CS 325 - Homework 7 - Solutions

1. **(6 points 1 pt each)** Let X and Y be two decision problems. Suppose we know that X reduces to Y in polynomial time. Which of the following can we infer? Explain

   a. If Y is NP-complete then so is X. False cannot be inferred
   b. If X is NP-complete then so is Y. False cannot be inferred
   c. If Y is NP-complete and X is in NP then X is NP-complete. False cannot be inferred
   d. If X is NP-complete and Y is in NP then Y is NP-complete. TRUE
   e. If X is in P, then Y is in P. False cannot be inferred
   f. If Y is in P, then X is in P. TRUE

2. **(4 points 1 pt each)** Clearly explain whether or not each of the following statements follows from that fact that COMPOSITE is in NP and SUBSET-SUM is NP-complete:

a. SUBSET-SUM $\leq_p$ COMPOSITE.

No. SUBSET-SUM is NP-complete and so may be reduced to any other NP-complete problem. However, we don't know that COMPOSITE is NP-complete, only that it is in NP.

b. If there is an $O(n^3)$ algorithm for SUBSET-SUM, then there is a polynomial time algorithm for COMPOSITE.

Yes. The given running time is polynomial. Since SUBSET-SUM is NP-complete, this implies P = NP. Hence, every algorithm in NP, including COMPOSITE, would have a polynomial-time algorithm.

c. If there is a polynomial algorithm for COMPOSITE, then P = NP.

No. COMPOSITE is in NP, but it is not known if it is in NP-complete.

d. If P $\neq$ NP, then ***no*** problem in NP can be solved in polynomial time.

No All problems in P are also in NP and can be solved in polynomial time. Proving P 6= NP would show only that NP-complete problems cannot be solved in polynomial time.

3. **(8 points)** A Hamiltonian path in a graph is a simple path that visits every vertex exactly once. Show that HAM-PATH = { (G, u, v): there is a Hamiltonian path from u to v in G} is NP-complete. You may use the fact that HAM-CYCLE is NP-complete

3 points

1) show HAM-PATH ∈NP

Given a graph G with n vertices, and a path from u to v, we can verify in polynomial time that path is a simple path with n vertices, by checking the adjacency list to verify the vertices are adjacent, and that there are n vertices.

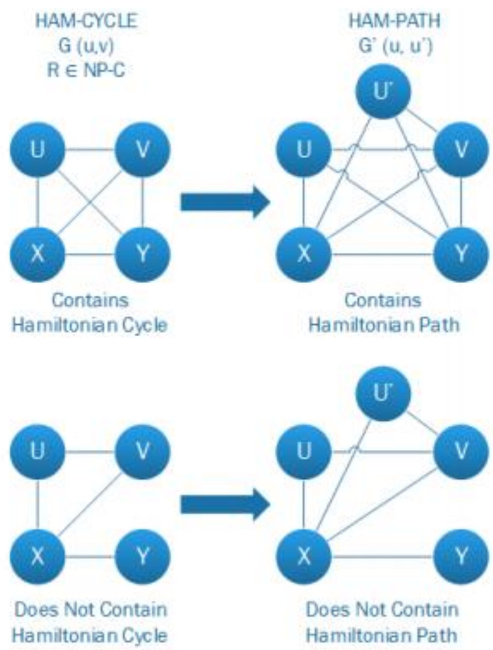
2) Show that R ≤p HAM-PATH for some R ∈NP-Complete **5 points**

a) Select R = HAM-CYCLE because it has a similar structure to HAM-PATH and we know HAM-CYCLE is NP-Complete, and therefore in NP.

b) Show that HAM-CYCLE reduces to HAM-PATH

Let HC = HAM-CYCLE and HP = HAM-PATH

HC (u-v) reduces to HP (u-u'). Given a graph G(u-v) having a Hamiltonian Cycle, where (u-v) is a set of vertices, we produce a new graph G'(u-u') by duplicating arbitrary vertex u along with all of it's connecting edges and naming it u'. This new graph, G'(u, u') now has a Hamiltonian Path from u to u'. This reduction occurs in polynomial time simply by adding the list of edges for u' to the edge list of G. See image below:



c) If G' has a Hamiltonian Path from u to u', then G has a Hamiltonian Cycle and conversely if G has a Hamiltonian Cycle, then G' has a Hamiltonian Path. Also IF G does not have a Hamiltonian Cycle, then G' does not have a Hamiltonian Path.

d) Since HC is NP-Complete, HP must be in NP-Hard.

Since 1 and 2 are true, HAM-PATH is NP-Complete.

CS 325 - Homework 7 - Solutions

Alternative proof.

$\text{HAM-PATH} \in NP$ **2 points**

Let $p = \{u, ..., v\}$ be a certificate path.

Traverse p, and mark the number of times a vertex is visited (initially zero).

Confirm that every vertex $i \in V$ is visited exactly once, and each traversed edge $(i, j) \in E$.

$\text{HAM-PATH} \in NP\text{-hard}$ **3 points**

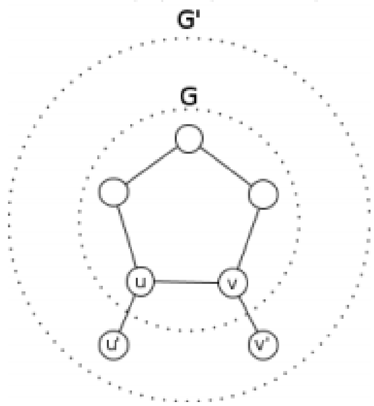
Let $G = (V, E)$ be an instance of the HAM-CYCLE problem ($\text{HAM-CYCLE} \in NP\text{-complete}$).

Let $G' = (V', E')$ be an instance of the HAM-PATH problem.

$V' = V \cup \{u', v'\}$

$E' = E \cup \{(u', u'), (v', v)\}$ for an edge $(u, v) \in E$.

Adding two vertices and two edges transforms $\langle G \rangle$ to $\langle G', u', v' \rangle$ in polynomial time.



Suppose that G has a Hamiltonian cycle.

A simple path $p = \{u, ..., v\}$visits each vertex in V exactly once, and $(u, v) \in E$.

A simple path $p' = \{u', u, ..., v, v'\}$ visits each vertex in V' exactly once.

Therefore, G' has a Hamiltonian path.

Suppose that G' has a Hamiltonian path from u' to v'.

A simple path $p' = \{u', u, ..., v, v'\}$ visits each vertex in V' exactly once.

A simple path $p = \{u, ..., v\}$is a subpath of p'.

p visits each vertex in V exactly once, and $(u, v) \in E$.

Therefore, G has a Hamiltonian cycle.

Having shown that $\text{HAM-PATH} \in NP$ and $\text{HAM-PATH} \in NP\text{-hard}$, this completes the proof that $\text{HAM-PATH} \in NP\text{-complete}$.

CS 325 - Homework 7 - Solutions

4. **Graph-Coloring**. **(12 points)**

Give an efficient algorithm to determine a 2-coloring of a graph, if one exists.

Several correct solutions. . **(4 points) 3 for algorithm + 1 running time O(E+V)**
Modify BFS or DFS

a. Give an efficient algorithm to determine a 2-coloring of a graph, if one exists.

```
While there exist uncolored vertices:
  Choose the next uncolored vertex in graph G, and color it black.
    While neighbors exist:
    Examine neighbors:
      if neighbor is colored the same as it's parent, stop and return false.
      else set color opposite to parent Then Examine its neighbors
Return True
```

Let $G = (V, E)$ be the graph to which a 2-coloring is applied.
Let C be an array indexed as $C[0] \leftarrow color1$ and $C[1] \leftarrow color2$.

2-COLOR(G, C)
- **for** $v \in V$
  - $v.visited \leftarrow false$
  - $v.color \leftarrow none$
- **for** $v \in V$
  - **if** $v.visited == false$
    - TWO-COLOR-VISIT(v, 0, C)

2-COLOR-VISIT(v, i, C)
- $v.visited \leftarrow true$
- $v.color \leftarrow C[i]$
- **Let** N be the set of vertices adjacent to v.
- **for** $n \in N$
  - **if** $v.visited == true$
    - **if** $v.color == n.color$
      - **return** $false$
  - **else**
    - 2-COLOR-VISIT(v, 1 - i, C)
- **return** $true$

A return value of $true$ indicates that a 2-coloring was assigned successfully. Like DFS, the above algorithm runs in $O(V + E)$ time.

b. It has been proven that 3-COLOR is NP-complete by using a reduction from SAT. Use the fact that 3-COLOR is NP-complete to prove that 4-COLOR is NP-complete. (8 points)

<u>Step 1:</u> **(3 points) Show that 4-COLOR is in NP. Give a polynomial time algorithm to verify solution.**

Given a Graph G=(V,E) and a 4-coloring certificate function c: V -> {1, 2, 3, 4} we can verify if c is a "legal" coloring function in polynomial time. To verify the solution, for each vertex u in V we must check the colors of the adjacent vertices. All colors of adjacent vertices must be different. If for any $(u, w) \in E$, $c(u) = c(w)$ then c is not a 4-COLORING of G. The verification of the 4-coloring is polynomial in n (the number of vertices) since $4 \leq n$ and the time required to look at all edges in G is **$O(n^2)$.**

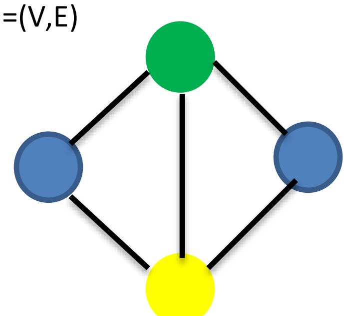
<u>Step 2:</u> (5 points) Show that there is a polynomial reduction from 3-COLOR to 4-COLOR.

Reduce an instance G of 3-COLOR to an instance G' of 4-COLOR in polynomial time, and show that there is a 3-COLOR in G iff there is a 4-COLOR in G'. Let G=(V,E) be an instance of 3-COLOR transform G into G' by adding a new vertex w' that is connect t every other vertex. That is

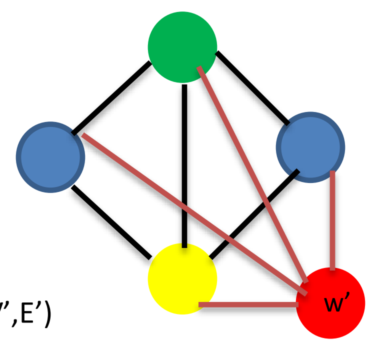
$G'=(V', E')$ where $V'= V \cup \{w'\}$ and $E' = E \cup \{ (w', u)$ for all $u \in G\}$

This reduction can be done in polynomial time since we are adding one vertex and at most n edges

G=(V,E)



G'=(V',E')



blue = 1, yellow = 2, green = 3, red = 4.

If G has a 3-COLORing then G' has a 4-COLORing. Assume G has a 3-COLORing then there exists a function c: V -> {1, 2, 3} such that for all $u, w \in V$ if $(u,w) \in E$ then $c(u) \neq c(w)$. Now define the 4-coloring function c' for G'

$$c'(u) = \begin{cases} c(u), & if\ u \in V \\ 4, & if\ u \notin V \ \ (u = w') \end{cases}$$

Therefore, if there is a 3-COLORing in G then there is a 4-COLORing in G'

If G' has a 4-COLORing then G has a 3-COLORing. Assume G' has a 4-COLORing, since w' is adjacent to all other vertices in G' then w' must be a different color. Let c' be the coloring function for G', without loss of generality we can say that c'(w')= 4 and $c(u) \neq 4$ for all $u \in (V' - \{w\})$. However, $(V' - \{w\}) = (V \cup \{w'\} - \{w\}) = V$. So we have colored all of the original vertices in

V using only colors 1, 2 and 3 proving that G is 3-COLORable. Thus, the 4-Color problem is NP-Hard

Since it was shown in Part 1 that 4-COLOR is in NP, and by Step 2 NP-Hard, 4-COLOR is NP-Complete.