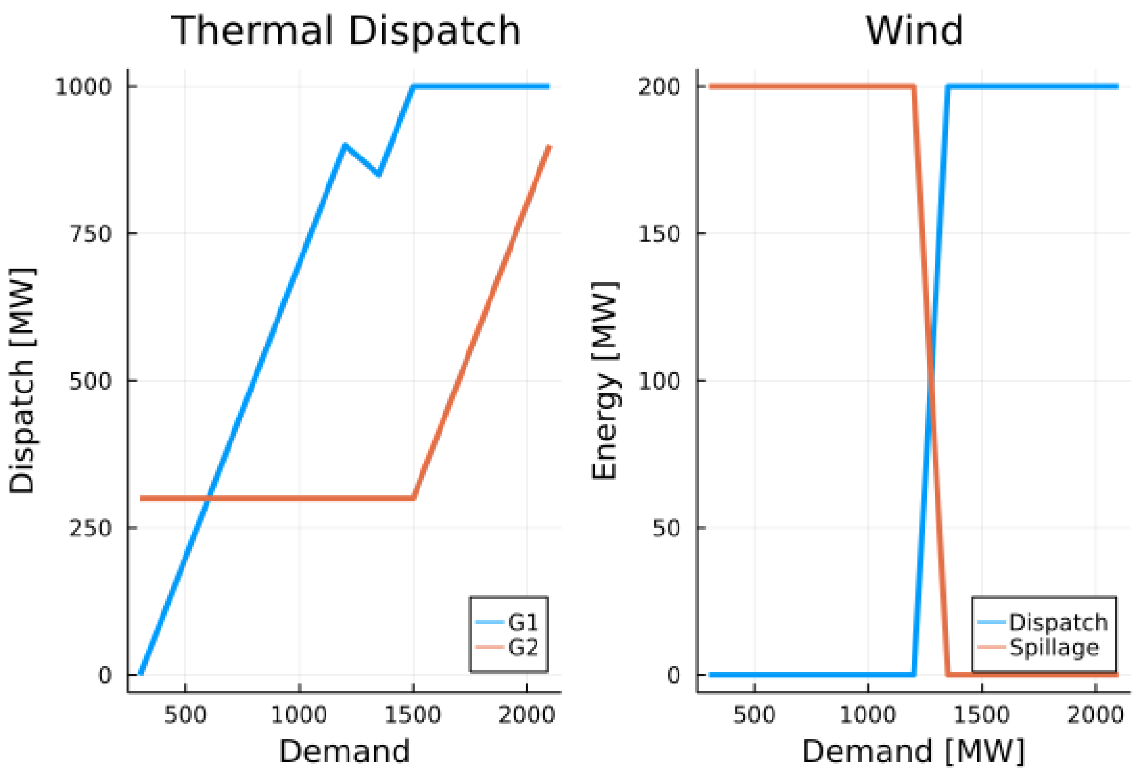
```
demand_scale_df = DataFrames.DataFrame(
    demand = Float64[],
    dispatch_G1 = Float64[],
    dispatch_G2 = Float64[],
    dispatch_wind = Float64[],
    spillage_wind = Float64[],
    total_cost = Float64[],
)

function scale_demand(scenario, scale)
    return Scenario(scale * scenario.demand, scenario.wind)
end

for demand_scale in 0.2:0.1:1.4
    new_scenario = scale_demand(scenario, demand_scale)
    sol = solve_ed(generators, wind_generator, new_scenario)
```

```
    push!(
        demand_scale_df,
        (
            new_scenario.demand,
            sol.g[1],
            sol.g[2],
            sol.w,
            sol.wind_spill,
            sol.total_cost,
        ),
    )
end

demand_scale_df
```

	demand	dispatch_G1	dispatch_G2	dispatch_wind	spillage_wind	total_cost
	Float64	Float64	Float64	Float64	Float64	Float64
1	300.0	0.0	300.0	0.0	200.0	30000.0
2	450.0	150.0	300.0	0.0	200.0	37500.0
3	600.0	300.0	300.0	0.0	200.0	45000.0
4	750.0	450.0	300.0	0.0	200.0	52500.0
5	900.0	600.0	300.0	0.0	200.0	60000.0
6	1050.0	750.0	300.0	0.0	200.0	67500.0
7	1200.0	900.0	300.0	0.0	200.0	75000.0
8	1350.0	850.0	300.0	200.0	0.0	82500.0
9	1500.0	1000.0	300.0	200.0	0.0	90000.0
10	1650.0	1000.0	450.0	200.0	0.0	105000.0
11	1800.0	1000.0	600.0	200.0	0.0	120000.0
12	1950.0	1000.0	750.0	200.0	0.0	135000.0
13	2100.0	1000.0	900.0	200.0	0.0	150000.0

```
dispatch_plot = StatsPlots.@df(
    demand_scale_df,
    Plots.plot(
        :demand,
        [:dispatch_G1, :dispatch_G2],
        labels = ["G1" "G2"],
        title = "Thermal Dispatch",
        legend = :bottomright,
        linewidth = 3,
        xlabel = "Demand",
        ylabel = "Dispatch [MW]",
    ),
)

wind_plot = StatsPlots.@df(
    demand_scale_df,
    Plots.plot(
        :demand,
        [:dispatch_wind, :spillage_wind],
        labels = ["Dispatch" "Spillage"],
        title = "Wind",
        legend = :bottomright,
        linewidth = 3,
```

```
        xlabel = "Demand [MW]",
        ylabel = "Energy [MW]",
    ),
)

Plots.plot(dispatch_plot, wind_plot)
```



This particular drawback can be overcome by introducing binary decisions on the "on/off" status of generators. This model is called unit commitment and considered later in these notes.

For further reading on the interplay between wind generation and the minimum power output constraints of generators, we refer interested readers to R. Baldick, "Wind and Energy Markets: A Case Study of Texas," IEEE Systems Journal, vol. 6, pp. 27-34, 2012.

Unit commitment

The Unit Commitment (UC) model can be obtained from ED model by introducing binary variable associated with each generator. This binary variable can attain two values: if it is "1", the generator is synchronized and, thus, can be dispatched, otherwise, i.e. if the binary variable is "0", that generator is not synchronized and its power output is set to 0.

To obtain the mathematical formulation of the UC model, we will modify the constraints of the ED model as follows:

$$g_i^{\min} \cdot u_{t,i} \leq g_i \leq g_i^{\max} \cdot u_{t,i},$$

where $u_i \in \{0, 1\}$. In this constraint, if $u_i = 0$, then $g_i = 0$. On the other hand, if $u_i = 1$, then $g_i^{min} \leq g_i \leq g_i^{max}$.

For further reading on the UC problem we refer interested readers to G. Morales-Espana, J. M. Latorre, and A. Ramos, "Tight and Compact MILP Formulation for the Thermal Unit Commitment Problem," IEEE Transactions on Power Systems, vol. 28, pp. 4897-4908, 2013.

In the following example we convert the ED model explained above to the UC model.

```
function solve_uc(generators::Vector, wind, scenario)
    uc = Model(HiGHS.Optimizer)
    set_silent(uc)
    N = length(generators)
    @variable(uc, generators[i].min <= g[i = 1:N] <= generators[i].max)
    @variable(uc, 0 <= w <= scenario.wind)
    @constraint(uc, sum(g[i] for i in 1:N) + w == scenario.demand)
    # !!! New: add binary on-off variables for each generator
    @variable(uc, u[i = 1:N], Bin)
    @constraint(uc, [i = 1:N], g[i] <= generators[i].max * u[i])
    @constraint(uc, [i = 1:N], g[i] >= generators[i].min * u[i])
    @objective(
        uc,
        Min,
        sum(generators[i].variable_cost * g[i] for i in 1:N) +
        wind.variable_cost * w +
        # !!! new
        sum(generators[i].fixed_cost * u[i] for i in 1:N)
    )
    optimize!(uc)
    status = termination_status(uc)
    if status != OPTIMAL
        return (status = status,)
    end
    return (
        status = status,
        g = value.(g),
        w = value(w),
        wind_spill = scenario.wind - value(w),
        u = value.(u),
        total_cost = objective_value(uc),
    )
end
```

```
solve_uc (generic function with 1 method)
```

Solve the economic dispatch problem

```
solution = solve_uc(generators, wind_generator, scenario)

println("Dispatch of Generators: ", solution.g, " MW")
println("Commitments of Generators: ", solution.u)
println("Dispatch of Wind: ", solution.w, " MW")
println("Wind spillage: ", solution.wind_spill, " MW")
println("Total cost: \$", solution.total_cost)
```

```
Dispatch of Generators: [1000.0, 300.0] MW
Commitments of Generators: [1.0, 1.0]
Dispatch of Wind: 200.0 MW
Wind spillage: 0.0 MW
Total cost: $91000.0
```

Unit commitment as a function of demand

After implementing the UC model, we can now assess the interplay between the minimum power output constraints on generators and wind generation.

```julia
uc_df = DataFrames.DataFrame(
    demand = Float64[],
    commitment_G1 = Float64[],
    commitment_G2 = Float64[],
    dispatch_G1 = Float64[],
    dispatch_G2 = Float64[],
    dispatch_wind = Float64[],
    spillage_wind = Float64[],
    total_cost = Float64[],
)

for demand_scale in 0.2:0.1:1.4
    new_scenario = scale_demand(scenario, demand_scale)
    sol = solve_uc(generators, wind_generator, new_scenario)
    if sol.status == OPTIMAL
        push!(
            uc_df,
            (
                new_scenario.demand,
                sol.u[1],
                sol.u[2],
                sol.g[1],
                sol.g[2],
                sol.w,
                sol.wind_spill,
                sol.total_cost,
            ),
        )
    end
    println("Status: $(sol.status) for demand_scale = $(demand_scale)")
end
```

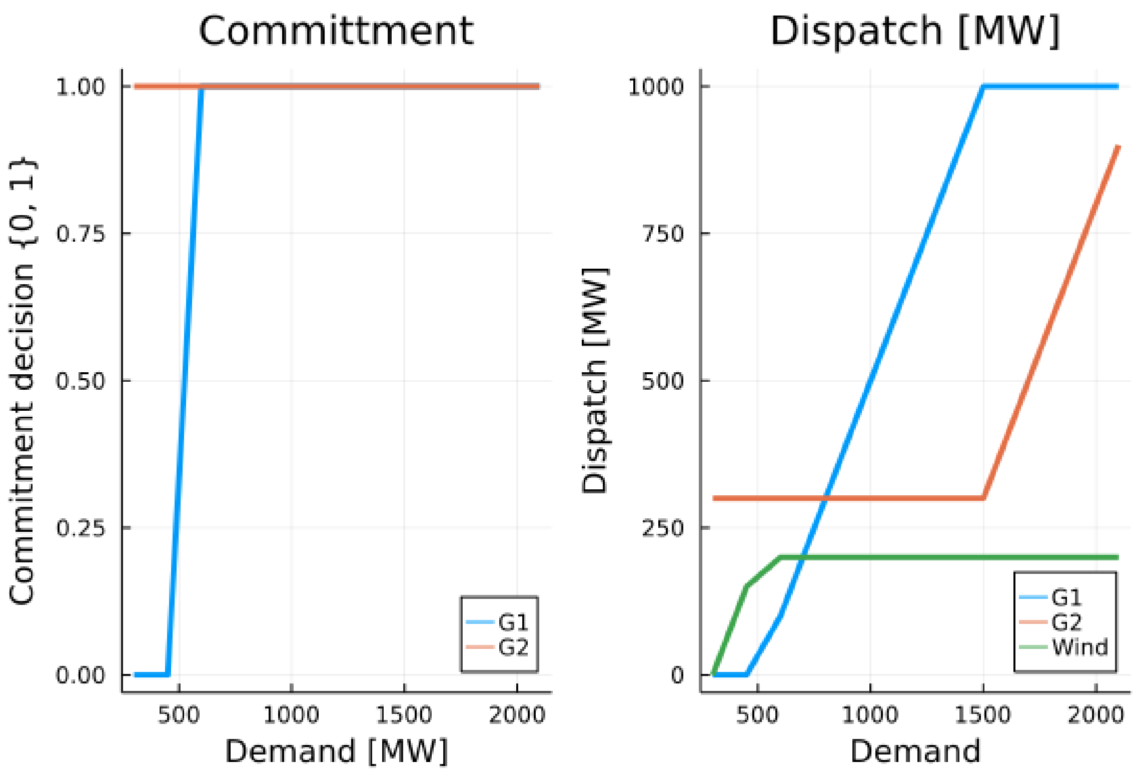
```
Status: OPTIMAL for demand_scale = 0.2
Status: OPTIMAL for demand_scale = 0.3
Status: OPTIMAL for demand_scale = 0.4
Status: OPTIMAL for demand_scale = 0.5
Status: OPTIMAL for demand_scale = 0.6
Status: OPTIMAL for demand_scale = 0.7
Status: OPTIMAL for demand_scale = 0.8
Status: OPTIMAL for demand_scale = 0.9
Status: OPTIMAL for demand_scale = 1.0
Status: OPTIMAL for demand_scale = 1.1
Status: OPTIMAL for demand_scale = 1.2
Status: OPTIMAL for demand_scale = 1.3
```

```
Status: OPTIMAL for demand_scale = 1.4
```

```
uc_df
```

	demand	commitment_G1	commitment_G2	dispatch_G1	dispatch_G2	dispatch_wind	spillage_wind	total_c
	Float64	Float64	Float64	Float64	Float64	Float64	Float64	Float
1	300.0	0.0	1.0	0.0	300.0	0.0	200.0	30000
2	450.0	0.0	1.0	0.0	300.0	150.0	50.0	37500
3	600.0	1.0	1.0	100.0	300.0	200.0	0.0	46000
4	750.0	1.0	1.0	250.0	300.0	200.0	0.0	53500
5	900.0	1.0	1.0	400.0	300.0	200.0	0.0	61000
6	1050.0	1.0	1.0	550.0	300.0	200.0	0.0	68500
7	1200.0	1.0	1.0	700.0	300.0	200.0	0.0	76000
8	1350.0	1.0	1.0	850.0	300.0	200.0	0.0	83500
9	1500.0	1.0	1.0	1000.0	300.0	200.0	0.0	91000
10	1650.0	1.0	1.0	1000.0	450.0	200.0	0.0	10600
11	1800.0	1.0	1.0	1000.0	600.0	200.0	0.0	12100
12	1950.0	1.0	1.0	1000.0	750.0	200.0	0.0	13600
13	2100.0	1.0	1.0	1000.0	900.0	200.0	0.0	15100

```
commitment_plot = StatsPlots.@df(
    uc_df,
    Plots.plot(
        :demand,
        [:commitment_G1, :commitment_G2],
        labels = ["G1" "G2"],
        title = "Committment",
        legend = :bottomright,
        linewidth = 3,
        xlabel = "Demand [MW]",
        ylabel = "Commitment decision {0, 1}",
    ),
)

dispatch_plot = StatsPlots.@df(
    uc_df,
    Plots.plot(
        :demand,
        [:dispatch_G1, :dispatch_G2, :dispatch_wind],
        labels = ["G1" "G2" "Wind"],
        title = "Dispatch [MW]",
        legend = :bottomright,
        linewidth = 3,
        xlabel = "Demand",
        ylabel = "Dispatch [MW]",
    ),
)

Plots.plot(commitment_plot, dispatch_plot)
```



Nonlinear economic dispatch

As a final example, we modify our economic dispatch problem in two ways:

- The thermal cost function is user-defined
- The output of the wind is only the square-root of the dispatch

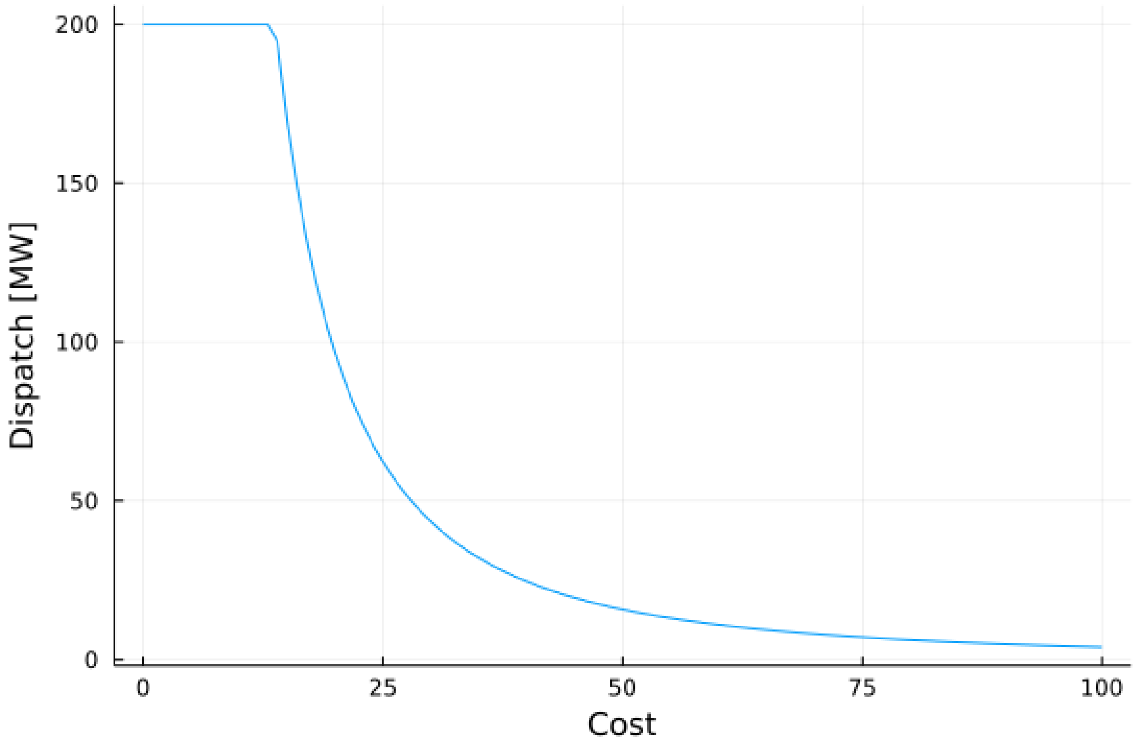
```
import Ipopt

"""
    thermal_cost_function(g)

A user-defined thermal cost function in pure-Julia! You can include
nonlinearities, and even things like control flow.

!!! warning
    It's still up to you to make sure that the function has a meaningful
    derivative.
"""
function thermal_cost_function(g)
    if g <= 500
        return g
    else
        return g + 1e-2 * (g - 500)^2
    end
end
```

```
function solve_nonlinear_ed(
    generators::Vector,
    wind,
    scenario;
    silent::Bool = false,
)
    model = Model(Ipopt.Optimizer)
    if silent
        set_silent(model)
    end
    register(model, :tcf, 1, thermal_cost_function; autodiff = true)
    N = length(generators)
    @variable(model, generators[i].min <= g[i = 1:N] <= generators[i].max)
    @variable(model, 0 <= w <= scenario.wind)
    @NLobjective(
        model,
        Min,
        sum(generators[i].variable_cost * tcf(g[i]) for i in 1:N) +
        wind.variable_cost * w,
    )
    @NLconstraint(model, sum(g[i] for i in 1:N) + sqrt(w) == scenario.demand)
    optimize!(model)
    return (
        g = value.(g),
        w = value(w),
        wind_spill = scenario.wind - value(w),
        total_cost = objective_value(model),
    )
end

solution = solve_nonlinear_ed(generators, wind_generator, scenario)
```

```
(g = [847.3509933774712, 648.6754966887423], w = 15.788781193899027, wind_spill = 184.211218806101,
     total_cost = 190455.298013245)
```

Now let's see how the wind is dispatched as a function of the cost:

```
wind_cost = 0.0:1:100
wind_dispatch = Float64[]
for c in wind_cost
    sol = solve_nonlinear_ed(
        generators,
        WindGenerator(c),
        scenario;
        silent = true,
    )
    push!(wind_dispatch, sol.w)
end

Plots.plot(
    wind_cost,
    wind_dispatch,
    xlabel = "Cost",
    ylabel = "Dispatch [MW]",
    label = false,
)
```



Tip

This tutorial was generated using Literate.jl. View the source `.jl` file on GitHub.

Part III

Manual

Chapter 10

Models

JuMP models are the fundamental building block that we use to construct optimization problems. They hold things like the variables and constraints, as well as which solver to use and even solution information.

Info

JuMP uses "optimizer" as a synonym for "solver." Our convention is to use "solver" to refer to the underlying software, and use "optimizer" to refer to the Julia object that wraps the solver. For example, `HiGHS` is a solver, and `HiGHS.Optimizer` is an optimizer.

Tip

See Supported solvers for a list of available solvers.

10.1 Create a model

Create a model by passing an optimizer to `Model`:

```
julia> model = Model(HiGHS.Optimizer)
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

If you don't know which optimizer you will be using at creation time, create a model without an optimizer, and then call `set_optimizer` at any time prior to `optimize!`:

```
julia> model = Model()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> set_optimizer(model, HiGHS.Optimizer)
```

Tip

Don't know what the fields `Model mode` and `CachingOptimizer state` mean? Read the Backends section.

What is the difference?

For most models, there is no difference between passing the optimizer to `Model`, and calling `set_optimizer`.

However, if an optimizer does not support a constraint in the model, the timing of when an error will be thrown can differ:

- If you pass an optimizer, an error will be thrown when you try to add the constraint.
- If you call `set_optimizer`, an error will be thrown when you try to solve the model via `optimize!`.

Therefore, most users should pass an optimizer to `Model` because it provides the earliest warning that your solver is not suitable for the model you are trying to build. However, if you are modifying a problem by adding and deleting different constraint types, you may need to use `set_optimizer`. See Switching optimizer for the relaxed problem for an example of when this is useful.

Reducing time-to-first-solve latency

By default, JuMP uses bridges to reformulate the model you are building into an equivalent model supported by the solver.

However, if your model is already supported by the solver, bridges add latency (read The "time-to-first-solve" issue). This is particularly noticeable for small models.

To reduce the "time-to-first-solve", try passing `add_bridges = false`.

```
julia> model = Model(HiGHS.Optimizer; add_bridges = false);
```

or

```
julia> model = Model();

julia> set_optimizer(model, HiGHS.Optimizer; add_bridges = false)
```

However, be wary! If your model and solver combination needs bridges, an error will be thrown:

```
julia> model = Model(SCS.Optimizer; add_bridges = false);


julia> @variable(model, x)
x

julia> @constraint(model, 2x <= 1)
ERROR: Constraints of type
↪  MathOptInterface.ScalarAffineFunction{Float64}-in-MathOptInterface.LessThan{Float64} are not
↪  supported by the solver.

If you expected the solver to support your problem, you may have an error in your formulation.
↪  Otherwise, consider using a different solver.
```

```
The list of available solvers, along with the problem types they support, is available at
↪  https://jump.dev/JuMP.jl/stable/installation/#Supported-solvers.
[...]
```

Solvers which expect environments

Some solvers accept (or require) positional arguments such as a license environment or a path to a binary executable. For these solvers, you can pass a function to `Model` which takes zero arguments and returns an instance of the optimizer.

A common use-case for this is passing an environment to Gurobi:

```
julia> grb_env = Gurobi.Env();

julia> model = Model(() -> Gurobi.Optimizer(grb_env))
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: Gurobi
```

10.2 Solver options

JuMP uses "attribute" as a synonym for "option." Use `optimizer_with_attributes` to create an optimizer with some attributes initialized:

```
julia> model = Model(
           optimizer_with_attributes(HiGHS.Optimizer, "output_flag" => false),
       )
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS
```

Alternatively, use `set_optimizer_attribute` to set an attribute after the model has been created:

```
julia> model = Model(HiGHS.Optimizer);

julia> set_optimizer_attribute(model, "output_flag", false)

julia> get_optimizer_attribute(model, "output_flag")
false
```

10.3 Print the model

By default, `show(model)` will print a summary of the problem:

```
julia> model = Model(); @variable(model, x >= 0); @objective(model, Max, x);
```

```
julia> model
A JuMP Model
Maximization problem with:
Variable: 1
Objective function type: VariableRef
`VariableRef`-in-`MathOptInterface.GreaterThan{Float64}`: 1 constraint
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
Names registered in the model: x
```

Use `print` to print the formulation of the model (in IJulia, this will render as LaTeX.

```
julia> print(model)
Max x
Subject to
 x ≥ 0.0
```

Warning

This format is specific to JuMP and may change in any future release. It is not intended to be an instance format. To write the model to a file, use `write_to_file` instead.

Use `latex_formulation` to display the model in LaTeX form.

```
julia> latex_formulation(model)
$$ \begin{aligned}
\max\quad & x\\
\text{Subject to} \quad & x \geq 0.0\\
\end{aligned} $$
```

In IJulia (and Documenter), ending a cell in with `latex_formulation` will render the model in LaTeX!

```
latex_formulation(model)
```

$$\begin{aligned}
\max\quad & x\\
\text{Subject to} \quad & x \geq 0.0\\
\end{aligned}$$

10.4 Turn off output

Use `set_silent` and `unset_silent` to disable or enable printing output from the solver.

```
julia> model = Model(HiGHS.Optimizer);

julia> set_silent(model)

julia> unset_silent(model)
```

Tip

Most solvers will also have a solver-specific option to provide finer-grained control over the output. Consult their README's for details.

10.5 Set a time limit

Use set_time_limit_sec, unset_time_limit_sec, and time_limit_sec to manage time limits.

```
julia> model = Model(HiGHS.Optimizer);

julia> set_time_limit_sec(model, 60.0)


julia> time_limit_sec(model)
60.0

julia> unset_time_limit_sec(model)

julia> time_limit_sec(model)
Inf
```

> **Info**
>
> Some solvers do not support time limits. In these cases, an error will be thrown.

10.6 Write a model to file

JuMP can write models to a variety of file-formats using write_to_file and Base.write.

For most common file formats, the file type will be detected from the extension. For example, here is how to write an MPS file:

```
julia> write_to_file(model, "model.mps")
```

To write to a specific io::IO, use Base.write. Specify the file type by passing a MOI.FileFormats.FileFormat enum.

```
julia> write(io, model; format = MOI.FileFormats.FORMAT_MPS)
```

10.7 Read a model from file

JuMP models can be created from file formats using read_from_file and Base.read.

```
julia> model = read_from_file("model.mps")
A JuMP Model
Minimization problem with:
Variables: 0
Objective function type: AffExpr
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> seekstart(io);

julia> model2 = read(io, Model; format = MOI.FileFormats.FORMAT_MPS)
A JuMP Model
Minimization problem with:
```

```
Variables: 0
Objective function type: AffExpr
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
```

Note

Because file formats do not serialize the containers of JuMP variables and constraints, the names in the model will not be registered. Therefore, you cannot access named variables and constraints via `model[:x]`. Instead, use `variable_by_name` or `constraint_by_name` to access specific variables or constraints.

10.8 Relax integrality

Use `relax_integrality` to remove any integrality constraints from the model, such as integer and binary restrictions on variables. `relax_integrality` returns a function that can be later called with zero arguments to re-add the removed constraints:

```
julia> model = Model();

julia> @variable(model, x, Int)
x

julia> num_constraints(model, VariableRef, MOI.Integer)
1

julia> undo = relax_integrality(model);

julia> num_constraints(model, VariableRef, MOI.Integer)
0

julia> undo()

julia> num_constraints(model, VariableRef, MOI.Integer)
1
```

Switching optimizer for the relaxed problem

A common reason for relaxing integrality is to compute dual variables of the relaxed problem. However, some mixed-integer linear solvers (for example, Cbc) do not return dual solutions, even if the problem does not have integrality restrictions.

Therefore, after `relax_integrality` you should call `set_optimizer` with a solver that does support dual solutions, such as Clp.

For example, instead of:

```
using JuMP, Cbc
model = Model(Cbc.Optimizer)
@variable(model, x, Int)
undo = relax_integrality(model)
optimize!(model)
reduced_cost(x)  # Errors
```

do:

```
using JuMP, Cbc, Clp
model = Model(Cbc.Optimizer)
@variable(model, x, Int)
undo = relax_integrality(model)
set_optimizer(model, Clp.Optimizer)
optimize!(model)
reduced_cost(x)  # Works
```

10.9 Backends

Info

This section discusses advanced features of JuMP. For new users, you may want to skip this section. You don't need to know how JuMP manages problems behind the scenes to create and solve JuMP models.

A JuMP `Model` is a thin layer around a backend of type `MOI.ModelLike` that stores the optimization problem and acts as the optimization solver.

However, if you construct a model like `Model(HiGHS.Optimizer)`, the backend is not a `HiGHS.Optimizer`, but a more complicated object.

From JuMP, the MOI backend can be accessed using the `backend` function. Let's see what the `backend` of a JuMP `Model` is:

```
julia> model = Model(HiGHS.Optimizer);

julia> b = backend(model)
MOIU.CachingOptimizer{MOIB.LazyBridgeOptimizer{HiGHS.Optimizer},
↪  MOIU.UniversalFallback{MOIU.Model{Float64}}}
in state EMPTY_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.UniversalFallback{MOIU.Model{Float64}}
  fallback for MOIU.Model{Float64}
with optimizer MOIB.LazyBridgeOptimizer{HiGHS.Optimizer}
  with 0 variable bridges
  with 0 constraint bridges
  with 0 objective bridges
  with inner model A HiGHS model with 0 columns and 0 rows.
```

Uh oh! Even though we passed a `HiGHS.Optimizer`, the backend is a much more complicated object.

CachingOptimizer

A `MOIU.CachingOptimizer` is a layer that abstracts the difference between solvers that support incremental modification (for example, they support adding variables one-by-one), and solvers that require the entire problem in a single API call (for example, they only accept the A, b and c matrices of a linear program).

It has two parts:

1. A cache, where the model can be built and modified incrementally

```
julia> b.model_cache
MOIU.UniversalFallback{MOIU.Model{Float64}}
fallback for MOIU.Model{Float64}
```

2. An optimizer, which is used to solve the problem

```
julia> b.optimizer
MOIB.LazyBridgeOptimizer{HiGHS.Optimizer}
with 0 variable bridges
with 0 constraint bridges
with 0 objective bridges
with inner model A HiGHS model with 0 columns and 0 rows.
```

Info

The LazyBridgeOptimizer section explains what a `LazyBridgeOptimizer` is.

The `CachingOptimizer` has logic to decide when to copy the problem from the cache to the optimizer, and when it can efficiently update the optimizer in-place.

A `CachingOptimizer` may be in one of three possible states:

- `NO_OPTIMIZER`: The CachingOptimizer does not have any optimizer.
- `EMPTY_OPTIMIZER`: The CachingOptimizer has an empty optimizer, and it is not synchronized with the cached model.
- `ATTACHED_OPTIMIZER`: The CachingOptimizer has an optimizer, and it is synchronized with the cached model.

A `CachingOptimizer` has two modes of operation:

- `AUTOMATIC`: The `CachingOptimizer` changes its state when necessary. For example, `optimize!` will automatically call `attach_optimizer` (an optimizer must have been previously set). Attempting to add a constraint or perform a modification not supported by the optimizer results in a drop to `EMPTY_OPTIMIZER` mode.
- `MANUAL`: The user must change the state of the `CachingOptimizer` using `MOIU.reset_optimizer(::JuMP.Model)`, `MOIU.drop_optimizer(::JuMP.Model)`, and `MOIU.attach_optimizer(::JuMP.Model)`. Attempting to perform an operation in the incorrect state results in an error.

By default `Model` will create a `CachingOptimizer` in `AUTOMATIC` mode.

LazyBridgeOptimizer

The second layer that JuMP applies automatically is a `LazyBridgeOptimizer`. A `LazyBridgeOptimizer` is an MOI layer that attempts to transform the problem from the formulation provided by the user into an equivalent problem supported by the solver. This may involve adding new variables and constraints to the optimizer. The transformations are selected from a set of known recipes called bridges.

A common example of a bridge is one that splits an interval constraint like `@constraint(model, 1 <= x + y <= 2)` into two constraints, `@constraint(model, x + y >= 1)` and `@constraint(model, x + y <= 2)`.

Use the `add_bridges = false` keyword to remove the bridging layer:

```
julia> model = Model(HiGHS.Optimizer; add_bridges = false)
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: EMPTY_OPTIMIZER
Solver name: HiGHS

julia> backend(model)
MOIU.CachingOptimizer{HiGHS.Optimizer, MOIU.UniversalFallback{MOIU.Model{Float64}}}
in state EMPTY_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.UniversalFallback{MOIU.Model{Float64}}
  fallback for MOIU.Model{Float64}
with optimizer A HiGHS model with 0 columns and 0 rows.
```

Unsafe backend

In some advanced use-cases, it is necessary to work with the inner optimization model directly. To access this model, use `unsafe_backend`:

```
julia> backend(model)
MOIU.CachingOptimizer{MOIB.LazyBridgeOptimizer{HiGHS.Optimizer},
↪  MOIU.UniversalFallback{MOIU.Model{Float64}}}
in state EMPTY_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.UniversalFallback{MOIU.Model{Float64}}
  fallback for MOIU.Model{Float64}
with optimizer MOIB.LazyBridgeOptimizer{HiGHS.Optimizer}
  with 0 variable bridges
  with 0 constraint bridges
  with 0 objective bridges
  with inner model A HiGHS model with 0 columns and 0 rows.

julia> unsafe_backend(model)
A HiGHS model with 0 columns and 0 rows.
```

Warning

`backend` and `unsafe_backend` are advanced routines. Read their docstrings to understand the caveats of their usage, and only call them if you wish to access low-level solver-specific functions.

10.10 Direct mode

Using a `CachingOptimizer` results in an additional copy of the model being stored by JuMP in the `.model_cache` field. To avoid this overhead, create a JuMP model using `direct_model`:

```
julia> model = direct_model(HiGHS.Optimizer())
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: DIRECT
Solver name: HiGHS
```

Warning

Solvers that do not support incremental modification do not support `direct_model`. An error will be thrown, telling you to use a `CachingOptimizer` instead.

The benefit of using `direct_model` is that there are no extra layers (for example, `Cachingoptimizer` or `LazyBridgeOptimizer`) between `model` and the provided optimizer:

```
julia> backend(model)
A HiGHS model with 0 columns and 0 rows.
```

A downside of direct mode is that there is no bridging layer. Therefore, only constraints which are natively supported by the solver are supported. For example, `HiGHS.jl` does not implement quadratic constraints:

```
julia> model = direct_model(HiGHS.Optimizer());

julia> @variable(model, x[1:2]);

julia> @constraint(model, x[1]^2 + x[2]^2 <= 2)
ERROR: Constraints of type
↪  MathOptInterface.MathOptInterface.ScalarQuadraticFunction{Float64}-in-MathOptInterface.LessThan{Float64}
↪  are not supported by the solver.

If you expected the solver to support your problem, you may have an error in your formulation.
↪  Otherwise, consider using a different solver.

The list of available solvers, along with the problem types they support, is available at
↪  https://jump.dev/JuMP.jl/stable/installation/#Supported-solvers.
[...]
```

Warning

Another downside of direct mode is that the behavior of querying solution information after modifying the problem is solver-specific. This can lead to errors, or the solver silently returning an incorrect value. See OptimizeNotCalled errors for more information.

Chapter 11

Variables

The term variable in mathematical optimization has many meanings. For example, optimization variables (also called decision variables) are the unknowns x that we are solving for in the problem:

$$\min_{x \in \mathbb{R}^n} \quad f_0(x) \tag{11.1}$$

$$\text{s.t.} \quad f_i(x) \in \mathcal{S}_i \quad i = 1 \ldots m \tag{11.2}$$

To complicate things, Julia uses variable to mean a binding between a name and a value. For example, in the statement:

```
julia> x = 1
1
```

x is a variable that stores the value 1.

JuMP uses variable in a third way, to mean an instance of the `VariableRef` struct. JuMP variables are the link between Julia and the optimization variables inside a JuMP model.

This page explains how to create and manage JuMP variables in a variety of contexts.

11.1 Create a variable

Create variables using the `@variable` macro. When creating a variable, you can also specify variable bounds:

```
model = Model()
@variable(model, x_free)
@variable(model, x_lower >= 0)
@variable(model, x_upper <= 1)
@variable(model, 2 <= x_interval <= 3)
@variable(model, x_fixed == 4)
print(model)

# output

Feasibility
Subject to
 x_fixed = 4.0
```

```
x_lower ≥ 0.0
x_interval ≥ 2.0
x_upper ≤ 1.0
x_interval ≤ 3.0
```

Warning

When creating a variable with a single lower- or upper-bound, and the value of the bound is not a numeric literal (for example, `1` or `1.0`), the name of the variable must appear on the left-hand side. Putting the name on the right-hand side is an error. For example, to create a variable `x`:

```
a = 1
@variable(model, x >= 1)      # ✓ Okay
@variable(model, 1.0 <= x)    # ✓ Okay
@variable(model, x >= a)      # ✓ Okay
@variable(model, a <= x)      # × Not okay
@variable(model, x >= 1 / 2)  # ✓ Okay
@variable(model, 1 / 2 <= x)  # × Not okay
```

Containers of variables

The `@variable` macro also supports creating collections of JuMP variables. We'll cover some brief syntax here; read the Variable containers section for more details.

You can create arrays of JuMP variables:

```
julia> @variable(model, x[1:2, 1:2])
2×2 Matrix{VariableRef}:
 x[1,1]  x[1,2]
 x[2,1]  x[2,2]

julia> x[1, 2]
x[1,2]
```

Index sets can be named, and bounds can depend on those names:

```
julia> @variable(model, sqrt(i) <= x[i = 1:3] <= i^2)
3-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]

julia> x[2]
x[2]
```

Sets can be any Julia type that supports iteration:

```
julia> @variable(model, x[i = 2:3, j = 1:2:3, ["red", "blue"]] >= 0)
3-dimensional DenseAxisArray{VariableRef,3,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, 1:2:3
    Dimension 3, ["red", "blue"]
And data, a 2×2×2 Array{VariableRef, 3}:
[:, :, "red"] =
```

```
 x[2,1,red]  x[2,3,red]
 x[3,1,red]  x[3,3,red]

[:, :, "blue"] =
 x[2,1,blue]  x[2,3,blue]
 x[3,1,blue]  x[3,3,blue]

julia> x[2, 1, "red"]
x[2,1,red]
```

Sets can depend upon previous indices:

```
julia> @variable(model, u[i = 1:2, j = i:3])
JuMP.Containers.SparseAxisArray{VariableRef, 2, Tuple{Int64, Int64}} with 5 entries:
  [1, 1]  =  u[1,1]
  [1, 2]  =  u[1,2]
  [1, 3]  =  u[1,3]
  [2, 2]  =  u[2,2]
  [2, 3]  =  u[2,3]
```

and we can filter elements in the sets using the `;` syntax:

```
julia> @variable(model, v[i = 1:9; mod(i, 3) == 0])
JuMP.Containers.SparseAxisArray{VariableRef, 1, Tuple{Int64}} with 3 entries:
  [3]  =  v[3]
  [6]  =  v[6]
  [9]  =  v[9]
```

11.2 Registered variables

When you create variables, JuMP registers them inside the model using their corresponding symbol. Get a registered name using `model[:key]`:

```
julia> model = Model()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> @variable(model, x)
x

julia> model
A JuMP Model
Feasibility problem with:
Variable: 1
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
Names registered in the model: x

julia> model[:x] === x
true
```

Registered names are most useful when you start to write larger models and want to break up the model construction into functions:

```
julia> function set_objective(model::Model)
           @objective(model, Min, 2 * model[:my_x] + 1)
           return
       end
set_objective (generic function with 1 method)

julia> model = Model();

julia> @variable(model, my_x);

julia> set_objective(model)

julia> print(model)
Min 2 my_x + 1
Subject to
```

11.3 Anonymous variables

To reduce the likelihood of accidental bugs, and because JuMP registers variables inside a model, creating two variables with the same name is an error:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @variable(model, x)
ERROR: An object of name x is already attached to this model. If this
    is intended, consider using the anonymous construction syntax, e.g.,
    `x = @variable(model, [1:N], ...)` where the name of the object does
    not appear inside the macro.

    Alternatively, use `unregister(model, :x)` to first unregister
    the existing name from the model. Note that this will not delete the
    object; it will just remove the reference at `model[:x]`.
[...]
```

A common reason for encountering this error is adding variables in a loop.

As a work-around, JuMP provides anonymous variables. Create a scalar valued anonymous variable by omitting the name argument:

```
julia> x = @variable(model)
_[1]
```

Anonymous variables get printed as an underscore followed by a unique index of the variable.

> **Warning**
>
> The index of the variable may not correspond to the column of the variable in the solver!

Create a container of anonymous JuMP variables by dropping the name in front of the `[`:

```
julia> y = @variable(model, [1:2])
2-element Vector{VariableRef}:
 _[1]
 _[2]
```

The <= and >= short-hand cannot be used to set bounds on scalar-valued anonymous JuMP variables. Instead, use the `lower_bound` and `upper_bound` keywords:

```
julia> x_lower = @variable(model, lower_bound = 1.0)
_[1]

julia> x_upper = @variable(model, upper_bound = 2.0)
_[2]

julia> x_interval = @variable(model, lower_bound = 3.0, upper_bound = 4.0)
_[3]
```

11.4 Variable names

In addition to the symbol that variables are registered with, JuMP variables have a `String` name that is used for printing and writing to file formats.

Get and set the name of a variable using `name` and `set_name`:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> name(x)
"x"

julia> set_name(x, "my_x_name")

julia> x
my_x_name
```

Override the default choice of name using the `base_name` keyword:

```
julia> model = Model();

julia> @variable(model, x[i=1:2], base_name = "my_var")
2-element Vector{VariableRef}:
 my_var[1]
 my_var[2]
```

Note that names apply to each element of the container, not to the container of variables:

```
julia> name(x[1])
"my_var[1]"

julia> set_name(x[1], "my_x")

julia> x
2-element Vector{VariableRef}:
 my_x
 my_var[2]
```

Retrieve a variable by name

Retrieve a variable from a model using `variable_by_name`:

```
julia> variable_by_name(model, "my_x")
my_x
```

If the name is not present, `nothing` will be returned:

```
julia> variable_by_name(model, "bad_name")
```

You can only look up individual variables using `variable_by_name`. Something like this will not work:

```
julia> model = Model();

julia> @variable(model, [i = 1:2], base_name = "my_var")
2-element Vector{VariableRef}:
 my_var[1]
 my_var[2]

julia> variable_by_name(model, "my_var")
```

To look up a collection of variables, do not use `variable_by_name`. Instead, register them using the `model[:key] = value` syntax:

```
julia> model = Model();

julia> model[:x] = @variable(model, [i = 1:2], base_name = "my_var")
2-element Vector{VariableRef}:
 my_var[1]
 my_var[2]

julia> model[:x]
2-element Vector{VariableRef}:
 my_var[1]
 my_var[2]
```

11.5 String names, symbolic names, and bindings

It's common for new users to experience confusion relating to JuMP variables. Part of the problem is the overloaded use of "variable" in mathematical optimization, along with the difference between the name that a variable is registered under and the `String` name used for printing.

Here's a summary of the differences:

- JuMP variables are created using `@variable`.
- JuMP variables can be named or anonymous.
- Named JuMP variables have the form `@variable(model, x)`. For named variables:
  - The `String` name of the variable is set to `"x"`.
  - A Julia variable `x` is created that binds `x` to the JuMP variable.
  - The name `:x` is registered as a key in the model with the value `x`.
- Anonymous JuMP variables have the form `x = @variable(model)`. For anonymous variables:
  - The `String` name of the variable is set to `""`. When printed, this is replaced with `"_[i]"` where `i` is the index of the variable.
  - You control the name of the Julia variable used as the binding.
  - No name is registered as a key in the model.
- The `base_name` keyword can override the `String` name of the variable.
- You can manually register names in the model via `model[:key] = value`

Here's an example that should make things clearer:

```
julia> model = Model();

julia> x_binding = @variable(model, base_name = "x")
x

julia> model
A JuMP Model
Feasibility problem with:
Variable: 1
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> x
ERROR: UndefVarError: x not defined

julia> x_binding
x

julia> name(x_binding)
"x"

julia> model[:x_register] = x_binding
x

julia> model
A JuMP Model
Feasibility problem with:
Variable: 1
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
```

```
Solver name: No optimizer attached.
Names registered in the model: x_register

julia> model[:x_register]
x

julia> model[:x_register] === x_binding
true

julia> x
ERROR: UndefVarError: x not defined
```

11.6 Create, delete, and modify variable bounds

Query whether a variable has a bound using `has_lower_bound`, `has_upper_bound`, and `is_fixed`:

```
julia> has_lower_bound(x_free)
false

julia> has_upper_bound(x_upper)
true

julia> is_fixed(x_fixed)
true
```

If a variable has a particular bound, query the value of it using `lower_bound`, `upper_bound`, and `fix_value`:

```
julia> lower_bound(x_interval)
2.0

julia> upper_bound(x_interval)
3.0

julia> fix_value(x_fixed)
4.0
```

Querying the value of a bound that does not exist will result in an error.

Delete variable bounds using `delete_lower_bound`, `delete_upper_bound`, and `unfix`:

```
julia> delete_lower_bound(x_lower)

julia> has_lower_bound(x_lower)
false

julia> delete_upper_bound(x_upper)

julia> has_upper_bound(x_upper)
false

julia> unfix(x_fixed)

julia> is_fixed(x_fixed)
false
```

Set or update variable bounds using `set_lower_bound`, `set_upper_bound`, and `fix`:

```
julia> set_lower_bound(x_lower, 1.1)

julia> set_upper_bound(x_upper, 2.1)

julia> fix(x_fixed, 4.1)
```

Fixing a variable with existing bounds will throw an error. To delete the bounds prior to fixing, use `fix(variable, value; force = true)`.

```
julia> model = Model();

julia> @variable(model, x >= 1)
x

julia> fix(x, 2)
ERROR: Unable to fix x to 2 because it has existing variable bounds. Consider calling
↪  `JuMP.fix(variable, value; force=true)` which will delete existing bounds before fixing the
↪  variable.

julia> fix(x, 2; force = true)


julia> fix_value(x)
2.0
```

Tip

Use `fix` instead of `@constraint(model, x == 2)`. The former modifies variable bounds, while the latter adds a new linear constraint to the problem.

11.7 Binary variables

Binary variables are constrained to the set $x \in \{0, 1\}$.

Create a binary variable by passing `Bin` as an optional positional argument:

```
julia> @variable(model, x, Bin)
x
```

Check if a variable is binary using `is_binary`:

```
julia> is_binary(x)
true
```

Delete a binary constraint using `unset_binary`:

```
julia> unset_binary(x)

julia> is_binary(x)
false
```

Binary variables can also be created by setting the `binary` keyword to `true`:

```
julia> @variable(model, x, binary=true)
x
```

or by using `set_binary`:

```
julia> @variable(model, x)
x

julia> set_binary(x)
```

11.8 Integer variables

Integer variables are constrained to the set $x \in \mathbb{Z}$.

Create an integer variable by passing `Int` as an optional positional argument:

```
julia> @variable(model, x, Int)
x
```

Check if a variable is integer using `is_integer`:

```
julia> is_integer(x)
true
```

Delete an integer constraint using `unset_integer`.

```
julia> unset_integer(x)

julia> is_integer(x)
false
```

Integer variables can also be created by setting the `integer` keyword to `true`:

```
julia> @variable(model, x, integer=true)
x
```

or by using `set_integer`:

```
julia> @variable(model, x)
x

julia> set_integer(x)
```

Tip

The `relax_integrality` function relaxes all integrality constraints in the model, returning a function that can be called to undo the operation later on.

11.9 Start values

There are two ways to provide a primal starting solution (also called MIP-start or a warmstart) for each variable:

- using the `start` keyword in the `@variable` macro
- using `set_start_value`

The starting value of a variable can be queried using `start_value`. If no start value has been set, `start_value` will return `nothing`.

```
julia> @variable(model, x)
x

julia> start_value(x)

julia> @variable(model, y, start = 1)
y

julia> start_value(y)
1.0

julia> set_start_value(y, 2)

julia> start_value(y)
2.0
```

Warning

Some solvers do not support start values. If a solver does not support start values, an `MathOptInterface.UnsupportedAttr` error will be thrown.

Tip

To set the optimal solution from a previous solve as a new starting value, use `all_variables` to get a vector of all the variables in the model, then run:

```
x = all_variables(model)
x_solution = value.(x)
set_start_value.(x, x_solution)
```

11.10 Delete a variable

Use `delete` to delete a variable from a model. Use `is_valid` to check if a variable belongs to a model and has not been deleted.

```
julia> @variable(model, x)
x

julia> is_valid(model, x)
true

julia> delete(model, x)

julia> is_valid(model, x)
false
```

Deleting a variable does not unregister the corresponding name from the model. Therefore, creating a new variable of the same name will throw an error:

```
julia> @variable(model, x)
ERROR: An object of name x is already attached to this model. If this
    is intended, consider using the anonymous construction syntax, e.g.,
    `x = @variable(model, [1:N], ...)` where the name of the object does
    not appear inside the macro.

    Alternatively, use `unregister(model, :x)` to first unregister
    the existing name from the model. Note that this will not delete the
    object; it will just remove the reference at `model[:x]`.
[...]
```

After calling `delete`, call `unregister` to remove the symbolic reference:

```
julia> unregister(model, :x)

julia> @variable(model, x)
x
```

Info

`delete` does not automatically `unregister` because we do not distinguish between names that are automatically registered by JuMP macros and names that are manually registered by the user by setting values in `object_dictionary`. In addition, deleting a variable and then adding a new variable of the same name is an easy way to introduce bugs into your code.

11.11 Variable containers

JuMP provides a mechanism for creating collections of variables in three types of data structures, which we refer to as containers.

The three types are `Arrays`, `DenseAxisArrays`, and `SparseAxisArrays`. We explain each of these in the following.

Tip

You can read more about containers in the Containers section.

Arrays

We have already seen the creation of an array of JuMP variables with the `x[1:2]` syntax. This can be extended to create multi-dimensional arrays of JuMP variables. For example:

```
julia> @variable(model, x[1:2, 1:2])
2×2 Matrix{VariableRef}:
 x[1,1]  x[1,2]
 x[2,1]  x[2,2]
```

Arrays of JuMP variables can be indexed and sliced as follows:

```
julia> x[1, 2]
x[1,2]

julia> x[2, :]
2-element Vector{VariableRef}:
 x[2,1]
 x[2,2]
```

Variable bounds can depend upon the indices:

```
julia> @variable(model, x[i=1:2, j=1:2] >= 2i + j)
2×2 Matrix{VariableRef}:
 x[1,1]  x[1,2]
 x[2,1]  x[2,2]

julia> lower_bound.(x)
2×2 Matrix{Float64}:
 3.0  4.0
 5.0  6.0
```

JuMP will form an `Array` of JuMP variables when it can determine at compile time that the indices are one-based integer ranges. Therefore `x[1:b]` will create an `Array` of JuMP variables, but `x[a:b]` will not. If JuMP cannot determine that the indices are one-based integer ranges (for example, in the case of `x[a:b]`), JuMP will create a `DenseAxisArray` instead.

DenseAxisArrays

We often want to create arrays where the indices are not one-based integer ranges. For example, we may want to create a variable indexed by the name of a product or a location. The syntax is the same as that above, except with an arbitrary vector as an index as opposed to a one-based range. The biggest difference is that instead of returning an `Array` of JuMP variables, JuMP will return a `DenseAxisArray`. For example:

```
julia> @variable(model, x[1:2, [:A,:B]])
2-dimensional DenseAxisArray{VariableRef,2,...} with index sets:
    Dimension 1, Base.OneTo(2)
    Dimension 2, [:A, :B]
And data, a 2×2 Matrix{VariableRef}:
 x[1,A]  x[1,B]
 x[2,A]  x[2,B]
```

DenseAxisArrays can be indexed and sliced as follows:

```
julia> x[1, :A]
x[1,A]

julia> x[2, :]
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, [:A, :B]
And data, a 2-element Vector{VariableRef}:
 x[2,A]
 x[2,B]
```

Bounds can depend upon indices:

```
julia> @variable(model, x[i=2:3, j=1:2:3] >= 0.5i + j)
2-dimensional DenseAxisArray{VariableRef,2,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, 1:2:3
And data, a 2×2 Matrix{VariableRef}:
 x[2,1]  x[2,3]
 x[3,1]  x[3,3]

julia> lower_bound.(x)
2-dimensional DenseAxisArray{Float64,2,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, 1:2:3
And data, a 2×2 Matrix{Float64}:
 2.0  4.0
 2.5  4.5
```

SparseAxisArrays

The third container type that JuMP natively supports is `SparseAxisArray`. These arrays are created when the indices do not form a rectangular set. For example, this applies when indices have a dependence upon previous indices (called triangular indexing). JuMP supports this as follows:

```
julia> @variable(model, x[i=1:2, j=i:2])
JuMP.Containers.SparseAxisArray{VariableRef, 2, Tuple{Int64, Int64}} with 3 entries:
  [1, 1]  =  x[1,1]
  [1, 2]  =  x[1,2]
  [2, 2]  =  x[2,2]
```

We can also conditionally create variables via a JuMP-specific syntax. This syntax appends a comparison check that depends upon the named indices and is separated from the indices by a semi-colon (`;`). For example:

```
julia> @variable(model, x[i=1:4; mod(i, 2)==0])
JuMP.Containers.SparseAxisArray{VariableRef, 1, Tuple{Int64}} with 2 entries:
  [2]  =  x[2]
  [4]  =  x[4]
```

Performance considerations

When using the semi-colon as a filter, JuMP iterates over all indices and evaluates the conditional for each combination. If there are many index dimensions and a large amount of sparsity, this can be inefficient.

For example:

```
julia> N = 10
10

julia> S = [(1, 1, 1), (N, N, N)]
2-element Vector{Tuple{Int64, Int64, Int64}}:
 (1, 1, 1)
 (10, 10, 10)

julia> @time @variable(model, x1[i=1:N, j=1:N, k=1:N; (i, j, k) in S])
  0.203861 seconds (392.22 k allocations: 23.977 MiB, 99.10% compilation time)
JuMP.Containers.SparseAxisArray{VariableRef, 3, Tuple{Int64, Int64, Int64}} with 2 entries:
```

```
  [1, 1, 1   ]  =  x1[1,1,1]
  [10, 10, 10]  =  x1[10,10,10]

julia> @time @variable(model, x2[S])
  0.045407 seconds (65.24 k allocations: 3.771 MiB, 99.15% compilation time)
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, [(1, 1, 1), (10, 10, 10)]
And data, a 2-element Vector{VariableRef}:
 x2[(1, 1, 1)]
 x2[(10, 10, 10)]
```

The first option is slower because it is equivalent to:

```
x1 = Dict()
for i in 1:N
    for j in 1:N
        for k in 1:N
            if (i, j, k) in S
                x1[i, j, k] = @variable(model)
            end
        end
    end
end
```

If performance is a concern, explicitly construct the set of indices instead of using the filtering syntax.

Forcing the container type

When creating a container of JuMP variables, JuMP will attempt to choose the tightest container type that can store the JuMP variables. Thus, it will prefer to create an Array before a DenseAxisArray and a DenseAxisArray before a SparseAxisArray. However, because this happens at compile time, JuMP does not always make the best choice. To illustrate this, consider the following example:

```
julia> A = 1:2
1:2

julia> @variable(model, x[A])
1-dimensional DenseAxisArray{VariableRef,1,...} with index sets:
    Dimension 1, 1:2
And data, a 2-element Vector{VariableRef}:
 x[1]
 x[2]
```

Since the value (and type) of `A` is unknown at parsing time, JuMP is unable to infer that `A` is a one-based integer range. Therefore, JuMP creates a `DenseAxisArray`, even though it could store these two variables in a standard one-dimensional `Array`.

We can share our knowledge that it is possible to store these JuMP variables as an array by setting the `container` keyword:

```
julia> @variable(model, y[A], container=Array)
2-element Vector{VariableRef}:
 y[1]
 y[2]
```

JuMP now creates a vector of JuMP variables instead of a DenseAxisArray. Choosing an invalid container type will throw an error.

User-defined containers

In addition to the built-in container types, you can create your own collections of JuMP variables.

> **Tip**
>
> This is a point that users often overlook: you are not restricted to the built-in container types in JuMP.

For example, the following code creates a dictionary with symmetric matrices as the values:

```
julia> variables = Dict{Symbol,Array{VariableRef,2}}(
           key => @variable(model, [1:2, 1:2], Symmetric, base_name = "$(key)")
           for key in [:A, :B]
       )
Dict{Symbol, Matrix{VariableRef}} with 2 entries:
  :A => [A[1,1] A[1,2]; A[1,2] A[2,2]]
  :B => [B[1,1] B[1,2]; B[1,2] B[2,2]]
```

Another common scenario is a request to add variables to existing containers, for example:

```
using JuMP
model = Model()
@variable(model, x[1:2] >= 0)
# Later I want to add
@variable(model, x[3:4] >= 0)
```

This is not possible with the built-in JuMP container types. However, you can use regular Julia types instead:

```
model = Model()
x = model[:x] = @variable(model, [1:2], lower_bound = 0, base_name = "x")
append!(x, @variable(model, [1:2], lower_bound = 0, base_name = "y"))
model[:x]

# output

4-element Vector{VariableRef}:
 x[1]
 x[2]
 y[1]
 y[2]
```

11.12 Semidefinite variables

A square symmetric matrix X is positive semidefinite if all eigenvalues are nonnegative.

Declare a matrix of JuMP variables to be positive semidefinite by passing PSD as an optional positional argument:

```
julia> @variable(model, x[1:2, 1:2], PSD)
2×2 LinearAlgebra.Symmetric{VariableRef, Matrix{VariableRef}}:
 x[1,1]  x[1,2]
 x[1,2]  x[2,2]
```

> **Note**
>
> x must be a square 2-dimensional `Array` of JuMP variables; it cannot be a DenseAxisArray or a SparseAxisArray.

11.13 Symmetric variables

Declare a square matrix of JuMP variables to be symmetric by passing `Symmetric` as an optional positional argument:

```
julia> @variable(model, x[1:2, 1:2], Symmetric)
2×2 LinearAlgebra.Symmetric{VariableRef, Matrix{VariableRef}}:
 x[1,1]  x[1,2]
 x[1,2]  x[2,2]
```

11.14 The `@variables` macro

If you have many `@variable` calls, JuMP provides the macro `@variables` that can improve readability:

```
julia> @variables(model, begin
           x
           y[i=1:2] >= i, (start = i, base_name = "Y_$i")
           z, Bin
       end)
(x, VariableRef[Y_1[1], Y_2[2]], z)

julia> print(model)
Feasibility
Subject to
 Y_1[1] ≥ 1.0
 Y_2[2] ≥ 2.0
 z binary
```

The `@variables` macro returns a tuple of the variables that were defined.

> **Note**
>
> Keyword arguments must be contained within parentheses.

11.15 Variables constrained on creation

All uses of the `@variable` macro documented so far translate into separate calls for variable creation and the adding of any bound or integrality constraints.

For example, `@variable(model, x >= 0, Int)`, is equivalent to:

```
@variable(model, x)
set_lower_bound(x, 0.0)
set_integer(x)
```

Importantly, the bound and integrality constraints are added after the variable has been created.

However, some solvers require a set specifying the variable domain to be given when the variable is first created. We say that these variables are constrained on creation.

Use `in` within `@variable` to access the special syntax for constraining variables on creation.

For example, the following creates a vector of variables that belong to the `SecondOrderCone`:

```
julia> @variable(model, y[1:3] in SecondOrderCone())
3-element Vector{VariableRef}:
 y[1]
 y[2]
 y[3]
```

For contrast, the standard syntax is as follows:

```
julia> @variable(model, x[1:3])
3-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]

julia> @constraint(model, x in SecondOrderCone())
[x[1], x[2], x[3]] ∈ MathOptInterface.SecondOrderCone(3)
```

An alternate syntax to `x in Set` is to use the `set` keyword of `@variable`. This is most useful when creating anonymous variables:

```
x = @variable(model, [1:3], set = SecondOrderCone())
```

Note

You cannot delete the constraint associated with a variable constrained on creation.

Example: positive semidefinite variables

An alternative to the syntax in Semidefinite variables, declare a matrix of JuMP variables to be positive semidefinite using `PSDCone`:

```
julia> @variable(model, x[1:2, 1:2] in PSDCone())
2×2 LinearAlgebra.Symmetric{VariableRef, Matrix{VariableRef}}:
 x[1,1]  x[1,2]
 x[1,2]  x[2,2]
```

Example: symmetric variables

As an alternative to the syntax in Symmetric variables, declare a matrix of JuMP variables to be symmetric using `SymmetricMatrixSpace`:

```
julia> @variable(model, x[1:2, 1:2] in SymmetricMatrixSpace())
2×2 LinearAlgebra.Symmetric{VariableRef, Matrix{VariableRef}}:
 x[1,1]  x[1,2]
 x[1,2]  x[2,2]
```

Example: skew-symmetric variables

Declare a matrix of JuMP variables to be skew-symmetric using `SkewSymmetricMatrixSpace`:

```
julia> @variable(model, x[1:2, 1:2] in SkewSymmetricMatrixSpace())
2×2 Matrix{AffExpr}:
 0        x[1,2]
 -x[1,2]  0
```

Note

Even though x is a 2 by 2 matrix, only one decision variable is added to `model`; the remaining elements in x are linear transformations of the single variable.

Because the returned matrix x is `Matrix{AffExpr}`, you cannot use variable-related functions on its elements:

```
julia> set_lower_bound(x[1, 2], 0.0)
ERROR: MethodError: no method matching set_lower_bound(::AffExpr, ::Float64)
[...]
```

However, you can convert the matrix into one in which the upper triangular elements are `VariableRef` and the lower triangular elements are `AffExpr` as follows:

```
julia> y = Union{VariableRef,AffExpr}[
           j > i ? first(keys(x[i, j].terms)) : x[i, j]
           for i in 1:size(x, 1), j in 1:size(x, 2)
       ]
2×2 Matrix{Union{VariableRef, AffExpr}}:
 0        x[1,2]
 -x[1,2]  0

julia> set_lower_bound(y[1, 2], 0.0)
```

Why use variables constrained on creation?

For most users, it does not matter if you use the constrained on creation syntax. Therefore, use whatever syntax you find most convenient.

However, if you use `direct_model`, you may be forced to use the constrained on creation syntax.

The technical difference between variables constrained on creation and the standard JuMP syntax is that variables constrained on creation calls `MOI.add_constrained_variables`, while the standard JuMP syntax calls `MOI.add_variables` and then `MOI.add_constraint`.

Consult the implementation of solver package you are using to see if your solver requires `MOI.add_constrained_variables`.

Chapter 12

Constraints

JuMP is based on the MathOptInterface (MOI) API. Because of this, JuMP uses the following standard form to represent problems:

$$\min_{x\in\mathbb{R}^n} \quad f_0(x) \tag{12.1}$$

$$\text{s.t.} \quad f_i(x) \in \mathcal{S}_i \qquad i = 1\ldots m \tag{12.2}$$

Each constraint, $f_i(x) \in \mathcal{S}_i$, is composed of a function and a set. For example, instead of calling $a^\top x \leq b$ a less-than-or-equal-to constraint, we say that it is a scalar-affine-in-less-than constraint, where the function $a^\top x$ belongs to the less-than set $(-\infty, b]$. We use the shorthand function-in-set to refer to constraints composed of different types of functions and sets.

This page explains how to write various types of constraints in JuMP. For nonlinear constraints, see Nonlinear Modeling instead.

12.1 Add a constraint

Add a constraint to a JuMP model using the `@constraint` macro. The syntax to use depends on the type of constraint you wish to add.

Add a linear constraint

Create linear constraints using the `@constraint` macro:

```
model = Model()
@variable(model, x[1:3])
@constraint(model, c1, sum(x) <= 1)
@constraint(model, c2, x[1] + 2 * x[3] >= 2)
@constraint(model, c3, sum(i * x[i] for i in 1:3) == 3)
@constraint(model, c4, 4 <= 2 * x[2] <= 5)
print(model)

# output

Feasibility
Subject to
 c3 : x[1] + 2 x[2] + 3 x[3] = 3.0
```

```
c2 : x[1] + 2 x[3] ≥ 2.0
c1 : x[1] + x[2] + x[3] ≤ 1.0
c4 : 2 x[2] ∈ [4.0, 5.0]
```

Normalization

JuMP normalizes constraints by moving all of the terms containing variables to the left-hand side and all of the constant terms to the right-hand side. Thus, we get:

```
julia> @constraint(model, c, 2x + 1 <= 4x + 4)
c : -2 x <= 3.0
```

Add a quadratic constraint

In addition to affine functions, JuMP also supports constraints with quadratic terms. For example:

```
julia> @variable(model, x[i=1:2])
2-element Vector{VariableRef}:
 x[1]
 x[2]

julia> @variable(model, t >= 0)
t

julia> @constraint(model, my_q, x[1]^2 + x[2]^2 <= t^2)
my_q : x[1]² + x[2]² - t² <= 0.0
```

> **Tip**
>
> Because solvers can take advantage of the knowledge that a constraint is quadratic, prefer adding quadratic constraints using `@constraint`, rather than `@NLconstraint`.

Vectorized constraints

You can also add constraints to JuMP using vectorized linear algebra. For example:

```
julia> @variable(model, x[i=1:2])
2-element Vector{VariableRef}:
 x[1]
 x[2]

julia> A = [1 2; 3 4]
2×2 Matrix{Int64}:
 1  2
 3  4

julia> b = [5, 6]
2-element Vector{Int64}:
 5
 6

julia> @constraint(model, con, A * x .== b)
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.EqualTo{Float64}}, ScalarShape}}:
```

```
con : x[1] + 2 x[2] = 5.0
con : 3 x[1] + 4 x[2] = 6.0
```

Note

Make sure to use Julia's dot syntax in front of the comparison operators (for example, `.==`, `.>=`, and `.<=`). If you use a comparison without the dot, an error will be thrown.

Containers of constraints

The `@constraint` macro supports creating collections of constraints. We'll cover some brief syntax here; read the Constraint containers section for more details:

Create arrays of constraints:

```
julia> @constraint(model, c[i=1:3], x[i] <= i^2)
3-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 c[1] : x[1] ≤ 1.0
 c[2] : x[2] ≤ 4.0
 c[3] : x[3] ≤ 9.0

julia> c[2]
c[2] : x[2] ≤ 4.0
```

Sets can be any Julia type that supports iteration:

```
julia> @constraint(model, c[i=2:3, ["red", "blue"]], x[i] <= i^2)
2-dimensional DenseAxisArray{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape},2,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, ["red", "blue"]
And data, a 2×2 Matrix{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 c[2,red] : x[2] ≤ 4.0  c[2,blue] : x[2] ≤ 4.0
 c[3,red] : x[3] ≤ 9.0  c[3,blue] : x[3] ≤ 9.0

julia> c[2, "red"]
c[2,red] : x[2] ≤ 4.0
```

Sets can depend upon previous indices:

```
julia> @constraint(model, c[i=1:3, j=i:3], x[i] <= j)
JuMP.Containers.SparseAxisArray{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}, 2, Tuple{Int64, Int64}} with 6 entries:
  [1, 1]  =  c[1,1] : x[1] ≤ 1.0
  [1, 2]  =  c[1,2] : x[1] ≤ 2.0
  [1, 3]  =  c[1,3] : x[1] ≤ 3.0
  [2, 2]  =  c[2,2] : x[2] ≤ 2.0
  [2, 3]  =  c[2,3] : x[2] ≤ 3.0
  [3, 3]  =  c[3,3] : x[3] ≤ 3.0
```

and you can filter elements in the sets using the ; syntax:

```
julia> @constraint(model, c[i=1:9; mod(i, 3) == 0], x[i] <= i)
JuMP.Containers.SparseAxisArray{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}, 1, Tuple{Int64}} with 3 entries:
  [3]  =  c[3] : x[3] ≤ 3.0
  [6]  =  c[6] : x[6] ≤ 6.0
  [9]  =  c[9] : x[9] ≤ 9.0
```

12.2 Registered constraints

When you create constraints, JuMP registers them inside the model using their corresponding symbol. Get a registered name using `model[:key]`:

```
julia> model = Model()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> @variable(model, x)
x

julia> @constraint(model, my_c, 2x <= 1)
my_c : 2 x ≤ 1.0

julia> model
A JuMP Model
Feasibility problem with:
Variable: 1
`AffExpr`-in-`MathOptInterface.LessThan{Float64}`: 1 constraint
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
Names registered in the model: my_c, x

julia> model[:my_c] === my_c
true
```

12.3 Anonymous constraints

To reduce the likelihood of accidental bugs, and because JuMP registers constraints inside a model, creating two constraints with the same name is an error:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @constraint(model, c, 2x <= 1)
```

```
c : 2 x <= 1.0

julia> @constraint(model, c, 2x <= 1)
ERROR: An object of name c is already attached to this model. If this
    is intended, consider using the anonymous construction syntax, e.g.,
    `x = @variable(model, [1:N], ...)` where the name of the object does
    not appear inside the macro.

    Alternatively, use `unregister(model, :c)` to first unregister
    the existing name from the model. Note that this will not delete the
    object; it will just remove the reference at `model[:c]`.
[...]
```

A common reason for encountering this error is adding constraints in a loop.

As a work-around, JuMP provides anonymous constraints. Create an anonymous constraint by omitting the name argument:

```
julia> c = @constraint(model, 2x <= 1)
2 x <= 1.0
```

Create a container of anonymous constraints by dropping the name in front of the `[`:

```
julia> c = @constraint(model, [i = 1:3], x[i] <= i)
3-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 x[1] ≤ 1.0
 x[2] ≤ 2.0
 x[3] ≤ 3.0
```

12.4 Constraint names

In addition to the symbol that constraints are registered with, constraints have a `String` name that is used for printing and writing to file formats.

Get and set the name of a constraint using `name(::JuMP.ConstraintRef)` and `set_name(::JuMP.ConstraintRef, ::String)`:

```
julia> model = Model(); @variable(model, x);

julia> @constraint(model, con, x <= 1)
con : x <= 1.0

julia> name(con)
"con"

julia> set_name(con, "my_con_name")

julia> con
my_con_name : x <= 1.0
```

Override the default choice of name using the `base_name` keyword:

```
julia> model = Model(); @variable(model, x);

julia> con = @constraint(model, [i=1:2], x <= i, base_name = "my_con")
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 my_con[1] : x ≤ 1.0
 my_con[2] : x ≤ 2.0
```

Note that names apply to each element of the container, not to the container of constraints:

```
julia> name(con[1])
"my_con[1]"

julia> set_name(con[1], "c")

julia> con
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 c : x ≤ 1.0
 my_con[2] : x ≤ 2.0
```

Retrieve a constraint by name

Retrieve a constraint from a model using `constraint_by_name`:

```
julia> constraint_by_name(model, "c")
c : x ≤ 1.0
```

If the name is not present, `nothing` will be returned:

```
julia> constraint_by_name(model, "bad_name")
```

You can only look up individual constraints using `constraint_by_name`. Something like this will not work:

```
julia> model = Model(); @variable(model, x);

julia> con = @constraint(model, [i=1:2], x <= i, base_name = "my_con")
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 my_con[1] : x ≤ 1.0
 my_con[2] : x ≤ 2.0

julia> constraint_by_name(model, "my_con")
```

To look up a collection of constraints, do not use `constraint_by_name`. Instead, register them using the `model[:key] = value` syntax:

```
julia> model = Model(); @variable(model, x);

julia> model[:con] = @constraint(model, [i=1:2], x <= i, base_name = "my_con")
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 my_con[1] : x ≤ 1.0
 my_con[2] : x ≤ 2.0

julia> model[:con]
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 my_con[1] : x ≤ 1.0
 my_con[2] : x ≤ 2.0
```

12.5 String names, symbolic names, and bindings

It's common for new users to experience confusion relating to constraints. Part of the problem is the difference between the name that a constraint is registered under and the `String` name used for printing.

Here's a summary of the differences:

- Constraints are created using `@constraint`.
- Constraints can be named or anonymous.
- Named constraints have the form `@constraint(model, c, expr)`. For named constraints:
  - The `String` name of the constraint is set to `"c"`.
  - A Julia variable `c` is created that binds `c` to the JuMP constraint.
  - The name `:c` is registered as a key in the model with the value `c`.
- Anonymous constraints have the form `c = @constraint(model, expr)`. For anonymous constraints:
  - The `String` name of the constraint is set to `""`.
  - You control the name of the Julia variable used as the binding.
  - No name is registered as a key in the model.
- The `base_name` keyword can override the `String` name of the constraint.
- You can manually register names in the model via `model[:key] = value`.

Here's an example of the differences:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> c_binding = @constraint(model, 2x <= 1, base_name = "c")
c : 2 x <= 1.0
```

```
julia> model
A JuMP Model
Feasibility problem with:
Variable: 1
`AffExpr`-in-`MathOptInterface.LessThan{Float64}`: 1 constraint
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
Names registered in the model: x

julia> c
ERROR: UndefVarError: c not defined

julia> c_binding
c : 2 x <= 1.0

julia> name(c_binding)
"c"

julia> model[:c_register] = c_binding
c : 2 x <= 1.0

julia> model
A JuMP Model
Feasibility problem with:
Variable: 1
`AffExpr`-in-`MathOptInterface.LessThan{Float64}`: 1 constraint
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
Names registered in the model: c_register, x

julia> model[:c_register]
c : 2 x <= 1.0

julia> model[:c_register] === c_binding
true

julia> c
ERROR: UndefVarError: c not defined
```

12.6 The `@constraints` macro

If you have many `@constraint` calls, use the `@constraints` macro to improve readability:

```
julia> @constraints(model, begin
           2x <= 1
           c, x >= -1
       end)
(2 x ≤ 1.0, c : x ≥ -1.0)

julia> print(model)
Feasibility
Subject to
```

```
c : x ≥ -1.0
2 x ≤ 1.0
```

The @constraints macro returns a tuple of the constraints that were defined.

12.7 Duality

JuMP adopts the notion of conic duality from MathOptInterface. For linear programs, a feasible dual on a >= constraint is nonnegative and a feasible dual on a <= constraint is nonpositive. If the constraint is an equality constraint, it depends on which direction is binding.

> **Warning**
>
> JuMP's definition of duality is independent of the objective sense. That is, the sign of feasible duals associated with a constraint depends on the direction of the constraint and not whether the problem is maximization or minimization. **This is a different convention from linear programming duality in some common textbooks.** If you have a linear program, and you want the textbook definition, you probably want to use `shadow_price` and `reduced_cost` instead.

The dual value associated with a constraint in the most recent solution can be accessed using the `dual` function. Use `has_duals` to check if the model has a dual solution available to query. For example:

```
julia> model = Model(HiGHS.Optimizer);

julia> set_silent(model)

julia> @variable(model, x)
x

julia> @constraint(model, con, x <= 1)
con : x <= 1.0

julia> @objective(model, Min, -2x)
-2 x

julia> has_duals(model)
false

julia> optimize!(model)

julia> has_duals(model)
true

julia> dual(con)
-2.0

julia> @objective(model, Max, 2x)
2 x

julia> optimize!(model)

julia> dual(con)
-2.0
```

To help users who may be less familiar with conic duality, JuMP provides shadow_price, which returns a value that can be interpreted as the improvement in the objective in response to an infinitesimal relaxation (on the scale of one unit) in the right-hand side of the constraint. shadow_price can be used only on linear constraints with a <=, >=, or == comparison operator.

In the example above, dual(con) returned -2.0 regardless of the optimization sense. However, in the second case when the optimization sense is Max, shadow_price returns:

```
julia> shadow_price(con)
2.0
```

Duals of variable bounds

To query the dual variables associated with a variable bound, first obtain a constraint reference using one of UpperBoundRef, LowerBoundRef, or FixRef, and then call dual on the returned constraint reference. The reduced_cost function may simplify this process as it returns the shadow price of an active bound of a variable (or zero, if no active bound exists).

```
julia> model = Model(HiGHS.Optimizer);

julia> set_silent(model)

julia> @variable(model, x <= 1)
x

julia> @objective(model, Min, -2x)
-2 x

julia> optimize!(model)

julia> dual(UpperBoundRef(x))
-2.0

julia> reduced_cost(x)
-2.0
```

12.8 Modify a constant term

This section explains how to modify the constant term in a constraint. There are multiple ways to achieve this goal; we explain three options.

Option 1: change the right-hand side

Use set_normalized_rhs to modify the right-hand side (constant) term of a linear or quadratic constraint. Use normalized_rhs to query the right-hand side term.

```
julia> @constraint(model, con, 2x <= 1)
con : 2 x <= 1.0

julia> set_normalized_rhs(con, 3)

julia> con
con : 2 x <= 3.0
```

```
julia> normalized_rhs(con)
3.0
```

Warning

set_normalized_rhs sets the right-hand side term of the normalized constraint. See Normalization for more details.

Option 2: use fixed variables

If constraints are complicated, for example, they are composed of a number of components, each of which has a constant term, then it may be difficult to calculate what the right-hand side term is in the standard form.

For this situation, JuMP includes the ability to fix variables to a value using the `fix` function. Fixing a variable sets its lower and upper bound to the same value. Thus, changes in a constant term can be simulated by adding a new variable and fixing it to different values. Here is an example:

```
julia> @variable(model, const_term)
const_term

julia> @constraint(model, con, 2x <= const_term + 1)
con : 2 x - const_term <= 1.0

julia> fix(const_term, 1.0)
```

The constraint con is now equivalent to 2x <= 2.

Warning

Fixed variables are not replaced with constants when communicating the problem to a solver. Therefore, even though `const_term` is fixed, it is still a decision variable, and so `const_term * x` is bilinear.

Option 3: modify the function's constant term

The third option is to use `add_to_function_constant`. The constant given is added to the function of a `func-in-set` constraint. In the following example, adding 2 to the function has the effect of removing 2 to the right-hand side:

```
julia> @constraint(model, con, 2x <= 1)
con : 2 x <= 1.0

julia> add_to_function_constant(con, 2)

julia> con
con : 2 x <= -1.0

julia> normalized_rhs(con)
-1.0
```

In the case of interval constraints, the constant is removed from each bound:

```
julia> @constraint(model, con, 0 <= 2x + 1 <= 2)
con : 2 x ∈ [-1.0, 1.0]

julia> add_to_function_constant(con, 3)

julia> con
con : 2 x ∈ [-4.0, -2.0]
```

12.9 Modify a variable coefficient

To modify the coefficients for a linear term (modifying the coefficient of a quadratic term is not supported) in a constraint, use `set_normalized_coefficient`. To query the current coefficient, use `normalized_coefficient`.

```
julia> @constraint(model, con, 2x[1] + x[2] <= 1)
con : 2 x[1] + x[2] ≤ 1.0

julia> set_normalized_coefficient(con, x[2], 0)

julia> con
con : 2 x[1] ≤ 1.0

julia> normalized_coefficient(con, x[2])
0.0
```

Warning

`set_normalized_coefficient` sets the coefficient of the normalized constraint. See Normalization for more details.

12.10 Delete a constraint

Use `delete` to delete a constraint from a model. Use `is_valid` to check if a constraint belongs to a model and has not been deleted.

```
julia> @constraint(model, con, 2x <= 1)
con : 2 x <= 1.0

julia> is_valid(model, con)
true

julia> delete(model, con)

julia> is_valid(model, con)
false
```

Deleting a constraint does not unregister the symbolic reference from the model. Therefore, creating a new constraint of the same name will throw an error:

```
julia> @constraint(model, con, 2x <= 1)
ERROR: An object of name con is already attached to this model. If this
    is intended, consider using the anonymous construction syntax, e.g.,
    `x = @variable(model, [1:N], ...)` where the name of the object does
    not appear inside the macro.
```

```
    Alternatively, use `unregister(model, :con)` to first unregister
    the existing name from the model. Note that this will not delete the
    object; it will just remove the reference at `model[:con]`.
[...]
```

After calling `delete`, call `unregister` to remove the symbolic reference:

```
julia> unregister(model, :con)

julia> @constraint(model, con, 2x <= 1)
con : 2 x <= 1.0
```

Info

`delete` does not automatically `unregister` because we do not distinguish between names that are automatically registered by JuMP macros, and names that are manually registered by the user by setting values in `object_dictionary`. In addition, deleting a constraint and then adding a new constraint of the same name is an easy way to introduce bugs into your code.

12.11 Start values

Provide a starting value (also called warmstart) for a constraint's primal and dual solutions using `set_start_value` and `set_dual_start_value`.

Query the starting value for a constraint's primal and dual solution using `start_value` and `dual_start_value`. If no start value has been set, the methods will return `nothing`.

```
julia> @variable(model, x)
x

julia> @constraint(model, con, x >= 10)
con : x ≥ 10.0

julia> start_value(con)

julia> set_start_value(con, 10.0)

julia> start_value(con)
10.0

julia> dual_start_value(con)

julia> set_dual_start_value(con, 2)

julia> dual_start_value(con)
2.0
```

Vector-valued constraints require a vector:

```
julia> @variable(model, x[1:3])
3-element Vector{VariableRef}:
 x[1]
```

```
 x[2]
 x[3]

julia> @constraint(model, con, x in SecondOrderCone())
con : [x[1], x[2], x[3]] in MathOptInterface.SecondOrderCone(3)

julia> dual_start_value(con)

julia> set_dual_start_value(con, [1.0, 2.0, 3.0])

julia> dual_start_value(con)
3-element Vector{Float64}:
 1.0
 2.0
 3.0
```

Tip

For more information, check out the Primal and dual warm-starts tutorial.

12.12 Constraint containers

Like Variable containers, JuMP provides a mechanism for building groups of constraints compactly. References to these groups of constraints are returned in containers. Three types of constraint containers are supported: `Array`s, `DenseAxisArray`s, and `SparseAxisArray`s. We explain each of these in the following.

Tip

You can read more about containers in the Containers section.

Arrays

One way of adding a group of constraints compactly is the following:

```
julia> @constraint(model, con[i = 1:3], i * x <= i + 1)
3-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 con[1] : x ≤ 2.0
 con[2] : 2 x ≤ 3.0
 con[3] : 3 x ≤ 4.0
```

JuMP returns references to the three constraints in an `Array` that is bound to the Julia variable `con`. This array can be accessed and sliced as you would with any Julia array:

```
julia> con[1]
con[1] : x <= 2.0

julia> con[2:3]
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 con[2] : 2 x ≤ 3.0
 con[3] : 3 x ≤ 4.0
```

Anonymous containers can also be constructed by dropping the name (for example, con) before the square brackets:

```
julia> con = @constraint(model, [i = 1:2], i * x <= i + 1)
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 x ≤ 2.0
 2 x ≤ 3.0
```

Just like `@variable`, JuMP will form an `Array` of constraints when it can determine at parse time that the indices are one-based integer ranges. Therefore `con[1:b]` will create an `Array`, but `con[a:b]` will not. A special case is `con[Base.OneTo(n)]` which will produce an `Array`. If JuMP cannot determine that the indices are one-based integer ranges (for example, in the case of `con[a:b]`), JuMP will create a `DenseAxisArray` instead.

DenseAxisArrays

The syntax for constructing a `DenseAxisArray` of constraints is very similar to the syntax for constructing a `DenseAxisArray` of variables.

```
julia> @constraint(model, con[i = 1:2, j = 2:3], i * x <= j + 1)
2-dimensional DenseAxisArray{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape},2,...} with index sets:
    Dimension 1, Base.OneTo(2)
    Dimension 2, 2:3
And data, a 2×2 Matrix{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}}:
 con[1,2] : x ≤ 3.0    con[1,3] : x ≤ 4.0
 con[2,2] : 2 x ≤ 3.0  con[2,3] : 2 x ≤ 4.0
```

SparseAxisArrays

The syntax for constructing a `SparseAxisArray` of constraints is very similar to the syntax for constructing a `SparseAxisArray` of variables.

```
julia> @constraint(model, con[i = 1:2, j = 1:2; i != j], i * x <= j + 1)
JuMP.Containers.SparseAxisArray{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}, ScalarShape}, 2, Tuple{Int64, Int64}} with 2 entries:
  [1, 2]  =  con[1,2] : x ≤ 3.0
  [2, 1]  =  con[2,1] : 2 x ≤ 2.0
```

> **Warning**
>
> If you have many index dimensions and a large amount of sparsity, read Performance considerations.

Forcing the container type

When creating a container of constraints, JuMP will attempt to choose the tightest container type that can store the constraints. However, because this happens at parse time, it does not always make the best choice. Just like in `@variable`, you can force the type of container using the `container` keyword. For syntax and the reason behind this, take a look at the variable docs.

Constraints with similar indices

Containers are often used to create constraints over a set of indices. However, you'll often have cases in which you are repeating the indices:

```
julia> @constraints(model, begin
           [i=1:2, j=1:2, k=1:2], i * x[j] <= k
           [i=1:2, j=1:2, k=1:2], i * y[j] <= k
       end);
```

This is hard to read and leads to a lot of copy-paste. A more readable way is to use a for-loop:

```
julia> for i=1:2, j=1:2, k=1:2
           @constraints(model, begin
               i * x[j] <= k
               i * y[j] <= k
           end)
       end
```

12.13 Accessing constraints from a model

Query the types of function-in-set constraints in a model using `list_of_constraint_types`:

```
julia> model = Model();

julia> @variable(model, x[i=1:2] >= i, Int);

julia> @constraint(model, x[1] + x[2] <= 1);

julia> list_of_constraint_types(model)
3-element Vector{Tuple{Type, Type}}:
 (AffExpr, MathOptInterface.LessThan{Float64})
 (VariableRef, MathOptInterface.GreaterThan{Float64})
 (VariableRef, MathOptInterface.Integer)
```

For a given combination of function and set type, use `num_constraints` to access the number of constraints and `all_constraints` to access a list of their references:

```
julia> num_constraints(model, VariableRef, MOI.Integer)
2

julia> cons = all_constraints(model, VariableRef, MOI.Integer)
2-element Vector{ConstraintRef{Model,
↪  MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex, MathOptInterface.Integer},
↪  ScalarShape}}:
 x[1] integer
 x[2] integer
```

You can also count the total number of constraints in the model, but you must explicitly choose whether to count `VariableRef` constraints such as bound and integrality constraints:

```
julia> num_constraints(model; count_variable_in_set_constraints = true)
5

julia> num_constraints(model; count_variable_in_set_constraints = false)
1
```

If you need finer-grained control on which constraints to include, use a variant of:

```
julia> sum(
           num_constraints(model, F, S) for
           (F, S) in list_of_constraint_types(model) if F != VariableRef
       )
1
```

Use `constraint_object` to get an instance of an `AbstractConstraint` object that stores the constraint data:

```
julia> con = constraint_object(cons[1])
ScalarConstraint{VariableRef, MathOptInterface.Integer}(x[1], MathOptInterface.Integer())

julia> con.func
x[1]

julia> con.set
MathOptInterface.Integer()
```

12.14 MathOptInterface constraints

Because JuMP is based on MathOptInterface, you can add any constraints supported by MathOptInterface using the function-in-set syntax. For a list of supported functions and sets, read Standard form problem.

> **Note**
>
> We use `MOI` as an alias for the `MathOptInterface` module. This alias is defined by `using JuMP`. You may also define it in your code as follows:
>
> ```
> import MathOptInterface
> const MOI = MathOptInterface
> ```

For example, the following two constraints are equivalent:

```
julia> @constraint(model, 2 * x[1] <= 1)
2 x[1] ≤ 1.0

julia> @constraint(model, 2 * x[1] in MOI.LessThan(1.0))
2 x[1] ≤ 1.0
```

You can also use any set defined by MathOptInterface:

```
julia> @constraint(model, x - [1; 2; 3] in MOI.Nonnegatives(3))
[x[1] - 1, x[2] - 2, x[3] - 3] ∈ MathOptInterface.Nonnegatives(3)

julia> @constraint(model, x in MOI.ExponentialCone())
[x[1], x[2], x[3]] ∈ MathOptInterface.ExponentialCone()
```

Info

Similar to how JuMP defines the <= and >= syntax as a convenience way to specify `MOI.LessThan` and `MOI.GreaterThan` constraints, the remaining sections in this page describe functions and syntax that have been added for the convenience of common modeling situations.

12.15 Set inequality syntax

For modeling convenience, the syntax `@constraint(model, x >= y, Set())` is short-hand for `@constraint(model, x - y in Set())`. Therefore, the following calls are equivalent:

```
julia> model = Model();

julia> @variable(model, x[1:2]);

julia> y = [0.5, 0.75];

julia> @constraint(model, x >= y, MOI.Nonnegatives(2))
[x[1] - 0.5, x[2] - 0.75] ∈ MathOptInterface.Nonnegatives(2)

julia> @constraint(model, y <= x, MOI.Nonnegatives(2))
[x[1] - 0.5, x[2] - 0.75] ∈ MathOptInterface.Nonnegatives(2)

julia> @constraint(model, x - y in MOI.Nonnegatives(2))
[x[1] - 0.5, x[2] - 0.75] ∈ MathOptInterface.Nonnegatives(2)
```

Non-zero constants are not supported in this syntax:

```
julia> @constraint(model, x >= 1, MOI.Nonnegatives(2))
ERROR: Operation `sub_mul` between `Vector{VariableRef}` and `Int64` is not allowed. You should use
↪  broadcast.
Stacktrace:
[...]
```

Use instead:

```
julia> @constraint(model, x .- 1 >= 0, MOI.Nonnegatives(2))
[x[1] - 1, x[2] - 1] ∈ MathOptInterface.Nonnegatives(2)
```

12.16 Second-order cone constraints

A `SecondOrderCone` constrains the variables `t` and `x` to the set:

$$||x||_2 \leq t,$$

and $t \geq 0$. It can be added as follows:

```
julia> model = Model();

julia> @variable(model, t)
```

```
t

julia> @variable(model, x[1:2])
2-element Vector{VariableRef}:
 x[1]
 x[2]

julia> @constraint(model, [t; x] in SecondOrderCone())
[t, x[1], x[2]] ∈ MathOptInterface.SecondOrderCone(3)
```

12.17 Rotated second-order cone constraints

A `RotatedSecondOrderCone` constrains the variables `t`, `u`, and `x` to the set:

$$||x||_2^2 \le 2t \cdot u$$

and $t, u \ge 0$. It can be added as follows:

```
julia> model = Model();

julia> @variable(model, t)
t

julia> @variable(model, u)
u

julia> @variable(model, x[1:2])
2-element Vector{VariableRef}:
 x[1]
 x[2]

julia> @constraint(model, [t; u; x] in RotatedSecondOrderCone())
[t, u, x[1], x[2]] ∈ MathOptInterface.RotatedSecondOrderCone(4)
```

12.18 Semi-integer and semi-continuous variables

Semi-continuous variables are constrained to the set $x \in \{0\} \cup [l, u]$.

Create a semi-continuous variable using the `MOI.Semicontinuous` set:

```
julia> @constraint(model, x in MOI.Semicontinuous(1.5, 3.5))
x in MathOptInterface.Semicontinuous{Float64}(1.5, 3.5)
```

Semi-integer variables are constrained to the set x
$in\{0\} \cup \{l, l+1, \dots, u\}$.

Create a semi-integer variable using the `MOI.Semiinteger` set:

```
julia> @constraint(model, x in MOI.Semiinteger(1.0, 3.0))
x in MathOptInterface.Semiinteger{Float64}(1.0, 3.0)
```

12.19 Special Ordered Sets of Type 1

In a Special Ordered Set of Type 1 (often denoted SOS-I or SOS1), at most one element can take a non-zero value.

Construct SOS-I constraints using the `SOS1` set:

```
julia> @variable(model, x[1:3])
3-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]

julia> @constraint(model, x in SOS1())
[x[1], x[2], x[3]] in MathOptInterface.SOS1{Float64}([1.0, 2.0, 3.0])
```

Although not required for feasibility, solvers can benefit from an ordering of the variables (for example, the variables represent different factories to build, at most one factory can be built, and the factories can be ordered according to cost). To induce an ordering, a vector of weights can be provided, and the variables are ordered according to their corresponding weight.

For example, in the constraint:

```
julia> @constraint(model, x in SOS1([3.1, 1.2, 2.3]))
[x[1], x[2], x[3]] in MathOptInterface.SOS1{Float64}([3.1, 1.2, 2.3])
```

the variables `x` have precedence `x[2]`, `x[3]`, `x[1]`.

12.20 Special Ordered Sets of Type 2

In a Special Ordered Set of Type 2 (SOS-II), at most two elements can be non-zero, and if there are two non-zeros, they must be consecutive according to the ordering induced by a weight vector.

Construct SOS-II constraints using the `SOS2` set:

```
julia> @constraint(model, x in SOS2([3.0, 1.0, 2.0]))
[x[1], x[2], x[3]] in MathOptInterface.SOS2{Float64}([3.0, 1.0, 2.0])
```

The possible non-zero pairs are (`x[1]`, `x[3]`) and (`x[2]`, `x[3]`):

If the weight vector is omitted, JuMP induces an ordering from `1:length(x)`:

```
julia> @constraint(model, x in SOS2())
[x[1], x[2], x[3]] in MathOptInterface.SOS2{Float64}([1.0, 2.0, 3.0])
```

12.21 Indicator constraints

Indicator constraints consist of a binary variable and a linear constraint. The constraint holds when the binary variable takes the value 1. The constraint may or may not hold when the binary variable takes the value `0`.

To enforce the constraint `x + y <= 1` when the binary variable `a` is 1, use:

```
julia> @variable(model, x)
x

julia> @variable(model, y)
y

julia> @variable(model, a, Bin)
a

julia> @constraint(model, a => {x + y <= 1})
a => {x + y ≤ 1.0}
```

If the constraint must hold when a is zero, add ! or ¬ before the binary variable;

```
julia> @constraint(model, !a => {x + y <= 1})
!a => {x + y ≤ 1.0}
```

12.22 Semidefinite constraints

To constrain a matrix to be positive semidefinite (PSD), use PSDCone:

```
julia> @variable(model, X[1:2, 1:2])
2×2 Matrix{VariableRef}:
 X[1,1]  X[1,2]
 X[2,1]  X[2,2]

julia> @constraint(model, X >= 0, PSDCone())
[X[1,1]  X[1,2];
 X[2,1]  X[2,2]] ∈ PSDCone()
```

Tip

Where possible, prefer constructing a matrix of Semidefinite variables using the `@variable` macro, rather than adding a constraint like `@constraint(model, X >= 0, PSDCone())`. In some solvers, adding the constraint via `@constraint` is less efficient, and can result in additional intermediate variables and constraints being added to the model.

The inequality `X >= Y` between two square matrices X and Y is understood as constraining `X - Y` to be positive semidefinite.

```
julia> Y = [1 2; 2 1]
2×2 Matrix{Int64}:
 1  2
 2  1

julia> @constraint(model, X >= Y, PSDCone())
[X[1,1] - 1  X[1,2] - 2;
 X[2,1] - 2  X[2,2] - 1] ∈ PSDCone()
```

Symmetry

Solvers supporting PSD constraints usually expect to be given a matrix that is symbolically symmetric, that is, for which the expression in corresponding off-diagonal entries are the same. In our example, the expressions of entries `(1, 2)` and `(2, 1)` are respectively `X[1,2] - 2` and `X[2,1] - 2` which are different.

To bridge the gap between the constraint modeled and what the solver expects, solvers may add an equality constraint `X[1,2] - 2 == X[2,1] - 2` to force symmetry. Use `LinearAlgebra.Symmetric` to explicitly tell the solver that the matrix is symmetric:

```
julia> import LinearAlgebra

julia> Z = [X[1, 1] X[1, 2]; X[1, 2] X[2, 2]]
2×2 Matrix{VariableRef}:
 X[1,1]  X[1,2]
 X[1,2]  X[2,2]

julia> @constraint(model, LinearAlgebra.Symmetric(Z) >= 0, PSDCone())
[X[1,1]  X[1,2];
 X[1,2]  X[2,2]] ∈ PSDCone()
```

Note that the lower triangular entries are ignored even if they are different so use it with caution:

```
julia> @constraint(model, LinearAlgebra.Symmetric(X) >= 0, PSDCone())
[X[1,1]  X[1,2];
 X[1,2]  X[2,2]] ∈ PSDCone()
```

(Note the `(2, 1)` element of the constraint is `X[1,2]`, not `X[2,1]`.)

12.23 Complementarity constraints

A mixed complementarity constraint `F(x) ⊥ x` consists of finding `x` in the interval `[lb, ub]`, such that the following holds:

- `F(x) == 0` if `lb < x < ub`
- `F(x) >= 0` if `lb == x`
- `F(x) <= 0` if `x == ub`

JuMP supports mixed complementarity constraints via `complements(F(x), x)` or `F(x) ⊥ x` in the `@constraint` macro. The interval set `[lb, ub]` is obtained from the variable bounds on `x`.

For example, to define the problem `2x - 1 ⊥ x` with `x ∈ [0, ∞)`, do:

```
julia> @variable(model, x >= 0)
x

julia> @constraint(model, 2x - 1 ⊥ x)
[2 x - 1, x] ∈ MathOptInterface.Complements(2)
```

This problem has a unique solution at `x = 0.5`.

The perp operator ⊥ can be entered in most editors (and the Julia REPL) by typing `\perp<tab>`.

An alternative approach that does not require the ⊥ symbol uses the `complements` function as follows:

```
julia> @constraint(model, complements(2x - 1, x))
[2 x - 1, x] ∈ MathOptInterface.Complements(2)
```

In both cases, the mapping `F(x)` is supplied as the first argument, and the matching variable `x` is supplied as the second.

Vector-valued complementarity constraints are also supported:

```
julia> @variable(model, -2 <= y[1:2] <= 2)
2-element Vector{VariableRef}:
 y[1]
 y[2]

julia> M = [1 2; 3 4]
2×2 Matrix{Int64}:
 1  2
 3  4

julia> q = [5, 6]
2-element Vector{Int64}:
 5
 6

julia> @constraint(model, M * y + q ⊥ y)
[y[1] + 2 y[2] + 5, 3 y[1] + 4 y[2] + 6, y[1], y[2]] ∈ MathOptInterface.Complements(4)
```

Chapter 13

Expressions

JuMP has three types of expressions: affine, quadratic, and nonlinear. These expressions can be inserted into constraints or into the objective. This is particularly useful if an expression is used in multiple places in the model.

13.1 Affine expressions

There are four ways of constructing an affine expression in JuMP: with the `@expression` macro, with operator overloading, with the `AffExpr` constructor, and with `add_to_expression!`.

Macros

The recommended way to create an affine expression is via the `@expression` macro.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = @expression(model, 2x + y - 1)

# output

2 x + y - 1
```

This expression can be used in the objective or added to a constraint. For example:

```
@objective(model, Min, 2 * ex - 1)
objective_function(model)

# output

4 x + 2 y - 3
```

Just like variables and constraints, named expressions can also be created. For example

```
model = Model()
@variable(model, x[i = 1:3])
@expression(model, expr[i = 1:3], i * sum(x[j] for j in i:3))
expr
```

```
# output

3-element Vector{AffExpr}:
 x[1] + x[2] + x[3]
 2 x[2] + 2 x[3]
 3 x[3]
```

Tip

You can read more about containers in the Containers section.

Operator overloading

Expressions can also be created without macros. However, note that in some cases, this can be much slower that constructing an expression using macros.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = 2x + y - 1

# output

2 x + y - 1
```

Constructors

A third way to create an affine expression is by the `AffExpr` constructor. The first argument is the constant term, and the remaining arguments are variable-coefficient pairs.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = AffExpr(-1.0, x => 2.0, y => 1.0)

# output

2 x + y - 1
```

`add_to_expression!`

The fourth way to create an affine expression is by using `add_to_expression!`. Compared to the operator overloading method, this approach is faster because it avoids constructing temporary objects. The `@expression` macro uses `add_to_expression!` behind-the-scenes.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = AffExpr(-1.0)
add_to_expression!(ex, 2.0, x)
add_to_expression!(ex, 1.0, y)

# output
```

```
2 x + y - 1
```

Warning

Read the section Initializing arrays for some cases to be careful about when using `add_to_expression!`.

Removing zero terms

Use `drop_zeros!` to remove terms from an affine expression with a `0` coefficient.

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @expression(model, ex, x + 1 - x)
0 x + 1

julia> drop_zeros!(ex)

julia> ex
1
```

Coefficients

Use `coefficient` to return the coefficient associated with a variable in an affine expression.

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @variable(model, y)
y

julia> @expression(model, ex, 2x + 1)
2 x + 1

julia> coefficient(ex, x)
2.0

julia> coefficient(ex, y)
0.0
```

13.2 Quadratic expressions

Like affine expressions, there are four ways of constructing a quadratic expression in JuMP: macros, operator overloading, constructors, and `add_to_expression!`.

Macros

The `@expression` macro can be used to create quadratic expressions by including quadratic terms.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = @expression(model, x^2 + 2 * x * y + y^2 + x + y - 1)

# output

x² + 2 y*x + y² + x + y - 1
```

Operator overloading

Operator overloading can also be used to create quadratic expressions. The same performance warning (discussed in the affine expression section) applies.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = x^2 + 2 * x * y + y^2 + x + y - 1

# output

x² + 2 x*y + y² + x + y - 1
```

Constructors

Quadratic expressions can also be created using the `QuadExpr` constructor. The first argument is an affine expression, and the remaining arguments are pairs, where the first term is a `JuMP.UnorderedPair` and the second term is the coefficient.

```
model = Model()
@variable(model, x)
@variable(model, y)
aff_expr = AffExpr(-1.0, x => 1.0, y => 1.0)
quad_expr = QuadExpr(aff_expr, UnorderedPair(x, x) => 1.0,
                     UnorderedPair(x, y) => 2.0, UnorderedPair(y, y) => 1.0)

# output

x² + 2 x*y + y² + x + y - 1
```

`add_to_expression!`

Finally, `add_to_expression!` can also be used to add quadratic terms.

```
model = Model()
@variable(model, x)
@variable(model, y)
ex = QuadExpr(x + y - 1.0)
add_to_expression!(ex, 1.0, x, x)
add_to_expression!(ex, 2.0, x, y)
add_to_expression!(ex, 1.0, y, y)

# output

x² + 2 x*y + y² + x + y - 1
```

Warning

Read the section Initializing arrays for some cases to be careful about when using `add_to_expression!`.

Removing zero terms

Use `drop_zeros!` to remove terms from a quadratic expression with a `0` coefficient.

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @expression(model, ex, x^2 + x + 1 - x^2)
0 x² + x + 1

julia> drop_zeros!(ex)

julia> ex
x + 1
```

Coefficients

Use `coefficient` to return the coefficient associated with a pair of variables in a quadratic expression.

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @variable(model, y)
y

julia> @expression(model, ex, 2*x*y + 3*x)
2 x*y + 3 x

julia> coefficient(ex, x, y)
2.0

julia> coefficient(ex, x, x)
0.0

julia> coefficient(ex, y, x)
2.0

julia> coefficient(ex, x)
3.0
```

13.3 Nonlinear expressions

Nonlinear expressions can be constructed only using the `@NLexpression` macro and can be used only in `@NLobjective`, `@NLconstraint`, and other `@NLexpression`s. Moreover, quadratic and affine expressions cannot be used in the nonlinear macros. For more details, see the Nonlinear Modeling section.

13.4 Initializing arrays

JuMP implements `zero(AffExpr)` and `one(AffExpr)` to support various functions in `LinearAlgebra` (for example, accessing the off-diagonal of a `Diagonal` matrix).

```
julia> zero(AffExpr)
0

julia> one(AffExpr)
1
```

However, this can result in a subtle bug if you call `add_to_expression!` or the MutableArithmetics API on an element created by `zeros` or `ones`:

```
julia> x = zeros(AffExpr, 2)
2-element Vector{AffExpr}:
 0
 0

julia> add_to_expression!(x[1], 1.1)
1.1

julia> x
2-element Vector{AffExpr}:
 1.1
 1.1
```

Notice how we modified `x[1]`, but we also changed `x[2]`!

This happened because `zeros(AffExpr, 2)` calls `zero(AffExpr)` once to obtain a zero element, and then creates an appropriately sized array filled with the same element.

This also happens with broadcasting calls containing a conversion of `0` or `1`:

```
julia> x = Vector{AffExpr}(undef, 2)
2-element Vector{AffExpr}:
 #undef
 #undef

julia> x .= 0
2-element Vector{AffExpr}:
 0
 0

julia> add_to_expression!(x[1], 1.1)
1.1

julia> x
2-element Vector{AffExpr}:
 1.1
 1.1
```

The recommended way to create an array of empty expressions is as follows:

```
julia> x = Vector{AffExpr}(undef, 2)
2-element Vector{AffExpr}:
 #undef
 #undef

julia> for i in eachindex(x)
           x[i] = AffExpr(0.0)
       end

julia> add_to_expression!(x[1], 1.1)
1.1

julia> x
2-element Vector{AffExpr}:
 1.1
 0
```

Alternatively, use non-mutating operation to avoid updating `x[1]` in-place:

```
julia> x = zeros(AffExpr, 2)
2-element Vector{AffExpr}:
 0
 0

julia> x[1] += 1.1
1.1

julia> x
2-element Vector{AffExpr}:
 1.1
 0
```

Note that for large expressions this will be slower due to the allocation of additional temporary objects.

Chapter 14

Objectives

This page describes macros and functions related to linear and quadratic objective functions only, unless otherwise indicated. For nonlinear objective functions, see Nonlinear Modeling.

14.1 Set a linear objective

Use the `@objective` macro to set a linear objective function.

Use `Min` to create a minimization objective:

```
julia> @objective(model, Min, 2x + 1)
2 x + 1
```

Use `Max` to create a maximization objective:

```
julia> @objective(model, Max, 2x + 1)
2 x + 1
```

14.2 Set a quadratic objective

Use the `@objective` macro to set a quadratic objective function.

Use ^2 to have a variable squared:

```
julia> @objective(model, Min, x^2 + 2x + 1)
x² + 2 x + 1
```

You can also have bilinear terms between variables:

```
julia> @variable(model, x)
x

julia> @variable(model, y)
y

julia> @objective(model, Max, x * y + x + y)
x*y + x + y
```

14.3 Query the objective function

Use `objective_function` to return the current objective function.

```
julia> @objective(model, Min, 2x + 1)
2 x + 1

julia> objective_function(model)
2 x + 1
```

14.4 Evaluate the objective function at a point

Use `value` to evaluate an objective function at a point specifying values for variables.

```
julia> @variable(model, x[1:2]);

julia> @objective(model, Min, 2x[1]^2 + x[1] + 0.5*x[2])
2 x[1]² + x[1] + 0.5 x[2]

julia> f = objective_function(model)
2 x[1]² + x[1] + 0.5 x[2]

julia> point = Dict(x[1] => 2.0, x[2] => 1.0);

julia> value(z -> point[z], f)
10.5
```

14.5 Query the objective sense

Use `objective_sense` to return the current objective sense.

```
julia> @objective(model, Min, 2x + 1)
2 x + 1

julia> objective_sense(model)
MIN_SENSE::OptimizationSense = 0
```

14.6 Modify an objective

To modify an objective, call `@objective` with the new objective function.

```
julia> @objective(model, Min, 2x)
2 x

julia> @objective(model, Max, -2x)
-2 x
```

Alternatively, use `set_objective_function`.

```
julia> @objective(model, Min, 2x)
2 x
```

```
julia> new_objective = @expression(model, -2 * x)
-2 x

julia> set_objective_function(model, new_objective)
```

14.7 Modify an objective coefficient

Use `set_objective_coefficient` to modify an objective coefficient.

```
julia> @objective(model, Min, 2x)
2 x

julia> set_objective_coefficient(model, x, 3)

julia> objective_function(model)
3 x
```

Info

There is no way to modify the coefficient of a quadratic term. Set a new objective instead.

14.8 Modify the objective sense

Use `set_objective_sense` to modify the objective sense.

```
julia> @objective(model, Min, 2x)
2 x

julia> objective_sense(model)
MIN_SENSE::OptimizationSense = 0

julia> set_objective_sense(model, MAX_SENSE);

julia> objective_sense(model)
MAX_SENSE::OptimizationSense = 1
```

Alternatively, call `@objective` and pass the existing objective function.

```
julia> @objective(model, Min, 2x)
2 x

julia> @objective(model, Max, objective_function(model))
2 x
```

Chapter 15

Containers

JuMP provides specialized containers similar to AxisArrays that enable multi-dimensional arrays with non-integer indices.

These containers are created automatically by JuMP's macros. Each macro has the same basic syntax:

```
@macroname(model, name[key1=index1, index2; optional_condition], other stuff)
```

The containers are generated by the `name[key1=index1, index2; optional_condition]` syntax. Everything else is specific to the particular macro.

Containers can be named, for example, `name[key=index]`, or unnamed, for example, `[key=index]`. We call unnamed containers anonymous.

We call the bits inside the square brackets and before the `;` the index sets. The index sets can be named, for example, `[i = 1:4]`, or they can be unnamed, for example, `[1:4]`.

We call the bit inside the square brackets and after the `;` the condition. Conditions are optional.

In addition to the standard JuMP macros like `@variable` and `@constraint`, which construct containers of variables and constraints respectively, you can use `Containers.@container` to construct containers with arbitrary elements.

We will use this macro to explain the three types of containers that are natively supported by JuMP: `Array`, `Containers.DenseAxisArray`, and `Containers.SparseAxisArray`.

15.1 Array

An `Array` is created when the index sets are rectangular and the index sets are of the form `1:n`.

```
julia> Containers.@container(x[i = 1:2, j = 1:3], (i, j))
2×3 Matrix{Tuple{Int64, Int64}}:
 (1, 1)  (1, 2)  (1, 3)
 (2, 1)  (2, 2)  (2, 3)
```

The result is a normal Julia `Array`, so you can do all the usual things.

Slicing

Arrays can be sliced

```
julia> x[:, 1]
2-element Vector{Tuple{Int64, Int64}}:
 (1, 1)
 (2, 1)

julia> x[2, :]
3-element Vector{Tuple{Int64, Int64}}:
 (2, 1)
 (2, 2)
 (2, 3)
```

Looping

Use `eachindex` to loop over the elements:

```
julia> for key in eachindex(x)
           println(x[key])
       end
(1, 1)
(2, 1)
(1, 2)
(2, 2)
(1, 3)
(2, 3)
```

Get the index sets

Use `axes` to obtain the index sets:

```
julia> axes(x)
(Base.OneTo(2), Base.OneTo(3))
```

Broadcasting

Broadcasting over an Array returns an Array

```
julia> swap(x::Tuple) = (last(x), first(x))
swap (generic function with 1 method)

julia> swap.(x)
2×3 Matrix{Tuple{Int64, Int64}}:
 (1, 1)  (2, 1)  (3, 1)
 (1, 2)  (2, 2)  (3, 2)
```

15.2 DenseAxisArray

A `Containers.DenseAxisArray` is created when the index sets are rectangular, but not of the form `1:n`. The index sets can be of any type.

```
julia> x = Containers.@container([i = 1:2, j = [:A, :B]], (i, j))
2-dimensional DenseAxisArray{Tuple{Int64, Symbol},2,...} with index sets:
    Dimension 1, Base.OneTo(2)
    Dimension 2, [:A, :B]
And data, a 2×2 Matrix{Tuple{Int64, Symbol}}:
 (1, :A)  (1, :B)
 (2, :A)  (2, :B)
```

Slicing

DenseAxisArrays can be sliced

```
julia> x[:, :A]
1-dimensional DenseAxisArray{Tuple{Int64, Symbol},1,...} with index sets:
    Dimension 1, Base.OneTo(2)
And data, a 2-element Vector{Tuple{Int64, Symbol}}:
 (1, :A)
 (2, :A)

julia> x[1, :]
1-dimensional DenseAxisArray{Tuple{Int64, Symbol},1,...} with index sets:
    Dimension 1, [:A, :B]
And data, a 2-element Vector{Tuple{Int64, Symbol}}:
 (1, :A)
 (1, :B)
```

Looping

Use eachindex to loop over the elements:

```
julia> for key in eachindex(x)
           println(x[key])
       end
(1, :A)
(2, :A)
(1, :B)
(2, :B)
```

Get the index sets

Use axes to obtain the index sets:

```
julia> axes(x)
(Base.OneTo(2), [:A, :B])
```

Broadcasting

Broadcasting over a DenseAxisArray returns a DenseAxisArray

```
julia> swap(x::Tuple) = (last(x), first(x))
swap (generic function with 1 method)

julia> swap.(x)
2-dimensional DenseAxisArray{Tuple{Symbol, Int64},2,...} with index sets:
```

```
    Dimension 1, Base.OneTo(2)
    Dimension 2, [:A, :B]
And data, a 2×2 Matrix{Tuple{Symbol, Int64}}:
 (:A, 1)  (:B, 1)
 (:A, 2)  (:B, 2)
```

Access internal data

Use `Array(x)` to copy the internal data array into a new `Array`:

```
julia> Array(x)
2×2 Matrix{Tuple{Int64, Symbol}}:
 (1, :A)  (1, :B)
 (2, :A)  (2, :B)
```

To access the internal data without a copy, use `x.data`.

```
julia> x.data
2×2 Matrix{Tuple{Int64, Symbol}}:
 (1, :A)  (1, :B)
 (2, :A)  (2, :B)
```

15.3 SparseAxisArray

A `Containers.SparseAxisArray` is created when the index sets are non-rectangular. This occurs in two circumstances:

An index depends on a prior index:

```
julia> Containers.@container([i = 1:2, j = i:2], (i, j))
JuMP.Containers.SparseAxisArray{Tuple{Int64, Int64}, 2, Tuple{Int64, Int64}} with 3 entries:
  [1, 1]  =  (1, 1)
  [1, 2]  =  (1, 2)
  [2, 2]  =  (2, 2)
```

The `[indices; condition]` syntax is used:

```
julia> x = Containers.@container([i = 1:3, j = [:A, :B]; i > 1 && j == :B], (i, j))
JuMP.Containers.SparseAxisArray{Tuple{Int64, Symbol}, 2, Tuple{Int64, Symbol}} with 2 entries:
  [2, B]  =  (2, :B)
  [3, B]  =  (3, :B)
```

Here we have the index sets `i = 1:3, j = [:A, :B]`, followed by `;`, and then a condition, which evaluates to `true` or `false`: `i > 1 && j == :B`.

Slicing

Slicing is not supported.

```
julia> x[:, :B]
ERROR: ArgumentError: Indexing with `:` is not supported by Containers.SparseAxisArray
[...]
```

Looping

Use eachindex to loop over the elements:

```
julia> for key in eachindex(x)
           println(x[key])
       end
(2, :B)
(3, :B)
```

Broadcasting

Broadcasting over a SparseAxisArray returns a SparseAxisArray

```
julia> swap(x::Tuple) = (last(x), first(x))
swap (generic function with 1 method)

julia> swap.(x)
JuMP.Containers.SparseAxisArray{Tuple{Symbol, Int64}, 2, Tuple{Int64, Symbol}} with 2 entries:
  [2, B]  =  (:B, 2)
  [3, B]  =  (:B, 3)
```

15.4 Forcing the container type

Pass `container = T` to use T as the container. For example:

```
julia> Containers.@container([i = 1:2, j = 1:2], i + j, container = Array)
2×2 Matrix{Int64}:
 2  3
 3  4

julia> Containers.@container([i = 1:2, j = 1:2], i + j, container = Dict)
Dict{Tuple{Int64, Int64}, Int64} with 4 entries:
  (1, 2) => 3
  (1, 1) => 2
  (2, 2) => 4
  (2, 1) => 3
```

You can also pass `DenseAxisArray` or `SparseAxisArray`.

15.5 How different container types are chosen

If the compiler can prove at compile time that the index sets are rectangular, and indexed by a compact set of integers that start at 1, `Containers.@container` will return an array. This is the case if your index sets are visible to the macro as `1:n`:

```
julia> Containers.@container([i=1:3, j=1:5], i + j)
3×5 Matrix{Int64}:
 2  3  4  5  6
 3  4  5  6  7
 4  5  6  7  8
```

or an instance of `Base.OneTo`:

```
julia> set = Base.OneTo(3)
Base.OneTo(3)

julia> Containers.@container([i=set, j=1:5], i + j)
3×5 Matrix{Int64}:
 2  3  4  5  6
 3  4  5  6  7
 4  5  6  7  8
```

If the compiler can prove that the index set is rectangular, but not necessarily of the form `1:n` at compile time, then a `Containers.DenseAxisArray` will be constructed instead:

```
julia> set = 1:3
1:3

julia> Containers.@container([i=set, j=1:5], i + j)
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, 1:3
    Dimension 2, Base.OneTo(5)
And data, a 3×5 Matrix{Int64}:
 2  3  4  5  6
 3  4  5  6  7
 4  5  6  7  8
```

> **Info**
>
> What happened here? Although we know that `set` contains `1:3`, at compile time the `typeof(set)` is a `UnitRange{Int}`. Therefore, Julia can't prove that the range starts at 1 (it only finds this out at runtime), and it defaults to a `DenseAxisArray`. The case where we explicitly wrote `i = 1:3` worked because the macro can "see" the 1 at compile time.

However, if you know that the indices do form an `Array`, you can force the container type with `container = Array`:

```
julia> set = 1:3
1:3

julia> Containers.@container([i=set, j=1:5], i + j, container = Array)
3×5 Matrix{Int64}:
 2  3  4  5  6
 3  4  5  6  7
 4  5  6  7  8
```

Here's another example with something similar:

```
julia> a = 1
1

julia> Containers.@container([i=a:3, j=1:5], i + j)
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, 1:3
    Dimension 2, Base.OneTo(5)
And data, a 3×5 Matrix{Int64}:
```

```
 2  3  4  5  6
 3  4  5  6  7
 4  5  6  7  8

julia> Containers.@container([i=1:a, j=1:5], i + j)
1×5 Matrix{Int64}:
 2  3  4  5  6
```

Finally, if the compiler cannot prove that the index set is rectangular, a `Containers.SparseAxisArray` will be created.

This occurs when some indices depend on a previous one:

```
julia> Containers.@container([i=1:3, j=1:i], i + j)
JuMP.Containers.SparseAxisArray{Int64, 2, Tuple{Int64, Int64}} with 6 entries:
  [1, 1]  =  2
  [2, 1]  =  3
  [2, 2]  =  4
  [3, 1]  =  4
  [3, 2]  =  5
  [3, 3]  =  6
```

or if there is a condition on the index sets:

```
julia> Containers.@container([i = 1:5; isodd(i)], i^2)
JuMP.Containers.SparseAxisArray{Int64, 1, Tuple{Int64}} with 3 entries:
  [1]  =  1
  [3]  =  9
  [5]  =  25
```

The condition can depend on multiple indices, the only requirement is that it is an expression that returns `true` or `false`:

```
julia> condition(i, j) = isodd(i) && iseven(j)
condition (generic function with 1 method)

julia> Containers.@container([i = 1:2, j = 1:4; condition(i, j)], i + j)
JuMP.Containers.SparseAxisArray{Int64, 2, Tuple{Int64, Int64}} with 2 entries:
  [1, 2]  =  3
  [1, 4]  =  5
```

Chapter 16

Solutions

This section of the manual describes how to access a solved solution to a problem. It uses the following model as an example:

```
model = Model(HiGHS.Optimizer)
set_silent(model)
@variable(model, x >= 0)
@variable(model, y[[:a, :b]] <= 1)
@objective(model, Max, -12x - 20y[:a])
@expression(model, my_expr, 6x + 8y[:a])
@constraint(model, my_expr >= 100)
@constraint(model, c1, 7x + 12y[:a] >= 120)
optimize!(model)
print(model)

# output

Max -12 x - 20 y[a]
Subject to
 6 x + 8 y[a] ≥ 100.0
 c1 : 7 x + 12 y[a] ≥ 120.0
 x ≥ 0.0
 y[a] ≤ 1.0
 y[b] ≤ 1.0
```

16.1 Solutions summary

`solution_summary` can be used for checking the summary of the optimization solutions.

```
julia> solution_summary(model)
* Solver : HiGHS

* Status
  Termination status : OPTIMAL
  Primal status      : FEASIBLE_POINT
  Dual status        : FEASIBLE_POINT
  Message from the solver:
  "kHighsModelStatusOptimal"

* Candidate solution
```

```
  Objective value      : -2.05143e+02
  Objective bound      : -0.00000e+00
  Dual objective value : -2.05143e+02

* Work counters
  Solve time (sec)   : 8.31557e-04
  Simplex iterations : 2
  Barrier iterations : 0

julia> solution_summary(model, verbose=true)
* Solver : HiGHS

* Status
  Termination status : OPTIMAL
  Primal status      : FEASIBLE_POINT
  Dual status        : FEASIBLE_POINT
  Result count       : 1
  Has duals          : true
  Message from the solver:
  "kHighsModelStatusOptimal"

* Candidate solution
  Objective value      : -2.05143e+02
  Objective bound      : -0.00000e+00
  Dual objective value : -2.05143e+02
  Primal solution :
    x : 1.54286e+01
    y[a] : 1.00000e+00
    y[b] : 1.00000e+00
  Dual solution :
    c1 : 1.71429e+00

* Work counters
  Solve time (sec)   : 8.31557e-04
  Simplex iterations : 2
  Barrier iterations : 0
```

16.2 Why did the solver stop?

Use`termination_status` to understand why the solver stopped.

```
julia> termination_status(model)
OPTIMAL::TerminationStatusCode = 1
```

The `MOI.TerminationStatusCode` enum describes the full list of statuses that could be returned.

Common return values include `OPTIMAL`, `LOCALLY_SOLVED`, `INFEASIBLE`, `DUAL_INFEASIBLE`, and `TIME_LIMIT`.

Info

A return status of `OPTIMAL` means the solver found (and proved) a globally optimal solution. A return status of `LOCALLY_SOLVED` means the solver found a locally optimal solution (which may also be globally optimal, but it could not prove so).

Warning

> A return status of DUAL_INFEASIBLE does not guarantee that the primal is unbounded. When the dual is infeasible, the primal is unbounded if there exists a feasible primal solution.

Use raw_status to get a solver-specific string explaining why the optimization stopped:

```
julia> raw_status(model)
"kHighsModelStatusOptimal"
```

16.3 Primal solutions

Primal solution status

Use primal_status to return an MOI.ResultStatusCode enum describing the status of the primal solution.

```
julia> primal_status(model)
FEASIBLE_POINT::ResultStatusCode = 1
```

Other common returns are NO_SOLUTION, and INFEASIBILITY_CERTIFICATE. The first means that the solver doesn't have a solution to return, and the second means that the primal solution is a certificate of dual infeasibility (a primal unbounded ray).

You can also use has_values, which returns true if there is a solution that can be queried, and false otherwise.

```
julia> has_values(model)
true
```

Objective values

The objective value of a solved problem can be obtained via objective_value. The best known bound on the optimal objective value can be obtained via objective_bound. If the solver supports it, the value of the dual objective can be obtained via dual_objective_value.

```
julia> objective_value(model)
-205.14285714285714

julia> objective_bound(model)  # HiGHS only implements objective bound for MIPs
-0.0

julia> dual_objective_value(model)
-205.1428571428571
```

Primal solution values

If the solver has a primal solution to return, use value to access it:

```
julia> value(x)
15.428571428571429
```

Broadcast value over containers:

```
julia> value.(y)
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, Symbol[:a, :b]
And data, a 2-element Array{Float64,1}:
 1.0
 1.0
```

value also works on expressions:

```
julia> value(my_expr)
100.57142857142857
```

and constraints:

```
julia> value(c1)
120.0
```

Info

Calling value on a constraint returns the constraint function evaluated at the solution.

16.4 Dual solutions

Dual solution status

Use dual_status to return an MOI.ResultStatusCode enum describing the status of the dual solution.

```
julia> dual_status(model)
FEASIBLE_POINT::ResultStatusCode = 1
```

Other common returns are NO_SOLUTION, and INFEASIBILITY_CERTIFICATE. The first means that the solver doesn't have a solution to return, and the second means that the dual solution is a certificate of primal infeasibility (a dual unbounded ray).

You can also use has_duals, which returns true if there is a solution that can be queried, and false otherwise.

```
julia> has_duals(model)
true
```

Dual solution values

If the solver has a dual solution to return, use dual to access it:

```
julia> dual(c1)
1.7142857142857142
```

Query the duals of variable bounds using LowerBoundRef, UpperBoundRef, and FixRef:

```
julia> dual(LowerBoundRef(x))
0.0

julia> dual.(UpperBoundRef.(y))
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, Symbol[:a, :b]
And data, a 2-element Array{Float64,1}:
 -0.5714285714285694
  0.0
```

Warning

JuMP's definition of duality is independent of the objective sense. That is, the sign of feasible duals associated with a constraint depends on the direction of the constraint and not whether the problem is maximization or minimization. **This is a different convention from linear programming duality in some common textbooks.** If you have a linear program, and you want the textbook definition, you probably want to use `shadow_price` and `reduced_cost` instead.

```
julia> shadow_price(c1)
1.7142857142857142

julia> reduced_cost(x)
0.0

julia> reduced_cost.(y)
1-dimensional DenseAxisArray{Float64,1,...} with index sets:
    Dimension 1, Symbol[:a, :b]
And data, a 2-element Array{Float64,1}:
  0.5714285714285694
 -0.0
```

16.5 Recommended workflow

The recommended workflow for solving a model and querying the solution is something like the following:

```
if termination_status(model) == OPTIMAL
    println("Solution is optimal")
elseif termination_status(model) == TIME_LIMIT && has_values(model)
    println("Solution is suboptimal due to a time limit, but a primal solution is available")
else
    error("The model was not solved correctly.")
end
println("  objective value = ", objective_value(model))
if primal_status(model) == FEASIBLE_POINT
    println("  primal solution: x = ", value(x))
end
if dual_status(model) == FEASIBLE_POINT
    println("  dual solution: c1 = ", dual(c1))
end

# output

Solution is optimal
  objective value = -205.14285714285714
  primal solution: x = 15.428571428571429
  dual solution: c1 = 1.7142857142857142
```

16.6 OptimizeNotCalled errors

Due to differences in how solvers cache solutions internally, modifying a model after calling `optimize!` will reset the model into the `MOI.OPTIMIZE_NOT_CALLED` state. If you then attempt to query solution information, an `OptimizeNotCalled` error will be thrown.

If you are iteratively querying solution information and modifying a model, query all the results first, then modify the problem.

For example, instead of:

```julia
julia> model = Model(HiGHS.Optimizer);

julia> set_silent(model)

julia> @variable(model, x >= 0);

julia> optimize!(model)

julia> termination_status(model)
OPTIMAL::TerminationStatusCode = 1

julia> set_upper_bound(x, 1)

julia> x_val = value(x)
ERROR: OptimizeNotCalled()
Stacktrace:
[...]

julia> termination_status(model)
OPTIMIZE_NOT_CALLED::TerminationStatusCode = 0
```

do

```julia
julia> model = Model(HiGHS.Optimizer);

julia> set_silent(model)

julia> @variable(model, x >= 0);

julia> optimize!(model);

julia> x_val = value(x)
0.0

julia> termination_status(model)
OPTIMAL::TerminationStatusCode = 1

julia> set_upper_bound(x, 1)

julia> set_lower_bound(x, x_val)

julia> termination_status(model)
OPTIMIZE_NOT_CALLED::TerminationStatusCode = 0
```

If you know that your particular solver supports querying solution information after modifications, you can use `direct_model` to bypass the `MOI.OPTIMIZE_NOT_CALLED` state:

```
julia> model = direct_model(HiGHS.Optimizer());

julia> set_silent(model)

julia> @variable(model, x >= 0);

julia> optimize!(model)

julia> termination_status(model)
OPTIMAL::TerminationStatusCode = 1

julia> set_upper_bound(x, 1)

julia> x_val = value(x)
0.0

julia> set_lower_bound(x, x_val)

julia> termination_status(model)
OPTIMAL::TerminationStatusCode = 1
```

Warning

Be careful doing this! If your particular solver does not support querying solution information after modification, it may silently return incorrect solutions or throw an error.

16.7 Accessing attributes

MathOptInterface defines many model attributes that can be queried. Some attributes can be directly accessed by getter functions. These include:

- `solve_time`
- `relative_gap`
- `simplex_iterations`
- `barrier_iterations`
- `node_count`

16.8 Sensitivity analysis for LP

Given an LP problem and an optimal solution corresponding to a basis, we can question how much an objective coefficient or standard form right-hand side coefficient (c.f., `normalized_rhs`) can change without violating primal or dual feasibility of the basic solution.

Note that not all solvers compute the basis, and for sensitivity analysis, the solver interface must implement `MOI.ConstraintBasisStatus`.

Tip

Read the Sensitivity analysis of a linear program for more information on sensitivity analysis.

To give a simple example, we could analyze the sensitivity of the optimal solution to the following (non-degenerate) LP problem:

```
model = Model(HiGHS.Optimizer)
set_silent(model)
@variable(model, x[1:2])
set_lower_bound(x[2], -0.5)
set_upper_bound(x[2], 0.5)
@constraint(model, c1, x[1] + x[2] <= 1)
@constraint(model, c2, x[1] - x[2] <= 1)
@objective(model, Max, x[1])
print(model)

# output

Max x[1]
Subject to
 c1 : x[1] + x[2] ≤ 1.0
 c2 : x[1] - x[2] ≤ 1.0
 x[2] ≥ -0.5
 x[2] ≤ 0.5
```

To analyze the sensitivity of the problem we could check the allowed perturbation ranges of, for example, the cost coefficients and the right-hand side coefficient of the constraint c1 as follows:

```
julia> optimize!(model)

julia> value.(x)
2-element Vector{Float64}:
  1.0
 -0.0

julia> report = lp_sensitivity_report(model);

julia> x1_lo, x1_hi = report[x[1]]
(-1.0, Inf)

julia> println("The objective coefficient of x[1] could decrease by $(x1_lo) or increase by
↪  $(x1_hi).")
The objective coefficient of x[1] could decrease by -1.0 or increase by Inf.

julia> x2_lo, x2_hi = report[x[2]]
(-1.0, 1.0)

julia> println("The objective coefficient of x[2] could decrease by $(x2_lo) or increase by
↪  $(x2_hi).")
The objective coefficient of x[2] could decrease by -1.0 or increase by 1.0.

julia> c_lo, c_hi = report[c1]
(-1.0, 1.0)

julia> println("The RHS of c1 could decrease by $(c_lo) or increase by $(c_hi).")
The RHS of c1 could decrease by -1.0 or increase by 1.0.
```

The range associated with a variable is the range of the allowed perturbation of the corresponding objective coefficient. Note that the current primal solution remains optimal within this range; however the corresponding dual solution might change since a cost coefficient is perturbed. Similarly, the range associated with a constraint is the range of the allowed perturbation of the corresponding right-hand side coefficient. In this range the current dual solution remains optimal, but the optimal primal solution might change.

If the problem is degenerate, there are multiple optimal bases and hence these ranges might not be as intuitive and seem too narrow, for example, a larger cost coefficient perturbation might not invalidate the optimality of the current primal solution. Moreover, if a problem is degenerate, due to finite precision, it can happen that, for example, a perturbation seems to invalidate a basis even though it doesn't (again providing too narrow ranges). To prevent this, increase the `atol` keyword argument to `lp_sensitivity_report`. Note that this might make the ranges too wide for numerically challenging instances. Thus, do not blindly trust these ranges, especially not for highly degenerate or numerically unstable instances.

16.9 Conflicts

When the model you input is infeasible, some solvers can help you find the cause of this infeasibility by offering a conflict, that is, a subset of the constraints that create this infeasibility. Depending on the solver, this can also be called an IIS (irreducible inconsistent subsystem).

The function `compute_conflict!` is used to trigger the computation of a conflict. Once this process is finished, the attribute `MOI.ConflictStatus` returns a `MOI.ConflictStatusCode`.

If there is a conflict, you can query from each constraint whether it participates in the conflict or not using the attribute `MOI.ConstraintConflictStatus`, which returns a `MOI.ConflictParticipationStatusCode`.

To create a new model containing only the constraints that participate in the conflict, use `copy_conflict`. It may be helpful to write this model to a file for easier debugging using `write_to_file`.

For instance, this is how you can use this functionality:

```
using JuMP
model = Model() # You must use a solver that supports conflict refining/IIS
# computation, like CPLEX or Gurobi
# for example, using Gurobi; model = Model(Gurobi.Optimizer)
@variable(model, x >= 0)
@constraint(model, c1, x >= 2)
@constraint(model, c2, x <= 1)
optimize!(model)

# termination_status(model) will likely be INFEASIBLE,
# depending on the solver

compute_conflict!(model)
if MOI.get(model, MOI.ConflictStatus()) != MOI.CONFLICT_FOUND
    error("No conflict could be found for an infeasible model.")
end

# Both constraints participate in the conflict.
MOI.get(model, MOI.ConstraintConflictStatus(), c1)
MOI.get(model, MOI.ConstraintConflictStatus(), c2)

# Get a copy of the model with only the constraints in the conflict.
new_model, reference_map = copy_conflict(model)
```

Conflicting constraints can be collected in a list and printed as follows:

```
conflict_constraint_list = ConstraintRef[]
for (F, S) in list_of_constraint_types(model)
    for con in all_constraints(model, F, S)
        if MOI.get(model, MOI.ConstraintConflictStatus(), con) == MOI.IN_CONFLICT
            push!(conflict_constraint_list, con)
            println(con)
        end
    end
end
```

16.10 Multiple solutions

Some solvers support returning multiple solutions. You can check how many solutions are available to query using `result_count`.

Functions for querying the solutions, for example, `primal_status` and `value`, all take an additional keyword argument `result` which can be used to specify which result to return.

> **Warning**
>
> Even if `termination_status` is `OPTIMAL`, some of the returned solutions may be suboptimal! However, if the solver found at least one optimal solution, then `result = 1` will always return an optimal solution. Use `objective_value` to assess the quality of the remaining solutions.

```
using JuMP
model = Model()
@variable(model, x[1:10] >= 0)
# ... other constraints ...
optimize!(model)

if termination_status(model) != OPTIMAL
    error("The model was not solved correctly.")
end

an_optimal_solution = value.(x; result = 1)
optimal_objective = objective_value(model; result = 1)
for i in 2:result_count(model)
    @assert has_values(model; result = i)
    println("Solution $(i) = ", value.(x; result = i))
    obj = objective_value(model; result = i)
    println("Objective $(i) = ", obj)
    if isapprox(obj, optimal_objective; atol = 1e-8)
        print("Solution $(i) is also optimal!")
    end
end
```

16.11 Checking feasibility of solutions

To check the feasibility of a primal solution, use `primal_feasibility_report`, which takes a `model`, a dictionary mapping each variable to a primal solution value (defaults to the last solved solution), and a tolerance `atol` (defaults to `0.0`).

The function returns a dictionary which maps the infeasible constraint references to the distance between the primal value of the constraint and the nearest point in the corresponding set. A point is classed as infeasible if the distance is greater than the supplied tolerance `atol`.

```
julia> model = Model(HiGHS.Optimizer);

julia> set_silent(model)

julia> @variable(model, x >= 1, Int);

julia> @variable(model, y);

julia> @constraint(model, c1, x + y <= 1.95);

julia> point = Dict(x => 1.9, y => 0.06);

julia> primal_feasibility_report(model, point)
Dict{Any, Float64} with 2 entries:
  x integer         => 0.1
  c1 : x + y ≤ 1.95 => 0.01

julia> primal_feasibility_report(model, point; atol = 0.02)
Dict{Any, Float64} with 1 entry:
  x integer => 0.1
```

If the point is feasible, an empty dictionary is returned:

```
julia> primal_feasibility_report(model, Dict(x => 1.0, y => 0.0))
Dict{Any, Float64}()
```

To use the primal solution from a solve, omit the `point` argument:

```
julia> optimize!(model)

julia> primal_feasibility_report(model; atol = 0.0)
Dict{Any, Float64}()
```

Calling `primal_feasibility_report` without the `point` argument is useful when `primal_status` is `FEASIBLE_POINT` or `NEARLY_FEASIBLE_POINT`, and you want to assess the solution quality.

> **Warning**
>
> To apply `primal_feasibility_report` to infeasible models, you must also provide a candidate point (solvers generally do not provide one). To diagnose the source of infeasibility, see Conflicts.

Pass `skip_mising = true` to skip constraints which contain variables that are not in `point`:

```
julia> primal_feasibility_report(model, Dict(x => 2.1); skip_missing = true)
Dict{Any, Float64} with 1 entry:
  x integer => 0.1
```

You can also use the functional form, where the first argument is a function that maps variables to their primal values:

```
julia> optimize!(model)

julia> primal_feasibility_report(v -> value(v), model)
Dict{Any, Float64}()
```

Chapter 17

Nonlinear Modeling

JuMP has support for general smooth nonlinear (convex and nonconvex) optimization problems. JuMP is able to provide exact, sparse second-order derivatives to solvers. This information can improve solver accuracy and performance.

There are three main changes to solve nonlinear programs in JuMP.

- Use `@NLobjective` instead of `@objective`
- Use `@NLconstraint` instead of `@constraint`
- Use `@NLexpression` instead of `@expression`

> **Info**
>
> There are some restrictions on what syntax you can use in the `@NLxxx` macros. Make sure to read the Syntax notes.

17.1 Set a nonlinear objective

Use `@NLobjective` to set a nonlinear objective.

```
julia> @NLobjective(model, Min, exp(x[1]) - sqrt(x[2]))
```

17.2 Add a nonlinear constraint

Use `@NLconstraint` to add a nonlinear constraint.

```
julia> @NLconstraint(model, exp(x[1]) <= 1)
exp(x[1]) - 1.0 ≤ 0

julia> @NLconstraint(model, [i = 1:2], x[i]^i >= i)
2-element Vector{NonlinearConstraintRef{ScalarShape}}:
 x[1] ^ 1.0 - 1.0 ≥ 0
 x[2] ^ 2.0 - 2.0 ≥ 0

julia> @NLconstraint(model, con[i = 1:2], prod(x[j] for j = 1:i) == i)
2-element Vector{NonlinearConstraintRef{ScalarShape}}:
 (*)(x[1]) - 1.0 = 0
 x[1] * x[2] - 2.0 = 0
```

Info

You can only create nonlinear constraints with <=, >=, and ==. More general `Nonlinear`-in-`Set` constraints are not supported.

17.3 Create a nonlinear expression

Use `@NLexpression` to create nonlinear expression objects. The syntax is identical to `@expression`, except that the expression can contain nonlinear terms.

```
julia> expr = @NLexpression(model, exp(x[1]) + sqrt(x[2]))
subexpression[1]: exp(x[1]) + sqrt(x[2])

julia> my_anon_expr = @NLexpression(model, [i = 1:2], sin(x[i]))
2-element Vector{NonlinearExpression}:
 subexpression[2]: sin(x[1])
 subexpression[3]: sin(x[2])

julia> @NLexpression(model, my_expr[i = 1:2], sin(x[i]))
2-element Vector{NonlinearExpression}:
 subexpression[4]: sin(x[1])
 subexpression[5]: sin(x[2])
```

Nonlinear expression can be used in `@NLobjective`, `@NLconstraint`, and even nested in other `@NLexpression`s.

```
julia> @NLobjective(model, Min, expr^2 + 1)

julia> @NLconstraint(model, [i = 1:2], my_expr[i] <= i)
2-element Vector{NonlinearConstraintRef{ScalarShape}}:
 subexpression[4] - 1.0 ≤ 0
 subexpression[5] - 2.0 ≤ 0

julia> @NLexpression(model, nested[i = 1:2], sin(my_expr[i]))
2-element Vector{NonlinearExpression}:
 subexpression[6]: sin(subexpression[4])
 subexpression[7]: sin(subexpression[5])
```

17.4 Create a nonlinear parameter

For nonlinear models only, JuMP offers a syntax for explicit "parameter" objects, which are constants in the model that can be efficiently updated between solves.

Nonlinear parameters are declared by using the `@NLparameter` macro and may be indexed by arbitrary sets analogously to JuMP variables and expressions.

The initial value of the parameter must be provided on the right-hand side of the == sign.

```
julia> @NLparameter(model, p[i = 1:2] == i)
2-element Vector{NonlinearParameter}:
 parameter[1] == 1.0
 parameter[2] == 2.0
```

Create anonymous parameters using the `value` keyword:

```
julia> anon_parameter = @NLparameter(model, value = 1)
parameter[3] == 1.0
```

Info

A parameter is not an optimization variable. It must be fixed to a value with ==. If you want a parameter that is <= or >=, create a variable instead using `@variable`.

Use `value` and `set_value` to query or update the value of a parameter.

```
julia> value.(p)
2-element Vector{Float64}:
 1.0
 2.0

julia> set_value(p[2], 3.0)
3.0

julia> value.(p)
2-element Vector{Float64}:
 1.0
 3.0
```

Nonlinear parameters can be used within nonlinear macros only:

```
julia> @objective(model, Max, p[1] * x)
ERROR: MethodError: no method matching *(::NonlinearParameter, ::VariableRef)
[...]

julia> @NLobjective(model, Max, p[1] * x)

julia> @expression(model, my_expr, p[1] * x^2)
ERROR: MethodError: no method matching *(::NonlinearParameter, ::QuadExpr)
Closest candidates are:
[...]

julia> @NLexpression(model, my_nl_expr, p[1] * x^2)
subexpression[1]: parameter[1] * x ^ 2.0
```

When to use a parameter

Nonlinear parameters are useful when solving nonlinear models in a sequence:

```
using JuMP, Ipopt
model = Model(Ipopt.Optimizer)
set_silent(model)
@variable(model, z)
@NLparameter(model, x == 1.0)
@NLobjective(model, Min, (z - x)^2)
optimize!(model)
@show value(z) # Equals 1.0.

# Now, update the value of x to solve a different problem.
set_value(x, 5.0)
```

```
optimize!(model)
@show value(z) # Equals 5.0
```

```
value(z) = 1.0
value(z) = 5.0
```

Info

Using nonlinear parameters can be faster than creating a new model from scratch with updated data because JuMP is able to avoid repeating a number of steps in processing the model before handing it off to the solver.

17.5 Syntax notes

The syntax accepted in nonlinear macros is more restricted than the syntax for linear and quadratic macros. We note some important points below.

No operator overloading

There is no operator overloading provided to build up nonlinear expressions. For example, if x is a JuMP variable, the code 3x will return an `AffExpr` object that can be used inside of future expressions and linear constraints. However, the code `sin(x)` is an error. All nonlinear expressions must be inside of macros.

```
julia> expr = sin(x) + 1
ERROR: sin is not defined for type AbstractVariableRef. Are you trying to build a nonlinear problem?
↪  Make sure you use @NLconstraint/@NLobjective.
[...]

julia> expr = @NLexpression(model, sin(x) + 1)
subexpression[1]: sin(x) + 1.0
```

Scalar operations only

Except for the splatting syntax discussed below, all expressions must be simple scalar operations. You cannot use `dot`, matrix-vector products, vector slices, etc.

```
julia> @NLobjective(model, Min, c' * x + 3y)
ERROR: Unexpected array [1 2] in nonlinear expression. Nonlinear expressions may contain only scalar
↪  expressions.
[...]
```

Translate vector operations into explicit `sum()` operations:

```
julia> @NLobjective(model, Min, sum(c[i] * x[i] for i = 1:2) + 3y)
```

Or use an `@expression`:

```
julia> @expression(model, expr, c' * x)
x[1] + 2 x[2]

julia> @NLobjective(model, Min, expr + 3y)
```

Splatting

The splatting operator `...` is recognized in a very restricted setting for expanding function arguments. The expression splatted can be only a symbol. More complex expressions are not recognized.

```
julia> model = Model();

julia> @variable(model, x[1:3]);

julia> @NLconstraint(model, *(x...) <= 1.0)
x[1] * x[2] * x[3] - 1.0 ≤ 0

julia> @NLconstraint(model, *((x / 2)...) <= 0.0)
ERROR: LoadError: Unexpected expression in (*)(x / 2...). JuMP supports splatting only symbols. For
↪  example, x... is ok, but (x + 1)..., [x; y]... and g(f(y)...) are not.
```

17.6 User-defined Functions

JuMP's library of recognized univariate functions is derived from the Calculus.jl package.

In addition to this list of functions, it is possible to register custom user-defined nonlinear functions.

> **Tip**
>
> User-defined functions can be used anywhere in `@NLobjective`, `@NLconstraint`, and `@NLexpression`.

> **Tip**
>
> JuMP will attempt to automatically register functions it detects in your nonlinear expressions, which usually means manually registering a function is not needed. Two exceptions are if you want to provide custom derivatives, or if the function is not available in the scope of the nonlinear expression.

> **Warning**
>
> User-defined functions must return a scalar output. For a work-around, see User-defined functions with vector outputs.

Automatic differentiation

JuMP does not support black-box optimization, so all user-defined functions must provide derivatives in some form.

Fortunately, JuMP supports **automatic differentiation of user-defined functions**, a feature to our knowledge not available in any comparable modeling systems.

> **Info**
>
> Automatic differentiation is not finite differencing. JuMP's automatically computed derivatives are not subject to approximation error.

JuMP uses ForwardDiff.jl to perform automatic differentiation; see the ForwardDiff.jl documentation for a description of how to write a function suitable for automatic differentiation.

Warning

Get an error like `No method matching Float64(::ForwardDiff.Dual)`? Read this section, and see the guidelines at ForwardDiff.jl.

The most common error is that your user-defined function is not generic with respect to the number type, that is, don't assume that the input to the function is `Float64`.

```
f(x::Float64) = 2 * x  # This will not work.
f(x::Real)    = 2 * x  # This is good.
f(x)          = 2 * x  # This is also good.
```

Another reason you may encounter this error is if you create arrays inside your function which are `Float64`.

```
function bad_f(x...)
    y = zeros(length(x))  # This constructs an array of `Float64`!
    for i = 1:length(x)
        y[i] = x[i]^i
    end
    return sum(y)
end

function good_f(x::T...) where {T<:Real}
    y = zeros(T, length(x))  # Construct an array of type `T` instead!
    for i = 1:length(x)
        y[i] = x[i]^i
    end
    return sum(y)
end
```

Register a function

To register a user-defined function with derivatives computed by automatic differentiation, use the `register` method as in the following example:

```
square(x) = x^2
f(x, y) = (x - 1)^2 + (y - 2)^2

model = Model()

register(model, :square, 1, square; autodiff = true)
register(model, :my_f, 2, f; autodiff = true)

@variable(model, x[1:2] >= 0.5)
@NLobjective(model, Min, my_f(x[1], square(x[2])))
```

The above code creates a JuMP model with the objective function `(x[1] - 1)^2 + (x[2]^2 - 2)^2`. The arguments to `register` are:

1. The model for which the functions are registered.
2. A Julia symbol object which serves as the name of the user-defined function in JuMP expressions.
3. The number of input arguments that the function takes.

4. The Julia method which computes the function

5. A flag to instruct JuMP to compute exact gradients automatically.

Tip

The symbol `:my_f` doesn't have to match the name of the function `f`. However, it's more readable if it does. Make sure you use `my_f` and not `f` in the macros.

Warning

If you use multi-variate user-defined functions, JuMP will disable second-derivative information. This can lead to significant slow-downs in some cases. Only use a user-defined function if you cannot write out the expression algebraically in the macro.

Warning

User-defined functions cannot be re-registered and will not update if you modify the underlying Julia function. If you want to change a user-defined function between solves, rebuild the model or use a different name. To use a different name programmatically, see Raw expression input.

Register a function and gradient

Forward-mode automatic differentiation as implemented by ForwardDiff.jl has a computational cost that scales linearly with the number of input dimensions. As such, it is not the most efficient way to compute gradients of user-defined functions if the number of input arguments is large. In this case, users may want to provide their own routines for evaluating gradients.

Univariate functions

For univariate functions, the gradient function `∇f` returns a number that represents the first-order derivative:

```
f(x) = x^2
∇f(x) = 2x
model = Model()
register(model, :my_square, 1, f, ∇f; autodiff = true)
@variable(model, x >= 0)
@NLobjective(model, Min, my_square(x))
```

If `autodiff = true`, JuMP will use automatic differentiation to compute the hessian.

Multivariate functions

For multivariate functions, the gradient function `∇f` must take a gradient vector as the first argument that is filled in-place:

```
f(x, y) = (x - 1)^2 + (y - 2)^2
function ∇f(g::AbstractVector{T}, x::T, y::T) where {T}
    g[1] = 2 * (x - 1)
    g[2] = 2 * (y - 2)
    return
end

model = Model()
```

```
register(model, :my_square, 2, f, ∇f)
@variable(model, x[1:2] >= 0)
@NLobjective(model, Min, my_square(x[1], x[2]))
```

Warning

Make sure the first argument to `∇f` supports an `AbstractVector`, and do not assume the input is `Float64`.

Register a function, gradient, and hessian

Warning

The ability to explicitly register a hessian is only available for univariate functions.

Instead of automatically differentiating the hessian, you can instead pass a function which returns a number representing the second-order derivative.

```
f(x) = x^2
∇f(x) = 2x
∇²f(x) = 2
model = Model()
register(model, :my_square, 1, f, ∇f, ∇²f)
@variable(model, x >= 0)
@NLobjective(model, Min, my_square(x))
```

User-defined functions with vector inputs

User-defined functions which take vectors as input arguments (for example, `f(x::Vector)`) are not supported. Instead, use Julia's splatting syntax to create a function with scalar arguments. For example, instead of

```
f(x::Vector) = sum(x[i]^i for i in 1:length(x))
```

define:

```
f(x...) = sum(x[i]^i for i in 1:length(x))
```

This function `f` can be used in a JuMP model as follows:

```
model = Model()
@variable(model, x[1:5] >= 0)
f(x...) = sum(x[i]^i for i in 1:length(x))
register(model, :f, 5, f; autodiff = true)
@NLobjective(model, Min, f(x...))
```

Tip

Make sure to read the syntax restrictions of Splatting.

17.7 Factors affecting solution time

The execution time when solving a nonlinear programming problem can be divided into two parts, the time spent in the optimization algorithm (the solver) and the time spent evaluating the nonlinear functions and corresponding derivatives. Ipopt explicitly displays these two timings in its output, for example:

```
Total CPU secs in IPOPT (w/o function evaluations)   =      7.412
Total CPU secs in NLP function evaluations           =      2.083
```

For Ipopt in particular, one can improve the performance by installing advanced sparse linear algebra packages, see Installation Guide. For other solvers, see their respective documentation for performance tips.

The function evaluation time, on the other hand, is the responsibility of the modeling language. JuMP computes derivatives by using reverse-mode automatic differentiation with graph coloring methods for exploiting sparsity of the Hessian matrix [1]. As a conservative bound, JuMP's performance here currently may be expected to be within a factor of 5 of AMPL's.

17.8 Querying derivatives from a JuMP model

For some advanced use cases, one may want to directly query the derivatives of a JuMP model instead of handing the problem off to a solver. Internally, JuMP implements the `MOI.AbstractNLPEvaluator` interface. To obtain an NLP evaluator object from a JuMP model, use `NLPEvaluator`. `index` returns the `MOI.VariableIndex` corresponding to a JuMP variable. `MOI.VariableIndex` itself is a type-safe wrapper for `Int64` (stored in the `.value` field.)

For example:

```
raw_index(v::MOI.VariableIndex) = v.value
model = Model()
@variable(model, x)
@variable(model, y)
@NLobjective(model, Min, sin(x) + sin(y))
values = zeros(2)
x_index = raw_index(JuMP.index(x))
y_index = raw_index(JuMP.index(y))
values[x_index] = 2.0
values[y_index] = 3.0
d = NLPEvaluator(model)
MOI.initialize(d, [:Grad])
MOI.eval_objective(d, values) # == sin(2.0) + sin(3.0)

# output
1.0504174348855488
```

```
∇f = zeros(2)
MOI.eval_objective_gradient(d, ∇f, values)
(∇f[x_index], ∇f[y_index]) # == (cos(2.0), cos(3.0))

# output
(-0.4161468365471424, -0.9899924966004454)
```

Only nonlinear constraints (those added with `@NLconstraint`), and nonlinear objectives (added with `@NLobjective`) exist in the scope of the `NLPEvaluator`.

The `NLPEvaluator` does not evaluate derivatives of linear or quadratic constraints or objectives.

The index method applied to a nonlinear constraint reference object returns its index as a NonlinearConstraintIndex. The .value field of NonlinearConstraintIndex stores the raw integer index. For example:

```
julia> model = Model();

julia> @variable(model, x);

julia> @NLconstraint(model, cons1, sin(x) <= 1);

julia> @NLconstraint(model, cons2, x + 5 == 10);

julia> typeof(cons1)
NonlinearConstraintRef{ScalarShape} (alias for ConstraintRef{Model, NonlinearConstraintIndex,
↪  ScalarShape})

julia> index(cons1)
NonlinearConstraintIndex(1)

julia> index(cons2)
NonlinearConstraintIndex(2)
```

Note that for one-sided nonlinear constraints, JuMP subtracts any values on the right-hand side when computing expressions. In other words, one-sided nonlinear constraints are always transformed to have a right-hand side of zero.

This method of querying derivatives directly from a JuMP model is convenient for interacting with the model in a structured way, for example, for accessing derivatives of specific variables. For example, in statistical maximum likelihood estimation problems, one is often interested in the Hessian matrix at the optimal solution, which can be queried using the NLPEvaluator.

17.9 Raw expression input

> **Warning**
>
> This section requires advanced knowledge of Julia's Expr. You should read the Expressions and evaluation section of the Julia documentation first.

In addition to the @NLexpression, @NLobjective and @NLconstraint macros, it is also possible to provide Julia Expr objects directly by using add_nonlinear_expression, set_nonlinear_objective and add_nonlinear_constraint.

This input form may be useful if the expressions are generated programmatically.

Add a nonlinear expression

Use add_nonlinear_expression to add a nonlinear expression to the model.

```
julia> @variable(model, x)
x

julia> expr = :($(x) + sin($(x)^2))
:(x + sin(x ^ 2))

julia> expr_ref = add_nonlinear_expression(model, expr)
subexpression[1]: x + sin(x ^ 2.0)
```

This is equivalent to

```
julia> expr_ref = @NLexpression(model, x + sin(x^2))
subexpression[1]: x + sin(x ^ 2.0)
```

Note

You must interpolate the variables directly into the expression `expr`.

Set the objective function

Use `set_nonlinear_objective` to set a nonlinear objective.

```
julia> expr = :($(x) + $(x)^2)
:(x + x ^ 2)

julia> set_nonlinear_objective(model, MIN_SENSE, expr)
```

This is equivalent to

```
julia> @NLobjective(model, Min, x + x^2)
```

Note

You must use `MIN_SENSE` or `MAX_SENSE` instead of `Min` and `Max`.

Add a constraint

Use `add_nonlinear_constraint` to add a nonlinear constraint.

```
julia> expr = :($(x) + $(x)^2)
:(x + x ^ 2)

julia> add_nonlinear_constraint(model, :($(expr) <= 1))
(x + x ^ 2.0) - 1.0 ≤ 0
```

This is equivalent to

```
julia> @NLconstraint(model, Min, x + x^2 <= 1)
(x + x ^ 2.0) - 1.0 ≤ 0
```

More complicated examples

Raw expression input is most useful when the expressions are generated programmatically, often in conjunction with user-defined functions.

As an example, we construct a model with the nonlinear constraints `f(x) <= 1`, where `f(x) = x^2` and `f(x) = sin(x)^2`:

```
julia> function main(functions::Vector{Function})
           model = Model()
           @variable(model, x)
           for (i, f) in enumerate(functions)
               f_sym = Symbol("f_$(i)")
               register(model, f_sym, 1, f; autodiff = true)
               add_nonlinear_constraint(model, :($(f_sym)($(x)) <= 1))
           end
           print(model)
           return
       end
main (generic function with 1 method)

julia> main([x -> x^2, x -> sin(x)^2])
Feasibility
Subject to
 f_1(x) - 1.0 ≤ 0
 f_2(x) - 1.0 ≤ 0
```

As another example, we construct a model with the constraint `x^2 + sin(x)^2 <= 1`:

```
julia> function main(functions::Vector{Function})
           model = Model()
           @variable(model, x)
           expr = Expr(:call, :+)
           for (i, f) in enumerate(functions)
               f_sym = Symbol("f_$(i)")
               register(model, f_sym, 1, f; autodiff = true)
               push!(expr.args, :($(f_sym)($(x))))
           end
           add_nonlinear_constraint(model, :($(expr) <= 1))
           print(model)
           return
       end
main (generic function with 1 method)

julia> main([x -> x^2, x -> sin(x)^2])
Feasibility
Subject to
 (f_1(x) + f_2(x)) - 1.0 ≤ 0
```

[1] Dunning, Huchette, and Lubin, "JuMP: A Modeling Language for Mathematical Optimization", SIAM Review, PDF.

Chapter 18

Solver-independent Callbacks

Many mixed-integer (linear, conic, and nonlinear) programming solvers offer the ability to modify the solve process. Examples include changing branching decisions in branch-and-bound, adding custom cutting planes, providing custom heuristics to find feasible solutions, or implementing on-demand separators to add new constraints only when they are violated by the current solution (also known as lazy constraints).

While historically this functionality has been limited to solver-specific interfaces, JuMP provides solver-independent support for three types of callbacks:

1. lazy constraints
2. user-cuts
3. heuristic solutions

18.1 Available solvers

Solver-independent callback support is limited to a few solvers. This includes CPLEX, GLPK, Gurobi, and Xpress.

> **Warning**
>
> While JuMP provides a solver-independent way of accessing callbacks, you should not assume that you will see identical behavior when running the same code on different solvers. For example, some solvers may ignore user-cuts for various reasons, while other solvers may add every user-cut. Read the underlying solver's callback documentation to understand details specific to each solver.

> **Tip**
>
> This page discusses solver-independent callbacks. However, each solver listed above also provides a solver-dependent callback to provide access to the full range of solver-specific features. Consult the solver's README for an example of how to use the solver-dependent callback. This will require you to understand the C interface of the solver.

18.2 Things you can and cannot do during solver-independent callbacks

There is a limited range of things you can do during a callback. Only use the functions and macros explicitly stated in this page of the documentation, or in the Callbacks tutorial.

Using any other part of the JuMP API (for example, adding a constraint with `@constraint` or modifying a variable bound with `set_lower_bound`) is undefined behavior, and your solver may throw an error, return an incorrect solution, or result in a segfault that aborts Julia.

In each of the three solver-independent callbacks, there are two things you may query:

- `callback_node_status` returns an `MOI.CallbackNodeStatusCode` enum indicating if the current primal solution is integer feasible.
- `callback_value` returns the current primal solution of a variable.

If you need to query any other information, use a solver-dependent callback instead. Each solver supporting a solver-dependent callback has information on how to use it in the README of their GitHub repository.

If you want to modify the problem in a callback, you must use a lazy constraint.

> **Warning**
>
> You can only set each callback once. Calling `set` twice will over-write the earlier callback. In addition, if you use a solver-independent callback, you cannot set a solver-dependent callback.

18.3 Lazy constraints

Lazy constraints are useful when the full set of constraints is too large to explicitly include in the initial formulation. When a MIP solver reaches a new solution, for example with a heuristic or by solving a problem at a node in the branch-and-bound tree, it will give the user the chance to provide constraints that would make the current solution infeasible. For some more information about lazy constraints, see this blog post by Paul Rubin.

A lazy constraint callback can be set using the following syntax:

```
model = Model(GLPK.Optimizer)
@variable(model, x <= 10, Int)
@objective(model, Max, x)
function my_callback_function(cb_data)
    status = callback_node_status(cb_data, model)
    if status == MOI.CALLBACK_NODE_STATUS_FRACTIONAL
        # `callback_value(cb_data, x)` is not integer (to some tolerance).
        # If, for example, your lazy constraint generator requires an
        # integer-feasible primal solution, you can add a `return` here.
        return
    elseif status == MOI.CALLBACK_NODE_STATUS_INTEGER
        # `callback_value(cb_data, x)` is integer (to some tolerance).
    else
        @assert status == MOI.CALLBACK_NODE_STATUS_UNKNOWN
        # `callback_value(cb_data, x)` might be fractional or integer.
    end
    x_val = callback_value(cb_data, x)
    if x_val > 2 + 1e-6
        con = @build_constraint(x <= 2)
        MOI.submit(model, MOI.LazyConstraint(cb_data), con)
    end
end
MOI.set(model, MOI.LazyConstraintCallback(), my_callback_function)
```

Info

The lazy constraint callback may be called at fractional or integer nodes in the branch-and-bound tree. There is no guarantee that the callback is called at every primal solution.

Warning

Only add a lazy constraint if your primal solution violates the constraint. Adding the lazy constraint irrespective of feasibility may result in the solver returning an incorrect solution, or lead to many constraints being added, slowing down the solution process.

```
model = Model(GLPK.Optimizer)
@variable(model, x <= 10, Int)
@objective(model, Max, x)
function bad_callback_function(cb_data)
    # Don't do this!
    con = @build_constraint(x <= 2)
    MOI.submit(model, MOI.LazyConstraint(cb_data), con)
end
function good_callback_function(cb_data)
    if callback_value(x) > 2
        con = @build_constraint(x <= 2)
        MOI.submit(model, MOI.LazyConstraint(cb_data), con)
    end
end
MOI.set(model, MOI.LazyConstraintCallback(), good_callback_function)
```

Warning

During the solve, a solver may visit a point that was cut off by a previous lazy constraint, for example, because the earlier lazy constraint was removed during presolve. However, the solver will not stop until it reaches a solution that satisfies all added lazy constraints.

18.4 User cuts

User cuts, or simply cuts, provide a way for the user to tighten the LP relaxation using problem-specific knowledge that the solver cannot or is unable to infer from the model. Just like with lazy constraints, when a MIP solver reaches a new node in the branch-and-bound tree, it will give the user the chance to provide cuts to make the current relaxed (fractional) solution infeasible in the hopes of obtaining an integer solution. For more details about the difference between user cuts and lazy constraints see the aforementioned blog post.

A user-cut callback can be set using the following syntax:

```
model = Model(GLPK.Optimizer)
@variable(model, x <= 10.5, Int)
@objective(model, Max, x)
function my_callback_function(cb_data)
    x_val = callback_value(cb_data, x)
    con = @build_constraint(x <= floor(x_val))
    MOI.submit(model, MOI.UserCut(cb_data), con)
end
MOI.set(model, MOI.UserCutCallback(), my_callback_function)
```

Warning

User cuts must not change the set of integer feasible solutions. Equivalently, user cuts can only remove fractional solutions. If you add a cut that removes an integer solution (even one that is not optimal), the solver may return an incorrect solution.

Info

The user-cut callback may be called at fractional nodes in the branch-and-bound tree. There is no guarantee that the callback is called at every fractional primal solution.

18.5 Heuristic solutions

Integer programming solvers frequently include heuristics that run at the nodes of the branch-and-bound tree. They aim to find integer solutions quicker than plain branch-and-bound would to tighten the bound, allowing us to fathom nodes quicker and to tighten the integrality gap.

Some heuristics take integer solutions and explore their "local neighborhood" (for example, flipping binary variables, fix some variables and solve a smaller MILP) and others take fractional solutions and attempt to round them in an intelligent way.

You may want to add a heuristic of your own if you have some special insight into the problem structure that the solver is not aware of, for example, you can consistently take fractional solutions and intelligently guess integer solutions from them.

A heuristic solution callback can be set using the following syntax:

```
model = Model(GLPK.Optimizer)
@variable(model, x <= 10.5, Int)
@objective(model, Max, x)
function my_callback_function(cb_data)
    x_val = callback_value(cb_data, x)
    status = MOI.submit(
        model, MOI.HeuristicSolution(cb_data), [x], [floor(Int, x_val)]
    )
    println("I submitted a heuristic solution, and the status was: ", status)
end
MOI.set(model, MOI.HeuristicCallback(), my_callback_function)
```

The third argument to `submit` is a vector of JuMP variables, and the fourth argument is a vector of values corresponding to each variable.

`MOI.submit` returns an enum that depends on whether the solver accepted the solution. The possible return codes are:

- `MOI.HEURISTIC_SOLUTION_ACCEPTED`
- `MOI.HEURISTIC_SOLUTION_REJECTED`
- `MOI.HEURISTIC_SOLUTION_UNKNOWN`

Warning

Some solvers may accept partial solutions. Others require a feasible integer solution for every variable. If in doubt, provide a complete solution.

Info

The heuristic solution callback may be called at fractional nodes in the branch-and-bound tree. There is no guarantee that the callback is called at every fractional primal solution.

Part IV

API Reference

Chapter 19

Models

More information can be found in the Models section of the manual.

19.1 Constructors

`JuMP.Model` – Type.

```
Model
```

A mathematical model of an optimization problem.

source

`JuMP.direct_model` – Function.

```
direct_model(backend::MOI.ModelLike)
```

Return a new JuMP model using `backend` to store the model and solve it.

As opposed to the `Model` constructor, no cache of the model is stored outside of `backend` and no bridges are automatically applied to `backend`.

Notes

The absence of a cache reduces the memory footprint but, it is important to bear in mind the following implications of creating models using this direct mode:

- When `backend` does not support an operation, such as modifying constraints or adding variables/-constraints after solving, an error is thrown. For models created using the `Model` constructor, such situations can be dealt with by storing the modifications in a cache and loading them into the optimizer when `optimize!` is called.
- No constraint bridging is supported by default.
- The optimizer used cannot be changed the model is constructed.
- The model created cannot be copied.

source

```
direct_model(factory::MOI.OptimizerWithAttributes)
```

Create a `direct_model` using `factory`, a `MOI.OptimizerWithAttributes` object created by `optimizer_with_attributes`.

Example

```
model = direct_model(
    optimizer_with_attributes(
        Gurobi.Optimizer,
        "Presolve" => 0,
        "OutputFlag" => 1,
    )
)
```

is equivalent to:

```
model = direct_model(Gurobi.Optimizer())
set_optimizer_attribute(model, "Presolve", 0)
set_optimizer_attribute(model, "OutputFlag", 1)
```

source

19.2 Enums

`JuMP.ModelMode` – Type.

```
ModelMode
```

An enum to describe the state of the CachingOptimizer inside a JuMP model.

source

`JuMP.AUTOMATIC` – Constant.

`moi_backend` field holds a CachingOptimizer in AUTOMATIC mode.

source

`JuMP.MANUAL` – Constant.

`moi_backend` field holds a CachingOptimizer in MANUAL mode.

source

`JuMP.DIRECT` – Constant.

`moi_backend` field holds an AbstractOptimizer. No extra copy of the model is stored. The `moi_backend` must support `add_constraint` etc.

source

19.3 Basic functions

`JuMP.backend` – Function.

```
backend(model::Model)
```

Return the lower-level MathOptInterface model that sits underneath JuMP. This model depends on which operating mode JuMP is in (see `mode`).

- If JuMP is in `DIRECT` mode (i.e., the model was created using `direct_model`), the backend will be the optimizer passed to `direct_model`.

- If JuMP is in MANUAL or AUTOMATIC mode, the backend is a `MOI.Utilities.CachingOptimizer`.

This function should only be used by advanced users looking to access low-level MathOptInterface or solver-specific functionality.

Notes

If JuMP is not in DIRECT mode, the type returned by backend may change between any JuMP releases. Therefore, only use the public API exposed by MathOptInterface, and do not access internal fields. If you require access to the innermost optimizer, see `unsafe_backend`. Alternatively, use `direct_model` to create a JuMP model in DIRECT mode.

See also: `unsafe_backend`.

source

`JuMP.unsafe_backend` – Function.

```
unsafe_backend(model::Model)
```

Return the innermost optimizer associated with the JuMP model `model`.

This function should only be used by advanced users looking to access low-level solver-specific functionality. It has a high-risk of incorrect usage. We strongly suggest you use the alternative suggested below.

See also: `backend`.

Unsafe behavior

This function is unsafe for two main reasons.

First, the formulation and order of variables and constraints in the unsafe backend may be different to the variables and constraints in `model`. This can happen because of bridges, or because the solver requires the variables or constraints in a specific order. In addition, the variable or constraint index returned by `index` at the JuMP level may be different to the index of the corresponding variable or constraint in the `unsafe_backend`. There is no solution to this. Use the alternative suggested below instead.

Second, the `unsafe_backend` may be empty, or lack some modifications made to the JuMP model. Thus, before calling `unsafe_backend` you should first call `MOI.Utilities.attach_optimizer` to ensure that the backend is synchronized with the JuMP model.

```
MOI.Utilities.attach_optimizer(model)
inner = unsafe_backend(model)
```

Moreover, if you modify the JuMP model, the reference you have to the backend (i.e., `inner` in the example above) may be out-dated, and you should call `MOI.Utilities.attach_optimizer` again.

This function is also unsafe in the reverse direction: if you modify the unsafe backend, e.g., by adding a new constraint to `inner`, the changes may be silently discarded by JuMP when the JuMP `model` is modified or solved.

Alternative

Instead of `unsafe_backend`, create a model using `direct_model` and call `backend` instead.

For example, instead of:

```
model = Model(HiGHS.Optimizer)
@variable(model, x >= 0)
MOI.Utilities.attach_optimizer(model)
highs = unsafe_backend(model)
```

Use:

```
model = direct_model(HiGHS.Optimizer())
@variable(model, x >= 0)
highs = backend(model)  # No need to call `attach_optimizer`.
```

source

JuMP.name – Method.

```
name(model::AbstractModel)
```

Return the MOI.Name attribute of model's backend, or a default if empty.

source

JuMP.solver_name – Function.

```
solver_name(model::Model)
```

If available, returns the SolverName property of the underlying optimizer.

Returns "No optimizer attached" in AUTOMATIC or MANUAL modes when no optimizer is attached.

Returns "SolverName() attribute not implemented by the optimizer." if the attribute is not implemented.

source

Base.empty! – Method.

```
empty!(model::Model)::Model
```

Empty the model, that is, remove all variables, constraints and model attributes but not optimizer attributes. Always return the argument.

Note: removes extensions data.

source

Base.isempty – Method.

```
isempty(model::Model)
```

Verifies whether the model is empty, that is, whether the MOI backend is empty and whether the model is in the same state as at its creation apart from optimizer attributes.

source

JuMP.mode – Function.

```
mode(model::Model)
```

Return the ModelMode (DIRECT, AUTOMATIC, or MANUAL) of model.

source

JuMP.object_dictionary – Function.

```
object_dictionary(model::Model)
```

Return the dictionary that maps the symbol name of a variable, constraint, or expression to the corresponding object.

Objects are registered to a specific symbol in the macros. For example, `@variable(model, x[1:2, 1:2])` registers the array of variables `x` to the symbol `:x`.

This method should be defined for any subtype of `AbstractModel`.

source

`JuMP.unregister` – Function.

```
unregister(model::Model, key::Symbol)
```

Unregister the name key from `model` so that a new variable, constraint, or expression can be created with the same key.

Note that this will not delete the object `model[key]`; it will just remove the reference at `model[key]`. To delete the object, use

```
delete(model, model[key])
unregister(model, key)
```

See also: `object_dictionary`.

Examples

```
julia> @variable(model, x)
x

julia> @variable(model, x)
ERROR: An object of name x is already attached to this model. If
this is intended, consider using the anonymous construction syntax,
e.g., `x = @variable(model, [1:N], ...)` where the name of the object
does not appear inside the macro.

Alternatively, use `unregister(model, :x)` to first unregister the
existing name from the model. Note that this will not delete the object;
it will just remove the reference at `model[:x]`.
[...]

julia> num_variables(model)
1

julia> unregister(model, :x)

julia> @variable(model, x)
x

julia> num_variables(model)
2
```

source

`JuMP.latex_formulation` – Function.

```
latex_formulation(model::AbstractModel)
```

Wrap `model` in a type so that it can be pretty-printed as `text/latex` in a notebook like IJulia, or in Documenter.

To render the model, end the cell with `latex_formulation(model)`, or call `display(latex_formulation(model))` in to force the display of the model from inside a function.

source

19.4 Working with attributes

`JuMP.set_optimizer` – Function.

```
set_optimizer(
    model::Model,
    optimizer_factory;
    add_bridges::Bool = true,
)
```

Creates an empty `MathOptInterface.AbstractOptimizer` instance by calling `optimizer_factory()` and sets it as the optimizer of `model`. Specifically, `optimizer_factory` must be callable with zero arguments and return an empty `MathOptInterface.AbstractOptimizer`.

If `add_bridges` is true, constraints and objectives that are not supported by the optimizer are automatically bridged to equivalent supported formulation. Passing `add_bridges = false` can improve performance if the solver natively supports all of the elements in `model`.

See `set_optimizer_attributes` and `set_optimizer_attribute` for setting solver-specific parameters of the optimizer.

Examples

```
model = Model()
set_optimizer(model, HiGHS.Optimizer)
set_optimizer(model, HiGHS.Optimizer; add_bridges = false)
```

source

`JuMP.optimizer_with_attributes` – Function.

```
optimizer_with_attributes(optimizer_constructor, attrs::Pair...)
```

Groups an optimizer constructor with the list of attributes `attrs`. Note that it is equivalent to `MOI.OptimizerWithAttributes`.

When provided to the `Model` constructor or to `set_optimizer`, it creates an optimizer by calling `optimizer_constructor()`, and then sets the attributes using `set_optimizer_attribute`.

Example

```
model = Model(
    optimizer_with_attributes(
        Gurobi.Optimizer, "Presolve" => 0, "OutputFlag" => 1
    )
)
```

is equivalent to:

```
model = Model(Gurobi.Optimizer)
set_optimizer_attribute(model, "Presolve", 0)
set_optimizer_attribute(model, "OutputFlag", 1)
```

Note

The string names of the attributes are specific to each solver. One should consult the solver's documentation to find the attributes of interest.

See also: `set_optimizer_attribute`, `set_optimizer_attributes`, `get_optimizer_attribute`.

source

`JuMP.get_optimizer_attribute` – Function.

```
get_optimizer_attribute(model, name::String)
```

Return the value associated with the solver-specific attribute named `name`.

Note that this is equivalent to `get_optimizer_attribute(model, MOI.RawOptimizerAttribute(name))`.

Example

```
get_optimizer_attribute(model, "SolverSpecificAttributeName")
```

See also: `set_optimizer_attribute`, `set_optimizer_attributes`.

source

```
get_optimizer_attribute(
    model::Model, attr::MOI.AbstractOptimizerAttribute
)
```

Return the value of the solver-specific attribute `attr` in `model`.

Example

```
get_optimizer_attribute(model, MOI.Silent())
```

See also: `set_optimizer_attribute`, `set_optimizer_attributes`.

source

`JuMP.set_optimizer_attribute` – Function.

```
set_optimizer_attribute(model::Model, name::String, value)
```

Sets solver-specific attribute identified by `name` to `value`.

Note that this is equivalent to `set_optimizer_attribute(model, MOI.RawOptimizerAttribute(name), value)`.

Example

```
set_optimizer_attribute(model, "SolverSpecificAttributeName", true)
```

See also: `set_optimizer_attributes`, `get_optimizer_attribute`.

source

```
set_optimizer_attribute(
    model::Model,
    attr::MOI.AbstractOptimizerAttribute,
    value,
)
```

Set the solver-specific attribute `attr` in `model` to `value`.

Example

```
set_optimizer_attribute(model, MOI.Silent(), true)
```

See also: `set_optimizer_attributes`, `get_optimizer_attribute`.

source

`JuMP.set_optimizer_attributes` – Function.

```
set_optimizer_attributes(model::Model, pairs::Pair...)
```

Given a list of `attribute => value` pairs, calls `set_optimizer_attribute(model, attribute, value)` for each pair.

Example

```
model = Model(Ipopt.Optimizer)
set_optimizer_attributes(model, "tol" => 1e-4, "max_iter" => 100)
```

is equivalent to:

```
model = Model(Ipopt.Optimizer)
set_optimizer_attribute(model, "tol", 1e-4)
set_optimizer_attribute(model, "max_iter", 100)
```

See also: `set_optimizer_attribute`, `get_optimizer_attribute`.

source

`JuMP.set_silent` – Function.

```
set_silent(model::Model)
```

Takes precedence over any other attribute controlling verbosity and requires the solver to produce no output.

See also: `unset_silent`.

source

`JuMP.unset_silent` – Function.

```
unset_silent(model::Model)
```

Neutralize the effect of the `set_silent` function and let the solver attributes control the verbosity.

See also: `set_silent`.

source

`JuMP.set_time_limit_sec` – Function.

```
set_time_limit_sec(model::Model, limit::Float64)
```

Set the time limit (in seconds) of the solver.

Can be unset using `unset_time_limit_sec` or with `limit` set to `nothing`.

See also: `unset_time_limit_sec`, `time_limit_sec`.

source

`JuMP.unset_time_limit_sec` – Function.

```
unset_time_limit_sec(model::Model)
```

Unset the time limit of the solver.

See also: `set_time_limit_sec`, `time_limit_sec`.

source

`JuMP.time_limit_sec` – Function.

```
time_limit_sec(model::Model)
```

Return the time limit (in seconds) of the `model`.

Returns `nothing` if unset.

See also: `set_time_limit_sec`, `unset_time_limit_sec`.

source

19.5 Copying

`JuMP.ReferenceMap` – Type.

```
ReferenceMap
```

Mapping between variable and constraint reference of a model and its copy. The reference of the copied model can be obtained by indexing the map with the reference of the corresponding reference of the original model.

source

`JuMP.copy_model` – Function.

```
copy_model(model::Model; filter_constraints::Union{Nothing, Function}=nothing)
```

Return a copy of the model `model` and a `ReferenceMap` that can be used to obtain the variable and constraint reference of the new model corresponding to a given `model`'s reference. A `Base.copy(::AbstractModel)` method has also been implemented, it is similar to `copy_model` but does not return the reference map.

If the `filter_constraints` argument is given, only the constraints for which this function returns `true` will be copied. This function is given a constraint reference as argument.

Note

Model copy is not supported in `DIRECT` mode, i.e. when a model is constructed using the `direct_model` constructor instead of the `Model` constructor. Moreover, independently on whether an optimizer was provided at model construction, the new model will have no optimizer, i.e., an optimizer will have to be provided to the new model in the `optimize!` call.

Examples

In the following example, a model `model` is constructed with a variable `x` and a constraint `cref`. It is then copied into a model `new_model` with the new references assigned to `x_new` and `cref_new`.

```
model = Model()
@variable(model, x)
@constraint(model, cref, x == 2)

new_model, reference_map = copy_model(model)
x_new = reference_map[x]
cref_new = reference_map[cref]
```

source

`JuMP.copy_extension_data` – Function.

```
copy_extension_data(data, new_model::AbstractModel, model::AbstractModel)
```

Return a copy of the extension data `data` of the model `model` to the extension data of the new model `new_model`.

A method should be added for any JuMP extension storing data in the `ext` field.

Warning

Do not engage in type piracy by implementing this method for types of `data` that you did not define! JuMP extensions should store types that they define in `model.ext`, rather than regular Julia types.

source

`Base.copy` – Method.

```
copy(model::AbstractModel)
```

Return a copy of the model `model`. It is similar to `copy_model` except that it does not return the mapping between the references of `model` and its copy.

Note

Model copy is not supported in `DIRECT` mode, i.e. when a model is constructed using the `direct_model` constructor instead of the `Model` constructor. Moreover, independently on whether an optimizer was provided at model construction, the new model will have no optimizer, i.e., an optimizer will have to be provided to the new model in the `optimize!` call.

Examples

In the following example, a model `model` is constructed with a variable `x` and a constraint `cref`. It is then copied into a model `new_model` with the new references assigned to `x_new` and `cref_new`.

```
model = Model()
@variable(model, x)
@constraint(model, cref, x == 2)

new_model = copy(model)
x_new = model[:x]
cref_new = model[:cref]
```

source

19.6 I/O

JuMP.write_to_file – Function.

```
write_to_file(
    model::Model,
    filename::String;
    format::MOI.FileFormats.FileFormat = MOI.FileFormats.FORMAT_AUTOMATIC,
    kwargs...,
)
```

Write the JuMP model `model` to `filename` in the format `format`.

If the filename ends in `.gz`, it will be compressed using Gzip. If the filename ends in `.bz2`, it will be compressed using BZip2.

Other `kwargs` are passed to the `Model` constructor of the chosen format.

source

Base.write – Method.

```
Base.write(
    io::IO,
    model::Model;
    format::MOI.FileFormats.FileFormat = MOI.FileFormats.FORMAT_MOF,
    kwargs...,
)
```

Write the JuMP model `model` to `io` in the format `format`.

Other `kwargs` are passed to the `Model` constructor of the chosen format.

source

JuMP.read_from_file – Function.

```
read_from_file(
    filename::String;
    format::MOI.FileFormats.FileFormat = MOI.FileFormats.FORMAT_AUTOMATIC,
    kwargs...,
)
```

Return a JuMP model read from `filename` in the format `format`.

If the filename ends in `.gz`, it will be uncompressed using Gzip. If the filename ends in `.bz2`, it will be uncompressed using BZip2.

Other `kwargs` are passed to the `Model` constructor of the chosen format.

source

Base.read – Method.

```
Base.read(
    io::IO,
    ::Type{Model};
    format::MOI.FileFormats.FileFormat,
    kwargs...,
)
```

Return a JuMP model read from `io` in the format `format`.

Other `kwargs` are passed to the `Model` constructor of the chosen format.

source

19.7 Caching Optimizer

`MathOptInterface.Utilities.reset_optimizer` – Method.

```
MOIU.reset_optimizer(model::Model)
```

Call `MOIU.reset_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

`MathOptInterface.Utilities.drop_optimizer` – Method.

```
MOIU.drop_optimizer(model::Model)
```

Call `MOIU.drop_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

`MathOptInterface.Utilities.attach_optimizer` – Method.

```
MOIU.attach_optimizer(model::Model)
```

Call `MOIU.attach_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

19.8 Bridge tools

`JuMP.bridge_constraints` – Function.

```
bridge_constraints(model::Model)
```

When in direct mode, return `false`. When in manual or automatic mode, return a `Bool` indicating whether the optimizer is set and unsupported constraints are automatically bridged to equivalent supported constraints when an appropriate transformation is available.

source

`JuMP.print_bridge_graph` – Function.

```
print_bridge_graph([io::IO,] model::Model)
```

Print the hyper-graph containing all variable, constraint, and objective types that could be obtained by bridging the variables, constraints, and objectives that are present in the model.

Each node in the hyper-graph corresponds to a variable, constraint, or objective type.

- Variable nodes are indicated by `[ ]`

- Constraint nodes are indicated by ( )
- Objective nodes are indicated by | |

The number inside each pair of brackets is an index of the node in the hyper-graph.

Note that this hyper-graph is the full list of possible transformations. When the bridged model is created, we select the shortest hyper-path(s) from this graph, so many nodes may be un-used.

For more information, see Legat, B., Dowson, O., Garcia, J., and Lubin, M. (2020). "MathOptInterface: a data structure for mathematical optimization problems." URL: https://arxiv.org/abs/2002.03447

source

19.9 Extension tools

`JuMP.AbstractModel` – Type.

```
AbstractModel
```

An abstract type that should be subtyped for users creating JuMP extensions.

source

`JuMP.operator_warn` – Function.

```
operator_warn(model::AbstractModel)
operator_warn(model::Model)
```

This function is called on the model whenever two affine expressions are added together without using `destructive_add!`, and at least one of the two expressions has more than 50 terms.

For the case of `Model`, if this function is called more than 20,000 times then a warning is generated once.

source

`JuMP.error_if_direct_mode` – Function.

```
error_if_direct_mode(model::Model, func::Symbol)
```

Errors if `model` is in direct mode during a call from the function named `func`.

Used internally within JuMP, or by JuMP extensions who do not want to support models in direct mode.

source

Chapter 20

Variables

More information can be found in the Variables section of the manual.

20.1 Macros

`JuMP.@variable` – Macro.

```
@variable(model, expr, args..., kw_args...)
```

Add a variable to the model `model` described by the expression `expr`, the positional arguments `args` and the keyword arguments `kw_args`.

Anonymous and named variables

`expr` must be one of the forms:

- Omitted, like `@variable(model)`, which creates an anonymous variable
- A single symbol like `@variable(model, x)`
- A container expression like `@variable(model, x[i=1:3])`
- An anoymous container expression like `@variable(model, [i=1:3])`

Bounds

In addition, the expression can have bounds, such as:

- `@variable(model, x >= 0)`
- `@variable(model, x <= 0)`
- `@variable(model, x == 0)`
- `@variable(model, 0 <= x <= 1)`

and bounds can depend on the indices of the container expressions:

- `@variable(model, -i <= x[i=1:3] <= i)`

Sets

You can explicitly specify the set to which the variable belongs:

- `@variable(model, x in MOI.Interval(0.0, 1.0))`

For more information on this syntax, read Variables constrained on creation.

Positional arguments

The recognized positional arguments in `args` are the following:

- `Bin`: restricts the variable to the `MOI.ZeroOne` set, that is, `{0, 1}`. For example, `@variable(model, x, Bin)`. Note: you cannot use `@variable(model, Bin)`, use the `binary` keyword instead.
- `Int`: restricts the variable to the set of integers, that is, ..., -2, -1, 0, 1, 2, ... For example, `@variable(model, x, Int)`. Note: you cannot use `@variable(model, Int)`, use the `integer` keyword instead.
- `Symmetric`: Only available when creating a square matrix of variables, i.e., when `expr` is of the form `varname[1:n,1:n]` or `varname[i=1:n,j=1:n]`, it creates a symmetric matrix of variables.
- `PSD`: A restrictive extension to `Symmetric` which constraints a square matrix of variables to `Symmetric` and constrains to be positive semidefinite.

Keyword arguments

Three keyword arguments are useful in all cases:

- `base_name`: Sets the name prefix used to generate variable names. It corresponds to the variable name for scalar variable, otherwise, the variable names are set to `base_name[...]` for each index `...` of the axes axes.
- `start::Float64`: specify the value passed to `set_start_value` for each variable
- `container`: specify the container type. See Forcing the container type for more information.

Other keyword arguments are needed to disambiguate sitations with anonymous variables:

- `lower_bound::Float64`: an alternative to `x >= lb`, sets the value of the variable lower bound.
- `upper_bound::Float64`: an alternative to `x <= ub`, sets the value of the variable upper bound.
- `binary::Bool`: an alternative to passing `Bin`, sets whether the variable is binary or not.
- `integer::Bool`: an alternative to passing `Int`, sets whether the variable is integer or not.
- `set::MOI.AbstractSet`: an alternative to using `x in set`
- `variable_type`: used by JuMP extensions. See Extend `@variable` for more information.

Examples

The following are equivalent ways of creating a variable `x` of name `x` with lower bound 0:

```
model = Model()
@variable(model, x >= 0)
@variable(model, x, lower_bound = 0)
x = @variable(model, base_name = "x", lower_bound = 0)
```

Other examples:

```
model = Model()
@variable(model, x[i=1:3] <= i, Int, start = sqrt(i), lower_bound = -i)
@variable(model, y[i=1:3], container = DenseAxisArray, set = MOI.ZeroOne())
```

source

`JuMP.@variables` – Macro.

```
@variables(model, args...)
```

Adds multiple variables to model at once, in the same fashion as the `@variable` macro.

The model must be the first argument, and multiple variables can be added on multiple lines wrapped in a `begin ... end` block.

The macro returns a tuple containing the variables that were defined.

Examples

```
@variables(model, begin
    x
    y[i = 1:2] >= 0, (start = i)
    z, Bin, (start = 0, base_name = "Z")
end)
```

> **Note**
> Keyword arguments must be contained within parentheses (refer to the example above).

source

20.2 Basic utilities

`JuMP.VariableRef` – Type.

```
VariableRef <: AbstractVariableRef
```

Holds a reference to the model and the corresponding MOI.VariableIndex.

source

`JuMP.num_variables` – Function.

```
num_variables(model::Model)::Int64
```

Returns number of variables in `model`.

source

`JuMP.all_variables` – Function.

```
all_variables(model::Model)::Vector{VariableRef}
```

Returns a list of all variables currently in the model. The variables are ordered by creation time.

Example

```
model = Model()
@variable(model, x)
@variable(model, y)
all_variables(model)

# output

2-element Array{VariableRef,1}:
 x
 y
```

source

JuMP.owner_model – Function.

```
owner_model(s::AbstractJuMPScalar)
```

Return the model owning the scalar s.

source

JuMP.index – Method.

```
index(v::VariableRef)::MOI.VariableIndex
```

Return the index of the variable that corresponds to v in the MOI backend.

source

JuMP.optimizer_index – Method.

```
optimizer_index(v::VariableRef)::MOI.VariableIndex
```

Return the index of the variable that corresponds to v in the optimizer model. It throws NoOptimizer if no optimizer is set and throws an ErrorException if the optimizer is set but is not attached.

source

JuMP.check_belongs_to_model – Function.

```
check_belongs_to_model(func::AbstractJuMPScalar, model::AbstractModel)
```

Throw VariableNotOwned if the owner_model of one of the variables of the function func is not model.

```
check_belongs_to_model(constraint::AbstractConstraint, model::AbstractModel)
```

Throw VariableNotOwned if the owner_model of one of the variables of the constraint constraint is not model.

source

JuMP.VariableNotOwned – Type.

```
struct VariableNotOwned{V <: AbstractVariableRef} <: Exception
    variable::V
end
```

The variable variable was used in a model different to owner_model(variable).

source

JuMP.VariableConstrainedOnCreation – Type.

```
VariablesConstrainedOnCreation <: AbstractVariable
```

Variable scalar_variables constrained to belong to set. Adding this variable can be understood as doing:

```
function JuMP.add_variable(model::Model, variable::JuMP.VariableConstrainedOnCreation, names)
    var_ref = JuMP.add_variable(model, variable.scalar_variable, name)
    JuMP.add_constraint(model, JuMP.VectorConstraint(var_ref, variable.set))
    return var_ref
end
```

but adds the variables with `MOI.add_constrained_variable(model, variable.set)` instead. See the MOI documentation for the difference between adding the variables with `MOI.add_constrained_variable` and adding them with `MOI.add_variable` and adding the constraint separately.

source

`JuMP.VariablesConstrainedOnCreation` – Type.

```
VariablesConstrainedOnCreation <: AbstractVariable
```

Vector of variables `scalar_variables` constrained to belong to `set`. Adding this variable can be thought as doing:

```
function JuMP.add_variable(model::Model, variable::JuMP.VariablesConstrainedOnCreation, names)
    var_refs = JuMP.add_variable.(model, variable.scalar_variables,
                                  JuMP.vectorize(names, variable.shape))
    JuMP.add_constraint(model, JuMP.VectorConstraint(var_refs, variable.set))
    return JuMP.reshape_vector(var_refs, variable.shape)
end
```

but adds the variables with `MOI.add_constrained_variables(model, variable.set)` instead. See the MOI documentation for the difference between adding the variables with `MOI.add_constrained_variables` and adding them with `MOI.add_variables` and adding the constraint separately.

source

20.3 Names

`JuMP.name` – Method.

```
name(v::VariableRef)::String
```

Get a variable's name attribute.

source

`JuMP.set_name` – Method.

```
set_name(v::VariableRef, s::AbstractString)
```

Set a variable's name attribute.

source

`JuMP.variable_by_name` – Function.

```
variable_by_name(model::AbstractModel,
                 name::String)::Union{AbstractVariableRef, Nothing}
```

Returns the reference of the variable with name attribute `name` or `Nothing` if no variable has this name attribute. Throws an error if several variables have `name` as their name attribute.

```
julia> model = Model();

julia> @variable(model, x)
x
```

```
julia> variable_by_name(model, "x")
x

julia> @variable(model, base_name="x")
x

julia> variable_by_name(model, "x")
ERROR: Multiple variables have the name x.
Stacktrace:
 [1] error(::String) at ./error.jl:33
 [2] get(::MOIU.Model{Float64}, ::Type{MathOptInterface.VariableIndex}, ::String) at
↪  /home/blegat/.julia/dev/MathOptInterface/src/Utilities/model.jl:222
 [3] get at /home/blegat/.julia/dev/MathOptInterface/src/Utilities/universalfallback.jl:201
↪  [inlined]
 [4]
↪  get(::MathOptInterface.Utilities.CachingOptimizer{MathOptInterface.AbstractOptimizer,MathOptInterface.Utilitie
↪  ::Type{MathOptInterface.VariableIndex}, ::String) at
↪  /home/blegat/.julia/dev/MathOptInterface/src/Utilities/cachingoptimizer.jl:490
 [5] variable_by_name(::Model, ::String) at /home/blegat/.julia/dev/JuMP/src/variables.jl:268
 [6] top-level scope at none:0

julia> var = @variable(model, base_name="y")
y

julia> variable_by_name(model, "y")
y

julia> set_name(var, "z")

julia> variable_by_name(model, "y")

julia> variable_by_name(model, "z")
z

julia> @variable(model, u[1:2])
2-element Array{VariableRef,1}:
 u[1]
 u[2]

julia> variable_by_name(model, "u[2]")
u[2]
```

source

20.4 Start values

`JuMP.set_start_value` – Function.

```
set_start_value(con_ref::ConstraintRef, value)
```

Set the primal start value (`MOI.ConstraintPrimalStart`) of the constraint `con_ref` to `value`. To remove a primal start value set it to `nothing`.

See also `start_value`.

source

```
set_start_value(variable::VariableRef, value::Union{Real,Nothing})
```

Set the start value (MOI attribute `VariablePrimalStart`) of the `variable` to `value`.

Pass `nothing` to unset the start value.

Note: `VariablePrimalStarts` are sometimes called "MIP-starts" or "warmstarts".

See also `start_value`.

source

`JuMP.start_value` – Function.

```
start_value(con_ref::ConstraintRef)
```

Return the primal start value (`MOI.ConstraintPrimalStart`) of the constraint `con_ref`.

Note: If no primal start value has been set, `start_value` will return `nothing`.

See also `set_start_value`.

source

```
start_value(v::VariableRef)
```

Return the start value (MOI attribute `VariablePrimalStart`) of the variable `v`.

Note: `VariablePrimalStarts` are sometimes called "MIP-starts" or "warmstarts".

See also `set_start_value`.

source

20.5 Lower bounds

`JuMP.has_lower_bound` – Function.

```
has_lower_bound(v::VariableRef)
```

Return `true` if `v` has a lower bound. If `true`, the lower bound can be queried with `lower_bound`.

See also `LowerBoundRef`, `lower_bound`, `set_lower_bound`, `delete_lower_bound`.

source

`JuMP.lower_bound` – Function.

```
lower_bound(v::VariableRef)
```

Return the lower bound of a variable. Error if one does not exist.

See also `LowerBoundRef`, `has_lower_bound`, `set_lower_bound`, `delete_lower_bound`.

source

`JuMP.set_lower_bound` – Function.

```
set_lower_bound(v::VariableRef, lower::Number)
```

Set the lower bound of a variable. If one does not exist, create a new lower bound constraint.

See also `LowerBoundRef`, `has_lower_bound`, `lower_bound`, `delete_lower_bound`.

source

`JuMP.delete_lower_bound` – Function.

```
delete_lower_bound(v::VariableRef)
```

Delete the lower bound constraint of a variable.

See also `LowerBoundRef`, `has_lower_bound`, `lower_bound`, `set_lower_bound`.

`source`

`JuMP.LowerBoundRef` – Function.

```
LowerBoundRef(v::VariableRef)
```

Return a constraint reference to the lower bound constraint of `v`. Errors if one does not exist.

See also `has_lower_bound`, `lower_bound`, `set_lower_bound`, `delete_lower_bound`.

`source`

20.6 Upper bounds

`JuMP.has_upper_bound` – Function.

```
has_upper_bound(v::VariableRef)
```

Return `true` if `v` has a upper bound. If `true`, the upper bound can be queried with `upper_bound`.

See also `UpperBoundRef`, `upper_bound`, `set_upper_bound`, `delete_upper_bound`.

`source`

`JuMP.upper_bound` – Function.

```
upper_bound(v::VariableRef)
```

Return the upper bound of a variable. Error if one does not exist.

See also `UpperBoundRef`, `has_upper_bound`, `set_upper_bound`, `delete_upper_bound`.

`source`

`JuMP.set_upper_bound` – Function.

```
set_upper_bound(v::VariableRef,upper::Number)
```

Set the upper bound of a variable. If one does not exist, create an upper bound constraint.

See also `UpperBoundRef`, `has_upper_bound`, `upper_bound`, `delete_upper_bound`.

`source`

`JuMP.delete_upper_bound` – Function.

```
delete_upper_bound(v::VariableRef)
```

Delete the upper bound constraint of a variable.

See also `UpperBoundRef`, `has_upper_bound`, `upper_bound`, `set_upper_bound`.

`source`

JuMP.UpperBoundRef – Function.

```
UpperBoundRef(v::VariableRef)
```

Return a constraint reference to the upper bound constraint of v. Errors if one does not exist.

See also has_upper_bound, upper_bound, set_upper_bound, delete_upper_bound.

source

20.7 Fixed bounds

JuMP.is_fixed – Function.

```
is_fixed(v::VariableRef)
```

Return true if v is a fixed variable. If true, the fixed value can be queried with fix_value.

See also FixRef, fix_value, fix, unfix.

source

JuMP.fix_value – Function.

```
fix_value(v::VariableRef)
```

Return the value to which a variable is fixed. Error if one does not exist.

See also FixRef, is_fixed, fix, unfix.

source

JuMP.fix – Function.

```
fix(v::VariableRef, value::Number; force::Bool = false)
```

Fix a variable to a value. Update the fixing constraint if one exists, otherwise create a new one.

If the variable already has variable bounds and force=false, calling fix will throw an error. If force=true, existing variable bounds will be deleted, and the fixing constraint will be added. Note a variable will have no bounds after a call to unfix.

See also FixRef, is_fixed, fix_value, unfix.

source

JuMP.unfix – Function.

```
unfix(v::VariableRef)
```

Delete the fixing constraint of a variable.

See also FixRef, is_fixed, fix_value, fix.

source

JuMP.FixRef – Function.

```
FixRef(v::VariableRef)
```

Return a constraint reference to the constraint fixing the value of v. Errors if one does not exist.

See also is_fixed, fix_value, fix, unfix.

source

20.8 Integer variables

JuMP.is_integer – Function.

```
is_integer(v::VariableRef)
```

Return `true` if v is constrained to be integer.

See also IntegerRef, set_integer, unset_integer.

source

JuMP.set_integer – Function.

```
set_integer(variable_ref::VariableRef)
```

Add an integrality constraint on the variable `variable_ref`.

See also IntegerRef, is_integer, unset_integer.

source

JuMP.unset_integer – Function.

```
unset_integer(variable_ref::VariableRef)
```

Remove the integrality constraint on the variable `variable_ref`.

See also IntegerRef, is_integer, set_integer.

source

JuMP.IntegerRef – Function.

```
IntegerRef(v::VariableRef)
```

Return a constraint reference to the constraint constraining v to be integer. Errors if one does not exist.

See also is_integer, set_integer, unset_integer.

source

20.9 Binary variables

JuMP.is_binary – Function.

```
is_binary(v::VariableRef)
```

Return `true` if v is constrained to be binary.

See also BinaryRef, set_binary, unset_binary.

source

JuMP.set_binary – Function.

```
set_binary(v::VariableRef)
```

Add a constraint on the variable v that it must take values in the set $\{0, 1\}$.

See also BinaryRef, is_binary, unset_binary.

source

JuMP.unset_binary – Function.

```
unset_binary(variable_ref::VariableRef)
```

Remove the binary constraint on the variable `variable_ref`.

See also `BinaryRef`, `is_binary`, `set_binary`.

source

JuMP.BinaryRef – Function.

```
BinaryRef(v::VariableRef)
```

Return a constraint reference to the constraint constraining v to be binary. Errors if one does not exist.

See also `is_binary`, `set_binary`, `unset_binary`.

source

20.10 Integrality utilities

JuMP.relax_integrality – Function.

```
relax_integrality(model::Model)
```

Modifies `model` to "relax" all binary and integrality constraints on variables. Specifically,

- Binary constraints are deleted, and variable bounds are tightened if necessary to ensure the variable is constrained to the interval $[0, 1]$.
- Integrality constraints are deleted without modifying variable bounds.
- An error is thrown if semi-continuous or semi-integer constraints are present (support may be added for these in the future).
- All other constraints are ignored (left in place). This includes discrete constraints like SOS and indicator constraints.

Returns a function that can be called without any arguments to restore the original model. The behavior of this function is undefined if additional changes are made to the affected variables in the meantime.

Example

```
julia> model = Model();

julia> @variable(model, x, Bin);

julia> @variable(model, 1 <= y <= 10, Int);

julia> @objective(model, Min, x + y);

julia> undo_relax = relax_integrality(model);

julia> print(model)
Min x + y
Subject to
 x ≥ 0.0
 y ≥ 1.0
```

```
 x ≤ 1.0
 y ≤ 10.0

julia> undo_relax()

julia> print(model)
Min x + y
Subject to
 y ≥ 1.0
 y ≤ 10.0
 y integer
 x binary
```

source

20.11 Extensions

`JuMP.AbstractVariable` – Type.

```
AbstractVariable
```

Variable returned by `build_variable`. It represents a variable that has not been added yet to any model. It can be added to a given `model` with `add_variable`.

source

`JuMP.AbstractVariableRef` – Type.

```
AbstractVariableRef
```

Variable returned by `add_variable`. Affine (resp. quadratic) operations with variables of type `V<:AbstractVariableRef` and coefficients of type `T` create a `GenericAffExpr{T,V}` (resp. `GenericQuadExpr{T,V}`).

source

`JuMP.parse_one_operator_variable` – Function.

```
parse_one_operator_variable(_error::Function, infoexpr::_VariableInfoExpr, sense::Val{S}, value)
↪  where S
```

Update `infoexr` for a variable expression in the `@variable` macro of the form `variable name S value`.

source

Chapter 21

Expressions

More information can be found in the Expressions section of the manual.

21.1 Macros

`JuMP.@expression` – Macro.

```
@expression(args...)
```

Efficiently builds a linear or quadratic expression but does not add to model immediately. Instead, returns the expression which can then be inserted in other constraints. For example:

```
@expression(m, shared, sum(i*x[i] for i=1:5))
@constraint(m, shared + y >= 5)
@constraint(m, shared + z <= 10)
```

The `ref` accepts index sets in the same way as `@variable`, and those indices can be used in the construction of the expressions:

```
@expression(m, expr[i=1:3], i*sum(x[j] for j=1:3))
```

Anonymous syntax is also supported:

```
expr = @expression(m, [i=1:3], i*sum(x[j] for j=1:3))
```

source

`JuMP.@expressions` – Macro.

```
@expressions(model, args...)
```

Adds multiple expressions to model at once, in the same fashion as the `@expression` macro.

The model must be the first argument, and multiple expressions can be added on multiple lines wrapped in a `begin ... end` block.

The macro returns a tuple containing the expressions that were defined.

Examples

```
@expressions(model, begin
    my_expr, x^2 + y^2
    my_expr_1[i = 1:2], a[i] - z[i]
end)
```

source

21.2 Affine expressions

JuMP.GenericAffExpr – Type.

```
mutable struct GenericAffExpr{CoefType,VarType} <: AbstractJuMPScalar
    constant::CoefType
    terms::OrderedDict{VarType,CoefType}
end
```

An expression type representing an affine expression of the form: $\sum a_i x_i + c$.

Fields

- `.constant`: the constant c in the expression.
- `.terms`: an `OrderedDict`, with keys of `VarType` and values of `CoefType` describing the sparse vector a.

source

JuMP.AffExpr – Type.

```
AffExpr
```

Alias for `GenericAffExpr{Float64,VariableRef}`, the specific `GenericAffExpr` used by JuMP.

source

JuMP.linear_terms – Function.

```
linear_terms(aff::GenericAffExpr{C, V})
```

Provides an iterator over coefficient-variable tuples `(a_i::C, x_i::V)` in the linear part of the affine expression.

source

```
linear_terms(quad::GenericQuadExpr{C, V})
```

Provides an iterator over tuples `(coefficient::C, variable::V)` in the linear part of the quadratic expression.

source

21.3 Quadratic expressions

JuMP.GenericQuadExpr – Type.

```
mutable struct GenericQuadExpr{CoefType,VarType} <: AbstractJuMPScalar
    aff::GenericAffExpr{CoefType,VarType}
    terms::OrderedDict{UnorderedPair{VarType}, CoefType}
end
```

An expression type representing an quadratic expression of the form: $\sum q_{i,j} x_i x_j + \sum a_i x_i + c$.

Fields

- `.aff`: an `GenericAffExpr` representing the affine portion of the expression.
- `.terms`: an `OrderedDict`, with keys of `UnorderedPair{VarType}` and values of `CoefType`, describing the sparse list of terms q.

source

`JuMP.QuadExpr` – Type.

```
QuadExpr
```

An alias for `GenericQuadExpr{Float64,VariableRef}`, the specific `GenericQuadExpr` used by JuMP.

source

`JuMP.UnorderedPair` – Type.

```
UnorderedPair(a::T, b::T)
```

A wrapper type used by `GenericQuadExpr` with fields `.a` and `.b`.

source

`JuMP.quad_terms` – Function.

```
quad_terms(quad::GenericQuadExpr{C, V})
```

Provides an iterator over tuples `(coefficient::C, var_1::V, var_2::V)` in the quadratic part of the quadratic expression.

source

21.4 Utilities and modifications

`JuMP.constant` – Function.

```
constant(aff::GenericAffExpr{C, V})::C
```

Return the constant of the affine expression.

source

```
constant(aff::GenericQuadExpr{C, V})::C
```

Return the constant of the quadratic expression.

source

`JuMP.coefficient` – Function.

```
coefficient(v1::VariableRef, v2::VariableRef)
```

Return `1.0` if `v1 == v2`, and `0.0` otherwise.

This is a fallback for other `coefficient` methods to simplify code in which the expression may be a single variable.

source

```
coefficient(a::GenericAffExpr{C,V}, v::V) where {C,V}
```

Return the coefficient associated with variable v in the affine expression a.

source

```
coefficient(a::GenericAffExpr{C,V}, v1::V, v2::V) where {C,V}
```

Return the coefficient associated with the term `v1 * v2` in the quadratic expression a.

Note that `coefficient(a, v1, v2)` is the same as `coefficient(a, v2, v1)`.

source

```
coefficient(a::GenericQuadExpr{C,V}, v::V) where {C,V}
```

Return the coefficient associated with variable v in the affine component of a.

source

`JuMP.isequal_canonical` – Function.

```
isequal_canonical(
    aff::GenericAffExpr{C,V},
    other::GenericAffExpr{C,V}
) where {C,V}
```

Return `true` if `aff` is equal to `other` after dropping zeros and disregarding the order. Mainly useful for testing.

source

`JuMP.add_to_expression!` – Function.

```
add_to_expression!(expression, terms...)
```

Updates `expression` in place to `expression + (*)(terms...)`. This is typically much more efficient than `expression += (*)(terms...)`. For example, `add_to_expression!(expression, a, b)` produces the same result as `expression += a*b`, and `add_to_expression!(expression, a)` produces the same result as `expression += a`.

Only a few methods are defined, mostly for internal use, and only for the cases when (1) they can be implemented efficiently and (2) `expression` is capable of storing the result. For example, `add_to_expression!(::AffExpr, ::VariableRef, ::VariableRef)` is not defined because a `GenericAffExpr` cannot store the product of two variables.

source

`JuMP.drop_zeros!` – Function.

```
drop_zeros!(expr::GenericAffExpr)
```

Remove terms in the affine expression with `0` coefficients.

source

```
drop_zeros!(expr::GenericQuadExpr)
```

Remove terms in the quadratic expression with `0` coefficients.

source

JuMP.map_coefficients – Function.

```
map_coefficients(f::Function, a::GenericAffExpr)
```

Apply f to the coefficients and constant term of an GenericAffExpr a and return a new expression.

See also: map_coefficients_inplace!

Example

```
julia> a = GenericAffExpr(1.0, x => 1.0)
x + 1

julia> map_coefficients(c -> 2 * c, a)
2 x + 2

julia> a
x + 1
```

source

```
map_coefficients(f::Function, a::GenericQuadExpr)
```

Apply f to the coefficients and constant term of an GenericQuadExpr a and return a new expression.

See also: map_coefficients_inplace!

Example

```
julia> a = @expression(model, x^2 + x + 1)
x² + x + 1

julia> map_coefficients(c -> 2 * c, a)
2 x² + 2 x + 2

julia> a
x² + x + 1
```

source

JuMP.map_coefficients_inplace! – Function.

```
map_coefficients_inplace!(f::Function, a::GenericAffExpr)
```

Apply f to the coefficients and constant term of an GenericAffExpr a and update them in-place.

See also: map_coefficients

Example

```
julia> a = GenericAffExpr(1.0, x => 1.0)
x + 1

julia> map_coefficients_inplace!(c -> 2 * c, a)
2 x + 2

julia> a
2 x + 2
```

source

```
map_coefficients_inplace!(f::Function, a::GenericQuadExpr)
```

Apply f to the coefficients and constant term of an GenericQuadExpr a and update them in-place.

See also: map_coefficients

Example

```
julia> a = @expression(model, x^2 + x + 1)
x² + x + 1

julia> map_coefficients_inplace!(c -> 2 * c, a)
2 x² + 2 x + 2

julia> a
2 x² + 2 x + 2
```

source

21.5 JuMP-to-MOI converters

JuMP.variable_ref_type – Function.

```
variable_ref_type(::GenericAffExpr{C, V}) where {C, V}
```

A helper function used internally by JuMP and some JuMP extensions. Returns the variable type V from a GenericAffExpr

source

JuMP.jump_function – Function.

```
jump_function(x)
```

Given an MathOptInterface object x, return the JuMP equivalent.

See also: moi_function.

source

JuMP.jump_function_type – Function.

```
jump_function_type(::Type{T}) where {T}
```

Given an MathOptInterface object type T, return the JuMP equivalent.

See also: moi_function_type.

source

JuMP.moi_function – Function.

```
moi_function(x)
```

Given a JuMP object x, return the MathOptInterface equivalent.

See also: jump_function.

source

`JuMP.moi_function_type` – Function.

```
moi_function_type(::Type{T}) where {T}
```

Given a JuMP object type T, return the MathOptInterface equivalent.

See also: `jump_function_type`.

source

Chapter 22

Objectives

More information can be found in the Objectives section of the manual.

22.1 Objective functions

`JuMP.@objective` – Macro.

```
@objective(model::Model, sense, func)
```

Set the objective sense to `sense` and objective function to `func`. The objective sense can be either `Min`, `Max`, `MathOptInterface.MIN_SENSE`, `MathOptInterface.MAX_SENSE` or `MathOptInterface.FEASIBILITY_SENSE`; see `MathOptInterface.ObjectiveSense`. In order to set the sense programmatically, i.e., when `sense` is a Julia variable whose value is the sense, one of the three `MathOptInterface.ObjectiveSense` values should be used. The function `func` can be a single JuMP variable, an affine expression of JuMP variables or a quadratic expression of JuMP variables.

Examples

To minimize the value of the variable `x`, do as follows:

```
julia> model = Model()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> @variable(model, x)
x

julia> @objective(model, Min, x)
x
```

To maximize the value of the affine expression `2x - 1`, do as follows:

```
julia> @objective(model, Max, 2x - 1)
2 x - 1
```

To set a quadratic objective and set the objective sense programmatically, do as follows:

```
julia> sense = MIN_SENSE
MIN_SENSE::OptimizationSense = 0

julia> @objective(model, sense, x^2 - 2x + 1)
x² - 2 x + 1
```

source

JuMP.objective_function – Function.

```
objective_function(model::Model,
               T::Type{<:AbstractJuMPScalar}=objective_function_type(model))
```

Return an object of type T representing the objective function. Error if the objective is not convertible to type T.

Examples

```
julia> model = Model()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> @variable(model, x)
x

julia> @objective(model, Min, 2x + 1)
2 x + 1

julia> objective_function(model, AffExpr)
2 x + 1

julia> objective_function(model, QuadExpr)
2 x + 1

julia> typeof(objective_function(model, QuadExpr))
GenericQuadExpr{Float64,VariableRef}
```

We see with the last two commands that even if the objective function is affine, as it is convertible to a quadratic function, it can be queried as a quadratic function and the result is quadratic.

However, it is not convertible to a variable.

```
julia> objective_function(model, VariableRef)
ERROR: InexactError: convert(MathOptInterface.VariableIndex,
↪  MathOptInterface.ScalarAffineFunction{Float64}(MathOptInterface.ScalarAffineTerm{Float64}[MathOptInterface.Sca
↪  MathOptInterface.VariableIndex(1))], 1.0))
[...]
```

source

JuMP.set_objective_function – Function.

```
set_objective_function(
    model::Model,
    func::Union{AbstractJuMPScalar, MathOptInterface.AbstractScalarFunction})
```

Sets the objective function of the model to the given function. See `set_objective_sense` to set the objective sense. These are low-level functions; the recommended way to set the objective is with the `@objective` macro.

source

`JuMP.set_objective_coefficient` – Function.

```
set_objective_coefficient(model::Model, variable::VariableRef, coefficient::Real)
```

Set the linear objective coefficient associated with `Variable` to `coefficient`.

Note: this function will throw an error if a nonlinear objective is set.

source

`JuMP.set_objective` – Function.

```
set_objective(model::AbstractModel, sense::MOI.OptimizationSense, func)
```

The functional equivalent of the `@objective` macro.

Sets the objective sense and objective function simultaneously, and is equivalent to:

```
set_objective_sense(model, sense)
set_objective_function(model, func)
```

Examples

```
model = Model()
@variable(model, x)
set_objective(model, MIN_SENSE, x)
```

source

`JuMP.objective_function_type` – Function.

```
objective_function_type(model::Model)::AbstractJuMPScalar
```

Return the type of the objective function.

source

`JuMP.objective_function_string` – Function.

```
objective_function_string(mode, model::AbstractModel)::String
```

Return a `String` describing the objective function of the model.

source

`JuMP.show_objective_function_summary` – Function.

```
show_objective_function_summary(io::IO, model::AbstractModel)
```

Write to `io` a summary of the objective function type.

source

22.2 Objective sense

`JuMP.objective_sense` – Function.

```
objective_sense(model::Model)::MOI.OptimizationSense
```

Return the objective sense.

source

`JuMP.set_objective_sense` – Function.

```
set_objective_sense(model::Model, sense::MOI.OptimizationSense)
```

Sets the objective sense of the model to the given sense. See `set_objective_function` to set the objective function. These are low-level functions; the recommended way to set the objective is with the `@objective` macro.

source

Chapter 23

Constraints

More information can be found in the Constraints section of the manual.

23.1 Macros

`JuMP.@constraint` – Macro.

```
@constraint(m::Model, expr, kw_args...)
```

Add a constraint described by the expression `expr`.

```
@constraint(m::Model, ref[i=..., j=..., ...], expr, kw_args...)
```

Add a group of constraints described by the expression `expr` parametrized by `i`, `j`, ...

The expression `expr` can either be

- of the form `func in set` constraining the function `func` to belong to the set `set` which is either a `MOI.AbstractSet` or one of the JuMP shortcuts `SecondOrderCone`, `RotatedSecondOrderCone` and `PSDCone`, e.g. `@constraint(model, [1, x-1, y-2] in SecondOrderCone())` constrains the norm of `[x-1, y-2]` be less than 1;
- of the form `a sign b`, where `sign` is one of `==`, `≥`, `>=`, `≤` and `<=` building the single constraint enforcing the comparison to hold for the expression `a` and `b`, e.g. `@constraint(m, x^2 + y^2 == 1)` constrains `x` and `y` to lie on the unit circle;
- of the form `a ≤ b ≤ c` or `a ≥ b ≥ c` (where `≤` and `<=` (resp. `≥` and `>=`) can be used interchangeably) constraining the paired the expression `b` to lie between `a` and `c`;
- of the forms `@constraint(m, a .sign b)` or `@constraint(m, a .sign b .sign c)` which broadcast the constraint creation to each element of the vectors.

The recognized keyword arguments in `kw_args` are the following:

- `base_name`: Sets the name prefix used to generate constraint names. It corresponds to the constraint name for scalar constraints, otherwise, the constraint names are set to `base_name[...]` for each index `...` of the axes `axes`.
- `container`: Specify the container type.

Note for extending the constraint macro

Each constraint will be created using `add_constraint(m, build_constraint(_error, func, set))` where

- `_error` is an error function showing the constraint call in addition to the error message given as argument,
- `func` is the expression that is constrained
- and `set` is the set in which it is constrained to belong.

For `expr` of the first type (i.e. `@constraint(m, func in set)`), `func` and `set` are passed unchanged to `build_constraint` but for the other types, they are determined from the expressions and signs. For instance, `@constraint(m, x^2 + y^2 == 1)` is transformed into `add_constraint(m, build_constraint(_error, x^2 + y^2, MOI.EqualTo(1.0)))`.

To extend JuMP to accept new constraints of this form, it is necessary to add the corresponding methods to `build_constraint`. Note that this will likely mean that either `func` or `set` will be some custom type, rather than e.g. a `Symbol`, since we will likely want to dispatch on the type of the function or set appearing in the constraint.

For extensions that need to create constraints with more information than just `func` and `set`, an additional positional argument can be specified to `@constraint` that will then be passed on `build_constraint`. Hence, we can enable this syntax by defining extensions of `build_constraint(_error, func, set, my_arg; kw_args...)`. This produces the user syntax: `@constraint(model, ref[...], expr, my_arg, kw_args...)`.

source

`JuMP.@constraints` – Macro.

```
@constraints(model, args...)
```

Adds groups of constraints at once, in the same fashion as the `@constraint` macro.

The model must be the first argument, and multiple constraints can be added on multiple lines wrapped in a `begin ... end` block.

The macro returns a tuple containing the constraints that were defined.

Examples

```
@constraints(model, begin
    x >= 1
    y - w <= 2
    sum_to_one[i=1:3], z[i] + y == 1
end)
```

source

`JuMP.ConstraintRef` – Type.

```
ConstraintRef
```

Holds a reference to the model and the corresponding MOI.ConstraintIndex.

source

`JuMP.AbstractConstraint` – Type.

```
abstract type AbstractConstraint
```

An abstract base type for all constraint types. `AbstractConstraints` store the function and set directly, unlike `ConstraintRef`s that are merely references to constraints stored in a model. `AbstractConstraints` do not need to be attached to a model.

source

`JuMP.ScalarConstraint` – Type.

```
struct ScalarConstraint
```

The data for a scalar constraint. The `func` field contains a JuMP object representing the function and the `set` field contains the MOI set. See also the documentation on JuMP's representation of constraints for more background.

source

`JuMP.VectorConstraint` – Type.

```
struct VectorConstraint
```

The data for a vector constraint. The `func` field contains a JuMP object representing the function and the `set` field contains the MOI set. The shape field contains an `AbstractShape` matching the form in which the constraint was constructed (e.g., by using matrices or flat vectors). See also the documentation on JuMP's representation of constraints.

source

23.2 Names

`JuMP.name` – Method.

```
name(con_ref::ConstraintRef)
```

Get a constraint's name attribute.

source

`JuMP.set_name` – Method.

```
set_name(con_ref::ConstraintRef, s::AbstractString)
```

Set a constraint's name attribute.

source

`JuMP.constraint_by_name` – Function.

```
constraint_by_name(model::AbstractModel,
                   name::String)::Union{ConstraintRef, Nothing}
```

Return the reference of the constraint with name attribute `name` or `Nothing` if no constraint has this name attribute. Throws an error if several constraints have `name` as their name attribute.

```
constraint_by_name(model::AbstractModel,
                   name::String,
                   F::Type{<:Union{AbstractJuMPScalar,
                                   Vector{<:AbstractJuMPScalar},
                                   MOI.AbstactFunction}},
                   S::Type{<:MOI.AbstractSet})::Union{ConstraintRef, Nothing}
```

Similar to the method above, except that it throws an error if the constraint is not an F-in-S contraint where F is either the JuMP or MOI type of the function, and S is the MOI type of the set. This method is recommended if you know the type of the function and set since its returned type can be inferred while for the method above (i.e. without F and S), the exact return type of the constraint index cannot be inferred.

```
julia> using JuMP

julia> model = Model()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.

julia> @variable(model, x)
x

julia> @constraint(model, con, x^2 == 1)
con : x² = 1.0

julia> constraint_by_name(model, "kon")

julia> constraint_by_name(model, "con")
con : x² = 1.0

julia> constraint_by_name(model, "con", AffExpr, MOI.EqualTo{Float64})

julia> constraint_by_name(model, "con", QuadExpr, MOI.EqualTo{Float64})
con : x² = 1.0
```

source

23.3 Modification

JuMP.normalized_coefficient – Function.

```
normalized_coefficient(con_ref::ConstraintRef, variable::VariableRef)
```

Return the coefficient associated with `variable` in `constraint` after JuMP has normalized the constraint into its standard form. See also `set_normalized_coefficient`.

source

JuMP.set_normalized_coefficient – Function.

```
set_normalized_coefficient(con_ref::ConstraintRef, variable::VariableRef, value)
```

Set the coefficient of `variable` in the constraint `constraint` to `value`.

Note that prior to this step, JuMP will aggregate multiple terms containing the same variable. For example, given a constraint `2x + 3x <= 2`, `set_normalized_coefficient(con, x, 4)` will create the constraint `4x <= 2`.

```
model = Model()
@variable(model, x)
```

```
@constraint(model, con, 2x + 3x <= 2)
set_normalized_coefficient(con, x, 4)
con

# output

con : 4 x <= 2.0
```

source

JuMP.normalized_rhs – Function.

```
normalized_rhs(con_ref::ConstraintRef)
```

Return the right-hand side term of `con_ref` after JuMP has converted the constraint into its normalized form. See also `set_normalized_rhs`.

source

JuMP.set_normalized_rhs – Function.

```
set_normalized_rhs(con_ref::ConstraintRef, value)
```

Set the right-hand side term of `constraint` to `value`.

Note that prior to this step, JuMP will aggregate all constant terms onto the right-hand side of the constraint. For example, given a constraint `2x + 1 <= 2`, `set_normalized_rhs(con, 4)` will create the constraint `2x <= 4`, not `2x + 1 <= 4`.

```
julia> @constraint(model, con, 2x + 1 <= 2)
con : 2 x <= 1.0

julia> set_normalized_rhs(con, 4)

julia> con
con : 2 x <= 4.0
```

source

JuMP.add_to_function_constant – Function.

```
add_to_function_constant(constraint::ConstraintRef, value)
```

Add `value` to the function constant term.

Note that for scalar constraints, JuMP will aggregate all constant terms onto the right-hand side of the constraint so instead of modifying the function, the set will be translated by `-value`. For example, given a constraint `2x <= 3`, `add_to_function_constant(c, 4)` will modify it to `2x <= -1`.

Examples

For scalar constraints, the set is translated by `-value`:

```
julia> @constraint(model, con, 0 <= 2x - 1 <= 2)
con : 2 x ∈ [1.0, 3.0]

julia> add_to_function_constant(con, 4)

julia> con
con : 2 x ∈ [-3.0, -1.0]
```

For vector constraints, the constant is added to the function:

```
julia> @constraint(model, con, [x + y, x, y] in SecondOrderCone())
con : [x + y, x, y] ∈ MathOptInterface.SecondOrderCone(3)

julia> add_to_function_constant(con, [1, 2, 2])

julia> con
con : [x + y + 1, x + 2, y + 2] ∈ MathOptInterface.SecondOrderCone(3)
```

source

23.4 Deletion

`JuMP.delete` – Function.

```
delete(model::Model, con_ref::ConstraintRef)
```

Delete the constraint associated with `constraint_ref` from the model `model`.

Note that `delete` does not unregister the name from the model, so adding a new constraint of the same name will throw an error. Use `unregister` to unregister the name after deletion as follows:

```
@constraint(model, c, 2x <= 1)
delete(model, c)
unregister(model, :c)
```

See also: `unregister`

source

```
delete(model::Model, con_refs::Vector{<:ConstraintRef})
```

Delete the constraints associated with `con_refs` from the model `model`. Solvers may implement specialized methods for deleting multiple constraints of the same concrete type, i.e., when `isconcretetype(eltype(con_refs))`. These may be more efficient than repeatedly calling the single constraint delete method.

See also: `unregister`

source

```
delete(model::Model, variable_ref::VariableRef)
```

Delete the variable associated with `variable_ref` from the model `model`.

Note that `delete` does not unregister the name from the model, so adding a new variable of the same name will throw an error. Use `unregister` to unregister the name after deletion as follows:

```
@variable(model, x)
delete(model, x)
unregister(model, :x)
```

See also: `unregister`

source

```
delete(model::Model, variable_refs::Vector{VariableRef})
```

Delete the variables associated with `variable_refs` from the model `model`. Solvers may implement methods for deleting multiple variables that are more efficient than repeatedly calling the single variable delete method.

See also: `unregister`

source

`JuMP.is_valid` – Function.

```
is_valid(model::Model, con_ref::ConstraintRef{<:AbstractModel})
```

Return `true` if `constraint_ref` refers to a valid constraint in `model`.

source

```
is_valid(model::Model, variable_ref::VariableRef)
```

Return `true` if `variable` refers to a valid variable in `model`.

source

```
is_valid(model::Model, c::NonlinearConstraintRef)
```

Return `true` if c refers to a valid nonlinear constraint in `model`.

source

`JuMP.ConstraintNotOwned` – Type.

```
struct ConstraintNotOwned{C <: ConstraintRef} <: Exception
    constraint_ref::C
end
```

The constraint `constraint_ref` was used in a model different to `owner_model(constraint_ref)`.

source

23.5 Query constraints

`JuMP.list_of_constraint_types` – Function.

```
list_of_constraint_types(model::Model)::Vector{Tuple{Type,Type}}
```

Return a list of tuples of the form `(F, S)` where F is a JuMP function type and S is an MOI set type such that `all_constraints(model, F, S)` returns a nonempty list.

Example

```
julia> model = Model();

julia> @variable(model, x >= 0, Bin);

julia> @constraint(model, 2x <= 1);

julia> list_of_constraint_types(model)
3-element Array{Tuple{Type,Type},1}:
 (GenericAffExpr{Float64,VariableRef}, MathOptInterface.LessThan{Float64})
 (VariableRef, MathOptInterface.GreaterThan{Float64})
 (VariableRef, MathOptInterface.ZeroOne)
```

source

JuMP.all_constraints – Function.

```
all_constraints(model::Model, function_type, set_type)::Vector{<:ConstraintRef}
```

Return a list of all constraints currently in the model where the function has type `function_type` and the set has type `set_type`. The constraints are ordered by creation time.

See also `list_of_constraint_types` and `num_constraints`.

Example

```
julia> model = Model();

julia> @variable(model, x >= 0, Bin);

julia> @constraint(model, 2x <= 1);

julia> all_constraints(model, VariableRef, MOI.GreaterThan{Float64})
1-element
↪  Array{ConstraintRef{Model,MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,MathOptInterface.Gre
 x ≥ 0.0

julia> all_constraints(model, VariableRef, MOI.ZeroOne)
1-element
↪  Array{ConstraintRef{Model,MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,MathOptInterface.Zer
 x binary

julia> all_constraints(model, AffExpr, MOI.LessThan{Float64})
1-element
↪  Array{ConstraintRef{Model,MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},Math
 2 x ≤ 1.0
```

source

JuMP.num_constraints – Function.

```
num_constraints(model::Model, function_type, set_type)::Int64
```

Return the number of constraints currently in the model where the function has type `function_type` and the set has type `set_type`.

See also `list_of_constraint_types` and `all_constraints`.

Example

```
julia> model = Model();

julia> @variable(model, x >= 0, Bin);

julia> @variable(model, y);

julia> @constraint(model, y in MOI.GreaterThan(1.0));

julia> @constraint(model, y <= 1.0);

julia> @constraint(model, 2x <= 1);
```

```
julia> num_constraints(model, VariableRef, MOI.GreaterThan{Float64})
2

julia> num_constraints(model, VariableRef, MOI.ZeroOne)
1

julia> num_constraints(model, AffExpr, MOI.LessThan{Float64})
2
```

source

```
num_constraints(model::Model; count_variable_in_set_constraints::Bool)
```

Return the number of constraints in `model`.

If `count_variable_in_set_constraints == true`, then `VariableRef` constraints such as `VariableRef`-in-`Integer` are included. To count only the number of structural constraints (e.g., the rows in the constraint matrix of a linear program), pass `count_variable_in_set_constraints = false`.

Examples

```
julia> model = Model();

julia> @variable(model, x >= 0, Int);

julia> @constraint(model, 2x <= 1);

julia> num_constraints(model; count_variable_in_set_constraints = true)
3

julia> num_constraints(model; count_variable_in_set_constraints = false)
1
```

source

`JuMP.index` – Method.

```
index(cr::ConstraintRef)::MOI.ConstraintIndex
```

Return the index of the constraint that corresponds to `cr` in the MOI backend.

source

`JuMP.optimizer_index` – Method.

```
optimizer_index(cr::ConstraintRef{Model})::MOI.ConstraintIndex
```

Return the index of the constraint that corresponds to `cr` in the optimizer model. It throws `NoOptimizer` if no optimizer is set and throws an `ErrorException` if the optimizer is set but is not attached or if the constraint is bridged.

source

`JuMP.constraint_object` – Function.

```
constraint_object(con_ref::ConstraintRef)
```

Return the underlying constraint data for the constraint referenced by `ref`.

source

23.6 Start values

JuMP.set_dual_start_value – Function.

```
set_dual_start_value(con_ref::ConstraintRef, value)
```

Set the dual start value (MOI attribute `ConstraintDualStart`) of the constraint `con_ref` to `value`. To remove a dual start value set it to `nothing`.

See also dual_start_value.

source

JuMP.dual_start_value – Function.

```
dual_start_value(con_ref::ConstraintRef)
```

Return the dual start value (MOI attribute `ConstraintDualStart`) of the constraint `con_ref`.

Note: If no dual start value has been set, `dual_start_value` will return `nothing`.

See also set_dual_start_value.

source

23.7 Special sets

JuMP.SecondOrderCone – Type.

```
SecondOrderCone
```

Second order cone object that can be used to constrain the euclidean norm of a vector x to be less than or equal to a nonnegative scalar `t`. This is a shortcut for the `MOI.SecondOrderCone`.

Examples

The following constrains $\|(x-1, x-2)\|_2 \leq t$ and $t \geq 0$:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @variable(model, t)
t

julia> @constraint(model, [t, x-1, x-2] in SecondOrderCone())
[t, x - 1, x - 2] ∈ MathOptInterface.SecondOrderCone(3)
```

source

JuMP.RotatedSecondOrderCone – Type.

```
RotatedSecondOrderCone
```

Rotated second order cone object that can be used to constrain the square of the euclidean norm of a vector x to be less than or equal to $2tu$ where `t` and `u` are nonnegative scalars. This is a shortcut for the `MOI.RotatedSecondOrderCone`.

Examples

The following constrains $\|(x-1, x-2)\|_2^2 \leq 2tx$ and $t, x \geq 0$:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> @variable(model, t)
t

julia> @constraint(model, [t, x, x-1, x-2] in RotatedSecondOrderCone())
[t, x, x - 1, x - 2] ∈ MathOptInterface.RotatedSecondOrderCone(4)
```

source

`JuMP.PSDCone` – Type.

```
PSDCone
```

Positive semidefinite cone object that can be used to constrain a square matrix to be positive semidefinite in the `@constraint` macro. If the matrix has type `Symmetric` then the columns vectorization (the vector obtained by concatenating the columns) of its upper triangular part is constrained to belong to the `MOI.PositiveSemidefiniteConeTriangle` set, otherwise its column vectorization is constrained to belong to the `MOI.PositiveSemidefiniteConeSquare` set.

Examples

Consider the following example:

```
julia> model = Model();

julia> @variable(model, x)
x

julia> a = [ x 2x
            2x  x];

julia> b = [1 2
            2 4];

julia> cref = @constraint(model, a >= b, PSDCone())
[x - 1    2 x - 2;
 2 x - 2  x - 4  ] ∈ PSDCone()

julia> jump_function(constraint_object(cref))
4-element Array{GenericAffExpr{Float64,VariableRef},1}:
 x - 1
 2 x - 2
 2 x - 2
 x - 4

julia> moi_set(constraint_object(cref))
MathOptInterface.PositiveSemidefiniteConeSquare(2)
```

We see in the output of the last command that the matrix the vectorization of the matrix is constrained to belong to the `PositiveSemidefiniteConeSquare`.

```
julia> using LinearAlgebra # For Symmetric
```

```
julia> cref = @constraint(model, Symmetric(a - b) in PSDCone())
[x - 1    2 x - 2;
 2 x - 2  x - 4  ] ∈ PSDCone()

julia> jump_function(constraint_object(cref))
3-element Array{GenericAffExpr{Float64,VariableRef},1}:
 x - 1
 2 x - 2
 x - 4

julia> moi_set(constraint_object(cref))
MathOptInterface.PositiveSemidefiniteConeTriangle(2)
```

As we see in the output of the last command, the vectorization of only the upper triangular part of the matrix is constrained to belong to the `PositiveSemidefiniteConeSquare`.

source

`JuMP.SOS1` – Type.

```
SOS1
```

SOS1 (Special Ordered Sets type 1) object than can be used to constrain a vector x to a set where at most 1 variable can take a non-zero value, all others being at 0. The `weights`, when specified, induce an ordering of the variables; as such, they should be unique values. The kth element in the set corresponds to the kth weight in `weights`. See here for a description of SOS constraints and their potential uses. This is a shortcut for the `MathOptInterface.SOS1` set.

source

`JuMP.SOS2` – Type.

```
SOS2
```

SOS1 (Special Ordered Sets type 2) object than can be used to constrain a vector x to a set where at most 2 variables can take a non-zero value, all others being at 0. In addition, if two are non-zero these must be consecutive in their ordering. The `weights` induce an ordering of the variables; as such, they should be unique values. The kth element in the set corresponds to the kth weight in `weights`. See here for a description of SOS constraints and their potential uses. This is a shortcut for the `MathOptInterface.SOS2` set.

source

`JuMP.SkewSymmetricMatrixSpace` – Type.

```
SkewSymmetricMatrixSpace()
```

Use in the `@variable` macro to constrain a matrix of variables to be skew-symmetric.

Examples

```
@variable(model, Q[1:2, 1:2] in SkewSymmetricMatrixSpace())
```

source

`JuMP.SkewSymmetricMatrixShape` – Type.

```
SkewSymmetricMatrixShape
```

Shape object for a skew symmetric square matrix of `side_dimension` rows and columns. The vectorized form contains the entries of the upper-right triangular part of the matrix (without the diagonal) given column by column (or equivalently, the entries of the lower-left triangular part given row by row). The diagonal is zero.

source

`JuMP.SymmetricMatrixSpace` – Type.

```
SymmetricMatrixSpace()
```

Use in the `@variable` macro to constrain a matrix of variables to be symmetric.

Examples

```
julia> @variable(model, Q[1:2, 1:2] in SymmetricMatrixSpace())
2×2 LinearAlgebra.Symmetric{VariableRef,Array{VariableRef,2}}:
 Q[1,1]  Q[1,2]
 Q[1,2]  Q[2,2]
```

source

`JuMP.moi_set` – Function.

```
moi_set(constraint::AbstractConstraint)
```

Return the set of the constraint `constraint` in the function-in-set form as a `MathOptInterface.AbstractSet`.

```
moi_set(s::AbstractVectorSet, dim::Int)
```

Returns the MOI set of dimension `dim` corresponding to the JuMP set `s`.

source

23.8 Printing

`JuMP.function_string` – Function.

```
function_string(
    mode::MIME,
    func::Union{JuMP.AbstractJuMPScalar,Vector{<:JuMP.AbstractJuMPScalar}},
)
```

Return a `String` representing the function `func` using print mode `mode`.

source

`JuMP.constraints_string` – Function.

```
constraints_string(mode, model::AbstractModel)::Vector{String}
```

Return a list of `Strings` describing each constraint of the model.

source

`JuMP.in_set_string` – Function.

```
in_set_string(mode::MIME, set)
```

Return a `String` representing the membership to the set `set` using print mode `mode`.

source

`JuMP.show_constraints_summary` – Function.

```
show_constraints_summary(io::IO, model::AbstractModel)
```

Write to `io` a summary of the number of constraints.

source

Chapter 24

Containers

More information can be found in the Containers section of the manual.

`JuMP.Containers` – Module.

```
Containers
```

Module defining the containers `DenseAxisArray` and `SparseAxisArray` that behaves as a regular `AbstractArray` but with custom indexes that are not necessarily integers.

source

`JuMP.Containers.DenseAxisArray` – Type.

```
DenseAxisArray(data::Array{T, N}, axes...) where {T, N}
```

Construct a JuMP array with the underlying data specified by the `data` array and the given axes. Exactly `N` axes must be provided, and their lengths must match `size(data)` in the corresponding dimensions.

Example

```
julia> array = JuMP.Containers.DenseAxisArray([1 2; 3 4], [:a, :b], 2:3)
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, Symbol[:a, :b]
    Dimension 2, 2:3
And data, a 2×2 Array{Int64,2}:
 1  2
 3  4

julia> array[:b, 3]
4
```

source

```
DenseAxisArray{T}(undef, axes...) where T
```

Construct an uninitialized DenseAxisArray with element-type T indexed over the given axes.

Example

```
julia> array = JuMP.Containers.DenseAxisArray{Float64}(undef, [:a, :b], 1:2);

julia> fill!(array, 1.0)
```

```
2-dimensional DenseAxisArray{Float64,2,...} with index sets:
    Dimension 1, Symbol[:a, :b]
    Dimension 2, 1:2
And data, a 2×2 Array{Float64,2}:
 1.0  1.0
 1.0  1.0

julia> array[:a, 2] = 5.0
5.0

julia> array[:a, 2]
5.0

julia> array
2-dimensional DenseAxisArray{Float64,2,...} with index sets:
    Dimension 1, Symbol[:a, :b]
    Dimension 2, 1:2
And data, a 2×2 Array{Float64,2}:
 1.0  5.0
 1.0  1.0
```

source

`JuMP.Containers.SparseAxisArray` – Type.

```
struct SparseAxisArray{T,N,K<:NTuple{N, Any}} <: AbstractArray{T,N}
    data::Dict{K,T}
end
```

N-dimensional array with elements of type T where only a subset of the entries are defined. The entries with indices `idx = (i1, i2, ..., iN)` in `keys(data)` has value `data[idx]`. Note that as opposed to `SparseArrays.AbstractSparseArray`, the missing entries are not assumed to be `zero(T)`, they are simply not part of the array. This means that the result of `map(f, sa::SparseAxisArray)` or `f.(sa::SparseAxisArray)` has the same sparsity structure than sa even if `f(zero(T))` is not zero.

Examples

```
julia> dict = Dict((:a, 2) => 1.0, (:a, 3) => 2.0, (:b, 3) => 3.0)
Dict{Tuple{Symbol,Int64},Float64} with 3 entries:
  (:b, 3) => 3.0
  (:a, 2) => 1.0
  (:a, 3) => 2.0

julia> array = JuMP.Containers.SparseAxisArray(dict)
JuMP.Containers.SparseAxisArray{Float64,2,Tuple{Symbol,Int64}} with 3 entries:
  [b, 3]  =  3.0
  [a, 2]  =  1.0
  [a, 3]  =  2.0

julia> array[:b, 3]
3.0
```

source

`JuMP.Containers.container` – Function.

```
container(f::Function, indices, ::Type{C})
```

Create a container of type `C` with indices `indices` and values at given indices given by `f`.

```
container(f::Function, indices)
```

Create a container with indices `indices` and values at given indices given by `f`. The type of container used is determined by `default_container`.

Examples

```
julia> Containers.container((i, j) -> i + j, Containers.vectorized_product(Base.OneTo(3), Base.
     OneTo(3)))
3×3 Array{Int64,2}:
 2  3  4
 3  4  5
 4  5  6

julia> Containers.container((i, j) -> i + j, Containers.vectorized_product(1:3, 1:3))
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, 1:3
    Dimension 2, 1:3
And data, a 3×3 Array{Int64,2}:
 2  3  4
 3  4  5
 4  5  6

julia> Containers.container((i, j) -> i + j, Containers.vectorized_product(2:3, Base.OneTo(3)))
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, Base.OneTo(3)
And data, a 2×3 Array{Int64,2}:
 3  4  5
 4  5  6

julia> Containers.container((i, j) -> i + j, Containers.nested(() -> 1:3, i -> i:3, condition = (
     i, j) -> isodd(i) || isodd(j)))
SparseAxisArray{Int64,2,Tuple{Int64,Int64}} with 5 entries:
  [1, 2]  =  3
  [2, 3]  =  5
  [3, 3]  =  6
  [1, 1]  =  2
  [1, 3]  =  4
```

source

`JuMP.Containers.default_container` – Function.

```
default_container(indices)
```

If indices is a `NestedIterator`, return a `SparseAxisArray`. Otherwise, indices should be a `VectorizedProductIterator` and the function returns `Array` if all iterators of the product are `Base.OneTo` and retunrs `DenseAxisArray` otherwise.

source

`JuMP.Containers.@container` – Macro.

```
@container([i=..., j=..., ...], expr[, container = :Auto])
```

Create a container with indices i, j, ... and values given by expr that may depend on the value of the indices.

```
@container(ref[i=..., j=..., ...], expr[, container = :Auto])
```

Same as above but the container is assigned to the variable of name ref.

The type of container can be controlled by the container keyword.

> **Note**
> When the index set is explicitly given as 1:n for any expression n, it is transformed to Base.OneTo(n) before being given to container.

Examples

```
julia> Containers.@container([i = 1:3, j = 1:3], i + j)
3×3 Array{Int64,2}:
 2  3  4
 3  4  5
 4  5  6

julia> I = 1:3
1:3

julia> Containers.@container(x[i = I, j = I], i + j);

julia> x
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, 1:3
    Dimension 2, 1:3
And data, a 3×3 Array{Int64,2}:
 2  3  4
 3  4  5
 4  5  6

julia> Containers.@container([i = 2:3, j = 1:3], i + j)
2-dimensional DenseAxisArray{Int64,2,...} with index sets:
    Dimension 1, 2:3
    Dimension 2, Base.OneTo(3)
And data, a 2×3 Array{Int64,2}:
 3  4  5
 4  5  6

julia> Containers.@container([i = 1:3, j = 1:3; i <= j], i + j)
JuMP.Containers.SparseAxisArray{Int64,2,Tuple{Int64,Int64}} with 6 entries:
  [1, 2]  =  3
  [2, 3]  =  5
  [3, 3]  =  6
  [2, 2]  =  4
  [1, 1]  =  2
  [1, 3]  =  4
```

source

JuMP.Containers.VectorizedProductIterator – Type.

```
struct VectorizedProductIterator{T}
    prod::Iterators.ProductIterator{T}
end
```

A wrapper type for `Iterators.ProuctIterator` that discards shape information and returns a `Vector`.

Construct a `VectorizedProductIterator` using `vectorized_product`.

source

`JuMP.Containers.vectorized_product` – Function.

```
vectorized_product(iterators...)
```

Created a `VectorizedProductIterator`.

Examples

```
vectorized_product(1:2, ["A", "B"])
```

source

`JuMP.Containers.NestedIterator` – Type.

```
struct NestedIterator{T}
    iterators::T # Tuple of functions
    condition::Function
end
```

Iterators over the tuples that are produced by a nested for loop.

Construct a `NestedIterator` using `nested`.

Example

If `length(iterators) == 3`:

```
x = NestedIterator(iterators, condition)
for (i1, i2, i3) in x
    # produces (i1, i2, i3)
end
```

is the same as

```
for i1 in iterators[1]()
    for i2 in iterator[2](i1)
        for i3 in iterator[3](i1, i2)
            if condition(i1, i2, i3)
                # produces (i1, i2, i3)
            end
        end
    end
end
```

source

`JuMP.Containers.nested` – Function.

```
nested(iterators...; condition = (args...) -> true)
```

Create a NestedIterator.

Example

```
nested(1:2, ["A", "B"]; condition = (i, j) -> isodd(i) || j == "B")
```

source

For advanced users, the following functions are provided to aid the writing of macros that use the container functionality.

JuMP.Containers.build_ref_sets – Function.

```
build_ref_sets(_error::Function, expr)
```

Helper function for macros to construct container objects.

> **Warning**
>
> This function is for advanced users implementing JuMP extensions. See container_code for more details.

Arguments

- `_error`: a function that takes a `String` and throws an error, potentially annotating the input string with extra information such as from which macro it was thrown from. Use `error` if you do not want a modified error message.
- `expr`: an `Expr` that specifies the container, e.g., `:(x[i = 1:3, [:red, :blue], k = S; i + k <= 6])`

Returns

1. `index_vars`: a `Vector{Any}` of names for the index variables, e.g., `[:i, gensym(), :k]`. These may also be expressions, like `:((i, j))` from a call like `:(x[(i, j) in S])`.
2. `indices`: an iterator over the indices, e.g.,

   ```
   Containers.NestedIterators(
       (1:3, [:red, :blue], S),
       (i, _, k) -> i + k <= 6,
   )
   ```

Examples

See container_code for a worked example.

source

JuMP.Containers.container_code – Function.

```
container_code(
    index_vars::Vector{Any},
    indices::Expr,
    code,
    requested_container::Union{Symbol,Expr},
)
```

Used in macros to construct a call to `container`. This should be used in conjunction with `build_ref_sets`.

Arguments

- `index_vars::Vector{Any}`: a vector of names for the indices of the container. These may also be expressions, like `:((i, j))` from a call like `:(x[(i, j) in S])`.
- `indices::Expr`: an expression that evaluates to an iterator of the indices.
- `code`: an expression or literal constant for the value to be stored in the container as a function of the named `index_vars`.
- `requested_container`: passed to the third argument of `container`. For built-in JuMP types, choose one of `:Array`, `:DenseAxisArray`, `:SparseAxisArray`, or `:Auto`. For a user-defined container, this expression must evaluate to the correct type.

Warning

In most cases, you should `esc(code)` before passing it to `container_code`.

Examples

```
julia> macro foo(ref_sets, code)
           index_vars, indices = Containers.build_ref_sets(error, ref_sets)
           return Containers.container_code(
               index_vars,
               indices,
               esc(code),
               :Auto,
            )
       end
@foo (macro with 1 method)

julia> @foo(x[i=1:2, j=["A", "B"]], j^i)
2-dimensional DenseAxisArray{String,2,...} with index sets:
    Dimension 1, Base.OneTo(2)
    Dimension 2, ["A", "B"]
And data, a 2×2 Matrix{String}:
 "A"   "B"
 "AA"  "BB"
```

source

Chapter 25

Solutions

More information can be found in the Solutions section of the manual.

25.1 Basic utilities

`JuMP.optimize!` – Function.

```
optimize!(model::Model;
          ignore_optimize_hook=(model.optimize_hook === nothing),
          kwargs...)
```

Optimize the model. If an optimizer has not been set yet (see `set_optimizer`), a `NoOptimizer` error is thrown.

Keyword arguments `kwargs` are passed to the `optimize_hook`. An error is thrown if `optimize_hook` is `nothing` and keyword arguments are provided.

source

`JuMP.NoOptimizer` – Type.

```
struct NoOptimizer <: Exception end
```

No optimizer is set. The optimizer can be provided to the `Model` constructor or by calling `set_optimizer`.

source

`JuMP.OptimizeNotCalled` – Type.

```
struct OptimizeNotCalled <: Exception end
```

A result attribute cannot be queried before `optimize!` is called.

source

`JuMP.solution_summary` – Function.

```
solution_summary(model::Model; verbose::Bool = false)
```

Return a struct that can be used print a summary of the solution.

If `verbose=true`, write out the primal solution for every variable and the dual solution for every constraint, excluding those with empty names.

Examples

When called at the REPL, the summary is automatically printed:

```
julia> solution_summary(model)
[...]
```

Use `print` to force the printing of the summary from inside a function:

```
function foo(model)
    print(solution_summary(model))
    return
end
```

source

25.2 Termination status

`JuMP.termination_status` – Function.

```
termination_status(model::Model)
```

Return a `MOI.TerminationStatusCode` describing why the solver stopped (i.e., the `MOI.TerminationStatus` attribute).

source

`JuMP.raw_status` – Function.

```
raw_status(model::Model)
```

Return the reason why the solver stopped in its own words (i.e., the MathOptInterface model attribute `RawStatusString`).

source

`JuMP.result_count` – Function.

```
result_count(model::Model)
```

Return the number of results available to query after a call to `optimize!`.

source

25.3 Primal solutions

`JuMP.primal_status` – Function.

```
primal_status(model::Model; result::Int = 1)
```

Return a `MOI.ResultStatusCode` describing the status of the most recent primal solution of the solver (i.e., the `MOI.PrimalStatus` attribute) associated with the result index `result`.

See also: `result_count`.

source

`JuMP.has_values` – Function.

```
has_values(model::Model; result::Int = 1)
```

Return `true` if the solver has a primal solution in result index `result` available to query, otherwise return `false`.

See also `value` and `result_count`.

source

`JuMP.value` – Function.

```
value(con_ref::ConstraintRef; result::Int = 1)
```

Return the primal value of constraint `con_ref` associated with result index `result` of the most-recent solution returned by the solver.

That is, if `con_ref` is the reference of a constraint `func`-in-`set`, it returns the value of `func` evaluated at the value of the variables (given by `value(::VariableRef)`).

Use `has_values` to check if a result exists before asking for values.

See also: `result_count`.

Note

For scalar constraints, the constant is moved to the `set` so it is not taken into account in the primal value of the constraint. For instance, the constraint `@constraint(model, 2x + 3y + 1 == 5)` is transformed into `2x + 3y`-in-`MOI.EqualTo(4)` so the value returned by this function is the evaluation of `2x + 3y`. "'

source

```
value(var_value::Function, con_ref::ConstraintRef)
```

Evaluate the primal value of the constraint `con_ref` using `var_value(v)` as the value for each variable `v`.

source

```
value(v::VariableRef; result = 1)
```

Return the value of variable `v` associated with result index `result` of the most-recent returned by the solver.

Use `has_values` to check if a result exists before asking for values.

See also: `result_count`.

source

```
value(var_value::Function, v::VariableRef)
```

Evaluate the value of the variable `v` as `var_value(v)`.

source

```
value(var_value::Function, ex::GenericAffExpr)
```

Evaluate `ex` using `var_value(v)` as the value for each variable `v`.

source

```
value(v::GenericAffExpr; result::Int = 1)
```

Return the value of the `GenericAffExpr` v associated with result index `result` of the most-recent solution returned by the solver.

See also: `result_count`.

source

```
value(var_value::Function, ex::GenericQuadExpr)
```

Evaluate ex using `var_value(v)` as the value for each variable v.

source

```
value(v::GenericQuadExpr; result::Int = 1)
```

Return the value of the `GenericQuadExpr` v associated with result index `result` of the most-recent solution returned by the solver.

Replaces `getvalue` for most use cases.

See also: `result_count`.

source

```
value(p::NonlinearParameter)
```

Return the current value stored in the nonlinear parameter p.

Example

```
model = Model()
@NLparameter(model, p == 10)
value(p)

# output
10.0
```

source

```
value(var_value::Function, ex::NonlinearExpression)
```

Evaluate ex using `var_value(v)` as the value for each variable v.

source

```
value(ex::NonlinearExpression; result::Int = 1)
```

Return the value of the `NonlinearExpression` ex associated with result index `result` of the most-recent solution returned by the solver.

Replaces `getvalue` for most use cases.

See also: `result_count`.

source

25.4 Dual solutions

JuMP.dual_status – Function.

```
dual_status(model::Model; result::Int = 1)
```

Return a MOI.ResultStatusCode describing the status of the most recent dual solution of the solver (i.e., the MOI.DualStatus attribute) associated with the result index result.

See also: result_count.

source

JuMP.has_duals – Function.

```
has_duals(model::Model; result::Int = 1)
```

Return true if the solver has a dual solution in result index result available to query, otherwise return false.

See also dual, shadow_price, and result_count.

source

JuMP.dual – Function.

```
dual(con_ref::ConstraintRef; result::Int = 1)
```

Return the dual value of constraint con_ref associated with result index result of the most-recent solution returned by the solver.

Use has_dual to check if a result exists before asking for values.

See also: result_count, shadow_price.

source

```
dual(c::NonlinearConstraintRef)
```

Return the dual of the nonlinear constraint c.

source

JuMP.shadow_price – Function.

```
shadow_price(con_ref::ConstraintRef)
```

Return the change in the objective from an infinitesimal relaxation of the constraint.

This value is computed from dual and can be queried only when has_duals is true and the objective sense is MIN_SENSE or MAX_SENSE (not FEASIBILITY_SENSE). For linear constraints, the shadow prices differ at most in sign from the dual value depending on the objective sense.

See also reduced_cost.

Notes

- The function simply translates signs from dual and does not validate the conditions needed to guarantee the sensitivity interpretation of the shadow price. The caller is responsible, e.g., for checking whether the solver converged to an optimal primal-dual pair or a proof of infeasibility.

- The computation is based on the current objective sense of the model. If this has changed since the last solve, the results will be incorrect.
- Relaxation of equality constraints (and hence the shadow price) is defined based on which sense of the equality constraint is active.

source

`JuMP.reduced_cost` – Function.

```
reduced_cost(x::VariableRef)::Float64
```

Return the reduced cost associated with variable x.

Equivalent to querying the shadow price of the active variable bound (if one exists and is active).

See also: `shadow_price`.

source

25.5 Basic attributes

`JuMP.objective_value` – Function.

```
objective_value(model::Model; result::Int = 1)
```

Return the objective value associated with result index `result` of the most-recent solution returned by the solver.

See also: `result_count`.

source

`JuMP.objective_bound` – Function.

```
objective_bound(model::Model)
```

Return the best known bound on the optimal objective value after a call to `optimize!(model)`.

source

`JuMP.dual_objective_value` – Function.

```
dual_objective_value(model::Model; result::Int = 1)
```

Return the value of the objective of the dual problem associated with result index `result` of the most-recent solution returned by the solver.

Throws `MOI.UnsupportedAttribute{MOI.DualObjectiveValue}` if the solver does not support this attribute.

See also: `result_count`.

source

`JuMP.solve_time` – Function.

```
solve_time(model::Model)
```

If available, returns the solve time reported by the solver. Returns "ArgumentError: ModelLike of type `Solver.Optimizer` does not support accessing the attribute MathOptInterface.SolveTimeSec()" if the attribute is not implemented.

source

`JuMP.relative_gap` – Function.

```
relative_gap(model::Model)
```

Return the final relative optimality gap after a call to `optimize!(model)`. Exact value depends upon implementation of MathOptInterface.RelativeGap() by the particular solver used for optimization.

source

`JuMP.simplex_iterations` – Function.

```
simplex_iterations(model::Model)
```

Gets the cumulative number of simplex iterations during the most-recent optimization.

Solvers must implement `MOI.SimplexIterations()` to use this function.

source

`JuMP.barrier_iterations` – Function.

```
barrier_iterations(model::Model)
```

Gets the cumulative number of barrier iterations during the most recent optimization.

Solvers must implement `MOI.BarrierIterations()` to use this function.

source

`JuMP.node_count` – Function.

```
node_count(model::Model)
```

Gets the total number of branch-and-bound nodes explored during the most recent optimization in a Mixed Integer Program.

Solvers must implement `MOI.NodeCount()` to use this function.

source

25.6 Conflicts

`JuMP.compute_conflict!` – Function.

```
compute_conflict!(model::Model)
```

Compute a conflict if the model is infeasible. If an optimizer has not been set yet (see `set_optimizer`), a `NoOptimizer` error is thrown.

The status of the conflict can be checked with the `MOI.ConflictStatus` model attribute. Then, the status for each constraint can be queried with the `MOI.ConstraintConflictStatus` attribute.

source

JuMP.copy_conflict – Function.

```
copy_conflict(model::Model)
```

Return a copy of the current conflict for the model `model` and a `ReferenceMap` that can be used to obtain the variable and constraint reference of the new model corresponding to a given `model`'s reference.

This is a convenience function that provides a filtering function for `copy_model`.

Note

Model copy is not supported in `DIRECT` mode, i.e. when a model is constructed using the `direct_model` constructor instead of the `Model` constructor. Moreover, independently on whether an optimizer was provided at model construction, the new model will have no optimizer, i.e., an optimizer will have to be provided to the new model in the `optimize!` call.

Examples

In the following example, a model `model` is constructed with a variable `x` and two constraints `cref` and `cref2`. This model has no solution, as the two constraints are mutually exclusive. The solver is asked to compute a conflict with `compute_conflict!`. The parts of `model` participating in the conflict are then copied into a model `new_model`.

```
model = Model() # You must use a solver that supports conflict refining/IIS
# computation, like CPLEX or Gurobi
@variable(model, x)
@constraint(model, cref, x >= 2)
@constraint(model, cref2, x <= 1)

compute_conflict!(model)
if MOI.get(model, MOI.ConflictStatus()) != MOI.CONFLICT_FOUND
    error("No conflict could be found for an infeasible model.")
end

new_model, reference_map = copy_conflict(model)
```

source

25.7 Sensitivity

JuMP.lp_sensitivity_report – Function.

```
lp_sensitivity_report(model::Model; atol::Float64 = 1e-8)::SensitivityReport
```

Given a linear program `model` with a current optimal basis, return a `SensitivityReport` object, which maps:

- Every variable reference to a tuple `(d_lo, d_hi)::Tuple{Float64,Float64}`, explaining how much the objective coefficient of the corresponding variable can change by, such that the original basis remains optimal.
- Every constraint reference to a tuple `(d_lo, d_hi)::Tuple{Float64,Float64}`, explaining how much the right-hand side of the corresponding constraint can change by, such that the basis remains optimal.

Both tuples are relative, rather than absolute. So given a objective coefficient of `1.0` and a tuple `(-0.5, 0.5)`, the objective coefficient can range between `1.0 - 0.5` an `1.0 + 0.5`.

`atol` is the primal/dual optimality tolerance, and should match the tolerance of the solver used to compute the basis.

Note: interval constraints are NOT supported.

Example

```
model = Model(HiGHS.Optimizer)
@variable(model, -1 <= x <= 2)
@objective(model, Min, x)
optimize!(model)
report = lp_sensitivity_report(model; atol = 1e-7)
dx_lo, dx_hi = report[x]
println(
    "The objective coefficient of `x` can decrease by $dx_lo or " *
    "increase by $dx_hi."
)
c = LowerBoundRef(x)
dRHS_lo, dRHS_hi = report[c]
println(
    "The lower bound of `x` can decrease by $dRHS_lo or increase " *
    "by $dRHS_hi."
)
```

source

`JuMP.SensitivityReport` – Type.

```
SensitivityReport
```

See `lp_sensitivity_report`.

source

25.8 Feasibility

`JuMP.primal_feasibility_report` – Function.

```
primal_feasibility_report(
    model::Model,
    point::AbstractDict{VariableRef,Float64} = _last_primal_solution(model),
    atol::Float64 = 0.0,
    skip_missing::Bool = false,
)::Dict{Any,Float64}
```

Given a dictionary `point`, which maps variables to primal values, return a dictionary whose keys are the constraints with an infeasibility greater than the supplied tolerance `atol`. The value corresponding to each key is the respective infeasibility. Infeasibility is defined as the distance between the primal value of the constraint (see `MOI.ConstraintPrimal`) and the nearest point by Euclidean distance in the corresponding set.

Notes

- If `skip_missing = true`, constraints containing variables that are not in `point` will be ignored.

- If `skip_missing = false` and a partial primal solution is provided, an error will be thrown.
- If no point is provided, the primal solution from the last time the model was solved is used.

Examples

```
julia> model = Model();

julia> @variable(model, 0.5 <= x <= 1);

julia> primal_feasibility_report(model, Dict(x => 0.2))
Dict{Any,Float64} with 1 entry:
  x ≥ 0.5 => 0.3
```

source

```
primal_feasibility_report(
    point::Function,
    model::Model;
    atol::Float64 = 0.0,
    skip_missing::Bool = false,
)
```

A form of `primal_feasibility_report` where a function is passed as the first argument instead of a dictionary as the second argument.

Examples

```
julia> model = Model();

julia> @variable(model, 0.5 <= x <= 1);

julia> primal_feasibility_report(model) do v
           return value(v)
       end
Dict{Any,Float64} with 1 entry:
    x ≥ 0.5 => 0.3
```

source

Chapter 26

Nonlinear Modeling

More information can be found in the Nonlinear Modeling section of the manual.

26.1 Constraints

`JuMP.@NLconstraint` – Macro.

```
@NLconstraint(m::Model, expr)
```

Add a constraint described by the nonlinear expression `expr`. See also `@constraint`. For example:

```
@NLconstraint(model, sin(x) <= 1)
@NLconstraint(model, [i = 1:3], sin(i * x) <= 1 / i)
```

source

`JuMP.@NLconstraints` – Macro.

```
@NLconstraints(model, args...)
```

Adds multiple nonlinear constraints to model at once, in the same fashion as the `@NLconstraint` macro.

The model must be the first argument, and multiple constraints can be added on multiple lines wrapped in a `begin ... end` block.

The macro returns a tuple containing the constraints that were defined.

Examples

```
@NLconstraints(model, begin
    t >= sqrt(x^2 + y^2)
    [i = 1:2], z[i] <= log(a[i])
end)
```

source

`JuMP.NonlinearConstraintIndex` – Type.

```
NonlinearConstraintIndex(index::Int64)
```

A struct to refer to the 1-indexed nonlinear constraint `index`.

source

JuMP.num_nonlinear_constraints – Function.

```
num_nonlinear_constraints(model::Model)
```

Returns the number of nonlinear constraints associated with the `model`.

source

JuMP.add_nonlinear_constraint – Function.

```
add_nonlinear_constraint(model::Model, expr::Expr)
```

Add a nonlinear constraint described by the Julia expression ex to `model`.

This function is most useful if the expression ex is generated programmatically, and you cannot use `@NLconstraint`.

Notes

- You must interpolate the variables directly into the expression `expr`.

Examples

```
julia> add_nonlinear_constraint(model, :($(x) + $(x)^2 <= 1))
(x + x ^ 2.0) - 1.0 ≤ 0
```

source

JuMP.all_nonlinear_constraints – Function.

```
all_nonlinear_constraints(model::Model)
```

Return a vector of all nonlinear constraint references in the model in the order they were added to the model.

source

JuMP.nonlinear_dual_start_value – Function.

```
nonlinear_dual_start_value(model::Model)
```

Return the current value of the MOI attribute `MOI.NLPBlockDualStart`.

source

JuMP.set_nonlinear_dual_start_value – Function.

```
set_nonlinear_dual_start_value(
    model::Model,
    start::Union{Nothing,Vector{Float64}},
)
```

Set the value of the MOI attribute `MOI.NLPBlockDualStart`.

The start vector corresponds to the Lagrangian duals of the nonlinear constraints, in the order given by `all_nonlinear_constraints`. That is, you must pass a single start vector corresponding to all of the nonlinear constraints in a single function call; you cannot set the dual start value of nonlinear constraints one-by-one. The example below demonstrates how to use `all_nonlinear_constraints` to create a mapping between the nonlinear constraint references and the start vector.

Pass `nothing` to unset a previous start.

Examples

```
julia> model = Model();

julia> @variable(model, x[1:2]);

julia> nl1 = @NLconstraint(model, x[1] <= sqrt(x[2]));

julia> nl2 = @NLconstraint(model, x[1] >= exp(x[2]));

julia> start = Dict(nl1 => -1.0, nl2 => 1.0);

julia> start_vector = [start[con] for con in all_nonlinear_constraints(model)]
2-element Vector{Float64}:
 -1.0
  1.0

julia> set_nonlinear_dual_start_value(model, start_vector)

julia> nonlinear_dual_start_value(model)
2-element Vector{Float64}:
 -1.0
  1.0
```

source

26.2 Expressions

`JuMP.@NLexpression` – Macro.

```
@NLexpression(args...)
```

Efficiently build a nonlinear expression which can then be inserted in other nonlinear constraints and the objective. See also [`@expression`]. For example:

```
@NLexpression(model, my_expr, sin(x)^2 + cos(x^2))
@NLconstraint(model, my_expr + y >= 5)
@NLobjective(model, Min, my_expr)
```

Indexing over sets and anonymous expressions are also supported:

```
@NLexpression(m, my_expr_1[i=1:3], sin(i * x))
my_expr_2 = @NLexpression(m, log(1 + sum(exp(x[i])) for i in 1:2))
```

source

`JuMP.@NLexpressions` – Macro.

```
@NLexpressions(model, args...)
```

Adds multiple nonlinear expressions to model at once, in the same fashion as the `@NLexpression` macro.

The model must be the first argument, and multiple expressions can be added on multiple lines wrapped in a `begin ... end` block.

The macro returns a tuple containing the expressions that were defined.

Examples

```
@NLexpressions(model, begin
    my_expr, sqrt(x^2 + y^2)
    my_expr_1[i = 1:2], log(a[i]) - z[i]
end)
```

source

JuMP.NonlinearExpression – Type.

```
NonlinearExpression <: AbstractJuMPScalar
```

A struct to represent a nonlinear expression.

Create an expression using @NLexpression.

source

JuMP.add_nonlinear_expression – Function.

```
add_nonlinear_expression(model::Model, expr::Expr)
```

Add a nonlinear expression `expr` to `model`.

This function is most useful if the expression `expr` is generated programmatically, and you cannot use @NLexpression.

Notes

- You must interpolate the variables directly into the expression `expr`.

Examples

```
julia> add_nonlinear_expression(model, :($(x) + $(x)^2))
subexpression[1]: x + x ^ 2.0
```

source

26.3 Objectives

JuMP.@NLobjective – Macro.

```
@NLobjective(model, sense, expression)
```

Add a nonlinear objective to `model` with optimization sense `sense`. `sense` must be `Max` or `Min`.

Example

```
@NLobjective(model, Max, 2x + 1 + sin(x))
```

source

JuMP.set_nonlinear_objective – Function.

```
set_nonlinear_objective(
    model::Model,
    sense::MOI.OptimizationSense,
    expr::Expr,
)
```

Set the nonlinear objective of `model` to the expression `expr`, with the optimization sense `sense`.

This function is most useful if the expression `expr` is generated programmatically, and you cannot use `@NLobjective`.

Notes

- You must interpolate the variables directly into the expression `expr`.
- You must use `MIN_SENSE` or `MAX_SENSE` instead of `Min` and `Max`.

Examples

```
julia> set_nonlinear_objective(model, MIN_SENSE, :($(x) + $(x)^2))
```

source

26.4 Parameters

`JuMP.@NLparameter` – Macro.

```
@NLparameter(model, param == value)
```

Create and return a nonlinear parameter `param` attached to the model `model` with initial value set to `value`. Nonlinear parameters may be used only in nonlinear expressions.

Example

```
model = Model()
@NLparameter(model, x == 10)
value(x)

# output
10.0
```

```
@NLparameter(model, value = param_value)
```

Create and return an anonymous nonlinear parameter `param` attached to the model `model` with initial value set to `param_value`. Nonlinear parameters may be used only in nonlinear expressions.

Example

```
model = Model()
x = @NLparameter(model, value = 10)
value(x)

# output
10.0
```

```
@NLparameter(model, param_collection[...] == value_expr)
```

Create and return a collection of nonlinear parameters `param_collection` attached to the model `model` with initial value set to `value_expr` (may depend on index sets). Uses the same syntax for specifying index sets as `@variable`.

Example

```
model = Model()
@NLparameter(model, y[i = 1:10] == 2 * i)
value(y[9])

# output
18.0
```

```
@NLparameter(model, [...] == value_expr)
```

Create and return an anonymous collection of nonlinear parameters attached to the model `model` with initial value set to `value_expr` (may depend on index sets). Uses the same syntax for specifying index sets as `@variable`.

Example

```
model = Model()
y = @NLparameter(model, [i = 1:10] == 2 * i)
value(y[9])

# output
18.0
```

source

`JuMP.@NLparameters` – Macro.

```
@NLparameters(model, args...)
```

Create and return multiple nonlinear parameters attached to model `model`, in the same fashion as `@NLparameter` macro.

The model must be the first argument, and multiple parameters can be added on multiple lines wrapped in a `begin ... end` block. Distinct parameters need to be placed on separate lines as in the following example.

The macro returns a tuple containing the parameters that were defined.

Example

```
model = Model()
@NLparameters(model, begin
    x == 10
    b == 156
end)
value(x)

# output
10.0
```

source

`JuMP.NonlinearParameter` – Type.

```
NonlinearParameter <: AbstractJuMPScalar
```

A struct to represent a nonlinear parameter.

Create a parameter using @NLparameter.

source

JuMP.value – Method.

```
value(p::NonlinearParameter)
```

Return the current value stored in the nonlinear parameter p.

Example

```
model = Model()
@NLparameter(model, p == 10)
value(p)

# output
10.0
```

source

JuMP.set_value – Method.

```
set_value(p::NonlinearParameter, v::Number)
```

Store the value v in the nonlinear parameter p.

Example

```
model = Model()
@NLparameter(model, p == 0)
set_value(p, 5)
value(p)

# output
5.0
```

source

26.5 User-defined functions

JuMP.register – Function.

```
register(
    model::Model,
    s::Symbol,
    dimension::Integer,
    f::Function;
    autodiff:Bool = false,
)
```

Register the user-defined function `f` that takes `dimension` arguments in `model` as the symbol `s`.

The function `f` must support all subtypes of `Real` as arguments. Do not assume that the inputs are `Float64`.

Notes

- For this method, you must explicitly set `autodiff = true`, because no user-provided gradient function `∇f` is given.
- Second-derivative information is only computed if `dimension == 1`.
- `s` does not have to be the same symbol as `f`, but it is generally more readable if it is.

Examples

```
model = Model()
@variable(model, x)
f(x::T) where {T<:Real} = x^2
register(model, :foo, 1, f; autodiff = true)
@NLobjective(model, Min, foo(x))
```

```
model = Model()
@variable(model, x[1:2])
g(x::T, y::T) where {T<:Real} = x * y
register(model, :g, 2, g; autodiff = true)
@NLobjective(model, Min, g(x[1], x[2]))
```

source

```
register(
    model::Model,
    s::Symbol,
    dimension::Integer,
    f::Function,
    ∇f::Function;
    autodiff:Bool = false,
)
```

Register the user-defined function `f` that takes `dimension` arguments in `model` as the symbol `s`. In addition, provide a gradient function `∇f`.

The functions `f`and `∇f` must support all subtypes of `Real` as arguments. Do not assume that the inputs are `Float64`.

Notes

- If the function `f` is univariate (i.e., `dimension == 1`), `∇f` must return a number which represents the first-order derivative of the function `f`.
- If the function `f` is multi-variate, `∇f` must have a signature matching `∇f(g::AbstractVector{T}, args::T...) where {T<:Real}`, where the first argument is a vector `g` that is modified in-place with the gradient.
- If `autodiff = true` and `dimension == 1`, use automatic differentiation to compute the second-order derivative information. If `autodiff = false`, only first-order derivative information will be used.
- `s` does not have to be the same symbol as `f`, but it is generally more readable if it is.

Examples

```
model = Model()
@variable(model, x)
f(x::T) where {T<:Real} = x^2
∇f(x::T) where {T<:Real} = 2 * x
register(model, :foo, 1, f, ∇f; autodiff = true)
@NLobjective(model, Min, foo(x))
```

```
model = Model()
@variable(model, x[1:2])
g(x::T, y::T) where {T<:Real} = x * y
function ∇g(g::AbstractVector{T}, x::T, y::T) where {T<:Real}
    g[1] = y
    g[2] = x
    return
end
register(model, :g, 2, g, ∇g; autodiff = true)
@NLobjective(model, Min, g(x[1], x[2]))
```

source

```
register(
    model::Model,
    s::Symbol,
    dimension::Integer,
    f::Function,
    ∇f::Function,
    ∇²f::Function,
)
```

Register the user-defined function `f` that takes `dimension` arguments in `model` as the symbol `s`. In addition, provide a gradient function `∇f` and a hessian function `∇²f`.

`∇f` and `∇²f` must return numbers corresponding to the first- and second-order derivatives of the function `f` respectively.

Notes

- Because automatic differentiation is not used, you can assume the inputs are all `Float64`.
- This method will throw an error if `dimension > 1`.
- `s` does not have to be the same symbol as `f`, but it is generally more readable if it is.

Examples

```
model = Model()
@variable(model, x)
f(x::Float64) = x^2
∇f(x::Float64) = 2 * x
∇²f(x::Float64) = 2.0
register(model, :foo, 1, f, ∇f, ∇²f)
@NLobjective(model, Min, foo(x))
```

source

26.6 Derivatives

`JuMP.NLPEvaluator` – Type.

```
NLPEvaluator(m::Model)
```

Return an `MOI.AbstractNLPEvaluator` constructed from the model `model`.

Before using, you must initialize the evaluator using `MOI.initialize`.

source

Chapter 27

Callbacks

More information can be found in the Callbacks section of the manual.

27.1 Macros

JuMP.@build_constraint – Macro.

```
@build_constraint(constraint_expr)
```

Constructs a ScalarConstraint or VectorConstraint using the same machinery as @constraint but without adding the constraint to a model.

Constraints using broadcast operators like x .<= 1 are also supported and will create arrays of ScalarConstraint or VectorConstraint.

Examples

```
model = Model();
@variable(model, x);
@build_constraint(2x >= 1)

# output
ScalarConstraint{GenericAffExpr{Float64,VariableRef},MathOptInterface.GreaterThan{Float64}}(2 x,
↪  MathOptInterface.GreaterThan{Float64}(1.0))
```

source

27.2 Callback variable primal

JuMP.callback_value – Function.

```
callback_value(cb_data, x::VariableRef)
```

Return the primal solution of a variable inside a callback.

cb_data is the argument to the callback function, and the type is dependent on the solver.

source

```
callback_value(cb_data, expr::Union{GenericAffExpr, GenericQuadExpr})
```

Return the primal solution of an affine or quadratic expression inside a callback by getting the value for each variable appearing in the expression.

`cb_data` is the argument to the callback function, and the type is dependent on the solver.

source

27.3 Callback node status

`JuMP.callback_node_status` – Function.

```
callback_node_status(cb_data, model::Model)
```

Return an `MOI.CallbackNodeStatusCode` enum, indicating if the current primal solution available from `callback_value` is integer feasible.

source

Chapter 28

Extensions

More information can be found in the Extensions section of the manual.

28.1 Define a new set

`JuMP.AbstractVectorSet` – Type.

```
AbstractVectorSet
```

An abstract type for defining new sets in JuMP.

Implement `moi_set(::AbstractVectorSet, dim::Int)` to convert the type into an MOI set.

See also: `moi_set`.

source

28.2 Extend `@variable`

`JuMP.ScalarVariable` – Type.

```
ScalarVariable{S,T,U,V} <: AbstractVariable
```

A struct used when adding variables.

See also: `add_variable`.

source

`JuMP.VariableInfo` – Type.

```
VariableInfo{S,T,U,V}
```

A struct by JuMP internally when creating variables. This may also be used by JuMP extensions to create new types of variables.

See also: `ScalarVariable`.

source

`JuMP.add_variable` – Function.

```
add_variable(m::Model, v::AbstractVariable, name::String="")
```

Add a variable v to `Model m` and sets its name.

source

JuMP.build_variable – Function.

```
build_variable(_error::Function, variables, ::SymmetricMatrixSpace)
```

Return a `VariablesConstrainedOnCreation` of shape `SymmetricMatrixShape` creating variables in `MOI.Reals`, i.e. "free" variables unless they are constrained after their creation.

This function is used by the `@variable` macro as follows:

```
@variable(model, Q[1:2, 1:2], Symmetric)
```

source

```
build_variable(_error::Function, variables, ::SkewSymmetricMatrixSpace)
```

Return a `VariablesConstrainedOnCreation` of shape `SkewSymmetricMatrixShape` creating variables in `MOI.Reals`, i.e. "free" variables unless they are constrained after their creation.

This function is used by the `@variable` macro as follows:

```
@variable(model, Q[1:2, 1:2] in SkewSymmetricMatrixSpace())
```

source

```
build_variable(_error::Function, variables, ::PSDCone)
```

Return a `VariablesConstrainedOnCreation` of shape `SymmetricMatrixShape` constraining the variables to be positive semidefinite.

This function is used by the `@variable` macro as follows:

```
@variable(model, Q[1:2, 1:2], PSD)
```

source

```
build_variable(
    _error::Function,
    info::VariableInfo,
    args...;
    kwargs...,
)
```

Return a new `AbstractVariable` object.

This method should only be implemented by developers creating JuMP extensions. It should never be called by users of JuMP.

Arguments

- `_error`: a function to call instead of `error`. `_error` annotates the error message with additional information for the user.
- `info`: an instance of `VariableInfo`. This has a variety of fields relating to the variable such as `info.lower_bound` and `info.binary`.
- `args`: optional additional positional arguments for extending the `@variable` macro.

- kwargs: optional keyword arguments for extending the @variable macro.

See also: @variable

Warning

Extensions should define a method with ONE positional argument to dispatch the call to a different method. Creating an extension that relies on multiple positional arguments leads to MethodErrors if the user passes the arguments in the wrong order.

Examples

```
@variable(model, x, Foo)
```

will call

```
build_variable(_error::Function, info::VariableInfo, ::Type{Foo})
```

Passing special-case positional arguments such as Bin, Int, and PSD is okay, along with keyword arguments:

```
@variable(model, x, Int, Foo(), mykwarg = true)
# or
@variable(model, x, Foo(), Int, mykwarg = true)
```

will call

```
build_variable(_error::Function, info::VariableInfo, ::Foo; mykwarg)
```

and info.integer will be true.

Note that the order of the positional arguments does not matter.

source

28.3 Extend @constraint

JuMP.build_constraint – Function.

```
build_constraint(
    _error::Function,
    f::AbstractVector{<:AbstractJuMPScalar},
    s::MOI.GreaterThan,
    extra::Union{MOI.AbstractVectorSet,AbstractVectorSet},
)
```

A helper method that re-writes

```
@constraint(model, X >= Y, extra)
```

into

```
@constraint(model, X - Y in extra)
```

source

```
build_constraint(
    _error::Function,
    f::AbstractVector{<:AbstractJuMPScalar},
    s::MOI.LessThan,
    extra::Union{MOI.AbstractVectorSet,AbstractVectorSet},
)
```

A helper method that re-writes

```
@constraint(model, Y <= X, extra)
```

into

```
@constraint(model, X - Y in extra)
```

source

```
build_constraint(_error::Function, Q::Symmetric{V, M},
                 ::PSDCone) where {V <: AbstractJuMPScalar,
                                   M <: AbstractMatrix{V}}
```

Return a `VectorConstraint` of shape `SymmetricMatrixShape` constraining the matrix `Q` to be positive semidefinite.

This function is used by the `@constraint` macros as follows:

```
@constraint(model, Symmetric(Q) in PSDCone())
```

The form above is usually used when the entries of `Q` are affine or quadratic expressions, but it can also be used when the entries are variables to get the reference of the semidefinite constraint, e.g.,

```
@variable model Q[1:2,1:2] Symmetric
# The type of `Q` is `Symmetric{VariableRef, Matrix{VariableRef}}`
var_psd = @constraint model Q in PSDCone()
# The `var_psd` variable contains a reference to the constraint
```

source

```
build_constraint(
    _error::Function,
    Q::AbstractMatrix{<:AbstractJuMPScalar},
    ::PSDCone,
)
```

Return a `VectorConstraint` of shape `SquareMatrixShape` constraining the matrix `Q` to be symmetric and positive semidefinite.

This function is used by the `@constraint` macro as follows:

```
@constraint(model, Q in PSDCone())
```

source

`JuMP.add_constraint` – Function.

```
add_constraint(model::Model, con::AbstractConstraint, name::String="")
```

Add a constraint con to `Model model` and sets its name.

source

`JuMP.AbstractShape` – Type.

```
AbstractShape
```

Abstract vectorizable shape. Given a flat vector form of an object of shape shape, the original object can be obtained by `reshape_vector`.

source

`JuMP.shape` – Function.

```
shape(c::AbstractConstraint)::AbstractShape
```

Return the shape of the constraint c.

source

`JuMP.reshape_vector` – Function.

```
reshape_vector(vectorized_form::Vector, shape::AbstractShape)
```

Return an object in its original shape shape given its vectorized form `vectorized_form`.

Examples

Given a `SymmetricMatrixShape` of vectorized form `[1, 2, 3]`, the following code returns the matrix `Symmetric(Matrix[1 2; 2 3])`:

```
julia> reshape_vector([1, 2, 3], SymmetricMatrixShape(2))
2×2 LinearAlgebra.Symmetric{Int64,Array{Int64,2}}:
 1  2
 2  3
```

source

`JuMP.reshape_set` – Function.

```
reshape_set(vectorized_set::MOI.AbstractSet, shape::AbstractShape)
```

Return a set in its original shape shape given its vectorized form `vectorized_form`.

Examples

Given a `SymmetricMatrixShape` of vectorized form `[1, 2, 3] in MOI.PositiveSemidefinieConeTriangle(2)`, the following code returns the set of the original constraint `Symmetric(Matrix[1 2; 2 3]) in PSDCone()`:

```
julia> reshape_set(MOI.PositiveSemidefiniteConeTriangle(2), SymmetricMatrixShape(2))
PSDCone()
```

source

`JuMP.dual_shape` – Function.

```
dual_shape(shape::AbstractShape)::AbstractShape
```

Returns the shape of the dual space of the space of objects of shape shape. By default, the `dual_shape` of a shape is itself. See the examples section below for an example for which this is not the case.

Examples

Consider polynomial constraints for which the dual is moment constraints and moment constraints for which the dual is polynomial constraints. Shapes for polynomials can be defined as follows:

```
struct Polynomial
    coefficients::Vector{Float64}
    monomials::Vector{Monomial}
end
struct PolynomialShape <: AbstractShape
    monomials::Vector{Monomial}
end
JuMP.reshape_vector(x::Vector, shape::PolynomialShape) = Polynomial(x, shape.monomials)
```

and a shape for moments can be defined as follows:

```
struct Moments
    coefficients::Vector{Float64}
    monomials::Vector{Monomial}
end
struct MomentsShape <: AbstractShape
    monomials::Vector{Monomial}
end
JuMP.reshape_vector(x::Vector, shape::MomentsShape) = Moments(x, shape.monomials)
```

Then `dual_shape` allows the definition of the shape of the dual of polynomial and moment constraints:

```
dual_shape(shape::PolynomialShape) = MomentsShape(shape.monomials)
dual_shape(shape::MomentsShape) = PolynomialShape(shape.monomials)
```

source

`JuMP.ScalarShape` – Type.

```
ScalarShape
```

Shape of scalar constraints.

source

`JuMP.VectorShape` – Type.

```
VectorShape
```

Vector for which the vectorized form corresponds exactly to the vector given.

source

`JuMP.SquareMatrixShape` – Type.

```
SquareMatrixShape
```

Shape object for a square matrix of `side_dimension` rows and columns. The vectorized form contains the entries of the the matrix given column by column (or equivalently, the entries of the lower-left triangular part given row by row).

source

`JuMP.SymmetricMatrixShape` – Type.

```
SymmetricMatrixShape
```

Shape object for a symmetric square matrix of `side_dimension` rows and columns. The vectorized form contains the entries of the upper-right triangular part of the matrix given column by column (or equivalently, the entries of the lower-left triangular part given row by row).

source

`JuMP.operator_to_set` – Function.

```
operator_to_set(_error::Function, ::Val{sense_symbol})
```

Converts a sense symbol to a set `set` such that `@constraint(model, func sense_symbol 0)` is equivalent to `@constraint(model, func in set)` for any `func::AbstractJuMPScalar`.

Example

Once a custom set is defined you can directly create a JuMP constraint with it:

```
julia> struct CustomSet{T} <: MOI.AbstractScalarSet
           value::T
       end

julia> Base.copy(x::CustomSet) = CustomSet(x.value)

julia> model = Model();

julia> @variable(model, x)
x

julia> cref = @constraint(model, x in CustomSet(1.0))
x ∈ CustomSet{Float64}(1.0)
```

However, there might be an appropriate sign that could be used in order to provide a more convenient syntax:

```
julia> JuMP.operator_to_set(::Function, ::Val{:}) = CustomSet(0.0)

julia> MOIU.shift_constant(set::CustomSet, value) = CustomSet(set.value + value)

julia> cref = @constraint(model, x  1)
x ∈ CustomSet{Float64}(1.0)
```

Note that the whole function is first moved to the right-hand side, then the sign is transformed into a set with zero constant and finally the constant is moved to the set with `MOIU.shift_constant`.

source

`JuMP.parse_constraint` – Function.

```
parse_constraint(_error::Function, expr::Expr)
```

The entry-point for all constraint-related parsing.

Arguments

- The `_error` function is passed everywhere to provide better error messages
- `expr` comes from the `@constraint` macro. There are two possibilities:
  - `@constraint(model, expr)`
  - `@constraint(model, name[args], expr)`

  In both cases, `expr` is the main component of the constraint.

Supported syntax

JuMP currently supports the following `expr` objects:

- `lhs <= rhs`
- `lhs == rhs`
- `lhs >= rhs`
- `l <= body <= u`
- `u >= body >= l`
- `lhs ⊥ rhs`
- `lhs in rhs`
- `lhs ∈ rhs`
- `z => {constraint}`
- `!z => {constraint}`

as well as all broadcasted variants.

Extensions

The infrastructure behind `parse_constraint` is extendable. See `parse_constraint_head` and `parse_constraint_call` for details.

source

`JuMP.parse_constraint_head` – Function.

```
parse_constraint_head(_error::Function, ::Val{head}, args...)
```

Implement this method to intercept the parsing of an expression with head head.

> **Warning**
> Extending the constraint macro at parse time is an advanced operation and has the potential to interfere with existing JuMP syntax. Please discuss with the developer chatroom before publishing any code that implements these methods.

Arguments

- `_error`: a function that accepts a `String` and throws the string as an error, along with some descriptive information of the macro from which it was thrown.
- head: the `.head` field of the `Expr` to intercept
- `args...`: the `.args` field of the `Expr`.

Returns

This function must return:

- `is_vectorized::Bool`: whether the expression represents a broadcasted expression like `x .<= 1`
- `parse_code::Expr`: an expression containing any setup or rewriting code that needs to be called before `build_constraint`
- `build_code::Expr`: an expression that calls `build_constraint(` or `build_constraint.(` depending on `is_vectorized`.

Existing implementations

JuMP currently implements:

- `::Val{:call}`, which forwards calls to `parse_constraint_call`
- `::Val{:comparison}`, which handles the special case of `l <= body <= u`.

See also: `parse_constraint_call`, `build_constraint`

source

`JuMP.parse_constraint_call` – Function.

```
parse_constraint_call(
    _error::Function,
    is_vectorized::Bool,
    ::Val{op},
    args...,
)
```

Implement this method to intercept the parsing of a `:call` expression with operator `op`.

> **Warning**
> Extending the constraint macro at parse time is an advanced operation and has the potential to interfere with existing JuMP syntax. Please discuss with the developer chatroom before publishing any code that implements these methods.

Arguments

- `_error`: a function that accepts a `String` and throws the string as an error, along with some descriptive information of the macro from which it was thrown.
- `is_vectorized`: a boolean to indicate if `op` should be broadcast or not
- `op`: the first element of the `.args` field of the `Expr` to intercept
- `args...`: the `.args` field of the `Expr`.

Returns

This function must return:

- `parse_code::Expr`: an expression containing any setup or rewriting code that needs to be called before `build_constraint`
- `build_code::Expr`: an expression that calls `build_constraint(` or `build_constraint.(` depending on `is_vectorized`.

See also: `parse_constraint_head`, `build_constraint`

source

```
parse_constraint_call(
    _error::Function,
    vectorized::Bool,
    ::Val{op},
    lhs,
    rhs,
) where {op}
```

Fallback handler for binary operators. These might be infix operators like `@constraint(model, lhs op rhs)`, or normal operators like `@constraint(model, op(lhs, rhs))`.

In both cases, we rewrite as `lhs - rhs in operator_to_set(_error, op)`.

See `operator_to_set` for details.

source

Part V

Background Information

Chapter 29

Algebraic modeling languages

JuMP is an algebraic modeling language for mathematical optimization written in the Julia language. In this page, we explain what an algebraic modeling language actually is.

29.1 What is an algebraic modeling language?

If you have taken a class in mixed-integer linear programming, you will have seen a formulation like:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \\ & x_i \in \mathbb{Z}, \quad \forall i \in \mathcal{I} \end{aligned}$$

where c, A, and b are appropriately sized vectors and matrices of data, and $\mathcal{I}$ denotes the set of variables that are integer.

Solvers expect problems in a standard form like this because it limits the types of constraints that they need to consider. This makes writing a solver much easier.

> **What is a solver?**
>
> A solver is a software package that computes solutions to one or more classes of problems.
>
> For example, HiGHS is a solver for linear programming (LP) and mixed integer programming (MIP) problems. It incorporates algorithms such as the simplex method and the interior-point method.
>
> JuMP currently supports a number of open-source and commercial solvers, which can be viewed in the Supported-solvers table.

Despite the textbook view of a linear program, you probably formulated problems algebraically like so:

$$\begin{aligned} \max\ & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} w_i x_i \leq b \\ & x_i \geq 0 \quad \forall i = 1, \ldots, n \\ & x_i \in \mathbb{Z} \quad \forall i = 1, \ldots, n. \end{aligned}$$

Info

Do you recognize this formulation? It's the knapsack problem.

Users prefer to write problems in algebraic form because it is more convenient. For example, we used $\leq b$, even though the standard form only supported constraints of the form $Ax = b$.

We could convert our knapsack problem into the standard form by adding a new slack variable x_0:

$$\begin{aligned} \max \ & \sum_{i=1}^{n} c_i x_i \\ \text{s.t.} & x_0 + \sum_{i=1}^{n} w_i x_i = b \\ & x_i \geq 0 \quad \forall i = 0, \ldots, n \\ & x_i \in \mathbb{Z} \quad \forall i = 1, \ldots, n. \end{aligned}$$

However, as models get more complicated, this manual conversion becomes more and more error-prone.

An algebraic modeling language is a tool that simplifies the translation between the algebraic form of the modeler, and the standard form of the solver.

Each algebraic modeling language has two main parts:

1. A domain specific language for the user to write down problems in algebraic form.
2. A converter from the algebraic form into a standard form supported by the solver (and back again).

Part 2 is less trivial than it might seem, because each solver has a unique application programming interface (API) and data structure for representing optimization models and obtaining results.

JuMP uses the MathOptInterface.jl package to abstract these differences between solvers.

What is MathOptInterface?

MathOptInterface (MOI) is an abstraction layer designed to provide an interface to mathematical optimization solvers so that users do not need to understand multiple solver-specific APIs. MOI can be used directly, or through a higher-level modeling interface like JuMP.

There are three main parts to MathOptInterface:

1. A solver-independent API that abstracts concepts such as adding and deleting variables and constraints, setting and getting parameters, and querying results. For more information on the MathOptInterface API, read the documentation.
2. An automatic rewriting system based on equivalent formulations of a constraint. For more information on this rewriting system, read the LazyBridgeOptimizer section of the manual, and our paper on arXiv.
3. Utilities for managing how and when models are copied to solvers. For more information on this, read the CachingOptimizer section of the manual.

29.2 From user to solver

This section provides a brief summary of the steps that happen in order to translate the model that the user writes into a model that the solver understands.

Step I: writing in algebraic form

JuMP provides the first part of an algebraic modeling language using the `@variable`, `@objective`, and `@constraint` macros.

For example, here's how we write the knapsack problem in JuMP:

```
julia> using JuMP, HiGHS

julia> function algebraic_knapsack(c, w, b)
           n = length(c)
           model = Model(HiGHS.Optimizer)
           set_silent(model)
           @variable(model, x[1:n] >= 0, Int)
           @objective(model, Max, sum(c[i] * x[i] for i = 1:n))
           @constraint(model, sum(w[i] * x[i] for i = 1:n) <= b)
           optimize!(model)
           return value.(x)
       end
algebraic_knapsack (generic function with 1 method)

julia> algebraic_knapsack([1, 2], [0.5, 0.5], 1.25)
2-element Vector{Float64}:
 0.0
 2.0
```

This formulation is compact, and it closely matches the algebraic formulation of the model we wrote out above.

Step II: algebraic to functional

For the next step, JuMP's macros re-write the variables and constraints into a functional form. Here's what the JuMP code looks like after this step:

```
julia> using JuMP, HiGHS

julia> function nonalgebraic_knapsack(c, w, b)
           n = length(c)
           model = Model(HiGHS.Optimizer)
           set_silent(model)
           x = [VariableRef(model) for i = 1:n]
           for i = 1:n
               set_lower_bound(x[i], 0)
               set_integer(x[i])
               set_name(x[i], "x[$i]")
           end
           obj = AffExpr(0.0)
           for i = 1:n
               add_to_expression!(obj, c[i], x[i])
           end
           set_objective(model, MAX_SENSE, obj)
```

```
           lhs = AffExpr(0.0)
           for i = 1:n
               add_to_expression!(lhs, w[i], x[i])
           end
           con = build_constraint(error, lhs, MOI.LessThan(b))
           add_constraint(model, con)
           optimize!(model)
           return value.(x)
       end
nonalgebraic_knapsack (generic function with 1 method)

julia> nonalgebraic_knapsack([1, 2], [0.5, 0.5], 1.25)
2-element Vector{Float64}:
 0.0
 2.0
```

Hopefully you agree that the macro version is much easier to read!

Part III: JuMP to MathOptInterface

In the third step, JuMP converts the functional form of the problem, i.e., `nonalgebraic_knapsack`, into the MathOptInterface API:

```
julia> using MathOptInterface, HiGHS

julia> const MOI = MathOptInterface;

julia> function mathoptinterface_knapsack(optimizer, c, w, b)
           n = length(c)
           model = MOI.instantiate(optimizer)
           MOI.set(model, MOI.Silent(), true)
           x = MOI.add_variables(model, n)
           for i in 1:n
               MOI.add_constraint(model, x[i], MOI.GreaterThan(0.0))
               MOI.add_constraint(model, x[i], MOI.Integer())
               MOI.set(model, MOI.VariableName(), x[i], "x[$i]")
           end
           MOI.set(model, MOI.ObjectiveSense(), MOI.MAX_SENSE)
           obj = MOI.ScalarAffineFunction(MOI.ScalarAffineTerm.(c, x), 0.0)
           MOI.set(model, MOI.ObjectiveFunction{typeof(obj)}(), obj)
           MOI.add_constraint(
               model,
               MOI.ScalarAffineFunction(MOI.ScalarAffineTerm.(w, x), 0.0),
               MOI.LessThan(b),
           )
           MOI.optimize!(model)
           return MOI.get.(model, MOI.VariablePrimal(), x)
       end
mathoptinterface_knapsack (generic function with 1 method)

julia> mathoptinterface_knapsack(HiGHS.Optimizer, [1.0, 2.0], [0.5, 0.5], 1.25)
2-element Vector{Float64}:
 0.0
 2.0
```

The code is becoming more verbose and looking less like the mathematical formulation that we started with.

Step IV: MathOptInterface to HiGHS

As a final step, the HiGHS.jl package converts the MathOptInterface form, i.e., `mathoptinterface_knapsack`, into a HiGHS-specific API:

```
julia> using HiGHS

julia> function highs_knapsack(c, w, b)
           n = length(c)
           model = Highs_create()
           Highs_setBoolOptionValue(model, "output_flag", false)
           for i in 1:n
               Highs_addCol(model, c[i], 0.0, Inf, 0, C_NULL, C_NULL)
               Highs_changeColIntegrality(model, i-1, 1)
           end
           Highs_changeObjectiveSense(model, -1)
           Highs_addRow(
               model,
               -Inf,
               b,
               Cint(length(w)),
               collect(Cint(0):Cint(n-1)),
               w,
           )
           Highs_run(model)
           x = fill(NaN, 2)
           Highs_getSolution(model, x, C_NULL, C_NULL, C_NULL)
           Highs_destroy(model)
           return x
       end
highs_knapsack (generic function with 1 method)

julia> highs_knapsack([1.0, 2.0], [0.5, 0.5], 1.25)
2-element Vector{Float64}:
 0.0
 2.0
```

We've now gone from a algebraic model that looked identical to the mathematical model we started with, to a verbose function that uses HiGHS-specific functionality.

The difference between `algebraic_knapsack` and `highs_knapsack` highlights the benefit that algebraic modeling languages provide to users. Moreover, if we used a different solver, the solver-specific function would be entirely different. A key benefit of an algebraic modeling language is that you can change the solver without needing to rewrite the model.

Part VI

Developer Docs

Chapter 30

Contributing

30.1 How to contribute to JuMP

Welcome! This document explains some ways you can contribute to JuMP.

Code of Conduct

This project and everyone participating in it is governed by the JuMP Code of Conduct. By participating, you are expected to uphold this code.

Join the community forum

First up, join the community forum.

The forum is a good place to ask questions about how to use JuMP. You can also use the forum to discuss possible feature requests and bugs before raising a GitHub issue (more on this below).

Aside from asking questions, the easiest way you can contribute to JuMP is to help answer questions on the forum!

Join the developer chatroom

If you're interested in contributing code to JuMP, the next place to join is the developer chatroom. Let us know what you have in mind, and we can point you in the right direction.

Improve the documentation

Chances are, if you asked (or answered) a question on the community forum, then it is a sign that the documentation could be improved. Moreover, since it is your question, you are probably the best-placed person to improve it!

The docs are written in Markdown and are built using Documenter.jl. You can find the source of all the docs here.

If your change is small (like fixing typos, or one or two sentence corrections), the easiest way to do this is via GitHub's online editor. (GitHub has help on how to do this.)

If your change is larger, or touches multiple files, you will need to make the change locally and then use Git to submit a pull request. (See Contribute code to JuMP below for more on this.)

> **Tip**
>
> If you need any help, come join the developer chatroom and we will walk you through the process.

File a bug report

Another way to contribute to JuMP is to file bug reports.

Make sure you read the info in the box where you write the body of the issue before posting. You can also find a copy of that info here.

> **Tip**
>
> If you're unsure whether you have a real bug, post on the community forum first. Someone will either help you fix the problem, or let you know the most appropriate place to open a bug report.

Contribute code to JuMP

Finally, you can also contribute code to JuMP!

> **Warning**
>
> If you do not have experience with Git, GitHub, and Julia development, the first steps can be a little daunting. However, there are lots of tutorials available online, including these for:
>
> - GitHub
> - Git and GitHub
> - Git
> - Julia package development
>
> If you need any help, come join the developer chatroom and we will walk you through the process.

Once you are familiar with Git and GitHub, the workflow for contributing code to JuMP is similar to the following:

Step 1: decide what to work on

The first step is to find an open issue (or open a new one) for the problem you want to solve. Then, before spending too much time on it, discuss what you are planning to do in the issue to see if other contributors are fine with your proposed changes. Getting feedback early can improve code quality, and avoid time spent writing code that does not get merged into JuMP.

> **Tip**
>
> At this point, remember to be patient and polite; you may get a lot of comments on your issue! However, do not be afraid! Comments mean that people are willing to help you improve the code that you are contributing to JuMP.

Step 2: fork JuMP

Go to https://github.com/jump-dev/JuMP.jl and click the "Fork" button in the top-right corner. This will create a copy of JuMP under your GitHub account.

Step 3: install JuMP locally

Open Julia and run:

```
] dev JuMP
```

This will download the JuMP Git repository to `~/.julia/dev/JuMP`. If you're on Windows, this will be `C:\\Users\\<my_name>\\.jul`

Warning

] command means "first type] to enter the Julia pkg mode, then type the rest. Don't copy-paste the code directly.

Step 4: checkout a new branch

Note

In the following, replace any instance of `GITHUB_ACCOUNT` with your GitHub user name.

The next step is to checkout a development branch. In a terminal (or command prompt on Windows), run:

```
$ cd ~/.julia/dev/JuMP

$ git remote add GITHUB_ACCOUNT https://github.com/GITHUB_ACCOUNT/JuMP.jl.git

$ git checkout master

$ git pull

$ git checkout -b my_new_branch
```

Tip

Lines starting with $ mean "run these in a terminal (command prompt on Windows)."

Step 5: make changes

Now make any changes to the source code inside the `~/.julia/dev/JuMP` directory.

Make sure you:

- Follow the Style guide and run JuliaFormatter
- Add tests and documentation for any changes or new features

Tip

When you change the source code, you'll need to restart Julia for the changes to take effect. This is a pain, so install Revise.jl.

Step 6a: test your code changes

To test that your changes work, run the JuMP test-suite by opening Julia and running:

```
cd("~/.julia/dev/JuMP")
] activate .
] test
```

Warning

Running the tests might take a long time (~10–15 minutes).

Tip

If you're using Revise.jl, you can also run the tests by calling `include`:

```
include("test/runtests.jl")
```

This can be faster if you want to re-run the tests multiple times.

Step 6b: test your documentation changes

Open Julia, then run:

```
cd("~/.julia/dev/JuMP/docs")
] activate .
include("src/make.jl")
```

Warning

Building the documentation might take a long time (~10 minutes).

Tip

If there's a problem with the tests that you don't know how to fix, don't worry. Continue to step 5, and one of the JuMP contributors will comment on your pull request telling you how to fix things.

Step 7: make a pull request

Once you've made changes, you're ready to push the changes to GitHub. Run:

```
$ cd ~/.julia/dev/JuMP

$ git add .

$ git commit -m "A descriptive message of the changes"

$ git push -u GITHUB_ACCOUNT my_new_branch
```

Then go to https://github.com/jump-dev/JuMP.jl and follow the instructions that pop up to open a pull request.

Step 8: respond to comments

At this point, remember to be patient and polite; you may get a lot of comments on your pull request! However, do not be afraid! A lot of comments means that people are willing to help you improve the code that you are contributing to JuMP.

To respond to the comments, go back to step 5, make any changes, test the changes in step 6, and then make a new commit in step 7. Your PR will automatically update.

Step 9: cleaning up

Once the PR is merged, clean-up your Git repository ready for the next contribution!

```
$ cd ~/.julia/dev/JuMP

$ git checkout master

$ git pull
```

> **Note**
>
> If you have suggestions to improve this guide, please make a pull request! It's particularly helpful if you do this after your first pull request because you'll know all the parts that could be explained better.

Thanks for contributing to JuMP!

Chapter 31

Extensions

31.1 Extensions

JuMP provides a variety of ways to extend the basic modeling functionality.

> **Tip**
>
> This documentation in this section is still a work-in-progress. The best place to look for ideas and help when writing a new JuMP extension are existing JuMP extensions. Examples include:
>
> - BilevelJuMP.jl
> - Coluna.jl
> - InfiniteOpt.jl
> - Plasmo.jl
> - PolyJuMP.jl
> - SDDP.jl
> - StochasticPrograms.jl
> - SumOfSquares.jl
> - vOptGeneric.jl

Define a new set

To define a new set for JuMP, subtype `MOI.AbstractScalarSet` or `MOI.AbstractVectorSet` and implement `Base.copy` for the set. That's it!

```
struct _NewVectorSet <: MOI.AbstractVectorSet
    dimension::Int
end
Base.copy(x::_NewVectorSet) = x

model = Model()
@variable(model, x[1:2])
@constraint(model, x in _NewVectorSet(2))

# output

[x[1], x[2]] ∈ _NewVectorSet(2)
```

However, for vector-sets, this requires the user to specify the dimension argument to their set, even though we could infer it from the length of x!

You can make a more user-friendly set by subtyping AbstractVectorSet and implementing moi_set.

```
struct NewVectorSet <: JuMP.AbstractVectorSet end
JuMP.moi_set(::NewVectorSet, dim::Int) = _NewVectorSet(dim)

model = Model()
@variable(model, x[1:2])
@constraint(model, x in NewVectorSet())

# output

[x[1], x[2]] ∈ _NewVectorSet(2)
```

Extend @variable

Just as Bin and Int create binary and integer variables, you can extend the @variable macro to create new types of variables. Here is an explanation by example, where we create a AddTwice type, that creates a tuple of two JuMP variables instead of a single variable.

First, create a new struct. This can be anything. Our struct holds a VariableInfo object that stores bound information, and whether the variable is binary or integer.

```
julia> struct AddTwice
           info::JuMP.VariableInfo
       end
```

Second, implement build_variable, which takes ::Type{AddTwice} as an argument, and returns an instance of AddTwice. Note that you can also receive keyword arguments.

```
julia> function JuMP.build_variable(
           _err::Function,
           info::JuMP.VariableInfo,
           ::Type{AddTwice};
           kwargs...
       )
           println("Can also use $kwargs here.")
           return AddTwice(info)
       end
```

Third, implement add_variable, which takes the instance of AddTwice from the previous step, and returns something. Typically, you will want to call add_variable here. For example, our AddTwice call is going to add two JuMP variables.

```
julia> function JuMP.add_variable(
           model::JuMP.Model,
           duplicate::AddTwice,
           name::String,
       )
           a = JuMP.add_variable(
               model,
```

```
            JuMP.ScalarVariable(duplicate.info),
            name * "_a",
         )
        b = JuMP.add_variable(
            model,
            JuMP.ScalarVariable(duplicate.info),
            name * "_b",
         )
        return (a, b)
    end
```

Now AddTwice can be passed to @variable similar to Bin or Int, or through the variable_type keyword. However, now it adds two variables instead of one!

```
julia> model = Model();

julia> @variable(model, x[i=1:2], variable_type = AddTwice, kw = i)
Can also use Base.Iterators.Pairs(:kw => 1) here.
Can also use Base.Iterators.Pairs(:kw => 2) here.
2-element Vector{Tuple{VariableRef, VariableRef}}:
 (x[1]_a, x[1]_b)
 (x[2]_a, x[2]_b)

julia> num_variables(model)
4

julia> first(x[1])
x[1]_a

julia> last(x[2])
x[2]_b
```

Extend @constraint

The @constraint macro has three steps that can be intercepted and extended: parse time, build time, and add time.

Parse

To extend the @constraint macro at parse time, implement one of the following methods:

- parse_constraint_head
- parse_constraint_call

> **Warning**
>
> Extending the constraint macro at parse time is an advanced operation and has the potential to interfere with existing JuMP syntax. Please discuss with the developer chatroom before publishing any code that implements these methods.

parse_constraint_head should be implemented to intercept an expression based on the .head field of Base.Expr. For example:

```
julia> using JuMP

julia> const MutableArithmetics = JuMP._MA;

julia> model = Model(); @variable(model, x);

julia> function JuMP.parse_constraint_head(
           _error::Function,
           ::Val{:(:=)},
           lhs,
           rhs,
       )
           println("Rewriting := as ==")
           new_lhs, parse_code = MutableArithmetics.rewrite(lhs)
           build_code = :(
               build_constraint($(_error), $(new_lhs), MOI.EqualTo($(rhs)))
           )
           return false, parse_code, build_code
       end

julia> @constraint(model, x + x := 1.0)
Rewriting := as ==
2 x = 1.0
```

parse_constraint_call should be implemented to intercept an expression of the form `Expr(:call, op, args...)`. For example:

```
julia> using JuMP

julia> const MutableArithmetics = JuMP._MA;

julia> model = Model(); @variable(model, x);

julia> function JuMP.parse_constraint_call(
           _error::Function,
           is_vectorized::Bool,
           ::Val{:my_equal_to},
           lhs,
           rhs,
       )
           println("Rewriting my_equal_to to ==")
           new_lhs, parse_code = MutableArithmetics.rewrite(lhs)
           build_code = if is_vectorized
               :(build_constraint($(_error), $(new_lhs), MOI.EqualTo($(rhs)))
           )
           else
               :(build_constraint.($(_error), $(new_lhs), MOI.EqualTo($(rhs))))
           end
           return parse_code, build_code
       end

julia> @constraint(model, my_equal_to(x + x, 1.0))
Rewriting my_equal_to to ==
2 x = 1.0
```

Tip

When parsing a constraint you can recurse into sub-constraint (e.g., the `{expr}` in `z => {x <= 1}`) by calling `parse_constraint`.

Build

To extend the `@constraint` macro at build time, implement a new `build_constraint` method.

This may mean implementing a method for a specific function or set created at parse time, or it may mean implementing a method which handles additional positional arguments.

`build_constraint` must return an `AbstractConstraint`, which can either be an `AbstractConstraint` already supported by JuMP, e.g., `ScalarConstraint` or `VectorConstraint`, or a custom `AbstractConstraint` with a corresponding `add_constraint` method (see Add).

> **Tip**
>
> The easiest way to extend `@constraint` is via an additional positional argument to `build_constraint`.

Here is an example of adding extra arguments to `build_constraint`:

```
julia> model = Model(); @variable(model, x);

julia> struct MyConstrType end

julia> function JuMP.build_constraint(
           _error::Function,
           f::JuMP.GenericAffExpr,
           set::MOI.EqualTo,
           extra::Type{MyConstrType};
           d = 0,
       )
           new_set = MOI.LessThan(set.value + d)
           return JuMP.build_constraint(_error, f, new_set)
       end

julia> @constraint(model, my_con, x == 0, MyConstrType, d = 2)
my_con : x ≤ 2.0
```

> **Note**
>
> Only a single positional argument can be given to a particular constraint. Extensions that seek to pass multiple arguments (e.g., `Foo` and `Bar`) should combine them into one argument type (e.g., `FooBar`).

Add

`build_constraint` returns an `AbstractConstraint` object. To extend `@constraint` at add time, define a subtype of `AbstractConstraint`, implement `build_constraint` to return an instance of the new type, and then implement `add_constraint`.

Here is an example:

```
julia> model = Model(); @variable(model, x);

julia> struct MyTag
```

```
           name::String
       end

julia> struct MyConstraint{S} <: AbstractConstraint
           name::String
           f::AffExpr
           s::S
       end

julia> function JuMP.build_constraint(
            _error::Function,
            f::AffExpr,
            set::MOI.AbstractScalarSet,
            extra::MyTag,
       )
            return MyConstraint(extra.name, f, set)
       end

julia> function JuMP.add_constraint(
            model::Model,
            con::MyConstraint,
            name::String,
       )
            return add_constraint(
                model,
                ScalarConstraint(con.f, con.s),
                "$(con.name)[$(name)]",
            )
       end

julia> @constraint(model, my_con, 2x <= 1, MyTag("my_prefix"))
my_prefix[my_con] : 2 x - 1 ≤ 0.0
```

The extension dictionary

Every JuMP model has a field `.ext::Dict{Symbol,Any}` that can be used by extensions. This is useful if your extensions to `@variable` and `@constraint` need to store information between calls.

The most common way to initialize a model with information in the `.ext` dictionary is to provide a new constructor:

```
julia> function MyModel()
           model = Model()
           model.ext[:MyModel] = 1
           return model
       end
MyModel (generic function with 1 method)

julia> model = MyModel()
A JuMP Model
Feasibility problem with:
Variables: 0
Model mode: AUTOMATIC
CachingOptimizer state: NO_OPTIMIZER
Solver name: No optimizer attached.
```

```
julia> model.ext
Dict{Symbol, Any} with 1 entry:
  :MyModel => 1
```

If you define extension data, implement `copy_extension_data` to support `copy_model`.

Defining new JuMP models

If extending individual calls to `@variable` and `@constraint` is not sufficient, it is possible to implement a new model via a subtype of `AbstractModel`. You can also define new `AbstractVariableRef`s to create different types of JuMP variables.

> **Warning**
>
> Extending JuMP in this manner is an advanced operation. We strongly encourage you to consider how you can use the methods mentioned in the previous sections to achieve your aims instead of defining new model and variable types. Consult the developer chatroom before starting work on this.

If you define new types, you will need to implement a considerable number of methods, and doing so will require a detailed understanding of the JuMP internals. Therefore, the list of methods to implement is currently undocumented.

The easiest way to extend JuMP by defining a new model type is to follow an existing example. A simple example to follow is the JuMPExtension module in the JuMP test suite. The best example of an external JuMP extension that implements an `AbstractModel` is InfiniteOpt.jl.

Creating new container types

JuMP macros (for example, `@variable`) accept a `container` keyword argument to force the type of container that is chosen. By default, JuMP supports `container = Array`, `container = DenseAxisArray`, `container = SparseAxisArray` and `container = Auto`. You can extend support to user-defined types by implementing `Containers.container`.

For example, here is a container that reverses the order of the indices:

```
julia> struct Foo end

julia> function Containers.container(f::Function, indices, ::Type{Foo})
           return reverse([f(i...) for i in indices])
       end

julia> model = Model();

julia> @variable(model, x[1:3], container = Foo)
3-element Vector{VariableRef}:
 x[3]
 x[2]
 x[1]

julia> x[1]
x[3]

julia> @variable(model, y[1:3, 1:2], container = Foo)
```

```
3×2 Matrix{VariableRef}:
 y[3,2]  y[3,1]
 y[2,2]  y[2,1]
 y[1,2]  y[1,1]

julia> y[1, 1]
y[3,2]

julia> @variable(model, z[i=1:3; isodd(i)], container = Foo)
2-element Vector{VariableRef}:
 z[3]
 z[1]

julia> z[2]
z[1]
```

Warning

If you are a general user, you should not need to create a new container type. Instead, consider following User-defined containers and create a new container using standard Julia syntax. For example:

```
julia> model = Model();

julia> @variable(model, x[1:3])
3-element Vector{VariableRef}:
 x[1]
 x[2]
 x[3]

julia> y = reverse(x)
3-element Vector{VariableRef}:
 x[3]
 x[2]
 x[1]
```

Chapter 32

Custom binaries

32.1 How to use a custom binary

Many solvers are not written in Julia, but instead in languages like C or C++. JuMP interacts with these solvers through binary dependencies.

For many open-source solvers, we automatically install the appropriate binary when you run `Pkg.add("Solver")`. For example, `Pkg.add("ECOS")` will also install the ECOS binary.

This page explains how this installation works, and how you can use a custom binary.

> **Compat**
>
> These instructions require Julia 1.6 or later.

Background

Each solver that JuMP supports is structured as a Julia package. For example, the interface for the ECOS solver is provided by the ECOS.jl package.

> **Tip**
>
> This page uses the example of ECOS.jl because it is simple to compile. Other solvers follow similar conventions. For example, the interface to the Clp solver is provided by Clp.jl.

The ECOS.jl package provides an interface between the C API of ECOS and MathOptInterface. However, it does not handle the installation of the solver binary; that is the job for a JLL package.

A JLL is a Julia package that wraps a pre-compiled binary. Binaries are built using Yggdrasil (for example, ECOS) and hosted in the JuliaBinaryWrappers GitHub repository (for example, ECOS_jll.jl).

JLL packages contain little code. Their only job is to `dlopen` a dynamic library, along with any dependencies.

JLL packages manage their binary dependencies using Julia's artifact system. Each JLL package has an `Artifacts.toml` file which describes where to find each binary artifact for each different platform that it might be installed on. Here is the Artifacts.toml file for ECOS_jll.jl.

The binaries installed by the JLL package should be sufficient for most users. In rare cases, however, you may require a custom binary. The two main reasons to use a custom binary are:

- You want a binary with custom compilation settings (for example, debugging)

- You want a binary with a set of dependencies that are not available on Yggdrasil (for example, a commerial solver like Gurobi or CPLEX).

The following sections explain how to replace the binaries provided by a JLL package with the custom ones you have compiled. As a reminder, we use ECOS as an example for simplicity, but the steps are the same for other solvers.

Explore the JLL you want to override

The first step is to explore the structure and filenames of the JLL package we want to override.

Find the location of the files using `.artifact_dir`:

```
julia> using ECOS_jll

julia> ECOS_jll.artifact_dir
"/Users/oscar/.julia/artifacts/2addb75332eff5a1657b46bb6bf30d2410bc7ecf"
```

> **Tip**
>
> This path may be different on other machines.

Here is what it contains:

```
julia> readdir(ECOS_jll.artifact_dir)
4-element Vector{String}:
 "include"
 "lib"
 "logs"
 "share"

julia> readdir(joinpath(ECOS_jll.artifact_dir, "lib"))
1-element Vector{String}:
 "libecos.dylib"
```

Other solvers may have a `bin` directory containing executables. To use a custom binary of ECOS, we need to replace `/lib/libecos.dylib` with our custom binary.

Compile a custom binary

The next step is to compile a custom binary. Because ECOS is written in C with no dependencies, this is easy to do if you have a C compiler:

```
oscar@Oscars-MBP jll_example % git clone https://github.com/embotech/ecos.git
[... lines omitted ...]
oscar@Oscars-MBP jll_example % cd ecos
oscar@Oscars-MBP ecos % make shared
[... many lines omitted...]
oscar@Oscars-MBP ecos % mkdir lib
oscar@Oscars-MBP ecos % cp libecos.dylib lib
```

> **Warning**
>
> Compiling custom solver binaries is an advanced operation. Due to the complexities of compiling various solvers, the JuMP community is unable to help you diagnose and fix compilation issues.

After this compilation step, we now have a folder `/tmp/jll_example/ecos` that contains `lib` and `include` directories with the same files as `ECOS_jll`:

```
julia> readdir(joinpath("ecos", "lib"))
1-element Vector{String}:
 "libecos.dylib"
```

Overriding a single library

To override the `libecos` library, we need to know what `ECOS_jll` calls it. (In most cases, it will also be `libecos`, but not always.)

There are two ways you can check.

1. Check the bottom of the JLL's GitHub README. For example, ECOS_jll has a single `LibraryProduct` called `libecos`.

2. Type `ECOS_jll.` and the press the `[TAB]` key twice to auto-complete available options:

   ```
   julia> ECOS_jll.
   LIBPATH          PATH_list         best_wrapper      get_libecos_path   libecos_handle
   LIBPATH_list     __init__          dev_jll           is_available       libecos_path
   PATH             artifact_dir      find_artifact_dir libecos
   ```

   Here you can see there is `libecos`, and more usefully for us, `libecos_path`.

Once you know the name of the variable to override (the one that ends in `_path`), use Preferences.jl to specify a new path:

```
using Preferences
set_preferences!(
    "LocalPreferences.toml",
    "ECOS_jll",
    "libecos_path" => "/tmp/jll_example/ecos/lib/libecos"
)
```

This will create a file in your current directory called `LocalPreferences.toml` with the contents:

```
[ECOS_jll]
libecos_path = "/tmp/jll_example/ecos/lib/libecos"
```

Now if you restart Julia, you will see:

```
julia> using ECOS_jll

julia> ECOS_jll.libecos
"/tmp/jll_example/ecos/lib/libecos"
```

To go back to using the default library, just delete the `LocalPreferences.toml` file.

Overriding an entire artifact

Sometimes a solver may provide a number of libraries and executables, and specifying the path for each of the becomes tedious. In this case, we can use Julia's `Override.toml` to replace an entire artifact.

Overriding an entire artifact requires you to replicate the structure and contents of the JLL package that we explored above.

In most cases you need only reproduce the `include`, `lib`, and `bin` directories (if they exist). You can safely ignore any `logs` or `share` directories. Take careful note of what files each directory contains and what they are called.

For our ECOS example, we already reproduced the structure when we compiled ECOS.

So, now we need to tell Julia to use our custom installation instead of the default. We can do this by making an override file at `~/.julia/artifacts/Overrides.toml`.

`Overrides.toml` has the following content:

```
# Override for ECOS_jll
2addb75332eff5a1657b46bb6bf30d2410bc7ecf = "/tmp/jll_example/ecos"
```

where `2addb75332eff5a1657b46bb6bf30d2410bc7ecf` is the folder from the original `ECOS_jll.artifact_dir` and `"/tmp/jll_example/ecos"` is the location of our new installation. Replace these as appropriate for your system.

If you restart Julia after creating the override file, you will see:

```
julia> using ECOS_jll

julia> ECOS_jll.artifact_dir
"/tmp/jll_example/ecos"
```

Now when we use ECOS it will use our custom binary.

Using Cbc with a custom binary

As a second example, we demonstrate how to use Cbc.jl with a custom binary.

Explore the JLL you want to override

First, let's check where `Cbc_jll` is installed:

```
julia> using Cbc_jll

julia> Cbc_jll.artifact_dir
"/Users/oscar/.julia/artifacts/e481bc81db5e229ba1f52b2b4bd57484204b1b06"

julia> readdir(Cbc_jll.artifact_dir)
5-element Vector{String}:
 "bin"
 "include"
 "lib"
 "logs"
 "share"
```

```
julia> readdir(joinpath(Cbc_jll.artifact_dir, "bin"))
1-element Vector{String}:
 "cbc"

julia> readdir(joinpath(Cbc_jll.artifact_dir, "lib"))
10-element Vector{String}:
 "libCbc.3.10.5.dylib"
 "libCbc.3.dylib"
 "libCbc.dylib"
 "libCbcSolver.3.10.5.dylib"
 "libCbcSolver.3.dylib"
 "libCbcSolver.dylib"
 "libOsiCbc.3.10.5.dylib"
 "libOsiCbc.3.dylib"
 "libOsiCbc.dylib"
 "pkgconfig"
```

Compile a custom binary

Next, we need to compile Cbc. Cbc can be difficult to compile (it has a lot of dependencies), but for macOS users there is a homebrew recipe:

```
(base) oscar@Oscars-MBP jll_example % brew install cbc
[ ... lines omitted ... ]
(base) oscar@Oscars-MBP jll_example % brew list cbc
/usr/local/Cellar/cbc/2.10.5/bin/cbc
/usr/local/Cellar/cbc/2.10.5/include/cbc/ (76 files)
/usr/local/Cellar/cbc/2.10.5/lib/libCbc.3.10.5.dylib
/usr/local/Cellar/cbc/2.10.5/lib/libCbcSolver.3.10.5.dylib
/usr/local/Cellar/cbc/2.10.5/lib/libOsiCbc.3.10.5.dylib
/usr/local/Cellar/cbc/2.10.5/lib/pkgconfig/ (2 files)
/usr/local/Cellar/cbc/2.10.5/lib/ (6 other files)
/usr/local/Cellar/cbc/2.10.5/share/cbc/ (59 files)
/usr/local/Cellar/cbc/2.10.5/share/coin/ (4 files)
```

Override single libraries

To use `Preferences.jl` to override specific libraries we first check the names of each library in `Cbc_jll`:

```
julia> Cbc_jll.
LIBPATH             cbc                   get_libcbcsolver_path  libOsiCbc_path
LIBPATH_list        cbc_path              is_available           libcbcsolver
PATH                dev_jll               libCbc                 libcbcsolver_handle
PATH_list           find_artifact_dir     libCbc_handle          libcbcsolver_path
__init__            get_cbc_path          libCbc_path
artifact_dir        get_libCbc_path       libOsiCbc
best_wrapper        get_libOsiCbc_path    libOsiCbc_handle
```

Then we add the following to `LocalPreferences.toml`:

```
[Cbc_jll]
cbc_path = "/usr/local/Cellar/cbc/2.10.5/bin/cbc"
libCbc_path = "/usr/local/Cellar/cbc/2.10.5/lib/libCbc.3.10.5"
libOsiCbc_path = "/usr/local/Cellar/cbc/2.10.5/lib/libOsiCbc.3.10.5"
libcbcsolver_path = "/usr/local/Cellar/cbc/2.10.5/lib/libCbcSolver.3.10.5"
```

Info

Note that capitalization matters, so `libcbcsolver_path` corresponds to `libCbcSolver.3.10.5`.

Override entire artifact

To use the homebrew install as our custom binary we add the following to `~/.julia/artifacts/Overrides.toml`:

```
# Override for Cbc_jll
e481bc81db5e229ba1f52b2b4bd57484204b1b06 = "/usr/local/Cellar/cbc/2.10.5"
```

Chapter 33

Style Guide

33.1 Style guide and design principles

Style guide

This section describes the coding style rules that apply to JuMP code and that we recommend for JuMP models and surrounding Julia code. The motivations for a style guide include:

- conveying best practices for writing readable and maintainable code
- reducing the amount of time spent on bike-shedding by establishing basic naming and formatting conventions
- lowering the barrier for new contributors by codifying the existing practices (e.g., you can be more confident your code will pass review if you follow the style guide)

In some cases, the JuMP style guide diverges from the Julia style guide. All such cases will be explicitly noted and justified.

The JuMP style guide adopts many recommendations from the Google style guides.

> **Info**
>
> The style guide is always a work in progress, and not all JuMP code follows the rules. When modifying JuMP, please fix the style violations of the surrounding code (i.e., leave the code tidier than when you started). If large changes are needed, consider separating them into another PR.

JuliaFormatter

JuMP uses JuliaFormatter.jl as an autoformatting tool.

We use the options contained in `.JuliaFormatter.toml`.

To format code, cd to the JuMP directory, then run:

```
] add JuliaFormatter@0.22.2
using JuliaFormatter
format("src")
format("test")
```

Info

A continuous integration check verifies that all PRs made to JuMP have passed the formatter.

The following sections outline extra style guide points that are not fixed automatically by JuliaFormatter.

Whitespace

For conciseness, never use more than one blank line within a function, and never begin a function with a blank line.

Bad:

```julia
function foo(x)
    y = 2 * x


    return y
end

function foo(x)

    y = 2 * x
    return y
end
```

Juxtaposed multiplication

Only use juxtaposed multiplication when the right-hand side is a symbol.

Good:

```julia
2x  # Acceptable if there are space constraints.
2 * x  # This is preferred if space is not an issue.
2 * (x + 1)
```

Bad:

```julia
2(x + 1)
```

Empty vectors

For a type T, `T[]` and `Vector{T}()` are equivalent ways to create an empty vector with element type T. Prefer `T[]` because it is more concise.

Comments

For non-native speakers and for general clarity, comments in code must be proper English sentences with appropriate punctuation.

Good:

```julia
# This is a comment demonstrating a good comment.
```

Bad:

```julia
# a bad comment
```

JuMP macro syntax

For consistency, always use parentheses.

Good:

```
@variable(model, x >= 0)
```

Bad:

```
@variable model x >= 0
```

For consistency, always use `constant * variable` as opposed to `variable * constant`. This makes it easier to read models in ambiguous cases like `a * x`.

Good:

```
a = 4
@constraint(model, 3 * x <= 1)
@constraint(model, a * x <= 1)
```

Bad:

```
a = 4
@constraint(model, x * 3 <= 1)
@constraint(model, x * a <= 1)
```

In order to reduce boilerplate code, prefer the plural form of macros over lots of repeated calls to singular forms.

Good:

```
@variables(model, begin
    x >= 0
    y >= 1
    z <= 2
end)
```

Bad:

```
@variable(model, x >= 0)
@variable(model, y >= 1)
@variable(model, z <= 2)
```

An exception is made for calls with many keyword arguments, since these need to be enclosed in parentheses in order to parse properly.

Acceptable:

```
@variable(model, x >= 0, start = 0.0, base_name = "my_x")
@variable(model, y >= 1, start = 2.0)
@variable(model, z <= 2, start = -1.0)
```

Also acceptable:

```
@variables(model, begin
    x >= 0, (start = 0.0, base_name = "my_x")
    y >= 1, (start = 2.0)
    z <= 2, (start = -1.0)
end)
```

While we always use `in` for `for`-loops, it is acceptable to use = in the container declarations of JuMP macros.

Okay:

```
@variable(model, x[i=1:3])
```

Also okay:

```
@variable(model, x[i in 1:3])
```

Naming

```
module SomeModule end
function some_function end
const SOME_CONSTANT = ...
struct SomeStruct
  some_field::SomeType
end
@enum SomeEnum ENUM_VALUE_A ENUM_VALUE_B
some_local_variable = ...
some_file.jl # Except for ModuleName.jl.
```

Exported and non-exported names

Begin private module level functions and constants with an underscore. All other objects in the scope of a module should be exported. (See JuMP.jl for an example of how to do this.)

Names beginning with an underscore should only be used for distinguishing between exported (public) and non-exported (private) objects. Therefore, never begin the name of a local variable with an underscore.

```
module MyModule

export public_function, PUBLIC_CONSTANT

function _private_function()
    local_variable = 1
    return
end

function public_function end

const _PRIVATE_CONSTANT = 3.14159
const PUBLIC_CONSTANT = 1.41421

end
```

Use of underscores within names

The Julia style guide recommends avoiding underscores "when readable", for example, `haskey`, `isequal`, `remotecall`, and `remotecall_fetch`. This convention creates the potential for unnecessary bikeshedding and also forces the user to recall the presence/absence of an underscore, e.g., "was that argument named `basename` or `base_name`?". For consistency, always use underscores in variable names and function names to separate words.

Use of `!`

Julia has a convention of appending `!` to a function name if the function modifies its arguments. We recommend to:

- Omit `!` when the name itself makes it clear that modification is taking place, e.g., `add_constraint` and `set_name`. We depart from the Julia style guide because `!` does not provide a reader with any additional information in this case, and adherence to this convention is not uniform even in base Julia itself (consider `Base.println` and `Base.finalize`).
- Use `!` in all other cases. In particular it can be used to distinguish between modifying and non-modifying variants of the same function like `scale` and `scale!`.

Note that `!` is not a self-documenting feature because it is still ambiguous which arguments are modified when multiple arguments are present. Be sure to document which arguments are modified in the method's docstring.

See also the Julia style guide recommendations for ordering of function arguments.

Abbreviations

Abbreviate names to make the code more readable, not to save typing. Don't arbitrarily delete letters from a word to abbreviate it (e.g., `indx`). Use abbreviations consistently within a body of code (e.g., do not mix `con` and `constr`, `idx` and `indx`).

Common abbreviations:

- `num` for `number`
- `con` for `constraint`

No one-letter variable names

Where possible, avoid one-letter variable names.

Use `model = Model()` instead of `m = Model()`

Exceptions are made for indices in loops.

User-facing `MethodError`

Specifying argument types for methods is mostly optional in Julia, which means that it's possible to find out that you are working with unexpected types deep in the call chain. Avoid this situation or handle it with a helpful error message. A user should see a `MethodError` only for methods that they called directly.

Bad:

```julia
_internal_function(x::Integer) = x + 1
# The user sees a MethodError for _internal_function when calling
# public_function("a string"). This is not very helpful.
public_function(x) = _internal_function(x)
```

Good:

```julia
_internal_function(x::Integer) = x + 1
# The user sees a MethodError for public_function when calling
# public_function("a string"). This is easy to understand.
public_function(x::Integer) = _internal_function(x)
```

If it is hard to provide an error message at the top of the call chain, then the following pattern is also ok:

```julia
_internal_function(x::Integer) = x + 1
function _internal_function(x)
    error(
        "Internal error. This probably means that you called " *
        "public_function() with the wrong type.",
    )
end
public_function(x) = _internal_function(x)
```

`@enum` vs. `Symbol`

The `@enum` macro lets you define types with a finite number of values that are explicitly enumerated (like `enum` in C/C++). `Symbol`s are lightweight strings that are used to represent identifiers in Julia (for example, `:x`).

`@enum` provides type safety and can have docstrings attached to explain the possible values. Use `@enum`s when applicable, e.g., for reporting statuses. Use strings to provide long-form additional information like error messages.

Use of `Symbol` should typically be reserved for identifiers, e.g., for lookup in the JuMP model (`model[:my_variable]`).

`using` vs. `import`

`using ModuleName` brings all symbols exported by the module `ModuleName` into scope, while `import ModuleName` brings only the module itself into scope. (See the Julia manual) for examples and more details.

For the same reason that `from <module> import *` is not recommended in python (PEP 8), avoid `using ModuleName` except in throw-away scripts or at the REPL. The `using` statement makes it harder to track where symbols come from and exposes the code to ambiguities when two modules export the same symbol.

Prefer `using ModuleName: x, p` to `import ModuleName.x, ModuleName.p` and `import MyModule: x, p` because the `import` versions allow method extension without qualifying with the module name.

Similarly, `using ModuleName: ModuleName` is an acceptable substitute for `import ModuleName`, because it does not bring all symbols exported by `ModuleName` into scope. However, we prefer `import ModuleName` for consistency.

Documentation

This section describes the writing style that should be used when writing documentation for JuMP (and supporting packages).

We can recommend the documentation style guides by Divio, Google, and Write the Docs as general reading for those writing documentation. This guide delegates a thorough handling of the topic to those guides and instead elaborates on the points more specific to Julia and documentation that use Documenter.

- Be concise
- Use lists instead of long sentences
- Use numbered lists when describing a sequence, e.g., (1) do X, (2) then Y
- Use bullet points when the items are not ordered
- Example code should be covered by doctests
- When a word is a Julia symbol and not an English word, enclose it with backticks. In addition, if it has a docstring in this doc add a link using `@ref`. If it is a plural, add the "s" after the closing backtick. For example,

  ```
  [`VariableRef`](@ref)s
  ```

- Use `@meta` blocks for TODOs and other comments that shouldn't be visible to readers. For example,

  ````
  ```@meta
  # TODO: Mention also X, Y, and Z.
  ```
  ````

Docstrings

- Every exported object needs a docstring
- All examples in docstrings should be `jldoctests`
- Always use complete English sentences with proper punctuation
- Do not terminate lists with punctuation (e.g., as in this doc)

Here is an example:

````
"""
    signature(args; kwargs...)

Short sentence describing the function.

Optional: add a slightly longer paragraph describing the function.

## Notes

 - List any notes that the user should be aware of

## Examples

```jldoctest
julia> 1 + 1
2
```
"""
````

Testing

Use a module to encapsulate tests, and structure all tests as functions. This avoids leaking local variables between tests.

Here is a basic skeleton:

```julia
module TestPkg

using Test

_helper_function() = 2

function test_addition()
    @test 1 + 1 == _helper_function()
end

function runtests()
    for name in names(@__MODULE__; all = true)
        if startswith("$(name)", "test_")
            @testset "$(name)" begin
                getfield(@__MODULE__, name)()
            end
        end
    end
end

end # TestPkg

TestPkg.runtests()
```

Break the tests into multiple files, with one module per file, so that subsets of the codebase can be tested by calling `include` with the relevant file.

Design principles

TODO: How to structure and test large JuMP models, libraries that use JuMP.

For how to write a solver, see MOI.

Chapter 34

Roadmap

34.1 Development roadmap

This page is not JuMP documentation per se but are notes for the JuMP community. The JuMP developers have compiled this roadmap document to share their plans and goals. Contributions to roadmap issues are especially invited.

Most of these issues will require changes to both JuMP and MathOptInterface, and are non-trivial in their implementation. They are in no particular order, but represent broad themes that we see as areas in which JuMP could be improved.

- Make nonlinear programming a first-class citizen. There have been many issues and discussions about this: currently nonlinear constraints are handled through a `MOI.NLPBlock` and have various limitations and restrictions.
  - https://github.com/jump-dev/JuMP.jl/issues/1185
  - https://github.com/jump-dev/JuMP.jl/issues/1198
  - https://github.com/jump-dev/JuMP.jl/issues/2788
  - https://github.com/jump-dev/MathOptInterface.jl/issues/846
  - https://github.com/jump-dev/MathOptInterface.jl/issues/1397
- Add support for coefficient types other than `Float64`: https://github.com/jump-dev/JuMP.jl/issues/2025 Since the very beginning, JuMP has hard-coded the coefficient type as `Float64`. This has made it impossible to support solvers which can use other types such as `BigFloat` or `Rational{BigInt}`.
- Add support for constraint programming: https://github.com/jump-dev/JuMP.jl/issues/2227 JuMP has a strong focus on linear, conic and nonlinear optimization problems. We want to add better support for constraint programming.
- Add support for multiobjective problems: https://github.com/jump-dev/JuMP.jl/issues/2099 JuMP is restricted to problems with scalar-valued objectives. We want to extend this to vector-valued problems.
- Refactor the internal code of JuMP's macros. The code in `src/macros.jl` is some of the oldest part of JuMP and is difficult to read, modify, and extend. We should overhaul the internals of JuMP's macros–-without making user-visible breaking changes–-to improve their long-term maintainability.

Part VII

MathOptInterface

Chapter 35

Introduction

35.1 Introduction

Warning

This documentation in this section is a copy of the official MathOptInterface documentation available at https://jump.dev/MathOptInterface.jl/v1.1.1. It is included here to make it easier to link concepts between JuMP and MathOptInterface.

What is MathOptInterface?

MathOptInterface.jl (MOI) is an abstraction layer designed to provide a unified interface to mathematical optimization solvers so that users do not need to understand multiple solver-specific APIs.

Tip

This documentation is aimed at developers writing software interfaces to solvers and modeling languages using the MathOptInterface API. If you are a user interested in solving optimization problems, we encourage you instead to use MOI through a higher-level modeling interface like JuMP or Convex.jl.

How the documentation is structured

Having a high-level overview of how this documentation is structured will help you know where to look for certain things.

- The **Tutorials** section contains articles on how to use and implement the MathOptInteraface API. Look here if you want to write a model in MOI, or write an interface to a new solver.
- The **Manual** contains short code-snippets that explain how to use the MOI API. Look here for more details on particular areas of MOI.
- The **Background** section contains articles on the theory behind MathOptInterface. Look here if you want to understand why, rather than how.
- The **API Reference** contains a complete list of functions and types that comprise the MOI API. Look here is you want to know how to use (or implement) a particular function.
- The **Submodules** section contains stand-alone documentation for each of the submodules within MOI. These submodules are not required to interface a solver with MOI, but they make the job much easier.

Citing MathOptInterface

A paper describing the design and features of MathOptInterface is available on arXiv.

If you find MathOptInterface useful in your work, we kindly request that you cite the following paper:

```
@article{legat2021mathoptinterface,
    title={{MathOptInterface}: a data structure for mathematical optimization problems},
    author={Legat, Beno{\^\i}t and Dowson, Oscar and Garcia, Joaquim Dias and Lubin, Miles},
    journal={INFORMS Journal on Computing},
    year={2021},
    doi={10.1287/ijoc.2021.1067},
    publisher={INFORMS}
}
```

35.2 Motivation

MathOptInterface (MOI) is a replacement for MathProgBase, the first-generation abstraction layer for mathematical optimization previously used by JuMP and Convex.jl.

To address a number of limitations of MathProgBase, MOI is designed to:

- Be simple and extensible
  - unifying linear, quadratic, and conic optimization,
  - seamlessly facilitating extensions to essentially arbitrary constraints and functions (e.g., indicator constraints, complementarity constraints, and piecewise-linear functions)
- Be fast
  - by allowing access to a solver's in-memory representation of a problem without writing intermediate files (when possible)
  - by using multiple dispatch and avoiding requiring containers of nonconcrete types
- Allow a solver to return multiple results (e.g., a pool of solutions)
- Allow a solver to return extra arbitrary information via attributes (e.g., variable- and constraint-wise membership in an irreducible inconsistent subset for infeasibility analysis)
- Provide a greatly expanded set of status codes explaining what happened during the optimization procedure
- Enable a solver to more precisely specify which problem classes it supports
- Enable both primal and dual warm starts
- Enable adding and removing both variables and constraints by indices that are not required to be consecutive
- Enable any modification that the solver supports to an existing model
- Avoid requiring the solver wrapper to store an additional copy of the problem data

Chapter 36

Tutorials

36.1 Solving a problem using MathOptInterface

In this tutorial we demonstrate how to use MathOptInterface to solve the binary-constrained knapsack problem:

$$
\begin{aligned}
\max\ & c^\top x \\
s.t.\ & w^\top x \leq C \\
& x_i \in \{0,1\}, \quad \forall i = 1,\ldots,n
\end{aligned}
$$

Required packages

Load the MathOptInterface module and define the shorthand `MOI`:

```
using MathOptInterface
const MOI = MathOptInterface
```

As an optimizer, we choose GLPK:

```
using GLPK
optimizer = GLPK.Optimizer()
```

Define the data

We first define the constants of the problem:

```
julia> c = [1.0, 2.0, 3.0]
3-element Vector{Float64}:
 1.0
 2.0
 3.0

julia> w = [0.3, 0.5, 1.0]
3-element Vector{Float64}:
 0.3
 0.5
 1.0

julia> C = 3.2
3.2
```

Add the variables

```
julia> x = MOI.add_variables(optimizer, length(c));
```

Set the objective

```
julia> MOI.set(
           optimizer,
           MOI.ObjectiveFunction{MOI.ScalarAffineFunction{Float64}}(),
           MOI.ScalarAffineFunction(MOI.ScalarAffineTerm.(c, x), 0.0),
       );

julia> MOI.set(optimizer, MOI.ObjectiveSense(), MOI.MAX_SENSE)
```

> **Tip**
>
> `MOI.ScalarAffineTerm.(c, x)` is a shortcut for `[MOI.ScalarAffineTerm(c[i], x[i]) for i = 1:3]`. This is Julia's broadcast syntax in action, and is used quite often throughout MOI.

Add the constraints

We add the knapsack constraint and integrality constraints:

```
julia> MOI.add_constraint(
           optimizer,
           MOI.ScalarAffineFunction(MOI.ScalarAffineTerm.(w, x), 0.0),
           MOI.LessThan(C),
       );
```

Add integrality constraints:

```
julia> for x_i in x
           MOI.add_constraint(optimizer, x_i, MOI.ZeroOne())
       end
```

Optimize the model

```
julia> MOI.optimize!(optimizer)
```

Understand why the solver stopped

The first thing to check after optimization is why the solver stopped, e.g., did it stop because of a time limit or did it stop because it found the optimal solution?

```
julia> MOI.get(optimizer, MOI.TerminationStatus())
OPTIMAL::TerminationStatusCode = 1
```

Looks like we found an optimal solution!

Understand what solution was returned

```
julia> MOI.get(optimizer, MOI.ResultCount())
1

julia> MOI.get(optimizer, MOI.PrimalStatus())
FEASIBLE_POINT::ResultStatusCode = 1

julia> MOI.get(optimizer, MOI.DualStatus())
NO_SOLUTION::ResultStatusCode = 0
```

Query the objective

What is its objective value?

```
julia> MOI.get(optimizer, MOI.ObjectiveValue())
6.0
```

Query the primal solution

And what is the value of the variables x?

```
julia> MOI.get(optimizer, MOI.VariablePrimal(), x)
3-element Vector{Float64}:
 1.0
 1.0
 1.0
```

36.2 Implementing a solver interface

This guide outlines the basic steps to implement an interface to MathOptInterface for a new solver.

> **Danger**
>
> Implementing an interface to MathOptInterface for a new solver is a lot of work. Before starting, we recommend that you join the Developer chatroom and explain a little bit about the solver you are wrapping. If you have questions that are not answered by this guide, please ask them in the Developer chatroom so we can improve this guide!

A note on the API

The API of MathOptInterface is large and varied. In order to support the diversity of solvers and use-cases, we make heavy use of duck-typing. That is, solvers are not expected to implement the full API, nor is there a well-defined minimal subset of what must be implemented. Instead, you should implement the API as necessary in order to make the solver function as you require.

The main reason for using duck-typing is that solvers work in different ways and target different use-cases.

For example:

- Some solvers support incremental problem construction, support modification after a solve, and have native support for things like variable names.
- Other solvers are "one-shot" solvers that require all of the problem data to construct and solve the problem in a single function call. They do not support modification or things like variable names.

- Other "solvers" are not solvers at all, but things like file readers. These may only support functions like `read_from_file`, and may not even support the ability to add variables or constraints directly!
- Finally, some "solvers" are layers which take a problem as input, transform it according to some rules, and pass the transformed problem to an inner solver.

Preliminaries

Before starting on your wrapper, you should do some background research and make the solver accessible via Julia.

Decide if MathOptInterface is right for you

The first step in writing a wrapper is to decide whether implementing an interface is the right thing to do.

MathOptInterface is an abstraction layer for unifying constrained mathematical optimization solvers. If your solver doesn't fit in the category, i.e., it implements a derivative-free algorithm for unconstrained objective functions, MathOptInterface may not be the right tool for the job.

> **Tip**
>
> If you're not sure whether you should write an interface, ask in the Developer chatroom.

Find a similar solver already wrapped

The next step is to find (if possible) a similar solver that is already wrapped. Although not strictly necessary, this will be a good place to look for inspiration when implementing your wrapper.

The JuMP documentation has a good list of solvers, along with the problem classes they support.

> **Tip**
>
> If you're not sure which solver is most similar, ask in the Developer chatroom.

Create a low-level interface

Before writing a MathOptInterface wrapper, you first need to be able to call the solver from Julia.

Wrapping solvers written in Julia If your solver is written in Julia, there's nothing to do here! Go to the next section.

Wrapping solvers written in C Julia is well suited to wrapping solvers written in C.

> **Info**
>
> This is not true for C++. If you have a solver written in C++, first write a C interface, then wrap the C interface.

Before writing a MathOptInterface wrapper, there are a few extra steps.

Create a JLL If the C code is publicly available under an open-source license, create a JLL package via Yggdrasil. The easiest way to do this is to copy an existing solver. Good examples to follow are the COIN-OR solvers.

> **Warning**
>
> Building the solver via Yggdrasil is non-trivial. Please ask the Developer chatroom for help.

If the code is commercial or not publicly available, the user will need to manually install the solver. See Gurobi.jl or CPLEX.jl for examples of how to structure this.

Use Clang.jl to wrap the C API The next step is to use Clang.jl to automatically wrap the C API. The easiest way to do this is to follow an example. Good examples to follow are Cbc.jl and HiGHS.jl.

Sometimes, you will need to make manual modifications to the resulting files.

Solvers written in other languages Ask the Developer chatroom for advice. You may be able to use one of the JuliaInterop packages to call out to the solver.

For example, SeDuMi.jl uses MATLAB.jl to call the SeDuMi solver written in MATLAB.

Structuring the package

Structure your wrapper as a Julia package. Consult the Julia documentation if you haven't done this before.

MOI solver interfaces may be in the same package as the solver itself (either the C wrapper if the solver is accessible through C, or the Julia code if the solver is written in Julia, for example), or in a separate package which depends on the solver package.

> **Note**
>
> The JuMP core contributors request that you do not use "JuMP" in the name of your package without prior consent.

Your package should have the following structure:

```
/.github
    /workflows
        ci.yml
        format_check.yml
        TagBot.yml
/gen
    gen.jl  # Code to wrap the C API
/src
    NewSolver.jl
    /gen
        libnewsolver_api.jl
        libnewsolver_common.jl
    /MOI_wrapper
        MOI_wrapper.jl
        other_files.jl
/test
    runtests.jl
    /MOI_wrapper
        MOI_wrapper.jl
```

```
.gitignore
.JuliaFormatter.toml
README.md
LICENSE.md
Project.toml
```

- The `/.github` folder contains the scripts for GitHub actions. The easiest way to write these is to copy the ones from an existing solver.
- The `/gen` and `/src/gen` folders are only needed if you are wrapping a solver written in C.
- The `/src/MOI_wrapper` folder contains the Julia code for the MOI wrapper.
- The `/test` folder contains code for testing your package. See Setup tests for more information.
- The `.JuliaFormatter.toml` and `.github/workflows/format_check.yml` enforce code formatting using JuliaFormatter.jl. Check existing solvers or JuMP.jl for details.

Documentation

Your package must include documentation explaining how to use the package. The easiest approach is to include documentation in your `README.md`. A more involved option is to use Documenter.jl.

Examples of packages with README-based documentation include:

- Cbc.jl
- HiGHS.jl
- SCS.jl

Examples of packages with Documenter-based documentation include:

- Alpine.jl
- COSMO.jl
- Juniper.jl

Setup tests

The best way to implement an interface to MathOptInterface is via test-driven development.

The `MOI.Test` submodule contains a large test suite to help check that you have implemented things correctly.

Follow the guide How to test a solver to set up the tests for your package.

> **Tip**
>
> Run the tests frequently when developing. However, at the start there is going to be a lot of errors! Start by excluding large classes of tests (e.g., `exclude = ["test_basic_", "test_model_"]`, implement any missing methods until the tests pass, then remove an exclusion and repeat.

Initial code

By this point, you should have a package setup with tests, formatting, and access to the underlying solver. Now it's time to start writing the wrapper.

The `Optimizer` object

The first object to create is a subtype of `AbstractOptimizer`. This type is going to store everything related to the problem.

By convention, these optimizers should not be exported and should be named `PackageName.Optimizer`.

```
import MathOptInterface
const MOI = MathOptInterface

struct Optimizer <: MOI.AbstractOptimizer
    # Fields go here
end
```

Optimizer objects for C solvers

> **Warning**
>
> This section is important if you wrap a solver written in C.

Wrapping a solver written in C will require the use of pointers, and for you to manually free the solver's memory when the `Optimizer` is garbage collected by Julia.

Never pass a pointer directly to a Julia `ccall` function.

Instead, store the pointer as a field in your `Optimizer`, and implement `Base.cconvert` and `Base.unsafe_convert`. Then you can pass `Optimizer` to any `ccall` function that expects the pointer.

In addition, make sure you implement a `finalizer` for each model you create.

If `newsolver_createProblem()` is the low-level function that creates the problem pointer in C, and `newsolver_freeProblem(::Pt` is the low-level function that frees memory associated with the pointer, your `Optimizer()` function should look like this:

```
struct Optimizer <: MOI.AbstractOptimizer
    ptr::Ptr{Cvoid}

    function Optimizer()
        ptr = newsolver_createProblem()
        model = Optimizer(ptr)
        finalizer(model) do m
            newsolver_freeProblem(m)
            return
        end
        return model
    end
end

Base.cconvert(::Type{Ptr{Cvoid}}, model::Optimizer) = model
Base.unsafe_convert(::Type{Ptr{Cvoid}}, model::Optimizer) = model.ptr
```

Implement methods for `Optimizer`

All `Optimizers` must implement the following methods:

- `empty!`

- is_empty
- optimize!

Other methods, detailed below, are optional or depend on how you implement the interface.

Tip

For this and all future methods, read the docstrings to understand what each method does, what it expects as input, and what it produces as output. If it isn't clear, let us know and we will improve the docstrings! It is also very helpful to look at an existing wrapper for a similar solver.

You should also implement `Base.show(::IO, ::Optimizer)` to print a nice string when someone prints your model. For example

```
function Base.show(io::IO, model::Optimizer)
    return print(io, "NewSolver with the pointer $(model.ptr)")
end
```

Implement attributes

MathOptInterface uses attributes to manage different aspects of the problem.

For each attribute

- get gets the current value of the attribute
- set sets a new value of the attribute. Not all attributes can be set. For example, the user can't modify the SolverName.
- supports returns a `Bool` indicating whether the solver supports the attribute.

Info

Use attribute_value_type to check the value expected by a given attribute. You should make sure that your get function correctly infers to this type (or a subtype of it).

Each column in the table indicates whether you need to implement the particular method for each attribute.

Attribute	get	set	supports
SolverName	Yes	No	No
SolverVersion	Yes	No	No
RawSolver	Yes	No	No
Name	Yes	Yes	Yes
Silent	Yes	Yes	Yes
TimeLimitSec	Yes	Yes	Yes
RawOptimizerAttribute	Yes	Yes	Yes
NumberOfThreads	Yes	Yes	Yes

For example:

```julia
function MOI.get(model::Optimizer, ::MOI.Silent)
    return # true if MOI.Silent is set
end

function MOI.set(model::Optimizer, ::MOI.Silent, v::Bool)
    if v
        # Set a parameter to turn off printing
    else
        # Restore the default printing
    end
    return
end

MOI.supports(::Optimizer, ::MOI.Silent) = true
```

Define `supports_constraint`

The next step is to define which constraints and objective functions you plan to support.

For each function-set constraint pair, define `supports_constraint`:

```julia
function MOI.supports_constraint(
    ::Optimizer,
    ::Type{MOI.VariableIndex},
    ::Type{MOI.ZeroOne},
)
    return true
end
```

To make this easier, you may want to use `Union`s:

```julia
function MOI.supports_constraint(
    ::Optimizer,
    ::Type{MOI.VariableIndex},
    ::Type{<:Union{MOI.LessThan,MOI.GreaterThan,MOI.EqualTo}},
)
    return true
end
```

> **Tip**
>
> Only support a constraint if your solver has native support for it.

The big decision: copy-to or incremental modifications?

Now you need to decide whether to support incremental modification or not.

Incremental modification means that the user can add variables and constraints one-by-one without needing to rebuild the entire problem, and they can modify the problem data after an `optimize!` call. Supporting incremental modification means implementing functions like `add_variable` and `add_constraint`.

The alternative is to accept the problem data in a single `copy_to` function call, afterwhich it cannot be modified. Because `copy_to` sees all of the data at once, it can typically call a more efficient function to load data into the underlying solver.

Good examples of solvers supporting incremental modification are MILP solvers like GLPK.jl and Gurobi.jl. Examples of `copy_to` solvers are AmplNLWriter.jl and SCS.jl

It is possible to implement both approaches, but you should probably start with one for simplicity.

> **Tip**
>
> Only support incremental modification if your solver has native support for it.

In general, supporting incremental modification is more work, and it usually requires some extra book-keeping. However, it provides a more efficient interface to the solver if the problem is going to be resolved multiple times with small modifications. Moreover, once you've implemented incremental modification, it's usually not much extra work to add a `copy_to` interface. The converse is not true.

> **Tip**
>
> If this is your first time writing an interface, start with `copy_to`.

The `copy_to` interface

To implement the `copy_to` interface, implement the following function:

- `copy_to`

The incremental interface

> **Warning**
>
> Writing this interface is a lot of work. The easiest way is to consult the source code of a similar solver!

To implement the incremental interface, implement the following functions:

- `add_variable`
- `add_variables`
- `add_constraint`
- `add_constraints`
- `is_valid`
- `delete`

> **Info**
>
> Solvers do not have to support `AbstractScalarFunction` in `GreaterThan`, `LessThan`, `EqualTo`, or `Interval` with a nonzero constant in the function. Throw `ScalarFunctionConstantNotZero` if the function constant is not zero.

In addition, you should implement the following model attributes:

Variable-related attributes:

Constraint-related attributes:

Attribute	get	set	supports
ListOfModelAttributesSet	Yes	No	No
ObjectiveFunctionType	Yes	No	No
ObjectiveFunction	Yes	Yes	Yes
ObjectiveSense	Yes	Yes	Yes
Name	Yes	Yes	Yes

Attribute	get	set	supports
ListOfVariableAttributesSet	Yes	No	No
NumberOfVariables	Yes	No	No
ListOfVariableIndices	Yes	No	No

Attribute	get	set	supports
ListOfConstraintAttributesSet	Yes	No	No
NumberOfConstraints	Yes	No	No
ListOfConstraintTypesPresent	Yes	No	No
ConstraintFunction	Yes	Yes	No
ConstraintSet	Yes	Yes	No

Modifications If your solver supports modifying data in-place, implement `modify` for the following `AbstractModifications`:

- `ScalarConstantChange`
- `ScalarCoefficientChange`
- `VectorConstantChange`
- `MultirowChange`

Variables constrained on creation Some solvers require variables be associated with a set when they are created. This conflicts with the incremental modification approach, since you cannot first add a free variable and then constrain it to the set.

If this is the case, implement:

- `add_constrained_variable`
- `add_constrained_variables`
- `supports_add_constrained_variables`

By default, MathOptInterface assumes solvers support free variables. If your solver does not support free variables, define:

```
MOI.supports_add_constrained_variables(::Optimizer, ::Type{Reals}) = false
```

Incremental and `copy_to`

If you implement the incremental interface, you have the option of also implementing `copy_to`.

If you don't want to implement `copy_to`, e.g., because the solver has no API for building the problem in a single function call, define the following fallback:

```
MOI.supports_incremental_interface(::Optimizer) = true

function MOI.copy_to(dest::Optimizer, src::MOI.ModelLike)
    return MOI.Utilities.default_copy_to(dest, src)
end
```

Names

Regardless of which interface you implement, you have the option of implementing the Name attribute for variables and constraints:

Attribute	get	set	supports
VariableName	Yes	Yes	Yes
ConstraintName	Yes	Yes	Yes

If you implement names, you must also implement the following three methods:

```
function MOI.get(model::Optimizer, ::Type{MOI.VariableIndex}, name::String)
    return # The variable named `name`.
end

function MOI.get(model::Optimizer, ::Type{MOI.ConstraintIndex}, name::String)
    return # The constraint any type named `name`.
end

function MOI.get(
    model::Optimizer,
    ::Type{MOI.ConstraintIndex{F,S}},
    name::String,
) where {F,S}
    return # The constraint of type F-in-S named `name`.
end
```

These methods have the following rules:

- If there is no variable or constraint with the name, return nothing
- If there is a single variable or constraint with that name, return the variable or constraint
- If there are multiple variables or constraints with the name, throw an error.

Warning

You should not implement ConstraintName for VariableIndex constraints. If you implement ConstraintName for other constraints, you can add the following two methods to disable ConstraintName for VariableIndex constraints.

```
function MOI.supports(
    ::Optimizer,
    ::MOI.ConstraintName,
    ::Type{<:MOI.ConstraintIndex{MOI.VariableIndex,<:MOI.AbstractScalarSet}},
)
    return throw(MOI.VariableIndexConstraintNameError())
end
```

```
function MOI.set(
    ::Optimizer,
    ::MOI.ConstraintName,
    ::MOI.ConstraintIndex{MOI.VariableIndex,<:MOI.AbstractScalarSet},
    ::String,
)
    return throw(MOI.VariableIndexConstraintNameError())
end
```

Solutions

Implement `optimize!` to solve the model:

- `optimize!`

All `Optimizers` must implement the following attributes:

- `DualStatus`
- `PrimalStatus`
- `RawStatusString`
- `ResultCount`
- `TerminationStatus`

Info

You only need to implement `get` for solution attributes. Don't implement `set` or `supports`.

Note

Solver wrappers should document how the low-level statuses map to the MOI statuses. Statuses like `NEARLY_FEASIBLE_POINT` and `INFEASIBLE_POINT`, are designed to be used when the solver explicitly indicates that relaxed tolerances are satisfied or the returned point is infeasible, respectively.

You should also implement the following attributes:

- `ObjectiveValue`
- `SolveTimeSec`
- `VariablePrimal`

Tip

Attributes like `VariablePrimal` and `ObjectiveValue` are indexed by the result count. Use `MOI.check_result_index_bounc` `attr)` to throw an error if the attribute is not available.

If your solver returns dual solutions, implement:

- `ConstraintDual`

- `DualObjectiveValue`

For integer solvers, implement:

- `ObjectiveBound`
- `RelativeGap`

If applicable, implement:

- `SimplexIterations`
- `BarrierIterations`
- `NodeCount`

If your solver uses the Simplex method, implement:

- `ConstraintBasisStatus`

If your solver accepts primal or dual warm-starts, implement:

- `VariablePrimalStart`
- `ConstraintDualStart`

Other tips

Here are some other points to be aware of when writing your wrapper.

Unsupported constraints at runtime

In some cases, your solver may support a particular type of constraint (e.g., quadratic constraints), but only if the data meets some condition (e.g., it is convex).

In this case, declare that you `support` the constraint, and throw `AddConstraintNotAllowed`.

Dealing with multiple variable bounds

MathOptInterface uses `VariableIndex` constraints to represent variable bounds. Defining multiple variable bounds on a single variable is not allowed.

Throw `LowerBoundAlreadySet` or `UpperBoundAlreadySet` if the user adds a constraint that results in multiple bounds.

Only throw if the constraints conflict. It is okay to add `VariableIndex`-in-`GreaterThan` and then `VariableIndex`-in-`LessThan`, but not `VariableIndex`-in-`Interval` and then `VariableIndex`-in-`LessThan`,

Expect duplicate coefficients

Solvers must expect that functions such as `ScalarAffineFunction` and `VectorQuadraticFunction` may contain duplicate coefficents.

For example, `ScalarAffineFunction([ScalarAffineTerm(x, 1), ScalarAffineTerm(x, 1)], 0.0)`.

Use `Utilities.canonical` to return a new function with the duplicate coefficients aggregated together.

Don't modify user-data

All data passed to the solver must be copied immediately to internal data structures. Solvers may not modify any input vectors and must assume that input vectors will not be modified by users in the future.

This applies, for example, to the `terms` vector in `ScalarAffineFunction`. Vectors returned to the user, e.g., via `ObjectiveFunction` or `ConstraintFunction` attributes, must not be modified by the solver afterwards. The in-place version of `get!` can be used by users to avoid extra copies in this case.

Column Generation

There is no special interface for column generation. If the solver has a special API for setting coefficients in existing constraints when adding a new variable, it is possible to queue modifications and new variables and then call the solver's API once all of the new coefficients are known.

Solver-specific attributes

You don't need to restrict yourself to the attributes defined in the MathOptInterface.jl package.

Solver-specific attributes should be specified by creating an appropriate subtype of `AbstractModelAttribute`, `AbstractOptimizerAttribute`, `AbstractVariableAttribute`, or `AbstractConstraintAttribute`.

For example, Gurobi.jl adds attributes for multiobjective optimization by defining:

```
struct NumberOfObjectives <: MOI.AbstractModelAttribute end

function MOI.set(model::Optimizer, ::NumberOfObjectives, n::Integer)
    # Code to set NumberOfObjectives
    return
end

function MOI.get(model::Optimizer, ::NumberOfObjectives)
    n = # Code to get NumberOfObjectives
    return n
end
```

Then, the user can write:

```
model = Gurobi.Optimizer()
MOI.set(model, Gurobi.NumberofObjectives(), 3)
```

36.3 Transitioning from MathProgBase

MathOptInterface is a replacement for MathProgBase.jl. However, it is not a direct replacement.

Transitioning a solver interface

MathOptInterface is more extensive than MathProgBase which may make its implementation seem daunting at first. There are however numerous utilities in MathOptInterface that the simplify implementation process.

For more information, read Implementing a solver interface.

Transitioning the high-level functions

MathOptInterface doesn't provide replacements for the high-level interfaces in MathProgBase. We recommend you use JuMP as a modeling interface instead.

> **Tip**
>
> If you haven't used JuMP before, start with the tutorial Getting started with JuMP

linprog

Here is one way of transitioning from `linprog`:

```
using JuMP

function linprog(c, A, sense, b, l, u, solver)
    N = length(c)
    model = Model(solver)
    @variable(model, l[i] <= x[i=1:N] <= u[i])
    @objective(model, Min, c' * x)
    eq_rows, ge_rows, le_rows = sense .== '=', sense .== '>', sense .== '<'
    @constraint(model, A[eq_rows, :] * x .== b[eq_rows])
    @constraint(model, A[ge_rows, :] * x .>= b[ge_rows])
    @constraint(model, A[le_rows, :] * x .<= b[le_rows])
    optimize!(model)
    return (
        status = termination_status(model),
        objval = objective_value(model),
        sol = value.(x)
    )
end
```

mixintprog

Here is one way of transitioning from `mixintprog`:

```
using JuMP

function mixintprog(c, A, rowlb, rowub, vartypes, lb, ub, solver)
    N = length(c)
    model = Model(solver)
    @variable(model, lb[i] <= x[i=1:N] <= ub[i])
    for i in 1:N
        if vartypes[i] == :Bin
            set_binary(x[i])
        elseif vartypes[i] == :Int
            set_integer(x[i])
        end
    end
    @objective(model, Min, c' * x)
    @constraint(model, rowlb .<= A * x .<= rowub)
    optimize!(model)
    return (
        status = termination_status(model),
        objval = objective_value(model),
        sol = value.(x)
```

```
    )
end
```

quadprog

Here is one way of transitioning from `quadprog`:

```
using JuMP

function quadprog(c, Q, A, rowlb, rowub, lb, ub, solver)
    N = length(c)
    model = Model(solver)
    @variable(model, lb[i] <= x[i=1:N] <= ub[i])
    @objective(model, Min, c' * x + 0.5 * x' * Q * x)
    @constraint(model, rowlb .<= A * x .<= rowub)
    optimize!(model)
    return (
        status = termination_status(model),
        objval = objective_value(model),
        sol = value.(x)
    )
end
```

36.4 Implementing a constraint bridge

This guide outlines the basic steps to create a new bridge from a constraint expressed in the formalism `Function-in-Set`.

Preliminaries

First, decide on the set you want to bridge. Then, study its properties: the most important one is whether the set is scalar or vector, which impacts the dimensionality of the functions that can be used with the set.

- A scalar function only has one dimension. MOI defines three types of scalar functions: a variable (`VariableIndex`), an affine function (`ScalarAffineFunction`), or a quadratic function (`ScalarQuadraticFunction`).
- A vector function has several dimensions (at least one). MOI defines three types of vector functions: several variables (`VectorOfVariables`), an affine function (`VectorAffineFunction`), or a quadratic function (`VectorQuadraticFunction`). The main difference with scalar functions is that the order of dimensions can be very important: for instance, in an indicator constraint (`Indicator`), the first dimension indicates whether the constraint about the second dimension is active.

To explain how to implement a bridge, we present the example of `Bridges.Constraint.FlipSignBridge`. This bridge maps <= (`LessThan`) constraints to >= (`GreaterThan`) constraints. This corresponds to reversing the sign of the inequality. We focus on scalar affine functions (we disregard the cases of a single variable or of quadratic functions). This example is a simplified version of the code included in MOI.

Four mandatory parts in a constraint bridge

The first part of a constraint bridge is a new concrete subtype of `Bridges.Constraint.AbstractBridge`. This type must have fields to store all the new variables and constraints that the bridge will add. Typically, these types are parametrized by the type of the coefficients in the model.

Then, three sets of functions must be defined:

1. Bridges.Constraint.bridge_constraint: this function implements the bridge and creates the required variables and constraints.

2. supports_constraint: these functions must return `true` when the combination of function and set is supported by the bridge. By default, the base implementation always returns `false` and the bridge does not have to provide this implementation.

3. Bridges.added_constrained_variable_types and Bridges.added_constraint_types: these functions return the types of variables and constraints that this bridge adds. They are used to compute the set of other bridges that are required to use the one you are defining, if need be.

More functions can be implemented, for instance to retrieve properties from the bridge or deleting a bridged constraint.

1. Structure for the bridge

A typical `struct` behind a bridge depends on the type of the coefficients that are used for the model (typically `Float64`, but coefficients might also be integers or complex numbers).

This structure must hold a reference to all the variables and the constraints that are created as part of the bridge.

The type of this structure is used throughout MOI as an identifier for the bridge. It is passed as argument to most functions related to bridges.

The best practice is to have the name of this type end with `Bridge`.

In our example, the bridge maps any `ScalarAffineFunction{T}`-in-`LessThan{T}` constraint to a single `ScalarAffineFunction{T` in-`GreaterThan{T}` constraint. The affine function has coefficients of type T. The bridge is parametrized with T, so that the constraint that the bridge creates also has coefficients of type T.

```
struct SignBridge{T<:Number} <: Bridges.Constraint.AbstractBridge
    constraint::ConstraintIndex{ScalarAffineFunction{T}, GreaterThan{T}}
end
```

2. Bridge creation

The function Bridges.Constraint.bridge_constraint is called whenever the bridge is instantiated for a specific model, with the given function and set. The arguments to `bridge_constraint` are similar to add_constraint, with the exception of the first argument: it is the `Type` of the struct defined in the first step (for our example, `Type{SignBridge{T}}`).

`bridge_constraint` returns an instance of the struct defined in the first step. the first step.

In our example, the bridge constraint could be defined as:

```
function Bridges.Constraint.bridge_constraint(
    ::Type{SignBridge{T}}, # Bridge to use.
    model::ModelLike, # Model to which the constraint is being added.
    f::ScalarAffineFunction{T}, # Function to rewrite.
    s::LessThan{T}, # Set to rewrite.
) where {T}
    # Create the variables and constraints required for the bridge.
    con = add_constraint(model, -f, GreaterThan(-s.upper))

    # Return an instance of the bridge type with a reference to all the
```

```
    # variables and constraints that were created in this function.
    return SignBridge(con)
end
```

3. Supported constraint types

The function `supports_constraint` determines whether the bridge type supports a given combination of function and set.

This function must closely match `bridge_constraint`, because it will not be called if `supports_constraint` returns `false`.

```
function supports_constraint(
    ::Type{SignBridge{T}}, # Bridge to use.
    ::Type{ScalarAffineFunction{T}}, # Function to rewrite.
    ::Type{LessThan{T}}, # Set to rewrite.
) where {T}
    # Do some computation to ensure that the constraint is supported.
    # Typically, you can directly return true.
    return true
end
```

4. Metadata about the bridge

To determine whether a bridge can be used, MOI uses a shortest-path algorithm that uses the variable types and the constraints that the bridge can create. This information is communicated from the bridge to MOI using the functions `Bridges.added_constrained_variable_types` and `Bridges.added_constraint_types`. Both return lists of tuples: either a list of 1-tuples containing the variable types (typically, `ZeroOne` or `Integer`) or a list of 2-tuples contained the functions and sets (like `ScalarAffineFunction{T}`-`GreaterThan`).

For our example, the bridge does not create any constrained variables, and only `ScalarAffineFunction{T}`-in-`GreaterThan{T}` constraints:

```
function Bridges.added_constrained_variable_types(::Type{SignBridge{T}}) where {T}
    # The bridge does not create variables, return an empty list of tuples:
    return Tuple{Type}[]
end

function Bridges.added_constraint_types(::Type{SignBridge{T}}) where {T}
    return Tuple{Type,Type}[
        # One element per F-in-S the bridge creates.
        (ScalarAffineFunction{T}, GreaterThan{T}),
    ]
end
```

A bridge that creates binary variables would rather have this definition of `added_constrained_variable_types`:

```
function Bridges.added_constrained_variable_types(::Type{SomeBridge{T}}) where {T}
    # The bridge only creates binary variables:
    return Tuple{Type}[(ZeroOne,)]
end
```

Warning

> If you declare the creation of constrained variables in `added_constrained_variable_types`, the corresponding constraint type `VariableIndex` must not be indicated in `added_constraint_types`. This would restrict the use of the bridge to solvers that can add such a constraint after the variable is created.
>
> More concretely, if you declare in `added_constrained_variable_types` that your bridge creates binary variables (`ZeroOne`), and if you never add such a constraint afterward (you do not call `add_constraint(model, var, ZeroOne())`), then you must not list `(VariableIndex, ZeroOne)` in `added_constraint_types`.

Typically, the function `Bridges.Constraint.concrete_bridge_type` does not have to be defined for most bridges.

Bridge registration

For a bridge to be used by MOI, it must be known by MOI.

`SingleBridgeOptimizer`

The first way to do so is to create a single-bridge optimizer. This type of optimizer wraps another optimizer and adds the possibility to use only one bridge. It is especially useful when unit testing bridges.

It is common practice to use the same name as the type defined for the bridge (`SignBridge`, in our example) without the suffix `Bridge`.

```
const Sign{T,OT<: ModelLike} =
    SingleBridgeOptimizer{SignBridge{T}, OT}
```

In the context of unit tests, this bridge is used in conjunction with a `Utilities.MockOptimizer`:

```
mock = Utilities.MockOptimizer(
    Utilities.UniversalFallback(Utilities.Model{Float64}()),
)
bridged_mock = Sign{Float64}(mock)
```

New bridge for a `LazyBridgeOptimizer`

Typical user-facing models for MOI are based on `Bridges.LazyBridgeOptimizer`. For instance, this type of model is returned by `Bridges.full_bridge_optimizer`. These models can be added more bridges by using `Bridges.add_bridge`:

```
inner_optimizer = Utilities.Model{Float64}()
optimizer = Bridges.full_bridge_optimizer(inner_optimizer, Float64)
Bridges.add_bridge(optimizer, SignBridge{Float64})
```

Bridge improvements

Attribute retrieval

Like models, bridges have attributes that can be retrieved using `get` and `set`. The most important ones are the number of variables and constraints, but also the lists of variables and constraints.

In our example, we only have one constraint and only have to implement the `NumberOfConstraints` and `ListOfConstraintIndices` attributes:

```julia
function get(
    ::SignBridge{T},
    ::NumberOfConstraints{
        ScalarAffineFunction{T},
        GreaterThan{T},
    },
) where {T}
    return 1
end

function get(
    bridge::SignBridge{T},
    ::ListOfConstraintIndices{
        ScalarAffineFunction{T},
        GreaterThan{T},
    },
) where {T}
    return [bridge.constraint]
end
```

You must implement one such pair of functions for each type of constraint the bridge adds to the model.

> **Warning**
>
> Avoid returning a list from the bridge object without copying it. Users must be able to change the contents of the returned list without altering the bridge object.

For variables, the situation is simpler. If your bridge creates new variables, you must implement the `NumberOfVariables` and `ListOfVariableIndices` attributes. However, these attributes do not have parameters, unlike their constraint counterparts. Only two functions suffice:

```julia
function get(
    ::SignBridge{T},
    ::NumberOfVariables,
) where {T}
    return 0
end

function get(
    ::SignBridge{T},
    ::ListOfVariableIndices,
) where {T}
    return VariableIndex[]
end
```

Model modifications

To avoid copying the model when the user request to change a constraint, MOI provides `modify`. Bridges can also implement this API to allow certain changes, such as coefficient changes.

In our case, a modification of a coefficient in the original constraint (i.e. replacing the value of the coefficient of a variable in the affine function) must be transmitted to the constraint created by the bridge, but with a sign change.

```
function modify(
    model::ModelLike,
    bridge::SignBridge,
    change::ScalarCoefficientChange,
)
    modify(
        model,
        bridge.constraint,
        ScalarCoefficientChange(change.variable, -change.new_coefficient),
    )
    return
end
```

Bridge deletion

When a bridge is deleted, the constraints it added must be deleted too.

```
function delete(model::ModelLike, bridge::SignBridge)
    delete(model, bridge.constraint)
    return
end
```

36.5 Manipulating expressions

This guide highlights a syntactically appealing way to build expressions at the MOI level, but also to look at their contents. It may be especially useful when writing models or bridge code.

Creating functions

This section details the ways to create functions with MathOptInterface.

Creating scalar affine functions

The simplest scalar function is simply a variable:

```
julia> x = MOI.add_variable(model) # Create the variable x
MathOptInterface.VariableIndex(1)
```

This type of function is extremely simple; to express more complex functions, other types must be used. For instance, a ScalarAffineFunction is a sum of linear terms (a factor times a variable) and a constant. Such an object can be built using the standard constructor:

```
julia> f = MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1, x)], 2) # x + 2
MathOptInterface.ScalarAffineFunction{Int64}(MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffineTer
↪  MathOptInterface.VariableIndex(1))], 2)
```

However, you can also use operators to build the same scalar function:

```
julia> f = x + 2
MathOptInterface.ScalarAffineFunction{Int64}(MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffineTer
↪  MathOptInterface.VariableIndex(1))], 2)
```

Creating scalar quadratic functions

Scalar quadratic functions are stored in ScalarQuadraticFunction objects, in a way that is highly similar to scalar affine functions. You can obtain a quadratic function as a product of affine functions:

```
julia> 1 * x * x
MathOptInterface.ScalarQuadraticFunction{Int64}(MathOptInterface.ScalarQuadraticTerm{Int64}[MathOptInterface.ScalarQua
↪  MathOptInterface.VariableIndex(1), MathOptInterface.VariableIndex(1))],
↪  MathOptInterface.ScalarAffineTerm{Int64}[], 0)

julia> f * f  # (x + 2)²
MathOptInterface.ScalarQuadraticFunction{Int64}(MathOptInterface.ScalarQuadraticTerm{Int64}[MathOptInterface.ScalarQua
↪  MathOptInterface.VariableIndex(1), MathOptInterface.VariableIndex(1))],
↪  MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffineTerm{Int64}(2,
↪  MathOptInterface.VariableIndex(1)), MathOptInterface.ScalarAffineTerm{Int64}(2,
↪  MathOptInterface.VariableIndex(1))], 4)

julia> f^2  # (x + 2)² too
MathOptInterface.ScalarQuadraticFunction{Int64}(MathOptInterface.ScalarQuadraticTerm{Int64}[MathOptInterface.ScalarQua
↪  MathOptInterface.VariableIndex(1), MathOptInterface.VariableIndex(1))],
↪  MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffineTerm{Int64}(2,
↪  MathOptInterface.VariableIndex(1)), MathOptInterface.ScalarAffineTerm{Int64}(2,
↪  MathOptInterface.VariableIndex(1))], 4)
```

Creating vector functions

A vector function is a function with several values, irrespective of the number of input variables. Similarly to scalar functions, there are three main types of vector functions: VectorOfVariables, VectorAffineFunction, and VectorQuadraticFunction.

The easiest way to create a vector function is to stack several scalar functions using Utilities.vectorize. It takes a vector as input, and the generated vector function (of the most appropriate type) has each dimension corresponding to a dimension of the vector.

```
julia> g = MOI.Utilities.vectorize([f, 2 * f])
MathOptInterface.VectorAffineFunction{Int64}(MathOptInterface.VectorAffineTerm{Int64}[MathOptInterface.VectorAffineTer
↪  MathOptInterface.ScalarAffineTerm{Int64}(1, MathOptInterface.VariableIndex(1))),
↪  MathOptInterface.VectorAffineTerm{Int64}(2, MathOptInterface.ScalarAffineTerm{Int64}(2,
↪  MathOptInterface.VariableIndex(1)))], [2, 4])
```

> **Warning**
>
> Utilities.vectorize only takes a vector of similar scalar functions: you cannot mix VariableIndex and ScalarAffineFunction, for instance. In practice, it means that `Utilities.vectorize([x, f])` does not work; you should rather use `Utilities.vectorize([1 * x, f])` instead to only have ScalarAffineFunction objects.

Canonicalizing functions

In more advanced use cases, you might need to ensure that a function is "canonical". Functions are stored as an array of terms, but there is no check that these terms are redundant: a ScalarAffineFunction object might have two terms with the same variable, like `x + x + 1`. These terms could be merged without changing the semantics of the function: `2x + 1`.

Working with these objects might be cumbersome. Canonicalization helps maintain redundancy to zero.

Utilities.is_canonical checks whether a function is already in its canonical form:

```
julia> MOI.Utilities.is_canonical(f + f) # (x + 2) + (x + 2) is stored as x + x + 4
false
```

`Utilities.canonical` returns the equivalent canonical version of the function:

```
julia> MOI.Utilities.canonical(f + f) # Returns 2x + 4
MathOptInterface.ScalarAffineFunction{Int64}(MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffineTer
↪  MathOptInterface.VariableIndex(1))], 4)
```

Exploring functions

At some point, you might need to dig into a function, for instance to map it into solver constructs.

Vector functions

`Utilities.scalarize` returns a vector of scalar functions from a vector function:

```
julia> MOI.Utilities.scalarize(g) # Returns a vector [f, 2 * f].
2-element Vector{MathOptInterface.ScalarAffineFunction{Int64}}:

↪  MathOptInterface.ScalarAffineFunction{Int64}(MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffin
↪  MathOptInterface.VariableIndex(1))], 2)

↪  MathOptInterface.ScalarAffineFunction{Int64}(MathOptInterface.ScalarAffineTerm{Int64}[MathOptInterface.ScalarAffin
↪  MathOptInterface.VariableIndex(1))], 4)
```

> **Note**
>
> `Utilities.eachscalar` returns an iterator on the dimensions, which serves the same purpose as `Utilities.scalarize`.

`output_dimension` returns the number of dimensions of the output of a function:

```
julia> MOI.output_dimension(g)
2
```

36.6 Latency

MathOptInterface suffers the "time-to-first-solve" problem of start-up latency.

This hurts both the user- and developer-experience of MathOptInterface. In the first case, because simple models have a multi-second delay before solving, and in the latter, because our tests take so long to run!

This page contains some advice on profiling and fixing latency-related problems in the MathOptInterface.jl repository.

Background

Before reading this part of the documentation, you should familiarize yourself with the reasons for latency in Julia and how to fix them.

- Read the blogposts on julialang.org on precompilation and SnoopCompile

- Read the SnoopCompile documentation.
- Watch Tim Holy's talk at JuliaCon 2021
- Watch the package development workshop at JuliaCon 2021

Causes

There are three main causes of latency in MathOptInterface:

1. A large number of types
2. Lack of method ownership
3. Type-instability in the bridge layer

A large number of types

Julia is very good at specializing method calls based on the input type. Each specialization has a compilation cost, but the benefit of faster run-time performance.

The best-case scenario is for a method to be called a large number of times with a single set of argument types. The worst-case scenario is for a method to be called a single time for a large set of argument types.

Because of MathOptInterface's function-in-set formulation, we fall into the worst-case situation.

This is a fundamental limitation of Julia, so there isn't much we can do about it. However, if we can precompile MathOptInterface, much of the cost can be shifted from start-up latency to the time it takes to precompile a package on installation.

However, there are two things which make MathOptInterface hard to precompile...

Lack of method ownership

Lack of method ownership happens when a call is made using a mix of structs and methods from different modules. Because of this, no single module "owns" the method that is being dispatched, and so it cannot be precompiled.

> **Tip**
>
> This is a slightly simplified explanation. Read the precompilation tutorial for a more in-depth discussion on back-edges.

Unfortunately, the design of MOI means that this is a frequent occurrence! We have a bunch of types in `MOI.Utilities` that wrap types defined in external packages (i.e., the `Optimizers`), which implement methods of functions defined in `MOI` (e.g., `add_variable`, `add_constraint`).

Here's a simple example of method-ownership in practice:

```
module MyMOI
struct Wrapper{T}
    inner::T
end
optimize!(x::Wrapper) = optimize!(x.inner)
end  # MyMOI
```

```
module MyOptimizer
using ..MyMOI
struct Optimizer end
MyMOI.optimize!(x::Optimizer) = 1
end  # MyOptimizer

using SnoopCompile
model = MyMOI.Wrapper(MyOptimizer.Optimizer())

julia> tinf = @snoopi_deep MyMOI.optimize!(model)
InferenceTimingNode: 0.008256/0.008543 on InferenceFrameInfo for Core.Compiler.Timings.ROOT() with 1
↪  direct children
```

The result is that there was one method that required type inference. If we visualize `tinf`:

```
using ProfileView
ProfileView.view(flamegraph(tinf))
```

we see a flamegraph with a large red-bar indicating that the method `MyMOI.optimize(MyMOI.Wrapper{MyOptimizer.Optimizer}` cannot be precompiled.

To fix this, we need to designate a module to "own" that method (i.e., create a back-edge). The easiest way to do this is for `MyOptimizer` to call `MyMOI.optimize(MyMOI.Wrapper{MyOptimizer.Optimizer})` during `using MyOptimizer`. Let's see that in practice:

```
module MyMOI
struct Wrapper{T}
    inner::T
end
optimize(x::Wrapper) = optimize(x.inner)
end  # MyMOI

module MyOptimizer
using ..MyMOI
struct Optimizer end
MyMOI.optimize(x::Optimizer) = 1
# The syntax of this let-while loop is very particular:
#  * `let ... end` keeps everything local to avoid polluting the MyOptimizer
#    namespace
#  * `while true ... break end` runs the code once, and forces Julia to compile
#    the inner loop, rather than interpret it.
let
    while true
        model = MyMOI.Wrapper(Optimizer())
        MyMOI.optimize(model)
        break
    end
end
end  # MyOptimizer

using SnoopCompile
model = MyMOI.Wrapper(MyOptimizer.Optimizer())

julia> tinf = @snoopi_deep MyMOI.optimize(model)
InferenceTimingNode: 0.006822/0.006822 on InferenceFrameInfo for Core.Compiler.Timings.ROOT() with 0
↪  direct children
```

There are now `0` direct children that required type inference because the method was already stored in `MyOptimizer`!

Unfortunately, this trick only works if the call-chain is fully inferrable. If there are breaks (due to type instability), then the benefit of doing this is reduced. And unfortunately for us, the design of MathOptInterface has a lot of type instabilities...

Type instability in the bridge layer

Most of MathOptInterface is pretty good at ensuring type-stability. However, a key component is not type stable, and that is the bridging layer.

In particular, the bridging layer defines `Bridges.LazyBridgeOptimizer`, which has fields like:

```
struct LazyBridgeOptimizer
    constraint_bridge_types::Vector{Any}
    constraint_node::Dict{Tuple{Type,Type},ConstraintNode}
    constraint_types::Vector{Tuple{Type,Type}}
end
```

This is because the `LazyBridgeOptimizer` needs to be able to deal with any function-in-set type passed to it, and we also allow users to pass additional bridges that they defined in external packages.

So to recap, MathOptInterface suffers package latency because:

1. there are a large number of types and functions...
2. and these are split between multiple modules, including external packages...
3. and there are type-instabilities like those in the bridging layer.

Resolutions

There are no magic solutions to reduce latency. Issue #1313 tracks progress on reducing latency in MathOptInterface.

A useful script is the following (replace GLPK as needed):

```
using MathOptInterface, GLPK
const MOI = MathOptInterface

function example_diet(optimizer, bridge)
    category_data = [
        1800.0 2200.0;
          91.0    Inf;
           0.0   65.0;
           0.0 1779.0
    ]
    cost = [2.49, 2.89, 1.50, 1.89, 2.09, 1.99, 2.49, 0.89, 1.59]
    food_data = [
        410 24 26 730;
        420 32 10 1190;
        560 20 32 1800;
        380  4 19 270;
        320 12 10 930;
        320 15 12 820;
        320 31 12 1230;
```

```
        100  8 2.5 125;
        330  8 10 180
    ]
    bridge_model = if bridge
        MOI.instantiate(optimizer; with_bridge_type=Float64)
    else
        MOI.instantiate(optimizer)
    end
    model = MOI.Utilities.CachingOptimizer(
        MOI.Utilities.UniversalFallback(MOI.Utilities.Model{Float64}()),
        MOI.Utilities.AUTOMATIC,
    )
    MOI.Utilities.reset_optimizer(model, bridge_model)
    MOI.set(model, MOI.Silent(), true)
    nutrition = MOI.add_variables(model, size(category_data, 1))
    for (i, v) in enumerate(nutrition)
        MOI.add_constraint(model, v, MOI.GreaterThan(category_data[i, 1]))
        MOI.add_constraint(model, v, MOI.LessThan(category_data[i, 2]))
    end
    buy = MOI.add_variables(model, size(food_data, 1))
    MOI.add_constraint.(model, buy, MOI.GreaterThan(0.0))
    MOI.set(model, MOI.ObjectiveSense(), MOI.MIN_SENSE)
    f = MOI.ScalarAffineFunction(MOI.ScalarAffineTerm.(cost, buy), 0.0)
    MOI.set(model, MOI.ObjectiveFunction{typeof(f)}(), f)
    for (j, n) in enumerate(nutrition)
        f = MOI.ScalarAffineFunction(
            MOI.ScalarAffineTerm.(food_data[:, j], buy),
            0.0,
        )
        push!(f.terms, MOI.ScalarAffineTerm(-1.0, n))
        MOI.add_constraint(model, f, MOI.EqualTo(0.0))
    end
    MOI.optimize!(model)
    term_status = MOI.get(model, MOI.TerminationStatus())
    @assert term_status == MOI.OPTIMAL
    MOI.add_constraint(
        model,
        MOI.ScalarAffineFunction(
            MOI.ScalarAffineTerm.(1.0, [buy[end-1], buy[end]]),
            0.0,
        ),
        MOI.LessThan(6.0),
    )
    MOI.optimize!(model)
    @assert MOI.get(model, MOI.TerminationStatus()) == MOI.INFEASIBLE
    return
end

if length(ARGS) > 0
    bridge = get(ARGS, 2, "") != "--no-bridge"
    println("Running: $(ARGS[1]) $(get(ARGS, 2, ""))")
    @time example_diet(GLPK.Optimizer, bridge)
    @time example_diet(GLPK.Optimizer, bridge)
    exit(0)
end
```

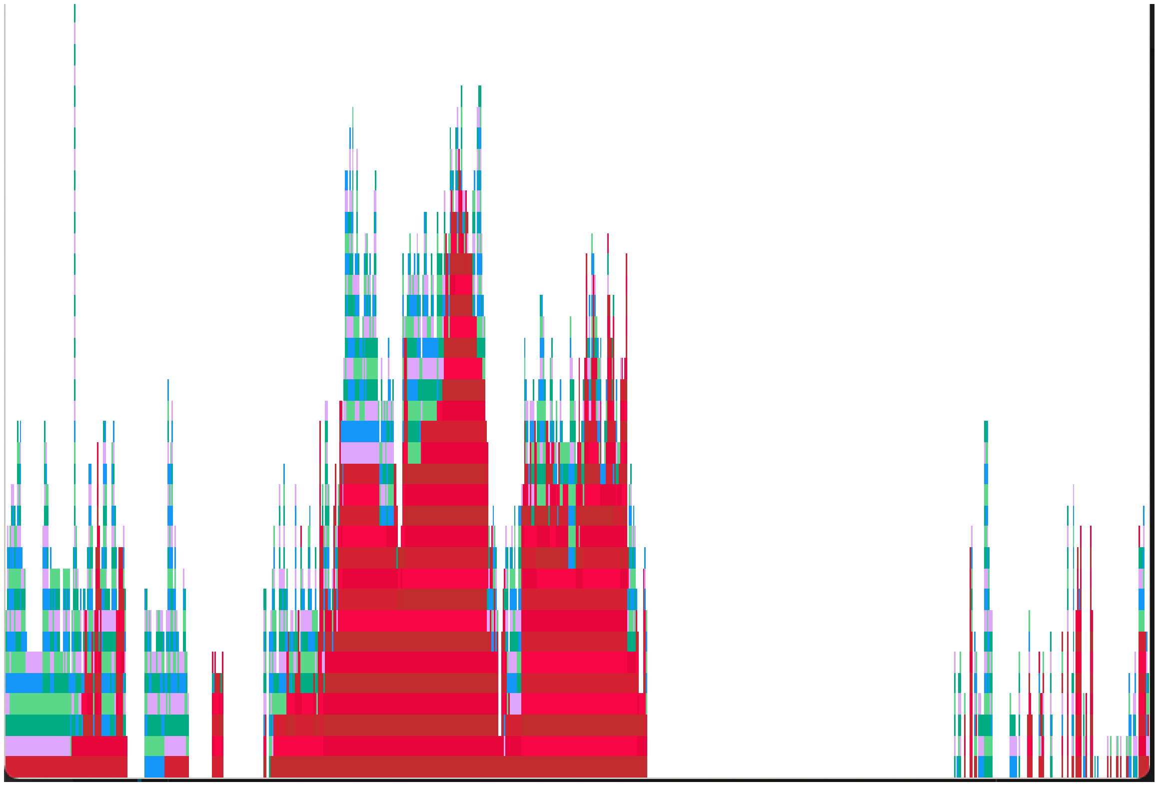


Figure 36.1: flamegraph

You can create a flame-graph via

```
using SnoopComile
tinf = @snoopi_deep example_diet(GLPK.Optimizer, true)
using ProfileView
ProfileView.view(flamegraph(tinf))
```

Here's how things looked in mid-August 2021:

There are a few opportunities for improvement (non-red flames, particularly on the right). But the main problem is a large red (non-precompilable due to method ownership) flame.

Chapter 37

Manual

37.1 Standard form problem

MathOptInterface represents optimization problems in the standard form:

$$\min_{x \in \mathbb{R}^n} \quad f_0(x) \tag{37.1}$$

$$\text{s.t.} \quad f_i(x) \in \mathcal{S}_i \qquad i = 1 \ldots m \tag{37.2}$$

where:

- the functions $f_0, f_1, \ldots, f_m$ are specified by `AbstractFunction` objects
- the sets $\mathcal{S}_1, \ldots, \mathcal{S}_m$ are specified by `AbstractSet` objects

Tip

For more information on this standard form, read our paper.

MOI defines some commonly used functions and sets, but the interface is extensible to other sets recognized by the solver.

Functions

The function types implemented in MathOptInterface.jl are:

- `VariableIndex`: x_j, i.e., projection onto a single coordinate defined by a variable index j.
- `VectorOfVariables`: projection onto multiple coordinates (i.e., extracting a subvector).
- `ScalarAffineFunction`: $a^T x + b$, where a is a vector and b scalar.
- `VectorAffineFunction`: $Ax + b$, where A is a matrix and b is a vector.
- `ScalarQuadraticFunction`: $\frac{1}{2}x^T Qx + a^T x + b$, where Q is a symmetric matrix, a is a vector, and b is a constant.
- `VectorQuadraticFunction`: a vector of scalar-valued quadratic functions.

Extensions for nonlinear programming are present but not yet well documented.

One-dimensional sets

The one-dimensional set types implemented in MathOptInterface.jl are:

- `LessThan(upper)`: $\{x \in \mathbb{R} : x \le \text{upper}\}$
- `GreaterThan(lower)`: $\{x \in \mathbb{R} : x \ge \text{lower}\}$
- `EqualTo(value)`: $\{x \in \mathbb{R} : x = \text{value}\}$
- `Interval(lower, upper)`: $\{x \in \mathbb{R} : x \in [\text{lower}, \text{upper}]\}$
- `Integer()`: $\mathbb{Z}$
- `ZeroOne()`: $\{0, 1\}$
- `Semicontinuous(lower,upper)`: $\{0\} \cup [\text{lower}, \text{upper}]$
- `Semiinteger(lower,upper)`: $\{0\} \cup \{\text{lower}, \text{lower} + 1, \ldots, \text{upper} - 1, \text{upper}\}$

Vector cones

The vector-valued set types implemented in MathOptInterface.jl are:

- `Reals(dimension)`: $\mathbb{R}^{\text{dimension}}$
- `Zeros(dimension)`: $0^{\text{dimension}}$
- `Nonnegatives(dimension)`: $\{x \in \mathbb{R}^{\text{dimension}} : x \ge 0\}$
- `Nonpositives(dimension)`: $\{x \in \mathbb{R}^{\text{dimension}} : x \le 0\}$
- `SecondOrderCone(dimension)`: $\{(t, x) \in \mathbb{R}^{\text{dimension}} : t \ge \|x\|_2\}$
- `RotatedSecondOrderCone(dimension)`: $\{(t, u, x) \in \mathbb{R}^{\text{dimension}} : 2tu \ge \|x\|_2^2, t, u \ge 0\}$
- `ExponentialCone()`: $\{(x, y, z) \in \mathbb{R}^3 : y \exp(x/y) \le z, y > 0\}$
- `DualExponentialCone()`: $\{(u, v, w) \in \mathbb{R}^3 : -u \exp(v/u) \le \exp(1) w, u < 0\}$
- `GeometricMeanCone(dimension)`: $\{(t, x) \in \mathbb{R}^{n+1} : x \ge 0, t \le \sqrt[n]{x_1 x_2 \cdots x_n}\}$ where n is dimension-1
- `PowerCone(exponent)`: $\{(x, y, z) \in \mathbb{R}^3 : x^{\text{exponent}} y^{1-\text{exponent}} \ge |z|, x, y \ge 0\}$
- `DualPowerCone(exponent)`: $\{(u, v, w) \in \mathbb{R}^3 : \frac{u}{\text{exponent}}^{\text{exponent}} \frac{v}{1-\text{exponent}}^{1-\text{exponent}} \ge |w|, u, v \ge 0\}$
- `NormOneCone(dimension)`: $\{(t, x) \in \mathbb{R}^{\text{dimension}} : t \ge \|x\|_1\}$ where $\|x\|_1 = \sum_i |x_i|$
- `NormInfinityCone(dimension)`: $\{(t, x) \in \mathbb{R}^{\text{dimension}} : t \ge \|x\|_\infty\}$ where $\|x\|_\infty = \max_i |x_i|$.
- `RelativeEntropyCone(dimension)`: $\{(u, v, w) \in \mathbb{R}^{\text{dimension}} : u \ge \sum_i w_i \log(\frac{w_i}{v_i}), v_i \ge 0, w_i \ge 0\}$

Matrix cones

The matrix-valued set types implemented in MathOptInterface.jl are:

- `RootDetConeTriangle(dimension)`: $\{(t, X) \in \mathbb{R}^{1+\text{dimension}(1+\text{dimension})/2} : t \le \det(X)^{1/\text{dimension}}, X$ is the upper
- `RootDetConeSquare(dimension)`: $\{(t, X) \in \mathbb{R}^{1+\text{dimension}^2} : t \le \det(X)^{1/\text{dimension}}, X$ is a PSD matrix$\}$
- `PositiveSemidefiniteConeTriangle(dimension)`: $\{X \in \mathbb{R}^{\text{dimension}(\text{dimension}+1)/2} : X$ is the upper triangle of a PSI
- `PositiveSemidefiniteConeSquare(dimension)`: $\{X \in \mathbb{R}^{\text{dimension}^2} : X$ is a PSD matrix$\}$
- `LogDetConeTriangle(dimension)`: $\{(t, u, X) \in \mathbb{R}^{2+\text{dimension}(1+\text{dimension})/2} : t \le u \log(\det(X/u)), X$ is the upper t $0\}$
- `LogDetConeSquare(dimension)`: $\{(t, u, X) \in \mathbb{R}^{2+\text{dimension}^2} : t \le u \log(\det(X/u)), X$ is a PSD matrix, $u > 0\}$
- `NormSpectralCone(row_dim, column_dim)`: $\{(t, X) \in \mathbb{R}^{1+\text{row\_dim}\times\text{column\_dim}} : t \ge \sigma_1(X), X$ is a matrix with row_c
- `NormNuclearCone(row_dim, column_dim)`: $\{(t, X) \in \mathbb{R}^{1+\text{row\_dim}\times\text{column\_dim}} : t \ge \sum_i \sigma_i(X), X$ is a matrix with row

Some of these cones can take two forms: `XXXConeTriangle` and `XXXConeSquare`.

In `XXXConeTriangle` sets, the matrix is assumed to be symmetric, and the elements are provided by a vector, in which the entries of the upper-right triangular part of the matrix are given column by column (or equivalently, the entries of the lower-left triangular part are given row by row).

In `XXXConeSquare` sets, the entries of the matrix are given column by column (or equivalently, row by row), and the matrix is constrained to be symmetric. As an example, given a 2-by-2 matrix of variables X and a one-dimensional variable `t`, we can specify a root-det constraint as `[t, X11, X12, X22] ∈ RootDetConeTriangle` or `[t, X11, X12, X21, X22] ∈ RootDetConeSquare`.

We provide both forms to enable flexibility for solvers who may natively support one or the other. Transformations between `XXXConeTriangle` and `XXXConeSquare` are handled by bridges, which removes the chance of conversion mistakes by users or solver developers.

Multi-dimensional sets with combinatorial structure

- `SOS1(weights)`: A special ordered set of Type I.
- `SOS2(weights)`: A special ordered set of Type II.
- `Indicator(set)`: A set to specify indicator constraints.
- `Complements(dimension)`: A set for mixed complementarity constraints.

37.2 Models

The most significant part of MOI is the definition of the **model API** that is used to specify an instance of an optimization problem (e.g., by adding variables and constraints). Objects that implement the model API must inherit from the `ModelLike` abstract type.

Notably missing from the model API is the method to solve an optimization problem. `ModelLike` objects may store an instance (e.g., in memory or backed by a file format) without being linked to a particular solver. In addition to the model API, MOI defines `AbstractOptimizer` and provides methods to solve the model and interact with solutions. See the Solutions section for more details.

Info

Throughout the rest of the manual, `model` is used as a generic `ModelLike`, and `optimizer` is used as a generic `AbstractOptimizer`.

Tip

MOI does not export functions, but for brevity we often omit qualifying names with the MOI module. Best practice is to have

```
using MathOptInterface
const MOI = MathOptInterface
```

and prefix all MOI methods with `MOI.` in user code. If a name is also available in base Julia, we always explicitly use the module prefix, for example, with `MOI.get`.

Attributes

Attributes are properties of the model that can be queried and modified. These include constants such as the number of variables in a model `NumberOfVariables`), and properties of variables and constraints such as the name of a variable (`VariableName`).

There are four types of attributes:

- Model attributes (subtypes of `AbstractModelAttribute`) refer to properties of a model.
- Optimizer attributes (subtypes of `AbstractOptimizerAttribute`) refer to properties of an optimizer.
- Constraint attributes (subtypes of `AbstractConstraintAttribute`) refer to properties of an individual constraint.
- Variable attributes (subtypes of `AbstractVariableAttribute`) refer to properties of an individual variable.

Some attributes are values that can be queried by the user but not modified, while other attributes can be modified by the user.

All interactions with attributes occur through the `get` and `set` functions.

Consult the docstsrings of each attribute for information on what it represents.

ModelLike API

The following attributes are available:

- `ListOfConstraintAttributesSet`
- `ListOfConstraintIndices`
- `ListOfConstraintTypesPresent`
- `ListOfModelAttributesSet`
- `ListOfVariableAttributesSet`
- `ListOfVariableIndices`

- `NumberOfConstraints`
- `NumberOfVariables`
- `Name`
- `ObjectiveFunction`
- `ObjectiveFunctionType`
- `ObjectiveSense`

AbstractOptimizer API

The following attributes are available:

- `DualStatus`
- `PrimalStatus`
- `RawStatusString`
- `ResultCount`
- `TerminationStatus`
- `BarrierIterations`
- `DualObjectiveValue`
- `NodeCount`
- `NumberOfThreads`
- `ObjectiveBound`
- `ObjectiveValue`
- `RelativeGap`
- `RawOptimizerAttribute`
- `RawSolver`
- `Silent`
- `SimplexIterations`
- `SolverName`
- `SolverVersion`
- `SolveTimeSec`
- `TimeLimitSec`

37.3 Variables

Add a variable

Use add_variable to add a single variable.

```
julia> x = MOI.add_variable(model)
MathOptInterface.VariableIndex(1)
```

add_variable returns a VariableIndex type, which is used to refer to the added variable in other calls.

Check if a VariableIndex is valid using is_valid.

```
julia> MOI.is_valid(model, x)
true
```

Use add_variables to add a number of variables.

```
julia> y = MOI.add_variables(model, 2)
2-element Vector{MathOptInterface.VariableIndex}:
 MathOptInterface.VariableIndex(2)
 MathOptInterface.VariableIndex(3)
```

> **Warning**
>
> The integer does not necessarily corresond to the column inside an optimizer!

Delete a variable

Delete a variable using delete.

```
julia> MOI.delete(model, x)

julia> MOI.is_valid(model, x)
false
```

> **Warning**
>
> Not all ModelLike models support deleting variables. A DeleteNotAllowed error is thrown if this is not supported.

Variable attributes

The following attributes are available for variables:

- VariableName
- VariablePrimalStart
- VariablePrimal

Get and set these attributes using get and set.

```
julia> MOI.set(model, MOI.VariableName(), x, "var_x")

julia> MOI.get(model, MOI.VariableName(), x)
"var_x"
```

37.4 Constraints

Add a constraint

Use `add_constraint` to add a single constraint.

```
julia> c = MOI.add_constraint(model, MOI.VectorOfVariables(x), MOI.Nonnegatives(2))
MathOptInterface.ConstraintIndex{MathOptInterface.VectorOfVariables,
↪  MathOptInterface.Nonnegatives}(1)
```

`add_constraint` returns a `ConstraintIndex` type, which is used to refer to the added constraint in other calls.

Check if a `ConstraintIndex` is valid using `is_valid`.

```
julia> MOI.is_valid(model, c)
true
```

Use `add_constraints` to add a number of constraints of the same type.

```
julia> c = MOI.add_constraints(
           model,
           [x[1], x[2]],
           [MOI.GreaterThan(0.0), MOI.GreaterThan(1.0)]
       )
2-element Vector{MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,
↪  MathOptInterface.GreaterThan{Float64}}}:
 MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,
↪  MathOptInterface.GreaterThan{Float64}}(1)
 MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,
↪  MathOptInterface.GreaterThan{Float64}}(2)
```

This time, a vector of `ConstraintIndex` are returned.

Use `supports_constraint` to check if the model supports adding a constraint type.

```
julia> MOI.supports_constraint(
           model,
           MOI.VariableIndex,
           MOI.GreaterThan{Float64},
       )
true
```

Delete a constraint

Use `delete` to delete a constraint.

```
julia> MOI.delete(model, c)

julia> MOI.is_valid(model, c)
false
```

Constraint attributes

The following attributes are available for constraints:

- ConstraintName
- ConstraintPrimalStart
- ConstraintDualStart
- ConstraintPrimal
- ConstraintDual
- ConstraintBasisStatus
- ConstraintFunction
- CanonicalConstraintFunction
- ConstraintSet

Get and set these attributes using get and set.

```
julia> MOI.set(model, MOI.ConstraintName(), c, "con_c")

julia> MOI.get(model, MOI.ConstraintName(), c)
"con_c"
```

Constraints by function-set pairs

Below is a list of common constraint types and how they are represented as function-set pairs in MOI. In the notation below, x is a vector of decision variables, x_i is a scalar decision variable, α, β are scalar constants, a, b are constant vectors, A is a constant matrix and $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) is the set of nonnegative (resp. nonpositive) real numbers.

Linear constraints

Mathematical Constraint	MOI Function	MOI Set
$a^T x \le \beta$	ScalarAffineFunction	LessThan
$a^T x \ge \alpha$	ScalarAffineFunction	GreaterThan
$a^T x = \beta$	ScalarAffineFunction	EqualTo
$\alpha \le a^T x \le \beta$	ScalarAffineFunction	Interval
$x_i \le \beta$	VariableIndex	LessThan
$x_i \ge \alpha$	VariableIndex	GreaterThan
$x_i = \beta$	VariableIndex	EqualTo
$\alpha \le x_i \le \beta$	VariableIndex	Interval
$Ax + b \in \mathbb{R}_+^n$	VectorAffineFunction	Nonnegatives
$Ax + b \in \mathbb{R}_-^n$	VectorAffineFunction	Nonpositives
$Ax + b = 0$	VectorAffineFunction	Zeros

By convention, solvers are not expected to support nonzero constant terms in the ScalarAffineFunctions the first four rows above, because they are redundant with the parameters of the sets. For example, encode $2x + 1 \le 2$ as $2x \le 1$.

Constraints with `VariableIndex` in `LessThan`, `GreaterThan`, `EqualTo`, or `Interval` sets have a natural interpretation as variable bounds. As such, it is typically not natural to impose multiple lower- or upper-bounds on the same variable, and the solver interfaces will throw respectively `LowerBoundAlreadySet` or `UpperBoundAlreadySet`.

Moreover, adding two `VariableIndex` constraints on the same variable with the same set is impossible because they share the same index as it is the index of the variable, see `ConstraintIndex`.

It is natural, however, to impose upper- and lower-bounds separately as two different constraints on a single variable. The difference between imposing bounds by using a single `Interval` constraint and by using separate `LessThan` and `GreaterThan` constraints is that the latter will allow the solver to return separate dual multipliers for the two bounds, while the former will allow the solver to return only a single dual for the interval constraint.

Conic constraints

Mathematical Constraint	MOI Function	MOI Set
$\|Ax+b\|_2 \leq c^T x + d$	`VectorAffineFunction`	`SecondOrderCone`
$y \geq \|x\|_2$	`VectorOfVariables`	`SecondOrderCone`
$2yz \geq \|x\|_2^2, y, z \geq 0$	`VectorOfVariables`	`RotatedSecondOrderCone`
$(a_1^T x + b_1, a_2^T x + b_2, a_3^T x + b_3) \in \mathcal{E}$	`VectorAffineFunction`	`ExponentialCone`
$A(x) \in \mathcal{S}_+$	`VectorAffineFunction`	`PositiveSemidefiniteConeTriangle`
$B(x) \in \mathcal{S}_+$	`VectorAffineFunction`	`PositiveSemidefiniteConeSquare`
$x \in \mathcal{S}_+$	`VectorOfVariables`	`PositiveSemidefiniteConeTriangle`
$x \in \mathcal{S}_+$	`VectorOfVariables`	`PositiveSemidefiniteConeSquare`

where $\mathcal{E}$ is the exponential cone (see `ExponentialCone`), $\mathcal{S}_+$ is the set of positive semidefinite symmetric matrices, A is an affine map that outputs symmetric matrices and B is an affine map that outputs square matrices.

Quadratic constraints

Mathematical Constraint	MOI Function	MOI Set
$x^T Qx + a^T x + b \geq 0$	`ScalarQuadraticFunction`	`GreaterThan`
$x^T Qx + a^T x + b \leq 0$	`ScalarQuadraticFunction`	`LessThan`
$x^T Qx + a^T x + b = 0$	`ScalarQuadraticFunction`	`EqualTo`
Bilinear matrix inequality	`VectorQuadraticFunction`	`PositiveSemidefiniteCone...`

Discrete and logical constraints

Mathematical Constraint	MOI Function	MOI Set
$x_i \in \mathbb{Z}$	`VariableIndex`	`Integer`
$x_i \in \{0,1\}$	`VariableIndex`	`ZeroOne`
$x_i \in \{0\} \cup [l,u]$	`VariableIndex`	`Semicontinuous`
$x_i \in \{0\} \cup \{l, l+1, \ldots, u-1, u\}$	`VariableIndex`	`Semiinteger`
At most one component of x can be nonzero	`VectorOfVariables`	`SOS1`
At most two components of x can be nonzero, and if so they must be adjacent components	`VectorOfVariables`	`SOS2`
$y = 1 \implies a^T x \in S$	`VectorAffineFunction`	`Indicator`

JuMP mapping

The following bullet points show examples of how JuMP constraints are translated into MOI function-set pairs:

- `@constraint(m, 2x + y <= 10)` becomes `ScalarAffineFunction-in-LessThan`
- `@constraint(m, 2x + y >= 10)` becomes `ScalarAffineFunction-in-GreaterThan`
- `@constraint(m, 2x + y == 10)` becomes `ScalarAffineFunction-in-EqualTo`
- `@constraint(m, 0 <= 2x + y <= 10)` becomes `ScalarAffineFunction-in-Interval`
- `@constraint(m, 2x + y in ArbitrarySet())` becomes `ScalarAffineFunction-in-ArbitrarySet`.

Variable bounds are handled in a similar fashion:

- `@variable(m, x <= 1)` becomes `VariableIndex-in-LessThan`
- `@variable(m, x >= 1)` becomes `VariableIndex-in-GreaterThan`

One notable difference is that a variable with an upper and lower bound is translated into two constraints, rather than an interval. i.e.:

- `@variable(m, 0 <= x <= 1)` becomes `VariableIndex-in-LessThan` and `VariableIndex-in-GreaterThan`.

37.5 Solutions

Solving and retrieving the results

Once an optimizer is loaded with the objective function and all of the constraints, we can ask the solver to solve the model by calling `optimize!`.

```
MOI.optimize!(optimizer)
```

Why did the solver stop?

The optimization procedure may terminate for a number of reasons. The `TerminationStatus` attribute of the optimizer returns a `TerminationStatusCode` object which explains why the solver stopped.

The termination statuses distinguish between proofs of optimality, infeasibility, local convergence, limits, and termination because of something unexpected like invalid problem data or failure to converge.

A typical usage of the `TerminationStatus` attribute is as follows:

```
status = MOI.get(optimizer, TerminationStatus())
if status == MOI.OPTIMAL
    # Ok, we solved the problem!
else
    # Handle other cases.
end
```

After checking the `TerminationStatus`, check `ResultCount`. This attribute returns the number of results that the solver has available to return. A result is defined as a primal-dual pair, but either the primal or the dual may be missing from the result. While the `OPTIMAL` termination status normally implies that at least one result is available, other statuses do not. For example, in the case of infeasiblity, a solver may return no result or a proof of infeasibility. The `ResultCount` attribute distinguishes between these two cases.

Primal solutions

Use the `PrimalStatus` optimizer attribute to return a `ResultStatusCode` describing the status of the primal solution.

Common returns are described below in the Common status situations section.

Query the primal solution using the `VariablePrimal` and `ConstraintPrimal` attributes.

Query the objective function value using the `ObjectiveValue` attribute.

Dual solutions

Warning

See Duality for a discussion of the MOI conventions for primal-dual pairs and certificates.

Use the `DualStatus` optimizer attribute to return a `ResultStatusCode` describing the status of the dual solution.

Query the dual solution using the `ConstraintDual` attribute.

Query the dual objective function value using the `DualObjectiveValue` attribute.

Common status situations

The sections below describe how to interpret typical or interesting status cases for three common classes of solvers. The example cases are illustrative, not comprehensive. Solver wrappers may provide additional information on how the solver's statuses map to MOI statuses.

Info

* in the tables indicate that multiple different values are possible.

Primal-dual convex solver

Linear programming and conic optimization solvers fall into this category.

What happened?	`TerminationStatus`	`ResultCount`	`PrimalStatus`	`DualStatus`
Proved optimality	`OPTIMAL`	1	`FEASIBLE_POINT`	`FEASIBLE_POINT`
Proved infeasible	`INFEASIBLE`	1	`NO_SOLUTION`	`INFEASIBILITY_CERTIFICATE`
Optimal within relaxed tolerances	`ALMOST_OPTIMAL`	1	`FEASIBLE_POINT`	`FEASIBLE_POINT`
Optimal within relaxed tolerances	`ALMOST_OPTIMAL`	1	`ALMOST_FEASIBLE_POINT`	`ALMOST_FEASIBLE_POINT`
Detected an unbounded ray of the primal	`DUAL_INFEASIBLE`	1	`INFEASIBILITY_CERTIFICATE`	`NO_SOLUTION`
Stall	`SLOW_PROGRESS`	1	*	*

Global branch-and-bound solvers

Mixed-integer programming solvers fall into this category.

Info

`CPXMIP_OPTIMAL_INFEAS` is a CPLEX status that indicates that a preprocessed problem was solved to optimality, but the solver was unable to recover a feasible solution to the original problem. Handling this status was one of the motivating drivers behind the design of MOI.

What happened?	TerminationStatus	ResultCount	PrimalStatus	DualStatus
Proved optimality	OPTIMAL	1	FEASIBLE_POINT	NO_SOLUTION
Presolve detected infeasibility or unboundedness	INFEASIBLE_OR_UNBOUNDED	0	NO_SOLUTION	NO_SOLUTION
Proved infeasibility	INFEASIBLE	0	NO_SOLUTION	NO_SOLUTION
Timed out (no solution)	TIME_LIMIT	0	NO_SOLUTION	NO_SOLUTION
Timed out (with a solution)	TIME_LIMIT	1	FEASIBLE_POINT	NO_SOLUTION
CPXMIP_OPTIMAL_INFEAS	ALMOST_OPTIMAL	1	INFEASIBLE_POINT	NO_SOLUTION

Local search solvers

Nonlinear programming solvers fall into this category. It also includes non-global tree search solvers like Juniper.

What happened?	TerminationStatus	ResultCount	PrimalStatus	DualStatus
Converged to a stationary point	LOCALLY_SOLVED	1	FEASIBLE_POINT	FEASIBLE_POINT
Completed a non-global tree search (with a solution)	LOCALLY_SOLVED	1	FEASIBLE_POINT	FEASIBLE_POINT
Converged to an infeasible point	LOCALLY_INFEASIBLE	1	INFEASIBLE_POINT	*
Completed a non-global tree search (no solution found)	LOCALLY_INFEASIBLE	0	NO_SOLUTION	NO_SOLUTION
Iteration limit	ITERATION_LIMIT	1	*	*
Diverging iterates	NORM_LIMIT or OBJECTIVE_LIMIT	1	*	*

Querying solution attributes

Some solvers will not implement every solution attribute. Therefore, a call like `MOI.get(model, MOI.SolveTimeSec())` may throw an `UnsupportedAttribute` error.

If you need to write code that is agnostic to the solver (for example, you are writing a library that an end-user passes their choice of solver to), you can work-around this problem using a `try-catch`:

```
function get_solve_time(model)
    try
        return MOI.get(model, MOI.SolveTimeSec())
    catch err
        if err isa MOI.UnsupportedAttribute
            return NaN  # Solver doesn't support. Return a placeholder value.
        end
        rethrow(err)  # Something else went wrong. Rethrow the error
    end
end
```

If, after careful profiling, you find that the `try-catch` is taking a significant portion of your runtime, you can improve performance by caching the result of the `try-catch`:

```
mutable struct CachedSolveTime{M}
    model::M
    supports_solve_time::Bool
    CachedSolveTime(model::M) where {M} = new(model, true)
end
```

```
function get_solve_time(model::CachedSolveTime)
    if !model.supports_solve_time
        return NaN
    end
    try
        return MOI.get(model, MOI.SolveTimeSec())
    catch err
        if err isa MOI.UnsupportedAttribute
            model.supports_solve_time = false
            return NaN
        end
        rethrow(err)  # Something else went wrong. Rethrow the error
    end
end
```

37.6 Problem modification

In addition to adding and deleting constraints and variables, MathOptInterface supports modifying, in-place, coefficients in the constraints and the objective function of a model.

These modifications can be grouped into two categories:

- modifications which replace the set of function of a constraint with a new set or function
- modifications which change, in-place, a component of a function

> **Warning**
>
> Solve `ModelLike` objects do not support problem modification.

Modify the set of a constraint

Use `set` and `ConstraintSet` to modify the set of a constraint by replacing it with a new instance of the same type.

```
julia> c = MOI.add_constraint(
           model,
           MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], 0.0),
           MOI.EqualTo(1.0),
       )
MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.EqualTo{Float64}}(1)

julia> MOI.set(model, MOI.ConstraintSet(), c, MOI.EqualTo(2.0));

julia> MOI.get(model, MOI.ConstraintSet(), c) == MOI.EqualTo(2.0)
true
```

However, the following will fail as the new set is of a different type to the original set:

```
julia> MOI.set(model, MOI.ConstraintSet(), c, MOI.GreaterThan(2.0))
ERROR: [...]
```

Special cases: set transforms

If our constraint is an affine inequality, then this corresponds to modifying the right-hand side of a constraint in linear programming.

In some special cases, solvers may support efficiently changing the set of a constraint (for example, from `LessThan` to `GreaterThan`). For these cases, MathOptInterface provides the `transform` method.

The `transform` function returns a new constraint index, and the old constraint index (i.e., c) is no longer valid.

```
julia> c = MOI.add_constraint(
           model,
           MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], 0.0),
           MOI.LessThan(1.0),
       )
MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.LessThan{Float64}}(1)

julia> new_c = MOI.transform(model, c, MOI.GreaterThan(2.0));

julia> MOI.is_valid(model, c)
false

julia> MOI.is_valid(model, new_c)
true
```

Note

`transform` cannot be called with a set of the same type. Use `set` instead.

Modify the function of a constraint

Use `set` and `ConstraintFunction` to modify the function of a constraint by replacing it with a new instance of the same type.

```
julia> c = MOI.add_constraint(
           model,
           MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], 0.0),
           MOI.EqualTo(1.0),
       )
MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.EqualTo{Float64}}(1)

julia> new_f = MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(2.0, x)], 1.0);

julia> MOI.set(model, MOI.ConstraintFunction(), c, new_f);

julia> MOI.get(model, MOI.ConstraintFunction(), c) ≈ new_f
true
```

However, the following will fail as the new function is of a different type to the original function:

```
julia> MOI.set(model, MOI.ConstraintFunction(), c, x)
ERROR: [...]
```

Modify constant term in a scalar function

Use `modify` and `ScalarConstantChange` to modify the constant term in a `ScalarAffineFunction` or `ScalarQuadraticFunction`.

```
julia> c = MOI.add_constraint(
           model,
           MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], 0.0),
           MOI.EqualTo(1.0),
       )
MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.EqualTo{Float64}}(1)

julia> MOI.modify(model, c, MOI.ScalarConstantChange(1.0));

julia> new_f = MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], 1.0);

julia> MOI.get(model, MOI.ConstraintFunction(), c) ≈ new_f
true
```

> **Tip**
>
> `ScalarConstantChange` can also be used to modify the objective function by passing an instance of `ObjectiveFunction` instead of the constraint index `c` as we saw above.

```
julia> MOI.set(
           model,
           MOI.ObjectiveFunction{MOI.ScalarAffineFunction{Float64}}(),
           new_f,
       );

julia> MOI.modify(
           model,
           MOI.ObjectiveFunction{MOI.ScalarAffineFunction{Float64}}(),
           MOI.ScalarConstantChange(-1.0)
       );

julia> MOI.get(
           model,
           MOI.ObjectiveFunction{MOI.ScalarAffineFunction{Float64}}(),
       ) ≈ MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], -1.0)
true
```

Modify constant terms in a vector function

Use `modify` and `VectorConstantChange` to modify the constant vector in a `VectorAffineFunction` or `VectorQuadraticFunction`

```
julia> c = MOI.add_constraint(
           model,
           MOI.VectorAffineFunction([
                   MOI.VectorAffineTerm(1, MOI.ScalarAffineTerm(1.0, x)),
                   MOI.VectorAffineTerm(2, MOI.ScalarAffineTerm(2.0, x)),
               ],
               [0.0, 0.0],
           ),
           MOI.Nonnegatives(2),
```

```
       )
MathOptInterface.ConstraintIndex{MathOptInterface.VectorAffineFunction{Float64},
↪  MathOptInterface.Nonnegatives}(1)

julia> MOI.modify(model, c, MOI.VectorConstantChange([3.0, 4.0]));

julia> new_f = MOI.VectorAffineFunction(
           [
        MOI.VectorAffineTerm(1, MOI.ScalarAffineTerm(1.0, x)),
        MOI.VectorAffineTerm(2, MOI.ScalarAffineTerm(2.0, x)),
           ],
           [3.0, 4.0],
       );

julia> MOI.get(model, MOI.ConstraintFunction(), c) ≈ new_f
true
```

Modify affine coefficients in a scalar function

Use `modify` and `ScalarCoefficientChange` to modify the affine coefficient of a `ScalarAffineFunction` or `ScalarQuadraticFunction`.

```
julia> c = MOI.add_constraint(
           model,
           MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(1.0, x)], 0.0),
           MOI.EqualTo(1.0),
       )
MathOptInterface.ConstraintIndex{MathOptInterface.ScalarAffineFunction{Float64},
↪  MathOptInterface.EqualTo{Float64}}(1)

julia> MOI.modify(model, c, MOI.ScalarCoefficientChange(x, 2.0));

julia> new_f = MOI.ScalarAffineFunction([MOI.ScalarAffineTerm(2.0, x)], 0.0);

julia> MOI.get(model, MOI.ConstraintFunction(), c) ≈ new_f
true
```

Tip

`ScalarCoefficientChange` can also be used to modify the objective function by passing an instance of `ObjectiveFunction` instead of the constraint index c as we saw above.

Modify affine coefficients in a vector function

Use `modify` and `MultirowChange` to modify a vector of affine coefficients in a `VectorAffineFunction` or a `VectorQuadraticFunction`.

```
julia> c = MOI.add_constraint(
           model,
           MOI.VectorAffineFunction([
                   MOI.VectorAffineTerm(1, MOI.ScalarAffineTerm(1.0, x)),
                   MOI.VectorAffineTerm(2, MOI.ScalarAffineTerm(2.0, x)),
               ],
               [0.0, 0.0],
           ),
```

```
           MOI.Nonnegatives(2),
       )
MathOptInterface.ConstraintIndex{MathOptInterface.VectorAffineFunction{Float64},
↪  MathOptInterface.Nonnegatives}(1)

julia> MOI.modify(model, c, MOI.MultirowChange(x, [(1, 3.0), (2, 4.0)]));

julia> new_f = MOI.VectorAffineFunction(
           [
        MOI.VectorAffineTerm(1, MOI.ScalarAffineTerm(3.0, x)),
        MOI.VectorAffineTerm(2, MOI.ScalarAffineTerm(4.0, x)),
           ],
           [0.0, 0.0],
       );

julia> MOI.get(model, MOI.ConstraintFunction(), c) ≈ new_f
true
```

Chapter 38

Background

38.1 Duality

Conic duality is the starting point for MOI's duality conventions. When all functions are affine (or coordinate projections), and all constraint sets are closed convex cones, the model may be called a conic optimization problem.

For a minimization problem in geometric conic form, the primal is:

$$\min_{x \in \mathbb{R}^n} \quad a_0^T x + b_0 \tag{38.1}$$

$$\text{s.t.} \quad A_i x + b_i \in \mathcal{C}_i \qquad i = 1 \dots m \tag{38.2}$$

and the dual is a maximization problem in standard conic form:

$$\max_{y_1, \dots, y_m} \quad -\sum_{i=1}^m b_i^T y_i + b_0 \tag{38.3}$$

$$\text{s.t.} \quad a_0 - \sum_{i=1}^m A_i^T y_i = 0 \tag{38.4}$$

$$y_i \in \mathcal{C}_i^* \qquad i = 1 \dots m \tag{38.5}$$

where each $\mathcal{C}_i$ is a closed convex cone and $\mathcal{C}_i^*$ is its dual cone.

For a maximization problem in geometric conic form, the primal is:

$$\max_{x \in \mathbb{R}^n} \quad a_0^T x + b_0 \tag{38.6}$$

$$\text{s.t.} \quad A_i x + b_i \in \mathcal{C}_i \qquad i = 1 \dots m \tag{38.7}$$

and the dual is a minimization problem in standard conic form:

$$\min_{y_1,\ldots,y_m} \quad \sum_{i=1}^m b_i^T y_i + b_0 \tag{38.8}$$

$$\text{s.t.} \quad a_0 + \sum_{i=1}^m A_i^T y_i = 0 \tag{38.9}$$

$$y_i \in \mathcal{C}_i^* \qquad i = 1 \ldots m \tag{38.10}$$

A linear inequality constraint $a^T x + b \geq c$ is equivalent to $a^T x + b - c \in \mathbb{R}_+$, and $a^T x + b \leq c$ is equivalent to $a^T x + b - c \in \mathbb{R}_-$. Variable-wise constraints are affine constraints with the appropriate identity mapping in place of A_i.

For the special case of minimization LPs, the MOI primal form can be stated as:

$$\min_{x \in \mathbb{R}^n} \quad a_0^T x + b_0 \tag{38.11}$$

$$\text{s.t.} \quad A_1 x \geq b_1 \tag{38.12}$$

$$A_2 x \leq b_2 \tag{38.13}$$

$$A_3 x = b_3 \tag{38.14}$$

By applying the stated transformations to conic form, taking the dual, and transforming back into linear inequality form, one obtains the following dual:

$$\max_{y_1,y_2,y_3} \quad b_1^T y_1 + b_2^T y_2 + b_3^T y_3 + b_0 \tag{38.15}$$

$$\text{s.t.} \quad A_1^T y_1 + A_2^T y_2 + A_3^T y_3 = a_0 \tag{38.16}$$

$$y_1 \geq 0 \tag{38.17}$$

$$y_2 \leq 0 \tag{38.18}$$

For maximization LPs, the MOI primal form can be stated as:

$$\max_{x \in \mathbb{R}^n} \quad a_0^T x + b_0 \tag{38.19}$$

$$\text{s.t.} \quad A_1 x \geq b_1 \tag{38.20}$$

$$A_2 x \leq b_2 \tag{38.21}$$

$$A_3 x = b_3 \tag{38.22}$$

and similarly, the dual is:

$$\min_{y_1,y_2,y_3} \quad -b_1^T y_1 - b_2^T y_2 - b_3^T y_3 + b_0 \tag{38.23}$$

$$\text{s.t.} \quad A_1^T y_1 + A_2^T y_2 + A_3^T y_3 = -a_0 \tag{38.24}$$

$$y_1 \geq 0 \tag{38.25}$$

$$y_2 \leq 0 \tag{38.26}$$

Warning

For the LP case, the signs of the feasible dual variables depend only on the sense of the corresponding primal inequality and not on the objective sense.

Duality and scalar product

The scalar product is different from the canonical one for the sets PositiveSemidefiniteConeTriangle, LogDetConeTriangle, RootDetConeTriangle.

If the set C_i of the section Duality is one of these three cones, then the rows of the matrix A_i corresponding to off-diagonal entries are twice the value of the `coefficients` field in the VectorAffineFunction for the corresponding rows. See PositiveSemidefiniteConeTriangle for details.

Dual for problems with quadratic functions

Quadratic Programs (QPs)

For quadratic programs with only affine conic constraints,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} & \quad \frac{1}{2}x^T Q_0 x + a_0^T x + b_0 \\ \text{s.t.} & \quad A_i x + b_i \in \mathcal{C}_i \qquad i = 1 \ldots m. \end{aligned}$$

with cones $\mathcal{C}_i \subseteq \mathbb{R}^{m_i}$ for $i = 1, \ldots, m$, consider the Lagrangian function

$$L(x, y) = \frac{1}{2}x^T Q_0 x + a_0^T x + b_0 - \sum_{i=1}^m y_i^T (A_i x + b_i).$$

Let $z(y)$ denote $\sum_{i=1}^m A_i^T y_i - a_0$, the Lagrangian can be rewritten as

$$L(x, y) = \frac{1}{2}x^T Q_0 x - z(y)^T x + b_0 - \sum_{i=1}^m y_i^T b_i.$$

The condition $\nabla_x L(x, y) = 0$ gives

$$0 = \nabla_x L(x, y) = Q_0 x + a_0 - \sum_{i=1}^m y_i^T b_i$$

which gives $Q_0 x = z(y)$. This allows to obtain that

$$\min_{x \in \mathbb{R}^n} L(x, y) = -\frac{1}{2}x^T Q_0 x + b_0 - \sum_{i=1}^m y_i^T b_i$$

so the dual problem is

$$\max_{y_i \in \mathcal{C}_i^*} \min_{x \in \mathbb{R}^n} -\frac{1}{2}x^T Q_0 x + b_0 - \sum_{i=1}^m y_i^T b_i.$$

If Q_0 is invertible, we have $x = Q_0^{-1} z(y)$ hence

$$\min_{x\in\mathbb{R}^n} L(x,y) = -\frac{1}{2} z(y)^T Q_0^{-1} z(y) + b_0 - \sum_{i=1}^m y_i^T b_i$$

so the dual problem is

$$\max_{y_i \in \mathcal{C}_i^*} -\frac{1}{2} z(y)^T Q_0^{-1} z(y) + b_0 - \sum_{i=1}^m y_i^T b_i.$$

Quadratically Constrained Quadratic Programs (QCQPs)

Given a problem with both quadratic function and quadratic objectives:

$$\begin{array}{lll} \min\limits_{x\in\mathbb{R}^n} & \frac{1}{2} x^T Q_0 x + a_0^T x + b_0 & \\ \text{s.t.} & \frac{1}{2} x^T Q_i x + a_i^T x + b_i \in \mathcal{C}_i & i = 1 \ldots m. \end{array}$$

with cones $\mathcal{C}_i \subseteq \mathbb{R}$ for $i = 1 \ldots m$, consider the Lagrangian function

$$L(x,y) = \frac{1}{2} x^T Q_0 x + a_0^T x + b_0 - \sum_{i=1}^m y_i (\frac{1}{2} x^T Q_i x + a_i^T x + b_i)$$

A pair of primal-dual variables $(x^\star, y^\star)$ is optimal if

- $x^\star$ is a minimizer of

  $$\min_{x\in\mathbb{R}^n} L(x, y^\star).$$

  That is,

  $$0 = \nabla_x L(x, y^\star) = Q_0 x + a_0 - \sum_{i=1}^m y_i^\star (Q_i x + a_i).$$

- and $y^\star$ is a maximizer of

  $$\max_{y_i\in\mathcal{C}_i^*} L(x^\star, y).$$

  That is, for all $i = 1, \ldots, m$, $\frac{1}{2} x^T Q_i x + a_i^T x + b_i$ is either zero or in the normal cone of $\mathcal{C}_i^*$ at $y^\star$. For instance, if $\mathcal{C}_i$ is $\{z \in \mathbb{R} : z \leq 0\}$, this means that if $\frac{1}{2} x^T Q_i x + a_i^T x + b_i$ is nonzero at $x^\star$ then $y_i^\star = 0$. This is the classical complementary slackness condition.

If $\mathcal{C}_i$ is a vector set, the discussion remains valid with $y_i(\frac{1}{2} x^T Q_i x + a_i^T x + b_i)$ replaced with the scalar product between y_i and the vector of scalar-valued quadratic functions.

38.2 Infeasibility certificates

When given a conic problem that is infeasible or unbounded, some solvers can produce a certificate of infeasibility. This page explains what a certificate of infeasibility is, and the related conventions that MathOptInterface adopts.

Conic duality

MathOptInterface uses conic duality to define infeasibility certificates. A full explanation is given in the section Duality, but here is a brief overview.

Minimization problems

For a minimization problem in geometric conic form, the primal is:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & a_0^\top x + b_0 & & \quad (38.27)\\ \text{s.t.} \quad & A_i x + b_i \in \mathcal{C}_i & i = 1 \ldots m, & \quad (38.28) \end{aligned}$$

and the dual is a maximization problem in standard conic form:

$$\begin{aligned} \max_{y_1, \ldots, y_m} \quad & -\sum_{i=1}^m b_i^\top y_i + b_0 & & \quad (38.29)\\ \text{s.t.} \quad & a_0 - \sum_{i=1}^m A_i^\top y_i = 0 & & \quad (38.30)\\ & y_i \in \mathcal{C}_i^* & i = 1 \ldots m, & \quad (38.31) \end{aligned}$$

where each $\mathcal{C}_i$ is a closed convex cone and $\mathcal{C}_i^*$ is its dual cone.

Maximization problems

For a maximization problem in geometric conic form, the primal is:

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & a_0^\top x + b_0 & & \quad (38.32)\\ \text{s.t.} \quad & A_i x + b_i \in \mathcal{C}_i & i = 1 \ldots m, & \quad (38.33) \end{aligned}$$

and the dual is a minimization problem in standard conic form:

$$\begin{aligned} \min_{y_1, \ldots, y_m} \quad & \sum_{i=1}^m b_i^\top y_i + b_0 & & \quad (38.34)\\ \text{s.t.} \quad & a_0 + \sum_{i=1}^m A_i^\top y_i = 0 & & \quad (38.35)\\ & y_i \in \mathcal{C}_i^* & i = 1 \ldots m. & \quad (38.36) \end{aligned}$$

Unbounded problems

A problem is unbounded if and only if:

1. there exists a feasible primal solution
2. the dual is infeasible.

A feasible primal solution—if one exists—can be obtained by setting `ObjectiveSense` to `FEASIBILITY_SENSE` before optimizing. Therefore, most solvers terminate after they prove the dual is infeasible via a certificate of dual infeasibility, but before they have found a feasible primal solution. This is also the reason that MathOpt-Interface defines the `DUAL_INFEASIBLE` status instead of `UNBOUNDED`.

A certificate of dual infeasibility is an improving ray of the primal problem. That is, there exists some vector d such that for all $\eta > 0$:

$$A_i(x + \eta d) + b_i \in \mathcal{C}_i, \;\; i = 1 \ldots m,$$

and (for minimization problems):

$$a_0^\top (x + \eta d) + b_0 < a_0^\top x + b_0,$$

for any feasible point x. The latter simplifies to $a_0^\top d < 0$. For maximization problems, the inequality is reversed, so that $a_0^\top d > 0$.

If the solver has found a certificate of dual infeasibility:

- `TerminationStatus` must be `DUAL_INFEASIBLE`
- `PrimalStatus` must be `INFEASIBILITY_CERTIFICATE`
- `VariablePrimal` must be the corresponding value of d
- `ConstraintPrimal` must be the corresponding value of $A_i d$
- `ObjectiveValue` must be the value $a_0^\top d$. Note that this is the value of the objective function at d, ignoring the constant `b_0`.

  Note

  The choice of whether to scale the ray d to have magnitude 1 is left to the solver.

Infeasible problems

A certificate of primal infeasibility is an improving ray of the dual problem. However, because infeasibility is independent of the objective function, we first homogenize the primal problem by removing its objective.

For a minimization problem, a dual improving ray is some vector d such that for all $\eta > 0$:

$$-\sum_{i=1}^{m} A_i^\top (y_i + \eta d_i) = 0 \tag{38.37}$$

$$(y_i + \eta d_i) \in \mathcal{C}_i^* \qquad i = 1 \ldots m, \tag{38.38}$$

and:

$$-\sum_{i=1}^{m} b_i^\top (y_i + \eta d_i) > -\sum_{i=1}^{m} b_i^\top y_i,$$

for any feasible dual solution y. The latter simplifies to $-\sum_{i=1}^m b_i^\top d_i > 0$. For a maximization problem, the inequality is $\sum_{i=1}^m b_i^\top d_i < 0$. (Note that these are the same inequality, modulo a `-` sign.)

If the solver has found a certificate of primal infeasibility:

- `TerminationStatus` must be `INFEASIBLE`
- `DualStatus` must be `INFEASIBILITY_CERTIFICATE`
- `ConstraintDual` must be the corresponding value of d
- `DualObjectiveValue` must be the value $-\sum_{i=1}^m b_i^\top d_i$ for minimization problems and $\sum_{i=1}^m b_i^\top d_i$ for maximization problems.

  Note

  The choice of whether to scale the ray d to have magnitude 1 is left to the solver.

Infeasibility certificates of variable bounds

Many linear solvers (e.g., Gurobi) do not provide explicit access to the primal infeasibility certificate of a variable bound. However, given a set of linear constraints:

$$l_A \le Ax \le u_A \tag{38.39}$$

$$l_x \le x \le u_x, \tag{38.40}$$

the primal certificate of the variable bounds can be computed using the primal certificate associated with the affine constraints, d. (Note that d will have one element for each row of the A matrix, and that some or all of the elements in the vectors l_A and u_A may be $\pm\infty$. If both l_A and u_A are finite for some row, the corresponding element in ‘d must be `0`.)

Given d, compute $\bar{d} = d^\top A$. If the bound is finite, a certificate for the lower variable bound of x_i is $\max\{\bar{d}_i, 0\}$, and a certificate for the upper variable bound is $\min\{\bar{d}_i, 0\}$.

38.3 Naming conventions

MOI follows several conventions for naming functions and structures. These should also be followed by packages extending MOI.

Sets

Sets encode the structure of constraints. Their names should follow the following conventions:

- Abstract types in the set hierarchy should begin with `Abstract` and end in `Set`, e.g., `AbstractScalarSet`, `AbstractVectorSet`.
- Vector-valued conic sets should end with `Cone`, e.g., `NormInfinityCone`, `SecondOrderCone`.
- Vector-valued Cartesian products should be plural and not end in `Cone`, e.g., `Nonnegatives`, not `NonnegativeCone`.
- Matrix-valued conic sets should provide two representations: `ConeSquare` and `ConeTriangle`, e.g., `RootDetConeTriangle` and `RootDetConeSquare`. See Matrix cones for more details.
- Scalar sets should be singular, not plural, e.g., `Integer`, not `Integers`.
- As much as possible, the names should follow established conventions in the domain where this set is used: for instance, convex sets should have names close to those of CVX, and constraint-programming sets should follow MiniZinc's constraints.

Chapter 39

API Reference

39.1 Standard form

Functions

MathOptInterface.AbstractFunction – Type.

```
AbstractFunction
```

Abstract supertype for function objects.

MathOptInterface.AbstractScalarFunction – Type.

```
AbstractScalarFunction
```

Abstract supertype for scalar-valued function objects.

MathOptInterface.AbstractVectorFunction – Type.

```
AbstractVectorFunction
```

Abstract supertype for vector-valued function objects.

MathOptInterface.VariableIndex – Type.

```
VariableIndex
```

A type-safe wrapper for `Int64` for use in referencing variables in a model. To allow for deletion, indices need not be consecutive.

MathOptInterface.VectorOfVariables – Type.

```
VectorOfVariables(variables)
```

The function that extracts the vector of variables referenced by `variables`, a `Vector{VariableIndex}`. This function is naturally be used for constraints that apply to groups of variables, such as an "all different" constraint, an indicator constraint, or a complementarity constraint.

MathOptInterface.ScalarAffineTerm – Type.

```
struct ScalarAffineTerm{T}
    coefficient::T
    variable::VariableIndex
end
```

Represents cx_i where c is `coefficient` and x_i is the variable identified by `variable`.

`MathOptInterface.ScalarAffineFunction` – Type.

```
ScalarAffineFunction{T}(terms, constant)
```

The scalar-valued affine function $a^T x + b$, where:

- a is a sparse vector specified by a list of `ScalarAffineTerm` structs.
- b is a scalar specified by `constant::T`

Duplicate variable indices in `terms` are accepted, and the corresponding coefficients are summed together.

`MathOptInterface.VectorAffineTerm` – Type.

```
struct VectorAffineTerm{T}
    output_index::Int64
    scalar_term::ScalarAffineTerm{T}
end
```

A `ScalarAffineTerm` plus its index of the output component of a `VectorAffineFunction` or `VectorQuadraticFunction`. `output_index` can also be interpreted as a row index into a sparse matrix, where the `scalar_term` contains the column index and coefficient.

`MathOptInterface.VectorAffineFunction` – Type.

```
VectorAffineFunction{T}(terms, constants)
```

The vector-valued affine function $Ax + b$, where:

- A is a sparse matrix specified by a list of `VectorAffineTerm` objects.
- b is a vector specified by `constants`

Duplicate indices in the A are accepted, and the corresponding coefficients are summed together.

`MathOptInterface.ScalarQuadraticTerm` – Type.

```
struct ScalarQuadraticTerm{T}
    coefficient::T
    variable_1::VariableIndex
    variable_2::VariableIndex
end
```

Represents $cx_i x_j$ where c is `coefficient`, x_i is the variable identified by `variable_1` and x_j is the variable identified by `variable_2`.

`MathOptInterface.ScalarQuadraticFunction` – Type.

```
ScalarQuadraticFunction{T}(quadratic_terms, affine_terms, constant)
```

The scalar-valued quadratic function $\frac{1}{2}x^T Qx + a^T x + b$, where:

- a is a sparse vector specified by a list of `ScalarAffineTerm` structs.
- b is a scalar specified by `constant`.
- Q is a symmetric matrix specified by a list of `ScalarQuadraticTerm` structs.

Duplicate indices in a or Q are accepted, and the corresponding coefficients are summed together. "Mirrored" indices `(q,r)` and `(r,q)` (where `r` and `q` are `VariableIndexes`) are considered duplicates; only one need be specified.

For example, for two scalar variables y, z, the quadratic expression $yz + y^2$ is represented by the terms `ScalarQuadraticTerm.([1.0, 2.0], [y, y], [z, y])`.

`MathOptInterface.VectorQuadraticTerm` – Type.

```
struct VectorQuadraticTerm{T}
    output_index::Int64
    scalar_term::ScalarQuadraticTerm{T}
end
```

A `ScalarQuadraticTerm` plus its index of the output component of a `VectorQuadraticFunction`. Each output component corresponds to a distinct sparse matrix Q_i.

`MathOptInterface.VectorQuadraticFunction` – Type.

```
VectorQuadraticFunction{T}(quadratic_terms, affine_terms, constants)
```

The vector-valued quadratic function with `i`th component ("output index") defined as $\frac{1}{2}x^TQ_ix + a_i^Tx + b_i$, where:

- a_i is a sparse vector specified by the `VectorAffineTerms` with `output_index == i`.
- b_i is a scalar specified by `constants[i]`
- Q_i is a symmetric matrix specified by the `VectorQuadraticTerm` with `output_index == i`.

Duplicate indices in a_i or Q_i are accepted, and the corresponding coefficients are summed together. "Mirrored" indices `(q,r)` and `(r,q)` (where `r` and `q` are `VariableIndexes`) are considered duplicates; only one need be specified.

Utilities

`MathOptInterface.output_dimension` – Function.

```
output_dimension(f::AbstractFunction)
```

Return 1 if `f` has a scalar output and the number of output components if `f` has a vector output.

`MathOptInterface.constant` – Method.

```
constant(f::Union{ScalarAffineFunction, ScalarQuadraticFunction})
```

Returns the constant term of the scalar function

`MathOptInterface.constant` – Method.

```
constant(f::Union{VectorAffineFunction, VectorQuadraticFunction})
```

Returns the vector of constant terms of the vector function

`MathOptInterface.constant` – Method.

```
constant(f::VariableIndex, ::Type{T}) where {T}
```

The constant term of a `VariableIndex` function is the zero value of the specified type T.

`MathOptInterface.constant` – Method.

```
constant(f::VectorOfVariables, ::Type{T}) where {T}
```

The constant term of a `VectorOfVariables` function is a vector of zero values of the specified type T.

Sets

`MathOptInterface.AbstractSet` – Type.

```
AbstractSet
```

Abstract supertype for set objects used to encode constraints. A set object should not contain any `VariableIndex` or `ConstraintIndex` as the set is passed unmodifed during `copy_to`.

`MathOptInterface.AbstractScalarSet` – Type.

```
AbstractScalarSet
```

Abstract supertype for subsets of $\mathbb{R}$.

`MathOptInterface.AbstractVectorSet` – Type.

```
AbstractVectorSet
```

Abstract supertype for subsets of $\mathbb{R}^n$ for some n.

Utilities

`MathOptInterface.dimension` – Function.

```
dimension(s::AbstractSet)
```

Return the `output_dimension` that an `AbstractFunction` should have to be used with the set s.

Examples

```
julia> dimension(Reals(4))
4

julia> dimension(LessThan(3.0))
1

julia> dimension(PositiveSemidefiniteConeTriangle(2))
3
```

`MathOptInterface.dual_set` – Function.

```
dual_set(s::AbstractSet)
```

Return the dual set of s, that is the dual cone of the set. This follows the definition of duality discussed in Duality.

See Dual cone for more information.

If the dual cone is not defined it returns an error.

Examples

```
julia> dual_set(Reals(4))
Zeros(4)

julia> dual_set(SecondOrderCone(5))
SecondOrderCone(5)

julia> dual_set(ExponentialCone())
DualExponentialCone()
```

`MathOptInterface.dual_set_type` – Function.

```
dual_set_type(S::Type{<:AbstractSet})
```

Return the type of dual set of sets of type S, as returned by `dual_set`. If the dual cone is not defined it returns an error.

Examples

```
julia> dual_set_type(Reals)
Zeros

julia> dual_set_type(SecondOrderCone)
SecondOrderCone

julia> dual_set_type(ExponentialCone)
DualExponentialCone
```

`MathOptInterface.constant` – Method.

```
constant(s::Union{EqualTo, GreaterThan, LessThan})
```

Returns the constant of the set.

`MathOptInterface.supports_dimension_update` – Function.

```
supports_dimension_update(S::Type{<:MOI.AbstractVectorSet})
```

Return a `Bool` indicating whether the elimination of any dimension of n-dimensional sets of type S give an `n-1`-dimensional set S. By default, this function returns `false` so it should only be implemented for sets that supports dimension update.

For instance, `supports_dimension_update(MOI.Nonnegatives}` is `true` because the elimination of any dimension of the n-dimensional nonnegative orthant gives the `n-1`-dimensional nonnegative orthant. However `supports_dimension_update(MOI.ExponentialCone}` is `false`.

`MathOptInterface.update_dimension` – Function.

```
update_dimension(s::AbstractVectorSet, new_dim)
```

Returns a set with the dimension modified to `new_dim`.

Scalar sets

List of recognized scalar sets.

`MathOptInterface.GreaterThan` – Type.

```
GreaterThan{T <: Real}(lower::T)
```

The set $[lower, \infty) \subseteq \mathbb{R}$.

MathOptInterface.LessThan – Type.

```
LessThan{T <: Real}(upper::T)
```

The set $(-\infty, upper] \subseteq \mathbb{R}$.

MathOptInterface.EqualTo – Type.

```
EqualTo{T <: Number}(value::T)
```

The set containing the single point $x \in \mathbb{R}$ where x is given by `value`.

MathOptInterface.Interval – Type.

```
Interval{T <: Real}(lower::T,upper::T)
```

The interval $[lower, upper] \subseteq \mathbb{R}$. If `lower` or `upper` is `-Inf` or `Inf`, respectively, the set is interpreted as a one-sided interval.

```
Interval(s::GreaterThan{<:AbstractFloat})
```

Construct a (right-unbounded) `Interval` equivalent to the given GreaterThan set.

```
Interval(s::LessThan{<:AbstractFloat})
```

Construct a (left-unbounded) `Interval` equivalent to the given LessThan set.

```
Interval(s::EqualTo{<:Real})
```

Construct a (degenerate) `Interval` equivalent to the given EqualTo set.

MathOptInterface.Integer – Type.

```
Integer()
```

The set of integers $\mathbb{Z}$.

MathOptInterface.ZeroOne – Type.

```
ZeroOne()
```

The set $\{0, 1\}$.

MathOptInterface.Semicontinuous – Type.

```
Semicontinuous{T <: Real}(lower::T,upper::T)
```

The set $\{0\} \cup [lower, upper]$.

MathOptInterface.Semiinteger – Type.

```
Semiinteger{T <: Real}(lower::T,upper::T)
```

The set $\{0\} \cup \{lower, lower + 1, \ldots, upper - 1, upper\}$.

Vector sets

List of recognized vector sets.

`MathOptInterface.Reals` – Type.

```
Reals(dimension)
```

The set $\mathbb{R}^{dimension}$ (containing all points) of dimension `dimension`.

`MathOptInterface.Zeros` – Type.

```
Zeros(dimension)
```

The set $\{0\}^{dimension}$ (containing only the origin) of dimension `dimension`.

`MathOptInterface.Nonnegatives` – Type.

```
Nonnegatives(dimension)
```

The nonnegative orthant $\{x \in \mathbb{R}^{dimension} : x \geq 0\}$ of dimension `dimension`.

`MathOptInterface.Nonpositives` – Type.

```
Nonpositives(dimension)
```

The nonpositive orthant $\{x \in \mathbb{R}^{dimension} : x \leq 0\}$ of dimension `dimension`.

`MathOptInterface.NormInfinityCone` – Type.

```
NormInfinityCone(dimension)
```

The ℓ_∞-norm cone $\{(t,x) \in \mathbb{R}^{dimension} : t \geq \|x\|_\infty = \max_i |x_i|\}$ of dimension `dimension`.

`MathOptInterface.NormOneCone` – Type.

```
NormOneCone(dimension)
```

The ℓ_1-norm cone $\{(t,x) \in \mathbb{R}^{dimension} : t \geq \|x\|_1 = \sum_i |x_i|\}$ of dimension `dimension`.

`MathOptInterface.SecondOrderCone` – Type.

```
SecondOrderCone(dimension)
```

The second-order cone (or Lorenz cone or ℓ_2-norm cone) $\{(t,x) \in \mathbb{R}^{dimension} : t \geq \|x\|_2\}$ of dimension `dimension`.

`MathOptInterface.RotatedSecondOrderCone` – Type.

```
RotatedSecondOrderCone(dimension)
```

The rotated second-order cone $\{(t,u,x) \in \mathbb{R}^{dimension} : 2tu \geq \|x\|_2^2, t, u \geq 0\}$ of dimension `dimension`.

`MathOptInterface.GeometricMeanCone` – Type.

```
GeometricMeanCone(dimension)
```

The geometric mean cone $\{(t,x) \in \mathbb{R}^{n+1} : x \ge 0, t \le \sqrt[n]{x_1 x_2 \cdots x_n}\}$, where `dimension = n + 1 >= 2`.

Duality note

The dual of the geometric mean cone is $\{(u,v) \in \mathbb{R}^{n+1} : u \le 0, v \ge 0, -u \le n \sqrt[n]{\prod_i v_i}\}$, where `dimension = n + 1 >= 2`.

`MathOptInterface.ExponentialCone` – Type.

```
ExponentialCone()
```

The 3-dimensional exponential cone $\{(x,y,z) \in \mathbb{R}^3 : y \exp(x/y) \le z, y > 0\}$.

`MathOptInterface.DualExponentialCone` – Type.

```
DualExponentialCone()
```

The 3-dimensional dual exponential cone $\{(u,v,w) \in \mathbb{R}^3 : -u \exp(v/u) \le \exp(1) w, u < 0\}$.

`MathOptInterface.PowerCone` – Type.

```
PowerCone{T <: Real}(exponent::T)
```

The 3-dimensional power cone $\{(x,y,z) \in \mathbb{R}^3 : x^{exponent} y^{1-exponent} \ge |z|, x \ge 0, y \ge 0\}$ with parameter `exponent`.

`MathOptInterface.DualPowerCone` – Type.

```
DualPowerCone{T <: Real}(exponent::T)
```

The 3-dimensional power cone $\{(u,v,w) \in \mathbb{R}^3 : (\frac{u}{exponent})^{exponent} (\frac{v}{1-exponent})^{1-exponent} \ge |w|, u \ge 0, v \ge 0\}$ with parameter `exponent`.

`MathOptInterface.RelativeEntropyCone` – Type.

```
RelativeEntropyCone(dimension)
```

The relative entropy cone $\{(u,v,w) \in \mathbb{R}^{1+2n} : u \ge \sum_{i=1}^n w_i \log(\frac{w_i}{v_i}), v_i \ge 0, w_i \ge 0\}$, where `dimension = 2n + 1 >= 1`.

Duality note

The dual of the relative entropy cone is $\{(u,v,w) \in \mathbb{R}^{1+2n} : \forall i, w_i \ge u(\log(\frac{u}{v_i}) - 1), v_i \ge 0, u > 0\}$ of dimension `dimension` $= 2n + 1$.

`MathOptInterface.NormSpectralCone` – Type.

```
NormSpectralCone(row_dim, column_dim)
```

The epigraph of the matrix spectral norm (maximum singular value function) $\{(t,X) \in \mathbb{R}^{1+row_dim \times column_dim} : t \ge \sigma_1(X)\}$, where σ_i is the ith singular value of the matrix X of row dimension `row_dim` and column dimension `column_dim`.

The matrix X is vectorized by stacking the columns, matching the behavior of Julia's vec function.

`MathOptInterface.NormNuclearCone` – Type.

```
NormNuclearCone(row_dim, column_dim)
```

The epigraph of the matrix nuclear norm (sum of singular values function) $\{(t, X) \in \mathbb{R}^{1+row_dim\times column_dim} : t \geq \sum_i \sigma_i(X)\}$, where σ_i is the ith singular value of the matrix X of row dimension `row_dim` and column dimension `column_dim`.

The matrix X is vectorized by stacking the columns, matching the behavior of Julia's vec function.

`MathOptInterface.SOS1` – Type.

```
SOS1{T <: Real}(weights::Vector{T})
```

The set corresponding to the special ordered set (SOS) constraint of type 1. Of the variables in the set, at most one can be nonzero. The `weights` induce an ordering of the variables; as such, they should be unique values. The kth element in the set corresponds to the kth weight in `weights`. See here for a description of SOS constraints and their potential uses.

`MathOptInterface.SOS2` – Type.

```
SOS2{T <: Real}(weights::Vector{T})
```

The set corresponding to the special ordered set (SOS) constraint of type 2. Of the variables in the set, at most two can be nonzero, and if two are nonzero, they must be adjacent in the ordering of the set. The `weights` induce an ordering of the variables; as such, they should be unique values. The kth element in the set corresponds to the kth weight in `weights`. See here for a description of SOS constraints and their potential uses.

`MathOptInterface.Indicator` – Type.

```
Indicator{A<:ActivationCondition,S<:AbstractScalarSet}(set::S)
```

The set corresponding to an indicator constraint.

When A is `ACTIVATE_ON_ZERO`, this means: $\{(y, x) \in \{0, 1\} \times \mathbb{R}^n : y = 0 \implies x \in set\}$

When A is `ACTIVATE_ON_ONE`, this means: $\{(y, x) \in \{0, 1\} \times \mathbb{R}^n : y = 1 \implies x \in set\}$

Notes

Most solvers expect that the first row of the function is interpretable as a variable index `x_i` (e.g., `1.0 * x + 0.0`). An error will be thrown if this is not the case.

Example

The constraint $\{(y, x) \in \{0, 1\} \times \mathbb{R}^2 : y = 1 \implies x_1 + x_2 \leq 9\}$ is defined as

```
f = MOI.VectorAffineFunction(
    [
        MOI.VectorAffineTerm(1, MOI.ScalarAffineTerm(1.0, y)),
        MOI.VectorAffineTerm(2, MOI.ScalarAffineTerm(1.0, x1)),
        MOI.VectorAffineTerm(2, MOI.ScalarAffineTerm(1.0, x2)),
    ],
    [0.0, 0.0],
)
s = MOI.Indicator{MOI.ACTIVATE_ON_ONE}(MOI.LessThan(9.0))
MOI.add_constraint(model, f, s)
```

`MathOptInterface.Complements` – Type.

```
Complements(dimension::Base.Integer)
```

The set corresponding to a mixed complementarity constraint.

Complementarity constraints should be specified with an `AbstractVectorFunction`-in-`Complements(dimension)` constraint.

The dimension of the vector-valued function `F` must be `dimension`. This defines a complementarity constraint between the scalar function `F[i]` and the variable in `F[i + dimension/2]`. Thus, `F[i + dimension/2]` must be interpretable as a single variable `x_i` (e.g., `1.0 * x + 0.0`), and `dimension` must be even.

The mixed complementarity problem consists of finding `x_i` in the interval `[lb, ub]` (i.e., in the set `Interval(lb, ub)`), such that the following holds:

1. `F_i(x) == 0` if `lb_i < x_i < ub_i`
2. `F_i(x) >= 0` if `lb_i == x_i`
3. `F_i(x) <= 0` if `x_i == ub_i`

Classically, the bounding set for `x_i` is `Interval(0, Inf)`, which recovers: `0 <= F_i(x) ⊥ x_i >= 0`, where the ⊥ operator implies `F_i(x) * x_i = 0`.

Examples

The problem:

```
x -in- Interval(-1, 1)
[-4 * x - 3, x] -in- Complements(2)
```

defines the mixed complementarity problem where the following holds:

1. `-4 * x - 3 == 0` if `-1 < x < 1`
2. `-4 * x - 3 >= 0` if `x == -1`
3. `-4 * x - 3 <= 0` if `x == 1`

There are three solutions:

1. `x = -3/4` with `F(x) = 0`
2. `x = -1` with `F(x) = 1`
3. `x = 1` with `F(x) = -7`

The function `F` can also be defined in terms of single variables. For example, the problem:

```
[x_3, x_4] -in- Nonnegatives(2)
[x_1, x_2, x_3, x_4] -in- Complements(4)
```

defines the complementarity problem where `0 <= x_1 ⊥ x_3 >= 0` and `0 <= x_2 ⊥ x_4 >= 0`.

Matrix sets

Matrix sets are vectorized in order to be subtypes of `AbstractVectorSet`.

For sets of symmetric matrices, storing both the `(i, j)` and `(j, i)` elements is redundant. Use the `AbstractSymmetricMatrixSe` set to represent only the vectorization of the upper triangular part of the matrix.

When the matrix of expressions constrained to be in the set is not symmetric, and hence additional constraints are needed to force the equality of the `(i, j)` and `(j, i)` elements, use the `AbstractSymmetricMatrixSetSquare` set.

The `Bridges.Constraint.SquareBridge` can transform a set from the square form to the `triangular_form` by adding appropriate constraints if the `(i, j)` and `(j, i)` expressions are different.

`MathOptInterface.AbstractSymmetricMatrixSetTriangle` – Type.

```
abstract type AbstractSymmetricMatrixSetTriangle <: AbstractVectorSet end
```

Abstract supertype for subsets of the (vectorized) cone of symmetric matrices, with `side_dimension` rows and columns. The entries of the upper-right triangular part of the matrix are given column by column (or equivalently, the entries of the lower-left triangular part are given row by row). A vectorized cone of `dimension` n corresponds to a square matrix with side dimension $\sqrt{1/4 + 2n} - 1/2$. (Because a $d \times d$ matrix has $d(d+1)/2$ elements in the upper or lower triangle.)

Examples

The matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 5 & 6 \end{bmatrix}$$

has `side_dimension` 3 and vectorization $(1, 2, 3, 4, 5, 6)$.

Note

Two packed storage formats exist for symmetric matrices, the respective orders of the entries are:

- upper triangular column by column (or lower triangular row by row);
- lower triangular column by column (or upper triangular row by row).

The advantage of the first format is the mapping between the `(i, j)` matrix indices and the `k` index of the vectorized form. It is simpler and does not depend on the side dimension of the matrix. Indeed,

- the entry of matrix indices `(i, j)` has vectorized index `k = div((j - 1) * j, 2) + i` if $i \leq j$ and `k = div((i - 1) * i, 2) + j` if $j \leq i$;
- and the entry with vectorized index `k` has matrix indices `i = div(1 + isqrt(8k - 7), 2)` and `j = k - div((i - 1) * i, 2)` or `j = div(1 + isqrt(8k - 7), 2)` and `i = k - div((j - 1) * j, 2)`.

Duality note

The scalar product for the symmetric matrix in its vectorized form is the sum of the pairwise product of the diagonal entries plus twice the sum of the pairwise product of the upper diagonal entries; see [p. 634, 1]. This has important consequence for duality.

Consider for example the following problem (`PositiveSemidefiniteConeTriangle` is a subtype of `AbstractSymmetricMatri`

$$\begin{aligned} \max_{x \in \mathbb{R}} \quad & x \\ \text{s.t.} \quad & (1, -x, 1) \in \text{PositiveSemidefiniteConeTriangle}(2). \end{aligned}$$

The dual is the following problem

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & y_1 + y_3 \\ \text{s.t.} \quad & 2y_2 = 1 \\ & y \in \text{PositiveSemidefiniteConeTriangle}(2). \end{aligned}$$

Why do we use $2y_2$ in the dual constraint instead of y_2 ? The reason is that $2y_2$ is the scalar product between y and the symmetric matrix whose vectorized form is $(0, 1, 0)$. Indeed, with our modified scalar products we have

$$\langle (0,1,0), (y_1, y_2, y_3) \rangle = \operatorname{trace} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \end{pmatrix} = 2y_2.$$

References

[1] Boyd, S. and Vandenberghe, L.. Convex optimization. Cambridge university press, 2004.

MathOptInterface.AbstractSymmetricMatrixSetSquare – Type.

```
abstract type AbstractSymmetricMatrixSetSquare <: AbstractVectorSet end
```

Abstract supertype for subsets of the (vectorized) cone of symmetric matrices, with side_dimension rows and columns. The entries of the matrix are given column by column (or equivalently, row by row). The matrix is both constrained to be symmetric and to have its triangular_form belong to the corresponding set. That is, if the functions in entries (i, j) and (j, i) are different, then a constraint will be added to make sure that the entries are equal.

Examples

PositiveSemidefiniteConeSquare is a subtype of AbstractSymmetricMatrixSetSquare and constraining the matrix

$$\begin{bmatrix} 1 & -y \\ -z & 0 \end{bmatrix}$$

to be symmetric positive semidefinite can be achieved by constraining the vector $(1, -z, -y, 0)$ (or $(1, -y, -z, 0)$) to belong to the PositiveSemidefiniteConeSquare(2). It both constrains $y = z$ and $(1, -y, 0)$ (or $(1, -z, 0)$) to be in PositiveSemidefiniteConeTriangle(2), since triangular_form(PositiveSemidefiniteConeSquare) is PositiveSemidefiniteConeTriangle.

MathOptInterface.side_dimension – Function.

```
side_dimension(set::Union{AbstractSymmetricMatrixSetTriangle,
                          AbstractSymmetricMatrixSetSquare})
```

Side dimension of the matrices in set. By convention, it should be stored in the side_dimension field but if it is not the case for a subtype of AbstractSymmetricMatrixSetTriangle, the method should be implemented for this subtype.

MathOptInterface.triangular_form – Function.

```
triangular_form(S::Type{<:AbstractSymmetricMatrixSetSquare})
triangular_form(set::AbstractSymmetricMatrixSetSquare)
```

Return the AbstractSymmetricMatrixSetTriangle corresponding to the vectorization of the upper triangular part of matrices in the AbstractSymmetricMatrixSetSquare set.

List of recognized matrix sets.

MathOptInterface.PositiveSemidefiniteConeTriangle – Type.

```
PositiveSemidefiniteConeTriangle(side_dimension) <: AbstractSymmetricMatrixSetTriangle
```

The (vectorized) cone of symmetric positive semidefinite matrices, with `side_dimension` rows and columns.

See AbstractSymmetricMatrixSetTriangle for more details on the vectorized form.

MathOptInterface.PositiveSemidefiniteConeSquare – Type.

```
PositiveSemidefiniteConeSquare(side_dimension) <: AbstractSymmetricMatrixSetSquare
```

The cone of symmetric positive semidefinite matrices, with side length `side_dimension`.

See AbstractSymmetricMatrixSetSquare for more details on the vectorized form.

The entries of the matrix are given column by column (or equivalently, row by row).

The matrix is both constrained to be symmetric and to be positive semidefinite. That is, if the functions in entries (i, j) and (j, i) are different, then a constraint will be added to make sure that the entries are equal.

Examples

Constraining the matrix

$$\begin{bmatrix} 1 & -y \\ -z & 0 \end{bmatrix}$$

to be symmetric positive semidefinite can be achieved by constraining the vector $(1, -z, -y, 0)$ (or $(1, -y, -z, 0)$) to belong to the `PositiveSemidefiniteConeSquare(2)`.

It both constrains $y = z$ and $(1, -y, 0)$ (or $(1, -z, 0)$) to be in `PositiveSemidefiniteConeTriangle(2)`.

MathOptInterface.LogDetConeTriangle – Type.

```
LogDetConeTriangle(side_dimension)
```

The log-determinant cone $\{(t, u, X) \in \mathbb{R}^{2+d(d+1)/2} : t \le u \log(\det(X/u)), u > 0\}$, where the matrix X is represented in the same symmetric packed format as in the `PositiveSemidefiniteConeTriangle`.

The argument `side_dimension` is the side dimension of the matrix X, i.e., its number of rows or columns.

MathOptInterface.LogDetConeSquare – Type.

```
LogDetConeSquare(side_dimension)
```

The log-determinant cone $\{(t, u, X) \in \mathbb{R}^{2+d^2} : t \le u \log(\det(X/u)), X \text{ symmetric}, u > 0\}$, where the matrix X is represented in the same format as in the PositiveSemidefiniteConeSquare.

Similarly to PositiveSemidefiniteConeSquare, constraints are added to ensure that X is symmetric.

The argument `side_dimension` is the side dimension of the matrix X, i.e., its number of rows or columns.

MathOptInterface.RootDetConeTriangle – Type.

```
RootDetConeTriangle(side_dimension)
```

The root-determinant cone $\{(t, X) \in \mathbb{R}^{1+d(d+1)/2} : t \le \det(X)^{1/d}\}$, where the matrix X is represented in the same symmetric packed format as in the `PositiveSemidefiniteConeTriangle`.

The argument `side_dimension` is the side dimension of the matrix X, i.e., its number of rows or columns.

`MathOptInterface.RootDetConeSquare` – Type.

```
RootDetConeSquare(side_dimension)
```

The root-determinant cone $\{(t, X) \in \mathbb{R}^{1+d^2} : t \le \det(X)^{1/d}, X \text{ symmetric}\}$, where the matrix X is represented in the same format as `PositiveSemidefiniteConeSquare`.

Similarly to `PositiveSemidefiniteConeSquare`, constraints are added to ensure that X is symmetric.

The argument `side_dimension` is the side dimension of the matrix X, i.e., its number of rows or columns.

39.2 Models

Attribute interface

`MathOptInterface.is_set_by_optimize` – Function.

```
is_set_by_optimize(::AnyAttribute)
```

Return a `Bool` indicating whether the value of the attribute is modified during an `optimize!` call, that is, the attribute is used to query the result of the optimization.

Important note when defining new attributes

This function returns `false` by default so it should be implemented for attributes that are modified by `optimize!`.

`MathOptInterface.is_copyable` – Function.

```
is_copyable(::AnyAttribute)
```

Return a `Bool` indicating whether the value of the attribute may be copied during `copy_to` using `set`.

Important note when defining new attributes

By default `is_copyable(attr)` returns `!is_set_by_optimize(attr)`. A specific method should be defined for attributes which are copied indirectly during `copy_to`. For instance, both `is_copyable` and `is_set_by_optimize` return `false` for the following attributes:

- `ListOfOptimizerAttributesSet`, `ListOfModelAttributesSet`, `ListOfConstraintAttributesSet` and `ListOfVariableAttributesSet`.
- `SolverName` and `RawSolver`: these attributes cannot be set.
- `NumberOfVariables` and `ListOfVariableIndices`: these attributes are set indirectly by `add_variable` and `add_variables`.
- `ObjectiveFunctionType`: this attribute is set indirectly when setting the `ObjectiveFunction` attribute.
- `NumberOfConstraints`, `ListOfConstraintIndices`, `ListOfConstraintTypesPresent`, `CanonicalConstraintFunctio` `ConstraintFunction` and `ConstraintSet`: these attributes are set indirectly by `add_constraint` and `add_constraints`.

`MathOptInterface.get` – Function.

```
get(optimizer::AbstractOptimizer, attr::AbstractOptimizerAttribute)
```

Return an attribute `attr` of the optimizer `optimizer`.

```
get(model::ModelLike, attr::AbstractModelAttribute)
```

Return an attribute `attr` of the model `model`.

```
get(model::ModelLike, attr::AbstractVariableAttribute, v::VariableIndex)
```

If the attribute `attr` is set for the variable v in the model `model`, return its value, return `nothing` otherwise. If the attribute `attr` is not supported by `model` then an error should be thrown instead of returning `nothing`.

```
get(model::ModelLike, attr::AbstractVariableAttribute, v::Vector{VariableIndex})
```

Return a vector of attributes corresponding to each variable in the collection `v` in the model `model`.

```
get(model::ModelLike, attr::AbstractConstraintAttribute, c::ConstraintIndex)
```

If the attribute `attr` is set for the constraint c in the model `model`, return its value, return `nothing` otherwise. If the attribute `attr` is not supported by `model` then an error should be thrown instead of returning `nothing`.

```
get(model::ModelLike, attr::AbstractConstraintAttribute, c::Vector{ConstraintIndex{F,S}})
```

Return a vector of attributes corresponding to each constraint in the collection `c` in the model `model`.

```
get(model::ModelLike, ::Type{VariableIndex}, name::String)
```

If a variable with name `name` exists in the model `model`, return the corresponding index, otherwise return `nothing`. Errors if two variables have the same name.

```
get(model::ModelLike, ::Type{ConstraintIndex{F,S}}, name::String) where {F<:AbstractFunction,S<:
    AbstractSet}
```

If an F-in-S constraint with name `name` exists in the model `model`, return the corresponding index, otherwise return `nothing`. Errors if two constraints have the same name.

```
get(model::ModelLike, ::Type{ConstraintIndex}, name::String)
```

If any constraint with name `name` exists in the model `model`, return the corresponding index, otherwise return `nothing`. This version is available for convenience but may incur a performance penalty because it is not type stable. Errors if two constraints have the same name.

Examples

```
get(model, ObjectiveValue())
get(model, VariablePrimal(), ref)
get(model, VariablePrimal(5), [ref1, ref2])
get(model, OtherAttribute("something specific to cplex"))
get(model, VariableIndex, "var1")
get(model, ConstraintIndex{ScalarAffineFunction{Float64},LessThan{Float64}}, "con1")
get(model, ConstraintIndex, "con1")
```

`MathOptInterface.get!` – Function.

```
get!(output, model::ModelLike, args...)
```

An in-place version of get.

The signature matches that of get except that the the result is placed in the vector `output`.

`MathOptInterface.set` – Function.

```
set(optimizer::AbstractOptimizer, attr::AbstractOptimizerAttribute, value)
```

Assign `value` to the attribute `attr` of the optimizer `optimizer`.

```
set(model::ModelLike, attr::AbstractModelAttribute, value)
```

Assign `value` to the attribute `attr` of the model `model`.

```
set(model::ModelLike, attr::AbstractVariableAttribute, v::VariableIndex, value)
```

Assign `value` to the attribute `attr` of variable `v` in model `model`.

```
set(model::ModelLike, attr::AbstractVariableAttribute, v::Vector{VariableIndex}, vector_of_values
    )
```

Assign a value respectively to the attribute `attr` of each variable in the collection `v` in model `model`.

```
set(model::ModelLike, attr::AbstractConstraintAttribute, c::ConstraintIndex, value)
```

Assign a value to the attribute `attr` of constraint `c` in model `model`.

```
set(model::ModelLike, attr::AbstractConstraintAttribute, c::Vector{ConstraintIndex{F,S}},
    vector_of_values)
```

Assign a value respectively to the attribute `attr` of each constraint in the collection `c` in model `model`.

An `UnsupportedAttribute` error is thrown if `model` does not support the attribute `attr` (see `supports`) and a `SetAttributeNotAllowed` error is thrown if it supports the attribute `attr` but it cannot be set.

Replace set in a constraint

```
set(model::ModelLike, ::ConstraintSet, c::ConstraintIndex{F,S}, set::S)
```

Change the set of constraint `c` to the new set `set` which should be of the same type as the original set.

Examples

If `c` is a `ConstraintIndex{F,Interval}`

```
set(model, ConstraintSet(), c, Interval(0, 5))
set(model, ConstraintSet(), c, GreaterThan(0.0))  # Error
```

Replace function in a constraint

```
set(model::ModelLike, ::ConstraintFunction, c::ConstraintIndex{F,S}, func::F)
```

Replace the function in constraint `c` with `func`. `F` must match the original function type used to define the constraint.

Note

Setting the constraint function is not allowed if `F` is `VariableIndex`, it throws a `SettingVariableIndexNotAllowed` error. Indeed, it would require changing the index `c` as the index of `VariableIndex` constraints should be the same as the index of the variable.

Examples

If `c` is a `ConstraintIndex{ScalarAffineFunction,S}` and `v1` and `v2` are `VariableIndex` objects,

```
set(model, ConstraintFunction(), c,
    ScalarAffineFunction(ScalarAffineTerm.([1.0, 2.0], [v1, v2]), 5.0))
set(model, ConstraintFunction(), c, v1) # Error
```

`MathOptInterface.supports` – Function.

```
supports(model::ModelLike, sub::AbstractSubmittable)::Bool
```

Return a `Bool` indicating whether `model` supports the submittable `sub`.

```
supports(model::ModelLike, attr::AbstractOptimizerAttribute)::Bool
```

Return a `Bool` indicating whether `model` supports the optimizer attribute `attr`. That is, it returns `false` if `copy_to(model, src)` shows a warning in case `attr` is in the `ListOfOptimizerAttributesSet` of `src`; see `copy_to` for more details on how unsupported optimizer attributes are handled in copy.

```
supports(model::ModelLike, attr::AbstractModelAttribute)::Bool
```

Return a `Bool` indicating whether `model` supports the model attribute `attr`. That is, it returns `false` if `copy_to(model, src)` cannot be performed in case `attr` is in the `ListOfModelAttributesSet` of `src`.

```
supports(model::ModelLike, attr::AbstractVariableAttribute, ::Type{VariableIndex})::Bool
```

Return a `Bool` indicating whether `model` supports the variable attribute `attr`. That is, it returns `false` if `copy_to(model, src)` cannot be performed in case `attr` is in the `ListOfVariableAttributesSet` of `src`.

```
supports(model::ModelLike, attr::AbstractConstraintAttribute, ::Type{ConstraintIndex{F,S}})::Bool
     where {F,S}
```

Return a `Bool` indicating whether `model` supports the constraint attribute `attr` applied to an `F`-in-`S` constraint. That is, it returns `false` if `copy_to(model, src)` cannot be performed in case `attr` is in the `ListOfConstraintAttributesSet` of `src`.

For all five methods, if the attribute is only not supported in specific circumstances, it should still return `true`.

Note that `supports` is only defined for attributes for which `is_copyable` returns `true` as other attributes do not appear in the list of attributes set obtained by `ListOf...AttributesSet`.

`MathOptInterface.attribute_value_type` – Function.

```
attribute_value_type(attr::AnyAttribute)
```

Given an attribute `attr`, return the type of value expected by `get`, or returned by `set`.

Notes

- Only implement this if it make sense to do so. If un-implemented, the default is `Any`.

Model interface

`MathOptInterface.ModelLike` – Type.

```
ModelLike
```

Abstract supertype for objects that implement the "Model" interface for defining an optimization problem.

`MathOptInterface.is_empty` – Function.

```
is_empty(model::ModelLike)
```

Returns `false` if the `model` has any model attribute set or has any variables or constraints.

Note that an empty model can have optimizer attributes set.

`MathOptInterface.empty!` – Function.

```
empty!(model::ModelLike)
```

Empty the model, that is, remove all variables, constraints and model attributes but not optimizer attributes.

`MathOptInterface.write_to_file` – Function.

```
write_to_file(model::ModelLike, filename::String)
```

Writes the current model data to the given file. Supported file types depend on the model type.

`MathOptInterface.read_from_file` – Function.

```
read_from_file(model::ModelLike, filename::String)
```

Read the file `filename` into the model `model`. If `model` is non-empty, this may throw an error.

Supported file types depend on the model type.

Note

Once the contents of the file are loaded into the model, users can query the variables via `get(model, ListOfVariableIndices())`. However, some filetypes, such as LP files, do not maintain an explicit ordering of the variables. Therefore, the returned list may be in an arbitrary order. To avoid depending on the order of the indices, users should look up each variable index by name: `get(model, VariableIndex, "name")`.

`MathOptInterface.supports_incremental_interface` – Function.

```
supports_incremental_interface(model::ModelLike)
```

Return a `Bool` indicating whether `model` supports building incrementally via `add_variable` and `add_constraint`.

The main purpose of this function is to determine whether a model can be loaded into `model` incrementally or whether it should be cached and copied at once instead.

`MathOptInterface.copy_to` – Function.

```
copy_to(dest::ModelLike, src::ModelLike)::IndexMap
```

Copy the model from `src` into `dest`.

The target `dest` is emptied, and all previous indices to variables and constraints in `dest` are invalidated.

Returns an `IndexMap` object that translates variable and constraint indices from the `src` model to the corresponding indices in the `dest` model.

Notes

- If a constraint that in `src` is not supported by `dest`, then an `UnsupportedConstraint` error is thrown.
- If an `AbstractModelAttribute`, `AbstractVariableAttribute`, or `AbstractConstraintAttribute` is set in `src` but not supported by `dest`, then an `UnsupportedAttribute` error is thrown.

AbstractOptimizerAttributes are not copied to the dest model.

IndexMap

Implementations of copy_to must return an IndexMap. For technical reasons, this type is defined in the Utilities submodule as MOI.Utilities.IndexMap. However, since it is an integral part of the MOI API, we provide MOI.IndexMap as an alias.

Example

```
# Given empty `ModelLike` objects `src` and `dest`.

x = add_variable(src)

is_valid(src, x)   # true
is_valid(dest, x)  # false (`dest` has no variables)

index_map = copy_to(dest, src)
is_valid(dest, x) # false (unless index_map[x] == x)
is_valid(dest, index_map[x]) # true
```

MathOptInterface.IndexMap – Type.

```
IndexMap()
```

The dictionary-like object returned by copy_to.

IndexMap

Implementations of copy_to must return an IndexMap. For technical reasons, the IndexMap type is defined in the Utilties submodule as MOI.Utilities.IndexMap. However, since it is an integral part of the MOI API, we provide this MOI.IndexMap as an alias.

Model attributes

MathOptInterface.AbstractModelAttribute – Type.

```
AbstractModelAttribute
```

Abstract supertype for attribute objects that can be used to set or get attributes (properties) of the model.

MathOptInterface.Name – Type.

```
Name()
```

A model attribute for the string identifying the model. It has a default value of "" if not set‘.

MathOptInterface.ObjectiveFunction – Type.

```
ObjectiveFunction{F<:AbstractScalarFunction}()
```

A model attribute for the objective function which has a type F<:AbstractScalarFunction. F should be guaranteed to be equivalent but not necessarily identical to the function type provided by the user. Throws an InexactError if the objective function cannot be converted to F, e.g. the objective function is quadratic and F is ScalarAffineFunction{Float64} or it has non-integer coefficient and F is ScalarAffineFunction{Int}.

MathOptInterface.ObjectiveFunctionType – Type.

```
ObjectiveFunctionType()
```

A model attribute for the type F of the objective function set using the `ObjectiveFunction{F}` attribute.

Examples

In the following code, `attr` should be equal to `MOI.VariableIndex`:

```
x = MOI.add_variable(model)
MOI.set(model, MOI.ObjectiveFunction{MOI.VariableIndex}(),
         x)
attr = MOI.get(model, MOI.ObjectiveFunctionType())
```

`MathOptInterface.ObjectiveSense` – Type.

```
ObjectiveSense()
```

A model attribute for the objective sense of the objective function, which must be an `OptimizationSense`: `MIN_SENSE`, `MAX_SENSE`, or `FEASIBILITY_SENSE`. The default is `FEASIBILITY_SENSE`.

`MathOptInterface.NumberOfVariables` – Type.

```
NumberOfVariables()
```

A model attribute for the number of variables in the model.

`MathOptInterface.ListOfVariableIndices` – Type.

```
ListOfVariableIndices()
```

A model attribute for the `Vector{VariableIndex}` of all variable indices present in the model (i.e., of length equal to the value of `NumberOfVariables()`) in the order in which they were added.

`MathOptInterface.ListOfConstraintTypesPresent` – Type.

```
ListOfConstraintTypesPresent()
```

A model attribute for the list of tuples of the form `(F,S)`, where F is a function type and S is a set type indicating that the attribute `NumberOfConstraints{F,S}()` has value greater than zero.

`MathOptInterface.NumberOfConstraints` – Type.

```
NumberOfConstraints{F,S}()
```

A model attribute for the number of constraints of the type F-in-S present in the model.

`MathOptInterface.ListOfConstraintIndices` – Type.

```
ListOfConstraintIndices{F,S}()
```

A model attribute for the `Vector{ConstraintIndex{F,S}}` of all constraint indices of type F-in-S in the model (i.e., of length equal to the value of `NumberOfConstraints{F,S}()`) in the order in which they were added.

`MathOptInterface.ListOfOptimizerAttributesSet` – Type.

```
ListOfOptimizerAttributesSet()
```

An optimizer attribute for the `Vector{AbstractOptimizerAttribute}` of all optimizer attributes that were set.

`MathOptInterface.ListOfModelAttributesSet` – Type.

```
ListOfModelAttributesSet()
```

A model attribute for the `Vector{AbstractModelAttribute}` of all model attributes `attr` such that 1) `is_copyable(attr)` returns `true` and 2) the attribute was set to the model.

`MathOptInterface.ListOfVariableAttributesSet` – Type.

```
ListOfVariableAttributesSet()
```

A model attribute for the `Vector{AbstractVariableAttribute}` of all variable attributes `attr` such that 1) `is_copyable(attr)` returns `true` and 2) the attribute was set to variables.

`MathOptInterface.ListOfConstraintAttributesSet` – Type.

```
ListOfConstraintAttributesSet{F, S}()
```

A model attribute for the `Vector{AbstractConstraintAttribute}` of all constraint attributes `attr` such that 1) `is_copyable(attr)` returns `true` and

2. the attribute was set to F-in-S constraints.

Note

The attributes `ConstraintFunction` and `ConstraintSet` should not be included in the list even if then have been set with `set`.

Optimizer interface

`MathOptInterface.AbstractOptimizer` – Type.

```
AbstractOptimizer <: ModelLike
```

Abstract supertype for objects representing an instance of an optimization problem tied to a particular solver. This is typically a solver's in-memory representation. In addition to `ModelLike`, `AbstractOptimizer` objects let you solve the model and query the solution.

`MathOptInterface.OptimizerWithAttributes` – Type.

```
struct OptimizerWithAttributes
    optimizer_constructor
    params::Vector{Pair{AbstractOptimizerAttribute,<:Any}}
end
```

Object grouping an optimizer constructor and a list of optimizer attributes. Instances are created with `instantiate`.

`MathOptInterface.optimize!` – Function.

```
optimize!(optimizer::AbstractOptimizer)
```

Optimize the problem contained in `optimizer`.

Before calling `optimize!`, the problem should first be constructed using the incremental interface (see `supports_incremental_interface`) or `copy_to`.

MathOptInterface.instantiate – Function.

```
instantiate(
    optimizer_constructor,
    with_bridge_type::Union{Nothing, Type} = nothing,
)
```

Creates an instance of optimizer by either:

- calling optimizer_constructor.optimizer_constructor() and setting the parameters in optimizer_constructor.p if optimizer_constructor is a OptimizerWithAttributes
- calling optimizer_constructor() if optimizer_constructor is callable.

If with_bridge_type is not nothing, it enables all the bridges defined in the MathOptInterface.Bridges submodule with coefficient type with_bridge_type.

If the optimizer created by optimizer_constructor does not support loading the problem incrementally (see supports_incremental_interface), then a Utilities.CachingOptimizer is added to store a cache of the bridged model.

MathOptInterface.default_cache – Function.

```
default_cache(optimizer::ModelLike, ::Type{T}) where {T}
```

Return a new instance of the default model type to be used as cache for optimizer in a Utilities.CachingOptimizer for holding constraints of coefficient type T. By default, this returns Utilities.UniversalFallback(Utilities.Model{T}()) If copying from a instance of a given model type is faster for optimizer then a new method returning an instance of this model type should be defined.

Optimizer attributes

MathOptInterface.AbstractOptimizerAttribute – Type.

```
AbstractOptimizerAttribute
```

Abstract supertype for attribute objects that can be used to set or get attributes (properties) of the optimizer.

Note

The difference between AbstractOptimizerAttribute and AbstractModelAttribute lies in the behavior of is_empty, empty! and copy_to. Typically optimizer attributes only affect how the model is solved.

MathOptInterface.SolverName – Type.

```
SolverName()
```

An optimizer attribute for the string identifying the solver/optimizer.

MathOptInterface.SolverVersion – Type.

```
SolverVersion()
```

An optimizer attribute for the string identifying the version of the solver.

Note

For solvers supporting semantic versioning, the `SolverVersion` should be a string of the form "vMAJOR.MINOR.PATCH", so that it can be converted to a Julia `VersionNumber` (e.g., 'VersionNumber("v1.2.3")).

We do not require Semantic Versioning because some solvers use alternate versioning systems. For example, CPLEX uses Calendar Versioning, so `SolverVersion` will return a string like `"202001"`.

`MathOptInterface.Silent` – Type.

```
Silent()
```

An optimizer attribute for silencing the output of an optimizer. When `set` to `true`, it takes precedence over any other attribute controlling verbosity and requires the solver to produce no output. The default value is `false` which has no effect. In this case the verbosity is controlled by other attributes.

Note

Every optimizer should have verbosity on by default. For instance, if a solver has a solver-specific log level attribute, the MOI implementation should set it to `1` by default. If the user sets `Silent` to `true`, then the log level should be set to `0`, even if the user specifically sets a value of log level. If the value of `Silent` is `false` then the log level set to the solver is the value given by the user for this solver-specific parameter or `1` if none is given.

`MathOptInterface.TimeLimitSec` – Type.

```
TimeLimitSec()
```

An optimizer attribute for setting a time limit for an optimization. When `set` to `nothing`, it deactivates the solver time limit. The default value is `nothing`. The time limit is in seconds.

`MathOptInterface.RawOptimizerAttribute` – Type.

```
RawOptimizerAttribute(name::String)
```

An optimizer attribute for the solver-specific parameter identified by `name`.

`MathOptInterface.NumberOfThreads` – Type.

```
NumberOfThreads()
```

An optimizer attribute for setting the number of threads used for an optimization. When set to `nothing` uses solver default. Values are positive integers. The default value is `nothing`.

`MathOptInterface.RawSolver` – Type.

```
RawSolver()
```

A model attribute for the object that may be used to access a solver-specific API for this optimizer.

List of attributes useful for optimizers

`MathOptInterface.TerminationStatus` – Type.

```
TerminationStatus()
```

A model attribute for the `TerminationStatusCode` explaining why the optimizer stopped.

`MathOptInterface.TerminationStatusCode` – Type.

```
TerminationStatusCode
```

An Enum of possible values for the `TerminationStatus` attribute. This attribute is meant to explain the reason why the optimizer stopped executing in the most recent call to `optimize!`.

If no call has been made to `optimize!`, then the `TerminationStatus` is:

- `OPTIMIZE_NOT_CALLED`: The algorithm has not started.

OK

These are generally OK statuses, i.e., the algorithm ran to completion normally.

- `OPTIMAL`: The algorithm found a globally optimal solution.
- `INFEASIBLE`: The algorithm concluded that no feasible solution exists.
- `DUAL_INFEASIBLE`: The algorithm concluded that no dual bound exists for the problem. If, additionally, a feasible (primal) solution is known to exist, this status typically implies that the problem is unbounded, with some technical exceptions.
- `LOCALLY_SOLVED`: The algorithm converged to a stationary point, local optimal solution, could not find directions for improvement, or otherwise completed its search without global guarantees.
- `LOCALLY_INFEASIBLE`: The algorithm converged to an infeasible point or otherwise completed its search without finding a feasible solution, without guarantees that no feasible solution exists.
- `INFEASIBLE_OR_UNBOUNDED`: The algorithm stopped because it decided that the problem is infeasible or unbounded; this occasionally happens during MIP presolve.

Solved to relaxed tolerances

- `ALMOST_OPTIMAL`: The algorithm found a globally optimal solution to relaxed tolerances.
- `ALMOST_INFEASIBLE`: The algorithm concluded that no feasible solution exists within relaxed tolerances.
- `ALMOST_DUAL_INFEASIBLE`: The algorithm concluded that no dual bound exists for the problem within relaxed tolerances.
- `ALMOST_LOCALLY_SOLVED`: The algorithm converged to a stationary point, local optimal solution, or could not find directions for improvement within relaxed tolerances.

Limits

The optimizer stopped because of some user-defined limit.

- `ITERATION_LIMIT`: An iterative algorithm stopped after conducting the maximum number of iterations.
- `TIME_LIMIT`: The algorithm stopped after a user-specified computation time.
- `NODE_LIMIT`: A branch-and-bound algorithm stopped because it explored a maximum number of nodes in the branch-and-bound tree.
- `SOLUTION_LIMIT`: The algorithm stopped because it found the required number of solutions. This is often used in MIPs to get the solver to return the first feasible solution it encounters.
- `MEMORY_LIMIT`: The algorithm stopped because it ran out of memory.

- OBJECTIVE_LIMIT: The algorithm stopped because it found a solution better than a minimum limit set by the user.
- NORM_LIMIT: The algorithm stopped because the norm of an iterate became too large.
- OTHER_LIMIT: The algorithm stopped due to a limit not covered by one of the above.

Problematic

This group of statuses means that something unexpected or problematic happened.

- SLOW_PROGRESS: The algorithm stopped because it was unable to continue making progress towards the solution.
- NUMERICAL_ERROR: The algorithm stopped because it encountered unrecoverable numerical error.
- INVALID_MODEL: The algorithm stopped because the model is invalid.
- INVALID_OPTION: The algorithm stopped because it was provided an invalid option.
- INTERRUPTED: The algorithm stopped because of an interrupt signal.
- OTHER_ERROR: The algorithm stopped because of an error not covered by one of the statuses defined above.

MathOptInterface.PrimalStatus – Type.

```
PrimalStatus(result_index::Int = 1)
```

A model attribute for the ResultStatusCode of the primal result result_index. If result_index is omitted, it defaults to 1.

See ResultCount for information on how the results are ordered.

If result_index is larger than the value of ResultCount then NO_SOLUTION is returned.

MathOptInterface.DualStatus – Type.

```
DualStatus(result_index::Int = 1)
```

A model attribute for the ResultStatusCode of the dual result result_index. If result_index is omitted, it defaults to 1.

See ResultCount for information on how the results are ordered.

If result_index is larger than the value of ResultCount then NO_SOLUTION is returned.

MathOptInterface.ResultStatusCode – Type.

```
ResultStatusCode
```

An Enum of possible values for the PrimalStatus and DualStatus attributes. The values indicate how to interpret the result vector.

- NO_SOLUTION: the result vector is empty.
- FEASIBLE_POINT: the result vector is a feasible point.
- NEARLY_FEASIBLE_POINT: the result vector is feasible if some constraint tolerances are relaxed.
- INFEASIBLE_POINT: the result vector is an infeasible point.

- INFEASIBILITY_CERTIFICATE: the result vector is an infeasibility certificate. If the PrimalStatus is INFEASIBILITY_CERTIFICATE, then the primal result vector is a certificate of dual infeasibility. If the DualStatus is INFEASIBILITY_CERTIFICATE, then the dual result vector is a proof of primal infeasibility.
- NEARLY_INFEASIBILITY_CERTIFICATE: the result satisfies a relaxed criterion for a certificate of infeasibility.
- REDUCTION_CERTIFICATE: the result vector is an ill-posed certificate; see this article for details. If the PrimalStatus is REDUCTION_CERTIFICATE, then the primal result vector is a proof that the dual problem is ill-posed. If the DualStatus is REDUCTION_CERTIFICATE, then the dual result vector is a proof that the primal is ill-posed.
- NEARLY_REDUCTION_CERTIFICATE: the result satisfies a relaxed criterion for an ill-posed certificate.
- UNKNOWN_RESULT_STATUS: the result vector contains a solution with an unknown interpretation.
- OTHER_RESULT_STATUS: the result vector contains a solution with an interpretation not covered by one of the statuses defined above.

MathOptInterface.RawStatusString – Type.

```
RawStatusString()
```

A model attribute for a solver specific string explaining why the optimizer stopped.

MathOptInterface.ResultCount – Type.

```
ResultCount()
```

A model attribute for the number of results available.

Order of solutions

A number of attributes contain an index, result_index, which is used to refer to one of the available results. Thus, result_index must be an integer between 1 and the number of available results.

As a general rule, the first result (result_index=1) is the most important result (e.g., an optimal solution or an infeasibility certificate). Other results will typically be alternate solutions that the solver found during the search for the first result.

If a (local) optimal solution is available, i.e., TerminationStatus is OPTIMAL or LOCALLY_SOLVED, the first result must correspond to the (locally) optimal solution. Other results may be alternative optimal solutions, or they may be other suboptimal solutions; use ObjectiveValue to distingiush between them.

If a primal or dual infeasibility certificate is available, i.e., TerminationStatus is INFEASIBLE or DUAL_INFEASIBLE and the corresponding PrimalStatus or DualStatus is INFEASIBILITY_CERTIFICATE, then the first result must be a certificate. Other results may be alternate certificates, or infeasible points.

MathOptInterface.ObjectiveValue – Type.

```
ObjectiveValue(result_index::Int = 1)
```

A model attribute for the objective value of the primal solution result_index.

If the solver does not have a primal value for the objective because the result_index is beyond the available solutions (whose number is indicated by the ResultCount attribute), getting this attribute must throw a ResultIndexBoundsError. Otherwise, if the result is unavailable for another reason (for instance, only a dual solution is available), the result is undefined. Users should first check PrimalStatus before accessing the ObjectiveValue attribute.

See ResultCount for information on how the results are ordered.

MathOptInterface.DualObjectiveValue – Type.

```
DualObjectiveValue(result_index::Int = 1)
```

A model attribute for the value of the objective function of the dual problem for the `result_index`th dual result.

If the solver does not have a dual value for the objective because the `result_index` is beyond the available solutions (whose number is indicated by the ResultCount attribute), getting this attribute must throw a ResultIndexBoundsError. Otherwise, if the result is unavailable for another reason (for instance, only a primal solution is available), the result is undefined. Users should first check DualStatus before accessing the `DualObjectiveValue` attribute.

See ResultCount for information on how the results are ordered.

MathOptInterface.ObjectiveBound – Type.

```
ObjectiveBound()
```

A model attribute for the best known bound on the optimal objective value.

MathOptInterface.RelativeGap – Type.

```
RelativeGap()
```

A model attribute for the final relative optimality gap.

> **Warning**
> The definition of this gap is solver-dependent. However, most solvers implementing this attribute define the relative gap as some variation of $\frac{|b-f|}{|f|}$, where b is the best bound and f is the best feasible objective value.

MathOptInterface.SolveTimeSec – Type.

```
SolveTimeSec()
```

A model attribute for the total elapsed solution time (in seconds) as reported by the optimizer.

MathOptInterface.SimplexIterations – Type.

```
SimplexIterations()
```

A model attribute for the cumulative number of simplex iterations during the optimization process. In particular, for a mixed-integer program (MIP), the total simplex iterations for all nodes.

MathOptInterface.BarrierIterations – Type.

```
BarrierIterations()
```

A model attribute for the cumulative number of barrier iterations while solving a problem.

MathOptInterface.NodeCount – Type.

```
NodeCount()
```

A model attribute for the total number of branch-and-bound nodes explored while solving a mixed-integer program (MIP).

Conflict Status

MathOptInterface.compute_conflict! – Function.

```
compute_conflict!(optimizer::AbstractOptimizer)
```

Computes a minimal subset of constraints such that the model with the other constraint removed is still infeasible.

Some solvers call a set of conflicting constraints an Irreducible Inconsistent Subsystem (IIS).

See also ConflictStatus and ConstraintConflictStatus.

Note

If the model is modified after a call to `compute_conflict!`, the implementor is not obliged to purge the conflict. Any calls to the above attributes may return values for the original conflict without a warning. Similarly, when modifying the model, the conflict can be discarded.

MathOptInterface.ConflictStatus – Type.

```
ConflictStatus()
```

A model attribute for the ConflictStatusCode explaining why the conflict refiner stopped when computing the conflict.

MathOptInterface.ConflictStatusCode – Type.

```
ConflictStatusCode
```

An Enum of possible values for the `ConflictStatus` attribute. This attribute is meant to explain the reason why the conflict finder stopped executing in the most recent call to compute_conflict!.

Possible values are:

- `COMPUTE_CONFLICT_NOT_CALLED`: the function compute_conflict! has not yet been called
- `NO_CONFLICT_EXISTS`: there is no conflict because the problem is feasible
- `NO_CONFLICT_FOUND`: the solver could not find a conflict
- `CONFLICT_FOUND`: at least one conflict could be found

MathOptInterface.ConstraintConflictStatus – Type.

```
ConstraintConflictStatus()
```

A constraint attribute indicating whether the constraint participates in the conflict. Its type is ConflictParticipationStatusC

MathOptInterface.ConflictParticipationStatusCode – Type.

```
ConflictParticipationStatusCode
```

An Enum of possible values for the ConstraintConflictStatus attribute. This attribute is meant to indicate whether a given constraint participates or not in the last computed conflict.

Possible values are:

- `NOT_IN_CONFLICT`: the constraint does not participate in the conflict
- `IN_CONFLICT`: the constraint participates in the conflict
- `MAYBE_IN_CONFLICT`: the constraint may participate in the conflict, the solver was not able to prove that the constraint can be excluded from the conflict

39.3 Variables

Functions

MathOptInterface.add_variable – Function.

```
add_variable(model::ModelLike)::VariableIndex
```

Add a scalar variable to the model, returning a variable index.

A AddVariableNotAllowed error is thrown if adding variables cannot be done in the current state of the model model.

MathOptInterface.add_variables – Function.

```
add_variables(model::ModelLike, n::Int)::Vector{VariableIndex}
```

Add n scalar variables to the model, returning a vector of variable indices.

A AddVariableNotAllowed error is thrown if adding variables cannot be done in the current state of the model model.

MathOptInterface.add_constrained_variable – Function.

```
add_constrained_variable(
    model::ModelLike,
    set::AbstractScalarSet
)::Tuple{MOI.VariableIndex,
         MOI.ConstraintIndex{MOI.VariableIndex, typeof(set)}}
```

Add to model a scalar variable constrained to belong to set, returning the index of the variable created and the index of the constraint constraining the variable to belong to set.

By default, this function falls back to creating a free variable with add_variable and then constraining it to belong to set with add_constraint.

MathOptInterface.add_constrained_variables – Function.

```
add_constrained_variables(
    model::ModelLike,
    sets::AbstractVector{<:AbstractScalarSet}
)::Tuple{
    Vector{MOI.VariableIndex},
    Vector{MOI.ConstraintIndex{MOI.VariableIndex,eltype(sets)}},
}
```

Add to model scalar variables constrained to belong to sets, returning the indices of the variables created and the indices of the constraints constraining the variables to belong to each set in sets. That is, if it returns variables and constraints, constraints[i] is the index of the constraint constraining variable[i] to belong to sets[i].

By default, this function falls back to calling add_constrained_variable on each set.

```
add_constrained_variables(
    model::ModelLike,
    set::AbstractVectorSet,
)::Tuple{
    Vector{MOI.VariableIndex},
    MOI.ConstraintIndex{MOI.VectorOfVariables,typeof(set)},
}
```

Add to model a vector of variables constrained to belong to set, returning the indices of the variables created and the index of the constraint constraining the vector of variables to belong to set.

By default, this function falls back to creating free variables with add_variables and then constraining it to belong to set with add_constraint.

MathOptInterface.supports_add_constrained_variable – Function.

```
supports_add_constrained_variable(
    model::ModelLike,
    S::Type{<:AbstractScalarSet}
)::Bool
```

Return a Bool indicating whether model supports constraining a variable to belong to a set of type S either on creation of the variable with add_constrained_variable or after the variable is created with add_constraint.

By default, this function falls back to supports_add_constrained_variables(model, Reals) && supports_constraint(mo
MOI.VariableIndex, S) which is the correct definition for most models.

Example

Suppose that a solver supports only two kind of variables: binary variables and continuous variables with a lower bound. If the solver decides not to support VariableIndex-in-Binary and VariableIndex-in-GreaterThan constraints, it only has to implement add_constrained_variable for these two sets which prevents the user to add both a binary constraint and a lower bound on the same variable. Moreover, if the user adds a VariableIndex-in-GreaterThan constraint, implementing this interface (i.e., supports_add_constrained_varia
enables the constraint to be transparently bridged into a supported constraint.

MathOptInterface.supports_add_constrained_variables – Function.

```
supports_add_constrained_variables(
    model::ModelLike,
    S::Type{<:AbstractVectorSet}
)::Bool
```

Return a Bool indicating whether model supports constraining a vector of variables to belong to a set of type S either on creation of the vector of variables with add_constrained_variables or after the variable is created with add_constraint.

By default, if S is Reals then this function returns true and otherwise, it falls back to supports_add_constrained_variables(
Reals) && supports_constraint(model, MOI.VectorOfVariables, S) which is the correct definition for most models.

Example

In the standard conic form (see Duality), the variables are grouped into several cones and the constraints are affine equality constraints. If Reals is not one of the cones supported by the solvers then it needs to implement supports_add_constrained_variables(::Optimizer, ::Type{Reals}) = false as free variables are not supported. The solvers should then implement supports_add_constrained_variables(::Optimizer, ::Type{<:SupportedCones}) = true where SupportedCones is the union of all cone types that are supported; it does not have to implement the method supports_constraint(::Type{VectorOfVariables}, Type{<:SupportedCones}) as it should return false and it's the default. This prevents the user to constrain the same variable in two different cones. When a VectorOfVariables-in-S is added, the variables of the vector have already been created so they already belong to given cones. If bridges are enabled, the constraint will therefore be bridged by adding slack variables in S and equality constraints ensuring that the slack variables are equal to the corresponding variables of the given constraint function.

Note that there may also be sets for which `!supports_add_constrained_variables(model, S)` and `supports_constraint(model, MOI.VectorOfVariables, S)`. For instance, suppose a solver supports positive semidefinite variable constraints and two types of variables: binary variables and nonnegative variables. Then the solver should support adding `VectorOfVariables`-in-`PositiveSemidefiniteConeTriangle` constraints, but it should not support creating variables constrained to belong to the `PositiveSemidefiniteConeTriangle` because the variables in `PositiveSemidefiniteConeTriangle` should first be created as either binary or non-negative.

`MathOptInterface.is_valid` – Method.

```
is_valid(model::ModelLike, index::Index)::Bool
```

Return a `Bool` indicating whether this index refers to a valid object in the model `model`.

`MathOptInterface.delete` – Method.

```
delete(model::ModelLike, index::Index)
```

Delete the referenced object from the model. Throw `DeleteNotAllowed` if if `index` cannot be deleted.

The following modifications also take effect if `Index` is `VariableIndex`:

- If `index` used in the objective function, it is removed from the function, i.e., it is substituted for zero.
- For each `func`-in-`set` constraint of the model:
  - If `func isa VariableIndex` and `func == index` then the constraint is deleted.
  - If `func isa VectorOfVariables` and `index in func.variables` then
    * if `length(func.variables) == 1` is one, the constraint is deleted;
    * if `length(func.variables) > 1` and `supports_dimension_update(set)` then then the variable is removed from `func` and `set` is replaced by `update_dimension(set, MOI.dimension(set) - 1)`.
    * Otherwise, a `DeleteNotAllowed` error is thrown.
  - Otherwise, the variable is removed from `func`, i.e., it is substituted for zero.

`MathOptInterface.delete` – Method.

```
delete(model::ModelLike, indices::Vector{R<:Index}) where {R}
```

Delete the referenced objects in the vector `indices` from the model. It may be assumed that `R` is a concrete type. The default fallback sequentially deletes the individual items in `indices`, although specialized implementations may be more efficient.

Attributes

`MathOptInterface.AbstractVariableAttribute` – Type.

```
AbstractVariableAttribute
```

Abstract supertype for attribute objects that can be used to set or get attributes (properties) of variables in the model.

`MathOptInterface.VariableName` – Type.

```
VariableName()
```

A variable attribute for a string identifying the variable. It is valid for two variables to have the same name; however, variables with duplicate names cannot be looked up using `get`. It has a default value of "" if not set'.

`MathOptInterface.VariablePrimalStart` – Type.

```
VariablePrimalStart()
```

A variable attribute for the initial assignment to some primal variable's value that the optimizer may use to warm-start the solve. May be a number or `nothing` (unset).

`MathOptInterface.VariablePrimal` – Type.

```
VariablePrimal(result_index::Int = 1)
```

A variable attribute for the assignment to some primal variable's value in result `result_index`. If `result_index` is omitted, it is 1 by default.

If the solver does not have a primal value for the variable because the `result_index` is beyond the available solutions (whose number is indicated by the `ResultCount` attribute), getting this attribute must throw a `ResultIndexBoundsError`. Otherwise, if the result is unavailable for another reason (for instance, only a dual solution is available), the result is undefined. Users should first check `PrimalStatus` before accessing the `VariablePrimal` attribute.

See `ResultCount` for information on how the results are ordered.

`MathOptInterface.VariableBasisStatus` – Type.

```
VariableBasisStatus(result_index::Int = 1)
```

A variable attribute for the `BasisStatusCode` of a variable in result `result_index`, with respect to an available optimal solution basis.

If the solver does not have a basis statue for the variable because the `result_index` is beyond the available solutions (whose number is indicated by the `ResultCount` attribute), getting this attribute must throw a `ResultIndexBoundsError`. Otherwise, if the result is unavailable for another reason (for instance, only a dual solution is available), the result is undefined. Users should first check `PrimalStatus` before accessing the `VariableBasisStatus` attribute.

See `ResultCount` for information on how the results are ordered.

39.4 Constraints

Types

`MathOptInterface.ConstraintIndex` – Type.

```
ConstraintIndex{F, S}
```

A type-safe wrapper for `Int64` for use in referencing F-in-S constraints in a model. The parameter F is the type of the function in the constraint, and the parameter S is the type of set in the constraint. To allow for deletion, indices need not be consecutive. Indices within a constraint type (i.e. F-in-S) must be unique, but non-unique indices across different constraint types are allowed. If F is `VariableIndex` then the index is equal to the index of the variable. That is for an `index::ConstraintIndex{VariableIndex}`, we always have

```
index.value == MOI.get(model, MOI.ConstraintFunction(), index).value
```

Functions

MathOptInterface.is_valid – Method.

```
is_valid(model::ModelLike, index::Index)::Bool
```

Return a Bool indicating whether this index refers to a valid object in the model model.

MathOptInterface.add_constraint – Function.

```
add_constraint(model::ModelLike, func::F, set::S)::ConstraintIndex{F,S} where {F,S}
```

Add the constraint $f(x) \in \mathcal{S}$ where f is defined by func, and $\mathcal{S}$ is defined by set.

```
add_constraint(model::ModelLike, v::VariableIndex, set::S)::ConstraintIndex{VariableIndex,S}
    where {S}
add_constraint(model::ModelLike, vec::Vector{VariableIndex}, set::S)::ConstraintIndex{
    VectorOfVariables,S} where {S}
```

Add the constraint $v \in \mathcal{S}$ where v is the variable (or vector of variables) referenced by v and $\mathcal{S}$ is defined by set.

- An UnsupportedConstraint error is thrown if model does not support F-in-S constraints,
- a AddConstraintNotAllowed error is thrown if it supports F-in-S constraints but it cannot add the constraint(s) in its current state and
- a ScalarFunctionConstantNotZero error may be thrown if func is an AbstractScalarFunction with nonzero constant and set is EqualTo, GreaterThan, LessThan or Interval.
- a LowerBoundAlreadySet error is thrown if F is a VariableIndex and a constraint was already added to this variable that sets a lower bound.
- a UpperBoundAlreadySet error is thrown if F is a VariableIndex and a constraint was already added to this variable that sets an upper bound.

MathOptInterface.add_constraints – Function.

```
add_constraints(model::ModelLike, funcs::Vector{F},
↪  sets::Vector{S})::Vector{ConstraintIndex{F,S}} where {F,S}
```

Add the set of constraints specified by each function-set pair in funcs and sets. F and S should be concrete types. This call is equivalent to add_constraint.(model, funcs, sets) but may be more efficient.

MathOptInterface.transform – Function.

Transform Constraint Set

```
transform(model::ModelLike, c::ConstraintIndex{F,S1}, newset::S2)::ConstraintIndex{F,S2}
```

Replace the set in constraint c with newset. The constraint index c will no longer be valid, and the function returns a new constraint index with the correct type.

Solvers may only support a subset of constraint transforms that they perform efficiently (for example, changing from a LessThan to GreaterThan set). In addition, set modification (where S1 = S2) should be performed via the modify function.

Typically, the user should delete the constraint and add a new one.

Examples

If c is a ConstraintIndex{ScalarAffineFunction{Float64},LessThan{Float64}},

```
c2 = transform(model, c, GreaterThan(0.0))
transform(model, c, LessThan(0.0)) # errors
```

MathOptInterface.supports_constraint – Function.

```
MOI.supports_constraint(
    BT::Type{<:AbstractBridge},
    F::Type{<:MOI.AbstractFunction},
    S::Type{<:MOI.AbstractSet},
)::Bool
```

Return a `Bool` indicating whether the bridges of type BT support bridging F-in-S constraints.

```
supports_constraint(
    model::ModelLike,
    ::Type{F},
    ::Type{S},
)::Bool where {F<:AbstractFunction,S<:AbstractSet}
```

Return a `Bool` indicating whether `model` supports F-in-S constraints, that is, `copy_to(model, src)` does not throw `UnsupportedConstraint` when `src` contains F-in-S constraints. If F-in-S constraints are only not supported in specific circumstances, e.g. F-in-S constraints cannot be combined with another type of constraint, it should still return `true`.

Attributes

MathOptInterface.AbstractConstraintAttribute – Type.

```
AbstractConstraintAttribute
```

Abstract supertype for attribute objects that can be used to set or get attributes (properties) of constraints in the model.

MathOptInterface.ConstraintName – Type.

```
ConstraintName()
```

A constraint attribute for a string identifying the constraint.

It is valid for constraints variables to have the same name; however, constraints with duplicate names cannot be looked up using `get`, regardless of whether they have the same F-in-S type.

`ConstraintName` has a default value of `""` if not set.

Notes

You should not implement `ConstraintName` for `VariableIndex` constraints.

MathOptInterface.ConstraintPrimalStart – Type.

```
ConstraintPrimalStart()
```

A constraint attribute for the initial assignment to some constraint's `ConstraintPrimal` that the optimizer may use to warm-start the solve.

May be `nothing` (unset), a number for `AbstractScalarFunction`, or a vector for `AbstractVectorFunction`.

MathOptInterface.ConstraintDualStart – Type.

```
ConstraintDualStart()
```

A constraint attribute for the initial assignment to some constraint's ConstraintDual that the optimizer may use to warm-start the solve.

May be `nothing` (unset), a number for AbstractScalarFunction, or a vector for AbstractVectorFunction.

MathOptInterface.ConstraintPrimal – Type.

```
ConstraintPrimal(result_index::Int = 1)
```

A constraint attribute for the assignment to some constraint's primal value(s) in result `result_index`.

If the constraint is `f(x) in S`, then in most cases the `ConstraintPrimal` is the value of `f`, evaluated at the correspondng VariablePrimal solution.

However, some conic solvers reformulate `b - Ax in S` to `s = b - Ax, s in S`. These solvers may return the value of `s` for `ConstraintPrimal`, rather than `b - Ax`. (Although these are constrained by an equality constraint, due to numerical tolerances they may not be identical.)

If the solver does not have a primal value for the constraint because the `result_index` is beyond the available solutions (whose number is indicated by the ResultCount attribute), getting this attribute must throw a ResultIndexBoundsError. Otherwise, if the result is unavailable for another reason (for instance, only a dual solution is available), the result is undefined. Users should first check PrimalStatus before accessing the `ConstraintPrimal` attribute.

If `result_index` is omitted, it is 1 by default. See ResultCount for information on how the results are ordered.

MathOptInterface.ConstraintDual – Type.

```
ConstraintDual(result_index::Int = 1)
```

A constraint attribute for the assignment to some constraint's dual value(s) in result `result_index`. If `result_index` is omitted, it is 1 by default.

If the solver does not have a dual value for the variable because the `result_index` is beyond the available solutions (whose number is indicated by the ResultCount attribute), getting this attribute must throw a ResultIndexBoundsError. Otherwise, if the result is unavailable for another reason (for instance, only a primal solution is available), the result is undefined. Users should first check DualStatus before accessing the `ConstraintDual` attribute.

See ResultCount for information on how the results are ordered.

MathOptInterface.ConstraintBasisStatus – Type.

```
ConstraintBasisStatus(result_index::Int = 1)
```

A constraint attribute for the `BasisStatusCode` of some constraint in result `result_index`, with respect to an available optimal solution basis. If `result_index` is omitted, it is 1 by default.

If the solver does not have a basis statue for the constraint because the `result_index` is beyond the available solutions (whose number is indicated by the ResultCount attribute), getting this attribute must throw a ResultIndexBoundsError. Otherwise, if the result is unavailable for another reason (for instance, only a dual solution is available), the result is undefined. Users should first check PrimalStatus before accessing the `ConstraintBasisStatus` attribute.

See ResultCount for information on how the results are ordered.

Notes

For the basis status of a variable, query `VariableBasisStatus`.

`ConstraintBasisStatus` does not apply to `VariableIndex` constraints. You can infer the basis status of a `VariableIndex` constraint by looking at the result of `VariableBasisStatus`.

`MathOptInterface.BasisStatusCode` – Type.

```
BasisStatusCode
```

An Enum of possible values for the `ConstraintBasisStatus` and `VariableBasisStatus` attributes, explaining the status of a given element with respect to an optimal solution basis.

Possible values are:

- `BASIC`: element is in the basis
- `NONBASIC`: element is not in the basis
- `NONBASIC_AT_LOWER`: element is not in the basis and is at its lower bound
- `NONBASIC_AT_UPPER`: element is not in the basis and is at its upper bound
- `SUPER_BASIC`: element is not in the basis but is also not at one of its bounds

Notes

- `NONBASIC_AT_LOWER` and `NONBASIC_AT_UPPER` should be used only for constraints with the `Interval` set. In this case, they are necessary to distinguish which side of the constraint is active. One-sided constraints (e.g., `LessThan` and `GreaterThan`) should use `NONBASIC` instead of the `NONBASIC_AT_*` values. This restriction does not apply to `VariableBasisStatus`, which should return `NONBASIC_AT_*` regardless of whether the alternative bound exists.
- In linear programs, `SUPER_BASIC` occurs when a variable with no bounds is not in the basis.

`MathOptInterface.ConstraintFunction` – Type.

```
ConstraintFunction()
```

A constraint attribute for the `AbstractFunction` object used to define the constraint. It is guaranteed to be equivalent but not necessarily identical to the function provided by the user.

`MathOptInterface.CanonicalConstraintFunction` – Type.

```
CanonicalConstraintFunction()
```

A constraint attribute for a canonical representation of the `AbstractFunction` object used to define the constraint. Getting this attribute is guaranteed to return a function that is equivalent but not necessarily identical to the function provided by the user.

By default, `MOI.get(model, MOI.CanonicalConstraintFunction(), ci)` fallbacks to `MOI.Utilities.canonical(MOI.get(model, MOI.ConstraintFunction(), ci))`. However, if `model` knows that the constraint function is canonical then it can implement a specialized method that directly return the function without calling `Utilities.canonical`. Therefore, the value returned **cannot** be assumed to be a copy of the function stored in `model`. Moreover, `Utilities.Model` checks with `Utilities.is_canonical` whether the function stored internally is already canonical and if it's the case, then it returns the function stored internally instead of a copy.

`MathOptInterface.ConstraintSet` – Type.

```
ConstraintSet()
```

A constraint attribute for the `AbstractSet` object used to define the constraint.

39.5 Modifications

MathOptInterface.modify – Function.

Constraint Function

```
modify(model::ModelLike, ci::ConstraintIndex, change::AbstractFunctionModification)
```

Apply the modification specified by change to the function of constraint ci.

An ModifyConstraintNotAllowed error is thrown if modifying constraints is not supported by the model model.

Examples

```
modify(model, ci, ScalarConstantChange(10.0))
```

Objective Function

```
modify(model::ModelLike, ::ObjectiveFunction, change::AbstractFunctionModification)
```

Apply the modification specified by change to the objective function of model. To change the function completely, call set instead.

An ModifyObjectiveNotAllowed error is thrown if modifying objectives is not supported by the model model.

Examples

```
modify(model, ObjectiveFunction{ScalarAffineFunction{Float64}}(), ScalarConstantChange(10.0))
```

MathOptInterface.AbstractFunctionModification – Type.

```
AbstractFunctionModification
```

An abstract supertype for structs which specify partial modifications to functions, to be used for making small modifications instead of replacing the functions entirely.

MathOptInterface.ScalarConstantChange – Type.

```
ScalarConstantChange{T}(new_constant::T)
```

A struct used to request a change in the constant term of a scalar-valued function. Applicable to ScalarAffineFunction and ScalarQuadraticFunction.

MathOptInterface.VectorConstantChange – Type.

```
VectorConstantChange{T}(new_constant::Vector{T})
```

A struct used to request a change in the constant vector of a vector-valued function. Applicable to VectorAffineFunction and VectorQuadraticFunction.

MathOptInterface.ScalarCoefficientChange – Type.

```
ScalarCoefficientChange{T}(variable::VariableIndex, new_coefficient::T)
```

A struct used to request a change in the linear coefficient of a single variable in a scalar-valued function. Applicable to ScalarAffineFunction and ScalarQuadraticFunction.

MathOptInterface.MultirowChange – Type.

```
MultirowChange{T}(variable::VariableIndex, new_coefficients::Vector{Tuple{Int64, T}})
```

A struct used to request a change in the linear coefficients of a single variable in a vector-valued function. New coefficients are specified by `(output_index, coefficient)` tuples. Applicable to `VectorAffineFunction` and `VectorQuadraticFunction`.

39.6 Nonlinear programming

Types

MathOptInterface.AbstractNLPEvaluator – Type.

```
AbstractNLPEvaluator
```

Abstract supertype for the callback object that is used to query function values, derivatives, and expression graphs. It is used in `NLPBlock`.

MathOptInterface.NLPBoundsPair – Type.

```
NLPBoundsPair(lower,upper)
```

A struct holding a pair of lower and upper bounds. `-Inf` and `Inf` can be used to indicate no lower or upper bound, respectively.

MathOptInterface.NLPBlockData – Type.

```
struct NLPBlockData
    constraint_bounds::Vector{NLPBoundsPair}
    evaluator::AbstractNLPEvaluator
    has_objective::Bool
end
```

A `struct` encoding a set of nonlinear constraints of the form $lb \leq g(x) \leq ub$ and, if `has_objective == true`, a nonlinear objective function $f(x)$. `constraint_bounds` holds the pairs of lb and ub elements. Nonlinear objectives override any objective set by using the `ObjectiveFunction` attribute. The `evaluator` is a callback object that is used to query function values, derivatives, and expression graphs. If `has_objective == false`, then it is an error to query properties of the objective function, and in Hessian-of-the-Lagrangian queries, σ must be set to zero.

> **Note**
>
> Throughout the evaluator, all variables are ordered according to ListOfVariableIndices. Hence, MOI copies of nonlinear problems should be done with attention.

Attributes

MathOptInterface.NLPBlock – Type.

```
NLPBlock()
```

Holds the `NLPBlockData` that represents a set of nonlinear constraints, and optionally a nonlinear objective.

MathOptInterface.NLPBlockDual – Type.

```
NLPBlockDual(result_index::Int)
NLPBlockDual()
```

The Lagrange multipliers on the constraints from the NLPBlock in result result_index. If result_index is omitted, it is 1 by default.

MathOptInterface.NLPBlockDualStart – Type.

```
NLPBlockDualStart()
```

An initial assignment of the Lagrange multipliers on the constraints from the NLPBlock that the solver may use to warm-start the solve.

Functions

MathOptInterface.initialize – Function.

```
initialize(d::AbstractNLPEvaluator, requested_features::Vector{Symbol})
```

Must be called before any other methods. The vector requested_features lists features requested by the solver. These may include :Grad for gradients of the obejctive, f, :Jac for explicit Jacobians of constraints, g, :JacVec for Jacobian-vector products, :HessVec for Hessian-vector and Hessian-of-Lagrangian-vector products, :Hess for explicit Hessians and Hessian-of-Lagrangians, and :ExprGraph for expression graphs.

MathOptInterface.features_available – Function.

```
features_available(d::AbstractNLPEvaluator)
```

Returns the subset of features available for this problem instance, as a vector of symbols in the same format as in initialize.

MathOptInterface.eval_objective – Function.

```
eval_objective(d::AbstractNLPEvaluator, x)
```

Evaluate the objective $f(x)$, returning a scalar value.

MathOptInterface.eval_constraint – Function.

```
eval_constraint(d::AbstractNLPEvaluator, g, x)
```

Evaluate the constraint function $g(x)$, storing the result in the vector g which must be of the appropriate size.

MathOptInterface.eval_objective_gradient – Function.

```
eval_objective_gradient(d::AbstractNLPEvaluator, df, x)
```

Evaluate $\nabla f(x)$ as a dense vector, storing the result in the vector df which must be of the appropriate size.

MathOptInterface.jacobian_structure – Function.

```
jacobian_structure(d::AbstractNLPEvaluator)::Vector{Tuple{Int64,Int64}}
```

Returns the sparsity structure of the Jacobian matrix $J_g(x) = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}$ where g_i is the ith component of g. The sparsity structure is assumed to be independent of the point x. Returns a vector of tuples, `(row, column)`, where each indicates the position of a structurally nonzero element. These indices are not required to be sorted and can contain duplicates, in which case the solver should combine the corresponding elements by adding them together.

`MathOptInterface.hessian_lagrangian_structure` – Function.

```
hessian_lagrangian_structure(d::AbstractNLPEvaluator)::Vector{Tuple{Int64,Int64}}
```

Returns the sparsity structure of the Hessian-of-the-Lagrangian matrix $\nabla^2 f + \sum_{i=1}^m \nabla^2 g_i$ as a vector of tuples, where each indicates the position of a structurally nonzero element. These indices are not required to be sorted and can contain duplicates, in which case the solver should combine the corresponding elements by adding them together. Any mix of lower and upper-triangular indices is valid. Elements `(i,j)` and `(j,i)`, if both present, should be treated as duplicates.

`MathOptInterface.eval_constraint_jacobian` – Function.

```
eval_constraint_jacobian(d::AbstractNLPEvaluator, J, x)
```

Evaluates the sparse Jacobian matrix $J_g(x) = \begin{bmatrix} \nabla g_1(x) \\ \nabla g_2(x) \\ \vdots \\ \nabla g_m(x) \end{bmatrix}$. The result is stored in the vector `J` in the same order as the indices returned by `jacobian_structure`.

`MathOptInterface.eval_constraint_jacobian_product` – Function.

```
eval_constraint_jacobian_product(d::AbstractNLPEvaluator, y, x, w)
```

Computes the Jacobian-vector product $J_g(x)w$, storing the result in the vector `y`.

`MathOptInterface.eval_constraint_jacobian_transpose_product` – Function.

```
eval_constraint_jacobian_transpose_product(d::AbstractNLPEvaluator, y, x, w)
```

Computes the Jacobian-transpose-vector product $J_g(x)^T w$, storing the result in the vector `y`.

`MathOptInterface.eval_hessian_lagrangian` – Function.

```
eval_hessian_lagrangian(d::AbstractNLPEvaluator, H, x, σ, μ)
```

Given scalar weight `σ` and vector of constraint weights `μ`, computes the sparse Hessian-of-the-Lagrangian matrix $\sigma\nabla^2 f(x) + \sum_{i=1}^m \mu_i \nabla^2 g_i(x)$, storing the result in the vector `H` in the same order as the indices returned by `hessian_lagrangian_structure`.

`MathOptInterface.eval_hessian_lagrangian_product` – Function.

```
eval_hessian_lagrangian_product(d::AbstractNLPEvaluator, h, x, v, σ, μ)
```

Given scalar weight `σ` and vector of constraint weights `μ`, computes the Hessian-of-the-Lagrangian-vector product $\left(\sigma\nabla^2 f(x) + \sum_{i=1}^m \mu_i \nabla^2 g_i(x)\right) v$, storing the result in the vector `h`.

MathOptInterface.objective_expr – Function.

```
objective_expr(d::AbstractNLPEvaluator)
```

Returns an expression graph for the objective function as a standard Julia `Expr` object. All sums and products are flattened out as simple `Expr(:+,...)` and `Expr(:*,...)` objects. The symbol `x` is used as a placeholder for the vector of decision variables. No other undefined symbols are permitted; coefficients are embedded as explicit values. For example, the expression $x_1 + \sin(x_2/\exp(x_3))$ would be represented as the Julia object `:(x[1] + sin(x[2]/exp(x[3])))`. Each integer index is wrapped in a VariableIndex. See the Julia manual for more information on the structure of `Expr` objects. There are currently no restrictions on recognized functions; typically these will be built-in Julia functions like `^`, `exp`, `log`, `cos`, `tan`, `sqrt`, etc., but modeling interfaces may choose to extend these basic functions.

MathOptInterface.constraint_expr – Function.

```
constraint_expr(d::AbstractNLPEvaluator, i)
```

Returns an expression graph for the ith constraint in the same format as described above, with an additional comparison operator indicating the sense of and bounds on the constraint. The right-hand side of the comparison must be a constant; that is, `:(x[1]^3 <= 1)` is allowed, while `:(1 <= x[1]^3)` is not valid. Double-sided constraints are allowed, in which case both the lower bound and upper bounds should be constants; for example, `:(-1 <= cos(x[1]) + sin(x[2]) <= 1)` is valid.

39.7 Callbacks

MathOptInterface.AbstractCallback – Type.

```
abstract type AbstractCallback <: AbstractModelAttribute end
```

Abstract type for a model attribute representing a callback function. The value set to subtypes of `AbstractCallback` is a function that may be called during optimize!. As optimize! is in progress, the result attributes (i.e, the attributes `attr` such that `is_set_by_optimize(attr)`) may not be accessible from the callback, hence trying to get result attributes might throw a OptimizeInProgress error.

At most one callback of each type can be registered. If an optimizer already has a function for a callback type, and the user registers a new function, then the old one is replaced.

The value of the attribute should be a function taking only one argument, commonly called `callback_data`, that can be used for instance in LazyConstraintCallback, HeuristicCallback and UserCutCallback.

MathOptInterface.AbstractSubmittable – Type.

```
AbstractSubmittable
```

Abstract supertype for objects that can be submitted to the model.

MathOptInterface.submit – Function.

```
submit(optimizer::AbstractOptimizer, sub::AbstractSubmittable,
       values...)::Nothing
```

Submit `values` to the submittable `sub` of the optimizer `optimizer`.

An UnsupportedSubmittable error is thrown if `model` does not support the attribute `attr` (see supports) and a SubmitNotAllowed error is thrown if it supports the submittable `sub` but it cannot be submitted.

Attributes

MathOptInterface.CallbackNodeStatus – Type.

```
CallbackNodeStatus(callback_data)
```

An optimizer attribute describing the (in)feasibility of the primal solution available from CallbackVariablePrimal during a callback identified by callback_data.

Returns a CallbackNodeStatusCode Enum.

MathOptInterface.CallbackNodeStatusCode – Type.

```
CallbackNodeStatusCode
```

An Enum of possible return values from calling get with CallbackNodeStatus.

Possible values are:

- CALLBACK_NODE_STATUS_INTEGER: the primal solution available from CallbackVariablePrimal is integer feasible.
- CALLBACK_NODE_STATUS_FRACTIONAL: the primal solution available from CallbackVariablePrimal is integer infeasible.
- CALLBACK_NODE_STATUS_UNKNOWN: the primal solution available from CallbackVariablePrimal might be integer feasible or infeasible.

MathOptInterface.CallbackVariablePrimal – Type.

```
CallbackVariablePrimal(callback_data)
```

A variable attribute for the assignment to some primal variable's value during the callback identified by callback_data.

Lazy constraints

MathOptInterface.LazyConstraintCallback – Type.

```
LazyConstraintCallback() <: AbstractCallback
```

The callback can be used to reduce the feasible set given the current primal solution by submitting a LazyConstraint. For instance, it may be called at an incumbent of a mixed-integer problem. Note that there is no guarantee that the callback is called at every feasible primal solution.

The current primal solution is accessed through CallbackVariablePrimal. Trying to access other result attributes will throw OptimizeInProgress as discussed in AbstractCallback.

Examples

```
x = MOI.add_variables(optimizer, 8)
MOI.set(optimizer, MOI.LazyConstraintCallback(), callback_data -> begin
    sol = MOI.get(optimizer, MOI.CallbackVariablePrimal(callback_data), x)
    if # should add a lazy constraint
        func = # computes function
        set = # computes set
        MOI.submit(optimizer, MOI.LazyConstraint(callback_data), func, set)
    end
end)
```

`MathOptInterface.LazyConstraint` – Type.

```
LazyConstraint(callback_data)
```

Lazy constraint `func`-in-`set` submitted as `func, set`. The optimal solution returned by `VariablePrimal` will satisfy all lazy constraints that have been submitted.

This can be submitted only from the `LazyConstraintCallback`. The field `callback_data` is a solver-specific callback type that is passed as the argument to the feasible solution callback.

Examples

Suppose x and y are `VariableIndex`s of `optimizer`. To add a `LazyConstraint` for `2x + 3y <= 1`, write

```
func = 2.0x + 3.0y
set = MOI.LessThan(1.0)
MOI.submit(optimizer, MOI.LazyConstraint(callback_data), func, set)
```

inside a `LazyConstraintCallback` of data `callback_data`.

User cuts

`MathOptInterface.UserCutCallback` – Type.

```
UserCutCallback() <: AbstractCallback
```

The callback can be used to submit `UserCut` given the current primal solution. For instance, it may be called at fractional (i.e., non-integer) nodes in the branch and bound tree of a mixed-integer problem. Note that there is not guarantee that the callback is called everytime the solver has an infeasible solution.

The infeasible solution is accessed through `CallbackVariablePrimal`. Trying to access other result attributes will throw `OptimizeInProgress` as discussed in `AbstractCallback`.

Examples

```
x = MOI.add_variables(optimizer, 8)
MOI.set(optimizer, MOI.UserCutCallback(), callback_data -> begin
    sol = MOI.get(optimizer, MOI.CallbackVariablePrimal(callback_data), x)
    if # can find a user cut
        func = # computes function
        set = # computes set
        MOI.submit(optimizer, MOI.UserCut(callback_data), func, set)
    end
end
```

`MathOptInterface.UserCut` – Type.

```
UserCut(callback_data)
```

Constraint `func`-to-`set` suggested to help the solver detect the solution given by `CallbackVariablePrimal` as infeasible. The cut is submitted as `func, set`. Typically `CallbackVariablePrimal` will violate integrality constraints, and a cut would be of the form `ScalarAffineFunction`-in-`LessThan` or `ScalarAffineFunction`-in-`GreaterThan`. Note that, as opposed to `LazyConstraint`, the provided constraint cannot modify the feasible set, the constraint should be redundant, e.g., it may be a consequence of affine and integrality constraints.

This can be submitted only from the `UserCutCallback`. The field `callback_data` is a solver-specific callback type that is passed as the argument to the infeasible solution callback.

Note that the solver may silently ignore the provided constraint.

Heuristic solutions

MathOptInterface.HeuristicCallback – Type.

```
HeuristicCallback() <: AbstractCallback
```

The callback can be used to submit HeuristicSolution given the current primal solution. For instance, it may be called at fractional (i.e., non-integer) nodes in the branch and bound tree of a mixed-integer problem. Note that there is not guarantee that the callback is called everytime the solver has an infeasible solution.

The current primal solution is accessed through CallbackVariablePrimal. Trying to access other result attributes will throw OptimizeInProgress as discussed in AbstractCallback.

Examples

```
x = MOI.add_variables(optimizer, 8)
MOI.set(optimizer, MOI.HeuristicCallback(), callback_data -> begin
    sol = MOI.get(optimizer, MOI.CallbackVariablePrimal(callback_data), x)
    if # can find a heuristic solution
        values = # computes heuristic solution
        MOI.submit(optimizer, MOI.HeuristicSolution(callback_data), x,
                   values)
    end
end
```

MathOptInterface.HeuristicSolutionStatus – Type.

```
HeuristicSolutionStatus
```

An Enum of possible return values for submit with HeuristicSolution. This informs whether the heuristic solution was accepted or rejected. Possible values are:

- HEURISTIC_SOLUTION_ACCEPTED: The heuristic solution was accepted.
- HEURISTIC_SOLUTION_REJECTED: The heuristic solution was rejected.
- HEURISTIC_SOLUTION_UNKNOWN: No information available on the acceptance.

MathOptInterface.HeuristicSolution – Type.

```
HeuristicSolution(callback_data)
```

Heuristically obtained feasible solution. The solution is submitted as `variables, values` where `values[i]` gives the value of `variables[i]`, similarly to set. The submit call returns a HeuristicSolutionStatus indicating whether the provided solution was accepted or rejected.

This can be submitted only from the HeuristicCallback. The field `callback_data` is a solver-specific callback type that is passed as the argument to the heuristic callback.

Some solvers require a complete solution, others only partial solutions.

39.8 Errors

When an MOI call fails on a model, precise errors should be thrown when possible instead of simply calling `error` with a message. The docstrings for the respective methods describe the errors that the implementation should throw in certain situations. This error-reporting system allows code to distinguish between internal errors (that should be shown to the user) and unsupported operations which may have automatic workarounds.

When an invalid index is used in an MOI call, an InvalidIndex is thrown:

MathOptInterface.InvalidIndex – Type.

```
struct InvalidIndex{IndexType<:Index} <: Exception
    index::IndexType
end
```

An error indicating that the index index is invalid.

When an invalid result index is used to retrieve an attribute, a ResultIndexBoundsError is thrown:

MathOptInterface.ResultIndexBoundsError – Type.

```
struct ResultIndexBoundsError{AttrType} <: Exception
    attr::AttrType
    result_count::Int
end
```

An error indicating that the requested attribute `attr` could not be retrieved, because the solver returned too few results compared to what was requested. For instance, the user tries to retrieve `VariablePrimal(2)` when only one solution is available, or when the model is infeasible and has no solution.

See also: check_result_index_bounds.

MathOptInterface.check_result_index_bounds – Function.

```
check_result_index_bounds(model::ModelLike, attr)
```

This function checks whether enough results are available in the `model` for the requested `attr`, using its `result_index` field. If the model does not have sufficient results to answer the query, it throws a ResultIndexBoundsError.

As discussed in JuMP mapping, for scalar constraint with a nonzero function constant, a ScalarFunctionConstantNotZero exception may be thrown:

MathOptInterface.ScalarFunctionConstantNotZero – Type.

```
struct ScalarFunctionConstantNotZero{T, F, S} <: Exception
    constant::T
end
```

An error indicating that the constant part of the function in the constraint F-in-S is nonzero.

Some VariableIndex constraints cannot be combined on the same variable:

MathOptInterface.LowerBoundAlreadySet – Type.

```
LowerBoundAlreadySet{S1, S2}
```

Error thrown when setting a `VariableIndex`-in-S2 when a `VariableIndex`-in-S1 has already been added and the sets S1, S2 both set a lower bound, i.e. they are EqualTo, GreaterThan, Interval, Semicontinuous or Semiinteger.

MathOptInterface.UpperBoundAlreadySet – Type.

```
UpperBoundAlreadySet{S1, S2}
```

Error thrown when setting a `VariableIndex`-in-`S2` when a `VariableIndex`-in-`S1` has already been added and the sets `S1`, `S2` both set an upper bound, i.e. they are `EqualTo`, `LessThan`, `Interval`, `Semicontinuous` or `Semiinteger`.

As discussed in `AbstractCallback`, trying to `get` attributes inside a callback may throw:

`MathOptInterface.OptimizeInProgress` – Type.

```
struct OptimizeInProgress{AttrType<:AnyAttribute} <: Exception
    attr::AttrType
end
```

Error thrown from optimizer when `MOI.get(optimizer, attr)` is called inside an `AbstractCallback` while it is only defined once `optimize!` has completed. This can only happen when `is_set_by_optimize(attr)` is `true`.

Trying to submit the wrong type of `AbstractSubmittable` inside an `AbstractCallback` (e.g., a `UserCut` inside a `LazyConstraintCallback`) will throw:

`MathOptInterface.InvalidCallbackUsage` – Type.

```
struct InvalidCallbackUsage{C, S} <: Exception
    callback::C
    submittable::S
end
```

An error indicating that `submittable` cannot be submitted inside `callback`.

For example, `UserCut` cannot be submitted inside `LazyConstraintCallback`.

The rest of the errors defined in MOI fall in two categories represented by the following two abstract types:

`MathOptInterface.UnsupportedError` – Type.

```
UnsupportedError <: Exception
```

Abstract type for error thrown when an element is not supported by the model.

`MathOptInterface.NotAllowedError` – Type.

```
NotAllowedError <: Exception
```

Abstract type for error thrown when an operation is supported but cannot be applied in the current state of the model.

The different `UnsupportedError` and `NotAllowedError` are the following errors:

`MathOptInterface.UnsupportedAttribute` – Type.

```
struct UnsupportedAttribute{AttrType} <: UnsupportedError
    attr::AttrType
    message::String
end
```

An error indicating that the attribute `attr` is not supported by the model, i.e. that `supports` returns `false`.

`MathOptInterface.SetAttributeNotAllowed` – Type.

```
struct SetAttributeNotAllowed{AttrType} <: NotAllowedError
    attr::AttrType
    message::String # Human-friendly explanation why the attribute cannot be set
end
```

An error indicating that the attribute `attr` is supported (see `supports`) but cannot be set for some reason (see the error string).

`MathOptInterface.AddVariableNotAllowed` – Type.

```
struct AddVariableNotAllowed <: NotAllowedError
    message::String # Human-friendly explanation why the attribute cannot be set
end
```

An error indicating that variables cannot be added to the model.

`MathOptInterface.UnsupportedConstraint` – Type.

```
struct UnsupportedConstraint{F<:AbstractFunction, S<:AbstractSet} <: UnsupportedError
    message::String # Human-friendly explanation why the attribute cannot be set
end
```

An error indicating that constraints of type F-in-S are not supported by the model, i.e. that `supports_constraint` returns `false`.

`MathOptInterface.AddConstraintNotAllowed` – Type.

```
struct AddConstraintNotAllowed{F<:AbstractFunction, S<:AbstractSet} <: NotAllowedError
    message::String # Human-friendly explanation why the attribute cannot be set
end
```

An error indicating that constraints of type F-in-S are supported (see `supports_constraint`) but cannot be added.

`MathOptInterface.ModifyConstraintNotAllowed` – Type.

```
struct ModifyConstraintNotAllowed{F<:AbstractFunction, S<:AbstractSet,
                                  C<:AbstractFunctionModification} <: NotAllowedError
    constraint_index::ConstraintIndex{F, S}
    change::C
    message::String
end
```

An error indicating that the constraint modification `change` cannot be applied to the constraint of index `ci`.

`MathOptInterface.ModifyObjectiveNotAllowed` – Type.

```
struct ModifyObjectiveNotAllowed{C<:AbstractFunctionModification} <: NotAllowedError
    change::C
    message::String
end
```

An error indicating that the objective modification `change` cannot be applied to the objective.

`MathOptInterface.DeleteNotAllowed` – Type.

```
struct DeleteNotAllowed{IndexType <: Index} <: NotAllowedError
    index::IndexType
    message::String
end
```

An error indicating that the index index cannot be deleted.

MathOptInterface.UnsupportedSubmittable – Type.

```
struct UnsupportedSubmittable{SubmitType} <: UnsupportedError
    sub::SubmitType
    message::String
end
```

An error indicating that the submittable sub is not supported by the model, i.e. that supports returns false.

MathOptInterface.SubmitNotAllowed – Type.

```
struct SubmitNotAllowed{SubmitTyp<:AbstractSubmittable} <: NotAllowedError
    sub::SubmitType
    message::String # Human-friendly explanation why the attribute cannot be set
end
```

An error indicating that the submittable sub is supported (see supports) but cannot be added for some reason (see the error string).

Note that setting the ConstraintFunction of a VariableIndex constraint is not allowed:

MathOptInterface.SettingVariableIndexNotAllowed – Type.

```
SettingVariableIndexNotAllowed()
```

Error type that should be thrown when the user calls set to change the ConstraintFunction of a VariableIndex constraint.

Chapter 40

Submodules

40.1 Benchmarks

Overview

The `Benchmarks` submodule

To aid the development of efficient solver wrappers, MathOptInterface provides benchmarking functionality. Benchmarking a wrapper follows a two-step process.

First, prior to making changes, run and save the benchmark results on a given benchmark suite as follows:

```
using SolverPackage  # Replace with your choice of solver.

using MathOptInterface
const MOI = MathOptInterface

suite = MOI.Benchmarks.suite() do
    SolverPackage.Optimizer()
end

MOI.Benchmarks.create_baseline(
    suite, "current"; directory = "/tmp", verbose = true
)
```

Use the exclude argument to `Benchmarks.suite` to exclude benchmarks that the solver doesn't support.

Second, after making changes to the package, re-run the benchmark suite and compare to the prior saved results:

```
using SolverPackage, MathOptInterface

const MOI = MathOptInterface

suite = MOI.Benchmarks.suite() do
    SolverPackage.Optimizer()
end

MOI.Benchmarks.compare_against_baseline(
    suite, "current"; directory = "/tmp", verbose = true
)
```

This comparison will create a report detailing improvements and regressions.

API Reference

Benchmarks Functions to help benchmark the performance of solver wrappers. See The Benchmarks submodule for more details.

MathOptInterface.Benchmarks.suite – Function.

```
suite(
    new_model::Function;
    exclude::Vector{Regex} = Regex[]
)
```

Create a suite of benchmarks. `new_model` should be a function that takes no arguments, and returns a new instance of the optimizer you wish to benchmark.

Use `exclude` to exclude a subset of benchmarks.

Examples

```
suite() do
    GLPK.Optimizer()
end
suite(exclude = [r"delete"]) do
    Gurobi.Optimizer(OutputFlag=0)
end
```

MathOptInterface.Benchmarks.create_baseline – Function.

```
create_baseline(suite, name::String; directory::String = ""; kwargs...)
```

Run all benchmarks in `suite` and save to files called `name` in `directory`.

Extra `kwargs` are based to `BenchmarkTools.run`.

Examples

```
my_suite = suite(() -> GLPK.Optimizer())
create_baseline(my_suite, "glpk_master"; directory = "/tmp", verbose = true)
```

MathOptInterface.Benchmarks.compare_against_baseline – Function.

```
compare_against_baseline(
    suite, name::String; directory::String = "",
    report_filename::String = "report.txt"
)
```

Run all benchmarks in `suite` and compare against files called `name` in `directory` that were created by a call to `create_baseline`.

A report summarizing the comparison is written to `report_filename` in `directory`.

Extra `kwargs` are based to `BenchmarkTools.run`.

Examples

```
my_suite = suite(() -> GLPK.Optimizer())
compare_against_baseline(
    my_suite, "glpk_master"; directory = "/tmp", verbose = true
)
```

40.2 Bridges

Overview

The Bridges submodule

The `Bridges` module simplifies the process of converting models between equivalent formulations.

> **Tip**
>
> Read our paper for more details on how bridges are implemented.

Why bridges? A constraint can often be written in a number of equivalent formulations. For example, the constraint $l \le a^\top x \le u$ (`ScalarAffineFunction`-in-`Interval`) could be re-formulated as two constraints: $a^\top x \ge l$ (`ScalarAffineFunction`-in-`GreaterThan`) and $a^\top x \le u$ (`ScalarAffineFunction`-in-`LessThan`). An alternative re-formulation is to add a dummy variable y with the constraints $l \le y \le u$ (`VariableIndex`-in-`Interval`) and $a^\top x - y = 0$ (`ScalarAffineFunction`-in-`EqualTo`).

To avoid each solver having to code these transformations manually, MathOptInterface provides bridges.

A bridge is a small transformation from one constraint type to another (potentially collection of) constraint type.

Because these bridges are included in MathOptInterface, they can be re-used by any optimizer. Some bridges also implement constraint modifications and constraint primal and dual translations.

Several bridges can be used in combination to transform a single constraint into a form that the solver may understand. Choosing the bridges to use takes the form of finding a shortest path in the hypergraph of bridges. The methodology is detailed in the MOI paper.

The three types of bridges There are three types of bridges in MathOptInterface:

1. Constraint bridges
2. Variable bridges
3. Objective bridges

Constraint bridges Constraint bridges convert constraints formulated by the user into an equivalent form supported by the solver. Constraint bridges are subtypes of `Bridges.Constraint.AbstractBridge`.

The equivalent formulation may add constraints (and possibly also variables) in the underlying model.

In particular, constraint bridges can focus on rewriting the function of a constraint, and do not change the set. Function bridges are subtypes of `Bridges.Constraint.AbstractFunctionConversionBridge`.

Read the list of implemented constraint bridges for more details on the types of transformations that are available. Function bridges are `Bridges.Constraint.ScalarFunctionizeBridge` and `Bridges.Constraint.VectorFunctionizeBri`

Variable bridges Variable bridges convert variables added by the user, either free with `add_variable`/`add_variables`, or constrained with `add_constrained_variable`/`add_constrained_variables`, into an equivalent form supported by the solver. Variable bridges are subtypes of `Bridges.Variable.AbstractBridge`.

The equivalent formulation may add constraints (and possibly also variables) in the underlying model.

Read the list of implemented variable bridges for more details on the types of transformations that are available.

Objective bridges Objective bridges convert the `ObjectiveFunction` set by the user into an equivalent form supported by the solver. Objective bridges are subtypes of `Bridges.Objective.AbstractBridge`.

The equivalent formulation may add constraints (and possibly also variables) in the underlying model.

Read the list of implemented objective bridges for more details on the types of transformations that are available.

Bridges.full_bridge_optimizer

> **Tip**
>
> Unless you have an advanced use-case, this is probably the only function you need to care about.

To enable the full power of MathOptInterface's bridges, wrap an `optimizer` in a `Bridges.full_bridge_optimizer`.

```
julia> inner_optimizer = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> optimizer = MOI.Bridges.full_bridge_optimizer(inner_optimizer, Float64)
MOIB.LazyBridgeOptimizer{MOIU.Model{Float64}}
with 0 variable bridges
with 0 constraint bridges
with 0 objective bridges
with inner model MOIU.Model{Float64}
```

That's all you have to do! Use `optimizer` as normal, and bridging will happen lazily behind the scenes. By lazily, we mean that bridging will only happen if the constraint is not supported by the `inner_optimizer`.

> **Info**
>
> Most bridges are added by default in `Bridges.full_bridge_optimizer`. However, for technical reasons, some bridges are not added by default. Three examples include `Bridges.Constraint.SOCtoPSDBridge`, `Bridges.Constraint.SOCtoNonConvexQuadBridge` and `Bridges.Constraint.RSOCtoNonConvexQuadBridge`. See the docs of those bridges for more information.

Add a single bridge If you don't want to use `Bridges.full_bridge_optimizer`, you can wrap an optimizer in a single bridge.

However, this will force the constraint to be bridged, even if the `inner_optimizer` supports it.

```
julia> inner_optimizer = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> optimizer = MOI.Bridges.Constraint.SplitInterval{Float64}(inner_optimizer)
MOIB.Constraint.SingleBridgeOptimizer{MOIB.Constraint.SplitIntervalBridge{Float64, F, S, LS, US}
↪  where {F<:MOI.AbstractFunction, S<:MOI.AbstractSet, LS<:MOI.AbstractSet, US<:MOI.AbstractSet},
↪  MOIU.Model{Float64}}
with 0 constraint bridges
with inner model MOIU.Model{Float64}

julia> x = MOI.add_variable(optimizer)
MOI.VariableIndex(1)

julia> MOI.add_constraint(optimizer, x, MOI.Interval(0.0, 1.0))
```

```
MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,
↪  MathOptInterface.Interval{Float64}}(1)

julia> MOI.get(optimizer, MOI.ListOfConstraintTypesPresent())
1-element Vector{Tuple{Type, Type}}:
 (MathOptInterface.VariableIndex, MathOptInterface.Interval{Float64})

julia> MOI.get(inner_optimizer, MOI.ListOfConstraintTypesPresent())
2-element Vector{Tuple{Type, Type}}:
 (MathOptInterface.VariableIndex, MathOptInterface.GreaterThan{Float64})
 (MathOptInterface.VariableIndex, MathOptInterface.LessThan{Float64})
```

Bridges.LazyBridgeOptimizer If you don't want to use `Bridges.full_bridge_optimizer`, but you need more than a single bridge (or you want the bridging to happen lazily), you can manually construct a `Bridges.LazyBridgeOptimize`

First, wrap an inner optimizer:

```
julia> inner_optimizer = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> optimizer = MOI.Bridges.LazyBridgeOptimizer(inner_optimizer)
MOIB.LazyBridgeOptimizer{MOIU.Model{Float64}}
with 0 variable bridges
with 0 constraint bridges
with 0 objective bridges
with inner model MOIU.Model{Float64}
```

Then use `Bridges.add_bridge` to add individual bridges:

```
julia> MOI.Bridges.add_bridge(optimizer, MOI.Bridges.Constraint.SplitIntervalBridge{Float64})

julia> MOI.Bridges.add_bridge(optimizer, MOI.Bridges.Objective.FunctionizeBridge{Float64})
```

Now the constraints will be bridged only if needed:

```
julia> x = MOI.add_variable(optimizer)
MOI.VariableIndex(1)

julia> MOI.add_constraint(optimizer, x, MOI.Interval(0.0, 1.0))
MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex,
↪  MathOptInterface.Interval{Float64}}(1)

julia> MOI.get(optimizer, MOI.ListOfConstraintTypesPresent())
1-element Vector{Tuple{Type, Type}}:
 (MathOptInterface.VariableIndex, MathOptInterface.Interval{Float64})

julia> MOI.get(inner_optimizer, MOI.ListOfConstraintTypesPresent())
1-element Vector{Tuple{Type, Type}}:
 (MathOptInterface.VariableIndex, MathOptInterface.Interval{Float64})
```

Implementation

Bridge interface

To be usable by a bridge optimizer, a bridge must implement the following functions:

MathOptInterface.Bridges.added_constrained_variable_types – Function.

```
added_constrained_variable_types(
    BT::Type{<:Variable.AbstractBridge},
)::Vector{Tuple{Type}}
```

Return a list of the types of constrained variables that bridges of concrete type BT add. This is used by the LazyBridgeOptimizer.

MathOptInterface.Bridges.added_constraint_types – Function.

```
added_constraint_types(
    BT::Type{<:Constraint.AbstractBridge},
)::Vector{Tuple{Type, Type}}
```

Return a list of the types of constraints that bridges of concrete type BT add. This is used by the LazyBridgeOptimizer.

Additionally, variable bridges must implement:

MathOptInterface.Bridges.Variable.supports_constrained_variable – Function.

```
supports_constrained_variable(
    ::Type{<:AbstractBridge},
    ::Type{<:MOI.AbstractSet},
)::Bool
```

Return a `Bool` indicating whether the bridges of type BT support bridging constrained variables in S.

MathOptInterface.Bridges.Variable.concrete_bridge_type – Function.

```
concrete_bridge_type(
    BT::Type{<:AbstractBridge},
    S::Type{<:MOI.AbstractSet},
)::Type
```

Return the concrete type of the bridge supporting variables in S constraints. This function can only be called if `MOI.supports_constrained_variable(BT, S)` is `true`.

Examples

As a variable in MathOptInterface.GreaterThan is bridged into variables in MathOptInterface.Nonnegatives by the VectorizeBridge:

```
MOI.Bridges.Variable.concrete_bridge_type(
    MOI.Bridges.Variable.VectorizeBridge{Float64},
    MOI.GreaterThan{Float64},
)

# output

MathOptInterface.Bridges.Variable.VectorizeBridge{Float64, MathOptInterface.Nonnegatives}
```

MathOptInterface.Bridges.Variable.bridge_constrained_variable – Function.

```
bridge_constrained_variable(
    BT::Type{<:AbstractBridge},
    model::MOI.ModelLike,
    set::MOI.AbstractSet,
)
```

Bridge the constrained variable in `set` using bridge BT to `model` and returns a bridge object of type BT. The bridge type BT should be a concrete type, that is, all the type parameters of the bridge should be set. Use `concrete_bridge_type` to obtain a concrete type for given set types.

constraint bridges must implement:

`MathOptInterface.supports_constraint` – Method.

```
MOI.supports_constraint(
    BT::Type{<:AbstractBridge},
    F::Type{<:MOI.AbstractFunction},
    S::Type{<:MOI.AbstractSet},
)::Bool
```

Return a `Bool` indicating whether the bridges of type BT support bridging F-in-S constraints.

`MathOptInterface.Bridges.Constraint.concrete_bridge_type` – Function.

```
concrete_bridge_type(
    BT::Type{<:AbstractBridge},
    F::Type{<:MOI.AbstractFunction},
    S::Type{<:MOI.AbstractSet}
)::Type
```

Return the concrete type of the bridge supporting F-in-S constraints. This function can only be called if `MOI.supports_constraint(BT, F, S)` is `true`.

Examples

As a `MathOptInterface.VariableIndex`-in-`MathOptInterface.Interval` constraint is bridged into a `MathOptInterface.Var` in-`MathOptInterface.GreaterThan` and a `MathOptInterface.VariableIndex`-in-`MathOptInterface.LessThan` by the `SplitIntervalBridge`:

```
MOI.Bridges.Constraint.concrete_bridge_type(
    MOI.Bridges.Constraint.SplitIntervalBridge{Float64},
    MOI.VariableIndex,
    MOI.Interval{Float64},
)

# output

MathOptInterface.Bridges.Constraint.SplitIntervalBridge{Float64, MathOptInterface.VariableIndex,
↪  MathOptInterface.Interval{Float64}, MathOptInterface.GreaterThan{Float64},
↪  MathOptInterface.LessThan{Float64}}
```

`MathOptInterface.Bridges.Constraint.bridge_constraint` – Function.

```
bridge_constraint(
    BT::Type{<:AbstractBridge},
    model::MOI.ModelLike,
    func::AbstractFunction,
    set::MOI.AbstractSet,
)
```

Bridge the constraint `func`-in-`set` using bridge BT to `model` and returns a bridge object of type BT. The bridge type BT should be a concrete type, that is, all the type parameters of the bridge should be set. Use `concrete_bridge_type` to obtain a concrete type for given function and set types.

and objective bridges must implement:

`MathOptInterface.Bridges.set_objective_function_type` – Function.

```
set_objective_function_type(
    BT::Type{<:Objective.AbstractBridge},
)::Type{<:MOI.AbstractScalarFunction}
```

Return the type of objective function that bridges of concrete type BT set. This is used by the `LazyBridgeOptimizer`.

`MathOptInterface.Bridges.Objective.concrete_bridge_type` – Function.

```
concrete_bridge_type(
    BT::Type{<:MOI.Bridges.Objective.AbstractBridge},
    F::Type{<:MOI.AbstractScalarFunction},
)::Type
```

Return the concrete type of the bridge supporting objective functions of type F. This function can only be called if `MOI.supports_objective_function(BT, F)` is `true`.

`MathOptInterface.Bridges.Objective.bridge_objective` – Function.

```
bridge_objective(
    BT::Type{<:MOI.Bridges.Objective.AbstractBridge},
    model::MOI.ModelLike,
    func::MOI.AbstractScalarFunction,
)
```

Bridge the objective function `func` using bridge BT to `model` and returns a bridge object of type BT. The bridge type BT should be a concrete type, that is, all the type parameters of the bridge should be set. Use `concrete_bridge_type` to obtain a concrete type for a given function type.

When querying the `NumberOfVariables`, `NumberOfConstraints` `ListOfVariableIndices`, and `ListOfConstraintIndices`, the variables and constraints created by the bridges in the underlying model are hidden by the bridge optimizer. For this purpose, the bridge must provide access to the variables and constraints it has created by implementing the following methods of `get`:

`MathOptInterface.get` – Method.

```
MOI.get(b::AbstractBridge, ::MOI.NumberOfVariables)
```

The number of variables created by the bridge b in the model.

`MathOptInterface.get` – Method.

```
MOI.get(b::AbstractBridge, ::MOI.ListOfVariableIndices)
```

The list of variables created by the bridge b in the model.

`MathOptInterface.get` – Method.

```
MOI.get(b::AbstractBridge, ::MOI.NumberOfConstraints{F, S}) where {F, S}
```

The number of constraints of the type F-in-S created by the bridge b in the model.

MathOptInterface.get – Method.

```
MOI.get(b::AbstractBridge, ::MOI.ListOfConstraintIndices{F, S}) where {F, S}
```

A Vector{ConstraintIndex{F,S}} with indices of all constraints of type F-inS created by the bride b in the model (i.e., of length equal to the value of NumberOfConstraints{F,S}()).

SetMap bridges

Implementing a constraint bridge relying on linear transformation between two sets is easier thanks to the SetMap interface. The bridge simply needs to be a subtype of [Bridges.Variable.SetMapBridge] for a variable bridge and [Bridges.Constraint.SetMapBridge] for a constraint bridge and the linear transformation is represented with Bridges.map_set, Bridges.map_function, Bridges.inverse_map_set, Bridges.inverse_map_function, Bridges.adjoint_map_function and Bridges.inverse_adjoint_map_function. Note that the implementing last 4 methods is optional in the sense that if they are not implemented, bridging constraint would still work but some features would be missing as described in the docstrings. See [L20, Section 2.1.2] for more details including [L20, Example 2.1.1] that illustrates the idea for Bridges.Variable.SOCtoRSOCBridge, Bridges.Variable.RSOCtoSOCBr Bridges.Constraint.SOCtoRSOCBridge and Bridges.Constraint.RSOCtoSOCBridge.

[L20] Legat, Benoît. Set Programming: Theory and Computation. PhD thesis. 2020.

API Reference

Bridges

MathOptInterface.Bridges.AbstractBridge – Type.

```
AbstractBridge
```

Represents a bridged constraint or variable in a MathOptInterface.Bridges.AbstractBridgeOptimizer. It contains the indices of the variables and constraints that it has created in the model. These can be obtained using MathOptInterface.NumberOfVariables, MathOptInterface.ListOfVariableIndices, MathOptInterface.Nu and MathOptInterface.ListOfConstraintIndices using MathOptInterface.get with the bridge in place of the MathOptInterface.ModelLike. Attributes of the bridged model such as MathOptInterface.ConstraintDual and MathOptInterface.ConstraintPrimal, can be obtained using MathOptInterface.get with the bridge in place of the constraint index. These calls are used by the MathOptInterface.Bridges.AbstractBridgeOptimizer to communicate with the bridge so they should be implemented by the bridge.

MathOptInterface.Bridges.AbstractBridgeOptimizer – Type.

```
AbstractBridgeOptimizer
```

A bridge optimizer applies given constraint bridges to a given optimizer thus extending the types of supported constraints. The attributes of the inner optimizer are automatically transformed to make the bridges transparent, e.g. the variables and constraints created by the bridges are hidden.

By convention, the inner optimizer should be stored in a model field and the dictionary mapping constraint indices to bridges should be stored in a bridges field. If a bridge optimizer deviates from these conventions, it should implement the functions MOI.optimize! and bridge respectively.

MathOptInterface.Bridges.LazyBridgeOptimizer – Type.

```
LazyBridgeOptimizer{OT<:MOI.ModelLike} <: AbstractBridgeOptimizer
```

The LazyBridgeOptimizer combines several bridges, which are added using the add_bridge function.

Whenever a constraint is added, it only attempts to bridge it if it is not supported by the internal model (hence its name Lazy).

When bridging a constraint, it selects the minimal number of bridges needed.

For example, if a constraint F-in-S can be bridged into a constraint F1-in-S1 (supported by the internal model) using bridge 1 or bridged into a constraint F2-in-S2 (unsupported by the internal model) using bridge 2 which can then be bridged into a constraint F3-in-S3 (supported by the internal model) using bridge 3, it will choose bridge 1 as it allows to bridge F-in-'S using only one bridge instead of two if it uses bridge 2 and 3.

`MathOptInterface.Bridges.add_bridge` – Function.

```
add_bridge(b::LazyBridgeOptimizer, BT::Type{<:AbstractBridge})
```

Enable the use of the bridges of type BT by b.

`MathOptInterface.Bridges.remove_bridge` – Function.

```
remove_bridge(b::LazyBridgeOptimizer, BT::Type{<:AbstractBridge})
```

Disable the use of the bridges of type BT by b.

`MathOptInterface.Bridges.has_bridge` – Function.

```
has_bridge(b::LazyBridgeOptimizer, BT::Type{<:AbstractBridge})
```

Return a `Bool` indicating whether the bridges of type BT are used by b.

`MathOptInterface.Bridges.full_bridge_optimizer` – Function.

```
full_bridge_optimizer(model::MOI.ModelLike, ::Type{T}) where {T}
```

Returns a `LazyBridgeOptimizer` bridging `model` for every bridge defined in this package (see below for the few exceptions) and for the coefficient type T in addition to the bridges in the list returned by `MOI.get(model, MOI.Bridges.ListOfNonstandardBridges{T}())`.

See also `ListOfNonstandardBridges`.

Note

The following bridges are not added by `full_bridge_optimizer` except if they are in the list returned by `MOI.get(model, MOI.Bridges.ListOfNonstandardBridges{T}())` (see the docstrings of the corresponding bridge for the reason they are not added):

- `Constraint.SOCtoNonConvexQuadBridge`, `Constraint.RSOCtoNonConvexQuadBridge` and `Constraint.SOCtoPSDBridge`.
- The subtypes of `Constraint.AbstractToIntervalBridge` (i.e. `Constraint.GreaterToIntervalBridge` and `Constraint.LessToIntervalBridge`) if T is not a subtype of `AbstractFloat`.

`MathOptInterface.Bridges.ListOfNonstandardBridges` – Type.

```
ListOfNonstandardBridges{T}() <: MOI.AbstractOptimizerAttribute
```

Any optimizer can be wrapped in a `LazyBridgeOptimizer` using `full_bridge_optimizer`. However, by default `LazyBridgeOptimizer` uses a limited set of bridges that are:

1. implemented in `MOI.Bridges`

2. generally applicable for all optimizers.

For some optimizers however, it is useful to add additional bridges, such as those that are implemented in external packages (e.g., within the solver package itself) or only apply in certain circumstances (e.g., `Constraint.SOCtoNonConvexQuadBridge`).

Such optimizers should implement the `ListOfNonstandardBridges` attribute to return a vector of bridge types that are added by `full_bridge_optimizer` in addition to the list of default bridges.

Note that optimizers implementing `ListOfNonstandardBridges` may require package-specific functions or sets to be used if the non-standard bridges are not added. Therefore, you are recommended to use `model = MOI.instantiate(Package.Optimizer; with_bridge_type = T)` instead of `model = MOI.instantiate(Package.Optimiz`
See `MathOptInterface.instantiate`.

Examples

An optimizer using a non-default bridge in `MOI.Bridges`

Solvers supporting `MOI.ScalarQuadraticFunction` can support `MOI.SecondOrderCone` and `MOI.RotatedSecondOrderCone` by defining:

```
function MOI.get(::MyQuadraticOptimizer, ::ListOfNonstandardBridges{Float64})
    return Type[
        MOI.Bridges.Constraint.SOCtoNonConvexQuadBridge{Float64},
        MOI.Bridges.Constraint.RSOCtoNonConvexQuadBridge{Float64},
    ]
end
```

An optimizer defining an internal bridge

Suppose an optimizer can exploit specific structure of a constraint, e.g., it can exploit the structure of the matrix A in the linear system of equations `A * x = b`.

The optimizer can define the function:

```
struct MatrixAffineFunction{T} <: MOI.AbstractVectorFunction
    A::SomeStructuredMatrixType{T}
    b::Vector{T}
end
```

and then a bridge

```
struct MatrixAffineFunctionBridge{T} <: MOI.Constraint.AbstractBridge
    # ...
end
# ...
```

from `VectorAffineFunction{T}` to the `MatrixAffineFunction`. Finally, it defines:

```
function MOI.get(::Optimizer{T}, ::ListOfNonstandardBridges{T}) where {T}
    return Type[MatrixAffineFunctionBridge{T}]
end
```

`MathOptInterface.Bridges.debug_supports_constraint` – Function.

```
debug_supports_constraint(
    b::LazyBridgeOptimizer,
    F::Type{<:MOI.AbstractFunction},
    S::Type{<:MOI.AbstractSet};
    io::IO = Base.stdout,
)
```

Prints to `io` explanations for the value of `MOI.supports_constraint` with the same arguments.

`MathOptInterface.Bridges.debug_supports` – Function.

```
debug_supports(
    b::LazyBridgeOptimizer,
    ::MOI.ObjectiveFunction{F};
    io::IO = Base.stdout,
) where F
```

Prints to `io` explanations for the value of `MOI.supports` with the same arguments.

`MathOptInterface.Bridges.bridged_variable_function` – Function.

```
bridged_variable_function(
    b::AbstractBridgeOptimizer,
    vi::MOI.VariableIndex,
)
```

Return a `MOI.AbstractScalarFunction` of variables of `b.model` that equals `vi`. That is, if the variable `vi` is bridged, it returns its expression in terms of the variables of `b.model`. Otherwise, it returns `vi`.

`MathOptInterface.Bridges.unbridged_variable_function` – Function.

```
unbridged_variable_function(
    b::AbstractBridgeOptimizer,
    vi::MOI.VariableIndex,
)
```

Return a `MOI.AbstractScalarFunction` of variables of `b` that equals `vi`. That is, if the variable `vi` is an internal variable of `b.model` created by a bridge but not visible to the user, it returns its expression in terms of the variables of bridged variables. Otherwise, it returns `vi`.

`MathOptInterface.Bridges.bridged_function` – Function.

```
bridged_function(b::AbstractBridgeOptimizer, value)::typeof(value)
```

Substitute any bridged `MOI.VariableIndex` in `value` by an equivalent expression in terms of variables of `b.model`.

`MathOptInterface.Bridges.Variable.unbridged_map` – Function.

unbridged_map(bridge::MOI.Bridges.Variable.AbstractBridge, vi::MOI.VariableIndex,)

For a bridged variable in a scalar set, return a tuple of pairs mapping the variables created by the bridge to an affine expression in terms of the bridged variable `vi`.

```
unbridged_map(
    bridge::MOI.Bridges.Variable.AbstractBridge,
    vis::Vector{MOI.VariableIndex},
)
```

For a bridged variable in a vector set, return a tuple of pairs mapping the variables created by the bridge to an affine expression in terms of the bridged variable `vis`. If this method is not implemented, it falls back to calling the following method for every variable of `vis`.

```
unbridged_map(
    bridge::MOI.Bridges.Variable.AbstractBridge,
    vi::MOI.VariableIndex,
    i::MOIB.IndexInVector,
)
```

For a bridged variable in a vector set, return a tuple of pairs mapping the variables created by the bridge to an affine expression in terms of the bridged variable `vi` corresponding to the `i`th variable of the vector.

If there is no way to recover the expression in terms of the bridged variable(s) `vi(s)`, return `nothing`. See `ZerosBridge` for an example of bridge returning `nothing`.

Constraint bridges `MathOptInterface.Bridges.Constraint.AbstractBridge` – Type.

```
AbstractBridge
```

Subtype of `MathOptInterface.Bridges.AbstractBridge` for constraint bridges.

`MathOptInterface.Bridges.Constraint.AbstractFunctionConversionBridge` – Type.

```
abstract type AbstractFunctionConversionBridge{F, S} <: AbstractBridge end
```

Bridge a constraint G-in-S into a constraint F-in-S where F and G are equivalent representations of the same function. By convention, the transformed function is stored in the `constraint` field.

`MathOptInterface.Bridges.Constraint.SingleBridgeOptimizer` – Type.

```
SingleBridgeOptimizer{BT<:AbstractBridge, OT<:MOI.ModelLike} <:
AbstractBridgeOptimizer
```

The `SingleBridgeOptimizer` bridges any constraint supported by the bridge BT. This is in contrast with the `MathOptInterface.Bridges.LazyBridgeOptimizer` which only bridges the constraints that are unsupported by the internal model, even if they are supported by one of its bridges.

`MathOptInterface.Bridges.Constraint.add_all_bridges` – Function.

```
add_all_bridges(bridged_model, ::Type{T}) where {T}
```

Add all bridges defined in the `Bridges.Constraint` submodule to `bridged_model`. The coefficient type used is T.

SetMap bridges `MathOptInterface.Bridges.Variable.SetMapBridge` – Type.

```
abstract type SetMapBridge{T,S1,S2} <: AbstractBridge end
```

Consider two type of sets S1, S2 and a linear mapping A that the image of a set of type S1 under A is a set of type S2. A `SetMapBridge{T,S1,S2}` is a bridge that substitutes constrained variables in S2 into the image through A of constrained variables in S1.

The linear map A is described by `MathOptInterface.Bridges.map_set`, `MathOptInterface.Bridges.map_function`. Implementing a method for these two functions is sufficient to bridge constrained variables. In order for the getters and setters of dual solutions, starting values, etc... to work as well a method for the following functions should be implemented as well: `MathOptInterface.Bridges.inverse_map_set`, `MathOptInterface.Bridges.inverse`

MathOptInterface.Bridges.adjoint_map_function and MathOptInterface.Bridges.inverse_adjoint_map_function. See the docstrings of the function to see which feature would be missing it it was not implemented for a given bridge.

MathOptInterface.Bridges.Constraint.SetMapBridge – Type.

```
abstract type SetMapBridge{T,S2,S1,F,G} <: AbstractBridge end
```

Consider two type of sets S1, S2 and a linear mapping A that the image of a set of type S1 under A is a set of type S2. A SetMapBridge{T,S2,S1,F,G} is a bridge that maps G-in-S2 constraints into F-in-S1 by mapping the function through A.

The linear map A is described by MathOptInterface.Bridges.map_set, MathOptInterface.Bridges.map_function. Implementing a method for these two functions is sufficient to bridge constraints. In order for the getters and setters of dual solutions, starting values, etc... to work as well a method for the following functions should be implemented as well: MathOptInterface.Bridges.inverse_map_set, MathOptInterface.Bridges.inverse_map MathOptInterface.Bridges.adjoint_map_function and MathOptInterface.Bridges.inverse_adjoint_map_function. See the docstrings of the function to see which feature would be missing it it was not implemented for a given bridge.

MathOptInterface.Bridges.map_set – Function.

```
map_set(::Type{BT}, set) where {BT}
```

Return the image of set through the linear map A defined in Variable.SetMapBridge and Constraint.SetMapBridge. This is used for bridging the constraint and setting the MathOptInterface.ConstraintSet.

MathOptInterface.Bridges.inverse_map_set – Function.

```
inverse_map_set(::Type{BT}, set) where {BT}
```

Return the preimage of set through the linear map A defined in Variable.SetMapBridge and Constraint.SetMapBridge. This is used for getting the MathOptInterface.ConstraintSet.

MathOptInterface.Bridges.map_function – Function.

```
map_function(::Type{BT}, func) where {BT}
```

Return the image of func through the linear map A defined in Variable.SetMapBridge and Constraint.SetMapBridge. This is used for getting the MathOptInterface.ConstraintPrimal of variable bridges. For constraint bridges, this is used for bridging the constraint, setting the MathOptInterface.ConstraintFunction and MathOptInterface.ConstraintPrimalStart and modifying the function with MathOptInterface.modify.

```
map_function(::Type{BT}, func, i::IndexInVector) where {BT}
```

Return the scalar function at the ith index of the vector function that would be returned by map_function(BT, func) except that it may compute the ith element. This is used by bridged_function and for getting the MathOptInterface.VariablePrimal and MathOptInterface.VariablePrimalStart of variable bridges.

MathOptInterface.Bridges.inverse_map_function – Function.

```
inverse_map_function(::Type{BT}, func) where {BT}
```

Return the image of func through the inverse of the linear map A defined in Variable.SetMapBridge and Constraint.SetMapBridge. This is used by Variable.unbridged_map and for setting the MathOptInterface.VariablePrima of variable bridges and for getting the MathOptInterface.ConstraintFunction, the MathOptInterface.ConstraintPrimal and the MathOptInterface.ConstraintPrimalStart of constraint bridges.

MathOptInterface.Bridges.adjoint_map_function – Function.

```
adjoint_map_function(::Type{BT}, func) where {BT}
```

Return the image of `func` through the adjoint of the linear map A defined in Variable.SetMapBridge and Constraint.SetMapBridge. This is used for getting the MathOptInterface.ConstraintDual and MathOptInterface.ConstraintDualStart of constraint bridges.

MathOptInterface.Bridges.inverse_adjoint_map_function – Function.

```
inverse_adjoint_map_function(::Type{BT}, func) where {BT}
```

Return the image of `func` through the inverse of the adjoint of the linear map A defined in Variable.SetMapBridge and Constraint.SetMapBridge. This is used for getting the MathOptInterface.ConstraintDual of variable bridges and setting the MathOptInterface.ConstraintDualStart of constraint bridges.

Bridges implemented MathOptInterface.Bridges.Constraint.FlipSignBridge – Type.

```
FlipSignBridge{T, S1, S2, F, G}
```

Bridge a G-in-S1 constraint into an F-in-S2 constraint by multiplying the function by `-1` and taking the point reflection of the set across the origin. The flipped F-in-S constraint is stored in the `constraint` field by convention.

MathOptInterface.Bridges.Constraint.AbstractToIntervalBridge – Type.

```
AbstractToIntervalBridge{T, S1, F}
```

Bridge a F-in-`Interval` constraint into an F-in-`Interval{T}` constraint where we have either:

- `S1 = MOI.GreaterThan{T}`
- `S1 = MOI.LessThan{T}`

The F-in-`Interval{T}` constraint is stored in the `constraint` field by convention.

> **Warning**
>
> It is required that T be a `AbstractFloat` type because otherwise `typemin` and `typemax` would either be not implemented (e.g. `BigInt`) or would not give infinite value (e.g. `Int`). For this reason, this bridge is only added to MathOptInterface.Bridges.full_bridge_optimizer.
>
> when T is a subtype of `AbstractFloat`.

MathOptInterface.Bridges.Constraint.GreaterToIntervalBridge – Type.

```
GreaterToIntervalBridge{T, F<:MOI.AbstractScalarFunction} <:
    AbstractToIntervalBridge{T, MOI.GreaterThan{T}, F}
```

Transforms a F-in-`GreaterThan{T}` constraint into an F-in-`Interval{T}` constraint.

MathOptInterface.Bridges.Constraint.LessToIntervalBridge – Type.

```
LessToIntervalBridge{T, F<:MOI.AbstractScalarFunction} <:
    AbstractToIntervalBridge{T, MOI.LessThan{T}, F}
```

Transforms a F-in-`LessThan{T}` constraint into an F-in-`Interval{T}` constraint.

`MathOptInterface.Bridges.Constraint.GreaterToLessBridge` – Type.

```
GreaterToLessBridge{
    T,
    F<:MOI.AbstractScalarFunction,
    G<:MOI.AbstractScalarFunction
} <: FlipSignBridge{T, MOI.GreaterThan{T}, MOI.LessThan{T}, F, G}
```

Transforms a G-in-`GreaterThan{T}` constraint into an F-in-`LessThan{T}` constraint.

`MathOptInterface.Bridges.Constraint.LessToGreaterBridge` – Type.

```
LessToGreaterBridge{
    T,
    F<:MOI.AbstractScalarFunction,
    G<:MOI.AbstractScalarFunction
} <: FlipSignBridge{T, MOI.LessThan{T}, MOI.GreaterThan{T}, F, G}
```

Transforms a G-in-`LessThan{T}` constraint into an F-in-`GreaterThan{T}` constraint.

`MathOptInterface.Bridges.Constraint.NonnegToNonposBridge` – Type.

```
NonnegToNonposBridge{
    T,
    F<:MOI.AbstractVectorFunction,
    G<:MOI.AbstractVectorFunction
} <: FlipSignBridge{T, MOI.Nonnegatives, MOI.Nonpositives, F, G}
```

Transforms a G-in-`Nonnegatives` constraint into a F-in-`Nonpositives` constraint.

`MathOptInterface.Bridges.Constraint.NonposToNonnegBridge` – Type.

```
NonposToNonnegBridge{
    T,
    F<:MOI.AbstractVectorFunction,
    G<:MOI.AbstractVectorFunction,
} <: FlipSignBridge{T, MOI.Nonpositives, MOI.Nonnegatives, F, G}
```

Transforms a G-in-`Nonpositives` constraint into a F-in-`Nonnegatives` constraint.

`MathOptInterface.Bridges.Constraint.VectorizeBridge` – Type.

```
VectorizeBridge{T,F,S,G}
```

Transforms a constraint G-in-`scalar_set_type(S, T)` where `S <: VectorLinearSet` to F-in-S.

Examples

The constraint `VariableIndex`-in-`LessThan{Float64}` becomes `VectorAffineFunction{Float64}`-in-`Nonpositives`, where `T = Float64`, `F = VectorAffineFunction{Float64}`, `S = Nonpositives`, and `G = VariableIndex`.

`MathOptInterface.Bridges.Constraint.ScalarizeBridge` – Type.

```
ScalarizeBridge{T, F, S}
```

Transforms a constraint `AbstractVectorFunction`-in-`vector_set_type(S)` where `S <: LPCone{T}` to F-in-S.

`MathOptInterface.Bridges.Constraint.ScalarSlackBridge` – Type.

```
ScalarSlackBridge{T, F, S}
```

The `ScalarSlackBridge` converts a constraint G-in-S where `G` is a function different from `VariableIndex` into the constraints F-in-`EqualTo{T}` and `VariableIndex`-in-S.

F is the result of subtracting a `VariableIndex` from G. Typically G is the same as F, but that is not mandatory.

`MathOptInterface.Bridges.Constraint.VectorSlackBridge` – Type.

```
VectorSlackBridge{T, F, S}
```

The `VectorSlackBridge` converts a constraint G-in-S where G is a function different from `VectorOfVariables` into the constraints Fin-`Zeros` and `VectorOfVariables`-in-S.

F is the result of subtracting a `VectorOfVariables` from G. Typically G is the same as F, but that is not mandatory.

`MathOptInterface.Bridges.Constraint.ScalarFunctionizeBridge` – Type.

```
ScalarFunctionizeBridge{T, S}
```

The `ScalarFunctionizeBridge` converts a constraint `VariableIndex`-in-S into the constraint `ScalarAffineFunction{T}`-in-S.

`MathOptInterface.Bridges.Constraint.VectorFunctionizeBridge` – Type.

```
VectorFunctionizeBridge{T, S}
```

The `VectorFunctionizeBridge` converts a constraint `VectorOfVariables`-in-S into the constraint `VectorAffineFunction{T}` in-S.

`MathOptInterface.Bridges.Constraint.SplitIntervalBridge` – Type.

```
SplitIntervalBridge{T, F, S, LS, US}
```

The `SplitIntervalBridge` splits a F-in-S constraint into a F-in-LS and a F-in-US constraint where we have either:

- `S = MOI.Interval{T}`, `LS = MOI.GreaterThan{T}` and `US = MOI.LessThan{T}`,
- `S = MOI.EqualTo{T}`, `LS = MOI.GreaterThan{T}` and `US = MOI.LessThan{T}`, or
- `S = MOI.Zeros`, `LS = MOI.Nonnegatives` and `US = MOI.Nonpositives`.

For instance, if F is `MOI.ScalarAffineFunction` and S is `MOI.Interval`, it transforms the constraint $la, x+ u$ into the constraints $a, x + l$ and $a, x + u$.

Note

If `T<:AbstractFloat` and S is `MOI.Interval{T}` then no lower (resp. upper) bound constraint is created if the lower (resp. upper) bound is `typemin(T)` (resp. `typemax(T)`). Similarly, when `MathOptInterface.ConstraintSet` is set, a lower or upper bound constraint may be deleted or created accordingly.

`MathOptInterface.Bridges.Constraint.SOCtoRSOCBridge` – Type.

```
SOCtoRSOCBridge{T, F, G}
```

We simply do the inverse transformation of RSOCtoSOCBridge. In fact, as the transformation is an involution, we do the same transformation.

MathOptInterface.Bridges.Constraint.RSOCtoSOCBridge – Type.

```
RSOCtoSOCBridge{T, F, G}
```

The `RotatedSecondOrderCone` is `SecondOrderCone` representable; see [BN01, p. 104]. Indeed, we have $2tu = (t/\surd 2 + u/\surd 2)^2 - (t/\surd 2 - u/\surd 2)^2$ hence

$$2tu \geq \|x\|_2^2$$

is equivalent to

$$(t/\surd 2 + u/\surd 2)^2 \geq \|x\|_2^2 + (t/\surd 2 - u/\surd 2)^2.$$

We can therefore use the transformation $(t, u, x) \mapsto (t/\surd 2 + u/\surd 2, t/\surd 2 - u/\surd 2, x)$. Note that the linear transformation is a symmetric involution (i.e. it is its own transpose and its own inverse). That means in particular that the norm of constraint primal and dual values are preserved by the tranformation.

[BN01] Ben-Tal, Aharon, and Nemirovski, Arkadi. Lectures on modern convex optimization: analysis, algorithms, and engineering applications. Society for Industrial and Applied Mathematics, 2001.

MathOptInterface.Bridges.Constraint.SOCtoNonConvexQuadBridge – Type.

```
SOCtoNonConvexQuadBridge{T}
```

Constraints of the form `VectorOfVariables`-in-`SecondOrderCone` can be transformed into a `ScalarQuadraticFunction`-in-`LessThan` and a `ScalarAffineFunction`-in-`GreaterThan`. Indeed, the definition of the second-order cone

$$t \geq \|x\|_2 \ (1)$$

is equivalent to

$$\sum x_i^2 \leq t^2 (2)$$

with $t \geq 0$. (3)

> **Warning**
> This transformation starts from a convex constraint (1) and creates a non-convex constraint (2), because the Q matrix associated with the constraint (2) has one negative eigenvalue. This might be wrongly interpreted by a solver. Some solvers can look at (2) and understand that it is a second order cone, but this is not a general rule. For these reasons this bridge is not automatically added by MOI.Bridges.full_bridge_optimizer. Care is recommended when adding this bridge to a optimizer.

MathOptInterface.Bridges.Constraint.RSOCtoNonConvexQuadBridge – Type.

```
RSOCtoNonConvexQuadBridge{T}
```

Constraints of the form `VectorOfVariables`-in-`SecondOrderCone` can be transformed into a `ScalarQuadraticFunction`-in-`LessThan` and a `ScalarAffineFunction`-in-`GreaterThan`. Indeed, the definition of the second-order cone

$$2tu \geq \|x\|_2^2, t, u \geq 0(1)$$

is equivalent to

$$\sum x_i^2 \leq 2tu(2)$$

with $t, u \geq 0$. (3)

WARNING This transformation starts from a convex constraint (1) and creates a non-convex constraint (2), because the Q matrix associated with the constraint 2 has two negative eigenvalues. This might be wrongly interpreted by a solver. Some solvers can look at (2) and understand that it is a rotated second order cone, but this is not a general rule. For these reasons, this bridge is not automatically added by `MOI.Bridges.full_bridge_optimizer`. Care is recommended when adding this bridge to an optimizer.

`MathOptInterface.Bridges.Constraint.QuadtoSOCBridge` – Type.

```
QuadtoSOCBridge{T}
```

The set of points x satisfying the constraint

$$\frac{1}{2}x^TQx + a^Tx + b \leq 0$$

is a convex set if `Q` is positive semidefinite and is the union of two convex cones if `a` and `b` are zero (i.e. homogeneous case) and `Q` has only one negative eigenvalue. Currently, only the non-homogeneous transformation is implemented, see the Note section below for more details.

Non-homogeneous case

If `Q` is positive semidefinite, there exists `U` such that $Q = U^TU$, the inequality can then be rewritten as

$$\|Ux\|_2^2 \leq 2(-a^Tx - b)$$

which is equivalent to the membership of `(1, -a^T x - b, Ux)` to the rotated second-order cone.

Homogeneous case

If `Q` has only one negative eigenvalue, the set of `x` such that $x^TQx \leq 0$ is the union of a convex cone and its opposite. We can choose which one to model by checking the existence of bounds on variables as shown below.

Second-order cone

If `Q` is diagonal and has eigenvalues `(1, 1, -1)`, the inequality $x^2 + x^2 \leq z^2$ combined with $z \geq 0$ defines the Lorenz cone (i.e. the second-order cone) but when combined with $z \leq 0$, it gives the opposite of the second order cone. Therefore, we need to check if the variable `z` has a lower bound 0 or an upper bound 0 in order to determine which cone is

Rotated second-order cone

The matrix `Q` corresponding to the inequality $x^2 \le 2yz$ has one eigenvalue 1 with eigenvectors `(1, 0, 0)` and `(0, 1, -1)` and one eigenvalue `-1` corresponding to the eigenvector `(0, 1, 1)`. Hence if we intersect this union of two convex cone with the halfspace $x + y \ge 0$, we get the rotated second-order cone and if we intersect it with the halfspace $x + y \le 0$ we get the opposite of the rotated second-order cone. Note that `y` and `z` have the same sign since `yz` is nonnegative hence $x + y \ge 0$ is equivalent to $x \ge 0$ and $y \ge 0$.

Note

The check for existence of bound can be implemented (but inefficiently) with the current interface but if bound is removed or transformed (e.g. `≤ 0` transformed into `≥ 0`) then the bridge is no longer valid. For this reason the homogeneous version of the bridge is not implemented yet.

`MathOptInterface.Bridges.Constraint.SOCtoPSDBridge` – Type.

The `SOCtoPSDBridge` transforms the second order cone constraint $\|x\| \le t$ into the semidefinite cone constraints

$$\begin{pmatrix} t & x^\top \\ x & tI \end{pmatrix} \succeq 0$$

Indeed by the Schur Complement, it is positive definite iff

$$\begin{aligned} tI &\succ 0 \\ t - x^\top (tI)^{-1} x &\succ 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} t &> 0 \\ t^2 &> x^\top x \end{aligned}$$

Warning

This bridge is not added by default by `MOI.Bridges.full_bridge_optimizer` as bridging second order cone constraints to semidefinite constraints can be achieved by the `SOCtoRSOCBridge` followed by the `RSOCtoPSDBridge` while creating a smaller semidefinite constraint.

`MathOptInterface.Bridges.Constraint.RSOCtoPSDBridge` – Type.

The `RSOCtoPSDBridge` transforms the second order cone constraint $\|x\| \le 2tu$ with $u \ge 0$ into the semidefinite cone constraints

$$\begin{pmatrix} t & x^\top \\ x & 2uI \end{pmatrix} \succeq 0$$

Indeed by the Schur Complement, it is positive definite iff

$$
\begin{aligned}
uI &\succ 0 \\
t - x^\top (2uI)^{-1} x &\succ 0
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
u &> 0 \\
2tu &> x^\top x
\end{aligned}
$$

MathOptInterface.Bridges.Constraint.NormInfinityBridge – Type.

```
NormInfinityBridge{T}
```

The NormInfinityCone is representable with LP constraints, since $t \geq \max_i |x_i|$ if and only if $t \geq x_i$ and $t \geq -x_i$ for all i.

MathOptInterface.Bridges.Constraint.NormOneBridge – Type.

```
NormOneBridge{T}
```

The NormOneCone is representable with LP constraints, since $t \geq \sum_i |x_i|$ if and only if there exists a vector y such that $t \geq \sum_i y_i$ and $y_i \geq x_i$, $y_i \geq -x_i$ for all i.

MathOptInterface.Bridges.Constraint.GeoMeantoRelEntrBridge – Type.

```
GeoMeantoRelEntrBridge{T}
```

The geometric mean cone is representable with a relative entropy constraint and a nonnegative auxiliary variable.

This is because $u \leq \prod_{i=1}^n w_i^{1/n}$ is equivalent to $y \geq 0$ and $0 \leq u + y \leq \prod_{i=1}^n w_i^{1/n}$, and the latter inequality is equivalent to $1 \leq \prod_{i=1}^n (\frac{w_i}{u+y})^{1/n}$, which is equivalent to $0 \leq \sum_{i=1}^n \log(\frac{w_i}{u+y})^{1/n}$, which is equivalent to $0 \geq \sum_{i=1}^n (u+y) \log(\frac{u+y}{w_i})$.

Thus $(u, w) \in GeometricMeanCone(1+n)$ is representable as $y \geq 0$, $(0, w, (u+y)e) \in RelativeEntropyCone(1+2n)$, where e is a vector of ones.

MathOptInterface.Bridges.Constraint.GeoMeanBridge – Type.

```
GeoMeanBridge{T, F, G, H}
```

The GeometricMeanCone is SecondOrderCone representable; see [1, p. 105].

The reformulation is best described in an example.

Consider the cone of dimension 4:

$$
t \leq \sqrt[3]{x_1 x_2 x_3}
$$

This can be rewritten as $\exists x_{21} \geq 0$ such that:

$$t \le x_{21},$$
$$x_{21}^4 \le x_1 x_2 x_3 x_{21}.$$

Note that we need to create x_{21} and not use t^4 directly as t is allowed to be negative. Now, this is equivalent to:

$$\begin{aligned} t &\le x_{21}/\sqrt{4}, \\ x_{21}^2 &\le 2x_{11}x_{12}, \\ x_{11}^2 &\le 2x_1x_2, & x_{12}^2 &\le 2x_3(x_{21}/\sqrt{4}). \end{aligned}$$

[1] Ben-Tal, Aharon, and Arkadi Nemirovski. Lectures on modern convex optimization: analysis, algorithms, and engineering applications. Society for Industrial and Applied Mathematics, 2001.

`MathOptInterface.Bridges.Constraint.RelativeEntropyBridge` – Type.

```
RelativeEntropyBridge{T}
```

The `RelativeEntropyCone` is representable with exponential cone and LP constraints, since $u \ge \sum_{i=1}^n w_i \log(\frac{w_i}{v_i})$ if and only if there exists a vector y such that $u \ge \sum_i y_i$ and $y_i \ge w_i \log(\frac{w_i}{v_i})$ or equivalently $v_i \ge w_i \exp(\frac{-y_i}{w_i})$ or equivalently $(-y_i, w_i, v_i) \in ExponentialCone$, for all i.

`MathOptInterface.Bridges.Constraint.NormSpectralBridge` – Type.

```
NormSpectralBridge{T}
```

The `NormSpectralCone` is representable with a PSD constraint, since $t \ge \sigma_1(X)$ if and only if $[tIX^\top; XtI] \succ 0$.

`MathOptInterface.Bridges.Constraint.NormNuclearBridge` – Type.

```
NormNuclearBridge{T}
```

The `NormNuclearCone` is representable with an SDP constraint and extra variables, since $t \ge \sum_i \sigma_i(X)$ if and only if there exists symmetric matrices U, V such that $[UX^\top; XV] \succ 0$ and $t \ge (tr(U) + tr(V))/2$.

`MathOptInterface.Bridges.Constraint.SquareBridge` – Type.

```
SquareBridge{T, F<:MOI.AbstractVectorFunction,
             G<:MOI.AbstractScalarFunction,
             TT<:MOI.AbstractSymmetricMatrixSetTriangle,
             ST<:MOI.AbstractSymmetricMatrixSetSquare} <: AbstractBridge
```

The `SquareBridge` reformulates the constraint of a square matrix to be in ST to a list of equality constraints for pair or off-diagonal entries with different expressions and a TT constraint the upper triangular part of the matrix.

For instance, the constraint for the matrix

$$\begin{pmatrix} 1 & 1+x & 2-3x \\ 1+x & 2+x & 3-x \\ 2-3x & 2+x & 2x \end{pmatrix}$$

to be PSD can be broken down to the constraint of the symmetric matrix

$$\begin{pmatrix} 1 & 1+x & 2-3x \\ \cdot & 2+x & 3-x \\ \cdot & \cdot & 2x \end{pmatrix}$$

and the equality constraint between the off-diagonal entries (2, 3) and (3, 2) $2x == 1$. Note that now symmetrization constraint need to be added between the off-diagonal entries (1, 2) and (2, 1) or between (1, 3) and (3, 1) since the expressions are the same.

`MathOptInterface.Bridges.Constraint.RootDetBridge` – Type.

```
RootDetBridge{T,F,G,H}
```

The `RootDetConeTriangle` is representable by a `PositiveSemidefiniteConeTriangle` and an `GeometricMeanCone` constraints; see [1, p. 149].

Indeed, $t \le \det(X)^{1/n}$ if and only if there exists a lower triangular matrix such that:

$$\begin{pmatrix} X & \\ ^\top & \mathrm{Diag}() \end{pmatrix} \succeq 0$$
$$t \le (_{11}{}_{22} \cdots {}_{nn})^{1/n}$$

[1] Ben-Tal, Aharon, and Arkadi Nemirovski. Lectures on modern convex optimization: analysis, algorithms, and engineering applications. Society for Industrial and Applied Mathematics, 2001.

`MathOptInterface.Bridges.Constraint.LogDetBridge` – Type.

```
LogDetBridge{T,F,G,H,I}
```

The `LogDetConeTriangle` is representable by a `PositiveSemidefiniteConeTriangle` and `ExponentialCone` constraints.

Indeed, $\log\det(X) = \log(\delta_1) + \cdots + \log(\delta_n)$ where δ_1, ..., δ_n are the eigenvalues of X.

Adapting the method from [1, p. 149], we see that $t \le u\log(\det(X/u))$ for $u > 0$ if and only if there exists a lower triangular matrix such that

$$\begin{pmatrix} X & \\ ^\top & \mathrm{Diag}() \end{pmatrix} \succeq 0$$
$$t \le u\log(_{11}/u) + u\log(_{22}/u) + \cdots + u\log(_{nn}/u)$$

[1] Ben-Tal, Aharon, and Arkadi Nemirovski. Lectures on modern convex optimization: analysis, algorithms, and engineering applications. Society for Industrial and Applied Mathematics, 2001. "‘

`MathOptInterface.Bridges.Constraint.IndicatorActiveOnFalseBridge` – Type.

```
IndicatorActiveOnFalseBridge{T}
```

The `IndicatorActiveOnFalseBridge` replaces an indicator constraint activated on 0 with a variable z_0 with the constraint activated on 1, with a variable z_1. It stores the added `variable` and added constraints:

- $z_1 \in \mathbb{B}$ in `zero_one_cons`

- $z_0 + z_1 == 1$ in ‘indisjunction_cons‘
- The added ACTIVATE_ON_ONE indicator constraint in indicator_cons_index.

MathOptInterface.Bridges.Constraint.IndicatorSOS1Bridge – Type.

```
IndicatorSOS1Bridge{T,S<:MOI.AbstractScalarSet}
```

The IndicatorSOS1Bridge replaces an indicator constraint of the following form: $z \in \mathbb{B}, z == 1 \implies f(x) \in S$ with a SOS1 constraint: $z \in \mathbb{B}, slack$ free, $f(x) + slack \in S, SOS1(slack, z)$.

MathOptInterface.Bridges.Constraint.SemiToBinaryBridge – Type.

```
SemiToBinaryBridge{T, S <: MOI.AbstractScalarSet}
```

The SemiToBinaryBridge replaces a Semicontinuous constraint: $x \in$ Semicontinuous(l, u) is replaced by: $z \in \{0, 1\}$, $x \leq z \cdot u$, $x \geq z \cdot l$.

The SemiToBinaryBridge replaces a Semiinteger constraint: $x \in Semiinteger(l, u)$ is replaced by: $z \in \{0, 1\}$, $x \in \mathbb{Z}$, $x \leq z \cdot u$, $x \geq z \cdot l$.

MathOptInterface.Bridges.Constraint.ZeroOneBridge – Type.

```
ZeroOneBridge{T}
```

The ZeroOneBridge splits a MOI.VariableIndex-in-MOI.ZeroOne constraint into a MOI.VariableIndex-in-MOI.Integer constraint and a MOI.VariableIndex-in-MOI.Interval(0, 1) constraint.

Variable bridges MathOptInterface.Bridges.Variable.AbstractBridge – Type.

```
AbstractBridge
```

Subtype of MathOptInterface.Bridges.AbstractBridge for variable bridges.

MathOptInterface.Bridges.Variable.SingleBridgeOptimizer – Type.

```
SingleBridgeOptimizer{BT<:AbstractBridge, OT<:MOI.ModelLike} <:
AbstractBridgeOptimizer
```

The SingleBridgeOptimizer bridges any constrained variables supported by the bridge BT. This is in contrast with the MathOptInterface.Bridges.LazyBridgeOptimizer which only bridges the constrained variables that are unsupported by the internal model, even if they are supported by one of its bridges.

> **Note**
>
> Two bridge optimizers using variable bridges cannot be used together as both of them assume that the underlying model only returns variable indices with nonnegative values.

MathOptInterface.Bridges.Variable.add_all_bridges – Function.

```
add_all_bridges(bridged_model, ::Type{T}) where {T}
```

Add all bridges defined in the Bridges.Variable submodule to bridged_model. The coefficient type used is T.

Bridges implemented `MathOptInterface.Bridges.Variable.FlipSignBridge` – Type.

```
FlipSignBridge{T, S1, S2}
```

Bridge constrained variables in S1 into constrained variables in S2 by multiplying the variables by `-1` and taking the point reflection of the set across the origin. The flipped `MOI.VectorOfVariables`-in-S constraint is stored in the `flipped_constraint` field by convention.

`MathOptInterface.Bridges.Variable.ZerosBridge` – Type.

```
ZerosBridge{T} <: Bridges.Variable.AbstractBridge
```

Transforms constrained variables in `MathOptInterface.Zeros` to zeros, which ends up creating no variables in the underlying model.

The bridged variables are therefore similar to parameters with zero values. Parameters with non-zero value can be created with constrained variables in `MOI.EqualTo` by combining a `VectorizeBridge` and this bridge. The functions cannot be unbridged, given a function, we cannot determine, if the bridged variables were used.

The dual values cannot be determined by the bridge but they can be determined by the bridged optimizer using `MathOptInterface.Utilities.get_fallback` if a `CachingOptimizer` is used (since `ConstraintFunction` cannot be got as functions cannot be unbridged).

`MathOptInterface.Bridges.Variable.FreeBridge` – Type.

```
FreeBridge{T} <: Bridges.Variable.AbstractBridge
```

Transforms constrained variables in `MOI.Reals` to the difference of constrained variables in `MOI.Nonnegatives`.

`MathOptInterface.Bridges.Variable.NonposToNonnegBridge` – Type.

```
NonposToNonnegBridge{T} <:
    FlipSignBridge{T, MOI.Nonpositives, MOI.Nonnegatives}
```

Transforms constrained variables in `Nonpositives` into constrained variables in `Nonnegatives`.

`MathOptInterface.Bridges.Variable.VectorizeBridge` – Type.

```
VectorizeBridge{T, S}
```

Transforms a constrained variable in `scalar_set_type(S, T)` where `S <: VectorLinearSet` into a constrained vector of one variable in S. For instance, `VectorizeBridge{Float64, MOI.Nonnegatives}` transforms a constrained variable in `MOI.GreaterThan{Float64}` into a constrained vector of one variable in `MOI.Nonnegatives`.

`MathOptInterface.Bridges.Variable.SOCtoRSOCBridge` – Type.

```
SOCtoRSOCBridge{T} <:
↪    Bridges.Variable.SetMapBridge{T,MOI.RotatedSecondOrderCone,MOI.SecondOrderCone}
```

Same transformation as `MOI.Bridges.Constraint.SOCtoRSOCBridge`.

`MathOptInterface.Bridges.Variable.RSOCtoSOCBridge` – Type.

```
RSOCtoSOCBridge{T} <:
↪    Bridges.Variable.SetMapBridge{T,MOI.SecondOrderCone,MOI.RotatedSecondOrderCone}
```

Same transformation as MOI.Bridges.Constraint.RSOCtoSOCBridge.

MathOptInterface.Bridges.Variable.RSOCtoPSDBridge – Type.

```
RSOCtoPSDBridge{T} <: Bridges.Variable.AbstractBridge
```

Transforms constrained variables in MathOptInterface.RotatedSecondOrderCone to constrained variables in MathOptInterface.PositiveSemidefiniteConeTriangle.

Objective bridges MathOptInterface.Bridges.Objective.AbstractBridge – Type.

```
AbstractBridge
```

Subtype of MathOptInterface.Bridges.AbstractBridge for objective bridges.

MathOptInterface.Bridges.Objective.SingleBridgeOptimizer – Type.

```
SingleBridgeOptimizer{BT<:AbstractBridge, OT<:MOI.ModelLike} <: AbstractBridgeOptimizer
```

The `SingleBridgeOptimizer` bridges any objective functions supported by the bridge BT. This is in contrast with the MathOptInterface.Bridges.LazyBridgeOptimizer which only bridges the objective functions that are unsupported by the internal model, even if they are supported by one of its bridges.

MathOptInterface.Bridges.Objective.add_all_bridges – Function.

```
add_all_bridges(bridged_model, ::Type{T}) where {T}
```

Add all bridges defined in the `Bridges.Objective` submodule to `bridged_model`. The coefficient type used is T.

Bridges implemented MathOptInterface.Bridges.Objective.SlackBridge – Type.

```
SlackBridge{T, F, G}
```

The `SlackBridge` converts an objective function of type `G` into a MOI.VariableIndex objective by creating a slack variable and a F-in-MOI.LessThan constraint for minimization or F-in-MOI.LessThan constraint for maximization where F is `MOI.Utilities.promote_operation(-, T, G, MOI.VariableIndex}`. Note that when using this bridge, changing the optimization sense is not supported. Set the sense to `MOI.FEASIBILITY_SENSE` first to delete the bridge in order to change the sense, then re-add the objective.

MathOptInterface.Bridges.Objective.FunctionizeBridge – Type.

```
FunctionizeBridge{T}
```

The `FunctionizeBridge` converts a `VariableIndex` objective into a `ScalarAffineFunction{T}` objective.

40.3 FileFormats

Overview

The FileFormats submodule

The `FileFormats` module provides functionality for reading and writing MOI models using write_to_file and read_from_file.

Supported file types You must read and write files to a `FileFormats.Model` object. Specifc the file-type by passing a `FileFormats.FileFormat` enum. For example:

The Conic Benchmark Format

```
julia> model = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_CBF)
A Conic Benchmark Format (CBF) model
```

The LP file format

```
julia> model = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_LP)
A .LP-file model
```

The MathOptFormat file format

```
julia> model = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_MOF)
A MathOptFormat Model
```

The MPS file format

```
julia> model = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_MPS)
A Mathematical Programming System (MPS) model
```

The NL file format

```
julia> model = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_NL)
An AMPL (.nl) model
```

The SDPA file format

```
julia> model = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_SDPA)
A SemiDefinite Programming Algorithm Format (SDPA) model
```

Write to file To write a model `src` to a MathOptFormat file, use:

```
julia> src = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> MOI.add_variable(src)
MathOptInterface.VariableIndex(1)

julia> dest = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_MOF)
A MathOptFormat Model

julia> MOI.copy_to(dest, src)
MathOptInterface.Utilities.IndexMap with 1 entry:
  VariableIndex(1) => VariableIndex(1)

julia> MOI.write_to_file(dest, "file.mof.json")

julia> print(read("file.mof.json", String))
```

```
{
  "name": "MathOptFormat Model",
  "version": {
    "major": 1,
    "minor": 0
  },
  "variables": [
    {
      "name": "x1"
    }
  ],
  "objective": {
    "sense": "feasibility"
  },
  "constraints": []
}
```

Read from file To read a MathOptFormat file, use:

```
julia> dest = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_MOF)
A MathOptFormat Model

julia> MOI.read_from_file(dest, "file.mof.json")

julia> MOI.get(dest, MOI.ListOfVariableIndices())
1-element Vector{MathOptInterface.VariableIndex}:
 MathOptInterface.VariableIndex(1)

julia> rm("file.mof.json")  # Clean up after ourselves.
```

Detecting the filetype automatically Instead of the `format` keyword, you can also use the `filename` keyword argument to `FileFormats.Model`. This will attempt to automatically guess the format from the file extension. For example:

```
julia> src = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> dest = MOI.FileFormats.Model(filename = "file.cbf.gz")
A Conic Benchmark Format (CBF) model

julia> MOI.copy_to(dest, src)
MathOptInterface.Utilities.IndexMap()

julia> MOI.write_to_file(dest, "file.cbf.gz")

julia> src_2 = MOI.FileFormats.Model(filename = "file.cbf.gz")
A Conic Benchmark Format (CBF) model

julia> src = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> dest = MOI.FileFormats.Model(filename = "file.cbf.gz")
A Conic Benchmark Format (CBF) model

julia> MOI.copy_to(dest, src)
```

```
MathOptInterface.Utilities.IndexMap()

julia> MOI.write_to_file(dest, "file.cbf.gz")

julia> src_2 = MOI.FileFormats.Model(filename = "file.cbf.gz")
A Conic Benchmark Format (CBF) model

julia> MOI.read_from_file(src_2, "file.cbf.gz")

julia> rm("file.cbf.gz")  # Clean up after ourselves.
```

Note how the compression format (GZip) is also automatically detected from the filename.

Unsupported constraints In some cases `src` may contain constraints that are not supported by the file format (e.g., the CBF format supports integer variables but not binary). If so, copy `src` to a bridged model using `Bridges.full_bridge_optimizer`:

```
src = MOI.Utilities.Model{Float64}()
x = MOI.add_variable(model)
MOI.add_constraint(model, x, MOI.ZeroOne())
dest = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_CBF)
bridged = MOI.Bridges.full_bridge_optimizer(dest, Float64)
MOI.copy_to(bridged, src)
MOI.write_to_file(dest, "my_model.cbf")
```

> **Note**
>
> Even after bridging, it may still not be possible to write the model to file because of unsupported constraints (e.g., PSD variables in the LP file format).

Read and write to io In addition to `write_to_file` and `read_from_file`, you can read and write directly from `IO` streams using `Base.write` and `Base.read!`:

```
julia> src = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}

julia> dest = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_MPS)
A Mathematical Programming System (MPS) model

julia> MOI.copy_to(dest, src)
MathOptInterface.Utilities.IndexMap()

julia> io = IOBuffer();

julia> write(io, dest)

julia> seekstart(io);

julia> src_2 = MOI.FileFormats.Model(format = MOI.FileFormats.FORMAT_MPS)
A Mathematical Programming System (MPS) model

julia> read!(io, src_2);
```

Validating MOF files MathOptFormat files are governed by a schema. Use JSONSchema.jl to check if a `.mof.json` file satisfies the schema.

First, construct the schema object as follows:

```
julia> import JSON, JSONSchema

julia> schema = JSONSchema.Schema(JSON.parsefile(MOI.FileFormats.MOF.SCHEMA_PATH))
A JSONSchema
```

Then, check if a model file is valid using `isvalid`:

```
julia> good_model = JSON.parse("""
       {
         "version": {
           "major": 1,
           "minor": 0
         },
         "variables": [{"name": "x"}],
         "objective": {"sense": "feasibility"},
         "constraints": []
       }
       """);

julia> isvalid(schema, good_model)
true
```

If we construct an invalid file, for example by mis-typing `name` as `NaMe`, the validation fails:

```
julia> bad_model = JSON.parse("""
       {
         "version": {
           "major": 1,
           "minor": 0
         },
         "variables": [{"NaMe": "x"}],
         "objective": {"sense": "feasibility"},
         "constraints": []
       }
       """);

julia> isvalid(schema, bad_model)
false
```

Use `JSONSchema.validate` to obtain more insight into why the validation failed:

```
julia> JSONSchema.validate(schema, bad_model)
Validation failed:
path:         [variables][1]
instance:     Dict{String, Any}("NaMe" => "x")
schema key:   required
schema value: Any["name"]
```

API Reference

File Formats Functions to help read and write MOI models to/from various file formats. See The FileFormats submodule for more details.

MathOptInterface.FileFormats.Model – Function.

```
Model(
    ;
    format::FileFormat = FORMAT_AUTOMATIC,
    filename::Union{Nothing, String} = nothing,
    kwargs...
)
```

Return model corresponding to the `FileFormat` `format`, or, if `format == FORMAT_AUTOMATIC`, guess the format from `filename`.

The `filename` argument is only needed if `format == FORMAT_AUTOMATIC`.

`kwargs` are passed to the underlying model constructor.

MathOptInterface.FileFormats.FileFormat – Type.

```
FileFormat
```

List of accepted export formats.

- `FORMAT_AUTOMATIC`: try to detect the file format based on the file name
- `FORMAT_CBF`: the Conic Benchmark format
- `FORMAT_LP`: the LP file format
- `FORMAT_MOF`: the MathOptFormat file format
- `FORMAT_MPS`: the MPS file format
- `FORMAT_NL`: the AMPL .nl file format
- `FORMAT_SDPA`: the SemiDefinite Programming Algorithm format

40.4 Utilities

Overview

The Utilities submodule

The Utilities submodule provides a variety of functionality for managing `MOI.ModelLike` objects.

Utilities.Model `Utilities.Model` provides an implementation of a `ModelLike` that efficiently supports all functions and sets defined within MOI. However, given the extensibility of MOI, this might not cover all use cases.

Create a model as follows:

```
julia> model = MOI.Utilities.Model{Float64}()
MOIU.Model{Float64}
```

Utilities.UniversalFallback `Utilities.UniversalFallback` is a layer that sits on top of any `ModelLike` and provides non-specialized (slower) fallbacks for constraints and attributes that the underlying `ModelLike` does not support.

For example, `Utilities.Model` doesn't support some variable attributes like `VariablePrimalStart`, so JuMP uses a combination of Universal fallback and `Utilities.Model` as a generic problem cache:

```
julia> model = MOI.Utilities.UniversalFallback(MOI.Utilities.Model{Float64}())
MOIU.UniversalFallback{MOIU.Model{Float64}}
fallback for MOIU.Model{Float64}
```

> **Warning**
>
> Adding a UniversalFallback means that your `model` will now support all constraints, even if the inner-model does not! This can lead to unexpected behavior.

Utilities.@model For advanced use cases that need efficient support for functions and sets defined outside of MOI (but still known at compile time), we provide the `Utilities.@model` macro.

The `@model` macro takes a name (for a new type, which must not exist yet), eight tuples specifying the types of constraints that are supported, and then a `Bool` indicating the type is a subtype of `MOI.AbstractOptimizer` (if `true`) or `MOI.ModelLike` (if `false`).

The eight tuples are in the following order:

1. Un-typed scalar sets, e.g., `Integer`
2. Typed scalar sets, e.g., `LessThan`
3. Un-typed vector sets, e.g., `Nonnegatives`
4. Typed vector sets, e.g., `PowerCone`
5. Un-typed scalar functions, e.g., `VariableIndex`
6. Typed scalar functions, e.g., `ScalarAffineFunction`
7. Un-typed vector functions, e.g., `VectorOfVariables`
8. Typed vector functions, e.g., `VectorAffineFunction`

The tuples can contain more than one element. Typed-sets must be specified without their type parameter, i.e., `MOI.LessThan`, not `MOI.LessThan{Float64}`.

Here is an example:

```
julia> MOI.Utilities.@model(
           MyNewModel,
           (MOI.Integer,),                  # Un-typed scalar sets
           (MOI.GreaterThan,),              # Typed scalar sets
           (MOI.Nonnegatives,),             # Un-typed vector sets
           (MOI.PowerCone,),                # Typed vector sets
           (MOI.VariableIndex,),            # Un-typed scalar functions
           (MOI.ScalarAffineFunction,),     # Typed scalar functions
           (MOI.VectorOfVariables,),        # Un-typed vector functions
           (MOI.VectorAffineFunction,),     # Typed vector functions
```

```
           true,                            # <:MOI.AbstractOptimizer?
       )
MathOptInterface.Utilities.GenericOptimizer{T, MathOptInterface.Utilities.ObjectiveContainer{T},
↪  MathOptInterface.Utilities.VariablesContainer{T}, MyNewModelFunctionConstraints{T}} where T

julia> model = MyNewModel{Float64}()
MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64}, MOIU.VariablesContainer{Float64},
↪  MyNewModelFunctionConstraints{Float64}}
```

Warning

`MyNewModel` supports every `VariableIndex`-in-Set constraint, as well as `VariableIndex`, `ScalarAffineFunction`, and `ScalarQuadraticFunction` objective functions. Implement `MOI.supports` as needed to forbid constraint and objective function combinations.

As another example, PATHSolver, which only supports `VectorAffineFunction`-in-`Complements` defines its optimizer as:

```
julia> MOI.Utilities.@model(
           PathOptimizer,
           (),  # Scalar sets
           (),  # Typed scalar sets
           (MOI.Complements,),  # Vector sets
           (),  # Typed vector sets
           (),  # Scalar functions
           (),  # Typed scalar functions
           (),  # Vector functions
           (MOI.VectorAffineFunction,),  # Typed vector functions
           true,  # is_optimizer
       )
MathOptInterface.Utilities.GenericOptimizer{T, MathOptInterface.Utilities.ObjectiveContainer{T},
↪  MathOptInterface.Utilities.VariablesContainer{T},
↪  MathOptInterface.Utilities.VectorOfConstraints{MathOptInterface.VectorAffineFunction{T},
↪  MathOptInterface.Complements}} where T
```

However, `PathOptimizer` does not support some `VariableIndex`-in-Set constraints, so we must explicitly define:

```
julia> function MOI.supports_constraint(
           ::PathOptimizer,
           ::Type{MOI.VariableIndex},
           ::Type{Union{<:MOI.Semiinteger,MOI.Semicontinuous,MOI.ZeroOne,MOI.Integer}}
       )
           return false
       end
```

Finally, PATH doesn't support an objective function, so we need to add:

```
julia> MOI.supports(::PathOptimizer, ::MOI.ObjectiveFunction) = false
```

Warning

This macro creates a new type, so it must be called from the top-level of a module, e.g., it cannot be called from inside a function.

Utilities.CachingOptimizer A [`Utilities.CachingOptimizer`] is an MOI layer that abstracts the difference between solvers that support incremental modification (e.g., they support adding variables one-by-one), and solvers that require the entire problem in a single API call (e.g., they only accept the `A`, `b` and `c` matrices of a linear program).

It has two parts:

1. A cache, where the model can be built and modified incrementally
2. An optimizer, which is used to solve the problem

```
julia> model = MOI.Utilities.CachingOptimizer(
           MOI.Utilities.Model{Float64}(),
           PathOptimizer{Float64}(),
       )
MOIU.CachingOptimizer{MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}, MOIU.Model{Float64}}
in state EMPTY_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.Model{Float64}
with optimizer MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}
```

A `Utilities.CachingOptimizer` may be in one of three possible states:

- `NO_OPTIMIZER`: The CachingOptimizer does not have any optimizer.
- `EMPTY_OPTIMIZER`: The CachingOptimizer has an empty optimizer, and it is not synchronized with the cached model. Modifications are forwarded to the cache, but not to the optimizer.
- `ATTACHED_OPTIMIZER`: The CachingOptimizer has an optimizer, and it is synchronized with the cached model. Modifications are forwarded to the optimizer. If the optimizer does not support modifications, and error will be thrown.

Use `Utilities.attach_optimizer` to go from `EMPTY_OPTIMIZER` to `ATTACHED_OPTIMIZER`:

```
julia> MOI.Utilities.attach_optimizer(model)

julia> model
MOIU.CachingOptimizer{MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}, MOIU.Model{Float64}}
in state ATTACHED_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.Model{Float64}
with optimizer MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}
```

Info

You must be in `ATTACHED_OPTIMIZER` to use `optimize!`.

Use `Utilities.reset_optimizer` to go from ATTACHED_OPTIMIZER to EMPTY_OPTIMIZER:

```
julia> MOI.Utilities.reset_optimizer(model)

julia> model
MOIU.CachingOptimizer{MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}, MOIU.Model{Float64}}
in state EMPTY_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.Model{Float64}
with optimizer MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}
```

Info

Calling `MOI.empty!(model)` also resets the state to EMPTY_OPTIMIZER. So after emptying a model, the modification will only be applied to the cache.

Use `Utilities.drop_optimizer` to go from any state to NO_OPTIMIZER:

```
julia> MOI.Utilities.drop_optimizer(model)

julia> model
MOIU.CachingOptimizer{MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}, MOIU.Model{Float64}}
in state NO_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.Model{Float64}
with optimizer nothing
```

Pass an empty optimizer to `Utilities.reset_optimizer` to go from NO_OPTIMIZER to EMPTY_OPTIMIZER:

```
julia> MOI.Utilities.reset_optimizer(model, PathOptimizer{Float64}())

julia> model
MOIU.CachingOptimizer{MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}, MOIU.Model{Float64}}
in state EMPTY_OPTIMIZER
in mode AUTOMATIC
with model cache MOIU.Model{Float64}
with optimizer MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}
```

Deciding when to attach and reset the optimizer is tedious, and you will often write code like this:

```
try
    # modification
catch
```

```
    MOI.Utilities.reset_optimizer(model)
    # Re-try modification
end
```

To make this easier, `Utilities.CachingOptimizer` has two modes of operation:

- `AUTOMATIC`: The `CachingOptimizer` changes its state when necessary. Attempting to add a constraint or perform a modification not supported by the optimizer results in a drop to `EMPTY_OPTIMIZER` mode.
- `MANUAL`: The user must change the state of the `CachingOptimizer`. Attempting to perform an operation in the incorrect state results in an error.

By default, `AUTOMATIC` mode is chosen. However, you can create a `CachingOptimizer` in `MANUAL` mode as follows:

```
julia> model = MOI.Utilities.CachingOptimizer(
           MOI.Utilities.Model{Float64}(),
           MOI.Utilities.MANUAL,
       )
MOIU.CachingOptimizer{MOI.AbstractOptimizer, MOIU.Model{Float64}}
in state NO_OPTIMIZER
in mode MANUAL
with model cache MOIU.Model{Float64}
with optimizer nothing

julia> MOI.Utilities.reset_optimizer(model, PathOptimizer{Float64}())

julia> model
MOIU.CachingOptimizer{MOI.AbstractOptimizer, MOIU.Model{Float64}}
in state EMPTY_OPTIMIZER
in mode MANUAL
with model cache MOIU.Model{Float64}
with optimizer MOIU.GenericOptimizer{Float64, MOIU.ObjectiveContainer{Float64},
↪  MOIU.VariablesContainer{Float64}, MOIU.VectorOfConstraints{MOI.VectorAffineFunction{Float64},
↪  MOI.Complements}}
```

Printing Use `print` to print the formulation of the model.

```
julia> model = MOI.Utilities.Model{Float64}();

julia> x = MOI.add_variable(model)
MathOptInterface.VariableIndex(1)

julia> MOI.set(model, MOI.VariableName(), x, "x_var")

julia> MOI.add_constraint(model, x, MOI.ZeroOne())
MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex, MathOptInterface.ZeroOne}(1)

julia> MOI.set(model, MOI.ObjectiveFunction{typeof(x)}(), x)

julia> MOI.set(model, MOI.ObjectiveSense(), MOI.MAX_SENSE)

julia> print(model)
Maximize VariableIndex:
```

```
 x_var

Subject to:

VariableIndex-in-ZeroOne
 x_var ∈ {0, 1}
```

Use `Utilities.latex_formulation` to display the model in LaTeX form:

```
julia> MOI.Utilities.latex_formulation(model)
$$ \begin{aligned}
\max\quad & x\_var \\
\text{Subject to}\\
 & \text{VariableIndex-in-ZeroOne} \\
 & x\_var \in \{0, 1\} \\
\end{aligned} $$
```

> **Tip**
>
> In IJulia, calling `print` or ending a cell with `Utilities.latex_formulation` will render the model in LaTeX.

Utilities.MatrixOfConstraints The constraints of `Utilities.Model` are stored as a vector of tuples of function and set in a `Utilities.VectorOfConstraints`. Other representations can be used by parametrizing the type `Utilities.GenericModel` (resp. `Utilities.GenericOptimizer`). For instance, if all non-`VariableIndex` constraints are affine, the coefficients of all the constraints can be stored in a single sparse matrix using `Utilities.MatrixOfConstraints`. The constraints storage can even be customized up to a point where it exactly matches the storage of the solver of interest, in which case `copy_to` can be implemented for the solver by calling `copy_to` to this custom model.

For instance, Clp defines the following model

```
MOI.Utilities.@product_of_scalar_sets(LP, MOI.EqualTo{T}, MOI.LessThan{T}, MOI.GreaterThan{T})
const Model = MOI.Utilities.GenericModel{
    Float64,
    MOI.Utilities.MatrixOfConstraints{
        Float64,
        MOI.Utilities.MutableSparseMatrixCSC{Float64,Cint,MOI.Utilities.ZeroBasedIndexing},
        MOI.Utilities.Hyperrectangle{Float64},
        LP{Float64},
    },
}
```

The `copy_to` operation can now be implemented as follows (assuming that the `Model` definition above is in the `Clp` module so that it can be referred to as `Model`, to be distinguished with `Utilities.Model`):

```
function _copy_to(dest::Optimizer, src::Model)
    @assert MOI.is_empty(dest)
    A = src.constraints.coefficients
    row_bounds = src.constraints.constants
    Clp_loadProblem(
        dest,
        A.n,
```

```
        A.m,
        A.colptr,
        A.rowval,
        A.nzval,
        src.lower_bound,
        src.upper_bound,
        # (...) objective vector (omitted),
        row_bounds.lower,
        row_bounds.upper,
    )
    # Set objective sense and constant (omitted)
    return
end

function MOI.copy_to(dest::Optimizer, src::Model)
    _copy_to(dest, src)
    return MOI.Utilities.identity_index_map(src)
end

function MOI.copy_to(
    dest::Optimizer,
    src::MOI.Utilities.UniversalFallback{Model},
)
    # Copy attributes from `src` to `dest` and error in case any unsupported
    # constraints or attributes are set in `UniversalFallback`.
    return MOI.copy_to(dest, src.model)
end

function MOI.copy_to(
    dest::Optimizer,
    src::MOI.ModelLike,
)
    model = Model()
    index_map = MOI.copy_to(model, src)
    _copy_to(dest, model)
    return index_map
end
```

ModelFilter Utilities provides `Utilities.ModelFilter` as a useful tool to copy a subset of a model. For example, given an infeasible model, we can copy the irreducible infeasible subsystem (for models implementing `ConstraintConflictStatus`) as follows:

```
my_filter(::Any) = true
function my_filter(ci::MOI.ConstraintIndex)
    status = MOI.get(dest, MOI.ConstraintConflictStatus(), ci)
    return status != MOI.NOT_IN_CONFLICT
end
filtered_src = MOI.Utilities.ModelFilter(my_filter, src)
index_map = MOI.copy_to(dest, filtered_src)
```

Fallbacks The value of some attributes can be inferred from the value of other attributes.

For example, the value of `ObjectiveValue` can be computed using `ObjectiveFunction` and `VariablePrimal`.

When a solver gives direct access to an attribute, it is better to return this value. However, if this is not the case, `Utilities.get_fallback` can be used instead. For example:

```
function MOI.get(model::Optimizer, attr::MOI.ObjectiveFunction)
    return MOI.Utilities.get_fallback(model, attr)
end
```

DoubleDicts When writing MOI interfaces, we often need to handle situations in which we map `ConstraintIndex`s to different values. For example, to a string for `ConstraintName`.

One option is to use a dictionary like `Dict{MOI.ConstraintIndex,String}`. However, this incurs a performance cost because the key is not a concrete type.

The DoubleDicts submodule helps this situation by providing two types main types `Utilities.DoubleDicts.DoubleDict` and `Utilities.DoubleDicts.IndexDoubleDict`. These types act like normal dictionaries, but internally they use more efficient dictionaries specialized to the type of the function-set pair.

The most common usage of a `DoubleDict` is in the `index_map` returned by `copy_to`. Performance can be improved, by using a function barrier. That is, instead of code like:

```
index_map = MOI.copy_to(dest, src)
for (F, S) in MOI.get(src, MOI.ListOfConstraintTypesPresent())
    for ci in MOI.get(src, MOI.ListOfConstraintIndices{F,S}())
        dest_ci = index_map[ci]
        # ...
    end
end
```

use instead:

```
function function_barrier(
    dest,
    src,
    index_map::MOI.Utilities.DoubleDicts.IndexDoubleDictInner{F,S},
) where {F,S}
    for ci in MOI.get(src, MOI.ListOfConstraintIndices{F,S}())
        dest_ci = index_map[ci]
        # ...
    end
    return
end

index_map = MOI.copy_to(dest, src)
for (F, S) in MOI.get(src, MOI.ListOfConstraintTypesPresent())
    function_barrier(dest, src, index_map[F, S])
end
```

API Reference

Utilities.Model `MathOptInterface.Utilities.Model` – Type.

An implementation of `ModelLike` that supports all functions and sets defined in MOI. It is parameterized by the coefficient type.

Examples

```
model = Model{Float64}()
x = add_variable(model)
```

Utilities.UniversalFallback `MathOptInterface.Utilities.UniversalFallback` – Type.

```
UniversalFallback
```

The `UniversalFallback` can be applied on a `MathOptInterface.ModelLike` `model` to create the model `UniversalFallback(model)` supporting any constraint and attribute. This allows to have a specialized implementation in `model` for performance critical constraints and attributes while still supporting other attributes with a small performance penalty. Note that `model` is unaware of constraints and attributes stored by `UniversalFallback` so this is not appropriate if `model` is an optimizer (for this reason, `MathOptInterface.optimize!` has not been implemented). In that case, optimizer bridges should be used instead.

Utilities.@model `MathOptInterface.Utilities.@model` – Macro.

```
macro model(
    model_name,
    scalar_sets,
    typed_scalar_sets,
    vector_sets,
    typed_vector_sets,
    scalar_functions,
    typed_scalar_functions,
    vector_functions,
    typed_vector_functions,
    is_optimizer = false
)
```

Creates a type `model_name` implementing the MOI model interface and containing `scalar_sets` scalar sets `typed_scalar_sets` typed scalar sets, `vector_sets` vector sets, `typed_vector_sets` typed vector sets, `scalar_functions` scalar functions, `typed_scalar_functions` typed scalar functions, `vector_functions` vector functions and `typed_vector_functions` typed vector functions. To give no set/function, write `()`, to give one set S, write `(S,)`.

The function `MathOptInterface.VariableIndex` should not be given in `scalar_functions`. The model supports `MathOptInterface.VariableIndex`-in-S constraints where S is `MathOptInterface.EqualTo`, `MathOptInterface.Gr` `MathOptInterface.LessThan`, `MathOptInterface.Interval`, `MathOptInterface.Integer`, `MathOptInterface.ZeroOne`, `MathOptInterface.Semicontinuous` or `MathOptInterface.Semiinteger`. The sets supported with the `MathOptInterface.VariableIndex` cannot be controlled from the macro, use the `UniversalFallback` to support more sets.

This macro creates a model specialized for specific types of constraint, by defining specialized structures and methods. To create a model that, in addition to be optimized for specific constraints, also support arbitrary constraints and attributes, use `UniversalFallback`.

If `is_optimizer = true`, the resulting struct is a of `GenericOptimizer`, which is a subtype of `MathOptInterface.AbstractOp` otherwise, it is a `GenericModel`, which is a subtype of `MathOptInterface.ModelLike`.

Examples

The model describing an linear program would be:

```
@model(LPModel,                                                # Name of model
      (),                                                      # untyped scalar sets
      (MOI.EqualTo, MOI.GreaterThan, MOI.LessThan, MOI.Interval), #   typed scalar sets
      (MOI.Zeros, MOI.Nonnegatives, MOI.Nonpositives),         # untyped vector sets
      (),                                                      #   typed vector sets
      (),                                                      # untyped scalar functions
```

```
    (MOI.ScalarAffineFunction,),                         #   typed scalar functions
    (MOI.VectorOfVariables,),                            # untyped vector functions
    (MOI.VectorAffineFunction,),                         #   typed vector functions
    false
)
```

Let MOI denote MathOptInterface, MOIU denote MOI.Utilities. The macro would create the following types with struct_of_constraint_code:

```
struct LPModelScalarConstraints{T, C1, C2, C3, C4} <: MOIU.StructOfConstraints
    moi_equalto::C1
    moi_greaterthan::C2
    moi_lessthan::C3
    moi_interval::C4
end
struct LPModelVectorConstraints{T, C1, C2, C3} <: MOIU.StructOfConstraints
    moi_zeros::C1
    moi_nonnegatives::C2
    moi_nonpositives::C3
end
struct LPModelFunctionConstraints{T} <: MOIU.StructOfConstraints
    moi_scalaraffinefunction::LPModelScalarConstraints{
        T,
        MOIU.VectorOfConstraints{MOI.ScalarAffineFunction{T}, MOI.EqualTo{T}},
        MOIU.VectorOfConstraints{MOI.ScalarAffineFunction{T}, MOI.GreaterThan{T}},
        MOIU.VectorOfConstraints{MOI.ScalarAffineFunction{T}, MOI.LessThan{T}},
        MOIU.VectorOfConstraints{MOI.ScalarAffineFunction{T}, MOI.Interval{T}}
    }
    moi_vectorofvariables::LPModelVectorConstraints{
        T,
        MOIU.VectorOfConstraints{MOI.VectorOfVariables, MOI.Zeros},
        MOIU.VectorOfConstraints{MOI.VectorOfVariables, MOI.Nonnegatives},
        MOIU.VectorOfConstraints{MOI.VectorOfVariables, MOI.Nonpositives}
    }
    moi_vectoraffinefunction::LPModelVectorConstraints{
        T,
        MOIU.VectorOfConstraints{MOI.VectorAffineFunction{T}, MOI.Zeros},
        MOIU.VectorOfConstraints{MOI.VectorAffineFunction{T}, MOI.Nonnegatives},
        MOIU.VectorOfConstraints{MOI.VectorAffineFunction{T}, MOI.Nonpositives}
    }
end
const LPModel{T} =
↪  MOIU.GenericModel{T,MOIU.ObjectiveContainer{T},MOIU.VariablesContainer{T},LPModelFunctionConstraints{T}}
```

The type LPModel implements the MathOptInterface API except methods specific to optimizers like optimize! or get with VariablePrimal.

MathOptInterface.Utilities.GenericModel – Type.

```
mutable struct GenericModel{T,O,V,C} <: AbstractModelLike{T}
```

Implements a model supporting coefficients of type T and:

- An objective function stored in .objective::O
- Variables and VariableIndex constraints stored in .variable_bounds::V

- F-in-S constraints (excluding `VariableIndex` constraints) stored in `.constraints::C`

All interactions should take place via the MOI interface, so the types `O`, `V`, and `C` should implement the API as needed for their functionality.

`MathOptInterface.Utilities.GenericOptimizer` – Type.

```
mutable struct GenericOptimizer{T,O,V,C} <: AbstractOptimizer{T}
```

Implements a model supporting coefficients of type `T` and:

- An objective function stored in `.objective::O`
- Variables and `VariableIndex` constraints stored in `.variable_bounds::V`
- F-in-S constraints (excluding `VariableIndex` constraints) stored in `.constraints::C`

All interactions should take place via the MOI interface, so the types `O`, `V`, and `C` should implement the API as needed for their functionality.

`.objective` `MathOptInterface.Utilities.ObjectiveContainer` – Type.

```
ObjectiveContainer{T}
```

A helper struct to simplify the handling of objective functions in Utilities.Model.

`.variables` `MathOptInterface.Utilities.VariablesContainer` – Type.

```
struct VariablesContainer{T} <: AbstractVectorBounds
    set_mask::Vector{UInt16}
    lower::Vector{T}
    upper::Vector{T}
end
```

A struct for storing variables and VariableIndex-related constraints. Used in `MOI.Utilities.Model` by default.

`MathOptInterface.Utilities.FreeVariables` – Type.

```
mutable struct FreeVariables <: MOI.ModelLike
    n::Int64
    FreeVariables() = new(0)
end
```

A struct for storing free variables that can be used as the `variables` field of `GenericModel` or `GenericModel`. It represents a model that does not support any constraint nor objective function.

Example

The following model type represents a conic model in geometric form. As opposed to `VariablesContainer`, `FreeVariables` does not support constraint bounds so they are bridged into an affine constraint in the `MathOptInterface.Nonnegatives` cone as expected for the geometric conic form.

```
julia> MOI.Utilities.@product_of_sets(
    Cones,
    MOI.Zeros,
    MOI.Nonnegatives,
    MOI.SecondOrderCone,
```

```
    MOI.PositiveSemidefiniteConeTriangle,
);

julia> const ConicModel{T} = MOI.Utilities.GenericOptimizer{
    T,
    MOI.Utilities.ObjectiveContainer{T},
    MOI.Utilities.FreeVariables,
    MOI.Utilities.MatrixOfConstraints{
        T,
        MOI.Utilities.MutableSparseMatrixCSC{
            T,
            Int,
            MOI.Utilities.OneBasedIndexing,
        },
        Vector{T},
        Cones{T},
    },
};

julia> model = MOI.instantiate(ConicModel{Float64}, with_bridge_type=Float64);

julia> x = MOI.add_variable(model)
MathOptInterface.VariableIndex(1)

julia> c = MOI.add_constraint(model, x, MOI.GreaterThan(1.0))
MathOptInterface.ConstraintIndex{MathOptInterface.VariableIndex, MathOptInterface.GreaterThan{
     Float64}}(1)

julia> MOI.Bridges.is_bridged(model, c)
true

julia> bridge = MOI.Bridges.bridge(model, c)
MathOptInterface.Bridges.Constraint.VectorizeBridge{Float64, MathOptInterface.
     VectorAffineFunction{Float64}, MathOptInterface.Nonnegatives, MathOptInterface.VariableIndex
     }(MathOptInterface.ConstraintIndex{MathOptInterface.VectorAffineFunction{Float64},
     MathOptInterface.Nonnegatives}(1), 1.0)

julia> bridge.vector_constraint
MathOptInterface.ConstraintIndex{MathOptInterface.VectorAffineFunction{Float64}, MathOptInterface
     .Nonnegatives}(1)

julia> MOI.Bridges.is_bridged(model, bridge.vector_constraint)
false
```

.constraints MathOptInterface.Utilities.VectorOfConstraints – Type.

```
mutable struct VectorOfConstraints{
    F<:MOI.AbstractFunction,
    S<:MOI.AbstractSet,
} <: MOI.ModelLike
    constraints::CleverDicts.CleverDict{
        MOI.ConstraintIndex{F,S},
        Tuple{F,S},
        typeof(CleverDicts.key_to_index),
        typeof(CleverDicts.index_to_key),
```

```
    }
end
```

A struct storing F-in-S constraints as a mapping between the constraint indices to the corresponding tuple of function and set.

MathOptInterface.Utilities.StructOfConstraints – Type.

```
abstract type StructOfConstraints <: MOI.ModelLike end
```

A struct storing a subfields other structs storing constraints of different types.

See `Utilities.@struct_of_constraints_by_function_types` and `Utilities.@struct_of_constraints_by_set_types`.

MathOptInterface.Utilities.@struct_of_constraints_by_function_types – Macro.

```
Utilities.@struct_of_constraints_by_function_types(name, func_types...)
```

Given a vector of n function types (`F1, F2,..., Fn`) in `func_types`, defines a subtype of `StructOfConstraints` of name name and which type parameters `{T, C1, C2, ..., Cn}`. It contains n field where the ith field has type `Ci` and stores the constraints of function type `Fi`.

The expression `Fi` can also be a union in which case any constraint for which the function type is in the union is stored in the field with type `Ci`.

MathOptInterface.Utilities.@struct_of_constraints_by_set_types – Macro.

```
Utilities.@struct_of_constraints_by_set_types(name, func_types...)
```

Given a vector of n set types (`S1, S2,..., Sn`) in `func_types`, defines a subtype of `StructOfConstraints` of name name and which type parameters `{T, C1, C2, ..., Cn}`. It contains n field where the ith field has type `Ci` and stores the constraints of set type `Si`. The expression `Si` can also be a union in which case any constraint for which the set type is in the union is stored in the field with type `Ci`. This can be useful if `Ci` is a `MatrixOfConstraints` in order to concatenate the coefficients of constraints of several different set types in the same matrix.

MathOptInterface.Utilities.struct_of_constraint_code – Function.

```
struct_of_constraint_code(struct_name, types, field_types = nothing)
```

Given a vector of n `Union{SymbolFun,_UnionSymbolFS{SymbolFun}}` or `Union{SymbolSet,_UnionSymbolFS{SymbolSet}}` in `types`, defines a subtype of `StructOfConstraints` of name `name` and which type parameters `{T, F1, F2, ..., Fn}` if `field_types` is `nothing` and a `{T}` otherwise. It contains `n` field where the ith field has type `Ci` if `field_types` is `nothing` and type `field_types[i]` otherwise. If `types` is vector of `Union{SymbolFun,_UnionSymbolFS{SymbolFun}}` (resp. `Union{SymbolSet,_UnionSymbolFS{SymbolSet}}`) then the constraints of that function (resp. set) type are stored in the corresponding field.

This function is used by the macros `@model`, `@struct_of_constraints_by_function_types` and `@struct_of_constraints_b`

Caching optimizer MathOptInterface.Utilities.CachingOptimizer – Type.

```
CachingOptimizer
```

`CachingOptimizer` is an intermediate layer that stores a cache of the model and links it with an optimizer. It supports incremental model construction and modification even when the optimizer doesn't.

Constructors

```
CachingOptimizer(cache::MOI.ModelLike, optimizer::AbstractOptimizer)
```

Creates a `CachingOptimizer` in `AUTOMATIC` mode, with the optimizer `optimizer`.

The type of the optimizer returned is `CachingOptimizer{typeof(optimizer), typeof(cache)}` so it does not support the function `reset_optimizer(::CachingOptimizer, new_optimizer)` if the type of `new_optimizer` is different from the type of `optimizer`.

```
CachingOptimizer(cache::MOI.ModelLike, mode::CachingOptimizerMode)
```

Creates a `CachingOptimizer` in the `NO_OPTIMIZER` state and mode `mode`.

The type of the optimizer returned is `CachingOptimizer{MOI.AbstractOptimizer,typeof(cache)}` so it does support the function `reset_optimizer(::CachingOptimizer, new_optimizer)` if the type of `new_optimizer` is different from the type of `optimizer`.

About the type

States

A `CachingOptimizer` may be in one of three possible states (`CachingOptimizerState`):

- `NO_OPTIMIZER`: The CachingOptimizer does not have any optimizer.
- `EMPTY_OPTIMIZER`: The CachingOptimizer has an empty optimizer. The optimizer is not synchronized with the cached model.
- `ATTACHED_OPTIMIZER`: The CachingOptimizer has an optimizer, and it is synchronized with the cached model.

Modes

A `CachingOptimizer` has two modes of operation (`CachingOptimizerMode`):

- `MANUAL`: The only methods that change the state of the `CachingOptimizer` are `Utilities.reset_optimizer`, `Utilities.drop_optimizer`, and `Utilities.attach_optimizer`. Attempting to perform an operation in the incorrect state results in an error.
- `AUTOMATIC`: The `CachingOptimizer` changes its state when necessary. For example, `optimize!` will automatically call `attach_optimizer` (an optimizer must have been previously set). Attempting to add a constraint or perform a modification not supported by the optimizer results in a drop to `EMPTY_OPTIMIZER` mode.

`MathOptInterface.Utilities.attach_optimizer` – Function.

```
attach_optimizer(model::CachingOptimizer)
```

Attaches the optimizer to `model`, copying all model data into it. Can be called only from the `EMPTY_OPTIMIZER` state. If the copy succeeds, the `CachingOptimizer` will be in state `ATTACHED_OPTIMIZER` after the call, otherwise an error is thrown; see `MathOptInterface.copy_to` for more details on which errors can be thrown.

```
MOIU.attach_optimizer(model::Model)
```

Call `MOIU.attach_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

MathOptInterface.Utilities.reset_optimizer – Function.

```
reset_optimizer(m::CachingOptimizer, optimizer::MOI.AbstractOptimizer)
```

Sets or resets `m` to have the given empty optimizer `optimizer`.

Can be called from any state. An assertion error will be thrown if `optimizer` is not empty.

The `CachingOptimizer` `m` will be in state `EMPTY_OPTIMIZER` after the call.

```
reset_optimizer(m::CachingOptimizer)
```

Detaches and empties the current optimizer. Can be called from `ATTACHED_OPTIMIZER` or `EMPTY_OPTIMIZER` state. The `CachingOptimizer` will be in state `EMPTY_OPTIMIZER` after the call.

```
MOIU.reset_optimizer(model::Model, optimizer::MOI.AbstractOptimizer)
```

Call `MOIU.reset_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

```
MOIU.reset_optimizer(model::Model)
```

Call `MOIU.reset_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

MathOptInterface.Utilities.drop_optimizer – Function.

```
drop_optimizer(m::CachingOptimizer)
```

Drops the optimizer, if one is present. Can be called from any state. The `CachingOptimizer` will be in state `NO_OPTIMIZER` after the call.

```
MOIU.drop_optimizer(model::Model)
```

Call `MOIU.drop_optimizer` on the backend of `model`.

Cannot be called in direct mode.

source

MathOptInterface.Utilities.state – Function.

```
state(m::CachingOptimizer)::CachingOptimizerState
```

Returns the state of the CachingOptimizer `m`. See Utilities.CachingOptimizer.

MathOptInterface.Utilities.mode – Function.

```
mode(m::CachingOptimizer)::CachingOptimizerMode
```

Returns the operating mode of the CachingOptimizer `m`. See Utilities.CachingOptimizer.

Mock optimizer `MathOptInterface.Utilities.MockOptimizer` – Type.

```
MockOptimizer
```

`MockOptimizer` is a fake optimizer especially useful for testing. Its main feature is that it can store the values that should be returned for each attribute.

Printing `MathOptInterface.Utilities.latex_formulation` – Function.

```
latex_formulation(model::MOI.ModelLike; kwargs...)
```

Wrap `model` in a type so that it can be pretty-printed as `text/latex` in a notebook like IJulia, or in Documenter.

To render the model, end the cell with `latex_formulation(model)`, or call `display(latex_formulation(model))` in to force the display of the model from inside a function.

Possible keyword arguments are:

- `simplify_coefficients` : Simplify coefficients if possible by omitting them or removing trailing zeros.
- `default_name` : The name given to variables with an empty name.
- `print_types` : Print the MOI type of each function and set for clarity.

Copy utilities `MathOptInterface.Utilities.default_copy_to` – Function.

```
default_copy_to(dest::MOI.ModelLike, src::MOI.ModelLike)
```

A default implementation of `MOI.copy_to(dest, src)` for models that implement the incremental interface, i.e., `MOI.supports_incremental_interface` returns `true`.

`MathOptInterface.Utilities.IndexMap` – Type.

```
IndexMap()
```

The dictionary-like object returned by `MathOptInterface.copy_to`.

`MathOptInterface.Utilities.identity_index_map` – Function.

```
identity_index_map(model::MOI.ModelLike)
```

Return an `IndexMap` that maps all variable and constraint indices of `model` to themselves.

`MathOptInterface.Utilities.ModelFilter` – Type.

```
ModelFilter(filter::Function, model::MOI.ModelLike)
```

A layer to filter out various components of `model`.

The filter function takes a single argument, which is eacy element from the list returned by the attributes below. It returns `true` if the element should be visible in the filtered model and `false` otherwise.

The components that are filtered are:

- Entire constraint types via:
  - `MOI.ListOfConstraintTypesPresent`

- Individual constraints via:
  - MOI.ListOfConstraintIndices{F,S}
- Specific attributes via:
  - MOI.ListOfModelAttributesSet
  - MOI.ListOfConstraintAttributesSet
  - MOI.ListOfVariableAttributesSet

Warning

The list of attributes filtered may change in a future release. You should write functions that are generic and not limited to the five types listed above. Thus, you should probably define a fallback `filter(::Any) = true`.

See below for examples of how this works.

Note

This layer has a limited scope. It is intended by be used in conjunction with `MOI.copy_to`.

Example: copy model excluding integer constraints

Use the do syntax to provide a single function.

```
filtered_src = MOI.Utilities.ModelFilter(src) do item
    return item != (MOI.VariableIndex, MOI.Integer)
end
MOI.copy_to(dest, filtered_src)
```

Example: copy model excluding names

Use type dispatch to simplify the implementation:

```
my_filter(::Any) = true  # Note the generic fallback!
my_filter(::MOI.VariableName) = false
my_filter(::MOI.ConstraintName) = false
filtered_src = MOI.Utilities.ModelFilter(my_filter, src)
MOI.copy_to(dest, filtered_src)
```

Example: copy irreducible infeasible subsystem

```
my_filter(::Any) = true  # Note the generic fallback!
function my_filter(ci::MOI.ConstraintIndex)
    status = MOI.get(dest, MOI.ConstraintConflictStatus(), ci)
    return status != MOI.NOT_IN_CONFLICT
end
filtered_src = MOI.Utilities.ModelFilter(my_filter, src)
MOI.copy_to(dest, filtered_src)
```

MatrixOfConstraints `MathOptInterface.Utilities.MatrixOfConstraints` – Type.

```
mutable struct MatrixOfConstraints{T,AT,BT,ST} <: MOI.ModelLike
    coefficients::AT
    constants::BT
    sets::ST
    caches::Vector{Any}
    are_indices_mapped::Vector{BitSet}
    final_touch::Bool
end
```

Represent `ScalarAffineFunction` and `VectorAffinefunction` constraints in a matrix form where the linear coefficients of the functions are stored in the `coefficients` field, the constants of the functions or sets are stored in the `constants` field. Additional information about the sets are stored in the `sets` field.

This model can only be used as the `constraints` field of a `MOI.Utilities.AbstractModel`.

When the constraints are added, they are stored in the `caches` field. They are only loaded in the `coefficients` and `constants` fields once `MOI.Utilities.final_touch` is called. For this reason, `MatrixOfConstraints` should not be used by an incremental interface. Use `MOI.copy_to` instead.

The constraints can be added in two different ways:

1. With `add_constraint`, in which case a canonicalized copy of the function is stored in `caches`.
2. With `pass_nonvariable_constraints`, in which case the functions and sets are stored themselves in `caches` without mapping the variable indices. The corresponding index in `caches` is added in `are_indices_mapped`. This avoids doing a copy of the function in case the getter of `CanonicalConstraintFunction` does not make a copy for the source model, e.g., this is the case of `VectorOfConstraints`.

We illustrate this with an example. Suppose a model is copied from a `src::MOI.Utilities.Model` to a bridged model with a `MatrixOfConstraints`. For all the types that are not bridged, the constraints will be copied with `pass_nonvariable_constraints`. Hence the functions stored in `caches` are exactly the same as the ones stored in `src`. This is ok since this is only during the `copy_to` operation during which `src` cannot be modified. On the other hand, for the types that are bridged, the functions added may contain duplicates even if the functions did not contain duplicates in `src` so duplicates are removed with `MOI.Utilities.canonical`.

Interface

The `.coefficients::AT` type must implement:

- `AT()`
- `MOI.empty(::AT)!`
- `MOI.Utilities.add_column`
- `MOI.Utilities.set_number_of_rows`
- `MOI.Utilities.allocate_terms`
- `MOI.Utilities.load_terms`
- `MOI.Utilities.final_touch`

The `.constants::BT` type must implement:

- `BT()`
- `Base.empty!(::BT)`
- `Base.resize(::BT)`
- `MOI.Utilities.load_constants`
- `MOI.Utilities.function_constants`
- `MOI.Utilities.set_from_constants`

The `.sets::ST` type must implement:

- `ST()`

- `MOI.is_empty(::ST)`
- `MOI.empty(::ST)`
- `MOI.dimension(::ST)`
- `MOI.is_valid(::ST, ::MOI.ConstraintIndex)`
- `MOI.get(::ST, ::MOI.ListOfConstraintTypesPresent)`
- `MOI.get(::ST, ::MOI.NumberOfConstraints)`
- `MOI.get(::ST, ::MOI.ListOfConstraintIndices)`
- `MOI.Utilities.set_types`
- `MOI.Utilities.set_index`
- `MOI.Utilities.add_set`
- `MOI.Utilities.rows`
- `MOI.Utilities.final_touch`

.coefficients `MathOptInterface.Utilities.add_column` – Function.

```
add_column(coefficients)::Nothing
```

Tell `coefficients` to pre-allocate datastructures as needed to store one column.

`MathOptInterface.Utilities.allocate_terms` – Function.

```
allocate_terms(coefficients, index_map, func)::Nothing
```

Tell `coefficients` that the terms of the function `func` where the variable indices are mapped with `index_map` will be loaded with `load_terms`.

The function `func` must be canonicalized before calling `allocate_terms`. See `is_canonical`.

`MathOptInterface.Utilities.set_number_of_rows` – Function.

```
set_number_of_rows(coefficients, n)::Nothing
```

Tell `coefficients` to pre-allocate datastructures as needed to store `n` rows.

`MathOptInterface.Utilities.load_terms` – Function.

```
load_terms(coefficients, index_map, func, offset)::Nothing
```

Loads the terms of `func` to `coefficients`, mapping the variable indices with `index_map`.

The `i`th dimension of `func` is loaded at the `(offset + i)`th row of `coefficients`.

The function must be allocated first with `allocate_terms`.

The function `func` must be canonicalized, see `is_canonical`.

`MathOptInterface.Utilities.final_touch` – Function.

```
final_touch(coefficients)::Nothing
```

Informs the `coefficients` that all functions have been added with `load_terms`. No more modification is allowed unless `MOI.empty!` is called.

```
final_touch(sets)::Nothing
```

Informs the `sets` that all functions have been added with `add_set`. No more modification is allowed unless `MOI.empty!` is called.

`MathOptInterface.Utilities.extract_function` – Function.

```
extract_function(coefficients, row::Integer, constant::T) where {T}
```

Return the `MOI.ScalarAffineFunction{T}` function corresponding to row `row` in `coefficients`.

```
extract_function(
    coefficients,
    rows::UnitRange,
    constants::Vector{T},
) where{T}
```

Return the `MOI.VectorAffineFunction{T}` function corresponding to rows `rows` in `coefficients`.

`MathOptInterface.Utilities.MutableSparseMatrixCSC` – Type.

```
mutable struct MutableSparseMatrixCSC{Tv,Ti<:Integer,I<:AbstractIndexing}
    indexing::I
    m::Int
    n::Int
    colptr::Vector{Ti}
    rowval::Vector{Ti}
    nzval::Vector{Tv}
    nz_added::Vector{Ti}
end
```

Matrix type loading sparse matrices in the Compressed Sparse Column format. The indexing used is `indexing`, see `AbstractIndexing`. The other fields have the same meaning than for `SparseArrays.SparseMatrixCSC` except that the indexing is different unless `indexing` is `OneBasedIndexing`. In addition, `nz_added` is used to cache the number of non-zero terms that have been added to each column due to the incremental nature of `load_terms`.

The matrix is loaded in 5 steps:

1. `MOI.empty!` is called.
2. `MOI.Utilities.add_column` and `MOI.Utilities.allocate_terms` are called in any order.
3. `MOI.Utilities.set_number_of_rows` is called.
4. `MOI.Utilities.load_terms` is called for each affine function.
5. `MOI.Utilities.final_touch` is called.

`MathOptInterface.Utilities.AbstractIndexing` – Type.

```
abstract type AbstractIndexing end
```

Indexing to be used for storing the row and column indices of `MutableSparseMatrixCSC`. See `ZeroBasedIndexing` and `OneBasedIndexing`.

`MathOptInterface.Utilities.ZeroBasedIndexing` – Type.

```
struct ZeroBasedIndexing <: AbstractIndexing end
```

Zero-based indexing: the ith row or column has index `i - 1`. This is useful when the vectors of row and column indices need to be communicated to a library using zero-based indexing such as C libraries.

MathOptInterface.Utilities.OneBasedIndexing – Type.

```
struct ZeroBasedIndexing <: AbstractIndexing end
```

One-based indexing: the ith row or column has index i. This enables an allocation-free conversion of MutableSparseMatrixCSC to SparseArrays.SparseMatrixCSC.

.constants MathOptInterface.Utilities.load_constants – Function.

```
load_constants(constants, offset, func_or_set)::Nothing
```

This function loads the constants of func_or_set in constants at an offset of offset. Where offset is the sum of the dimensions of the constraints already loaded. The storage should be preallocated with resize! before calling this function.

This function should be implemented to be usable as storage of constants for MatrixOfConstraints.

The constants are loaded in three steps:

1. Base.empty! is called.
2. Base.resize! is called with the sum of the dimensions of all constraints.
3. MOI.Utilities.load_constants is called for each function for vector constraint or set for scalar constraint.

MathOptInterface.Utilities.function_constants – Function.

```
function_constants(constants, rows)
```

This function returns the function constants that were loaded with load_constants at the rows rows.

This function should be implemented to be usable as storage of constants for MatrixOfConstraints.

MathOptInterface.Utilities.set_from_constants – Function.

```
set_from_constants(constants, S::Type, rows)::S
```

This function returns an instance of the set S for which the constants where loaded with load_constants at the rows rows.

This function should be implemented to be usable as storage of constants for MatrixOfConstraints.

MathOptInterface.Utilities.Hyperrectangle – Type.

```
struct Hyperrectangle{T} <: AbstractVectorBounds
    lower::Vector{T}
    upper::Vector{T}
end
```

A struct for the .constants field in MatrixOfConstraints.

.sets MathOptInterface.Utilities.set_index – Function.

```
set_index(sets, ::Type{S})::Union{Int,Nothing} where {S<:MOI.AbstractSet}
```

Return an integer corresponding to the index of the set type in the list given by `set_types`.

If S is not part of the list, return `nothing`.

MathOptInterface.Utilities.set_types – Function.

```
set_types(sets)::Vector{Type}
```

Return the list of the types of the sets allowed in `sets`.

MathOptInterface.Utilities.add_set – Function.

```
add_set(sets, i)::Int64
```

Add a scalar set of type index `i`.

```
add_set(sets, i, dim)::Int64
```

Add a vector set of type index `i` and dimension `dim`.

Both methods return a unique `Int64` of the set that can be used to reference this set.

MathOptInterface.Utilities.rows – Function.

```
rows(sets, ci::MOI.ConstraintIndex)::Union{Int,UnitRange{Int}}
```

Return the rows in `1:MOI.dimension(sets)` corresponding to the set of id `ci.value`.

For scalar sets, this returns an `Int`. For vector sets, this returns an `UnitRange{Int}`.

MathOptInterface.Utilities.num_rows – Function.

```
num_rows(sets::OrderedProductOfSets, ::Type{S}) where {S}
```

Return the number of rows corresponding to a set of type S. That is, it is the sum of the dimensions of the sets of type S.

MathOptInterface.Utilities.set_with_dimension – Function.

```
set_with_dimension(::Type{S}, dim) where {S<:MOI.AbstractVectorSet}
```

Returns the instance of S of `MathOptInterface.dimension` `dim`. This needs to be implemented for sets of type S to be useable with `MatrixOfConstraints`.

MathOptInterface.Utilities.ProductOfSets – Type.

```
abstract type ProductOfSets{T} end
```

Represents a cartesian product of sets of given types.

MathOptInterface.Utilities.MixOfScalarSets – Type.

```
abstract type MixOfScalarSets{T} <: ProductOfSets{T} end
```

Product of scalar sets in the order the constraints are added, mixing the constraints of different types.

Use `@mix_of_scalar_sets` to generate a new subtype.

MathOptInterface.Utilities.@mix_of_scalar_sets – Macro.

```
@mix_of_scalar_sets(name, set_types...)
```

Generate a new MixOfScalarSets subtype.

Example

```
@mix_of_scalar_sets(
    MixedIntegerLinearProgramSets,
    MOI.GreaterThan{T},
    MOI.LessThan{T},
    MOI.EqualTo{T},
    MOI.Integer,
)
```

MathOptInterface.Utilities.OrderedProductOfSets – Type.

```
abstract type OrderedProductOfSets{T} <: ProductOfSets{T} end
```

Product of sets in the order the constraints are added, grouping the constraints of the same types contiguously.

Use @product_of_sets to generate new subtypes.

MathOptInterface.Utilities.@product_of_sets – Macro.

```
@product_of_sets(name, set_types...)
```

Generate a new OrderedProductOfSets subtype.

Example

```
@product_of_sets(
    LinearOrthants,
    MOI.Zeros,
    MOI.Nonnegatives,
    MOI.Nonpositives,
    MOI.ZeroOne,
)
```

Fallbacks MathOptInterface.Utilities.get_fallback – Function.

```
get_fallback(model::MOI.ModelLike, ::MOI.ObjectiveValue)
```

Compute the objective function value using the `VariablePrimal` results and the `ObjectiveFunction` value.

```
get_fallback(model::MOI.ModelLike, ::MOI.DualObjectiveValue, T::Type)::T
```

Compute the dual objective value of type T using the `ConstraintDual` results and the `ConstraintFunction` and `ConstraintSet` values. Note that the nonlinear part of the model is ignored.

```
get_fallback(model::MOI.ModelLike, ::MOI.ConstraintPrimal,
             constraint_index::MOI.ConstraintIndex)
```

Compute the value of the function of the constraint of index `constraint_index` using the `VariablePrimal` results and the `ConstraintFunction` values.

```
get_fallback(model::MOI.ModelLike, attr::MOI.ConstraintDual,
             ci::MOI.ConstraintIndex{Union{MOI.VariableIndex,
                                            MOI.VectorOfVariables}})
```

Compute the dual of the constraint of index `ci` using the `ConstraintDual` of other constraints and the `ConstraintFunction` values. Throws an error if some constraints are quadratic or if there is one another `MOI.VariableIndex`-in-S or `MOI.VectorOfVariables`-in-S constraint with one of the variables in the function of the constraint `ci`.

Function utilities The following utilities are available for functions:

`MathOptInterface.Utilities.eval_variables` – Function.

```
eval_variables(varval::Function, f::AbstractFunction)
```

Returns the value of function `f` if each variable index `vi` is evaluated as `varval(vi)`. Note that `varval` should return a number, see `substitute_variables` for a similar function where `varval` returns a function.

`MathOptInterface.Utilities.map_indices` – Function.

```
map_indices(index_map::Function, attr::MOI.AnyAttribute, x::X)::X where {X}
```

Substitute any `MOI.VariableIndex` (resp. `MOI.ConstraintIndex`) in x by the `MOI.VariableIndex` (resp. `MOI.ConstraintIndex`) of the same type given by `index_map(x)`.

When to implement this method for new types `X`

This function is used by implementations of `MOI.copy_to` on constraint functions, attribute values and submittable values. If you define a new attribute whose values `x::X` contain variable or constraint indices, you must also implement this function.

```
map_indices(
    variable_map::AbstractDict{T,T},
    x::X,
)::X where {T<:MOI.Index,X}
```

Shortcut for `map_indices(vi -> variable_map[vi], x)`.

`MathOptInterface.Utilities.substitute_variables` – Function.

```
substitute_variables(variable_map::Function, x)
```

Substitute any `MOI.VariableIndex` in x by `variable_map(x)`. The `variable_map` function returns either `MOI.VariableIndex` or `MOI.ScalarAffineFunction`, see `eval_variables` for a similar function where `variable_map` returns a number.

This function is used by bridge optimizers on constraint functions, attribute values and submittable values when at least one variable bridge is used hence it needs to be implemented for custom types that are meant to be used as attribute or submittable value.

WARNING: Don't use `substitude_variables(::Function, ...)` because Julia will not specialize on this. Use instead `substitude_variables(::F, ...) where {F<:Function}`.

`MathOptInterface.Utilities.filter_variables` – Function.

```
filter_variables(keep::Function, f::AbstractFunction)
```

Return a new function `f` with the variable `vi` such that `!keep(vi)` removed.

WARNING: Don't define `filter_variables(::Function, ...)` because Julia will not specialize on this. Define instead `filter_variables(::F, ...) where {F<:Function}`.

`MathOptInterface.Utilities.remove_variable` – Function.

```
remove_variable(f::AbstractFunction, vi::VariableIndex)
```

Return a new function `f` with the variable vi removed.

```
remove_variable(f::MOI.AbstractFunction, s::MOI.AbstractSet, vi::MOI.VariableIndex)
```

Return a tuple `(g, t)` representing the constraint `f`-in-`s` with the variable `vi` removed. That is, the terms containing the variable `vi` in the function `f` are removed and the dimension of the set `s` is updated if needed (e.g. when `f` is a `VectorOfVariables` with `vi` being one of the variables).

`MathOptInterface.Utilities.all_coefficients` – Function.

```
all_coefficients(p::Function, f::MOI.AbstractFunction)
```

Determine whether predicate `p` returns `true` for all coefficients of `f`, returning `false` as soon as the first coefficient of `f` for which `p` returns `false` is encountered (short-circuiting). Similar to `all`.

`MathOptInterface.Utilities.unsafe_add` – Function.

```
unsafe_add(t1::MOI.ScalarAffineTerm, t2::MOI.ScalarAffineTerm)
```

Sums the coefficients of `t1` and `t2` and returns an output `MOI.ScalarAffineTerm`. It is unsafe because it uses the `variable` of `t1` as the `variable` of the output without checking that it is equal to that of `t2`.

```
unsafe_add(t1::MOI.ScalarQuadraticTerm, t2::MOI.ScalarQuadraticTerm)
```

Sums the coefficients of `t1` and `t2` and returns an output `MOI.ScalarQuadraticTerm`. It is unsafe because it uses the `variable`'s of `t1` as the `variable`'s of the output without checking that they are the same (up to permutation) to those of `t2`.

```
unsafe_add(t1::MOI.VectorAffineTerm, t2::MOI.VectorAffineTerm)
```

Sums the coefficients of `t1` and `t2` and returns an output `MOI.VectorAffineTerm`. It is unsafe because it uses the `output_index` and `variable` of `t1` as the `output_index` and `variable` of the output term without checking that they are equal to those of `t2`.

`MathOptInterface.Utilities.isapprox_zero` – Function.

```
isapprox_zero(f::MOI.AbstractFunction, tol)
```

Return a `Bool` indicating whether the function `f` is approximately zero using `tol` as a tolerance.

Important note

This function assumes that `f` does not contain any duplicate terms, you might want to first call `canonical` if that is not guaranteed. For instance, given

```
f = MOI.ScalarAffineFunction(MOI.ScalarAffineTerm.([1, -1], [x, x]), 0)`.
```

then `isapprox_zero(f)` is `false` but `isapprox_zero(MOIU.canonical(f))` is `true`.

`MathOptInterface.Utilities.modify_function` – Function.

```
modify_function(f::AbstractFunction, change::AbstractFunctionModification)
```

Return a new function `f` modified according to `change`.

`MathOptInterface.Utilities.zero_with_output_dimension` – Function.

```
zero_with_output_dimension(::Type{T}, output_dimension::Integer) where {T}
```

Create an instance of type T with the output dimension `output_dimension`.

This is mostly useful in Bridges, when code needs to be agnostic to the type of vector-valued function that is passed in.

The following functions can be used to canonicalize a function:

`MathOptInterface.Utilities.is_canonical` – Function.

```
is_canonical(f::Union{ScalarAffineFunction, VectorAffineFunction})
```

Returns a Bool indicating whether the function is in canonical form. See `canonical`.

```
is_canonical(f::Union{ScalarQuadraticFunction, VectorQuadraticFunction})
```

Returns a Bool indicating whether the function is in canonical form. See `canonical`.

`MathOptInterface.Utilities.canonical` – Function.

```
canonical(
    f::Union{
        ScalarAffineFunction,
        VectorAffineFunction,
        ScalarQuadraticFunction,
        VectorQuadraticFunction,
    },
)
```

Returns the function in a canonical form, i.e.

- A term appear only once.
- The coefficients are nonzero.
- The terms appear in increasing order of variable where there the order of the variables is the order of their value.
- For a `AbstractVectorFunction`, the terms are sorted in ascending order of output index.

The output of `canonical` can be assumed to be a copy of `f`, even for `VectorOfVariables`.

Examples

If x (resp. y, z) is `VariableIndex(1)` (resp. 2, 3). The canonical representation of `ScalarAffineFunction([y, x, z, x, z], [2, 1, 3, -2, -3], 5)` is `ScalarAffineFunction([x, y], [-1, 2], 5)`.

`MathOptInterface.Utilities.canonicalize!` – Function.

```
canonicalize!(f::Union{ScalarAffineFunction, VectorAffineFunction})
```

Convert a function to canonical form in-place, without allocating a copy to hold the result. See `canonical`.

```
canonicalize!(f::Union{ScalarQuadraticFunction, VectorQuadraticFunction})
```

Convert a function to canonical form in-place, without allocating a copy to hold the result. See canonical.

The following functions can be used to manipulate functions with basic algebra:

MathOptInterface.Utilities.scalar_type – Function.

```
scalar_type(F::Type{<:MOI.AbstractVectorFunction})
```

Type of functions obtained by indexing objects obtained by calling eachscalar on functions of type F.

MathOptInterface.Utilities.scalarize – Function.

```
scalarize(func::MOI.VectorOfVariables, ignore_constants::Bool = false)
```

Returns a vector of scalar functions making up the vector function in the form of a Vector{MOI.SingleVariable}.

See also eachscalar.

```
scalarize(func::MOI.VectorAffineFunction{T}, ignore_constants::Bool = false)
```

Returns a vector of scalar functions making up the vector function in the form of a Vector{MOI.ScalarAffineFunction{T}}.

See also eachscalar.

```
scalarize(func::MOI.VectorQuadraticFunction{T}, ignore_constants::Bool = false)
```

Returns a vector of scalar functions making up the vector function in the form of a Vector{MOI.ScalarQuadraticFunction{T}

See also eachscalar.

MathOptInterface.Utilities.eachscalar – Function.

```
eachscalar(f::MOI.AbstractVectorFunction)
```

Returns an iterator for the scalar components of the vector function.

See also scalarize.

```
eachscalar(f::MOI.AbstractVector)
```

Returns an iterator for the scalar components of the vector.

MathOptInterface.Utilities.promote_operation – Function.

```
promote_operation(
    op::Function,
    ::Type{T},
    ArgsTypes::Type{<:Union{T, MOI.AbstractFunction}}...,
) where {T}
```

Returns the type of the MOI.AbstractFunction returned to the call operate(op, T, args...) where the types of the arguments args are ArgsTypes.

MathOptInterface.Utilities.operate – Function.

```
operate(
    op::Function,
    ::Type{T},
    args::Union{T,MOI.AbstractFunction}...,
)::MOI.AbstractFunction where {T}
```

Returns an `MOI.AbstractFunction` representing the function resulting from the operation `op(args...)` on functions of coefficient type T. No argument can be modified.

`MathOptInterface.Utilities.operate!` – Function.

```
operate!(
    op::Function,
    ::Type{T},
    args::Union{T, MOI.AbstractFunction}...,
)::MOI.AbstractFunction where {T}
```

Returns an `MOI.AbstractFunction` representing the function resulting from the operation `op(args...)` on functions of coefficient type T. The first argument can be modified. The return type is the same than the method `operate(op, T, args...)` without `!`.

`MathOptInterface.Utilities.operate_output_index!` – Function.

```
operate_output_index!(
    op::Function,
    ::Type{T},
    output_index::Integer,
    func::MOI.AbstractVectorFunction
    args::Union{T, MOI.AbstractScalarFunction}...
)::MOI.AbstractFunction where {T}
```

Returns an `MOI.AbstractVectorFunction` where the function at `output_index` is the result of the operation op applied to the function at `output_index` of `func` and `args`. The functions at output index different to `output_index` are the same as the functions at the same output index in `func`. The first argument can be modified.

`MathOptInterface.Utilities.vectorize` – Function.

```
vectorize(x::AbstractVector{MOI.VariableIndex})
```

Returns the vector of scalar affine functions in the form of a `MOI.VectorAffineFunction{T}`.

```
vectorize(funcs::AbstractVector{MOI.ScalarAffineFunction{T}}) where T
```

Returns the vector of scalar affine functions in the form of a `MOI.VectorAffineFunction{T}`.

```
vectorize(funcs::AbstractVector{MOI.ScalarQuadraticFunction{T}}) where T
```

Returns the vector of scalar quadratic functions in the form of a `MOI.VectorQuadraticFunction{T}`.

Constraint utilities The following utilities are available for moving the function constant to the set for scalar constraints:

`MathOptInterface.Utilities.shift_constant` – Function.

```
shift_constant(set::MOI.AbstractScalarSet, offset)
```

Returns a new scalar set new_`set` such that `func`-in-`set` is equivalent to `func + offset`-in-new_`set`.

Only define this function if it makes sense to!

Use `supports_shift_constant` to check if the set supports shifting:

```
if supports_shift_constant(typeof(old_set))
    new_set = shift_constant(old_set, offset)
    f.constant = 0
    add_constraint(model, f, new_set)
else
    add_constraint(model, f, old_set)
end
```

See also `supports_shift_constant`.

Examples

The call `shift_constant(MOI.Interval(-2, 3), 1)` is equal to `MOI.Interval(-1, 4)`.

`MathOptInterface.Utilities.supports_shift_constant` – Function.

```
supports_shift_constant(::Type{S}) where {S<:MOI.AbstractSet}
```

Return `true` if `shift_constant` is defined for set S.

See also `shift_constant`.

`MathOptInterface.Utilities.normalize_and_add_constraint` – Function.

```
normalize_and_add_constraint(
    model::MOI.ModelLike,
    func::MOI.AbstractScalarFunction,
    set::MOI.AbstractScalarSet;
    allow_modify_function::Bool = false,
)
```

Adds the scalar constraint obtained by moving the constant term in `func` to the set in `model`. If `allow_modify_function` is `true` then the function `func` can be modified.

`MathOptInterface.Utilities.normalize_constant` – Function.

```
normalize_constant(
    func::MOI.AbstractScalarFunction,
    set::MOI.AbstractScalarSet;
    allow_modify_function::Bool = false,
)
```

Return the `func`-in-`set` constraint in normalized form. That is, if `func` is `MOI.ScalarQuadraticFunction` or `MOI.ScalarAffineFunction`, the constant is moved to the set. If `allow_modify_function` is `true` then the function `func` can be modified.

The following utility identifies those constraints imposing bounds on a given variable, and returns those bound values:

`MathOptInterface.Utilities.get_bounds` – Function.

```
get_bounds(model::MOI.ModelLike, ::Type{T}, x::MOI.VariableIndex)
```

Return a tuple `(lb, ub)` of type `Tuple{T, T}`, where `lb` and `ub` are lower and upper bounds, respectively, imposed on `x` in `model`.

The following utilities are useful when working with symmetric matrix cones.

`MathOptInterface.Utilities.is_diagonal_vectorized_index` – Function.

```
is_diagonal_vectorized_index(index::Base.Integer)
```

Return whether index is the index of a diagonal element in a `MOI.AbstractSymmetricMatrixSetTriangle` set.

`MathOptInterface.Utilities.side_dimension_for_vectorized_dimension` – Function.

```
side_dimension_for_vectorized_dimension(n::Integer)
```

Return the dimension d such that `MOI.dimension(MOI.PositiveSemidefiniteConeTriangle(d))` is n.

DoubleDicts `MathOptInterface.Utilities.DoubleDicts.DoubleDict` – Type.

```
DoubleDict{V}
```

An optimized dictionary to map `MOI.ConstraintIndex` to values of type `V`.

Works as a `AbstractDict{MOI.ConstraintIndex,V}` with minimal differences.

If `V` is also a `MOI.ConstraintIndex`, use `IndexDoubleDict`.

Note that `MOI.ConstraintIndex` is not a concrete type, opposed to `MOI.ConstraintIndex{MOI.VariableIndex, MOI.Integers}`, which is a concrete type.

When looping through multiple keys of the same Function-in-Set type, use

```
inner = dict[F, S]
```

to return a type-stable `DoubleDictInner`.

`MathOptInterface.Utilities.DoubleDicts.DoubleDictInner` – Type.

```
DoubleDictInner{F,S,V}
```

A type stable inner dictionary of `DoubleDict`.

`MathOptInterface.Utilities.DoubleDicts.IndexDoubleDict` – Type.

```
IndexDoubleDict
```

A specialized version of [`DoubleDict`] in which the values are of type `MOI.ConstraintIndex`

When looping through multiple keys of the same Function-in-Set type, use

```
inner = dict[F, S]
```

to return a type-stable `IndexDoubleDictInner`.

`MathOptInterface.Utilities.DoubleDicts.IndexDoubleDictInner` – Type.

```
IndexDoubleDictInner{F,S}
```

A type stable inner dictionary of `IndexDoubleDict`.

40.5 Test

Overview

The Test submodule

The `Test` submodule provides tools to help solvers implement unit tests in order to ensure they implement the MathOptInterface API correctly, and to check for solver-correctness.

We use a centralized repository of tests, so that if we find a bug in one solver, instead of adding a test to that particular repository, we add it here so that all solvers can benefit.

How to test a solver The skeleton below can be used for the wrapper test file of a solver named `FooBar`.

```julia
# ============================ /test/MOI_wrapper.jl ============================
module TestFooBar

import FooBar
using MathOptInterface
using Test

const MOI = MathOptInterface

const OPTIMIZER = MOI.instantiate(
    MOI.OptimizerWithAttributes(FooBar.Optimizer, MOI.Silent() => true),
)

const BRIDGED = MOI.instantiate(
    MOI.OptimizerWithAttributes(FooBar.Optimizer, MOI.Silent() => true),
    with_bridge_type = Float64,
)

# See the docstring of MOI.Test.Config for other arguments.
const CONFIG = MOI.Test.Config(
    # Modify tolerances as necessary.
    atol = 1e-6,
    rtol = 1e-6,
    # Use MOI.LOCALLY_SOLVED for local solvers.
    optimal_status = MOI.OPTIMAL,
    # Pass attributes or MOI functions to `exclude` to skip tests that
    # rely on this functionality.
    exclude = Any[MOI.VariableName, MOI.delete],
)

"""
    runtests()

This function runs all functions in the this Module starting with `test_`.
"""
function runtests()
    for name in names(@__MODULE__; all = true)
        if startswith("$(name)", "test_")
            @testset "$(name)" begin
                getfield(@__MODULE__, name)()
            end
        end
```

```julia
    end
end

"""
    test_runtests()

This function runs all the tests in MathOptInterface.Test.

Pass arguments to `exclude` to skip tests for functionality that is not
implemented or that your solver doesn't support.
"""
function test_runtests()
    MOI.Test.runtests(
        BRIDGED,
        CONFIG,
        exclude = [
            "test_attribute_NumberOfThreads",
            "test_quadratic_",
        ],
        # This argument is useful to prevent tests from failing on future
        # releases of MOI that add new tests. Don't let this number get too far
        # behind the current MOI release though! You should periodically check
        # for new tests in order to fix bugs and implement new features.
        exclude_tests_after = v"0.10.5",
    )
    return
end

"""
    test_SolverName()

You can also write new tests for solver-specific functionality. Write each new
test as a function with a name beginning with `test_`.
"""
function test_SolverName()
    @test MOI.get(FooBar.Optimizer(), MOI.SolverName()) == "FooBar"
    return
end

end # module TestFooBar

# This line at tne end of the file runs all the tests!
TestFooBar.runtests()
```

Then modify your `runtests.jl` file to include the `MOI_wrapper.jl` file:

```julia
# ============================ /test/runtests.jl ============================

using Test

@testset "MOI" begin
    include("test/MOI_wrapper.jl")
end
```

Info

> The optimizer `BRIDGED` constructed with `instantiate` automatically bridges constraints that are not supported by `OPTIMIZER` using the bridges listed in Bridges. It is recommended for an implementation of MOI to only support constraints that are natively supported by the solver and let bridges transform the constraint to the appropriate form. For this reason it is expected that tests may not pass if `OPTIMIZER` is used instead of `BRIDGED`.

How to debug a failing test When writing a solver, it's likely that you will initially fail many tests! Some failures will be bugs, but other failures you may choose to exclude.

There are two ways to exclude tests:

- Exclude tests whose names contain a string using:

  ```
  MOI.Test.runtests(
      model,
      config;
      exclude = String["test_to_exclude", "test_conic_"],
  )
  ```

  This will exclude tests whose name contains either of the two strings provided.

- Exclude tests which rely on specific functionality using:

  ```
  MOI.Test.Config(exclude = Any[MOI.VariableName, MOI.optimize!])
  ```

  This will exclude tests which use the `MOI.VariableName` attribute, or which call `MOI.optimize!`.

Each test that fails can be independently called as:

```
model = FooBar.Optimizer()
config = MOI.Test.Config()
MOI.empty!(model)
MOI.Test.test_category_name_that_failed(model, config)
```

You can look-up the source code of the test that failed by searching for it in the `src/Test/test_category.jl` file.

> **Tip**
>
> Each test function also has a docstring that explains what the test is for. Use `? MOI.Test.test_category_name_that_faile` from the REPL to read it.

How to add a test To detect bugs in solvers, we add new tests to `MOI.Test`.

As an example, ECOS errored calling `optimize!` twice in a row. (See ECOS.jl PR #72.) We could add a test to ECOS.jl, but that would only stop us from re-introducing the bug to ECOS.jl in the future, but it would not catch other solvers in the ecosystem with the same bug! Instead, if we add a test to `MOI.Test`, then all solvers will also check that they handle a double optimize call!

For this test, we care about correctness, rather than performance. therefore, we don't expect solvers to efficiently decide that they have already solved the problem, only that calling `optimize!` twice doesn't throw an error or give the wrong answer.

Step 1

Install the `MathOptInterface` julia package in dev mode (ref):

```
julia> ]
(@v1.6) pkg> dev MathOptInterface
```

Step 2

From here on, proceed with making the following changes in the `~/.julia/dev/MathOptInterface` folder (or equivalent dev path on your machine).

Step 3

Since the double-optimize error involves solving an optimization problem, add a new test to src/Test/UnitTests/solve.jl:

```
"""
    test_unit_optimize!_twice(model::MOI.ModelLike, config::Config)

Test that calling `MOI.optimize!` twice does not error.

This problem was first detected in ECOS.jl PR#72:
https://github.com/jump-dev/ECOS.jl/pull/72
"""
function test_unit_optimize!_twice(
    model::MOI.ModelLike,
    config::Config{T},
) where {T}
    # Use the `@requires` macro to check conditions that the test function
    # requires in order to run. Models failing this `@requires` check will
    # silently skip the test.
    @requires MOI.supports_constraint(
        model,
        MOI.VariableIndex,
        MOI.GreaterThan{Float64},
    )
    @requires _supports(config, MOI.optimize!)
    # If needed, you can test that the model is empty at the start of the test.
    # You can assume that this will be the case for tests run via `runtests`.
    # User's calling tests individually need to call `MOI.empty!` themselves.
    @test MOI.is_empty(model)
    # Create a simple model. Try to make this as simple as possible so that the
    # majority of solvers can run the test.
    x = MOI.add_variable(model)
    MOI.add_constraint(model, x, MOI.GreaterThan(one(T)))
    MOI.set(model, MOI.ObjectiveSense(), MOI.MIN_SENSE)
    MOI.set(
        model,
        MOI.ObjectiveFunction{MOI.VariableIndex}(),
        x,
    )
    # The main component of the test: does calling `optimize!` twice error?
    MOI.optimize!(model)
    MOI.optimize!(model)
    # Check we have a solution.
    @test MOI.get(model, MOI.TerminationStatus()) == MOI.OPTIMAL
    # There is a three-argument version of `Base.isapprox` for checking
    # approximate equality based on the tolerances defined in `config`:
    @test isapprox(MOI.get(model, MOI.VariablePrimal(), x), one(T), config)
```

```
    # For code-style, these tests should always `return` `nothing`.
    return
end
```

> **Info**
>
> Make sure the function is agnoistic to the number type T! Don't assume it is a `Float64` capable solver!

We also need to write a test for the test. Place this function immediately below the test you just wrote in the same file:

```
function setup_test(
    ::typeof(test_unit_optimize!_twice),
    model::MOI.Utilities.MockOptimizer,
    ::Config,
)
    MOI.Utilities.set_mock_optimize!(
        model,
        (mock::MOI.Utilities.MockOptimizer) -> MOIU.mock_optimize!(
            mock,
            MOI.OPTIMAL,
            (MOI.FEASIBLE_POINT, [1.0]),
        ),
    )
    return
end
```

Finally, you also need to implement `Test.version_added`. If we added this test when the latest released version of MOI was `v0.10.5`, define:

```
version_added(::typeof(test_unit_optimize!_twice)) = v"0.10.6"
```

Step 6

Commit the changes to git from `~/.julia/dev/MathOptInterface` and submit the PR for review.

> **Tip**
>
> If you need help writing a test, open an issue on GitHub, or ask the Developer Chatroom

API Reference

The Test submodule

Functions to help test implementations of MOI. See The Test submodule for more details.

`MathOptInterface.Test.Config` – Type.

```
Config(
    ::Type{T} = Float64;
    atol::Real = Base.rtoldefault(T),
    rtol::Real = Base.rtoldefault(T),
    optimal_status::MOI.TerminationStatusCode = MOI.OPTIMAL,
    infeasible_status::MOI.TerminationStatusCode = MOI.INFEASIBLE,
    exclude::Vector{Any} = Any[],
) where {T}
```

Return an object that is used to configure various tests.

Configuration arguments

- `atol::Real = Base.rtoldefault(T)`: Control the absolute tolerance used when comparing solutions.
- `rtol::Real = Base.rtoldefault(T)`: Control the relative tolerance used when comparing solutions.
- `optimal_status = MOI.OPTIMAL`: Set to `MOI.LOCALLY_SOLVED` if the solver cannot prove global optimality.
- `infeasible_status = MOI.INFEASIBLE`: Set to `MOI.LOCALLY_INFEASIBLE` if the solver cannot prove global infeasibility.
- `exclude = Vector{Any}`: Pass attributes or functions to `exclude` to skip parts of tests that require certain functionality. Common arguments include:
  - `MOI.delete` to skip deletion-related tests
  - `MOI.optimize!` to skip optimize-related tests
  - `MOI.ConstraintDual` to skip dual-related tests
  - `MOI.VariableName` to skip setting variable names
  - `MOI.ConstraintName` to skip setting constraint names

Examples

For a nonlinear solver that finds local optima and does not support finding dual variables or constraint names:

```
Config(
    Float64;
    optimal_status = MOI.LOCALLY_SOLVED,
    exclude = Any[
        MOI.ConstraintDual,
        MOI.VariableName,
        MOI.ConstraintName,
        MOI.delete,
    ],
)
```

`MathOptInterface.Test.runtests` – Function.

```
runtests(
    model::MOI.ModelLike,
    config::Config;
    include::Vector{String} = String[],
    exclude::Vector{String} = String[],
    warn_unsupported::Bool = false,
    exclude_tests_after::VersionNumber = v"999.0.0",
)
```

Run all tests in `MathOptInterface.Test` on `model`.

Configuration arguments

- `config` is a `Test.Config` object that can be used to modify the behavior of tests.
- If `include` is not empty, only run tests that contain an element from `include` in their name.

- If exclude is not empty, skip tests that contain an element from exclude in their name.
- exclude takes priority over include.
- If warn_unsupported is false, runtests will silently skip tests that fail with UnsupportedConstraint or UnsupportedAttribute. When warn_unsupported is true, a warning will be printed. For most cases the default behavior (false) is what you want, since these tests likely test functionality that is not supported by model. However, it can be useful to run warn_unsupported = true to check you are not skipping tests due to a missing supports_constraint method or equivalent.
- exclude_tests_after is a version number that excludes any tests to MOI added after that version number. This is useful for solvers who can declare a fixed set of tests, and not cause their tests to break if a new patch of MOI is released with a new test.

See also: setup_test.

Example

```
config = MathOptInterface.Test.Config()
MathOptInterface.Test.runtests(
    model,
    config;
    include = ["test_linear_"],
    exclude = ["VariablePrimalStart"],
    warn_unsupported = true,
    exclude_tests_after = v"0.10.5",
)
```

MathOptInterface.Test.setup_test – Function.

```
setup_test(::typeof(f), model::MOI.ModelLike, config::Config)
```

Overload this method to modify model before running the test function f on model with config. You can also modify the fields in config (e.g., to loosen the default tolerances).

This function should either return nothing, or return a function which, when called with zero arguments, undoes the setup to return the model to its previous state. You do not need to undo any modifications to config.

This function is most useful when writing new tests of the tests for MOI, but it can also be used to set test-specific tolerances, etc.

See also: runtests

Example

```
function MOI.Test.setup_test(
    ::typeof(MOI.Test.test_linear_VariablePrimalStart_partial),
    mock::MOIU.MockOptimizer,
    ::MOI.Test.Config,
)
    MOIU.set_mock_optimize!(
        mock,
        (mock::MOIU.MockOptimizer) -> MOIU.mock_optimize!(mock, [1.0, 0.0]),
    )
    mock.eval_variable_constraint_dual = false

    function reset_function()
```

```
        mock.eval_variable_constraint_dual = true
        return
    end
    return reset_function
end
```

MathOptInterface.Test.version_added – Function.

```
version_added(::typeof(function_name))
```

Returns the version of MOI in which the test `function_name` was added.

This method should be implemented for all new tests.

See the `exclude_tests_after` keyword of `runtests` for more details.

MathOptInterface.Test.@requires – Macro.

```
@requires(x)
```

Check that the condition x is `true`. Otherwise, throw an `RequirementUnmet` error to indicate that the model does not support something required by the test function.

Examples

```
@requires MOI.supports(model, MOI.Silent())
@test MOI.get(model, MOI.Silent())
```

MathOptInterface.Test.RequirementUnmet – Type.

```
RequirementUnmet(msg::String) <: Exception
```

An error for throwing in tests to indicate that the model does not support some requirement expected by the test function.