

# Theta Functions, Kronecker Functions, and Bilinear Relations

**Artyom Lisitsyn**

Riemann Surfaces  
in Mathematical Physics

# Diagram of plan



# Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

# Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

# Holomorphic differentials

## Definition and existence of holomorphic differentials

Definition:  $\omega = f_\alpha dz_\alpha = f_\beta dz_\beta$ ,  $f$  holomorphic

Existence:  $\dim \mathcal{H}^1 = g$  (genus of compact Riemann surface)

# Holomorphic differentials

## Definition and existence of holomorphic differentials

Definition:  $\omega = f_\alpha dz_\alpha = f_\beta dz_\beta$ ,  $f$  holomorphic

Existence:  $\dim \mathcal{H}^1 = g$  (genus of compact Riemann surface)

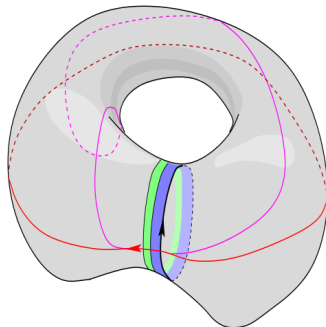
*Proof outline:*

- $\dim \mathcal{H}^1 \leq \# \text{ of a-cycles} = g$
- $\# \text{ of harmonic differentials} = \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

*Normalization & period matrix:*

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials  
Bertola 2006

# Abel's map

Bertola 2006 Section 4.2

## Formal definition of Abel's map

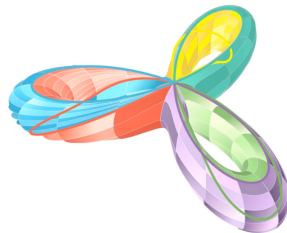
For a particular choice of a point  $P_0$  on the fundamental domain  $\mathcal{L}$ , using the normalized harmonic differentials  $\omega_i$ , we have Abel's map

$$\mathbf{u} : \mathcal{L} \rightarrow \mathbb{C}^g, \quad P \mapsto \begin{pmatrix} \int_{P_0}^P \omega_1 \\ \vdots \\ \int_{P_0}^P \omega_g \end{pmatrix}$$

*Analytic continuation beyond the fundamental domain:*

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$



Genus 3 surface

# Abel's map

Bertola 2006 Section 4.2

## Formal definition of Abel's map

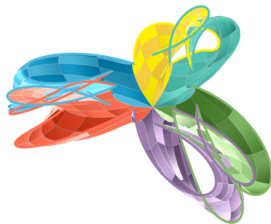
For a particular choice of a point  $P_0$  on the fundamental domain  $\mathcal{L}$ , using the normalized harmonic differentials  $\omega_i$ , we have Abel's map

$$\mathbf{u} : \mathcal{L} \rightarrow \mathbb{C}^g, \quad P \mapsto \begin{pmatrix} \int_{P_0}^P \omega_1 \\ \vdots \\ \int_{P_0}^P \omega_g \end{pmatrix}$$

*Analytic continuation beyond the fundamental domain:*

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$



Unfolding Genus 3 Surface



# Abel's map

Bertola 2006 Section 4.2

## Formal definition of Abel's map

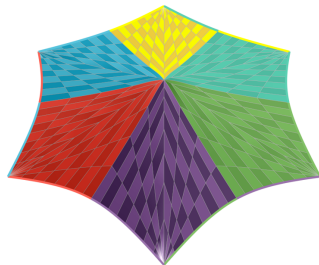
For a particular choice of a point  $P_0$  on the fundamental domain  $\mathcal{L}$ , using the normalized harmonic differentials  $\omega_i$ , we have Abel's map

$$\mathbf{u} : \mathcal{L} \rightarrow \mathbb{C}^g, \quad P \mapsto \begin{pmatrix} \int_{P_0}^P \omega_1 \\ \vdots \\ \int_{P_0}^P \omega_g \end{pmatrix}$$

*Analytic continuation beyond the fundamental domain:*

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$



Genus 3 fundamental domain

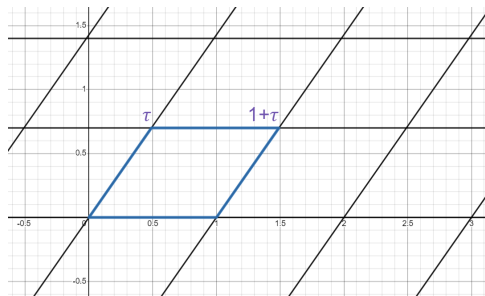
# Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1

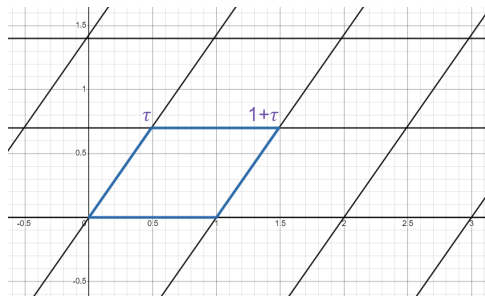
# Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1

What about higher genus?

- How do we represent the fundamental domain?
- What choice of holomorphic differentials can we make?
- What consequences does this have for Abel's map?

# Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

# Theta functions

Bertola 2006 Section 5.1

## Definition of the Theta function

Given a symmetric matrix  $\tau$  with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z}\right), \quad \mathbf{e}(z) = \exp(2\pi i z)$$

# Theta functions

Bertola 2006 Section 5.1

## Definition of the Theta function

Given a symmetric matrix  $\tau$  with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e} \left( \frac{1}{2} \vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z} \right), \quad \mathbf{e}(z) = \exp(2\pi i z)$$

*Properties:* For  $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \stackrel{\vec{n} \mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z} + \vec{\lambda}) = \sum_{\vec{n} \in \mathbb{Z}^g} \cancel{\mathbf{e}(\vec{n}^T \vec{\lambda})} \mathbf{e}(\dots) = \Theta(\vec{z})$$

$$\Theta(\vec{z} + \tau \vec{\lambda}) = \left[ \begin{array}{c} \text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry} \end{array} \right] = \mathbf{e} \left( -\frac{1}{2} \vec{\lambda}^T \tau \vec{\lambda} - \vec{\lambda}^T \vec{z} \right) \Theta(\vec{z})$$

# Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

## Definition of Theta function on a compact Riemann surface

For a compact Riemann surface  $\mathcal{M}$  of genus  $g$ , with period matrix  $\tau$  and Abel's map  $\mathbf{u}$ , we can identify

$$\begin{aligned}\theta : \mathcal{M} &\rightarrow \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P))\end{aligned}$$

# Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

## Definition of Theta function on a compact Riemann surface

For a compact Riemann surface  $\mathcal{M}$  of genus  $g$ , with period matrix  $\tau$  and Abel's map  $\mathbf{u}$ , we can identify

$$\begin{aligned}\theta : \mathcal{M} &\rightarrow \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P))\end{aligned}$$

*Properties:*

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \mathbf{e} \left( -\frac{1}{2} \tau_{ii} - \mathbf{u}_i(P) \right) \theta(P)$$



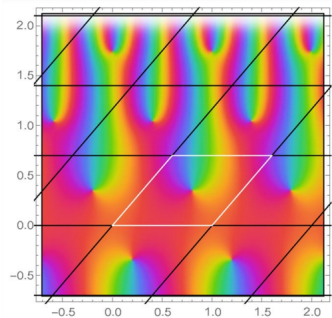
# Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \mathbf{e} \left( \frac{1}{2} n^2 \tau + n z \right)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e} \left( -\frac{1}{2} \tau - z \right) \theta(z)$$



Theta function for  $\tau = 0.7 + 0.6i$   
Chan 2022

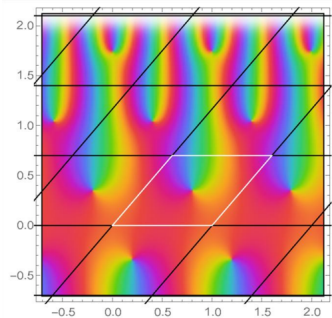
# Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \mathbf{e} \left( \frac{1}{2} n^2 \tau + n z \right)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e} \left( -\frac{1}{2} \tau - z \right) \theta(z)$$



Theta function for  $\tau = 0.7 + 0.6i$   
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

# Theta function with characteristics

Bertola 2006 Section 5.1

## Definition of Theta function with characteristics

Consider vectors  $\epsilon, \epsilon' \in \mathbb{R}^g$ . We can then define the Theta function with characteristics  $\epsilon, \epsilon'$  as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e} \left( \frac{1}{2} (\vec{n} + \epsilon/2)^T \tau (\vec{n} + \epsilon/2) + (\vec{n} + \epsilon/2)^T (\vec{z} + \epsilon'/2) \right)$$

*Properties:*

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left( \frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta}^T \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \alpha, \beta \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

# Theta function with characteristics

Bertola 2006 Section 5.1

## Definition of Theta function with characteristics

Consider vectors  $\epsilon, \epsilon' \in \mathbb{R}^g$ . We can then define the Theta function with characteristics  $\epsilon, \epsilon'$  as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \mathbf{e} \left( \frac{1}{8} \epsilon^T \tau \epsilon + \frac{1}{2} \epsilon^T \vec{z} + \frac{1}{4} \epsilon^T \epsilon' \right) \Theta \left( \vec{z} + \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right)$$

*Properties:*

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left( \frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta}^T \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

# Odd theta functions and zeros

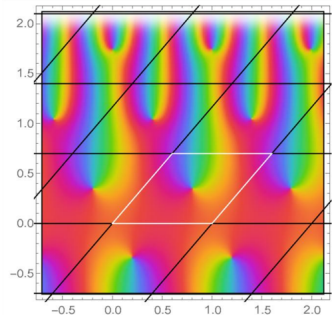
$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Rightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\Rightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0$$

$$\Rightarrow \Theta \left( \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0$$



Theta function for  $\tau = 0.7 + 0.6i$   
Chan 2022

# Odd theta functions and zeros

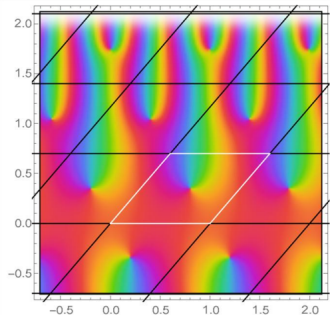
$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Rightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\Rightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0$$

$$\Rightarrow \Theta \left( \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0$$



Theta function for  $\tau = 0.7 + 0.6i$   
Chan 2022

## What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

# Odd theta function at genus 1

## Odd theta function at genus 1

We define

$$\theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$$

It has equivalent definition

$$\theta_1(z) = 2iq^{1/8} \sin(\pi z) \prod_{j>0} (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j), \quad q = \mathbf{e}(\tau), w = \mathbf{e}(z)$$

# (Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor  $(f) = \sum n_i P_i$

$$\left[ \begin{array}{l} \text{Find function } t(P, P') \\ \text{with simple zero at } P = P' \end{array} \right] \rightarrow \left[ \begin{array}{l} g(P) = \prod t(P, P_i)^{n_i} \\ \text{respecting possible periodicity} \end{array} \right] \rightarrow \left( \frac{f}{g} \right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that  $\deg((f)) = \sum n_i = 0$  for meromorphic functions, so extra factors can easily cancel.



# (Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor  $(f) = \sum n_i P_i$

$$\left[ \begin{array}{l} \text{Find function } t(P, P') \\ \text{with simple zero at } P = P' \end{array} \right] \rightarrow \left[ \begin{array}{l} g(P) = \prod t(P, P_i)^{n_i} \\ \text{respecting possible periodicity} \end{array} \right] \rightarrow \left( \frac{f}{g} \right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that  $\deg((f)) = \sum n_i = 0$  for meromorphic functions, so extra factors can easily cancel.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

# (Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor  $(f) = \sum n_i P_i$

$$\left[ \begin{array}{l} \text{Find function } t(P, P') \\ \text{with simple zero at } P = P' \end{array} \right] \rightarrow \left[ \begin{array}{l} g(P) = \prod t(P, P_i)^{n_i} \\ \text{respecting possible periodicity} \end{array} \right] \rightarrow \left( \frac{f}{g} \right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that  $\deg((f)) = \sum n_i = 0$  for meromorphic functions, so extra factors can easily cancel.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus 1:

- Decompose  $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left( \theta \begin{bmatrix} a_i + 1 \\ b_i + 1 \end{bmatrix} (z) \right)^{n_i}$

# (Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor  $(f) = \sum n_i P_i$

$$\left[ \begin{array}{l} \text{Find function } t(P, P') \\ \text{with simple zero at } P = P' \end{array} \right] \rightarrow \left[ \begin{array}{l} g(P) = \prod t(P, P_i)^{n_i} \\ \text{respecting possible periodicity} \end{array} \right] \rightarrow \left( \frac{f}{g} \right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that  $\deg((f)) = \sum n_i = 0$  for meromorphic functions, so extra factors can easily cancel.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus 1:

- Decompose  $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left( \theta \begin{bmatrix} a_i + 1 \\ b_i + 1 \end{bmatrix} (z) \right)^{n_i}$

Genus  $> 0$ :

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod (g_{P_i}(P))^{n_i}$

# Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

## Analogous function for genus 0

$$\tilde{F}(z, \alpha) = \frac{(z + \alpha)}{(z)(\alpha)}$$

## Analogous function for genus 0

$$\tilde{F}(z, \alpha) = \frac{(z + \alpha)}{(z)(\alpha)}$$

↓

$$\alpha \tilde{F}(z, \alpha) dz = \sum_{n=0,1} \alpha^n g^{(n)}(z) dz = dz + \alpha \frac{dz}{z}$$

## Analogous function for genus 0

$$\tilde{F}(z, \alpha) = \frac{(z + \alpha)}{(z)(\alpha)}$$

↓

$$\alpha \tilde{F}(z, \alpha) dz = \sum_{n=0,1} \alpha^n g^{(n)}(z) dz = dz + \alpha \frac{dz}{z}$$

↓

$$\begin{aligned}\tilde{F}(z_1, \alpha_1) \tilde{F}(z_2, \alpha_2) &= \tilde{F}(z_1, \alpha_1 + \alpha_2) \tilde{F}(z_2 - z_1, \alpha_2) + \tilde{F}(z_2, \alpha_1 + \alpha_2) \tilde{F}(z_1 - z_2, \alpha_1) \\ g^{(1)}(z_1) g^{(1)}(z_2) &= g^{(1)}(z_1) g^{(1)}(z_2 - z_1) + g^{(1)}(z_2) g^{(1)}(z_1 - z_2) \\ \frac{1}{(t-a)(t-b)} &= \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)}\end{aligned}$$

## Analogous function for genus 0

$$\tilde{F}(z, \alpha) = \frac{(z + \alpha)}{(z)(\alpha)}$$

↓

$$\alpha \tilde{F}(z, \alpha) dz = \sum_{n=0,1} \alpha^n g^{(n)}(z) dz = dz + \alpha \frac{dz}{z}$$

↓

$$\tilde{F}(z_1, \alpha_1) \tilde{F}(z_2, \alpha_2) = \tilde{F}(z_1, \alpha_1 + \alpha_2) \tilde{F}(z_2 - z_1, \alpha_2) + \tilde{F}(z_2, \alpha_1 + \alpha_2) \tilde{F}(z_1 - z_2, \alpha_1)$$

$$g^{(1)}(z_1) g^{(1)}(z_2) = g^{(1)}(z_1) g^{(1)}(z_2 - z_1) + g^{(1)}(z_2) g^{(1)}(z_1 - z_2)$$

$$\frac{1}{(t-a)(t-b)} = \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)}$$

↓

use differentials to calculate multiple polylogarithms



# Kronecker function

Brown and Levin 2013 Section 3.4

## Definitions of the Kronecker function

The Kronecker function  $F(z, \alpha, \tau)$  has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(z + \alpha)}{\theta_1(z)\theta_1(\alpha)}$$

2. In terms of a sum over exponentials

$$-2\pi i \left( \frac{\tilde{z}}{1 - \tilde{z}} + \frac{1}{1 - w} + \sum_{m,n>0} (\tilde{z}^m w^n - \tilde{z}^{-m} w^{-n}) q^{mn} \right), \quad \begin{pmatrix} \tilde{z} \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$

3. In terms of a sum over Eisenstein functions and series

$$\frac{1}{\alpha} \exp \left( - \sum_{j>0} \frac{(-\alpha)^j}{j} (E_j(z, \tau) - e_j(\tau)) \right)$$

# Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

*Periodicity Properties:*

$$F(z+1, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z, \alpha)$$

$$F(z+\tau, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+\tau)}{\theta_1(z+\tau)\theta_1(\alpha)} = \frac{\mathbf{e}(-z-\alpha)}{\mathbf{e}(-z)} F(z, \alpha)$$

# Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

*Periodicity Properties:*

$$F(z+1, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z, \alpha)$$

$$F(z+\tau, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+\tau)}{\theta_1(z+\tau)\theta_1(\alpha)} = \frac{\mathbf{e}(-z-\alpha)}{\mathbf{e}(-z)} F(z, \alpha)$$

The Fay identity

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

# Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

*Periodicity Properties:*

$$F(z+1, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z, \alpha)$$
$$F(z+\tau, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+\tau)}{\theta_1(z+\tau)\theta_1(\alpha)} = \frac{e(-z-\alpha)}{e(-z)} F(z, \alpha)$$

The Fay identity

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

What about higher genus?

- How can we define the Kronecker function at higher genus?

# Setup for derivation of the Fay identity

Matthes 2019

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

↓ rewrite using Theta functions ↓

$$\frac{\theta_1(z_1 + \alpha_1)\theta_1(z_2 + \alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1)\theta_1(z_2)\theta_1(\alpha_2)} = \frac{\theta_1(z_1 + \alpha_1 + \alpha_2)\theta_1(z_2 - z_1 + \alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1 + \alpha_2)\theta_1(z_2 - z_1)\theta_1(\alpha_2)} + \frac{\theta_1(z_2 + \alpha_1 + \alpha_2)\theta_1(z_1 - z_2 + \alpha_1)}{\theta_1(z_2)\theta_1(\alpha_1 + \alpha_2)\theta_1(z_1 - z_2)\theta_1(\alpha_1)}$$

↓ multiply common denominator and relabel ↓

$$\begin{aligned} &\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2 + \beta_1)\theta_1(\alpha_2 - \beta_1) + \\ &\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0 + \beta_2)\theta_1(\alpha_0 - \beta_2) + \\ &\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1 + \beta_0)\theta_1(\alpha_1 - \beta_0) = 0 \end{aligned}$$

↓ long process involving odd and even theta functions at genus 1 ↓

...

# Setup for derivation of the Fay identity

Matthes 2019

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

↓ rewrite using Theta functions ↓

$$\frac{\theta_1(z_1 + \alpha_1)\theta_1(z_2 + \alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1)\theta_1(z_2)\theta_1(\alpha_2)} = \frac{\theta_1(z_1 + \alpha_1 + \alpha_2)\theta_1(z_2 - z_1 + \alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1 + \alpha_2)\theta_1(z_2 - z_1)\theta_1(\alpha_2)} + \frac{\theta_1(z_2 + \alpha_1 + \alpha_2)\theta_1(z_1 - z_2 + \alpha_1)}{\theta_1(z_2)\theta_1(\alpha_1 + \alpha_2)\theta_1(z_1 - z_2)\theta_1(\alpha_1)}$$

↓ multiply common denominator and relabel ↓

$$\begin{aligned} &\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2 + \beta_1)\theta_1(\alpha_2 - \beta_1) + \\ &\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0 + \beta_2)\theta_1(\alpha_0 - \beta_2) + \\ &\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1 + \beta_0)\theta_1(\alpha_1 - \beta_0) = 0 \end{aligned}$$

↓ long process involving odd and even theta functions at genus 1 ↓

...

What about higher genus?

- What does the Fay identity look like at higher genus when theta functions are more complicated?

# Differentials from the Kronecker function

Broedel et al. 2015 Section 3.3.3

$$\alpha F(z, \alpha) dz = \sum_{n=0}^{\infty} g^{(n)}(z) dz \alpha^n$$

$$g^{(0)}(z) = 1$$

$$g^{(1)}(z) = \pi \cot(\pi z) + 4\pi \sum_{m=1}^{\infty} \sin(2\pi m z) \sum_{n=1}^{\infty} q^{mn}$$

$$g^{(2)}(z) = -2\zeta_2 + 8\pi^2 \sum_{m=1}^{\infty} \cos(2\pi m z) \sum_{n=1}^{\infty} n q^{mn}$$

$\vdots$

$$\boxed{g^{(n)}(-z) = (-1)^n g^{(n)}(z)}$$

## Fay identity for differentials

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

↓ decompose ↓

↓ match coefficients of  $\alpha_1^m \alpha_2^n$  ↓

↓ induction ↓

$$\begin{aligned} g^{(m)}(z_1)g^{(n)}(z_2) &= (-1)^{n+1}g^{(m+n)}(z_1 - z_2) \\ &+ \sum_{r=0}^m \binom{m+r-1}{r} g^{(m+r)}(z_1)g^{(n-r)}(z_2 - z_1) \\ &+ \sum_{r=0}^m \binom{n+r-1}{r} g^{(n+r)}(z_2)g^{(m-r)}(z_1 - z_2) \end{aligned}$$

$z_1 = t - x$  ;  $z_2 = t \implies$  repeated  $t$  dependence  $\rightarrow$  repeated  $x$  dependence



# Periodicity instead of holomorphicity

Broedel et al. 2015 Section 3.2.3

## Elliptic version of Kronecker function

$$\Omega(z, \alpha) = \mathbf{e} \left( \alpha \frac{\Im(z)}{\Im(\tau)} \right) F(z, \alpha)$$

$$\Omega(z + 1, \alpha) = \mathbf{e} \left( \alpha \frac{\Im(z + 1)}{\Im(\tau)} \right) F(z + 1, \alpha) = \Omega(z + 1, \alpha)$$

$$\Omega(z + \tau, \alpha) = \mathbf{e} \left( \alpha \frac{\Im(z + \tau)}{\Im(\tau)} \right) F(z + \tau, \alpha) = \mathbf{e}(\alpha) \mathbf{e} \left( \alpha \frac{\Im(z)}{\Im(\tau)} \right) \mathbf{e}(-\alpha) F(z, \alpha) = \Omega(z, \alpha)$$

Similarly, we find

$$\alpha \Omega(z, \alpha) = \sum_{n=0}^{\infty} f^{(n)}(z) dz \alpha^n$$

for perfectly elliptic, but non-holomorphic  $f$ .

# Independence of the differentials

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

# Independence of the differentials

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Let us assume that the first  $w$  differentials are not independent

$$\sum_{k \leq w} c_k f^{(k)}(z)dz = 0$$

# Independence of the differentials

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Let us assume that the first  $w$  differentials are not independent

$$\sum_{k \leq w} c_k f^{(k)}(z)dz = 0$$

Then, we find that

$$d\left(\sum_{k \leq w} c_k f^{(k)}(z)dz\right) = \nu \wedge \left(\sum_{k \leq w-1} c_{k+1} f^{(k)}(z)dz\right) = 0 \implies \sum_{k \leq w-1} c_k f^{(k)}(z)dz = 0$$

# Independence of the differentials

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Let us assume that the first  $w$  differentials are not independent

$$\sum_{k \leq w} c_k f^{(k)}(z)dz = 0$$

Then, we find that

$$d\left(\sum_{k \leq w} c_k f^{(k)}(z)dz\right) = \nu \wedge \left(\sum_{k \leq w-1} c_{k+1} f^{(k)}(z)dz\right) = 0 \implies \sum_{k \leq w-1} c_k f^{(k)}(z)dz = 0$$

Thus,

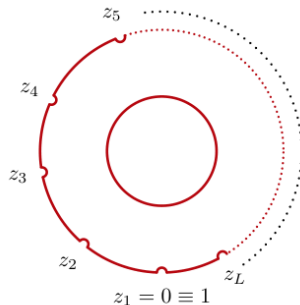
$$\sum_{k \leq w-1} c_k f^{(k)}(z)dz \neq 0 \implies \sum_{k \leq w} c_k f^{(k)}(z)dz \neq 0$$

and since  $c_0 f^{(0)}(z)dz \neq 0$ , all the differentials are independent by induction.

# Application of properties

## Properties of differentials:

- Periodic or quasi-periodic, with particular  $\tau$   
→ faithful to compact Riemann surface
- Integrability and independence  
→ suitable for homotopy-invariant integrals
- Constant ( $g^{(0)}$ ) and simple pole ( $g^{(1)}$ )  
→ constructing elliptic polylogarithms
- Fay identity  
→ rearranging dependence for integral evaluation

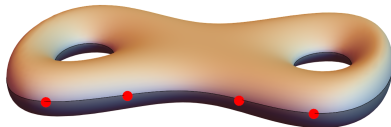
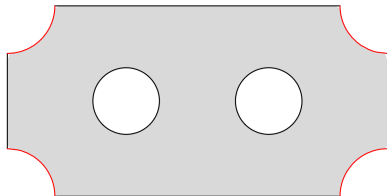
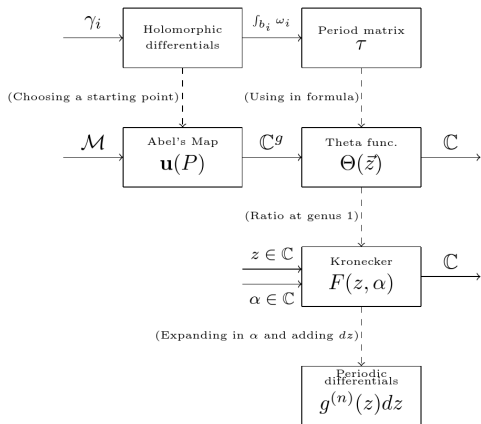


Annulus from open string  
Broedel and Kaderli 2022

# Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

# Big picture



Sketch of construction for genus 2 analogous to annulus



# Questions gathered so far

## What about higher genus?

- How do we represent the fundamental domain?
- What choice of holomorphic differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?
- How can we define the Kronecker function at higher genus?
- What does the Fay identity look like at higher genus when theta functions are more complicated?

# Schottky group

Bobenko and Klein 2011 and Chan 2022

## Schottky group

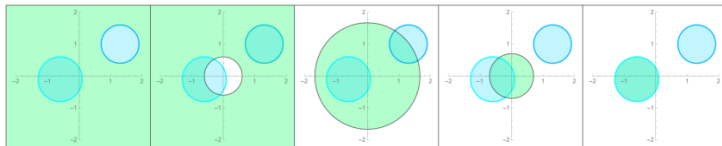
Choosing mutually disjoint discs  $\{D_i, D'_i\}$  with interiors  $\{\mathring{D}_i, \mathring{D}'_i\}$  on a Riemann sphere, we can choose mobius transformations  $\gamma_i$  such that the exterior of  $D_i$  is mapped to the interior of  $D'_i$

$$\gamma_i \in \mathrm{PSL}_2(\mathbb{C}), \quad \gamma_i : z \mapsto \frac{az + b}{cz + d}$$

$$\gamma_i(\bar{C} \setminus \mathring{D}_i) = D'_i$$

$$\gamma_i(\partial D_i) = \partial D'_i$$

The transformations formed by composition of  $\gamma_i$  form a group called a **Schottky group**, usually denoted as  $\Gamma$ .



Möbius transformations mapping outside of one disc to inside of another

Chan 2022

# Schottky cover

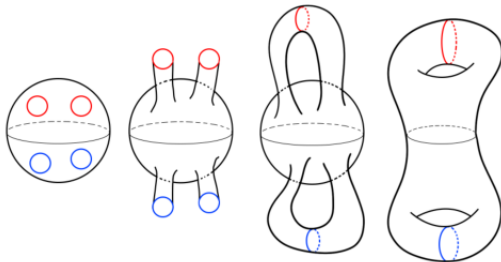
Bobenko and Klein 2011 and Chan 2022

## Schottky cover

Given a Schottky group  $\Gamma$  with associated discs  $\{D_i, D'_i\}_{i=1}^g$  we can define

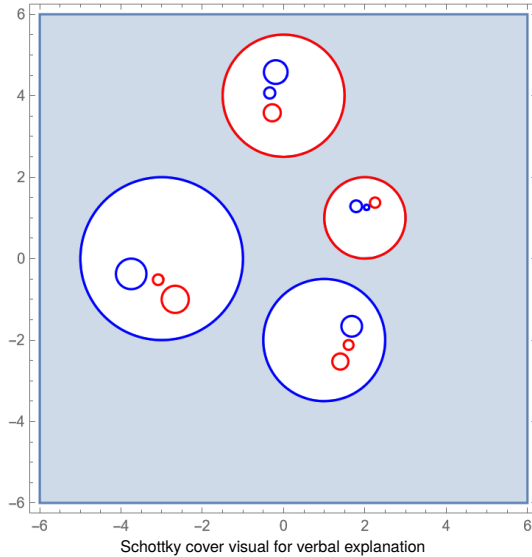
$$F := \bar{\mathbb{C}} \setminus \bigcup_i (\mathring{D}_i \cup \mathring{D}'_i) \quad ; \quad \Omega := \bigcup_{\gamma \in \Gamma} \gamma(F)$$

Then,  $\mathcal{M} := \Omega/\Gamma$  is a Riemann surface of genus  $g$  with fundamental domain  $F$ .

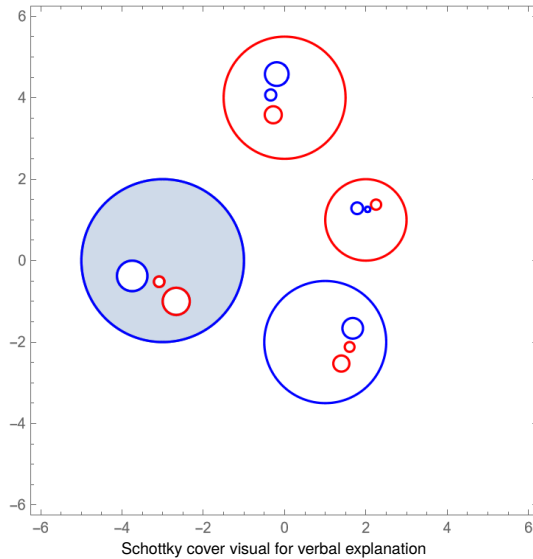


Schottky cover mapping to compact Riemann surface  
Chan 2022

# Schottky cover visual

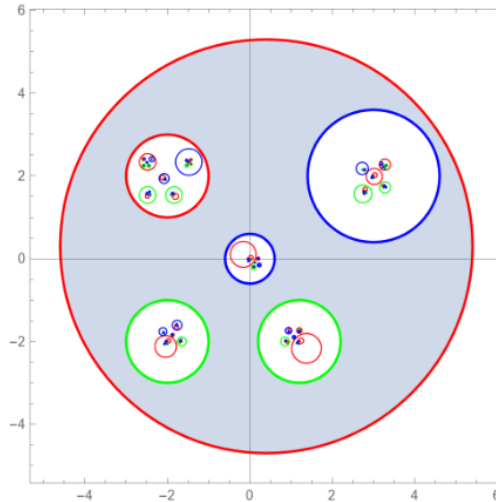


# Schottky cover visual



# Schottky cover visual

Chan 2022



Schottky cover visual for verbal explanation  
Chan 2022

# Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

We can define fixed points and cosets

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P'_i) = P'_i$$

$$\Gamma/\Gamma_i = \{\gamma_{j_1}^{n_1} \cdots \gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

# Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

We can define fixed points and cosets

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P'_i) = P'_i$$

$$\Gamma/\Gamma_i = \{\gamma_{j_1}^{n_1} \cdots \gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

And use these to define holomorphic differentials using fixed points  $P_i$

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left( \frac{1}{z - \gamma(P'_i)} - \frac{1}{z - \gamma(P_i)} \right) dz = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_i \setminus \Gamma} \left( \frac{1}{\gamma(z) - P'_i} - \frac{1}{\gamma(z) - P_i} \right) d(\gamma(z))$$



# Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

We can define fixed points and cosets

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P'_i) = P'_i$$

$$\Gamma/\Gamma_i = \{\gamma_{j_1}^{n_1} \cdots \gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

And use these to define holomorphic differentials using fixed points  $P_i$

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left( \frac{1}{z - \gamma(P'_i)} - \frac{1}{z - \gamma(P_i)} \right) dz = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_i \backslash \Gamma} \left( \frac{1}{\gamma(z) - P'_i} - \frac{1}{\gamma(z) - P_i} \right) d(\gamma(z))$$

Which can then be used to define Abel's map

$$u_i[p] = \int_{p_0}^p \omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \ln\{p, \gamma(P'_i), p_0, \gamma(P_i)\}$$

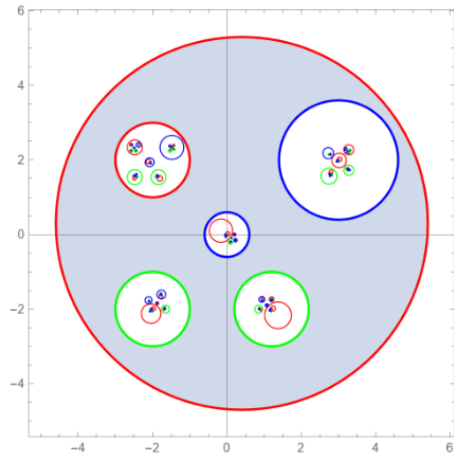
where

$$\{a, b, c, d\} = \frac{(a-b)(c-d)}{(a-d)(c-b)}$$

# Schottky differentials with visual

Bobenko and Klein 2011 and Chan 2022

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P'_i) = P'_i$$
$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left( \frac{1}{z - \gamma(P'_i)} - \frac{1}{z - \gamma(P_i)} \right) dz$$



Schottky cover visual for verbal explanation  
Chan 2022

# Attempt at a Kronecker function

Chan 2022

Focusing on three of the conditions:

1. Generalized Kronecker function should be quasi-periodic
2. Generalized Kronecker function should reduce to aforementioned genus 1 form
3. Generalized Kronecker function should satisfy integrability in a particular way

# Attempt at a Kronecker function

Chan 2022

Focusing on three of the conditions:

1. Generalized Kronecker function should be quasi-periodic
2. Generalized Kronecker function should reduce to aforementioned genus 1 form
3. Generalized Kronecker function should satisfy integrability in a particular way

$$K(z, \{w_1, \dots, w_g\} | \Gamma) = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{\gamma(z) - 1} w_1^{\text{ord}_1 \gamma} \dots w_g^{\text{ord}_g \gamma}$$

# Attempt at a Kronecker function

Chan 2022

Focusing on three of the conditions:

1. Generalized Kronecker function should be quasi-periodic
2. Generalized Kronecker function should reduce to aforementioned genus 1 form
3. Generalized Kronecker function should satisfy integrability in a particular way

$$K(z, \{w_1, \dots, w_g\} | \Gamma) = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{\gamma(z) - 1} w_1^{\text{ord}_1 \gamma} \dots w_g^{\text{ord}_g \gamma}$$

At Genus 1, for  $\gamma : z \mapsto \mathbf{e}(\tau)z = qz$

$$K(z, w | \Gamma) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n z - 1} w^n = \dots = \frac{1}{z} \left[ \frac{z}{1 - z} - \frac{1}{1 - w} - \sum_{m, n > 0} q^{mn} (z^m w^n - z^{-m} w^{-n}) \right]$$

# Attempt at a Kronecker function

Chan 2022

Focusing on three of the conditions:

1. Generalized Kronecker function should be quasi-periodic
2. Generalized Kronecker function should reduce to aforementioned genus 1 form
3. Generalized Kronecker function should satisfy integrability in a particular way

$$K(z, \{w_1, \dots, w_g\} | \Gamma) = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{\gamma(z) - 1} w_1^{\text{ord}_1 \gamma} \dots w_g^{\text{ord}_g \gamma}$$

At Genus 1, for  $\gamma : z \mapsto \mathbf{e}(\tau)z = qz$

$$K(z, w | \Gamma) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n z - 1} w^n = \dots = \frac{1}{z} \left[ \frac{z}{1 - z} - \frac{1}{1 - w} - \sum_{m, n > 0} q^{mn} (z^m w^n - z^{-m} w^{-n}) \right]$$

Recall

$$F(\xi, \alpha) = -2\pi i \left( \frac{z}{1 - z} + \frac{1}{1 - w} + \sum_{m, n > 0} q^{mn} (z^m w^n - z^{-m} w^{-n}) \right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$

# Open questions

## **Obtaining the Kronecker function in terms of theta functions**

- Choice of characters, description of theta's behavior
- Existing attempts inspiring representation with theta functions








## **More detail in Schottky cover description**

- Matching Schottky fundamental domain with usual fundamental polygon
- Alternative choices for generalized Kronecker function

## **Connection to algebraic curves**

- Mapping to other language of describing Riemann surfaces

# References

-  Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.
-  Bobenko, Alexander I. and Christian Klein (2011). *Computational Approach to Riemann Surfaces*. Lecture Notes in Mathematics. Springer Berlin Heidelberg.
-  Broedel, Johannes and Andre Kaderli (2022). “Amplitude recursions with an extra marked point”. In: *Communications in Number Theory and Physics*.
-  Broedel, Johannes et al. (2015). “Elliptic multiple zeta values and one-loop superstring amplitudes”. In: *Journal of High Energy Physics*.
-  Brown, Francis C. S. and Andrey Levin (2013). *Multiple Elliptic Polylogarithms*.
-  Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”.
-  Matthes, Nils (2019). “An algebraic characterization of the Kronecker function”. In: *Research in the Mathematical Sciences* 6.3, p. 24. ISSN: 2197-9847.



# Odd theta function at genus 1

*Jacobi Triple Product:*

$$f(x, y) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x, xy) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{(1 + xy^2)} f(x, y) = x^{-1}y^{-2} f(x, y)$$

# Odd theta function at genus 1

*Jacobi Triple Product:*

$$f(x, y) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x, xy) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{(1 + xy^2)} f(x, y) = x^{-1}y^{-2} f(x, y)$$

$$f(x, y) = \sum_{n=-\infty}^{\infty} c_n(x) y^{2n} \implies f(x, y) = xy^2 f(x, xy) = \sum_n c_n(x) x^{2n+1} y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1} c_n(x) \implies c_n(x) = c_0(x) x^{n^2} \implies f(x, y) = c_0(x) \sum_n x^{n^2} y^{2n}$$

This relates the two forms of the theta function :  $\prod_j (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2} \mathbf{e}(z)^n$

# Odd theta function at genus 1

*Jacobi Triple Product:*

$$f(x, y) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x, xy) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{(1 + xy^2)} f(x, y) = x^{-1}y^{-2} f(x, y)$$

$$f(x, y) = \sum_{n=-\infty}^{\infty} c_n(x) y^{2n} \implies f(x, y) = xy^2 f(x, xy) = \sum_n c_n(x) x^{2n+1} y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1} c_n(x) \implies c_n(x) = c_0(x) x^{n^2} \implies f(x, y) = c_0(x) \sum_n x^{n^2} y^{2n}$$

This relates the two forms of the theta function :  $\prod_j (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2} \mathbf{e}(z)^n$

What about higher genus?

- Are there similar Jacobi formulas for higher genus theta functions?