Graph Algorithms

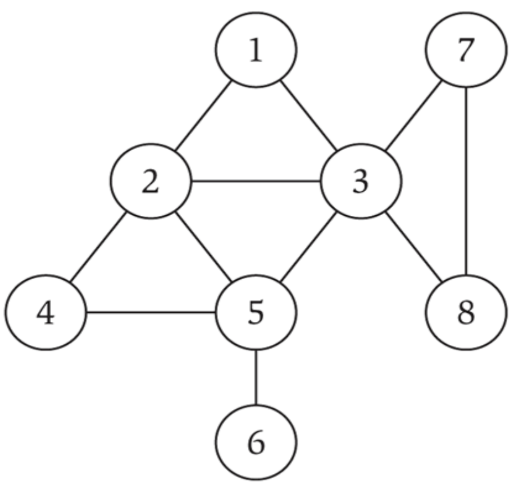
Part 1 BFS & DFS

CS 325

Introduction to graph theory

Graph – mathematical object consisting of a set of:

- Denoted by $G = (V, E)$.
- V = **vertices** (nodes, points). $V(G)$ and V_G
- E = **edges** (links, arcs) between pairs of vertices. Also denoted by $E(G)$ and E_G ; $E \subseteq V \times V$
- **Graph size** parameters: n = |V|, m = |E|.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { (1,2), (1,3), (2,3), (2,4), (2,5), (3,5), (3,7), (3,8), (4,5), (5,6) }

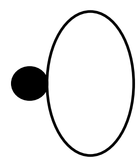
n = 8

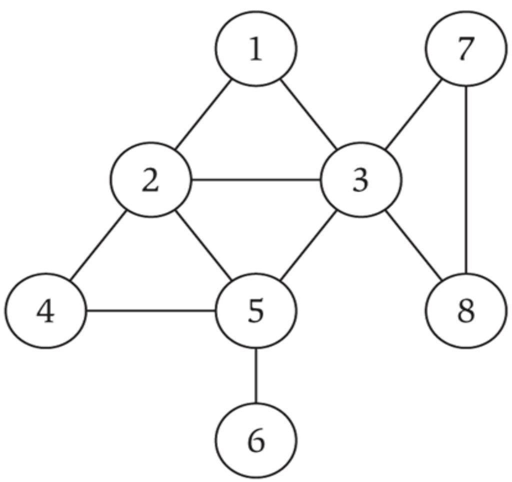
m = 11

Introduction to graph theory

For graph *G(V,E)*:

- If edge e=*(u,v)* ∈ *E(G)*, we say that *u* and *v* are **adjacent** or **neighbors**
- *u* and *v* are **incident** with *e*
- *u* and *v* are **end-vertices** of *e*
- An edge where the two end vertices are the same is called a **loop**, or a **self-loop**





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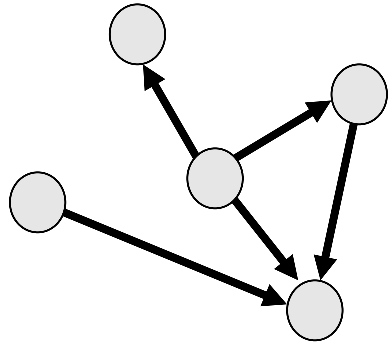
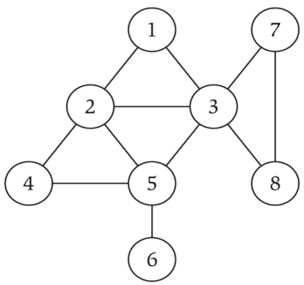
3

Directed graph (digraph)

- Directed edge - ordered pair of vertices $(\boldsymbol{u},\boldsymbol{v})$

- A graph with directed edges is called a directed graph or digraph



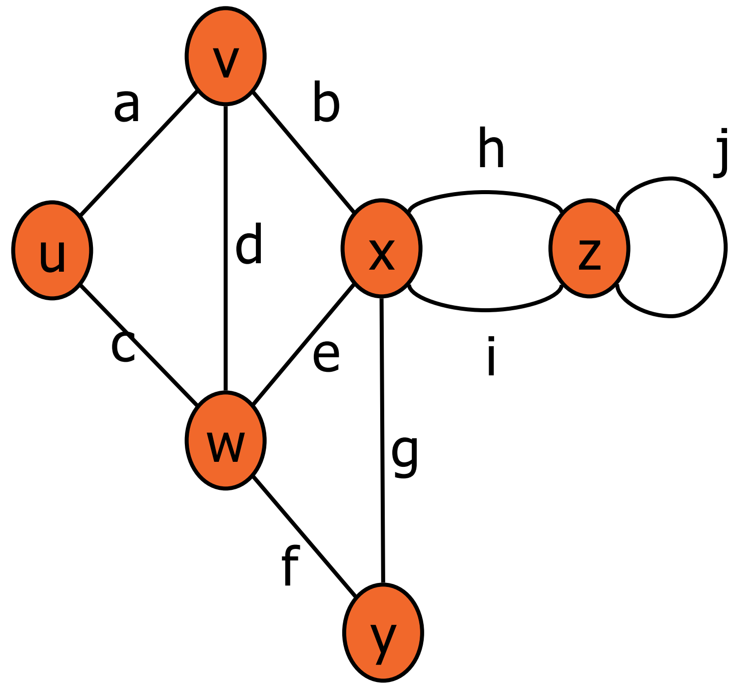


- Undirected edge- unordered pair of vertices $(\boldsymbol{u},\boldsymbol{v})$

- A graph with undirected edges is an undirected graph or simply a graph

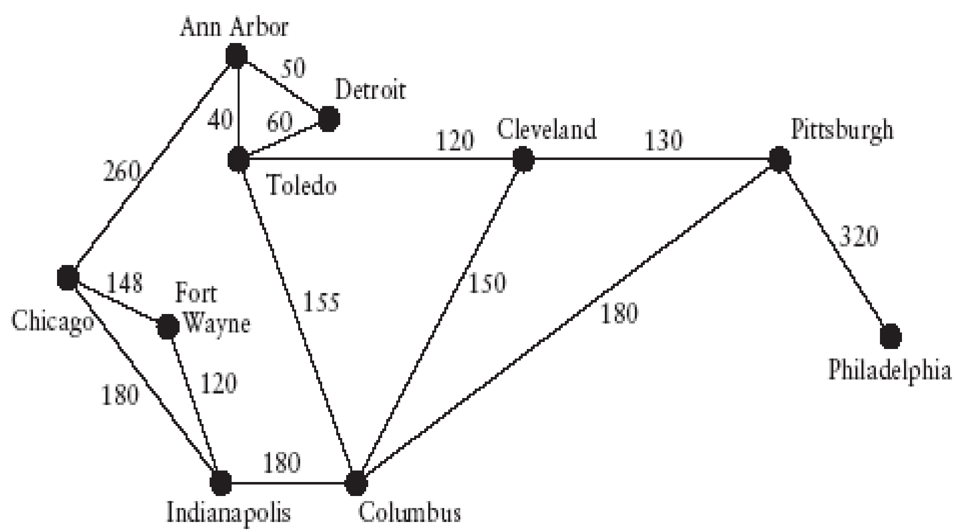
Terminology

- End vertices (or endpoints) of an edge
  - u and v are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on v
- Adjacent vertices
  - u and v are adjacent
- Degree of a vertex
  - x has degree 5
- Self-loop
  - j is a self-loop



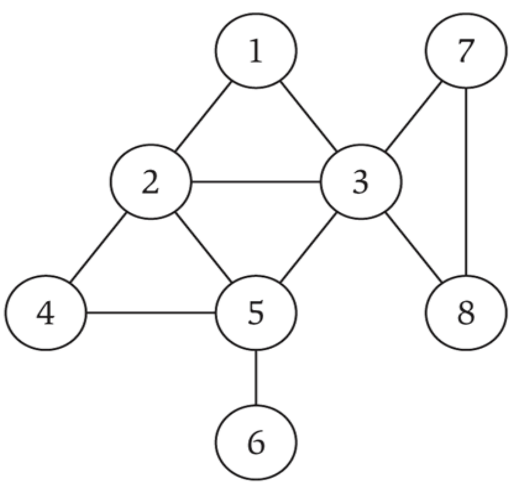
Weighted Graphs

- The edges in a graph may have values associated with them known as their weights
- A graph with weighted edges is known as a weighted graph

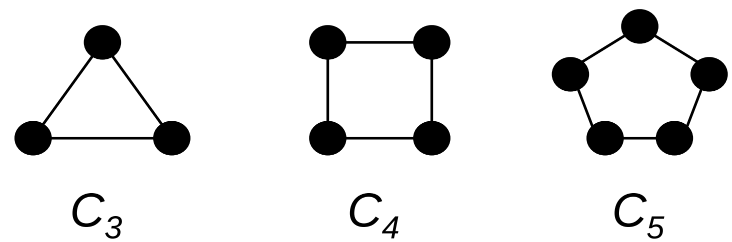


Terminology

- A **path** in an undirected graph G = (V, E) is a sequence P of vertices $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.
- A path is **simple** if all vertices are distinct.
- A **cycle** is a path in which the first and final vertices are the same
- A **cycle** is **simple** if all the vertices except the first and final are distinct
- Cycles can be denoted by C_k, where k is the number of vertices in the cycle

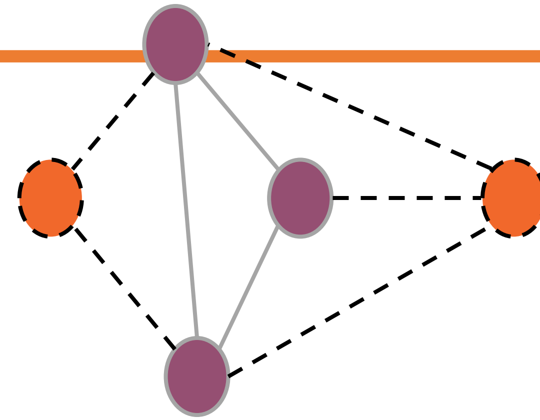


(1, 3, 7, 8, 3, 5) is a path
(6, 6, 3, 2) is a simple path
(1, 2, 4, 5, 2, 3, 1) is a cycle
(1, 2, 3, 1) is a simple cycle

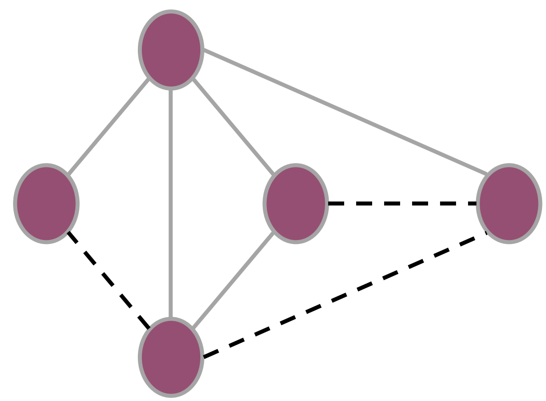


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



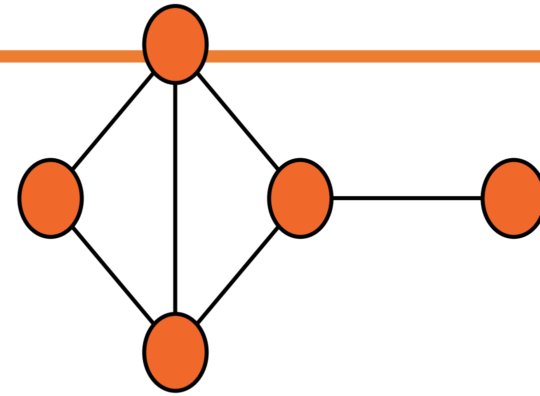
Subgraph



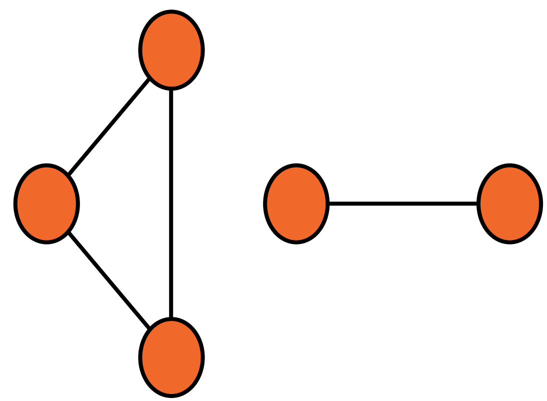
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G



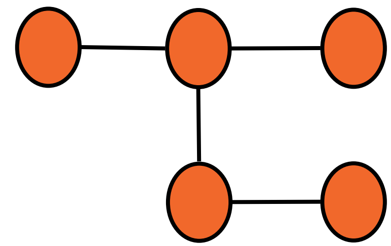
Connected graph



Non connected graph with two connected components

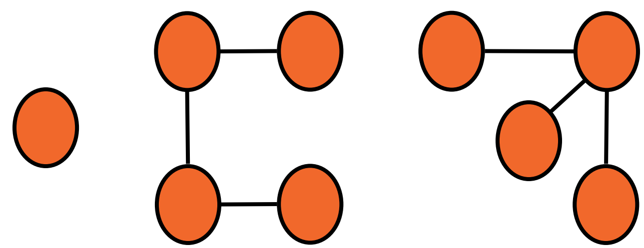
Trees and Forests

- A tree is an undirected graph T such that
  - T is connected
  - T has no cycles



Tree

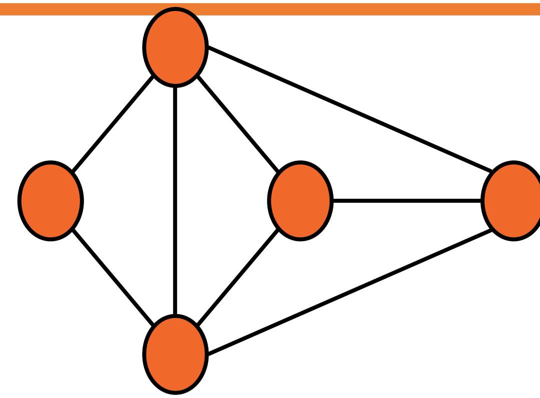
- A forest is an undirected graph without cycles
- The connected components of a forest are trees



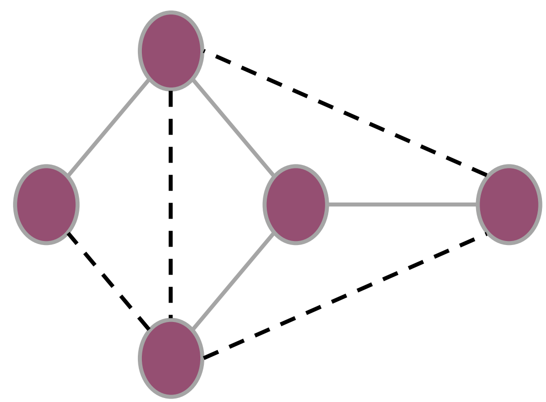
Forest

Spanning Trees and Forests

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree



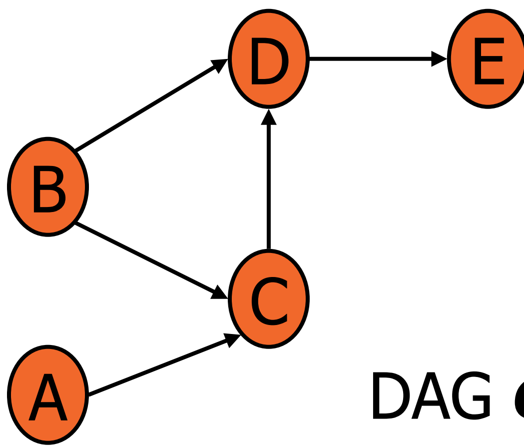
Graph



Spanning tree

DAG

- A directed acyclic graph (DAG) is a digraph that has no directed cycles



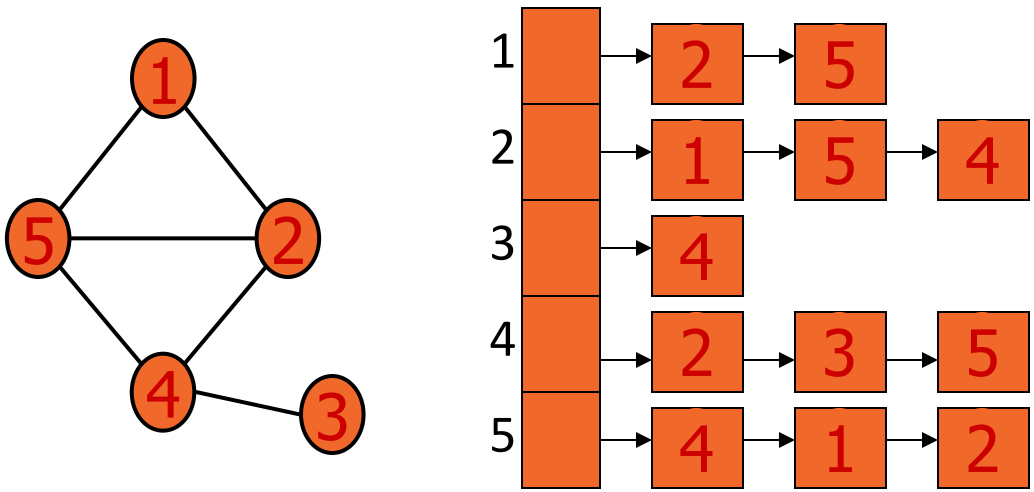
DAG G

Representation of Graphs

Two standard ways:

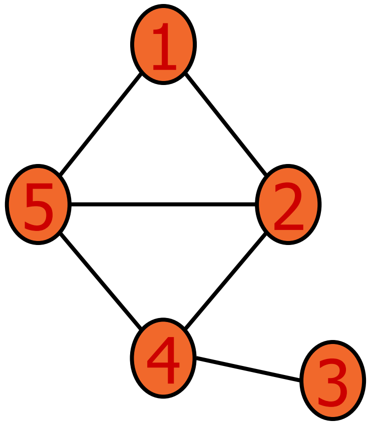
- Adjacency List
  - preferred for sparse graphs (|E| is much less than |V|^2)
  - Unless otherwise specified we will assume this representation
- Adjacency Matrix
  - Preferred for dense graphs

Adjacency List



- An array Adj of |V| lists, one per vertex
- For each vertex u in V,
  - Adj[u] contains all vertices v such that there is an edge (u,v) in E (i.e. all the vertices adjacent to u)
- Space required Θ(|V|+|E|) (Following CLRS, we will use V for |V| and E for |E|) thus Θ(V+E)

Adjacency Matrix



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	0	1	1
3	0	0	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Graph Traversals

- For solving most problems on graphs
  - Need to systematically visit all the vertices and edges of a graph

- Two major traversals
  - Breadth-First Search (BFS)
  - Depth-First Search(DFS)

BFS

- Starts at some source vertex s
- Discover every vertex that is reachable from s
- Also produces a BFS tree with root s and including all reachable vertices

- Discovers vertices in increasing order of distance from s
  - Distance between v and s is the minimum number of edges on a path from s to v

- i.e. discovers vertices in a series of layers

BFS : vertex colors stored in color[]

- Initially all undiscovered: white
- When first discovered: gray
  - They represent the frontier of vertices between discovered and undiscovered
  - Frontier vertices stored in a queue
  - Visits vertices across the entire breadth of this frontier
- When processed: black

Review: Breadth-First Search

- "Explore" a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find ("discover") its children, then their children, etc.

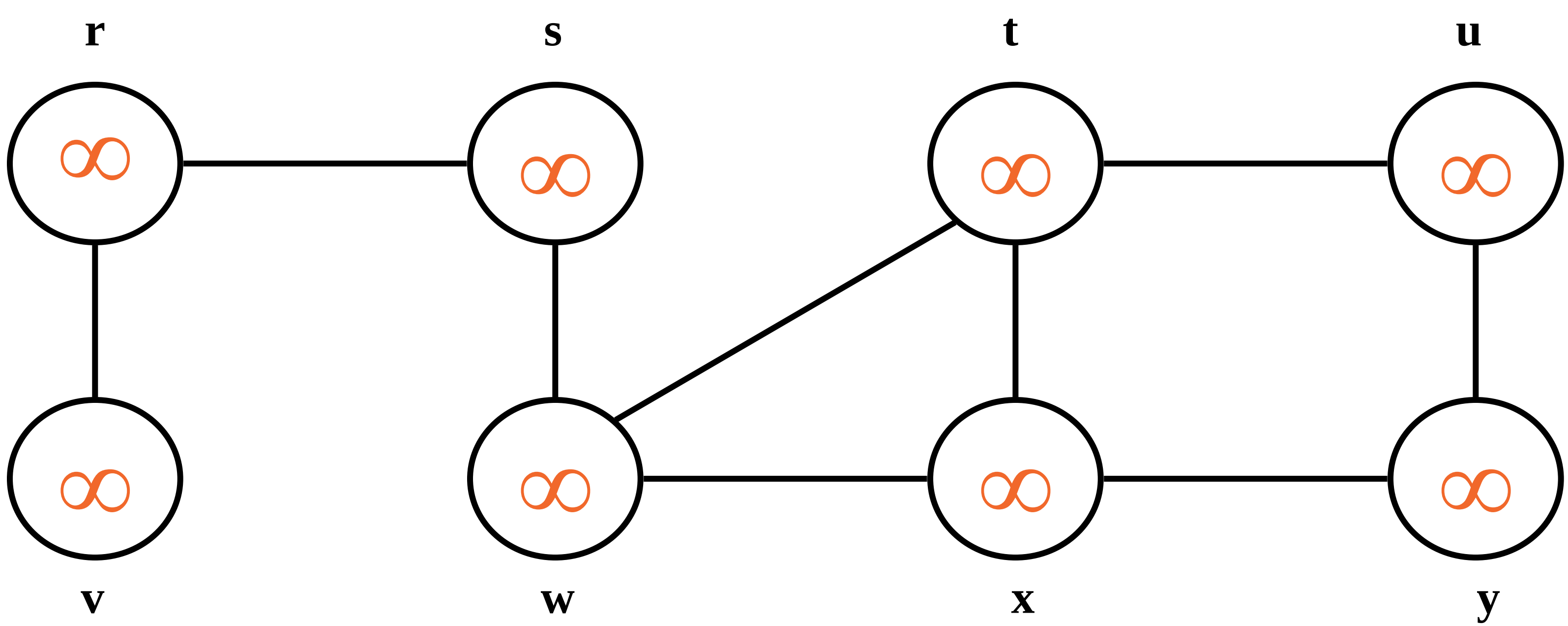
Breadth-First Search

- Again will associate vertex “colors” to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Grey vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - Black vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of grey vertices

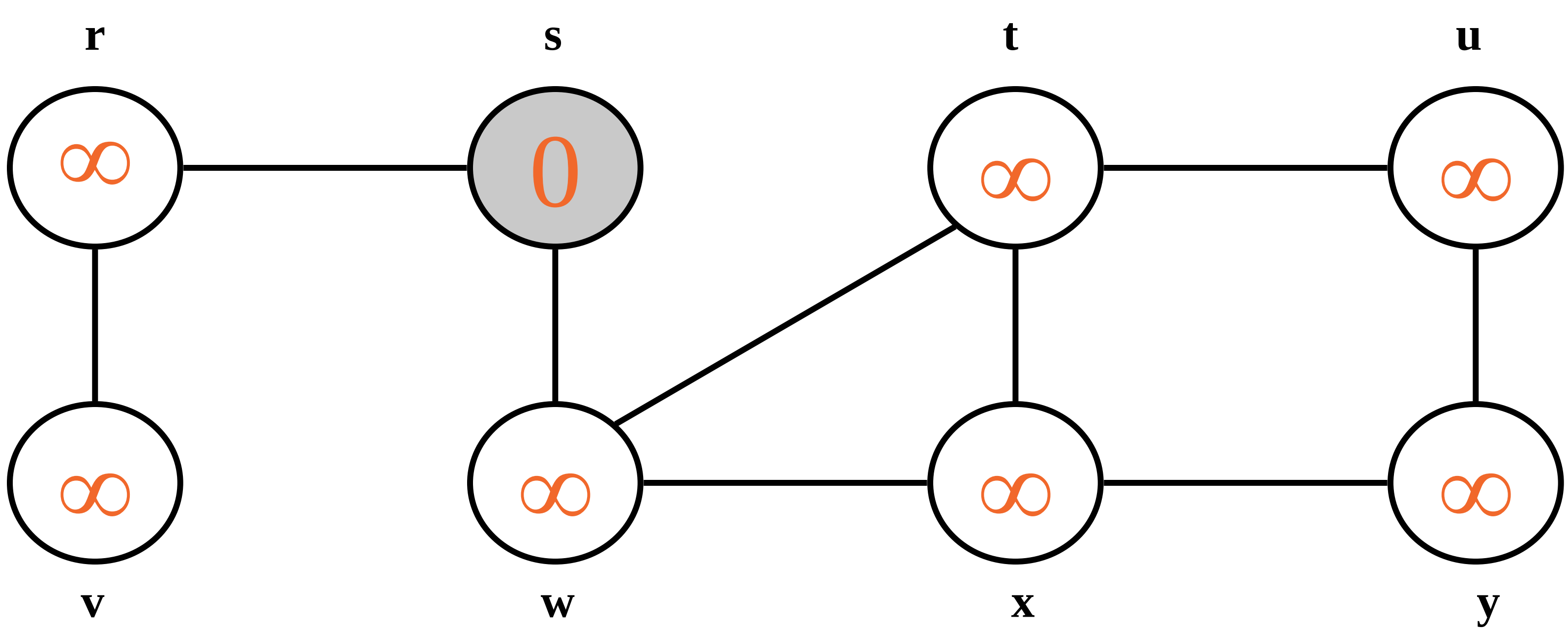
Review: Breadth-First Search

```
BFS(G, s) {
    initialize vertices;
    Q = {s};            // Q is a queue initialize to s
    while (Q not empty) {
        u = DEQUEUE(Q);
        for each v ∈ G.Adj[u] {
            if (v.color == WHITE)
                v.color = GREY;
                v.d = u.d + 1;
                v.p = u;
                ENQUEUE(Q, v);
        }
        u.color = BLACK;
    }
}
```

Breadth-First Search: Example

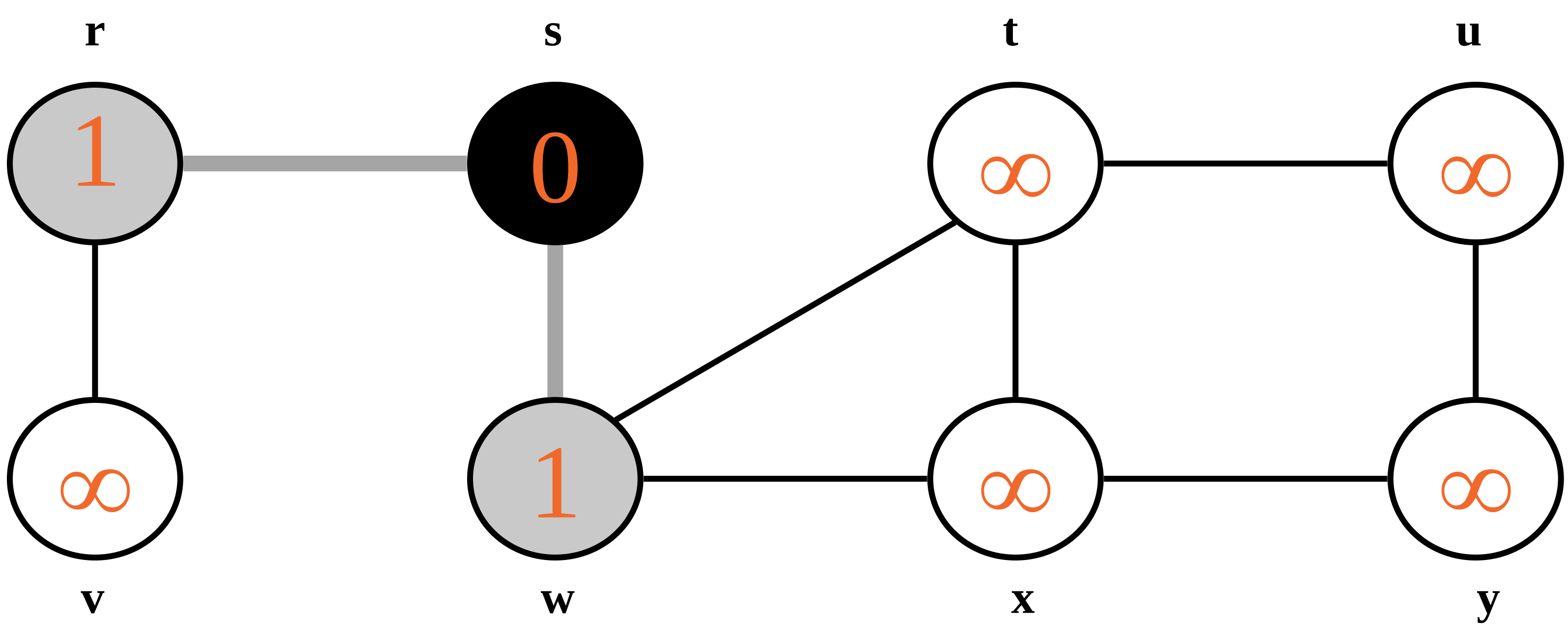


Breadth-First Search: Example



Q: s

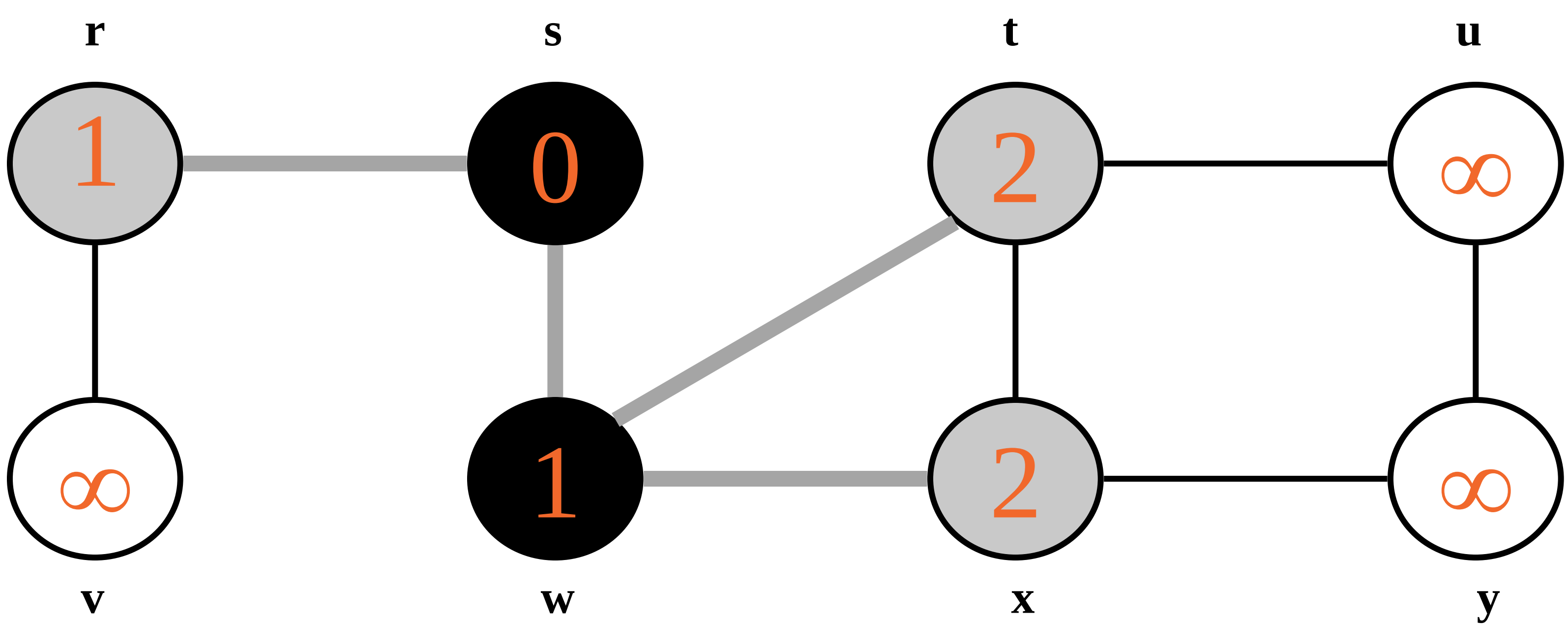
Breadth-First Search: Example



Q:

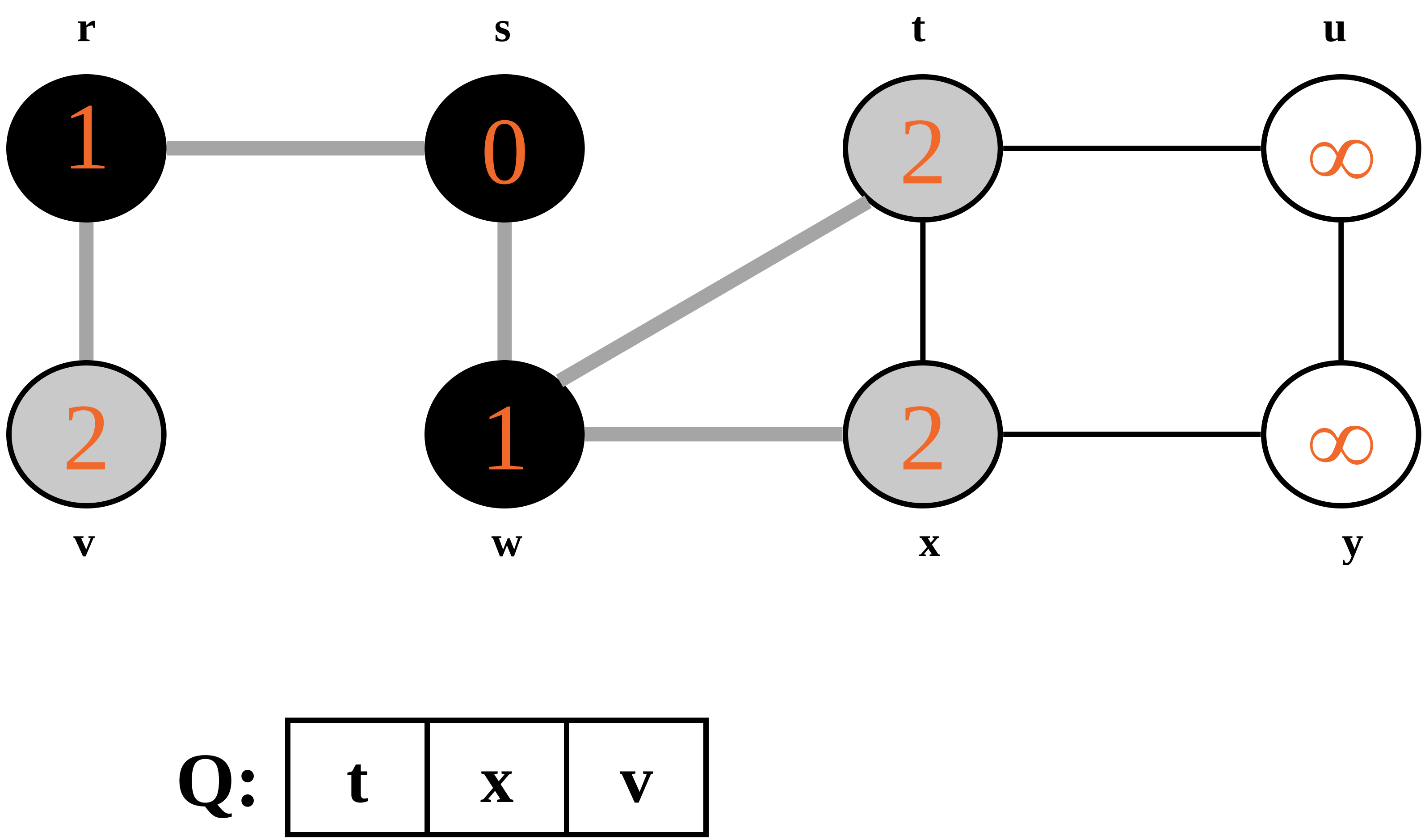
w	r

Breadth-First Search: Example

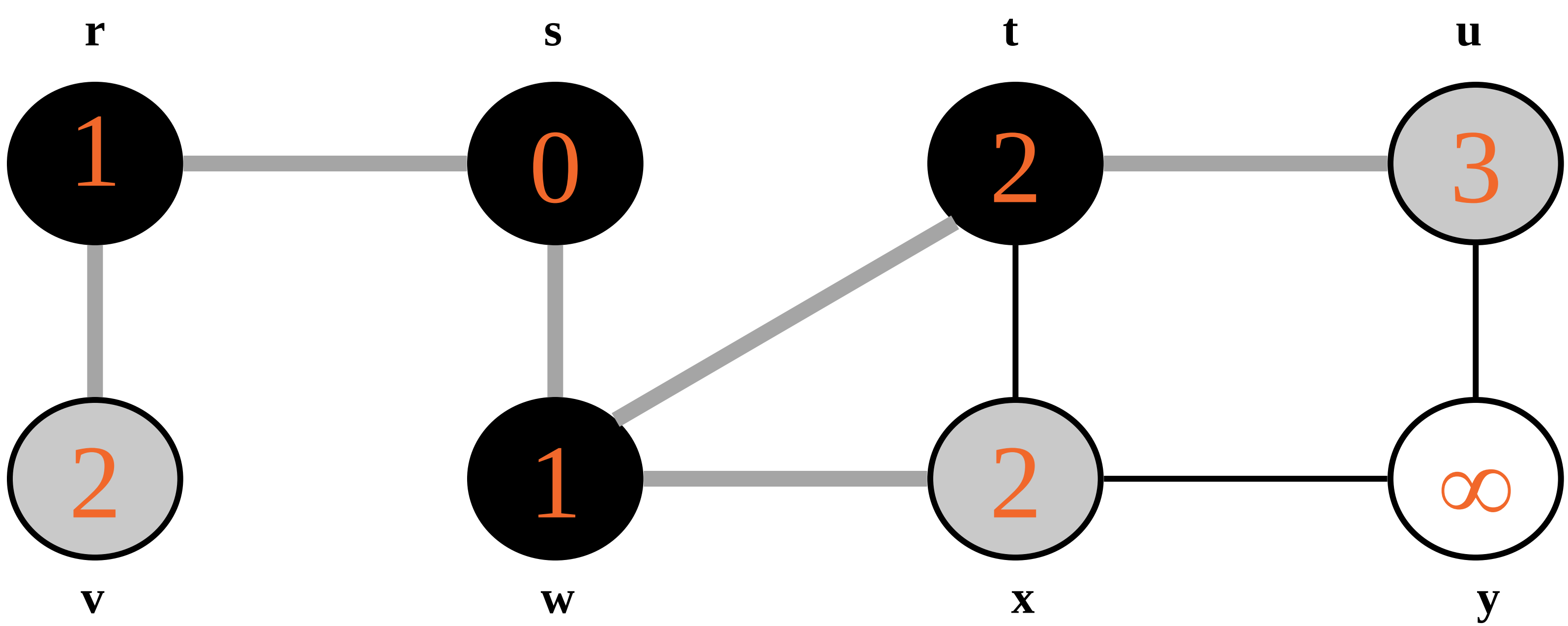


Q: | r | t | x |

Breadth-First Search: Example



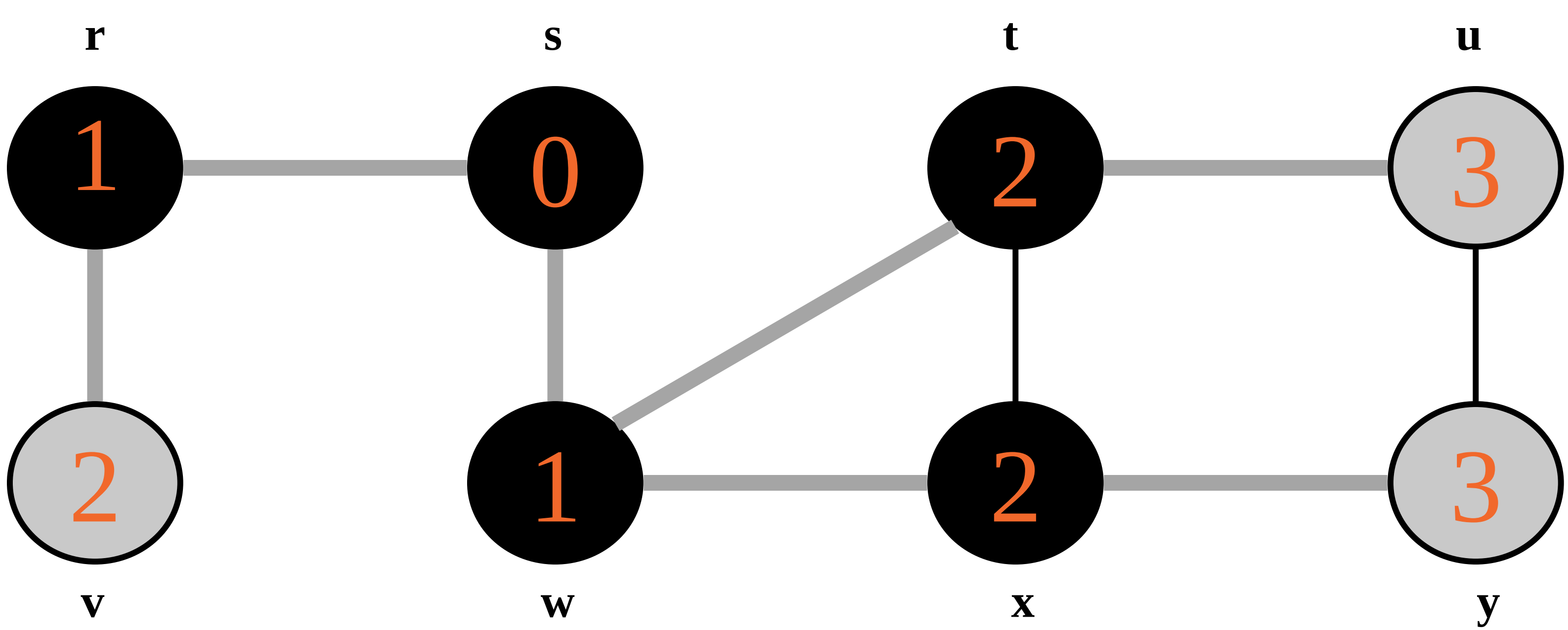
Breadth-First Search: Example



Q:

x	v	u

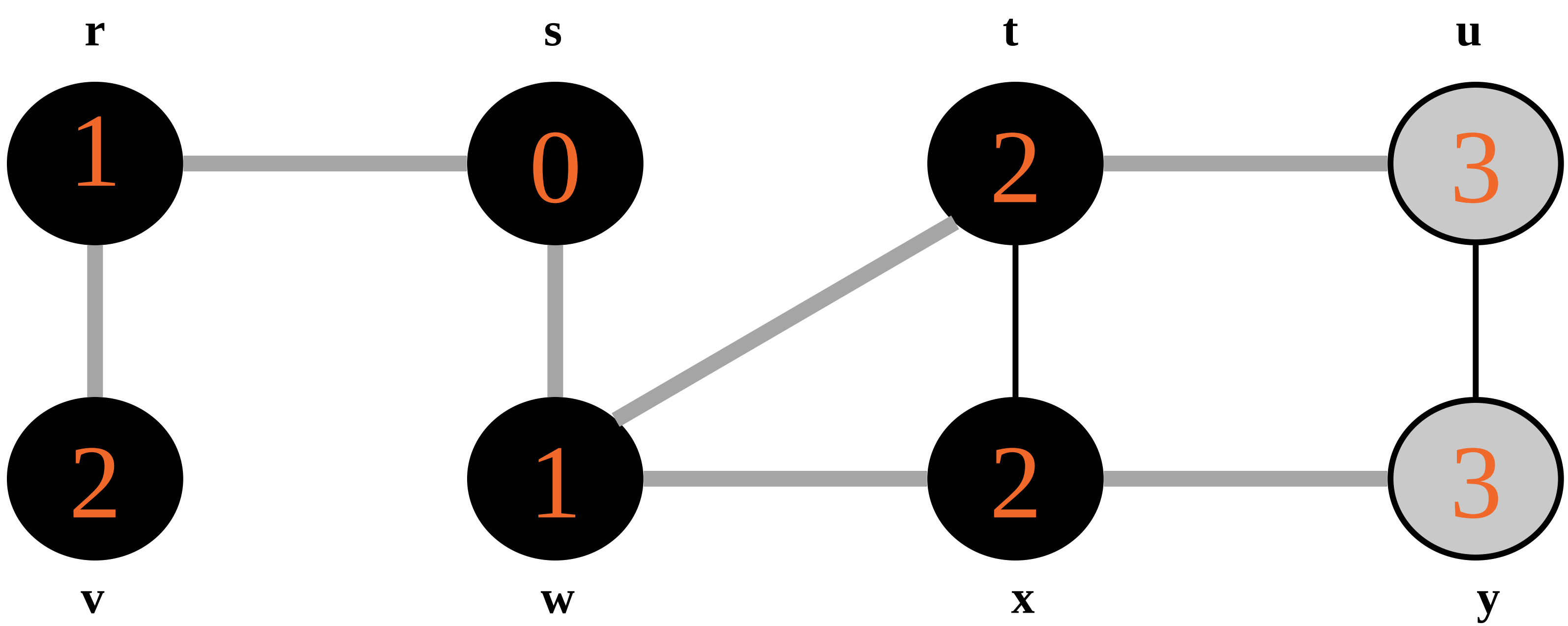
Breadth-First Search: Example



Q:

v	u	y

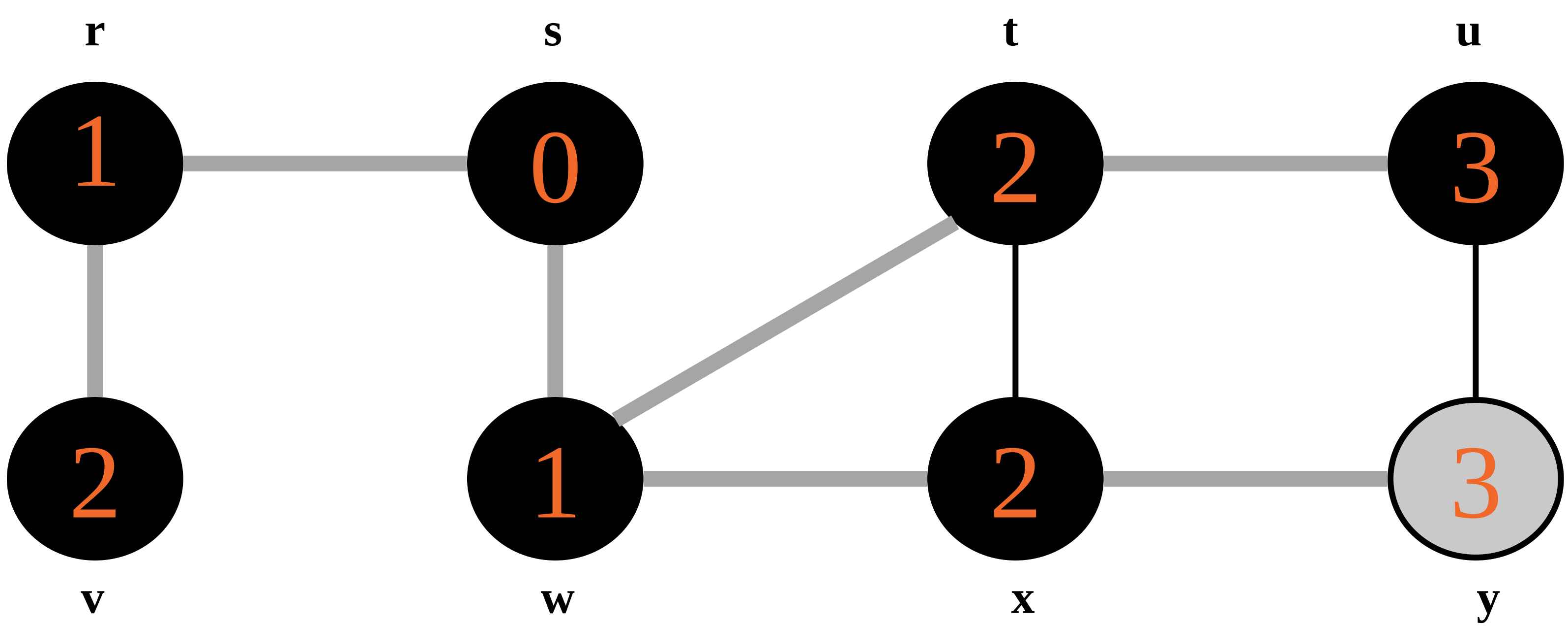
Breadth-First Search: Example



Q:

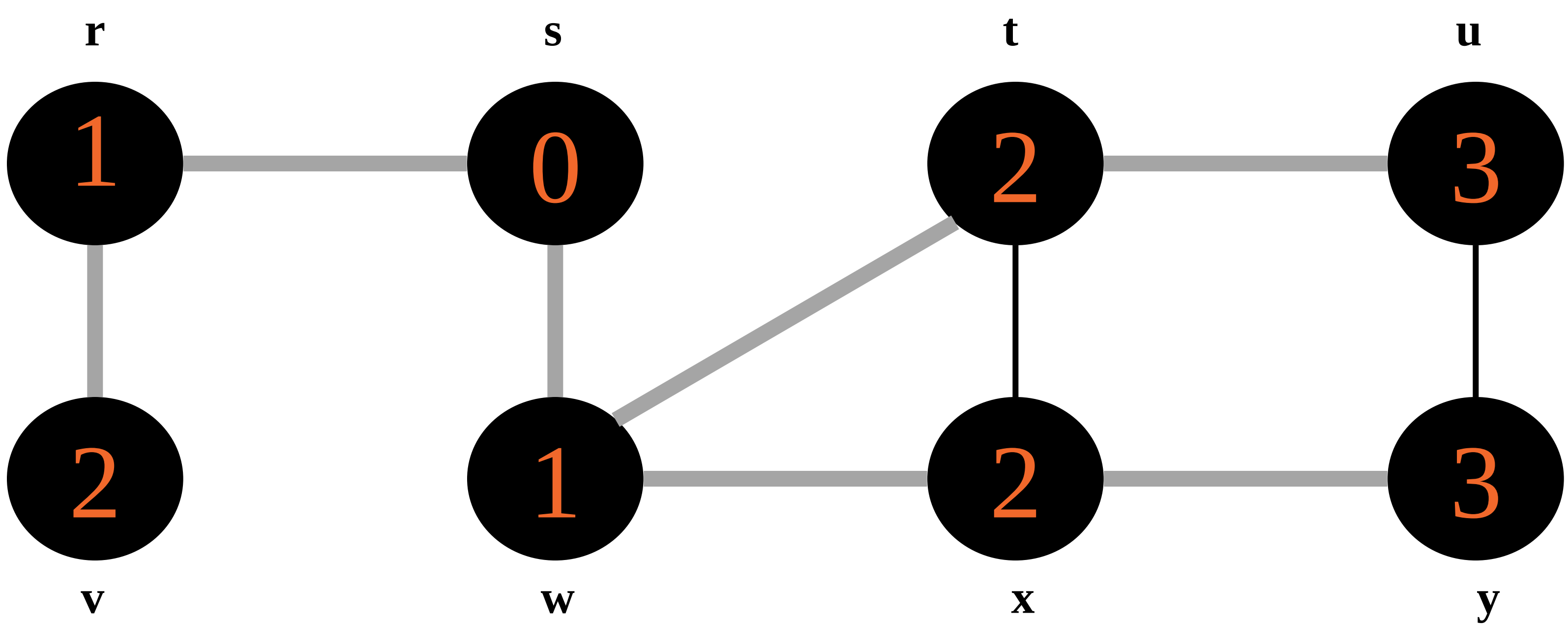
u	y

Breadth-First Search: Example



Q: y

Breadth-First Search: Example



Q: Ø

BFS: The Code Again

```
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    initialize vertices;
    Q = {s};              // Q is a queue initialize to s
    while (Q not empty) {
        u = DEQUEUE(Q);
        for each v ∈ G.Adj[u] {
            if (v.color == WHITE)
                v.color = GREY;
                v.d = u.d + 1;
                v.p = u;
                ENQUEUE(Q, v);
        }
        u.color = BLACK;
    }
}
```

Touch every vertex: O(V)

u = every vertex, but only once

So v = every vertex that appears in some other vert's adjacency list

What will be the running time?

Total running time: O(V+E)

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source vertex
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - Thus can use BFS to calculate shortest path from one vertex to another in O(V+E) time

Analysis

- Each vertex is enqueued once and dequeued once : O(V)

- Each adjacency list is traversed once:

$$\sum_{u \in V} \deg(u) = O(E)$$

- Total: O(V+E)

BFS and shortest paths

Theorem: Let G=(V,E) be a directed or undirected graph, and suppose BFS is run on G starting from vertex s. During its execution BFS discovers every vertex v in V that is reachable from s. Let δ(s,v) denote the number of edges on the shortest path form s to v. Upon termination of BFS, d[v] = δ(s,v) for all v in V.

Depth-First Search

Depth-first search is another strategy for exploring a graph

- Explore "deeper" in the graph whenever possible
- Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
- When all of *v*'s edges have been explored, backtrack to the vertex from which *v* was discovered
- recursive

Time stamps, color[u] and pred[u] as before

We store two time stamps:

- d[u] or u.d: the time vertex u is first discovered (discovery time)
- f[u] or u.f: the time we finish processing vertex u (finish time)

color[u] or u.color

- Undiscovered: white
- Discovered but not finished processing: gray
- Finished: black

pred[u] or u.π

- Pointer to the vertex that first discovered u

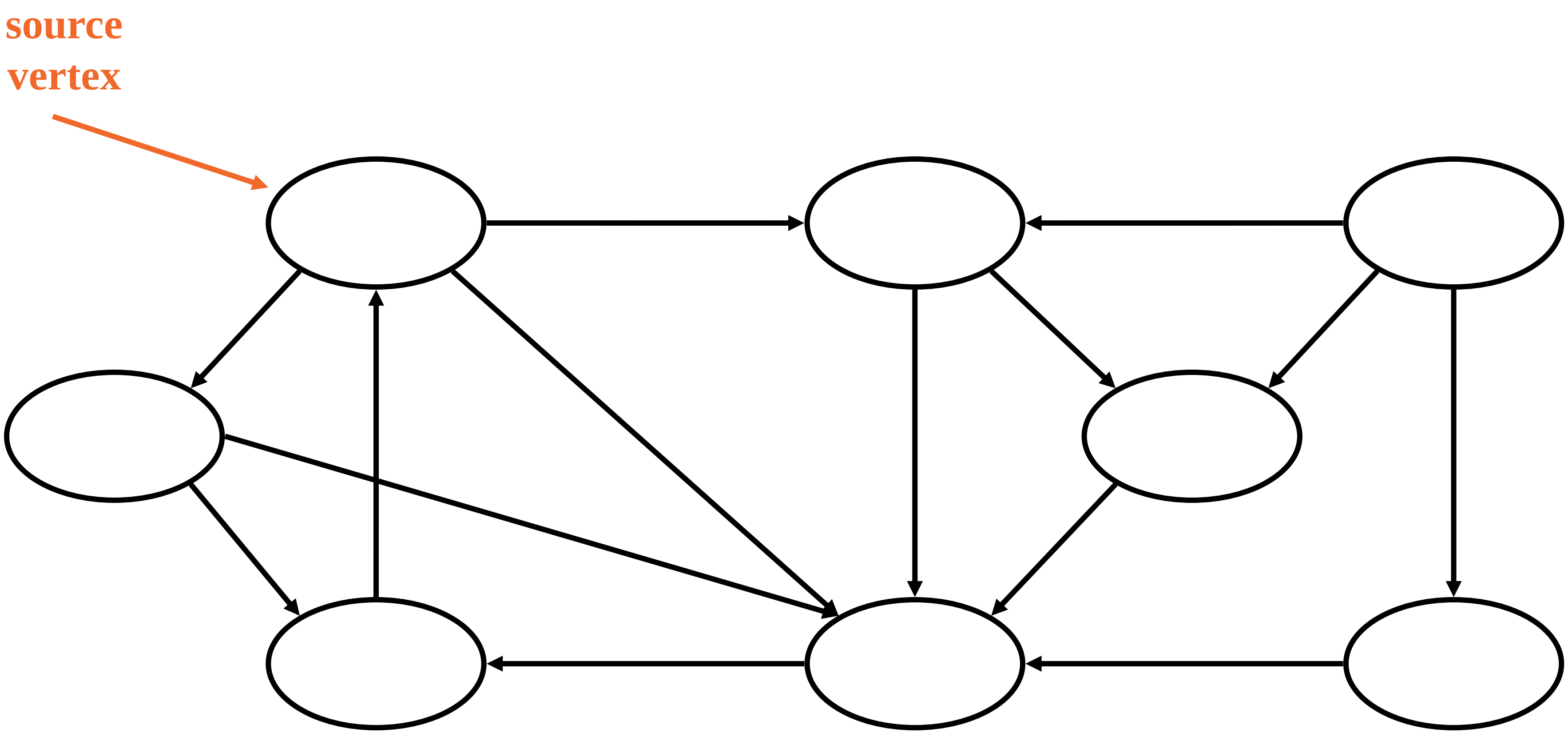
Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G.V
    {
        u.color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G.V
    {
        if (u.color == WHITE)
            DFS_Visit(G,u);
    }
}
```

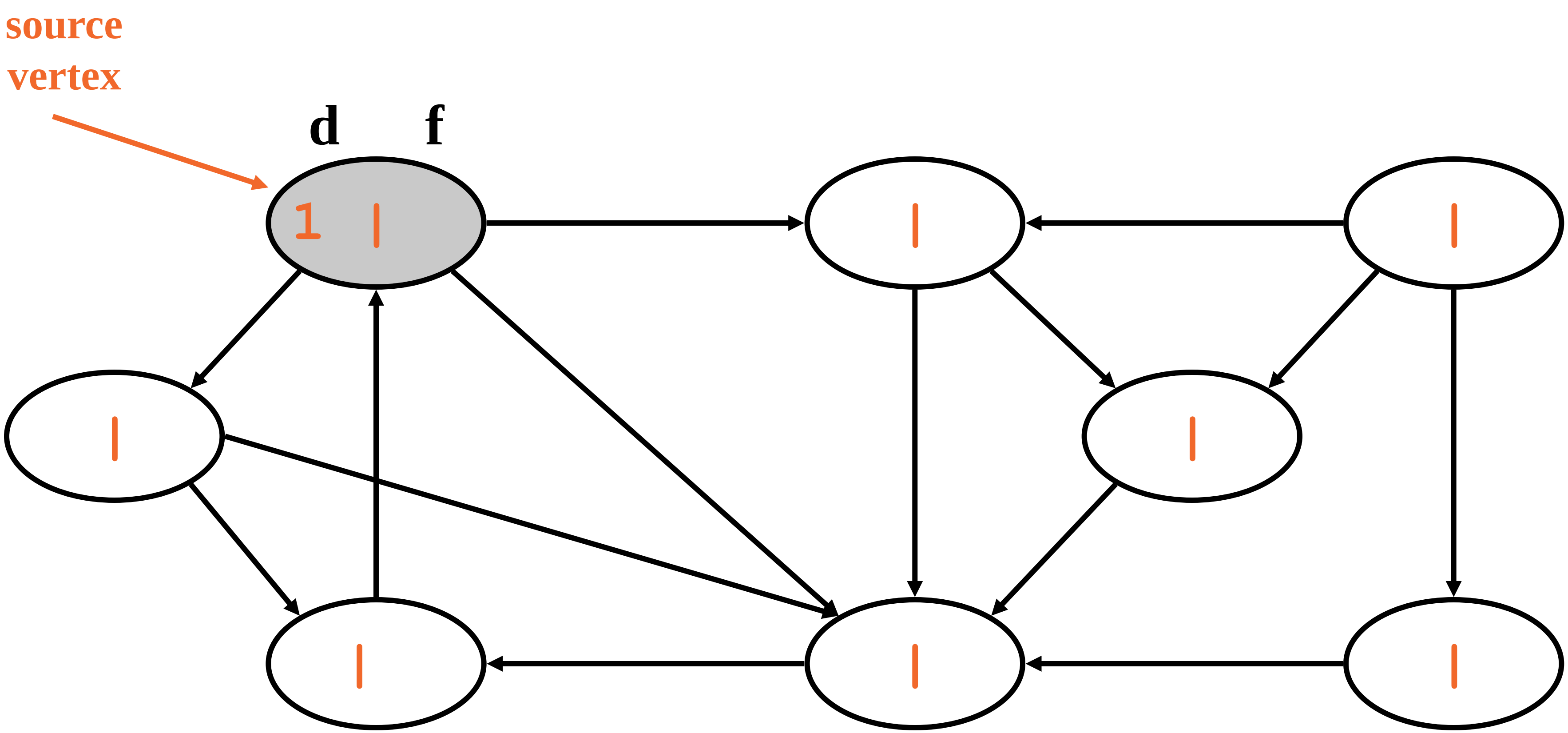
```
DFS_Visit(G, u)
{
    u.color = GREY;
    time = time+1;
    u.d = time;
    for each v ∈ G.Adj[u]
    {
        if (v.color == WHITE)
            DFS_Visit(G,v);
    }
    u.color = BLACK;
    time = time+1;
    u.f = time;
}
```

Running time: $\Theta(V+E) = \Theta(V^2)$ because call DFS_Visit on each vertex, and the loop over Adj[] can run as many as |V| times

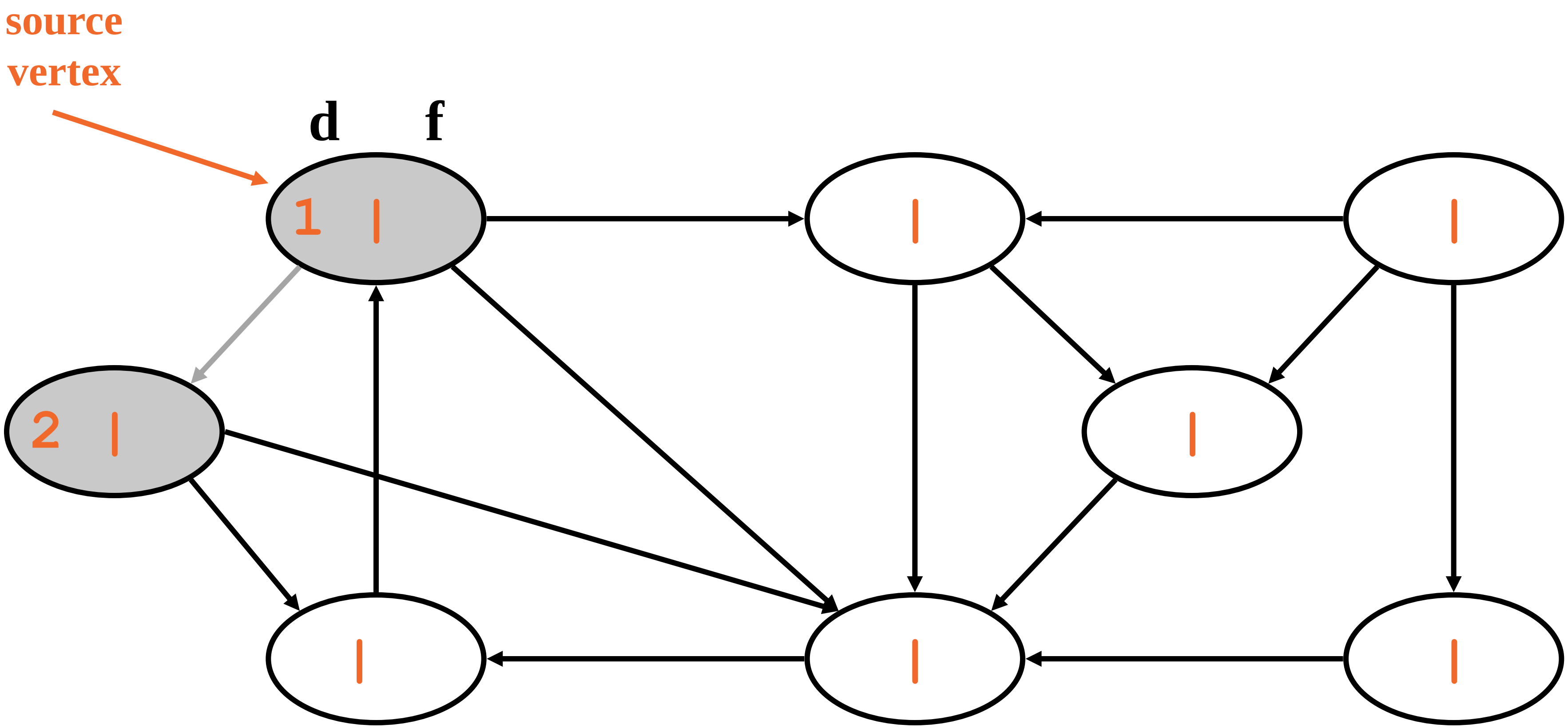
DFS Example



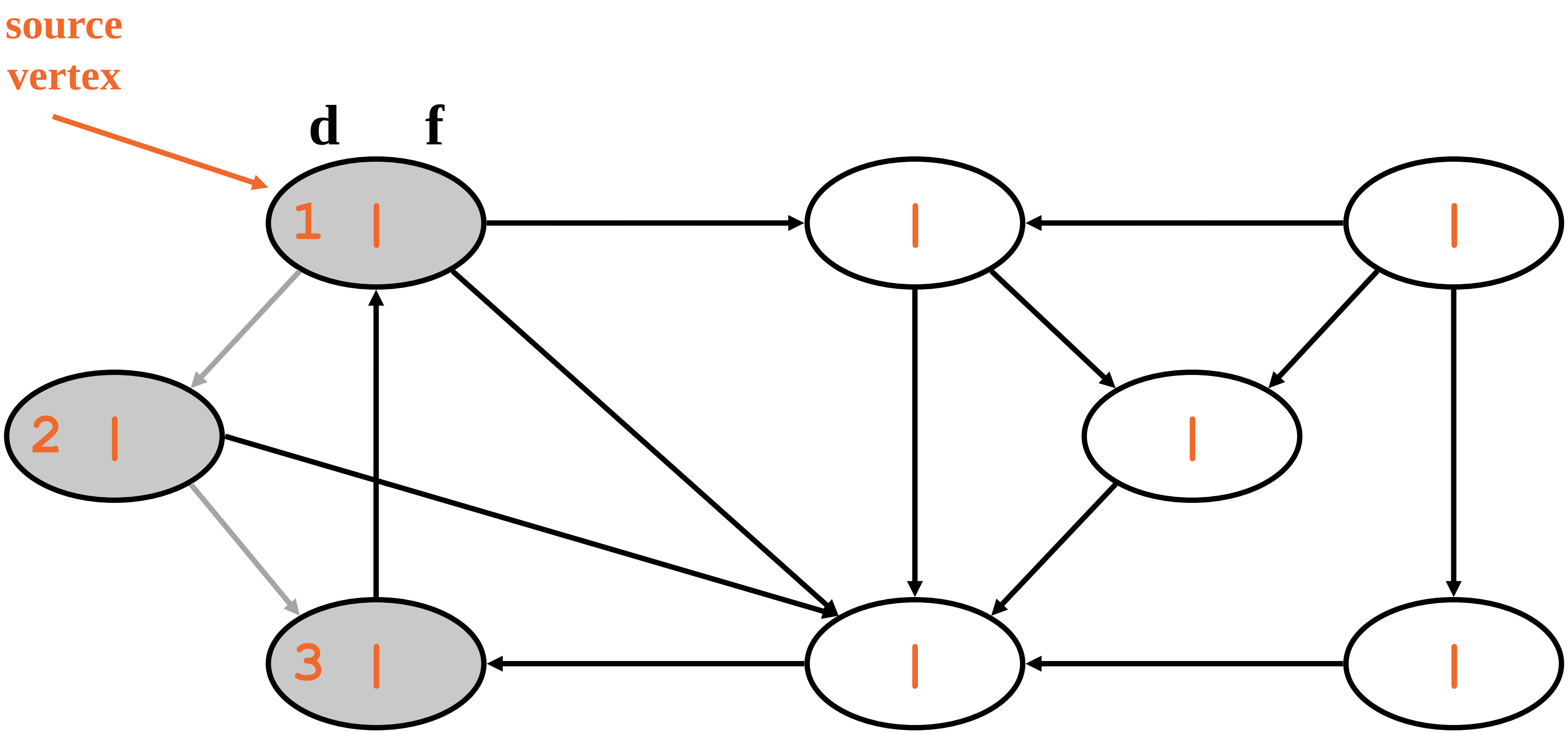
DFS Example



DFS Example



DFS Example



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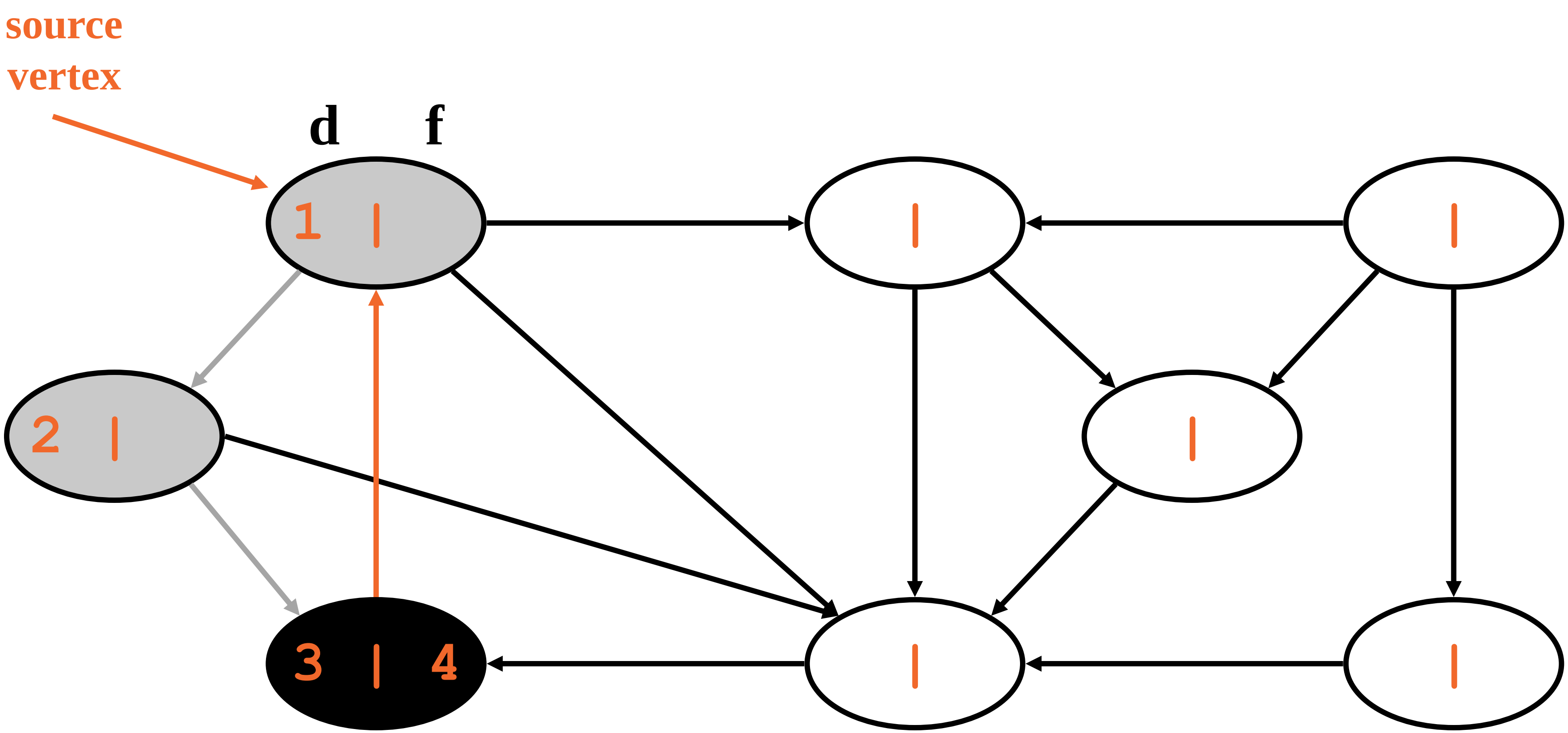

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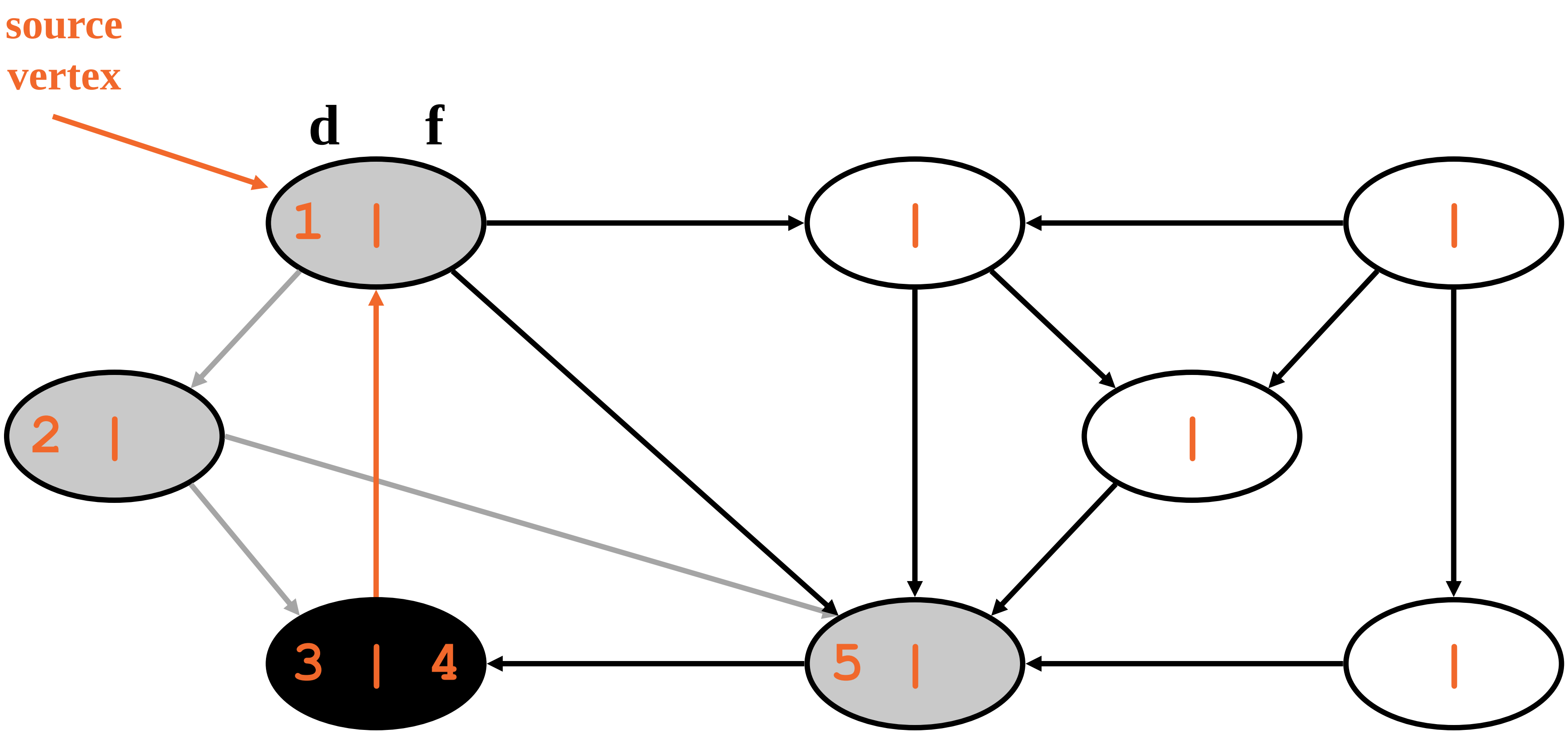

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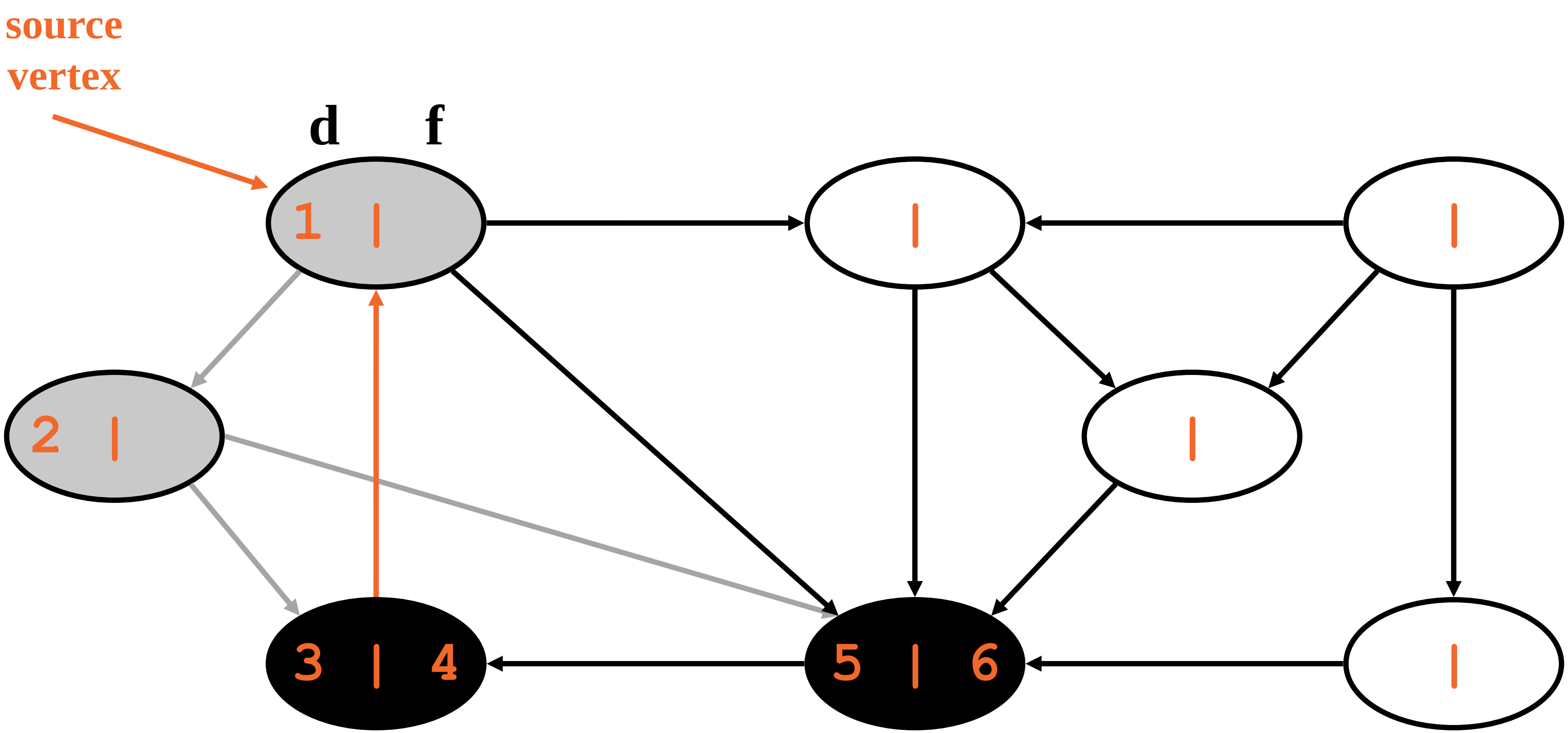

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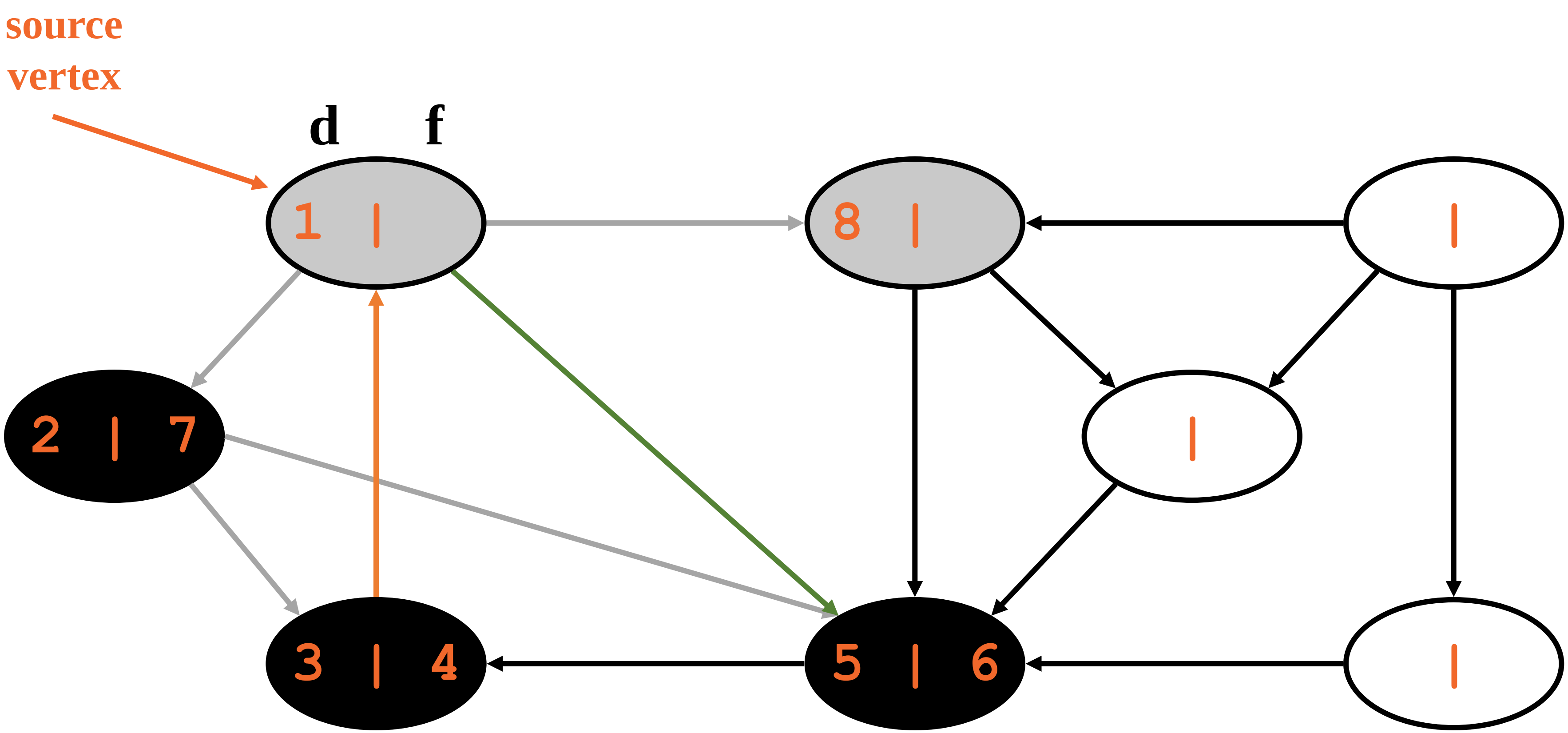
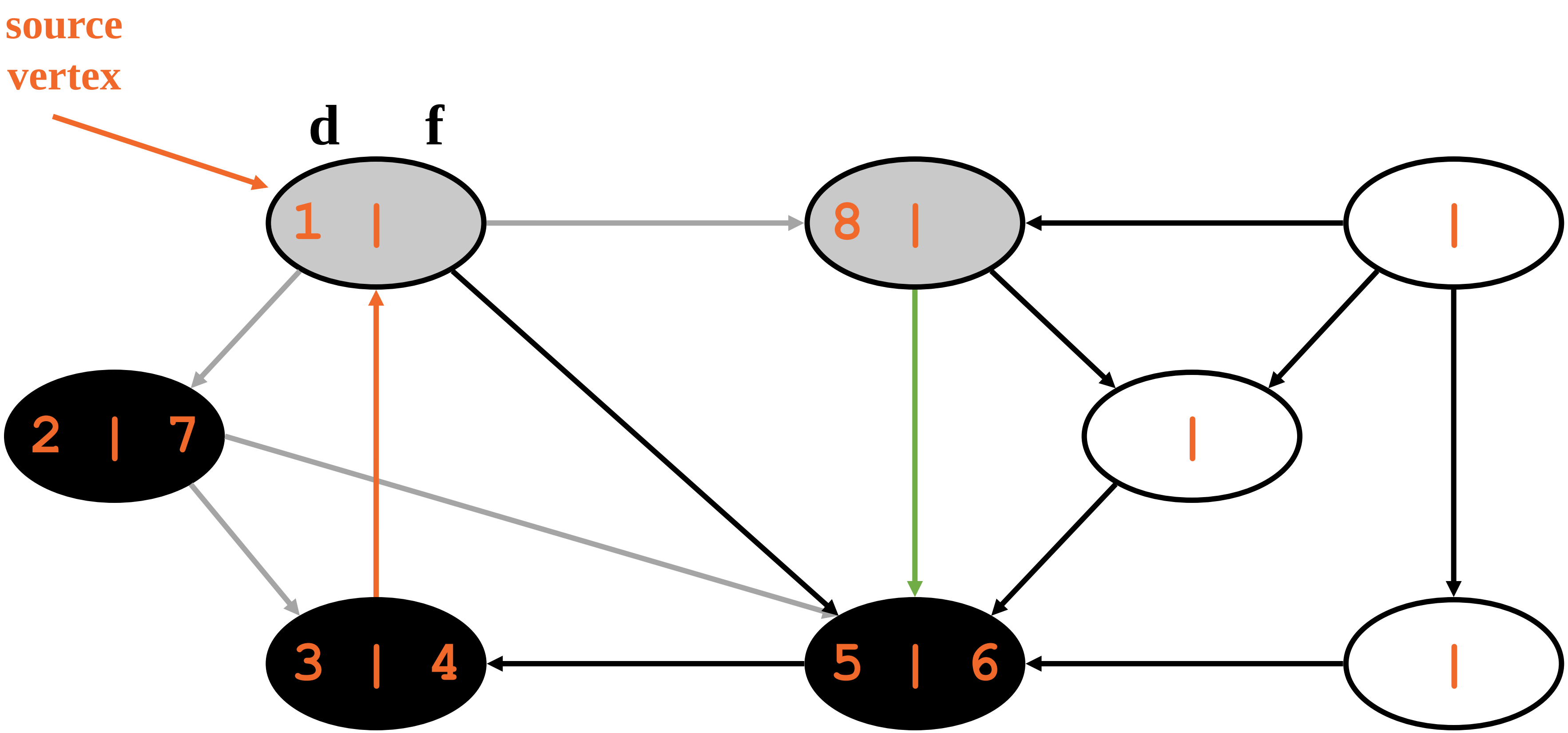
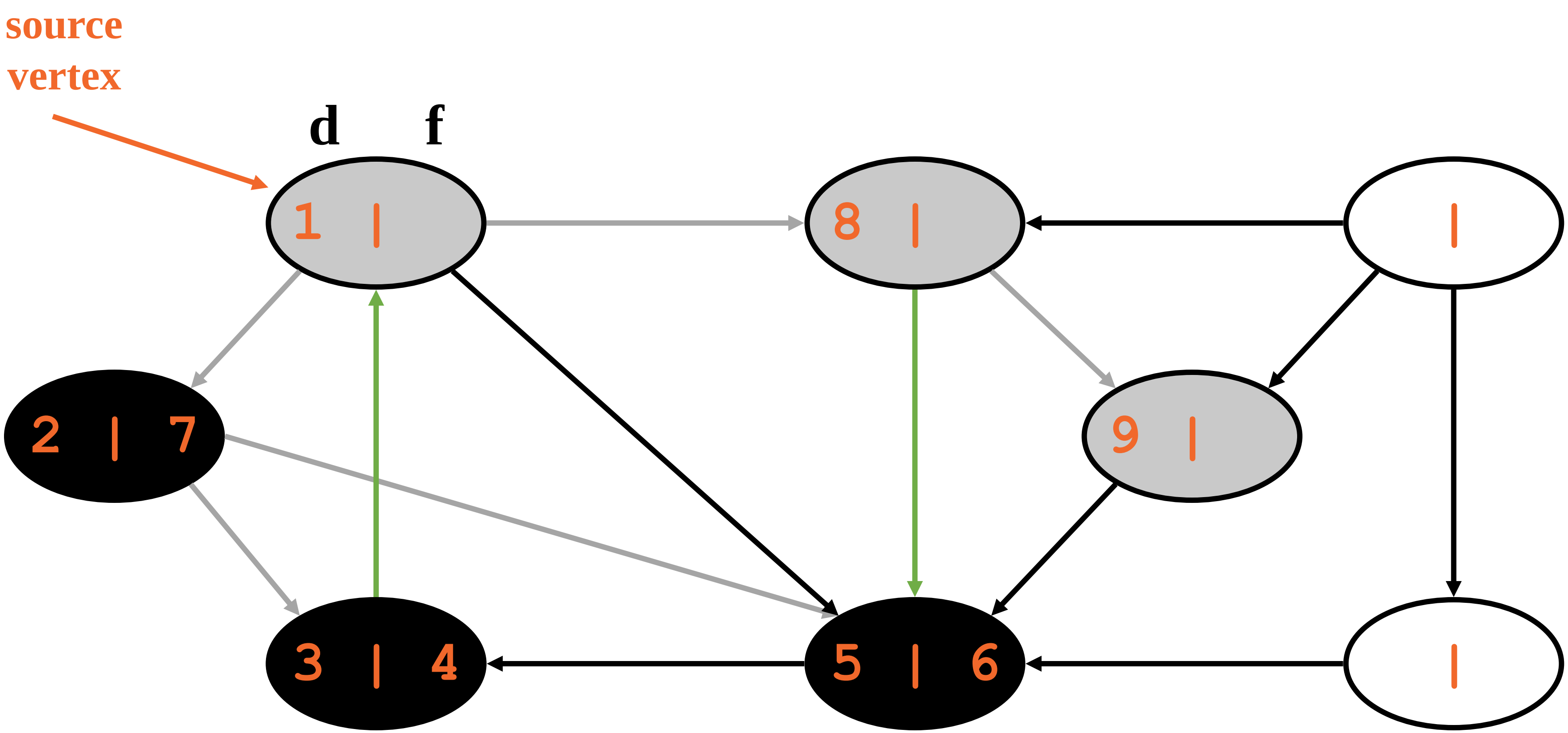


DFS Example

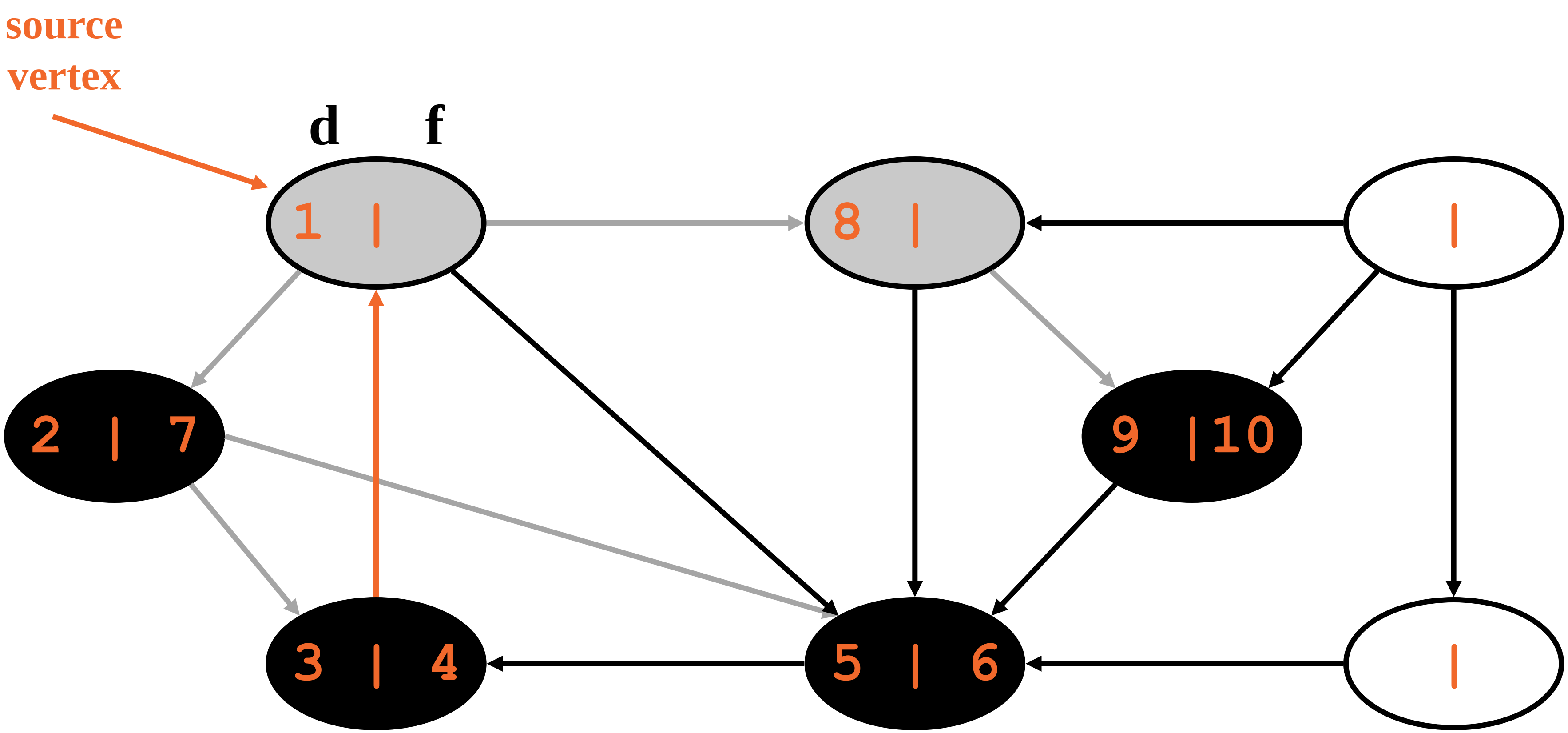


DFS Example

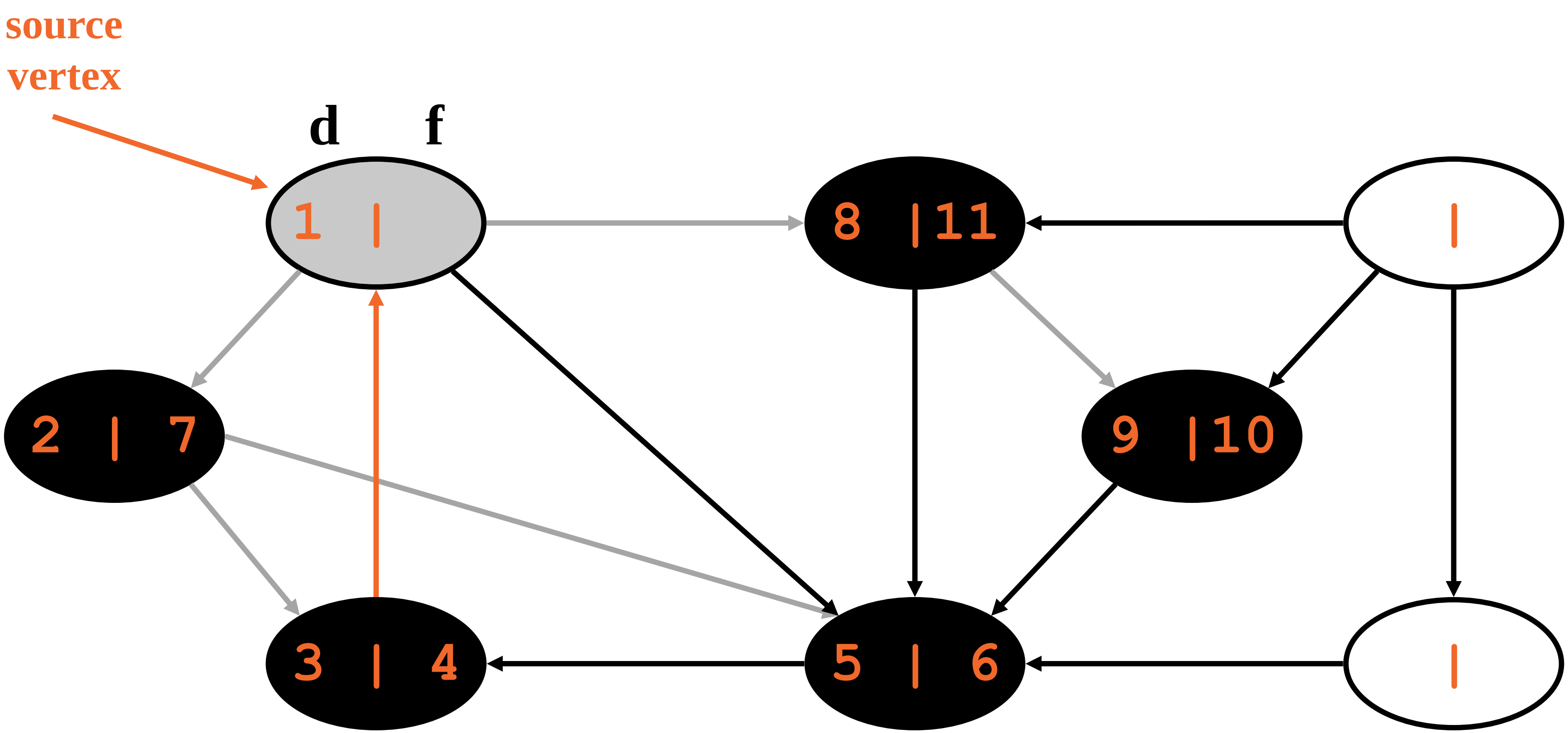


What is the structure of the grey vertices?
What do they represent?

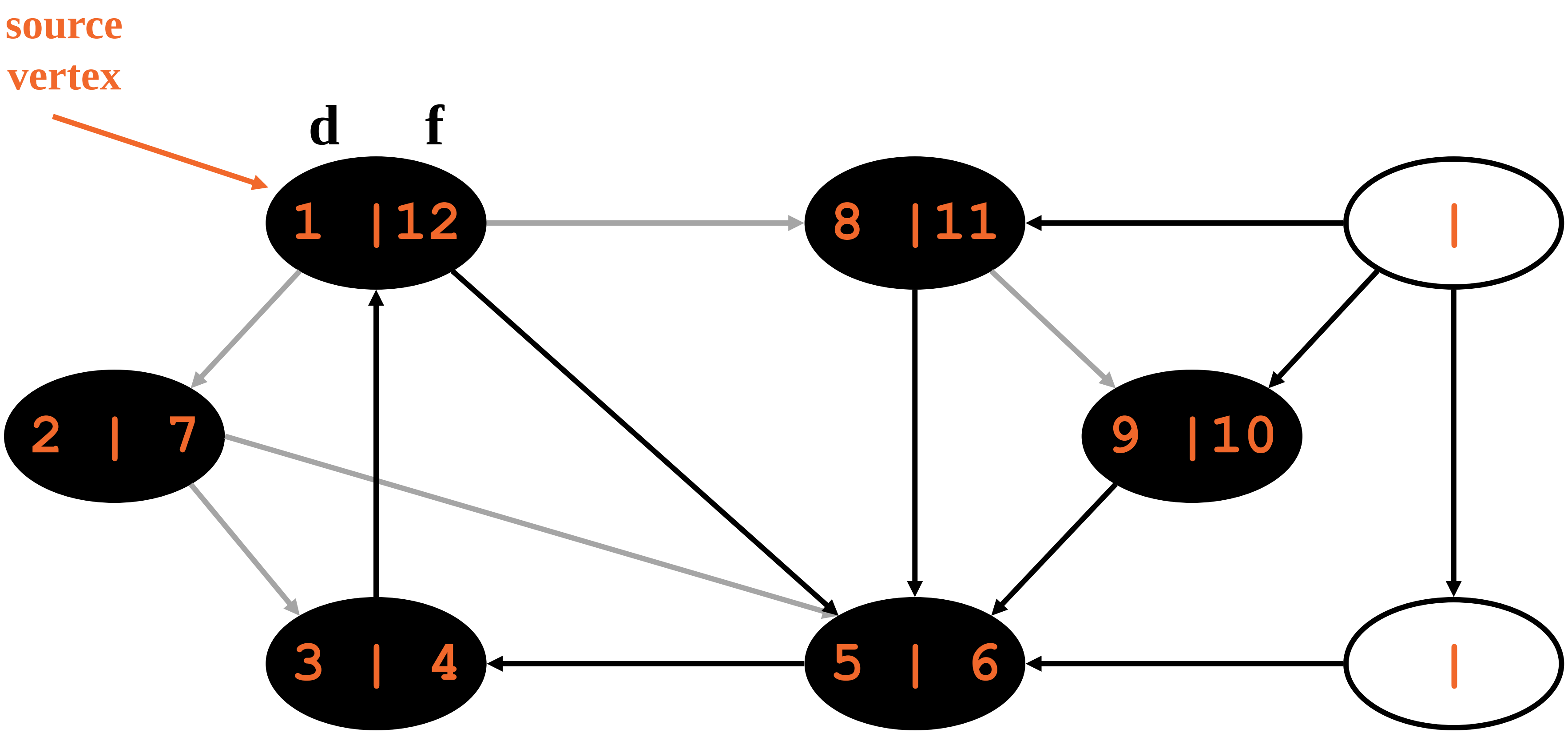
DFS Example



DFS Example



DFS Example



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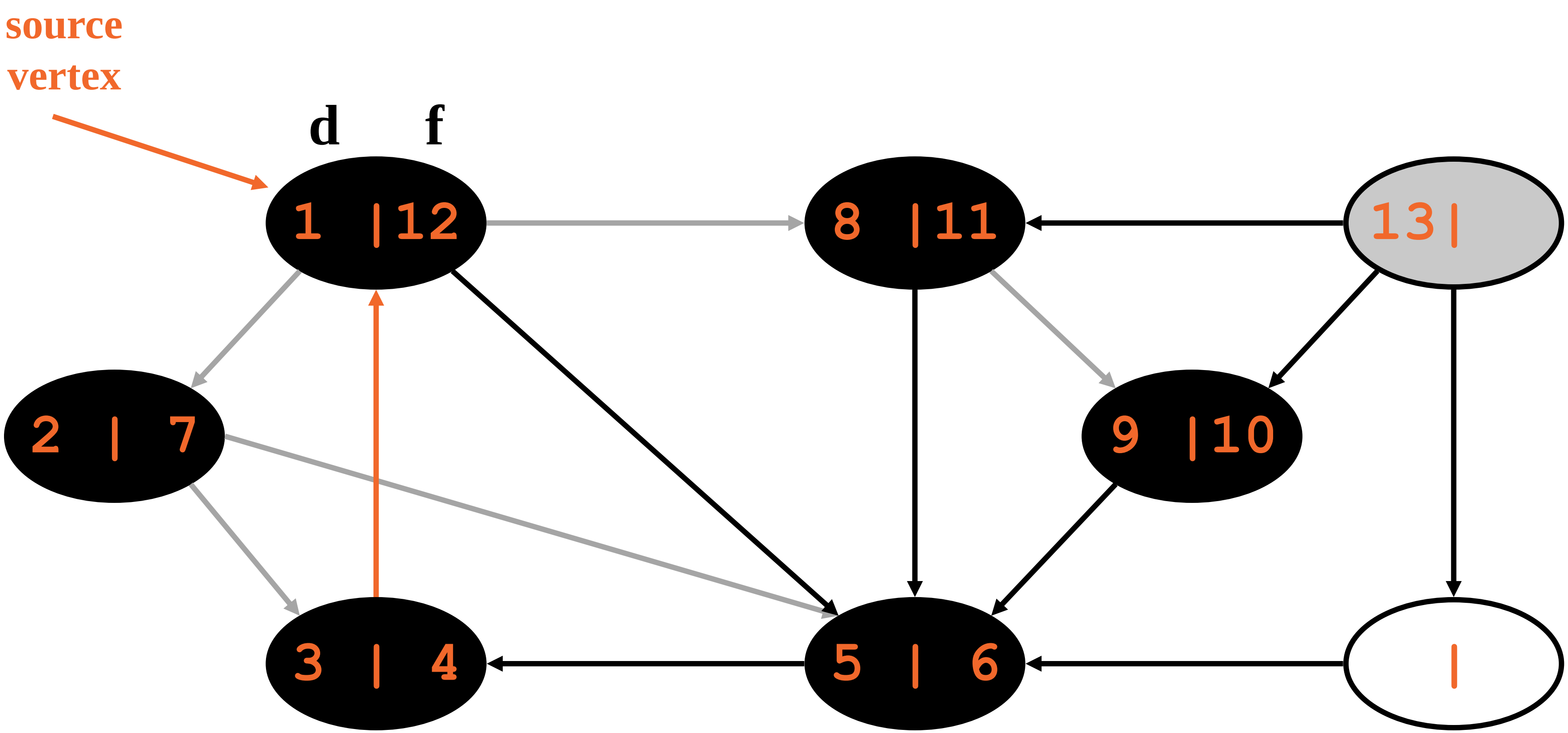

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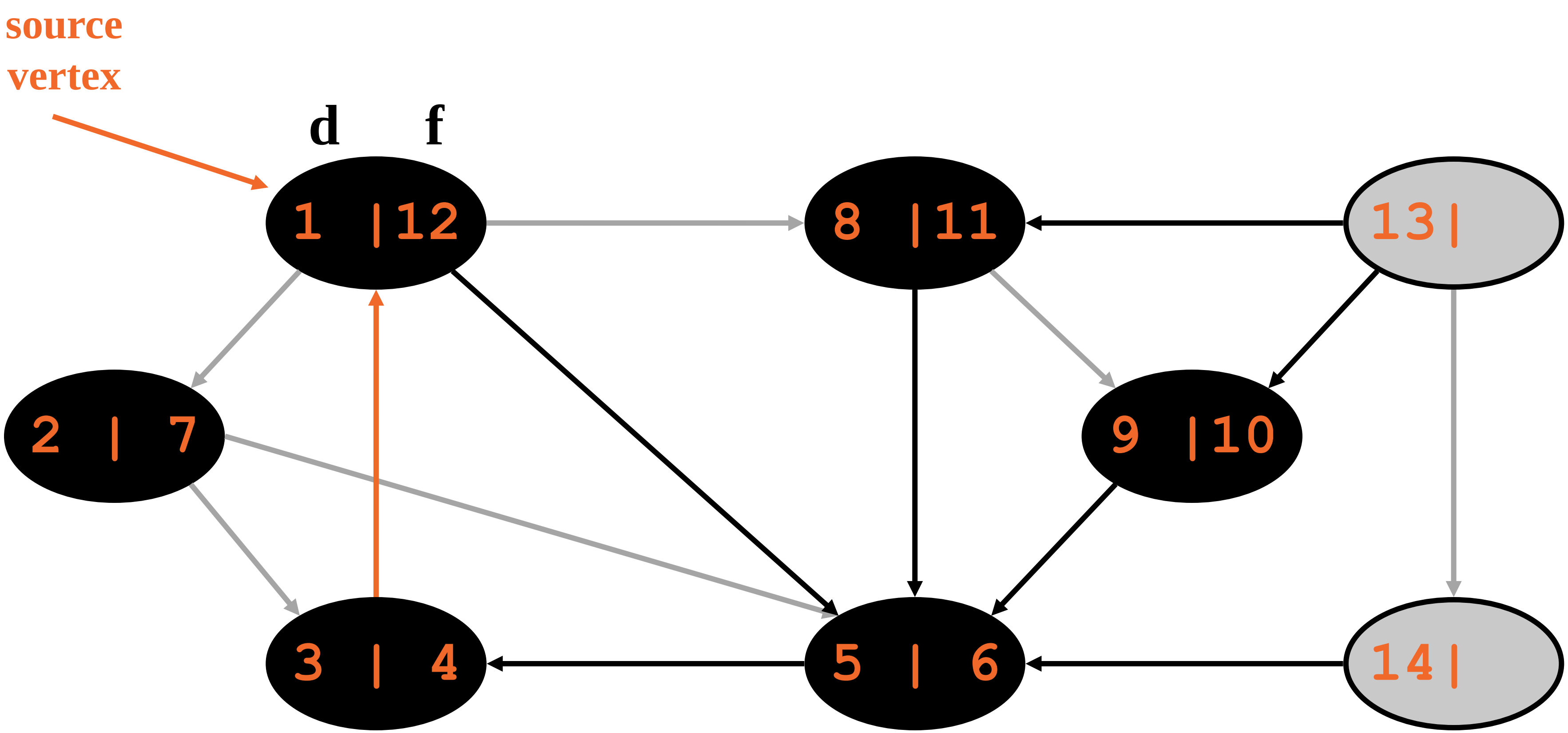

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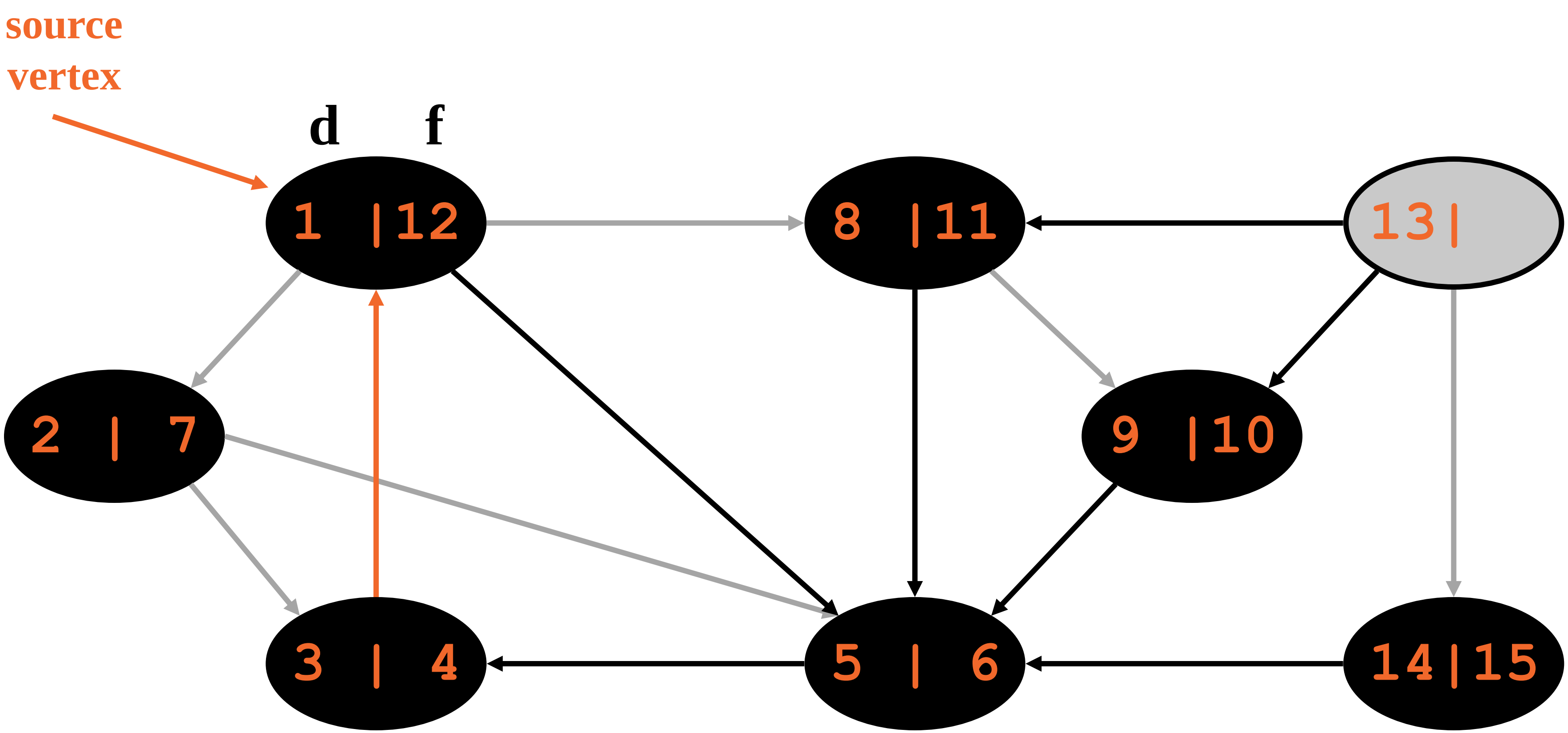

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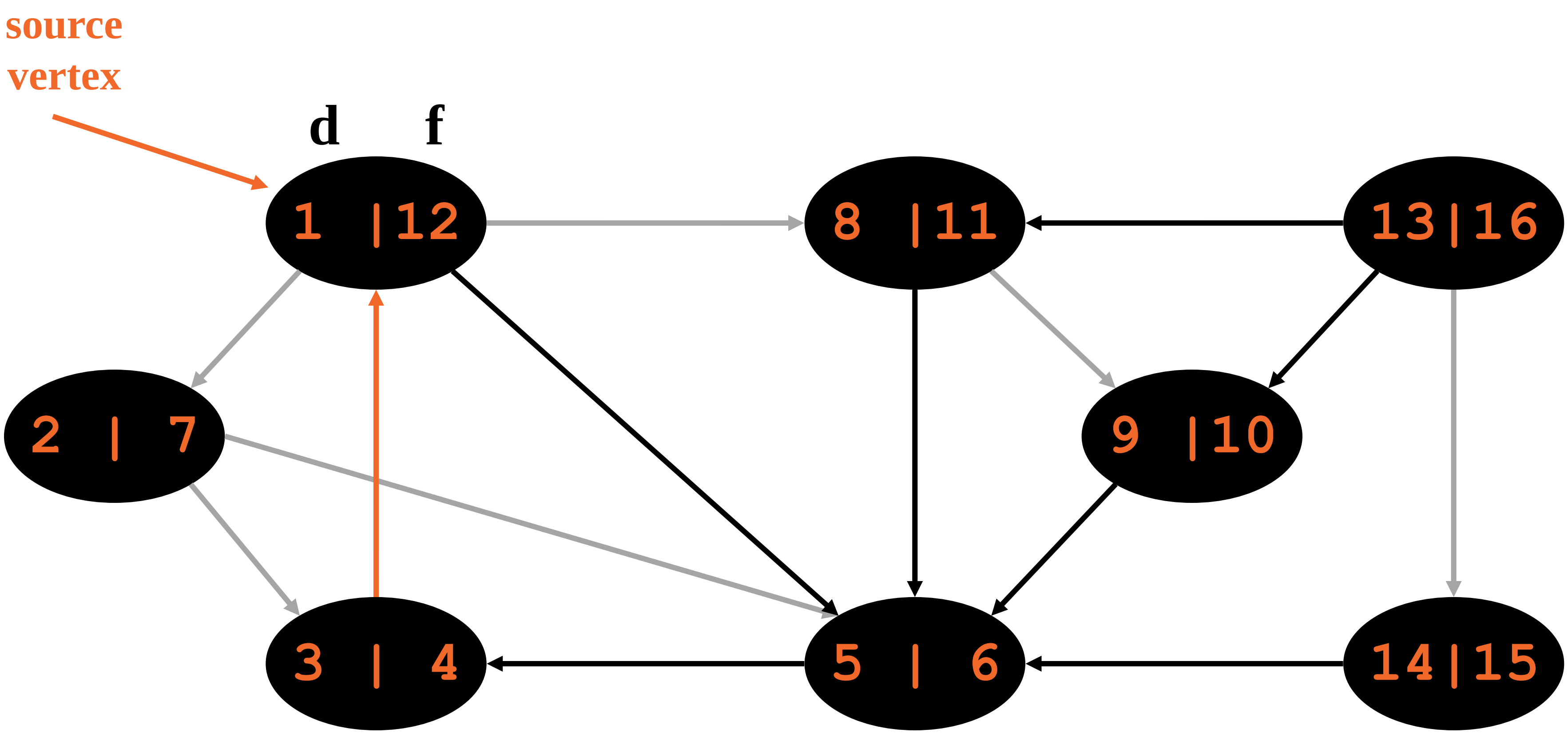
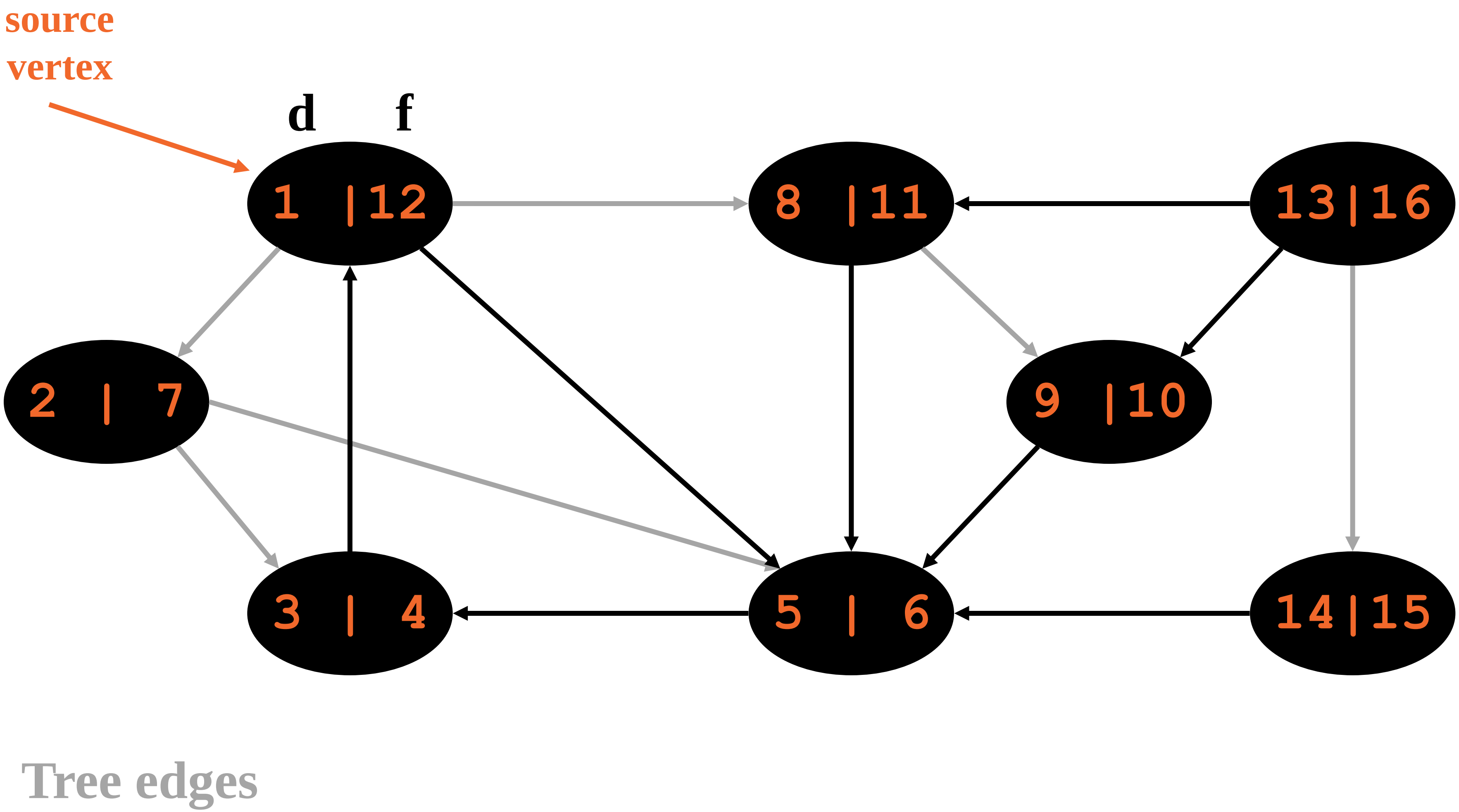
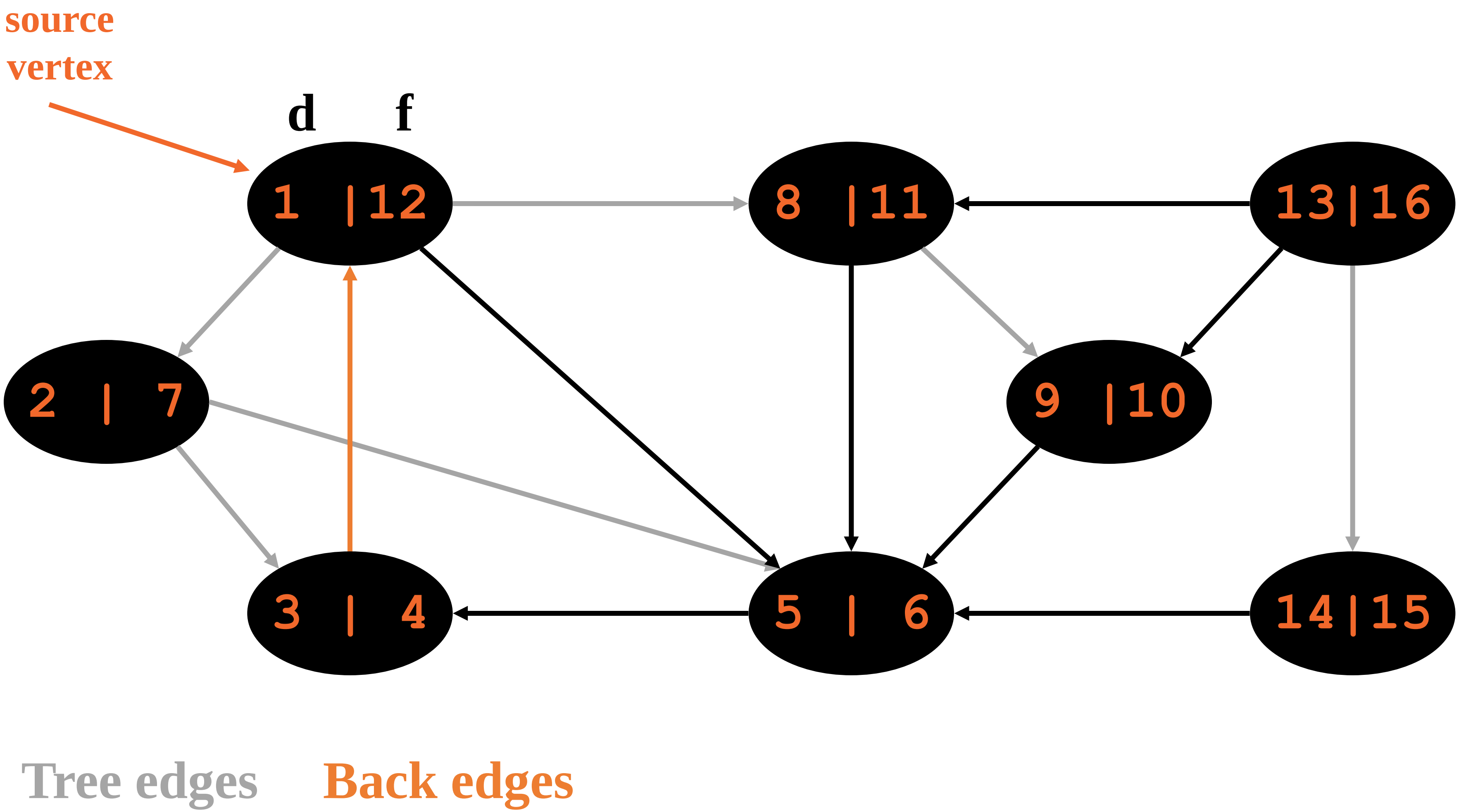


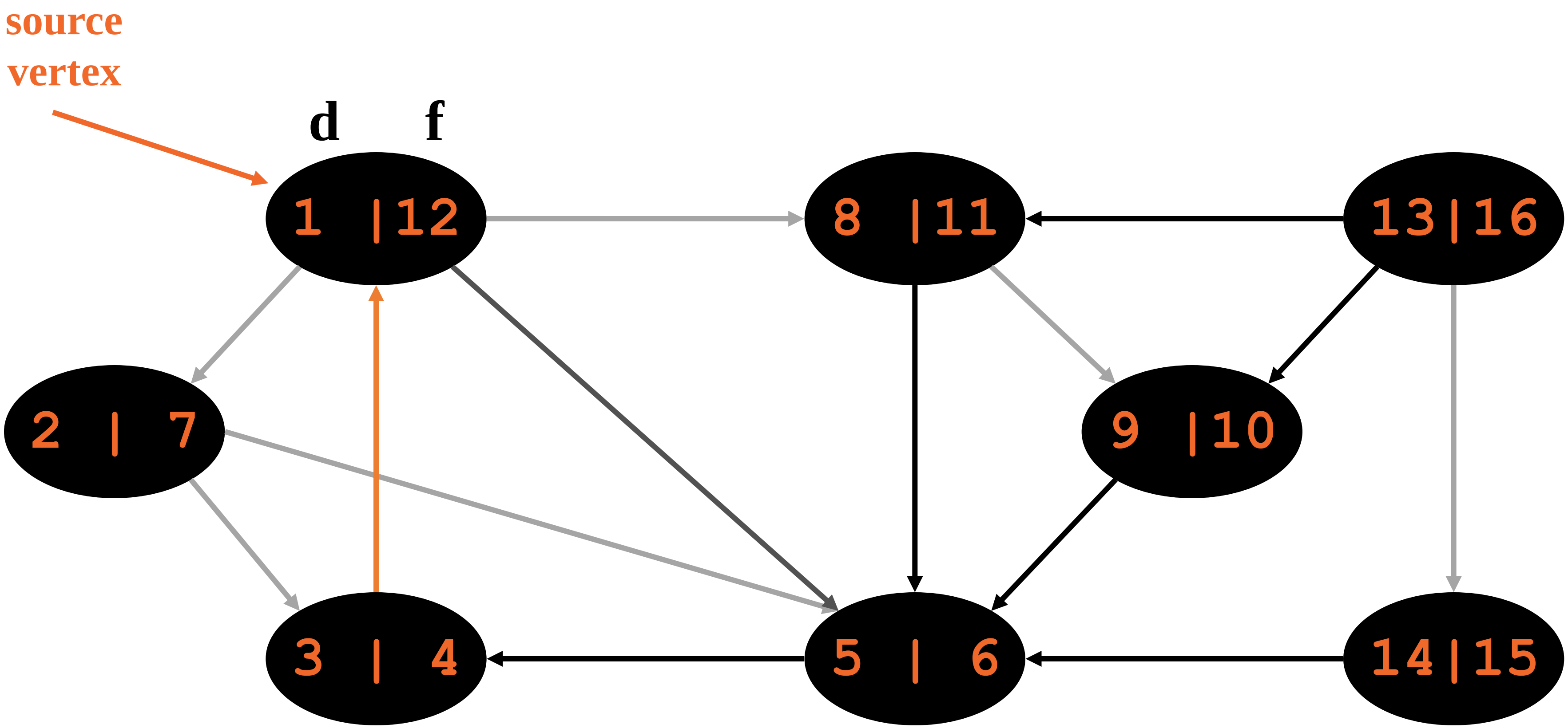
DFS Example



DFS Example


source
vertex
d
f
1 |12
8 |11
13|16
2 | 7
9 |10
3 | 4
5 | 6
14|15
Tree edges
Back edges

DFS Example

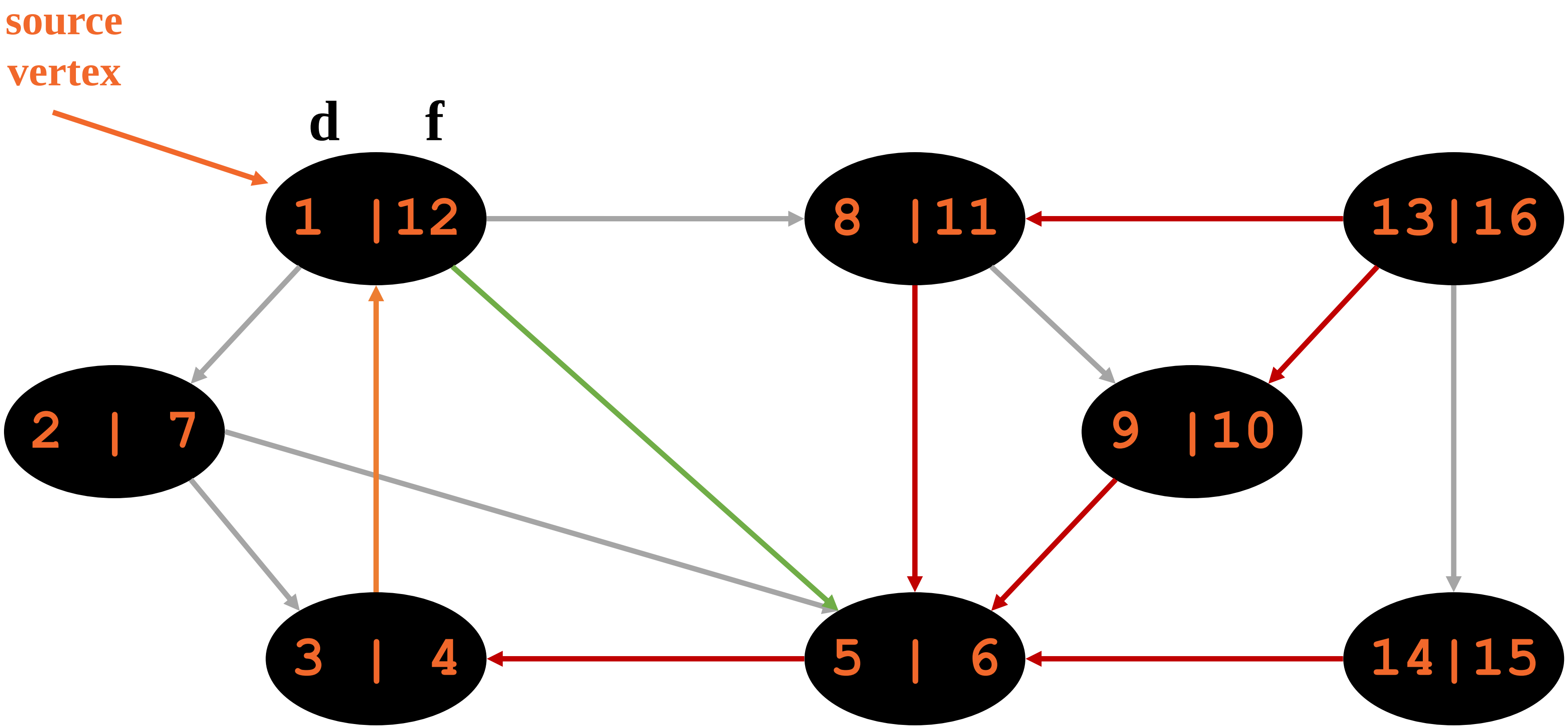


Tree edges Back edges Forward edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
- Note: tree & back edges are important; most algorithms don’t distinguish forward & cross

DFS Example



Tree edges Back edges Forward edges Cross edges

DFS And Graph Cycles

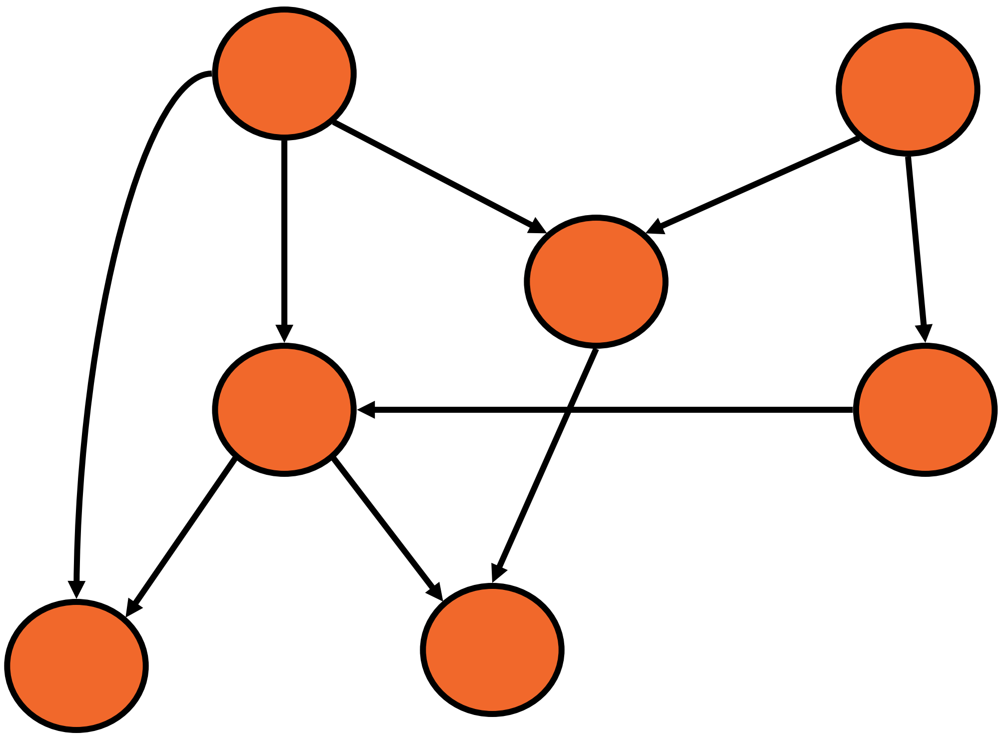
- Thm: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle
  - If no back edges, acyclic
    - No back edges implies only tree edges Only tree edges implies we have a tree or a forest
    - Which by definition is acyclic
- Thus, can run DFS to find whether a graph has a cycle

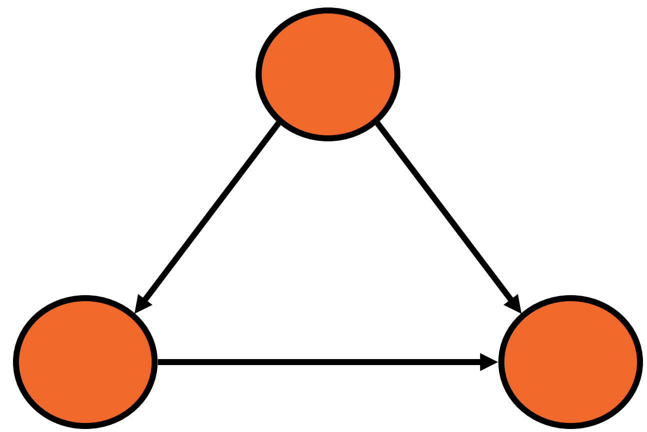
DFS And Cycles

- $\Theta(V+E)$
- We can actually determine if cycles exist in $\Theta(V)$ time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see $|V|$ distinct edges, must have seen a back edge along the way

Directed Acyclic Graphs

- A *directed acyclic graph* or *DAG* is a directed graph with no directed cycles:
- directed graph G is acyclic iff a DFS of G yields no back edges:



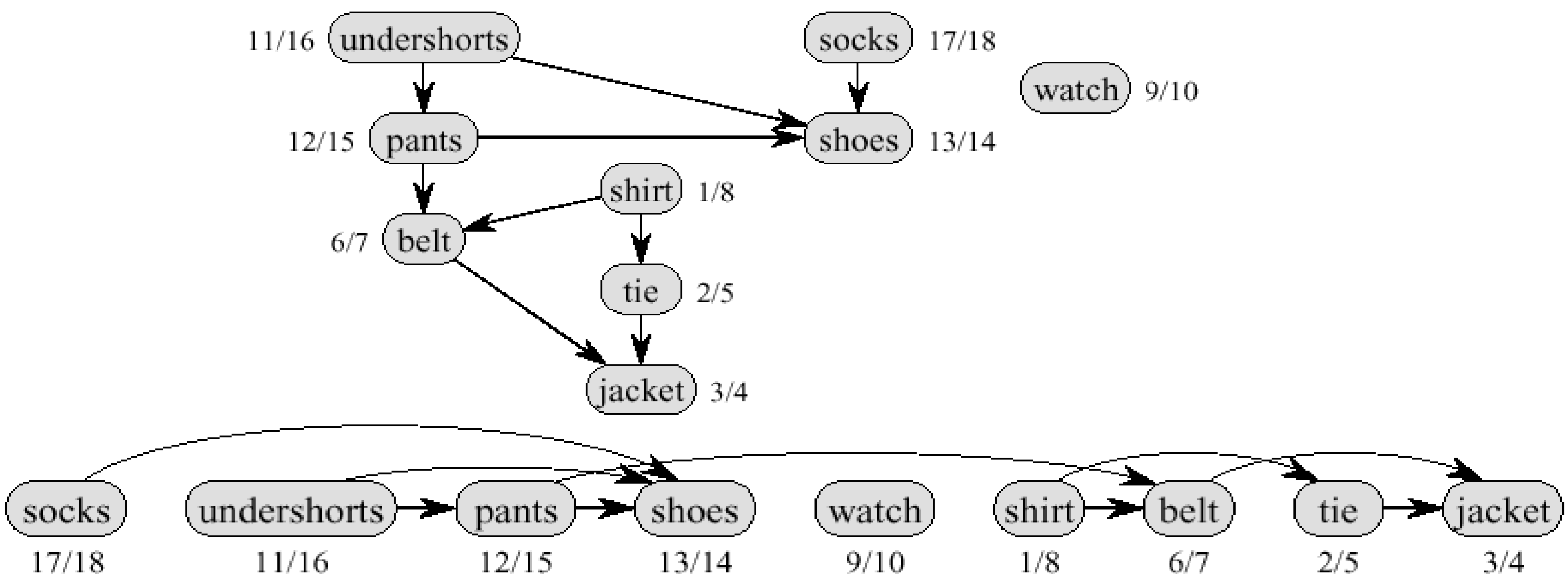


Topological Sort

- *Topological sort* of a DAG:
  - Linear ordering of all vertices in graph G such that vertex u comes before vertex v if edge $(u, v) \in G$
- Real-world example: getting dressed

Topological Sort Example

- Precedence relations: an edge from *x* to *y* means one must be done with *x* before one can do *y*
- Intuition: can schedule task only when all of its subtasks have been scheduled



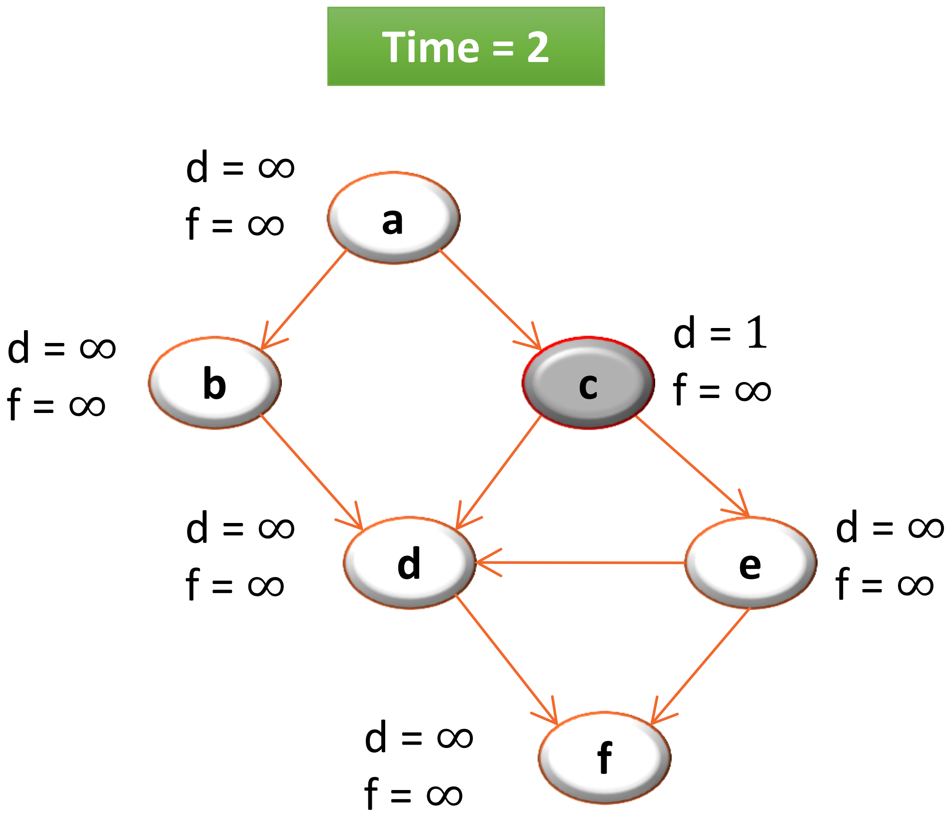
Topological Sort Algorithm

```
Topological-Sort()
{
   Run DFS
   When a vertex is finished, output it
   Vertices are output in reverse topological
     order
}
```

- Time: $\Theta(V+E)$

Topological Example

1) Call DFS(**G**) to compute the finishing times **f**[**v**]



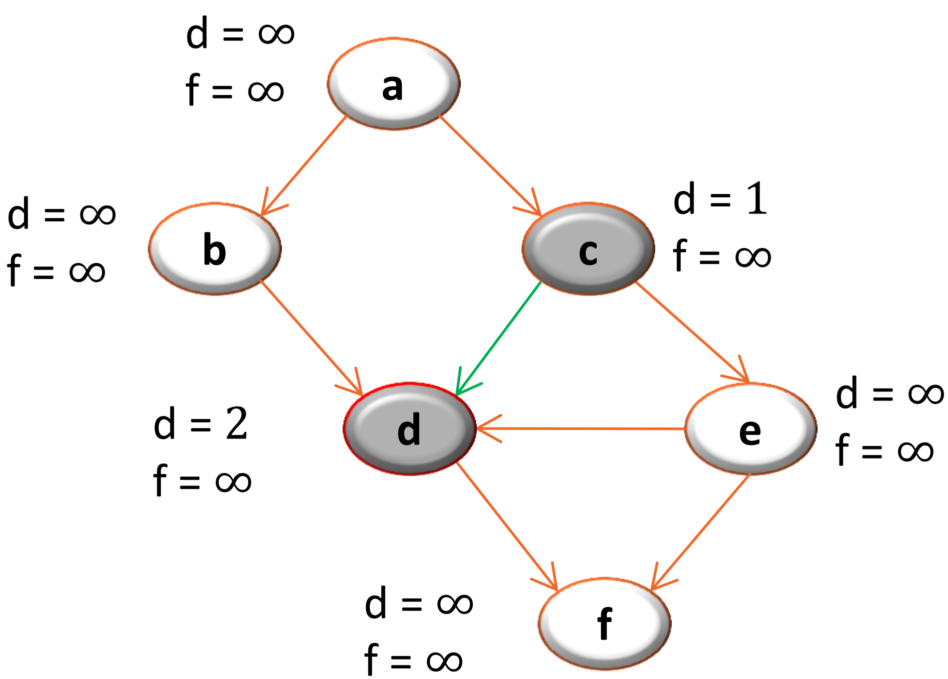
Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 3



Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

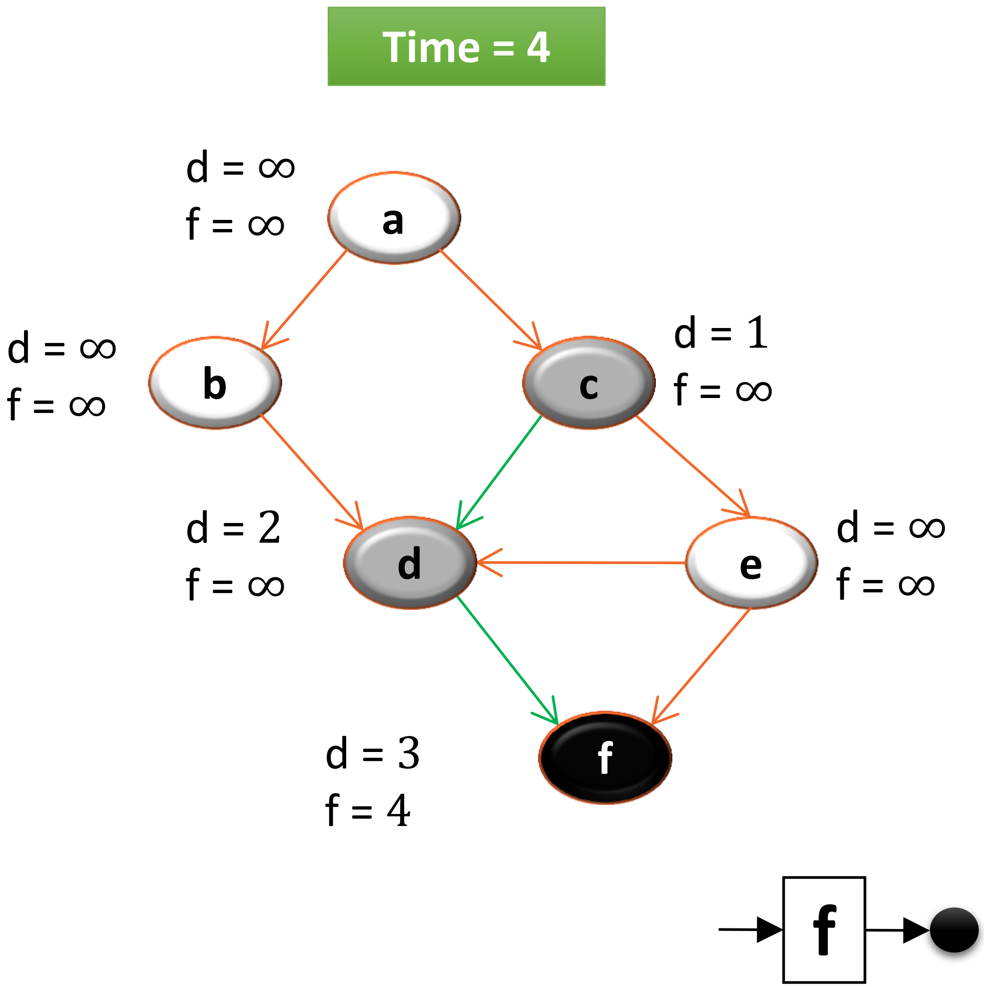
Topological sort

1) Call DFS(**G**) to compute the finishing times **f[v]**

2) as each vertex is finished, insert it onto the **front** of a linked list

Next we discover the vertex **f**

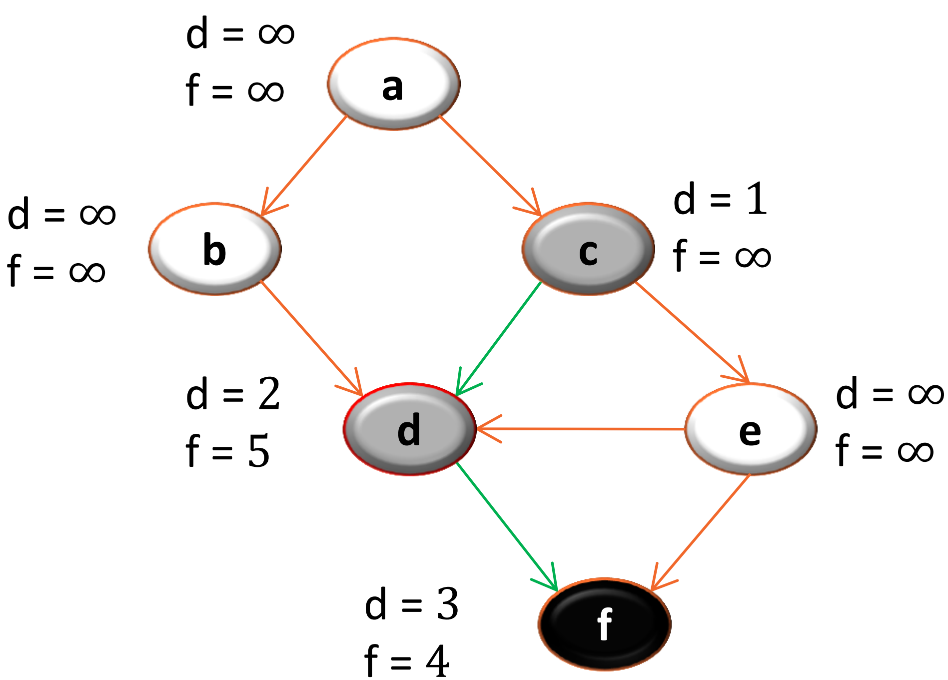
f is done, move back to **d**



Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 5



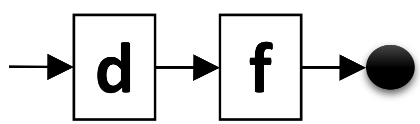
Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

Next we discover the vertex **f**

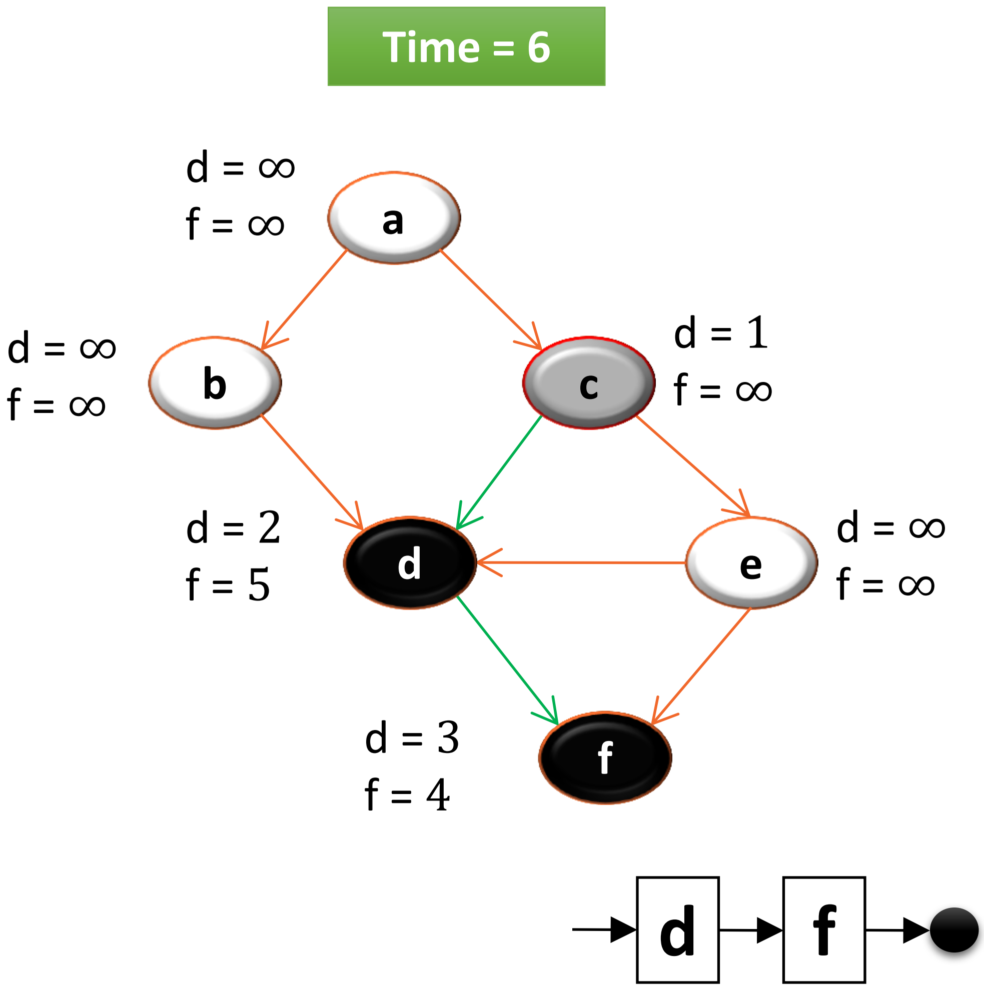
f is done, move back to **d**

d is done, move back to **c**



Topological sort

1) Call DFS(**G**) to compute the finishing times **f[v]**



Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

Next we discover the vertex **f**

f is done, move back to **d**

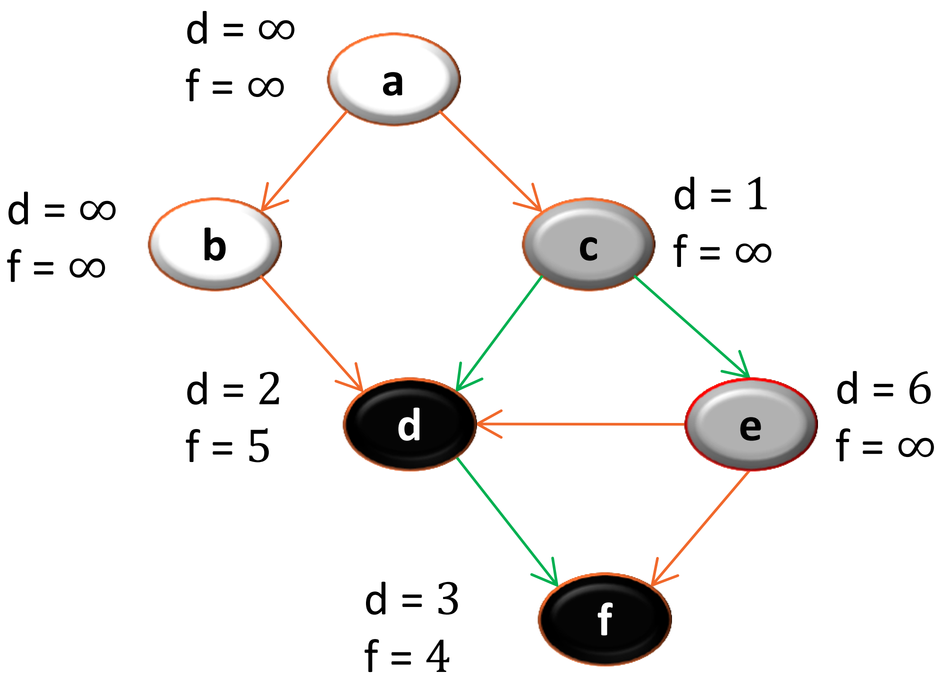
d is done, move back to **c**

Next we discover the vertex **e**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 7



Let's say we start the DFS from the vertex **c**

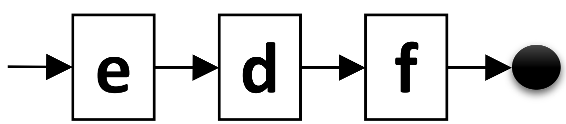
Next we discover the vertex **d**

Both edges from **e** are **cross edges**

d is done, move back to **c**

Next we discover the vertex **e**

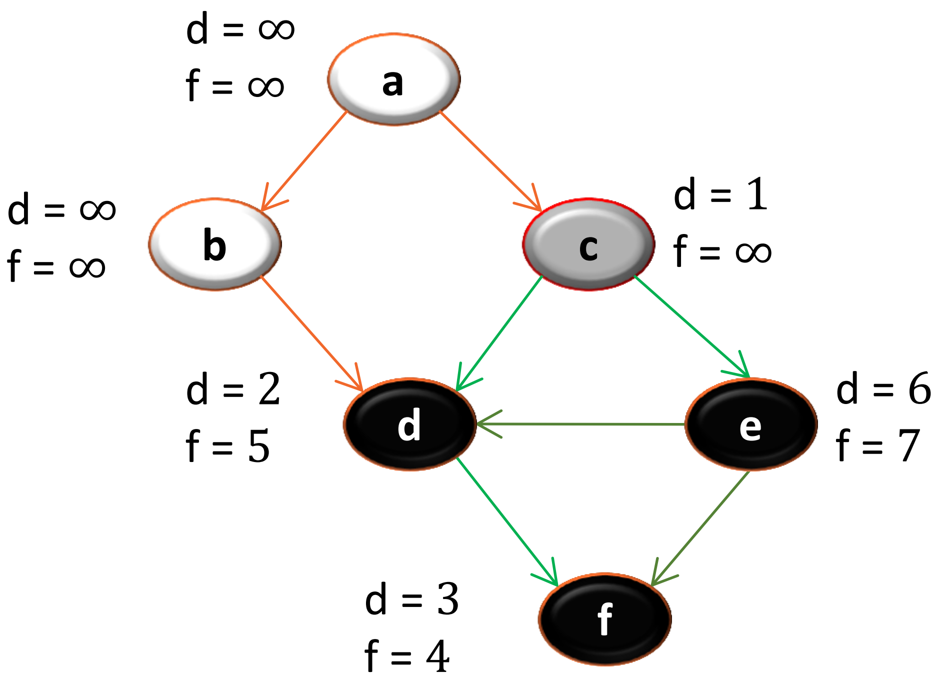
e is done, move back to **c**

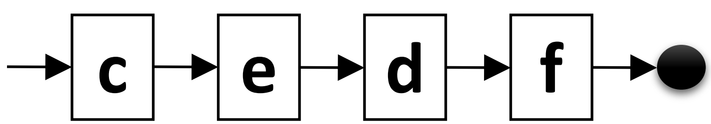


Topological sort

1) Call DFS(**G**) to compute the finishing times **f[v]**

Time = 8





Let's say we start the DFS from the vertex **c**

<u>Just a note:</u> If there was (**c**,**f**) edge in the graph, it would be classified as a **forward edge** (in this particular DFS run)

d is done, move back to **c**

Next we discover the vertex **e**

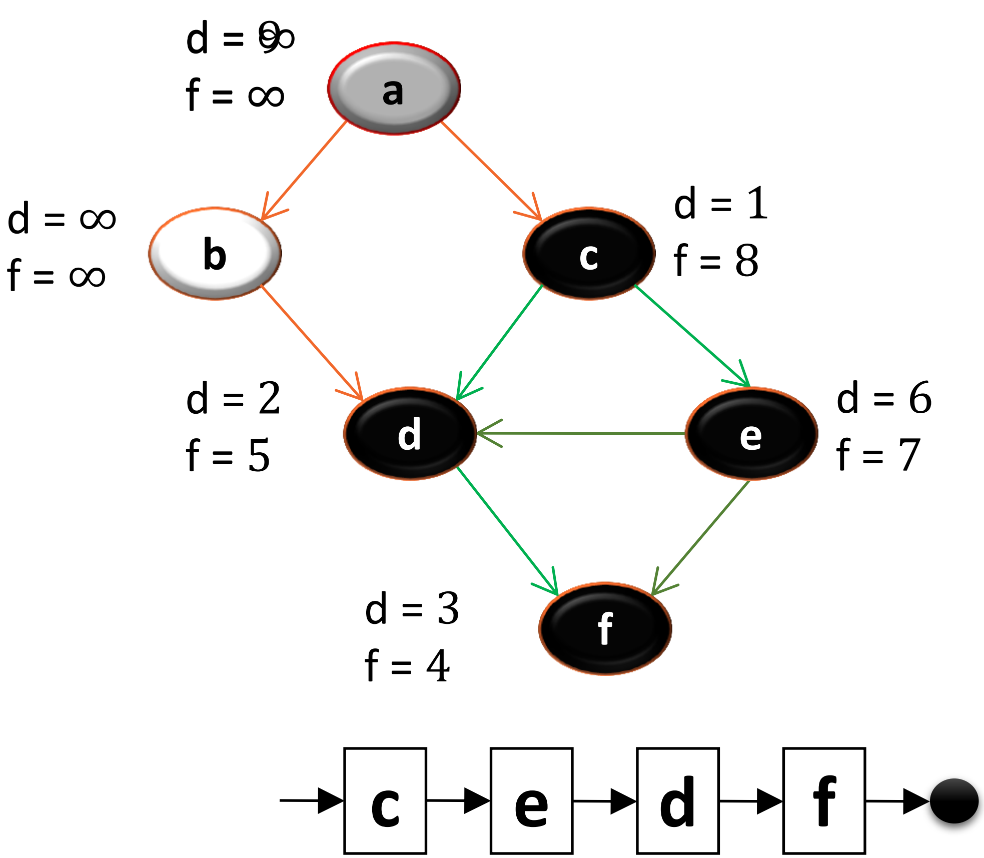
e is done, move back to **c**

c is done as well

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 10



Let's now call DFS visit from the vertex **a**

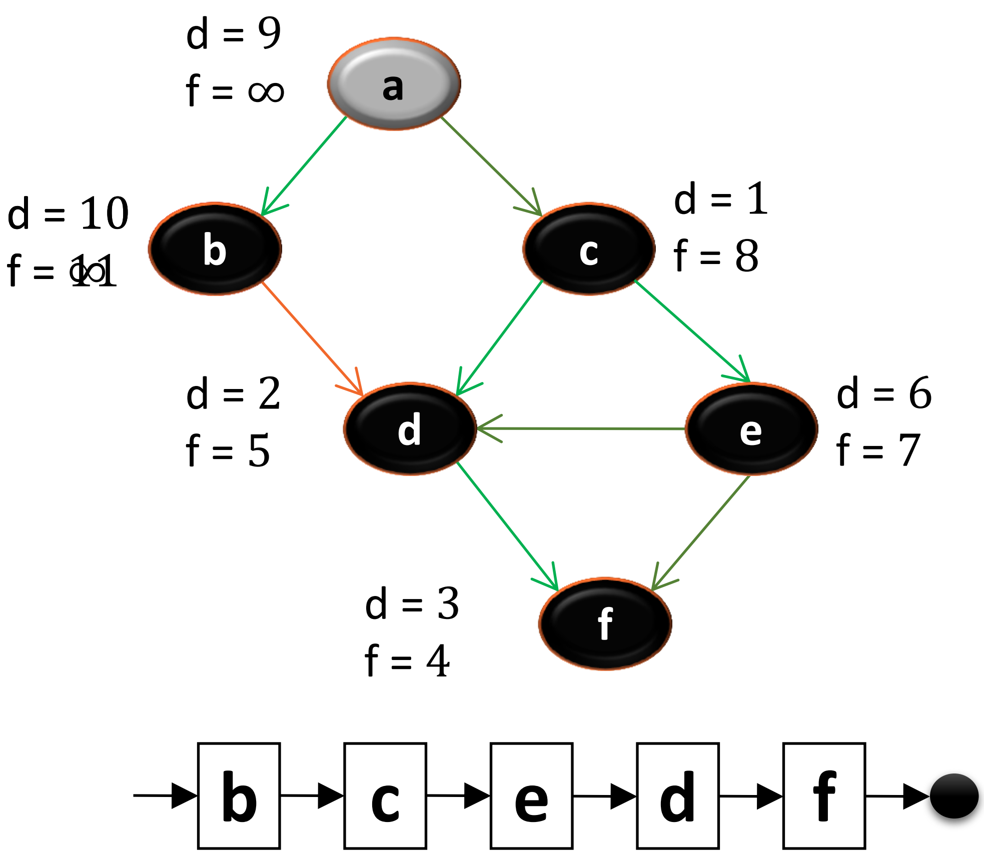
Next we discover the vertex **c**, but **c** was already processed => (**a**,**c**) is a cross edge

Next we discover the vertex **b**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 11



Let's now call DFS visit from the vertex **a**

Next we discover the vertex **c**, but **c** was already processed => (**a**,**c**) is a cross edge

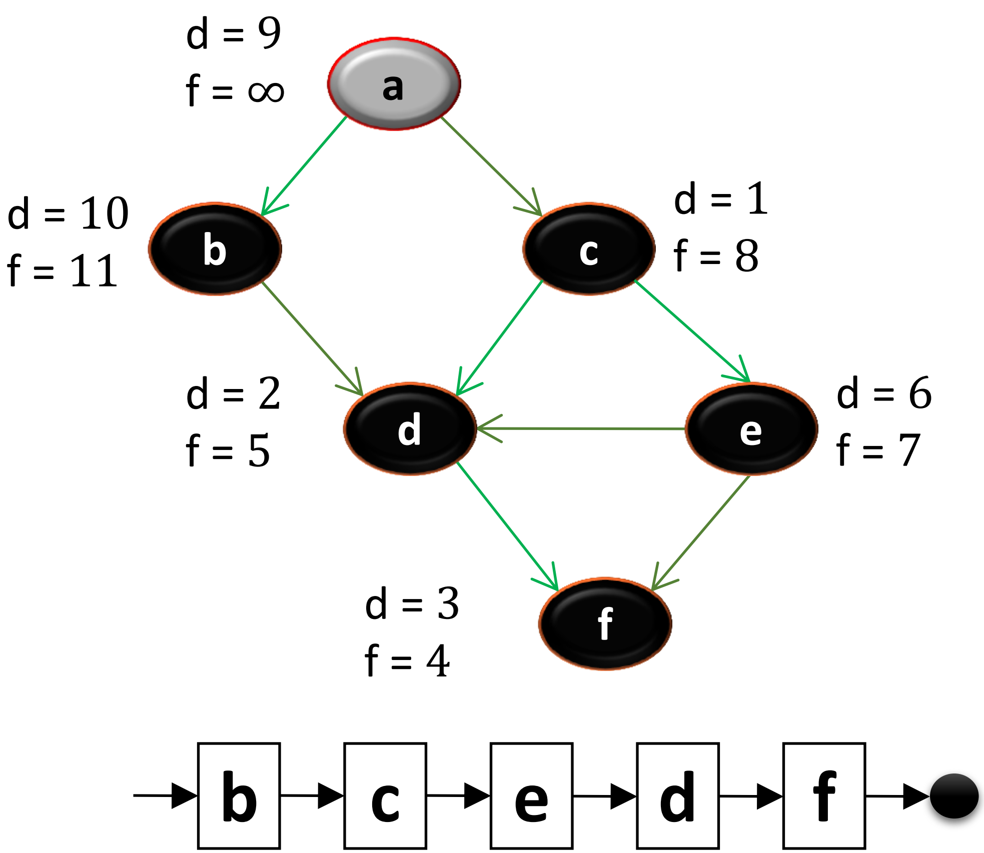
Next we discover the vertex **b**

b is done as (**b**,**d**) is a cross edge => now move back to **c**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 12



Let's now call DFS visit from the vertex **a**

Next we discover the vertex **c**, but **c** was already processed => (**a**,**c**) is a cross edge

Next we discover the vertex **b**

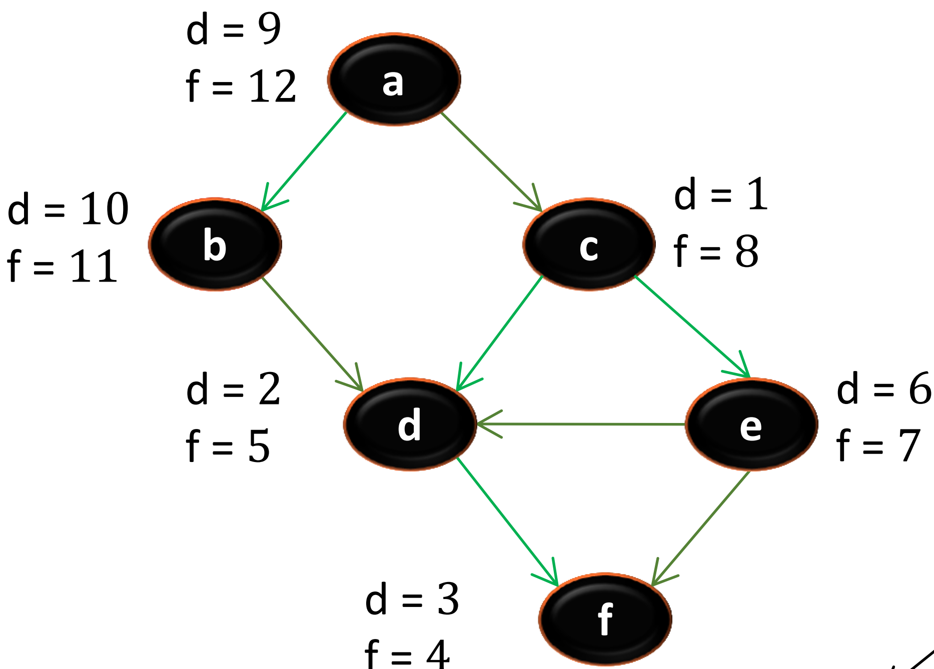
b is done as (**b**,**d**) is a cross edge => now move back to **c**

a is done as well

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 13



(**a**,**c**) is a cross edge

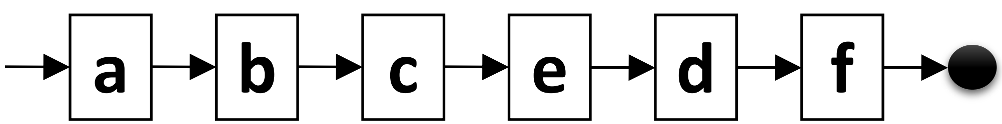
Next we discover the vertex **b**

b is done as (**b**,**d**) is a cross edge => now move back to **c**

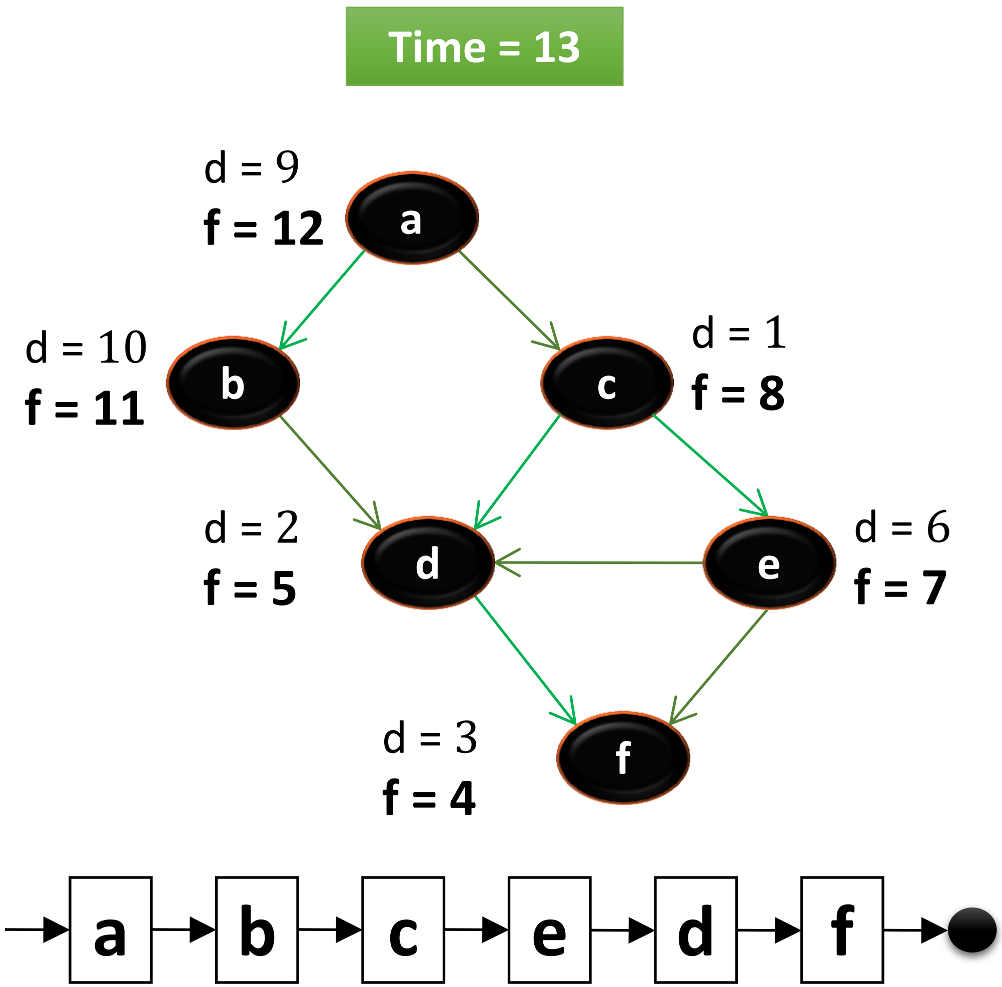
a is done as well

WE HAVE THE RESULT!

3) return the linked list of vertices



Topological sort



The linked list is sorted in **decreasing** order of finishing times **f**[]

Try yourself with different vertex order for DFS visit

Note: If you redraw the graph so that all vertices are in a line ordered by a valid topological sort, then all edges point „**from left to right**“