ETH zürich

D-PHYS

Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics

Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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Holomorphic Differentials

Existence of holomorphic differentials

The dimension of the space of holomorphic differentials is $\dim \mathcal{H}^1 = g$, the genus of the compact Riemann surface.

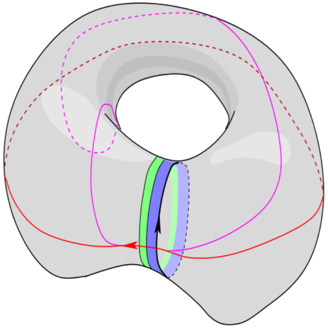
Proof outline:

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

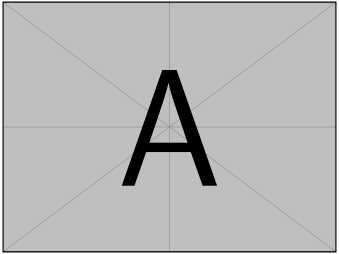
Abel's map

Bertola 2006 Section 4.2

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials ω_i, we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g, \quad P \qquad \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$



Fundamental domain
ImageSource

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

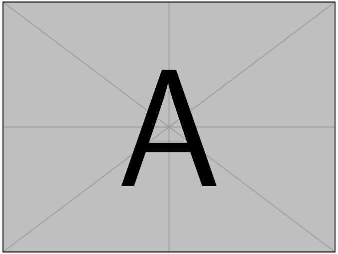
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1
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What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

Outline

Theta functions

Bertola 2006 Section 5.1

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp\left(2\pi i \left[\frac{1}{2}\vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z}\right]\right)$$

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Properties: For $\vec{\lambda}\in\mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n}\mapsto-\vec{n}}{=} \Theta(\vec{z})$$

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Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n} \mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z} + \vec{\lambda}) = \sum_{\vec{n} \in \mathbb{Z}^g} \cancelto{1}{\exp(2\pi i \vec{n}^T \vec{\lambda})} \exp(\ldots) = \Theta(\vec{z})$$

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$$\Theta(\vec{z}+\tau\vec{\lambda}) = \left[\begin{array}{c}\text{shift } \vec{n}\\ \text{use } \tau \text{ symmetry}\end{array}\right] = \exp\left(2\pi i\left[-\frac{1}{2}\vec{\lambda}^T\tau\lambda - \vec{\lambda}^T\vec{z}\right]\right)\Theta(\vec{z})$$

Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus g, with period matrix τ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

Properties:

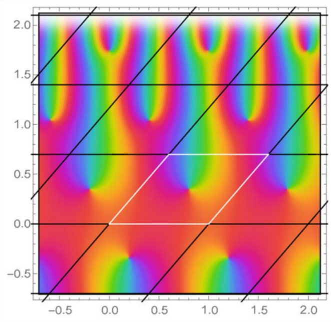
$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \exp\left(2\pi i \left[-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right]\right) \theta(P)$$

Theta function at genus 1

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(2\pi i[\frac{1}{2}n^2\tau + nz])$$

$$\theta(z) = \theta(-z)$$
$$\theta(z+1) = \theta(z)$$
$$\theta(z+\tau) = \theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

Theta function with characteristics

Bertola 2006 Section 5.1

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \exp\left(2\pi i \left[\frac{1}{8}\epsilon^T \tau \epsilon + \frac{1}{2}\epsilon^T \vec{z} + \frac{1}{4}\epsilon^T \epsilon'\right]\right) \Theta(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2})$$

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Properties:

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z} + \vec{\alpha} + \tau\vec{\beta}) = \exp\left(2\pi i\left[\frac{1}{2}(\epsilon^T\vec{\alpha} - \vec{\beta}^T\epsilon') - \frac{1}{2}\beta^T\tau\beta - \vec{\beta}\vec{z}\right]\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z})$$

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$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

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$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

Odd theta functions and zeros

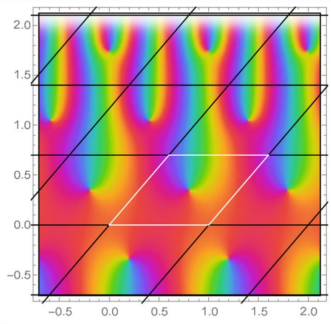
$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\lambda}' + \tau \vec{\lambda}) = 0$$

$$\implies \Theta(\frac{\epsilon'}{2} + \frac{\tau\epsilon}{2}) = 0$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix}\text{Find function } t(z) \\ \text{such that } t(0)=0\end{bmatrix} \to \begin{bmatrix} g(z) = \prod t(P-P_i)^{n_i} \\ \text{respecting possible periodicity}\end{bmatrix} \to \left(\frac{f}{g}\right) = \emptyset \to \frac{f}{g} = \text{const.}$$

Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions.

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Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus > 0:

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod \left(g_{P_i}(P)\right)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left(\theta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z)\right)^{n_i}$

Outline

Kronecker function

BL13 Section 3.4

Definitions of the Kronecker function

The Kronecker function $F(\xi, \eta, \tau)$ has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(\xi+\eta)}{\theta_1(\xi)\theta_1(\eta)} \quad , \quad \theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. In terms of a sum over exponentials

$$-2\pi i \left(\frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} (z^m w^n - z^{-m} w^{-n}) q^{mn} \right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \exp 2\pi i \begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix}$$

3. In terms of a sum over Eisenstain functions and series

$$\frac{1}{\eta} \exp \left(-\sum_{j>0} \frac{(-\eta)^j}{j} (E_j(\xi, \tau) - e_j(\tau)) \right)$$

Outline

Questions gathered so far

What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

References

- Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.
- Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”.