CS325: Linear programming example

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The Bicycle Problem

I need to get to Portland as quickly as possible (on my bicycle). The distance is 90 miles but I only have two Burgerville milkshakes (1000 calories and $3 each) to fuel my trip. I can bike 30 miles/hr, but that uses up 17 calories each minute. I could bike really slowly, at 10 miles/hr and only use 3 calories each minute. Or I could split the difference and travel at 20 miles/hr and use 10 calories each minute. What is the fastest way I can reach Portland without running out of energy?

Objective Let t_{30},t_{20} and t_{10} be the lengths of time (in minutes) that I will spend travelling 30, 20 and 10 miles/hr. Then, my goal is to:

$$\min t_{30} + t_{20} + t_{10}$$

Cover enough distance I need to make sure that those times will allow me to cover the 90 miles to Portland:

$$30 \cdot \frac{t_{30}}{60} + 20 \cdot \frac{t_{20}}{60} + 10 \cdot \frac{t_{10}}{60} = 90$$

Notice that we don't really need an equality, but that a $\geq$ will still allow us to find a feasible solution:

$$30 \cdot \frac{t_{30}}{60} + 20 \cdot \frac{t_{20}}{60} + 10 \cdot \frac{t_{10}}{60} \geq 90$$

Simplifying (this step isn't absolutely necessary):

$$3 \cdot t_{30} + 2 \cdot t_{20} + t_{10} \geq 540$$

Don't use more energy than we have I also need to make sure that I don't use up more energy than I have:

$$17 \cdot t_{30} + 10 \cdot t_{20} + 3 \cdot t_{10} \leq 2000$$

The linear program Putting it together with the additional observation that our times should not be negative:

$$\begin{array}{rc}
\min & t_{30} + t_{20} + t_{10} \\
s.t. & 3 \cdot t_{30} + 2 \cdot t_{20} + t_{10} \geq 540 \\
 & 17 \cdot t_{30} + 10 \cdot t_{20} + 3 \cdot t_{10} \leq 2000 \\
 & t_{30} \geq 0 \\
 & t_{20} \geq 0 \\
 & t_{10} \geq 0
\end{array}$$

Now, we just need to find the answer to this problem.

Using an LP-solver

We will use Matlab's linear programming solver `linprog` to solve this problem. The help page for `linprog` tells us

```
 linprog Linear programming.
    X = linprog(f,A,b) attempts to solve the linear programming problem:

             min f'*x    subject to:   A*x <= b
              x
```

which means that we need to get our LP into the form:

$$\begin{aligned} \min_x &\quad f'x \\ s.t. &\quad Ax \leq b \end{aligned}$$

where x, f and b are vectors and A is a matrix. Notice that the constraints are only of the form $\leq$. We first need to make that so by negating our $\geq$ constraints:

$$\begin{aligned} \min &\quad t_{30} + t_{15} + t_{10} \\ s.t. &\quad -3 \cdot t_{30} - 2 \cdot t_{15} - t_{10} \leq -540 \\ &\quad 17 \cdot t_{30} + 10 \cdot t_{15} + 3 \cdot t_{10} \leq 2000 \\ &\quad -t_{30} \leq 0 \\ &\quad -t_{15} \leq 0 \\ &\quad -t_{10} \leq 0 \end{aligned}$$

And turning it into matrix form:

$$\begin{aligned} \min &\quad [1\ 1\ 1] \begin{bmatrix} t_{30} \\ t_{15} \\ t_{10} \end{bmatrix} \\ s.t. &\quad \begin{bmatrix} -3 & -2 & -1 \\ 17 & 10 & 3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} t_{30} \\ t_{15} \\ t_{10} \end{bmatrix} \leq \begin{bmatrix} -540 \\ 2000 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Putting this into Matlab:

```
>> f = [1;1;1];
>> A = [-3 -2 -1; 17 10 3; -1 0 0; 0 -1 0; 0 0 -1];
>> b = [-540;2000;0;0;0];
>> t = linprog(f,A,b)
Optimization terminated.
t =
   29.5301
   35.9398
  379.5301
>> f.*t
ans =
   29.5301
   35.9398
  379.5301
```

Conclusion: I can get to Portland in 445 minutes (7 hours and 25 minutes) if I bike at 30 miles per hour for 29.5 minutes, 20 miles per hour for 36 minutes and 10 miles per hour for 379.5 minutes, and (if I had to guess) probably in that order. Not too bad for 2 milkshakes. And a lot tastier (and cheaper and more fun) than 5 gallons (or **3 million** calories) of gasoline.

Alternative tools

Matlab is not free. However, there are many open-source tools for solving linear programs, and it will not matter which one you use. The wikipedia page on linear programming maintains a list of open-source solves you may opt to use. While you are a student at OSU, though, you have access to Matlab through the College of Engineering.