ETH zürich

D-PHYS

# Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics

# Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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2. Theta functions
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# Holomorphic Differentials

## Existence of holomorphic differentials

The dimension of the space of holomorphic differentials is $\dim \mathcal{H}^1 = g$, the genus of the compact Riemann surface.

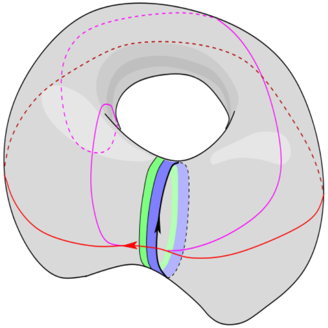
*Proof outline:*

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f\,dz + g\,d\bar{z} \implies \dim H = 2\dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

*Normalization & period matrix:*

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

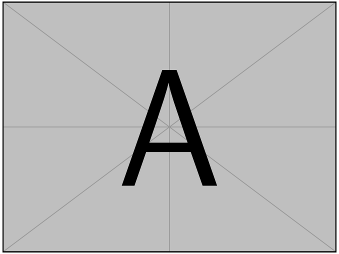
# Abel's map

Bertola 2006 Section 4.2

## Formal definition of Abel's map

For a particular choice of a point $P_0$ on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials $\omega_i$, we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g, \quad P \qquad \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$




Fundamental domain
**ImageSource**

*Analytic continuation beyond the fundamental domain:*

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

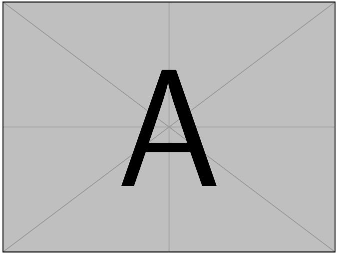
# Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$




Fundamental domain and continuation at genus 1
**ImageSource**

## What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

# Outline

# Theta functions

Bertola 2006 Section 5.1

## Definition of the Theta function

Given a symmetric matrix $\tau$ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z},\tau) := \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T\vec{z}\right) \quad , \quad \mathbf{e}(z) = \exp(2\pi i z)$$

*Properties:* For $\vec{\lambda}\in\mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n}\mapsto-\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z}+\vec{\lambda}) = \sum_{\vec{n}\in\mathbb{Z}^g} \cancelto{1}{\mathbf{e}(\vec{n}^T\vec{\lambda})}\mathbf{e}(\ldots) = \Theta(\vec{z})$$

$$\Theta(\vec{z}+\tau\vec{\lambda}) = \left[\begin{array}{c}\text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry}\end{array}\right] = \mathbf{e}\left(-\frac{1}{2}\vec{\lambda}^T\tau\lambda - \vec{\lambda}^T\vec{z}\right)\Theta(\vec{z})$$

# Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

## Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus $g$, with period matrix $\tau$ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

*Properties:*

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \mathbf{e}\left(-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right)\theta(P)$$

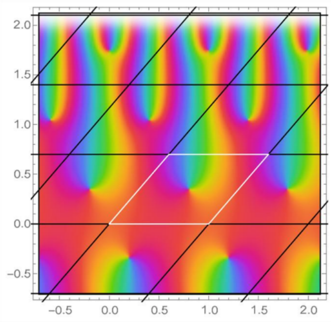
# Theta function at genus 1

$$\theta(z) = \sum_{n\in\mathbb{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e}(-\frac{1}{2}\tau - \xi)\theta(z)$$




Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

## What about higher genus?

- What does the Theta function look like at higher genus?

# Theta function with characteristics

Bertola 2006 Section 5.1

## Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics $\epsilon$, $\epsilon'$ as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \mathbf{e} \left( \frac{1}{8} \epsilon^T \tau \epsilon + \frac{1}{2} \epsilon^T \vec{z} + \frac{1}{4} \epsilon^T \epsilon' \right) \Theta(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2})$$

*Properties:*

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left( \frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta} \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

# Odd theta functions and zeros

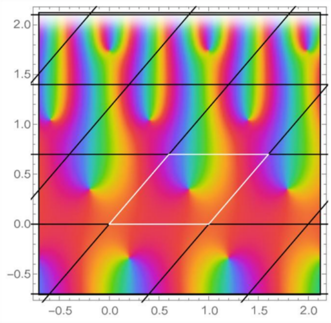
$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon'$ is odd

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\lambda}' + \tau\vec{\lambda}) = 0$$

$$\implies \Theta(\frac{\epsilon'}{2} + \frac{\tau\epsilon}{2}) = 0$$




Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

## What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

# (Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix}\text{Find function } t(z) \\ \text{such that } t(0) = 0\end{bmatrix} \to \begin{bmatrix} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity}\end{bmatrix} \to \left(\frac{f}{g}\right) = \emptyset \to \frac{f}{g} = \text{const.}$$

Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus $> 0$:

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod \left(g_{P_i}(P)\right)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left(\theta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z)\right)^{n_i}$

# Outline

# Kronecker function

Brown and Levin 2013 Section 3.4

## Definitions of the Kronecker function

The Kronecker function $F(\xi,\eta,\tau)$ has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(\xi+\eta)}{\theta_1(\xi)\theta_1(\eta)} \quad , \quad \theta_1(z) = -\theta\begin{bmatrix}1\\1\end{bmatrix}$$

2. In terms of a sum over exponentials

$$-2\pi i\left(\frac{z}{1-z}+\frac{1}{1-w}+\sum_{m,n>0}(z^m w^n - z^{-m}w^{-n})q^{mn}\right) \quad , \quad \begin{pmatrix}z\\w\\q\end{pmatrix} = \mathbf{e}\begin{pmatrix}\xi\\\eta\\\tau\end{pmatrix}$$

3. In terms of a sum over Eisenstein functions and series

$$\frac{1}{\eta}\exp\left(-\sum_{j>0}\frac{(-\eta)^j}{j}(E_j(\xi,\tau)-e_j(\tau))\right)$$

# Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

*Properties:*

$$F(\xi+1,\eta) = \frac{\theta_1'(0)\theta_1(\xi+\eta+1)}{\theta_1(\xi+1)\theta_1(\eta)} = F(\xi,\eta)$$

$$F(\xi+\tau,\eta) = \frac{\theta_1'(0)\theta_1(\xi+\eta+\tau)}{\theta_1(\xi+\tau)\theta_1(\eta)} = \frac{\mathbf{e}(-\xi-\eta)}{\mathbf{e}(-\xi)}F(\xi,\eta)$$

$$FAY RELATION$$

# Abridged derivation of the Fay relation

# Differentials from the Kronecker function

showing how the expansion is done

# Examples of differentials

[1412.5535] eqn 3.31 etc.

# Independence of the differentials

Brown and Levin (probably needs to be several slides)

# Fay relation for differentials

one line derivation one line statement

# Application of properties

big picture view of when these come into string theory

# Outline

# Why we care about higher genus

Connection to string theory

# Questions gathered so far

## What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel’s map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel’s map on compact Riemann surfaces of higher genus?

# Schottky cover definition

# Schottky group example

# Differentials and theta functions

# Attempt at a Kronecker function

Chan 2022

# Open questions

# References


- Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.
- Brown, Francis C. S. and Andrey Levin (2013). *Multiple Elliptic Polylogarithms*.
- Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”.