# Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics

Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

# Outline

## 1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

# Holomorphic Differentials

## Definition and existence of holomorphic differentials

Definition: $\omega = f_\alpha dz_\alpha = f_\beta dz_\beta$ , $f$ holomorphic

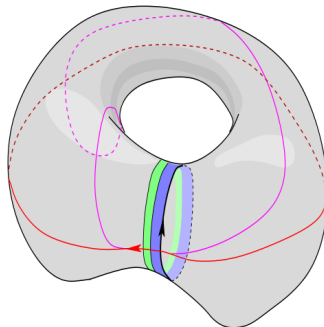Existence: $\dim \mathcal{H}^1 = g$ (genus of compact Riemann surface)

*Proof outline:*

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

*Normalization & period matrix:*

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

## Abel's map

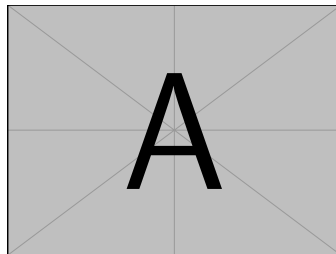Bertola 2006 Section 4.2

### Formal definition of Abel's map

For a particular choice of a point $P_0$ on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials $\omega_i$, we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g, \quad P \qquad \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$



Fundamental domain
**ImageSource**

*Analytic continuation beyond the fundamental domain:*

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$
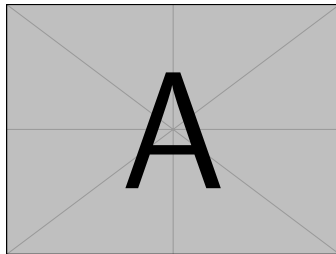
**ETH**zürich    D-PHYS

# Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1
**ImageSource**

## What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

# Outline

## Theta functions

Bertola 2006 Section 5.1

### Definition of the Theta function

Given a symmetric matrix $\tau$ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z}\right) \quad , \quad \mathbf{e}(z) = \exp(2\pi i z)$$

*Properties:* For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \stackrel{\vec{n} \mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z} + \vec{\lambda}) = \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e}(\vec{n}^T \vec{\lambda}) \overset{1}{\mathbf{e}}(\dots) = \Theta(\vec{z})$$

$$\Theta(\vec{z} + \tau \vec{\lambda}) = \begin{bmatrix} \text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry} \end{bmatrix} = \mathbf{e}\left(-\frac{1}{2}\vec{\lambda}^T \tau \lambda - \vec{\lambda}^T \vec{z}\right) \Theta(\vec{z})$$

# Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

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## Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus $g$, with period matrix $\tau$ and Abel's map $\mathbf{u}$, we can identify

$$\theta : \mathcal{M} \mapsto \mathbb{C}$$
$$P \mapsto \Theta(\mathbf{u}(P))$$

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*Properties:*

$$\theta(P + a_i) = \theta(P)$$

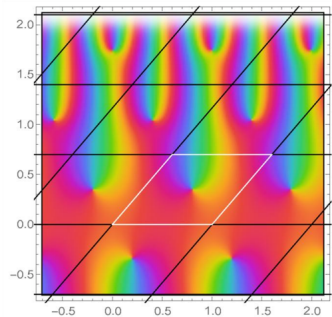$$\theta(P + b_i) = \mathbf{e}\left(-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right)\theta(P)$$

# Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e}(-\frac{1}{2}\tau - \xi)\theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

## What about higher genus?

- What does the Theta function look like at higher genus?

# Theta function with characteristics

Bertola 2006 Section 5.1

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics $\epsilon, \epsilon'$ as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \mathbf{e} \left( \frac{1}{8} \epsilon^T \tau \epsilon + \frac{1}{2} \epsilon^T \vec{z} + \frac{1}{4} \epsilon^T \epsilon' \right) \Theta \left( \vec{z} + \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right)$$

*Properties:*

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left( \frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta} \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$
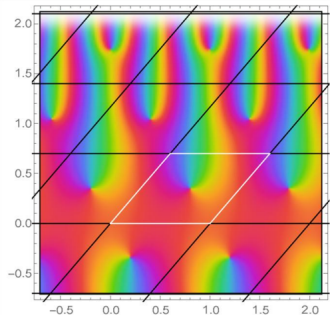
## Odd theta functions and zeros

$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0$$

$$\implies \Theta \left( \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

### What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

# Odd theta function at genus 1

## Odd theta function at genus 1

We define

$$\theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$$

It has equivalent definition

$$\theta_1(z) = 2iq^{1/8} \sin(\pi z) \prod_j (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j) \quad , \quad q = \mathbf{e}(\tau), w = \mathbf{e}z$$

*Jacobi Triple Product:*

$$f(x, y) = \prod_{j>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x, xy) = \prod_{j>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{1 + xy^2} f(x, y) = x^{-1}y^{-2} f(x, y)$$

## Odd theta function at genus 1

$$f(x,y) = \prod_{j>0}(1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x,xy) = \prod_{j>0}(1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{(1 + xy^2)}f(x,y) = x^{-1}y^{-2}f(x,y)$$

$$f(x,y) = \sum_{n=-\infty}^{\infty} c_n(x)y^{2n} \implies f(x,y) = xy^2 f(x,xy) = \sum_n c_n(x)x^{2n+1}y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1}c_n(x) \implies c_n(x) = c_0(x)x^{n^2} \implies f(x,y) = c_0(x)\sum_n x^{n^2}y^{2n}$$

This relates the two forms of the theta function : $\prod_j (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2}\mathbf{e}z^n$

### What about higher genus?

- Are there similar formulas for higher genus theta functions that make them easier to work with in some cases?

## (Application) Decomposing meromorphic functions

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix} \text{Find function } t(z) \\ \text{with known simple zero} \end{bmatrix} \to \begin{bmatrix} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity} \end{bmatrix} \to \left( \frac{f}{g} \right) = \emptyset \to \frac{f}{g} = \text{const.}$$

Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions, so extra factors can easily cancel.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus $> 0$:

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \\ \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod (g_{P_i}(P))^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left( \theta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z) \right)^{n_i}$

# Outline

# Kronecker function

Brown and Levin 2013 Section 3.4

## Definitions of the Kronecker function

The Kronecker function $F(\xi, \eta, \tau)$ has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(\xi + \eta)}{\theta_1(\xi)\theta_1(\eta)}$$

2. In terms of a sum over exponentials

$$-2\pi i \left( \frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} (z^m w^n - z^{-m} w^{-n}) q^{mn} \right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix}$$

3. In terms of a sum over Eisenstein functions and series

$$\frac{1}{\eta} \exp \left( - \sum_{j>0} \frac{(-\eta)^j}{j} (E_j(\xi, \tau) - e_j(\tau)) \right)$$

# Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

## What about higher genus?

- How can we define the Kronecker function at higher genus?

*Periodicity Properties:*

$$F(\xi + 1, \eta) = \frac{\theta_1'(0)\theta_1(\xi + \eta + 1)}{\theta_1(\xi + 1)\theta_1(\eta)} = F(\xi, \eta)$$

$$F(\xi + \tau, \eta) = \frac{\theta_1'(0)\theta_1(\xi + \eta + \tau)}{\theta_1(\xi + \tau)\theta_1(\eta)} = \frac{\mathbf{e}(-\xi - \eta)}{\mathbf{e}(-\xi)} F(\xi, \eta)$$

## The Fay Relation

$$F(\xi_1, \eta_1)F(\xi_2, \eta_2) = F(\xi_1, \eta_1 + \eta_2)F(\xi_2 - \xi_1, \eta_2) + F(\xi_2, \eta_1 + \eta_2)F(\xi_1 - \xi_2, \eta_1)$$

## Setup for derivation of the Fay relation

**Mat19**

$$F(\xi_1, \eta_1)F(\xi_2, \eta_2) = F(\xi_1, \eta_1 + \eta_2)F(\xi_2 - \xi_1, \eta_2) + F(\xi_2, \eta_1 + \eta_2)F(\xi_1 - \xi_2, \eta_1)$$

↓ rewrite using Theta functions ↓

$$\frac{\theta_1(\xi_1 + \eta_1)\theta_1(\xi_2 + \eta_2)}{\theta_1(\xi_1)\theta_1(\eta_1)\theta_1(\xi_2)\theta_1(\eta_2)} = \frac{\theta_1(\xi_1 + \eta_1 + \eta_2)\theta_1(\xi_2 - \xi_1 + \eta_2)}{\theta_1(\xi_1)\theta_1(\eta_1 + \eta_2)\theta_1(\xi_2 - \xi_1)\theta_1(\eta_2)} + \frac{\theta_1(\xi_1 + \eta_1 + \eta_2)\theta_1(\xi_1 - \xi_2 + \eta_1)}{\theta_1(\xi_1)\theta_1(\eta_1 + \eta_2)\theta_1(\xi_1 - \xi_2)\theta_1(\eta_1)}$$

↓ multiply common denominator and relabel ↓

$$\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2 + \beta_1)\theta_1(\alpha_2 - \beta_1) +$$
$$\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0 + \beta_2)\theta_1(\alpha_0 - \beta_2) +$$
$$\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1 + \beta_0)\theta_1(\alpha_1 - \beta_0) = 0$$

↓ long process involving odd and even theta functions at genus 1 ↓

...

### What about higher genus?

- What does the Fay identity look like at higher genus when theta functions are more complicated?

# Differentials from the Kronecker function

showing how the expansion is done

# Examples of differentials

[1412.5535] eqn 3.31 etc.

# Independence of the differentials

Brown and Levin (probably needs to be several slides)

# Fay relation for differentials

one line derivation one line statement

# Application of properties

big picture view of when these come into string theory

# Outline

4. Striving for higher genus

# Why we care about higher genus

Connection to string theory

## Questions gathered so far

### What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

Schottky cover definition

Schottky group example

# Differentials and theta functions

# Attempt at a Kronecker function

Chan 2022

# Open questions

# References

Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.

Brown, Francis C. S. and Andrey Levin (2013). *Multiple Elliptic Polylogarithms*.

Chan, Zhi Cong (2022). "Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers".