ETH zürich

D-PHYS

Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics

Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

Outline

Holomorphic Differentials

Existence of holomorphic differentials

The dimension of the space of holomorphic differentials is $\dim \mathcal{H}^1 = g$, the genus of the compact Riemann surface.

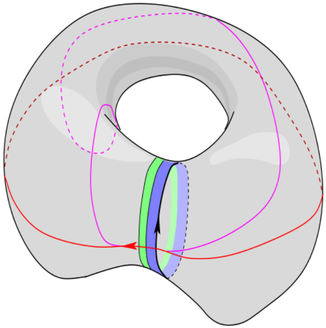
Proof outline:

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

Abel's map

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials ω_i, we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g$$
$$P \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

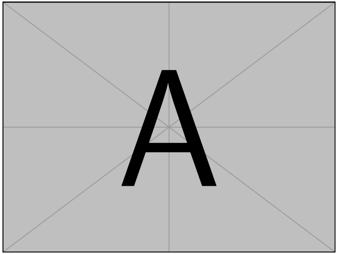
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1
ImageSource

What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

Outline

Theta functions

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z},\tau) := \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left(2\pi i\left[\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T\vec{z}\right]\right)$$

Properties: For $\vec{\lambda}\in\mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n}\mapsto-\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z}+\vec{\lambda}) = \sum_{\vec{n}\in\mathbb{Z}^g} \cancelto{1}{\exp(2\pi i\vec{n}^T\vec{\lambda})}\exp(\ldots) = \Theta(\vec{z})$$

$$\Theta(\vec{z}+\tau\vec{\lambda}) = \left[\begin{array}{c}\text{shift } \vec{n}\\ \text{use } \tau \text{ symmetry}\end{array}\right] = \exp\left(2\pi i\left[-\frac{1}{2}\vec{\lambda}^T\tau\lambda - \vec{\lambda}^T\vec{z}\right]\right)\Theta(\vec{z})$$

Theta function on a compact Riemann surface

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus g, with period matrix τ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

Properties:

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \exp\left(2\pi i \left[-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right]\right)\theta(P)$$

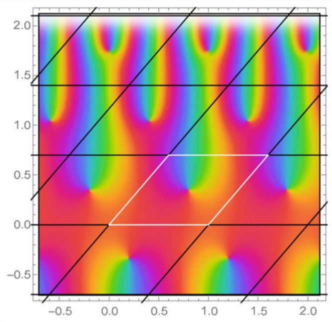
Theta function at genus 1

$$\theta(z) = \sum_{n\in\mathbb{Z}} \exp(2\pi i[\frac{1}{2}n^2\tau + nz])$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

(Application) Decomposing meromorphic functions

meromorphic

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kronecker

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motivation

References

Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.

Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”. In.