Report .1

Theta functions, Kronecker functions and bilinear relations

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1 Introduction

As seen in the previous two reports (**??** and **??**), the Theta and Kronecker functions are crucial to defining multiple elliptic polylogarithms. In this section we will analyze these mathematical tools in more detail, with the goal of considering open questions in generalizations to a higher genus can be approached.

In Section 2, we will start with a review of normalized holomorphic differentials on compact Riemann surfaces. These will be applied to define Abel's map, a function that takes assigns complex vectors to points on the Riemann surface, taking advantage of the differentials' properties to achieve an additively quasiperiodic result.

Then in Section 3, Theta functions will discussed in detail as they can be defined at an arbitrary genus. These functions use Abel's map to assign complex numbers to a Riemann surface in a quasiperiodic way. In particular, we will find through an example that odd versions of the Theta functions are essentially elliptic/hyperelliptic analogues of the monomial $(z - z_0)$ at genus 0.

In Section 4, we will define the familiar Kronecker function can be defined as a ratio of Theta functions at genus 1. This can be used to define quasiperiodic holomorphic differentials with properties including the bilinear Fay relation, which allows us to relate products of the differentials to products that may be simpler to evaluate.

Finally in Section 5, we will take a look at a particular way used today to attempt generalizations of the Kronecker function to higher genus. Schottky covers use mobius transformations to define a recursive structure on the complex plane that can be chosen to be a cover of a Riemann surface. With such covers, properties of holomorphic differentials, Abel's map, and Theta functions are easier to investigate directly, and attempts at a Kronecker function can be made.

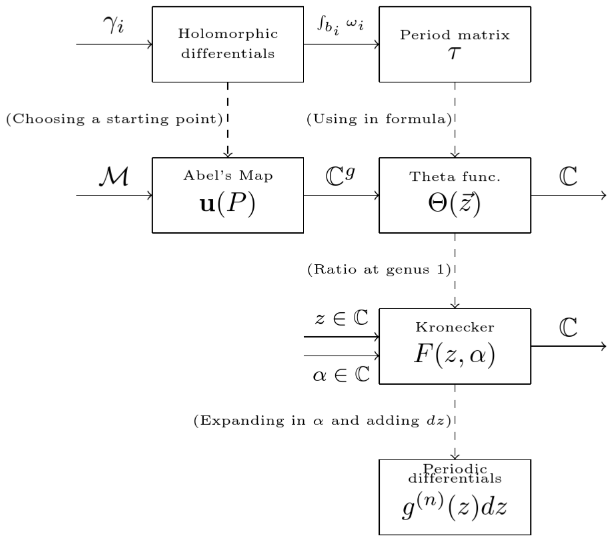

Figure .1.1: Diagramatic outline of the functions defined in this report. Dashed arrows downwards represent how concepts are used to built up to define new functions. Horizontal arrows represent inputs and outputs to the labeled objects.

2 Abel's Map

2.1 Holomorphic differentials

Holomorphic and harmonic differentials are described in detail in (REFERENCE TO A6 AND A7). However, for the readers convenience they will again be defined and their existence briefly shown here.

For more detail on the definitions and proofs shown, one may refer to [Ber06].

Definition 1 (Holomorphic differential)**.** *A holomorphic differential is a smooth, complex one-form, consisting of a collection of holomorphic functions f_α such that*

$$\omega = f_\alpha dz_\alpha \tag{.1|1}$$

is independent of chart.

Theorem 1 (Existence and normalization of holomorphic differentials on a compact Riemann surface)**.** *The dimension of the space of holomorphic differentials on a compact Riemann surface of genus g is*

$$\dim \mathcal{H}^1 = g. \tag{.1|2}$$

Proof. An abridged version of the proof is described here.

We can bound the number of holomorphic differentials above and below using the number of cycles.

Let us start with the lower bound, $\dim \mathcal{H}^1 \geq g$.

As was described in **??** and is illustrated in Figure .1.2, we are able to construct harmonic differentials from arbitrary cycles γ on a compact Riemann surface. Since there are $2g$ independent cycles, there are at least $2g$ harmonic one-forms

$$\dim H \geq 2g. \tag{.1|3}$$

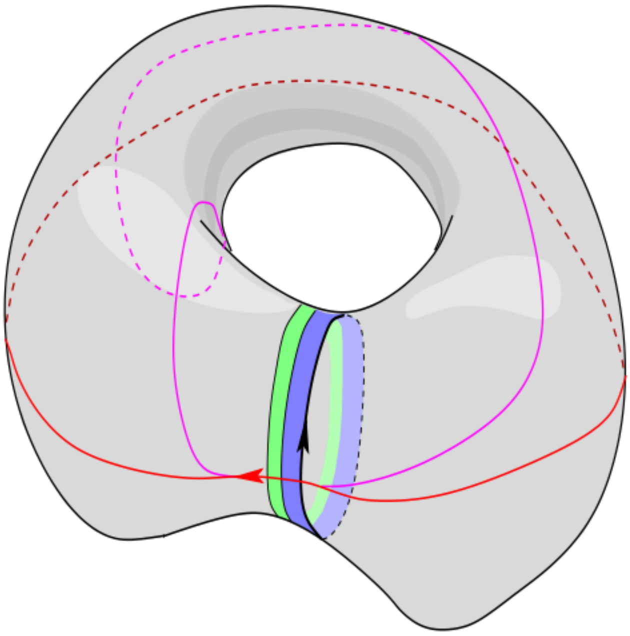

Figure .1.2: Figure 3.1 from [Ber06]. The black curve γ is an arbitrary cycle used to define a harmonic differential. We can define a function f_γ that is 1 on the blue belt, 0 outside of the colored belts, and smoothly connects 0 to 1 on the green belt. The exterior derivative $df_\gamma = dh + \eta_\gamma$ lets us identify the harmonic differential η_γ corresponding to the cycle γ.

Since harmonic differentials decompose into a holomorphic and antiholomorphic differentials, we find that

$$\dim H = 2\dim \mathcal{H}^1 \implies \mathcal{H}^1 \geq g \tag{.1|4}$$

Now, we will prove the upper bound, $\dim \mathcal{H}^1 \leq g$.

If there were more than g holomorphic differentials, it would be possible to find a non-zero linear combination thereof such that

$$\int_{a_i} \sum_j c_j \omega_j = 0 \quad \forall i. \tag{.1|5}$$

However, one could show that the only differentials that make all a-periods vanish in this way are $\omega \equiv 0$ (Corrolary 3.1.1 [Ber06]). Therefore, we reach a contradiction and indeed $\dim \mathcal{H}^1 \leq g$.

Combining the two inequalities, we find that $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$. □

Lemma 1 (Normalized basis of holomorphic differentials)**.** *For a compact Riemann surface of genus g, we can identify g independent holomorphic differentials normalized such that*

$$\int_{a_i} \omega_j = \delta_{ij}, \tag{.1|6}$$

$$\int_{b_i} \omega_j = \tau_{ij}, \tag{.1|7}$$

where τ is a symmetric matrix, referred to as the period matrix.

2.2 Definition of Abel's map

Now that the normalized holomorphic differentials are identified, we can all g of them to produce a function that outputs a $\mathbb{C}^g$ vector. Since the differentials have known integrals on a-cycles and b-cycles, the definition of the function can just be made on the fundamental domain, with further points being identified through the cycles.

Definition 2 (Abel's map)**.** *Given the normalized differentials ω_i for a Riemann surface $\mathcal{M}$ of genus g, we can define Abel's map on the fundamental domain $\mathcal{L}$ for some chosen basepoint P_0*

$$\mathbf{u} : \mathcal{L} \to \mathbb{C}^g \tag{.1|8}$$

$$P \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}. \tag{.1|9}$$

Such a mapping is well defined on the fundamental domain since the integration paths are limited to being homotopically equivalent. We can analytically continue Abel's map to beyond the fundamental domain using known results for integrations along a-cycles and b-cycles

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \\ \int_{a_i} \omega_g \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \\ \delta_{ig} \end{pmatrix}, \tag{.1|10}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{b_i} \omega_1 \\ \vdots \\ \int_{b_i} \omega_g \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \\ \tau_{ig} \end{pmatrix}. \tag{.1|11}$$

The simplest example of Abel's map is one we had implicitly already seen before. At genus 1, we can use the familiar cover in which the torus is mapped to the complex plane, where the fundamental domain is a parallelogram with vertices 0, 1, τ, $1+\tau$. There is a single holomorphic differential

$$\omega = dz. \tag{.1|12}$$

Using Abel's map, with the basepoint $P_0 = 0$, we find that

:AbelGenus1}

$$\mathbf{u}(z) = \int_0^z dz = z, \tag{.1|13}$$

with this result being kept as the map is continued analytically along a-cycles and b-cycles

$$\mathbf{u}(P + a_i) = \mathbf{u}(z + 1) = \int_0^z dz + \delta_{11} = z + 1, \tag{.1|14}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(z + \tau) = \int_0^z dz + \tau_{11} = z + \tau. \tag{.1|15}$$

Unfortunately, it is not so simple to define a fundamental domain and identify holomorphic differentials for higher genus. The resolution to this challenge is covered in Section 5.2.

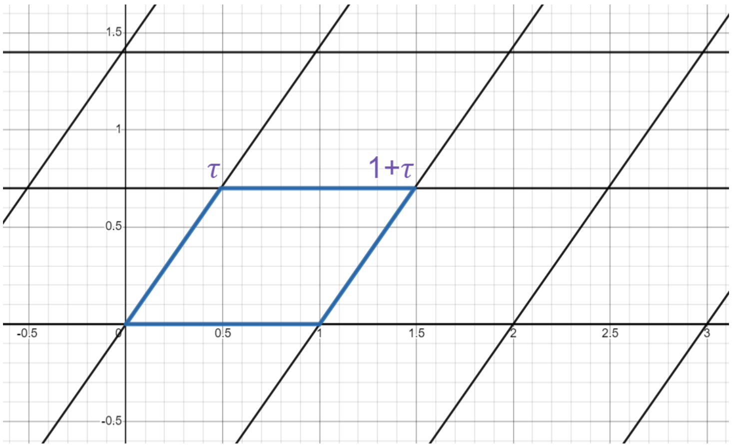

Figure .1.3: The fundamental domain at genus 1. Due to the trivial differential $\omega = dz$, we find that Abel's map at genus 1 is simply $\mathbf{u}(z) = z$, justifying the choice of cover.

3 Theta functions

With Abel's map, we have derived a additively quasiperiodic function that takes us from the manifold to vectors in $\mathbb{C}^g$. We seek to take advantage of this definition to find a function that instead embodies multiplicative quasiperiodicity. Then, the location of zeros on the Riemann surface will be homotopically invariant, since moving along a cycle and multiplying by some factor will not change whether the function vanishes.

The multiplicative quasiperiodicity will be facilitated by using properties of the exponential function, using the following notation to simplify some expressions

$$\mathbf{e}(z) = \exp(2\pi i z), \quad \mathbf{e}(z+1) = \mathbf{e}(z). \tag{.1|16}$$

3.1 Definition on $\mathbb{C}^g$

Definition 3 (Theta function)**.** *Given a $g \times g$ matrix τ that is symmetric ($\tau = \tau^T$) and has positive definite imaginary part ($\vec{n}^T(\operatorname{Im}\tau)\vec{n} > 0 \forall \vec{n} \in \mathbb{R}^g \setminus \vec{0}$), we can define the associated Theta function*

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z}\right) \tag{.1|17}$$ {eqnB12:Theta(

where $\vec{z} \in \mathbb{C}^g$. From this point forward the Theta functions explicit dependence on τ will be omitted, since the Theta functions are almost always discussed in the context of a τ chosen to be a constant.

Though we will omit a formal proof thereof, it is apparent that the series will converge due to the matrix τ having a positive definite imaginary part supressing terms with large $\vec{n}$. As we will go on to see in the following section, the choices made for the Theta function are made to fit naturally with the formalism we developed for Riemann surfaces, as the matrix τ will be filled by the period matrix and the input $\vec{z}$ will be the output of Abel's map.

Lemma 2 (Properties of the Theta function)**.** *The Theta function satisfies the properties*

$$\Theta(\vec{z}) = \Theta(-\vec{z}) \tag{.1|18}$$ {eqnB12:Theta(

$$\Theta(\vec{z}+\vec{\alpha}) = \Theta(\vec{z}) \quad \forall \vec{\alpha} \in \mathbb{Z}^g \tag{.1|19}$$ {eqnB12:Theta(

$$\Theta(\vec{z}+\tau\vec{\beta}) = \mathbf{e}\left(-\frac{1}{2}\vec{\beta}^T\tau\vec{\beta} - \vec{\beta}^T\vec{z}\right)\Theta(\vec{z}) \quad \forall \vec{\beta} \in \mathbb{Z}^g \tag{.1|20}$$ {eqnB12:Theta(

Proof. The simple proofs of the three properties are included for the convenience of the reader.

Proving that the Theta function is even (.1|18)

$$\Theta(-\vec{z}) = \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T(-\vec{z})\right) \overset{\vec{n}\mapsto-\vec{m}}{=} \sum_{\vec{m}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{m}^T\tau\vec{m} + \vec{m}^T\vec{z}\right) = \Theta(\vec{z}) \tag{.1|21}$$

Proving that the Theta function is invariant under integer shifts (.1|19)

$$\Theta(\vec{z}+\vec{\alpha}) = \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T(\vec{z}+\vec{\alpha})\right) \tag{.1|22}$$

$$= \sum_{\vec{n}\in\mathbb{Z}^g} \underbrace{\mathbf{e}\left(\vec{n}^T\vec{\alpha}\right)}_{=1}\mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T\vec{z}\right) = \Theta(\vec{z}) \tag{.1|23}$$

Proving that the Theta function is invariant under period matrix shifts (.1|20)

$$\Theta(\vec{z}+\tau\vec{\beta}) = \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T(\vec{z}+\tau\vec{\beta})\right) \tag{.1|24}$$

$$= \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(-\frac{1}{2}\vec{\beta}^T\tau\vec{\beta} - \vec{\beta}^T\vec{z}\right)\mathbf{e}\left(\frac{1}{2}(\vec{n}+\vec{\beta})^T\tau(\vec{n}+\vec{\beta}) + (\vec{n}+\vec{\beta})^T\vec{z}\right) \tag{.1|25}$$

$$= \mathbf{e}\left(-\frac{1}{2}\vec{\beta}^T\tau\vec{\beta} - \vec{\beta}^T\vec{z}\right)\Theta(\vec{z}) \tag{.1|26}$$

□

3.2 Definition on compact Riemann surface

Now, we can combine Abel's map (2) and the Theta function above (.1|17) to identify Theta functions directly on a compact Riemann surface. The properties of the two functions work perfectly together to produce desired quasiperiodic results on the Riemann surface.

Definition 4 (Theta function on a compact Riemann surface)**.** *Using Abel's map* $\mathbf{u} : \mathcal{M} \mapsto \mathbb{C}^g$ *with some basepoint* $P_0 \in \mathcal{M}$*, and the Theta function* $\Theta : \mathbb{C}^g \mapsto \mathbb{C}$ *with* τ *corresponding to the period matrix of* $\mathcal{M}$*, we identify*

$$\theta(P) = \Theta(\mathbf{u}(P), \tau) \tag{.1|27}$$

as a Theta function on the Riemann surface.

Lemma 3 (Properties of Theta functions on a compact Riemann surface)**.** *Using the analytic continuation provided by Abel's map and the quasiperiodic properties of the Theta function, we find that*

$$\theta(P + a_i) = \theta(P), \tag{.1|28}$$

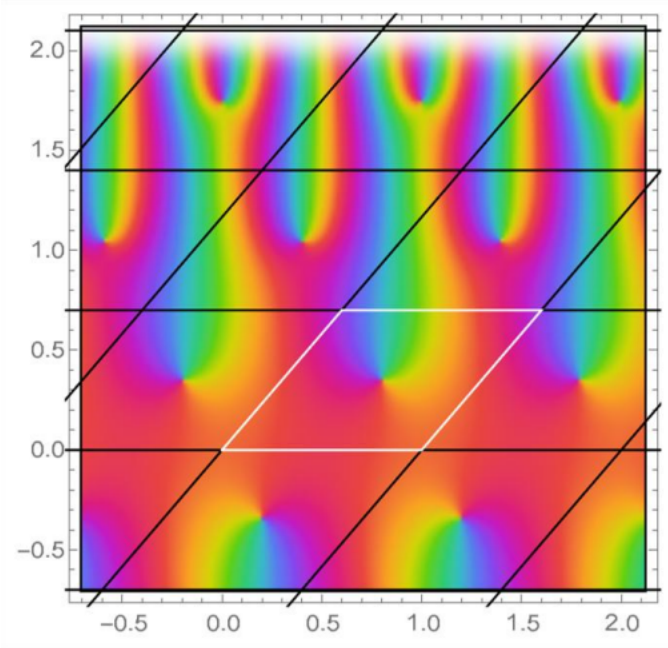

Figure .1.4: Complex plot of the Theta function on a genus 1 surface with $\tau = 0.7 + 0.6i$. For $\theta(z) = r\exp(i\phi)$, the color represents the argument ϕ. One can see the quasiperiodic properties: the plot is perfectly symmetric with respect to shifts by 1 (a-cycles), and quasiperiodic with respect to shifts by τ (b-cycles). The points around which the color wraps are simple zeros, the locations of which are proven in Theorem 2.

$$\theta(P + b_i) = \mathbf{e}\left(-\mathbf{u}_i(P) - \frac{1}{2}\tau_{ii}\right)\theta(P). \tag{.1|29}$$

Recall that Abel's map on a genus 1 Riemann surface is simply $\mathbf{u}(z) = z$ (Equation .1|13). We can then easily see that the Theta function on a genus 1 Riemann surface is simply

$$\theta(z) = \Theta(\mathbf{u}(z)) = \Theta(z) = \sum_{n\in\mathbb{Z}} \mathbf{e}\left(\frac{1}{2}n^2\tau + nz\right), \tag{.1|30}$$

a plot of which is seen in Figure .1.4.

3.3 Characteristics and zeros

Definition 5 (Theta function with characteristics)**.** *For vectors $\epsilon, \epsilon' \in \mathbb{C}^g$, the Theta function with characteristics is defined as*

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) := \mathbf{e}\left(\frac{1}{8}\epsilon^T\tau\epsilon + \frac{1}{2}\epsilon^T\vec{z} + \frac{1}{4}\epsilon^T\epsilon'\right)\Theta\left(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2}\right) \tag{.1|31}$$

$$= \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}(\vec{n} + \epsilon/2)^T\tau(\vec{n} + \epsilon/2) + (\vec{n} + \epsilon/2)^T(\vec{z} + \epsilon'/2)\right). \tag{.1|32}$$

Note that many sources (e.g. [Cha22] and [BK11]) use $\epsilon, \epsilon' \mapsto \epsilon/2, \epsilon'/2$ instead to simplify parts of the notation.

The choice of adding characteristics to the Theta functions in this way seems arbitrary, but it actually leads to a few key properties. First, it is apparent from the definition that the Theta function, and consequently its zeros, are shifted by some controlled amount. Second, the quasiperiodicity and parity properties of the Theta function will change in ways that we will be able to take advantage of.

Lemma 4 (Properties of the Theta functions with characteristics)**.** *The Theta function satisfies the properties*

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}+\vec{\alpha}+\tau\vec{\beta}) = \mathrm{e}\left(\frac{1}{2}(\epsilon^T\vec{\alpha}-\vec{\beta}^T\epsilon')-\frac{1}{2}\beta^T\tau\beta-\vec{\beta}\vec{z}\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \qquad (.1|33)$$

$$\Theta\begin{bmatrix}\epsilon+2\eta\\ \epsilon'+2\eta'\end{bmatrix}(\vec{z}) = \exp(\pi i\epsilon^T\eta')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}), \quad \eta,\eta'\in\mathbb{Z}^g \qquad (.1|34)$$

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-\vec{z}) = \exp(\pi i\epsilon^T\epsilon')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}), \quad \epsilon,\epsilon'\in\mathbb{Z}^g \qquad (.1|35)$$

The first property is analogous to the quasiperiodicity of the theta function before (.1|19 and .1|20).

The second property shows that, up to a sign, the characteristics are equivalent modulo 2.

The third property defines the parity of the resulting theta function; note that it is only true for integer characteristics.

Proof. The first and second properties are proved analogously to Lemma 2.

The third property uses

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) = \Theta\begin{bmatrix}-\epsilon\\ -\epsilon'\end{bmatrix}(-\vec{z}) \qquad (.1|36)$$

followed by applying the second property using $\eta = 2\epsilon$ and $\eta = 2\epsilon'$ to return to the original characteristics with a possible sign difference. □

The third property tells us that for characteristics $\epsilon,\epsilon'\in\mathbb{Z}^g$ such that $\epsilon^T\epsilon'$ is odd, the resulting Theta function will be odd. Since odd function vanish at the origin, this property can be used to identify the locations of zeros for the Theta function in general.

Theorem 2 (Location of zeros of the Theta function)**.** *The zeros of the Theta function are*

$$\Theta\left(\frac{\epsilon}{2}+\frac{\tau\epsilon'}{2}\right) = 0 \qquad (.1|37)$$

for all $\epsilon,\epsilon'\in\mathbb{Z}^g$ for which $\epsilon^T\epsilon'$ is odd.

Proof. For such ϵ and ϵ', we find that

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-\vec{z}) = -\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \implies \Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0) = 0. \qquad (.1|38)$$

Using the definition of the Theta function with characteristics

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0) = \overbrace{\mathbf{e}(\cdots)}^{\neq 0}\Theta\left(0+\frac{\epsilon'}{2}+\frac{\tau\epsilon}{2}\right) = 0, \qquad (.1|39)$$

we find that

$$\Theta\left(\frac{\epsilon'}{2}+\frac{\tau\epsilon}{2}\right)=0 \tag{.1|40}$$

as desired. □

Depending on the genus, the number of possible odd characteristics changes. Indeed, one can prove that the precise number is $2^{g-1}(2^g-1)$ choices out of the 2^{2g} possible integer characteristics modulo 2. At genus 1, this leaves only a single function, known as *the* odd Theta function at genus 1.

Definition 6 (The odd Theta function at genus 1)**.** *The odd Theta function at genus 1 can be identified as*

$$\theta_1(z)=-\Theta\begin{bmatrix}1\\1\end{bmatrix}(z), \tag{.1|41}$$

where the choice of characteristics $\epsilon=\epsilon'=1$ is the only possible odd choice, and the extra -1 factor exists by convention.

3.4 Decomposition of functions

To provide some intuition and illustration of the properties of Theta functions, this section will describe their use in the decomposition of meromorphic functions.

Parts of this proof have already been done in **??**, with results for decompositions of meromorphic functions at genus 0 (Equation **??**) and at genus 1 (Equation **??**).

Consider decomposing a function f with divisor (f) (see **??** for an overview of divisors). We can approach this by constructing a function g such that $(g)=(f)$. Then, the function f/g has divisor $(f/g)=(f)-(g)=0$, and is thus constant by Liouville's theorem (Theorem **??**). Consequently, the desired function f is off from our constructed function g by a constant.

Lemma 5. *Given a function f on the Riemann sphere, with divisor $(f)=\sum_i n_i z_i$, it is equal to*

$$f=c\prod_i(z-z_i)_i^n \tag{.1|42}$$

4 Kronecker function

With a better understanding and formalism of Theta functions, we can now return to the Kronecker function from prior reports (**??** and **??**). Rather than focusing on the consequences that the Kronecker function has for multiple elliptic polylogarithms [BL13] or its applications to string theory [BK22], we will focus on the properties that it has and how those properties are inherited by the functions used in its expansion.

4.1 Definition and properties of the Kronecker function

Definition 7. *The Kronecker function $F(z,\alpha,\tau)$ has equivalent definitions [BL13]*

1. *In terms of the odd theta function*

$$\frac{\theta_1'(0)\theta_1(z+\alpha)}{\theta_1(z)\theta_1(\alpha)}$$

2. *In terms of a sum over exponentials* $F(\xi, \alpha, \tau)$

$$-2\pi i\left(\frac{z}{1-z}+\frac{1}{1-w}+\sum_{m,n>0}(z^m w^n - z^{-m}w^{-n})q^{mn}\right), \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e}\begin{pmatrix} \xi \\ \alpha \\ \tau \end{pmatrix}$$

3. *In terms of a sum over Eisenstein functions and series*

$$\frac{1}{\alpha}\exp\left(-\sum_{j>0}\frac{(-\alpha)^j}{j}(E_j(z,\tau)-e_j(\tau))\right)$$

where $E_j(z,\tau)$ *and* $e_j(\tau)$ *are the Eisenstein functions and series, defined by*

$$E_j(\xi,\tau) = \lim_{N,M\to\infty}\sum_{n=-N}^{N}\sum_{m=-M}^{M}\frac{1}{(\xi+m+n\tau)^j}, \quad e_j(\tau) = \lim_{N,M\to\infty}\underbrace{\sum_{n=-N}^{N}\sum_{m=-M}^{M}}_{\text{except } (m,n)=(0,0)}\frac{1}{(m+n\tau)^j}. \tag{.1|43}$$

Proof. An explanation of the proof of equivalence of the definitions is given in [BL13]. □

4.2 Decomposition for differentials

4.3 Application

5 Schottky Covers

5.1 Definition of the Schottky group and cover

[Cha22] [BK11]

5.2 Differentials and Abel's map

[Cha22] [BK11]

5.3 Attepmts at a Kronecker function

[Cha22]

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