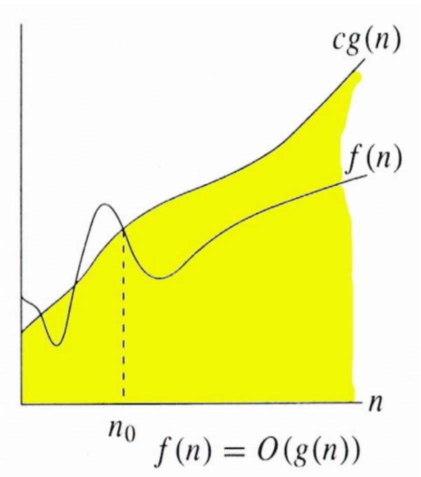
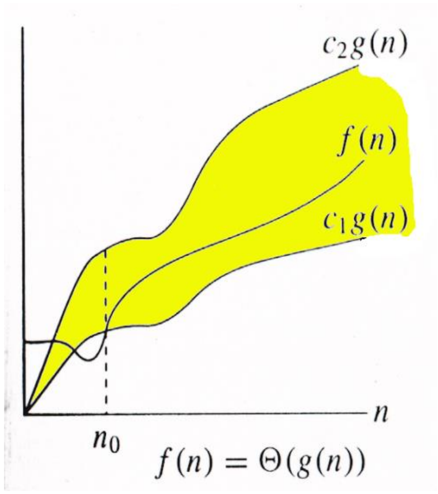
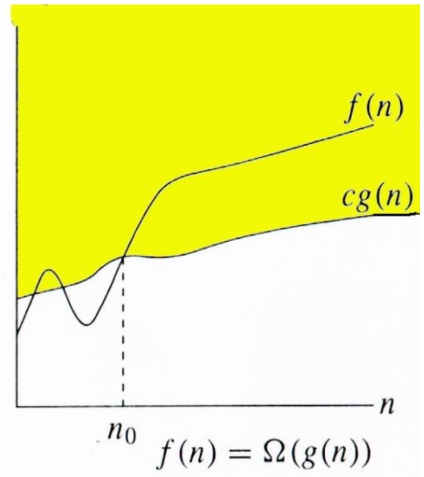
CS 325 – Asymptotic Analysis

Week 1 Part 3: Limits

Relations Between Θ, *O*, Ω







Rate of Growth

The low order terms in a function are relatively insignificant for **large** *n*

$$n^4 + 100n^2 + 10n + 50 \quad \sim \quad n^4$$

$$\lim_{n\to\infty} \frac{n^4 + 100n^2 + 10n + 50}{n^4} = 1$$

That is we say that $n^4 + 100n^2 + 10n + 50$ and n^4 have the same **rate of growth**

Mathematics a **tilde symbol** (~) indicating equivalency or similarity between two values.

Limit Method: The Process

Say we have functions f(n) and g(n). We set up a limit quotient between f and g as follows

$$\text{If } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & then\ f(n)\ is\ O(g(n)) \\ c > 0 & then\ f(n)\ is\ \Theta(g(n)) \\ \infty & then\ f(n)\ is\ \Omega(g(n)) \end{cases}$$

- The above can be proven using calculus, but for our purposes, the limit method is sufficient for showing asymptotic inclusions
- Always try to look for algebraic simplifications first
- If f and g both diverge or converge on zero or infinity, then you need to apply the l’Hôpital Rule

L'Hôpital Rule

Theorem (L'Hôpital Rule):

- Let f and g be two functions,
- if the limit between the quotient f(n)/g(n) exists,
- Then, it is equal to the limit of the derivative of the numerator and the denominator

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Limit Method: Example 1

- Example: Let $f(n) = 2^n$, $g(n) = 3^n$. Determine a tight inclusion of the form $f(n) \in \Delta(g(n))$
- What is your intuition in this case? Which function grows quicker?

Limit Method: Example1

Using algebra

$$\lim_{n\to\infty} 2^n/3^n = \lim_{n\to\infty} (2/3)^n$$

Now we use the following Theorem

$$\lim_{n\to\infty} \alpha^n = \begin{cases} 0 & \text{if } \alpha < 1 \\ 1 & \text{if } \alpha = 1 \\ \infty & \text{if } \alpha > 1 \end{cases}$$

$$\lim_{n\to\infty} 2^n/3^n = \lim_{n\to\infty} (2/3)^n$$
$$= 0$$

Therefore we conclude that 2^n is $O(3^n)$

Using Wolfram Alpha

https://www.wolframalpha.com/

lim(x->inf)(2^x/3^x)

Limit Method: Example 2

Example: Let $f(n) = \log_2 n$, $g(n) = \log_3 n^2$. Determine a tight inclusion of the form

$$f(n) \in \Delta(g(n))$$

What is your intuition in this case?

Limit Method: Example 2

We set up our limit

$$\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty} \log_2 n/\log_3 n^2$$
$$= \lim_{n\to\infty} \log_2 n/(2\log_3 n)$$

Here we use the change of base formula for logarithms:

$\log_3 n = \log_2 n \;/\; \log_2 3$

$$\lim_{n\to\infty} \log_2 n/(2\log_3 n) = \lim_{n\to\infty} (\log_2 n \log_2 3)\;/(2\log_2 n)$$
$$= \lim_{n\to\infty} (\log_2 3)/2$$
$$= (\log_2 3)\;/2$$
$$\approx 0.7924, \text{ a positive constant}$$

So we conclude that $f(n) \in \Theta(g(n))$ that is
$\log_2 n \in \Theta(\log_3 n^2)$ which implies that $\log_3 n^2 \in \Theta(\log_2 n)$

Using Wolfram Alpha

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lim(x->inf)(2^x/3^x)

Important Result

- All logs have the same asymptotic growth rate no what the base is.
- In many CS algorithms the base is 2.
- But we get sloppy since lg(n) is Θ(logn)

Properties

Theorem:

$$f(n) = \Theta(g(n)) \Leftrightarrow f = O(g(n)) \text{ and } f = \Omega(g(n))$$

- Transitivity:
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$
  - Same for O and Ω
- Reflexivity:
  - $f(n) = \Theta(f(n))$
  - Same for O and Ω
- Symmetry:
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$
- Transpose symmetry:
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

Let f, g and h be asymptotically positive functions. Prove or disprove each of the following conjectures.

Transitivity f(n) = Θ(g(n)) and g(n) = Θ(h(n)) $\Rightarrow$ f(n) = Θ(h(n))

1. By definition f(n) = Θ(g(n)) implies there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0$

2. By definition g(n) = Θ(h(n)) implies there exist positive constants c_3, c_4, and n_1 such that $0 \le c_3 h(n) \le g(n) \le c_4 h(n)$ for all $n \ge n_1$

3. Show f(n) = Θ(h(n)) that is there exist positive constants c_5, c_6, and n_2 such that $0 \le c_5 h(n) \le f(n) \le c_6 h(n)$ for all $n \ge n_2$

By combining 1 and 2: $c_1 c_3 h(n) \le c_1 g(n) \le f(n)$ let $c_5 = c_1 c_3$ so $c_5 h(n) \le f(n)$

Again from 1 and 2: $f(n) \le c_2 g(n) \le c_2 c_4 h(n)$ let $c_6 = c_2 c_4$ so $c_6 h(n) \le f(n)$

And let $n_2 = \max\{n_0, n_1\}$

Let f, g and h be asymptotically positive functions. Prove or disprove each of the following conjectures.

If f(n) = O(g(n)) and h(n) = O(g(n)), then f(n) = Θ(h(n)) ?

1. By definition f(n) = O(g(n)) implies there exist positive constants c_1 and n_0 such that $0 \le f(n) \le c_1 g(n)$ for all $n \ge n_0$

2. By definition h(n) = O(g(n)) implies there exist positive constants c_2 and n_1 such that $0 \le h(n) \le c_2 g(n)$ for all $n \ge n_1$

3. Show f(n) = Θ(h(n)) that is there exist positive constants c_3, c_3, and n_2 such that $0 £ c_3 h(n) \le f(n) \le c_4 h(n)$ for all $n \ge n_2$

?????????

Counter Example: Disprove conjecture

Let f(n) = n. g(n) = n^2 and h(n)= n^2