ETH zürich

D-PHYS

Theta Functions, Kronecker Functions, and Bilinear Relations

Artyom Lisitsyn

Riemann Surfaces
in Mathematical Physics

Outline

1. Abel’s map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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Holomorphic Differentials

Existence of holomorphic differentials

The dimension of the space of holomorphic differentials is $\dim \mathcal{H}^1 = g$, the genus of the compact Riemann surface.

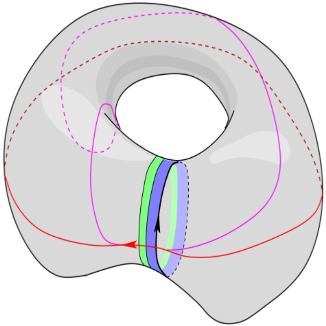
Proof outline:

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

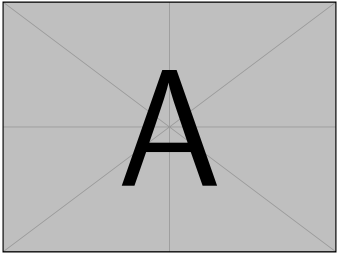
Abel's map

Bertola 2006 Section 4.2

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials ω_i, we have Abel's map

$$\mathbf{u}:\mathcal{L}\mapsto\mathbb{C}^g,\quad P \qquad \mapsto \begin{pmatrix}\int_{P_0}^{P}\omega_1\\ \vdots \\ \int_{P_0}^{P}\omega_g\end{pmatrix}$$



Fundamental domain
ImageSource

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P+a_i)=\mathbf{u}(P)+\begin{pmatrix}\int_{a_i}\omega_1\\ \vdots\end{pmatrix}=\mathbf{u}(P)+\begin{pmatrix}\delta_{i1}\\ \vdots\end{pmatrix}$$

$$\mathbf{u}(P+b_i)=\mathbf{u}(P)+\begin{pmatrix}\tau_{i1}\\ \vdots\end{pmatrix}$$

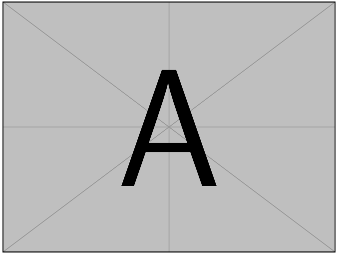
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1
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What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

Theta functions

Bertola 2006 Section 5.1

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z},\tau) := \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T\vec{z}\right) \quad , \quad \mathbf{e}(z) = \exp(2\pi i z)$$

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Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n}\mapsto-\vec{n}}{=} \Theta(\vec{z})$$

Theta functions

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$$\Theta(\vec{z}+\vec{\lambda}) = \sum_{\vec{n}\in\mathbb{Z}^g} \underbrace{\mathbf{e}(\vec{n}^T\vec{\lambda})}_{1}\mathbf{e}(\ldots) = \Theta(\vec{z})$$

Theta functions

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Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n} \mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z} + \vec{\lambda}) = \sum_{\vec{n} \in \mathbb{Z}^g} \cancelto{1}{\mathbf{e}(\vec{n}^T \vec{\lambda})} \mathbf{e}(\ldots) = \Theta(\vec{z})$$

$$\Theta(\vec{z} + \tau\vec{\lambda}) = \begin{bmatrix} \text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry} \end{bmatrix} = \mathbf{e}\left(-\frac{1}{2}\vec{\lambda}^T \tau \lambda - \vec{\lambda}^T \vec{z}\right) \Theta(\vec{z})$$

Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus g, with period matrix τ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

Properties:

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \mathbf{e}\left(-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right)\theta(P)$$

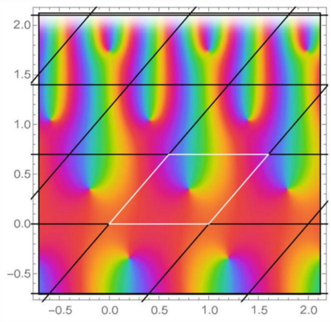
Theta function at genus 1

$$\theta(z) = \sum_{n\in\mathbb{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e}(-\frac{1}{2}\tau - \xi)\theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

Theta function with characteristics

Bertola 2006 Section 5.1

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) := \mathbf{e}\left(\frac{1}{8}\epsilon^T\tau\epsilon + \frac{1}{2}\epsilon^T\vec{z} + \frac{1}{4}\epsilon^T\epsilon'\right)\Theta(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2})$$

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Properties:

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau\vec{\beta}) = \mathbf{e}\left(\frac{1}{2}(\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2}\beta^T \tau \beta - \vec{\beta}\vec{z}\right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

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Properties:

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left(\frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta} \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

Theta function with characteristics

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Properties:

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z} + \vec{\alpha} + \tau\vec{\beta}) = \mathbf{e}\left(\frac{1}{2}(\epsilon^T\vec{\alpha} - \vec{\beta}^T\epsilon') - \frac{1}{2}\beta^T\tau\beta - \vec{\beta}\vec{z}\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z})$$

$$\Theta\begin{bmatrix}\epsilon + 2\eta\\ \epsilon' + 2\eta'\end{bmatrix}(\vec{z}) = \exp(\pi i\epsilon^T\eta')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-\vec{z}) = \exp(\pi i\epsilon^T\epsilon')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

Odd theta functions and zeros

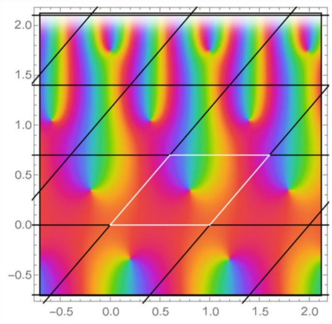
$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon'$ is odd

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\lambda}' + \tau \vec{\lambda}) = 0$$

$$\implies \Theta(\frac{\epsilon'}{2} + \frac{\tau \epsilon}{2}) = 0$$



Theta function for $\tau = 0.7 + 0.6i$
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What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix}\text{Find function } t(z) \\ \text{such that } t(0) = 0\end{bmatrix} \rightarrow \begin{bmatrix} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity}\end{bmatrix} \rightarrow \left(\frac{f}{g}\right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions.

(Application) Decomposing meromorphic functions

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Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus > 0:

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod \left(g_{P_i}(P)\right)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left(\theta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z)\right)^{n_i}$

Outline

Kronecker function

Brown and Levin 2013 Section 3.4

Definitions of the Kronecker function

The Kronecker function $F(\xi, \eta, \tau)$ has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(\xi+\eta)}{\theta_1(\xi)\theta_1(\eta)} \quad , \quad \theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. In terms of a sum over exponentials

$$-2\pi i \left(\frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} (z^m w^n - z^{-m} w^{-n}) q^{mn} \right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix}$$

3. In terms of a sum over Eisenstein functions and series

$$\frac{1}{\eta} \exp \left(- \sum_{j>0} \frac{(-\eta)^j}{j} (E_j(\xi, \tau) - e_j(\tau)) \right)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Properties:

$$F(\xi+1,\eta) = \frac{\theta_1'(0)\theta_1(\xi+\eta+1)}{\theta_1(\xi+1)\theta_1(\eta)} = F(\xi,\eta)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Properties:

$$F(\xi+1,\eta) = \frac{\theta_1'(0)\theta_1(\xi+\eta+1)}{\theta_1(\xi+1)\theta_1(\eta)} = F(\xi,\eta)$$

$$F(\xi+\tau,\eta) = \frac{\theta_1'(0)\theta_1(\xi+\eta+\tau)}{\theta_1(\xi+\tau)\theta_1(\eta)} = \frac{\mathbf{e}(-\xi-\eta)}{\mathbf{e}(-\xi)}F(\xi,\eta)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Properties:

$$F(\xi+1,\eta)=\frac{\theta_1'(0)\theta_1(\xi+\eta+1)}{\theta_1(\xi+1)\theta_1(\eta)}=F(\xi,\eta)$$

$$F(\xi+\tau,\eta)=\frac{\theta_1'(0)\theta_1(\xi+\eta+\tau)}{\theta_1(\xi+\tau)\theta_1(\eta)}=\frac{\mathbf{e}(-\xi-\eta)}{\mathbf{e}(-\xi)}F(\xi,\eta)$$

$$FAY\,RELATION$$

Abridged derivation of the Fay relation

Differentials from the Kronecker function

Examples of differentials

[1412.5535] eqn 3.31 etc.

Independence of the differentials

Fay relation for differentials

Application of properties

Calculation of integrals

Outline

Why we care about higher genus

Connection to string theory

Questions gathered so far

What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

Schottky cover definition

Schottky group example

Differentials and theta functions

Attempt at a Kronecker function

Chan 2022

Open questions

References

- Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.
- Brown, Francis C. S. and Andrey Levin (2013). *Multiple Elliptic Polylogarithms*.
- Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”.