CS 325 – Analysis of Sorting

Week 1 – Part 1

Getting Started

What is this class about? The theoretical study of design and analysis of computer algorithms

Basic goals for an algorithm:

- always correct
- always terminates
- This class: performance
  - Performance often draws the line between what is possible and what is impossible.

Design and Analysis of Algorithms

- ***Analysis:*** predict the cost of an algorithm in terms of resources and performance. *Theory*

- ***Design:*** design algorithms which minimize the cost

The Problem of Sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

Importance of Sorting

- Maintain a directory of names, phone book, sort by grades of students, …
- Find the median
- Binary Search assumes array is sorted.
- Greedy Algorithms

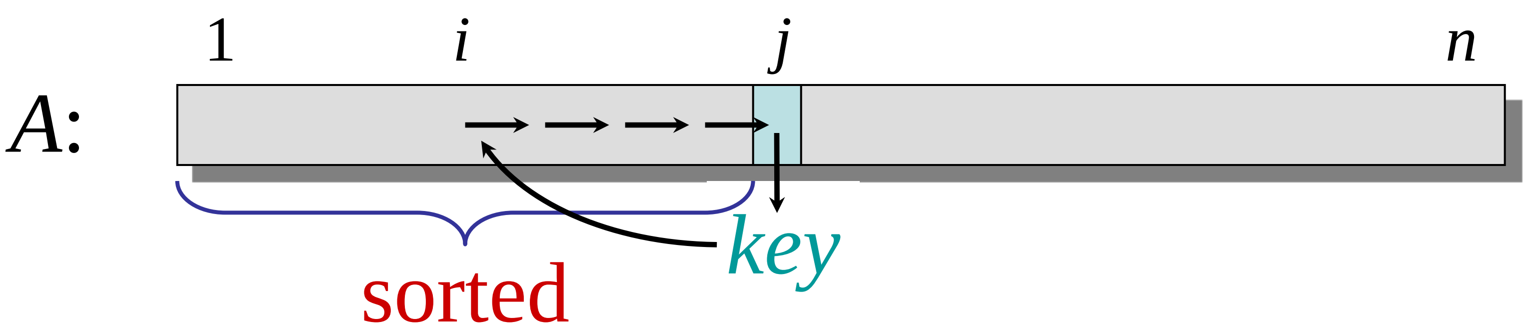
Problem vs Algorithm

- Many algorithms exist to solve the sorting problem.
- Running time is associated with an algorithms.
- Bounds on running times may also be associated with the problem.

Algorithm 1: Insertion sort

"pseudocode"

```
INSERTION-SORT (A, n)          ⊳ A[1 . . n]
    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
```



Example of insertion sort

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Example of insertion sort

8	2	4	9	3	6
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Example of insertion sort

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2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Major Simplifying Convention: Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.

  $T_A(n) =$ time of A on length n inputs
- Generally, we seek upper bounds on the running time, to have a guarantee of performance.

Kinds of Analyses

Worst-case: (usually)

- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (NEVER)

- Cheat with a slow algorithm that works fast on *some* input.

Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} j = \mathrm{O}\left(n^2\right)$$ [arithmetic series]

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

Insertion sort analysis

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} (j/2) = \Theta\left(n^2\right)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small *n*.
- Not at all, for large *n*.

Insertion sort analysis

Best Case: Already sorted. *Nearly Sorted??*

O(n)

Can we sort better?

Insertion upper bound $O(n^2)$

Are there other ways to sort??

Merge Sort

Sorting Problem: Sort a sequence of *n* elements into non-decreasing order.

- ***Divide:*** Divide the *n*-element sequence to be sorted into two subsequences of *n/2* elements each

- ***Conquer:*** Sort the two subsequences recursively using merge sort.

- ***Combine:*** Merge the two sorted subsequences to produce the sorted answer.

Merge sort – Pseudo code

MERGE-SORT $A[1 \,..\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

Merging two sorted arrays

20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
7	9
2	1

1

Merging two sorted arrays

20	12		20	12
13	11		13	11
7	9		7	9
2	1		2	

1

Merging two sorted arrays

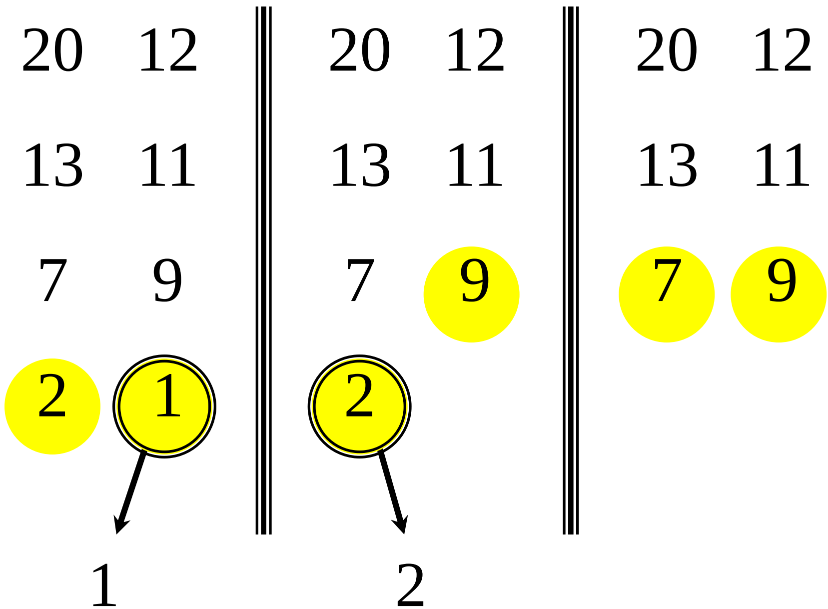
20	12
13	11
7	9
2	1

1

20	12
13	11
7	9
2	

2

Merging two sorted arrays



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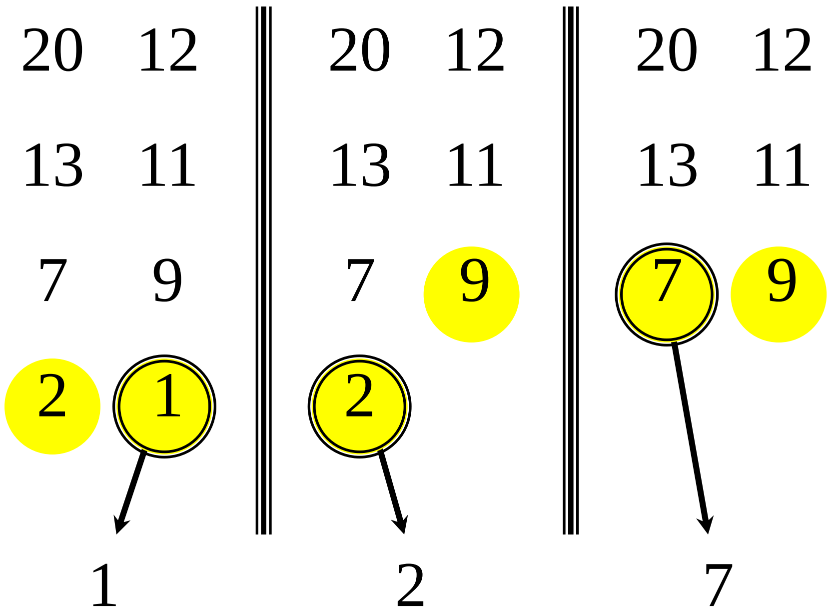

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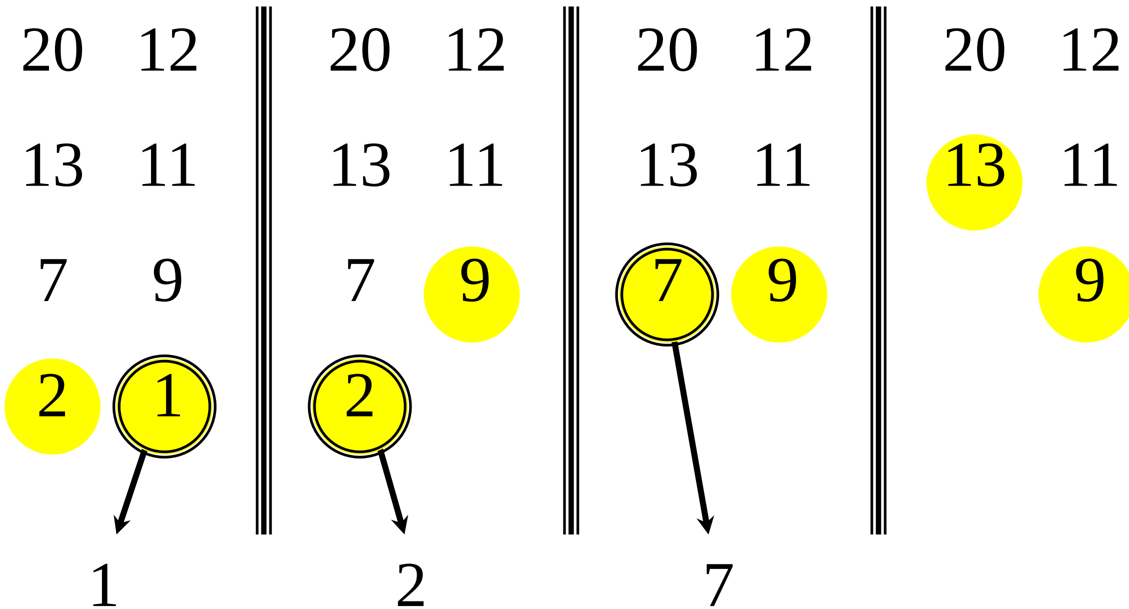

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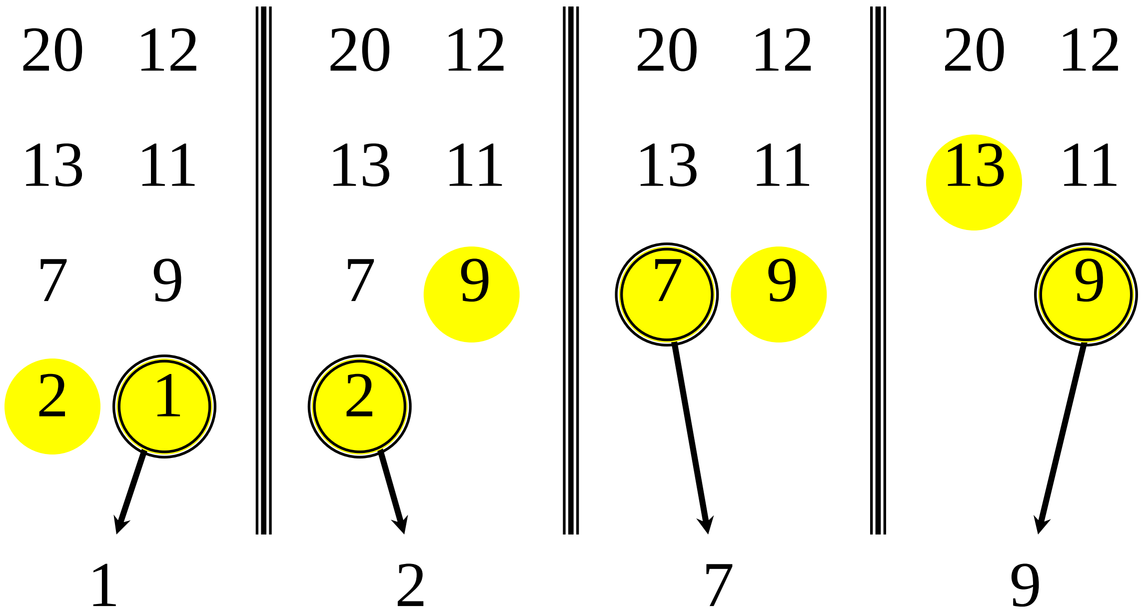

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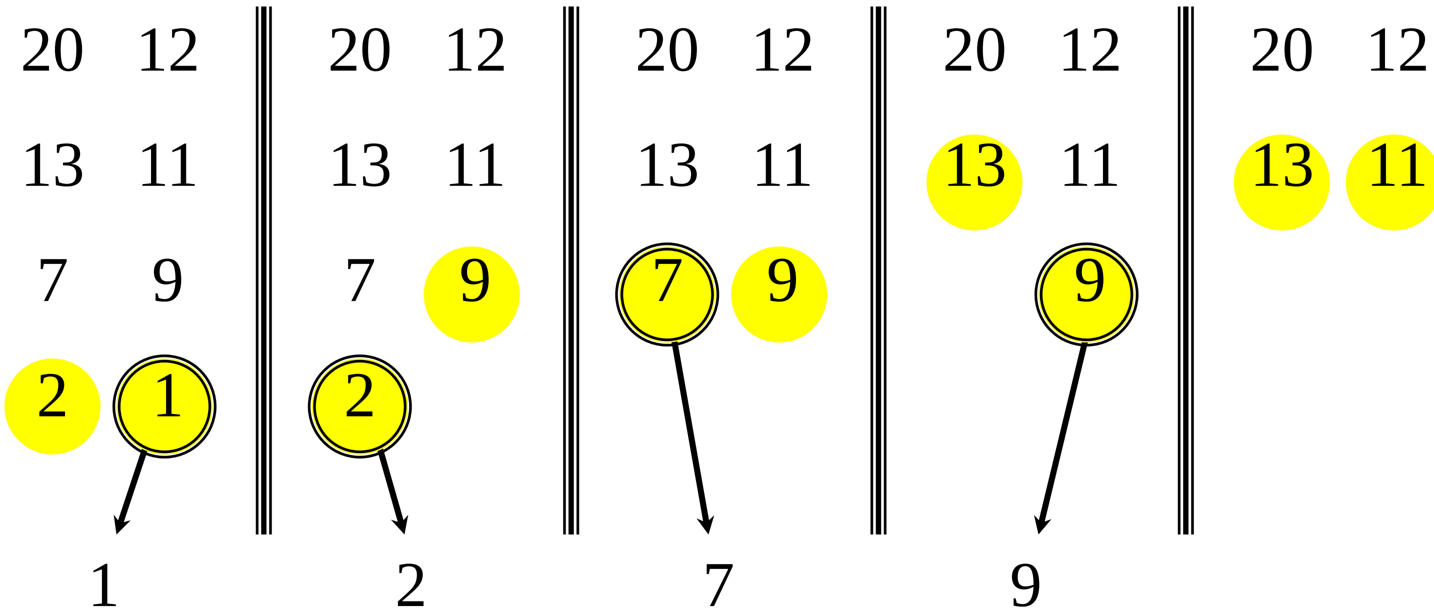

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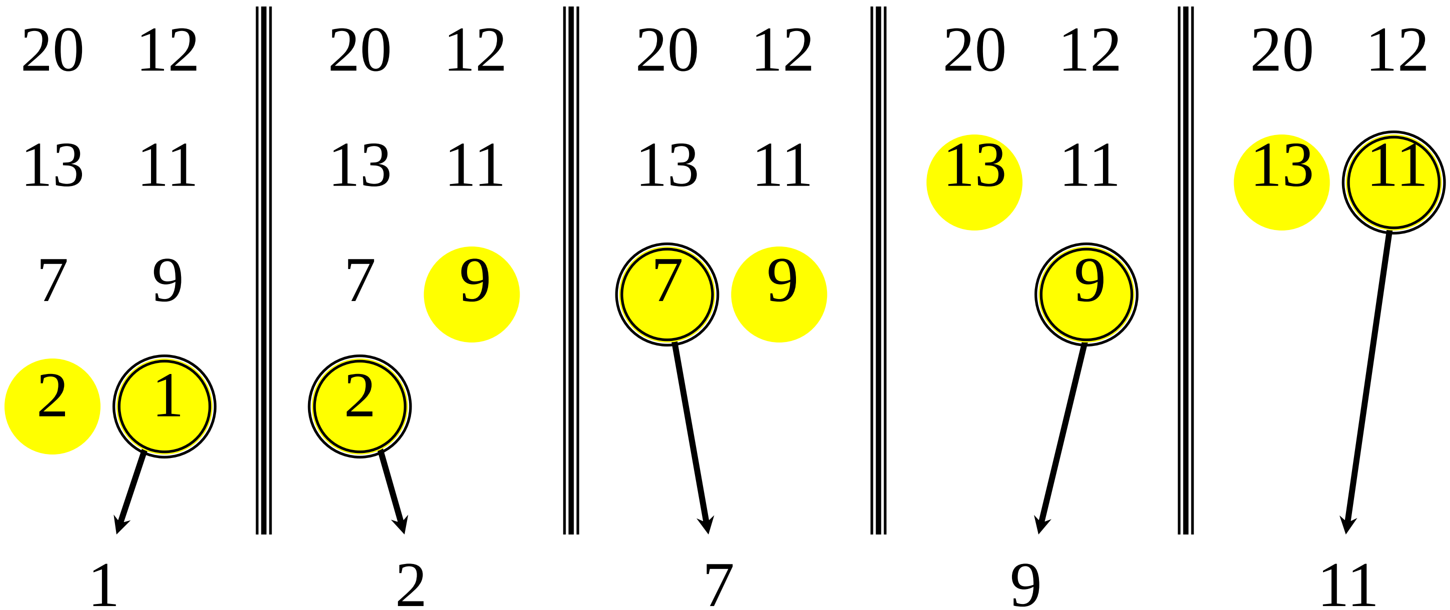

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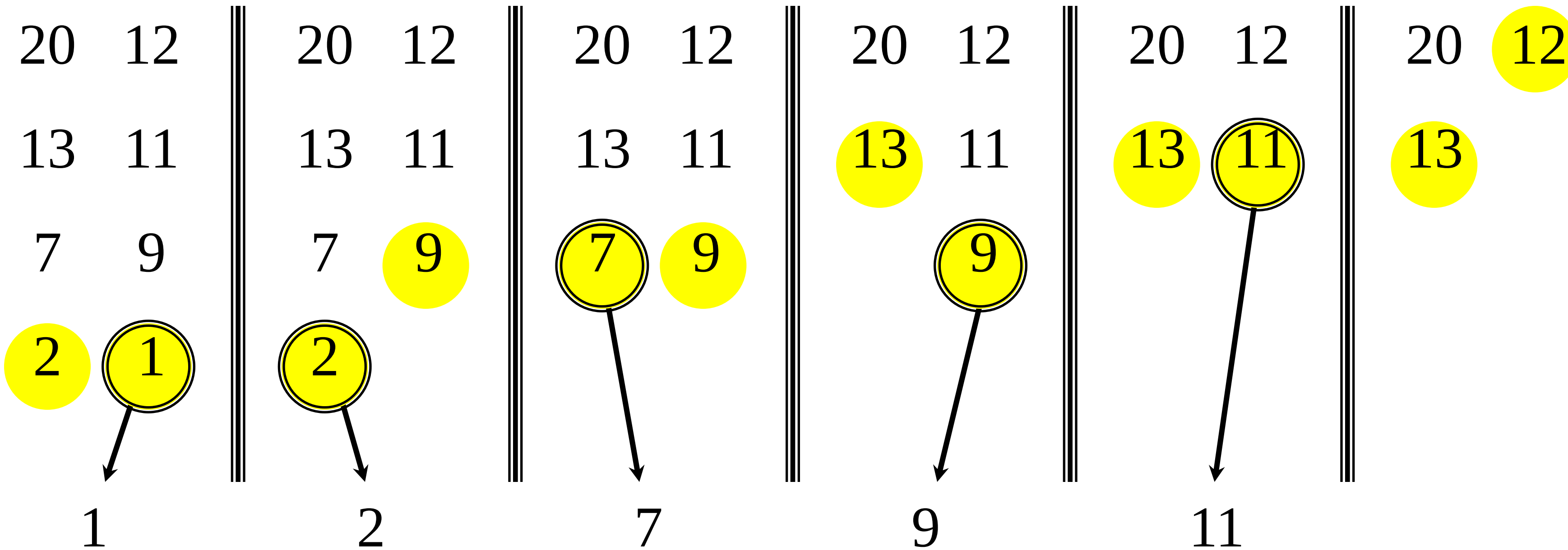

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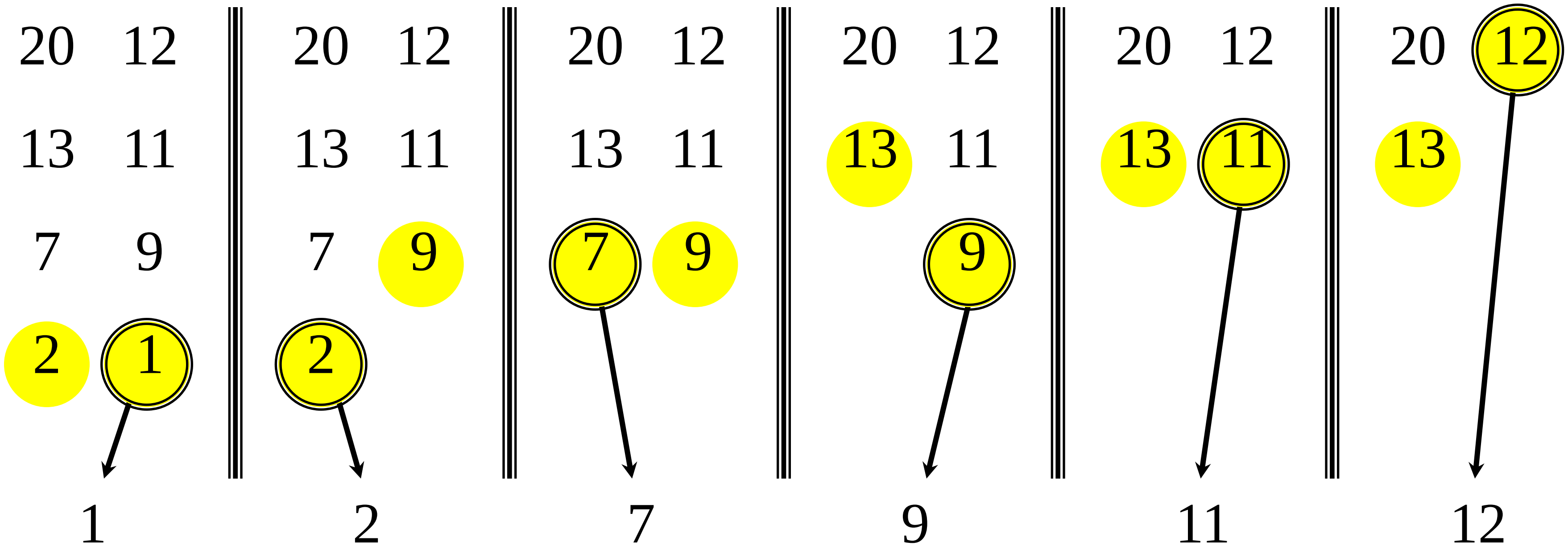
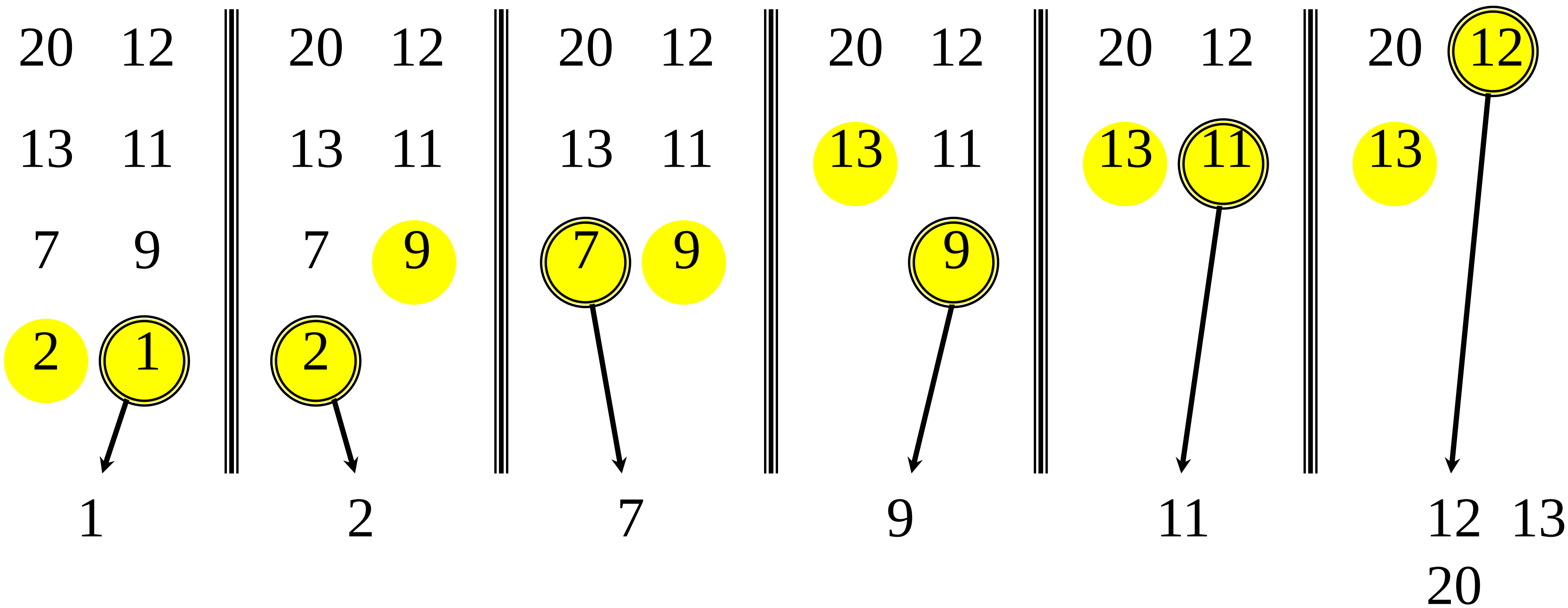
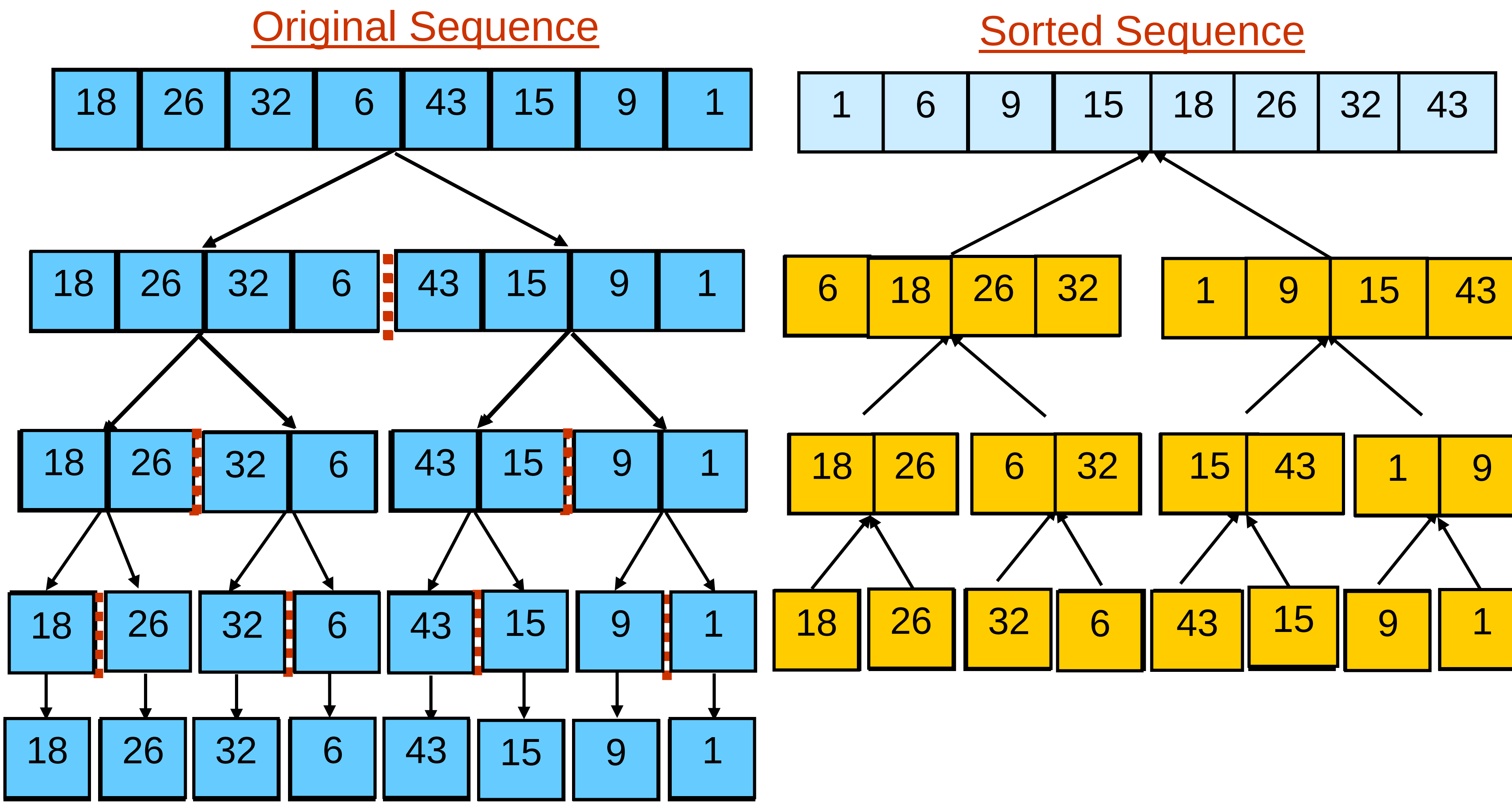


Merging two sorted arrays



Time = $O(n)$ to merge a total of n elements (linear time).

Merge Sort – Example



Analyzing merge sort

$T(n)$	**MERGE-SORT** $A[1\,..\,n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1\,..\,\lceil n/2\rceil\,]$ and $A[\,\lceil n/2\rceil+1\,..\,n\,]$.
$O(n)$	3. ***"Merge"*** the 2 sorted lists

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + O(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- Week 2 provides several ways to find a good upper bound on $T(n)$.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$cn$$

$$T(n/2) \qquad\qquad T(n/2)$$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$
\begin{array}{cccc}
 & cn & & \\
cn/2 & & & cn/2 \\
T(n/4) & T(n/4) & T(n/4) & T(n/4)
\end{array}
$$

Recursion tree

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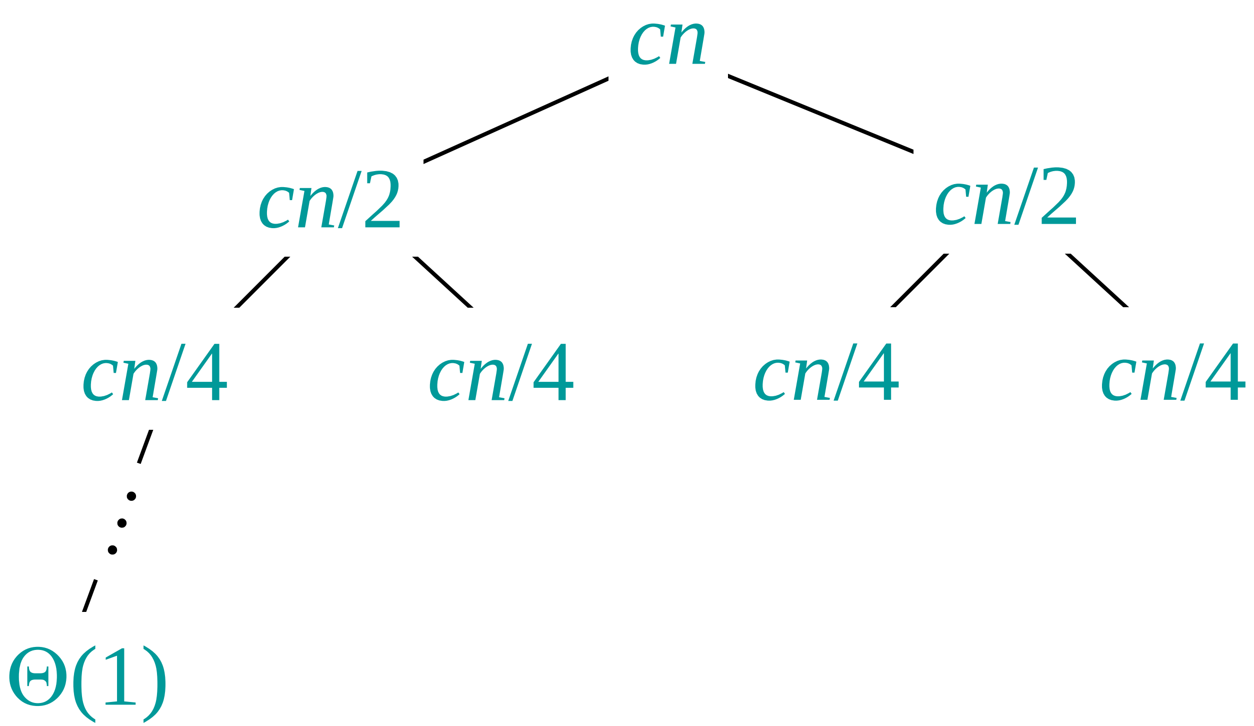

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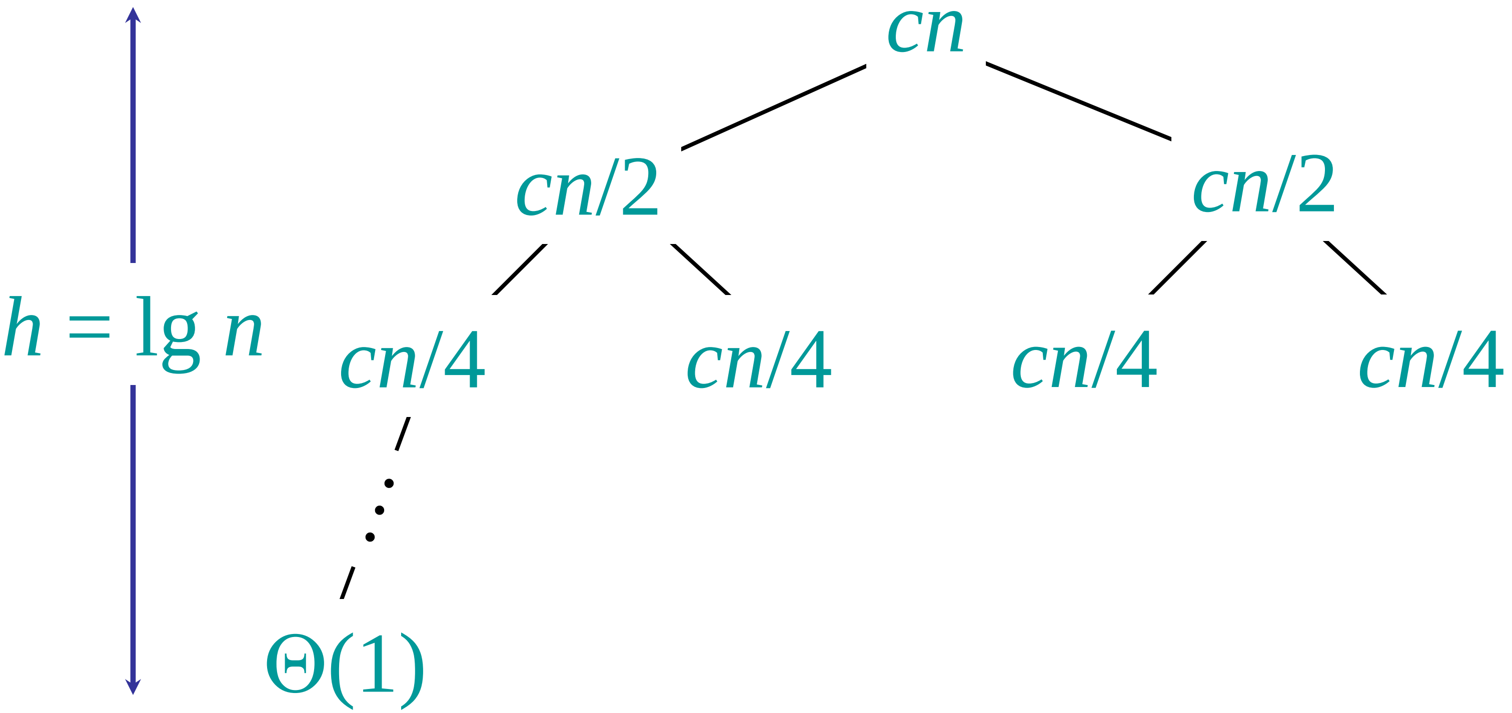

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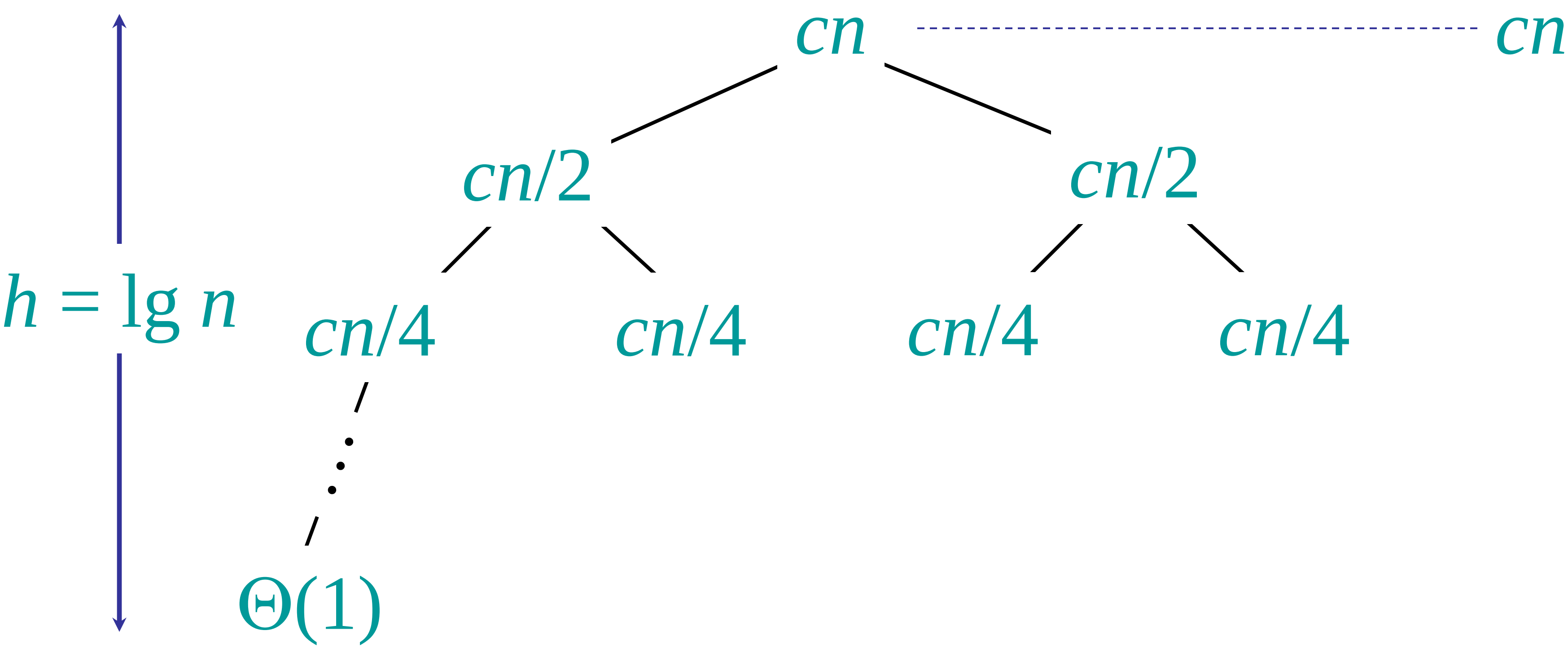

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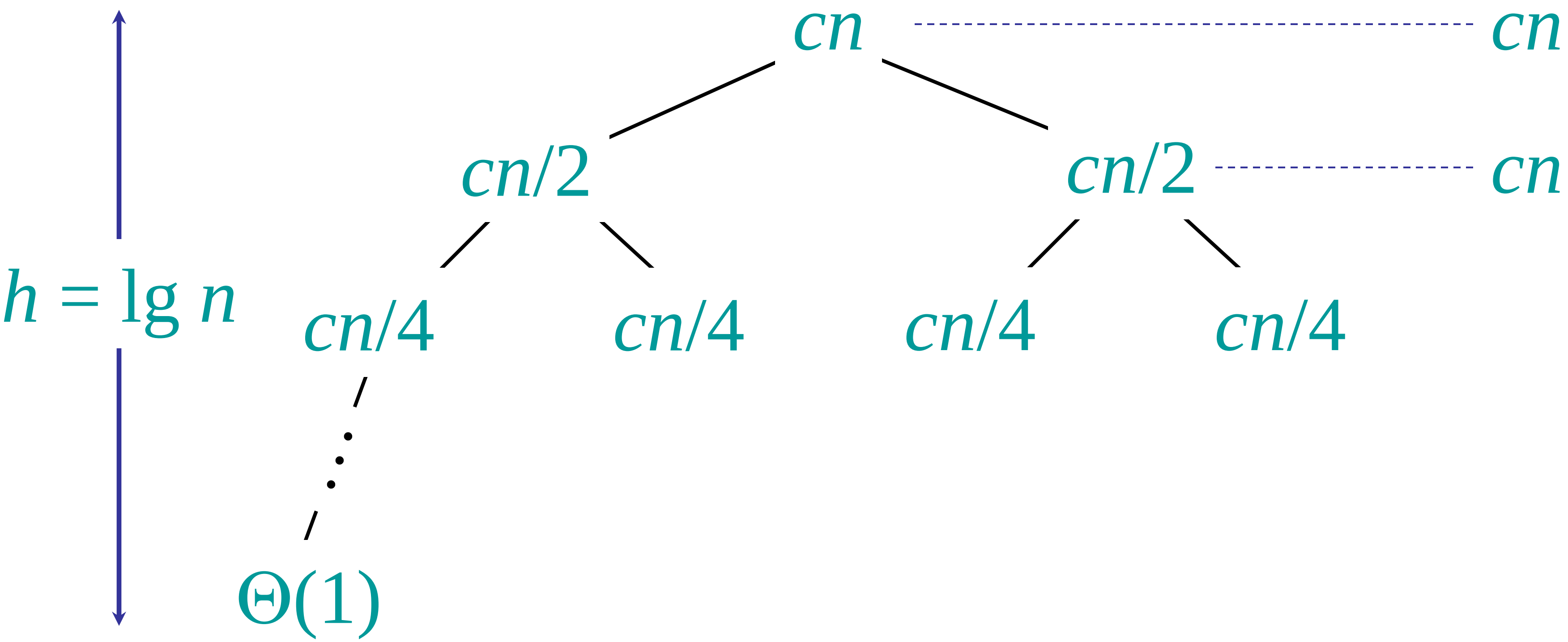

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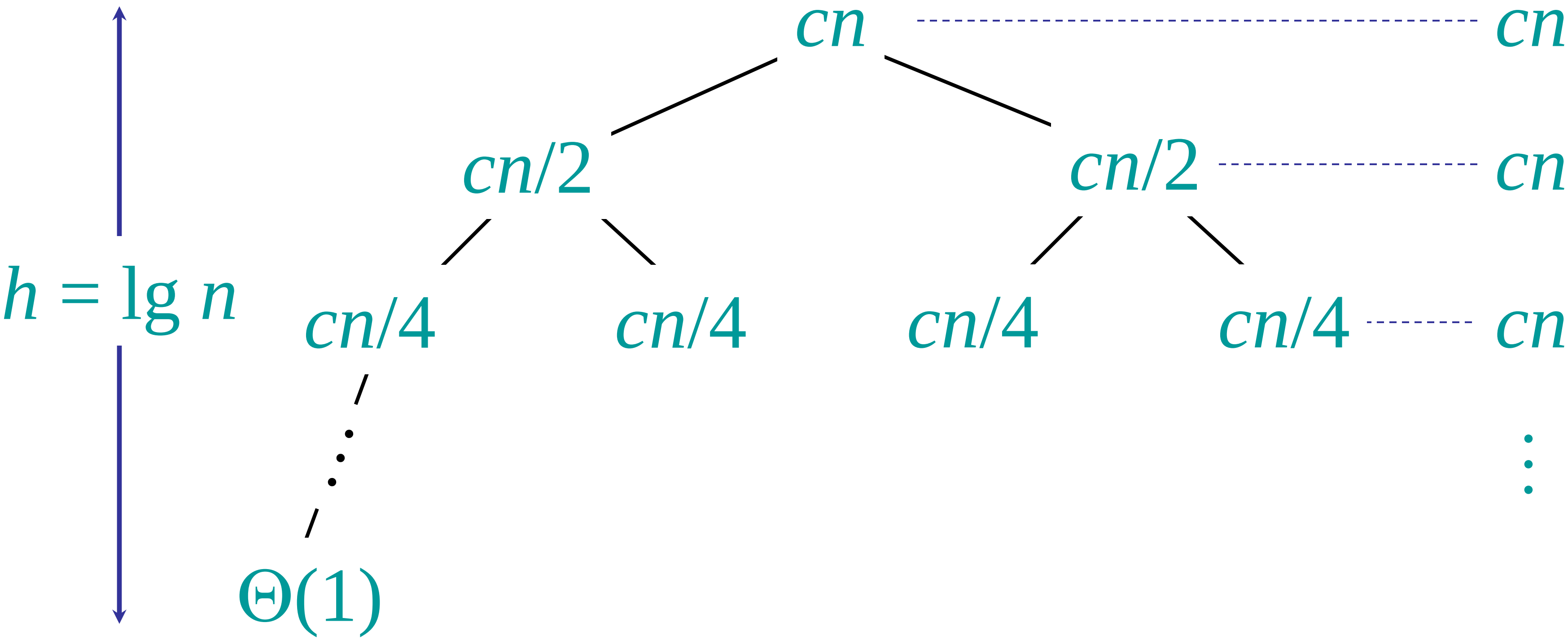

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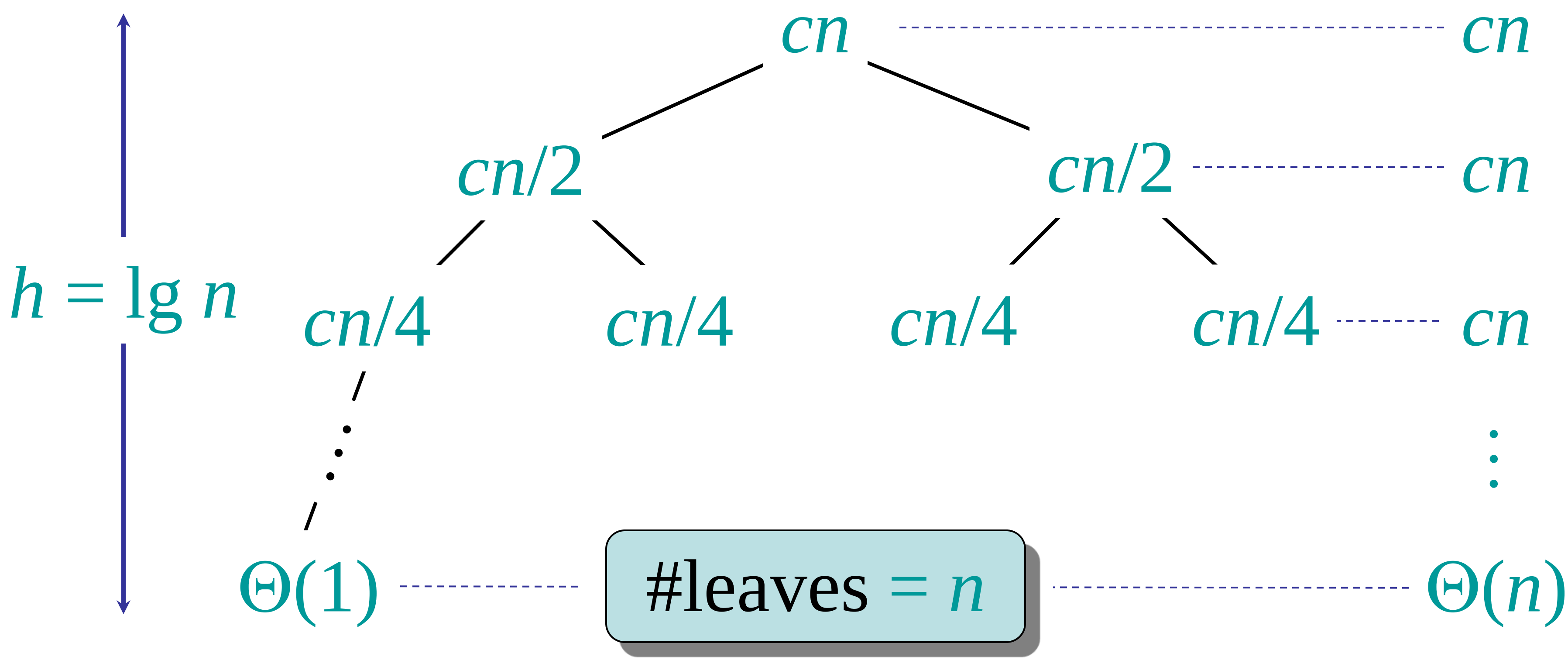

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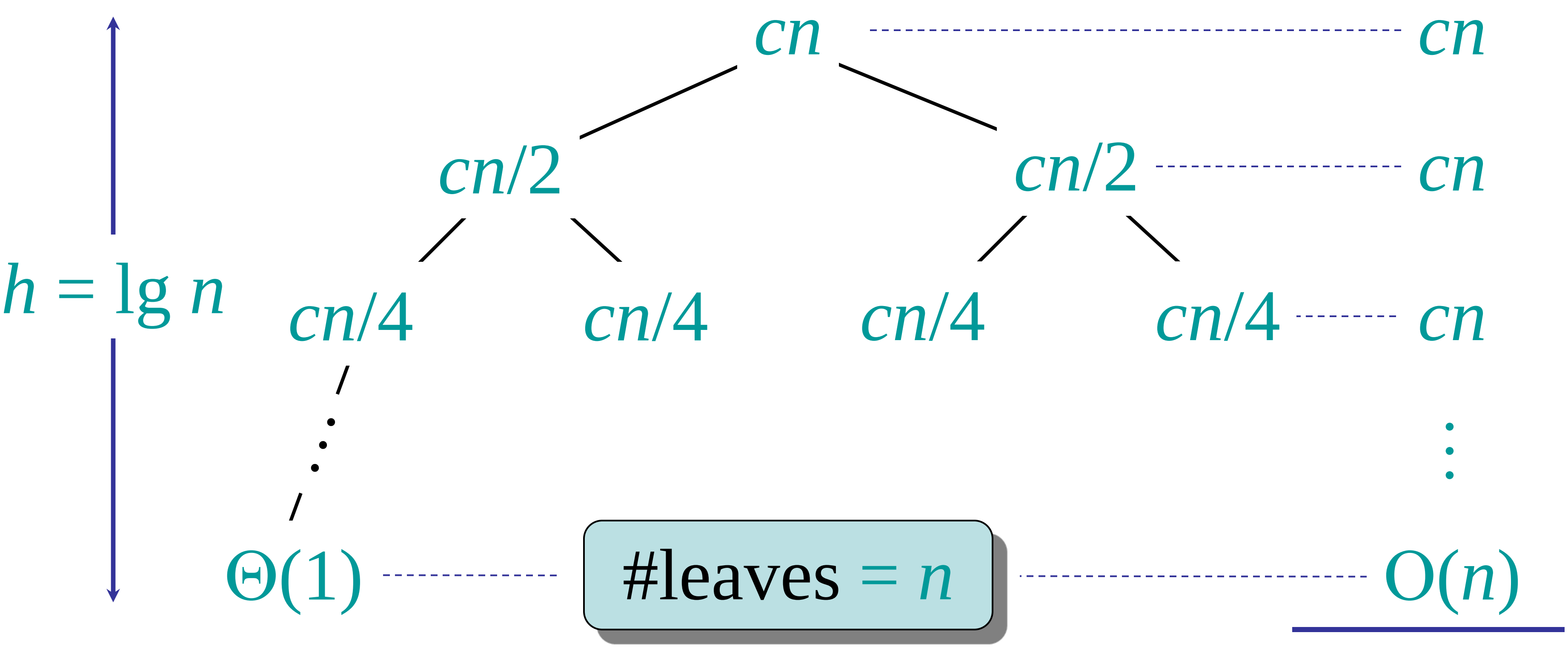

Total $= O(n \lg n)$

Conclusions

- $O(n \lg n)$ grows more slowly than $O(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.

- *Nearly Sorted??*

A tighter bound

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.

More about Theta Θ in the next lecture.