ETH zürich

D-PHYS

Theta Functions, Kronecker Functions, and Bilinear Relations

Artyom Lisitsyn
Riemann Surfaces
in Mathematical Physics

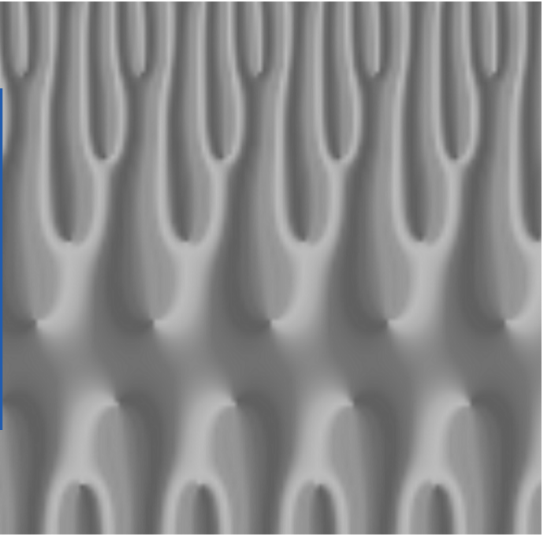
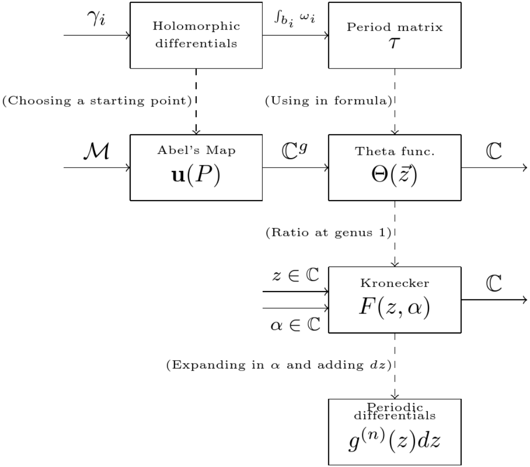


Diagram of plan



Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

Outline

Holomorphic differentials

Definition and existence of holomorphic differentials

Definition: $\omega = f_\alpha dz_\alpha = f_\beta dz_\beta, \quad f$ holomorphic

Existence: $\dim \mathcal{H}^1 = g$ (genus of compact Riemann surface)

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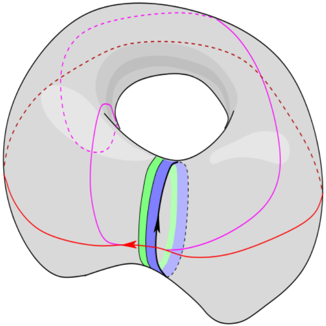
Proof outline:

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

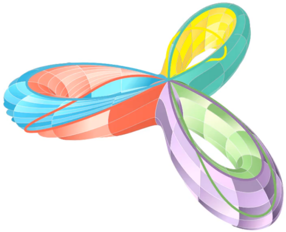
Abel's map

Bertola 2006 Section 4.2

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials ω_i, we have Abel's map

$$\mathbf{u} : \mathcal{L} \to \mathbb{C}^g, \quad P \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$



Genus 3 surface

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

Abel's map

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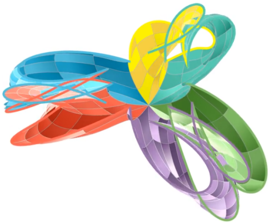

Unfolding Genus 3 Surface

Analytic continuation beyond the fundamental domain:

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$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

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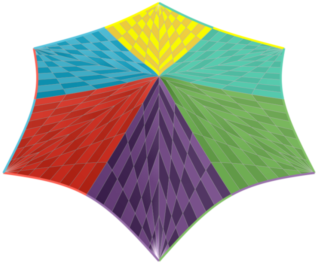


Genus 3 fundamental domain

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

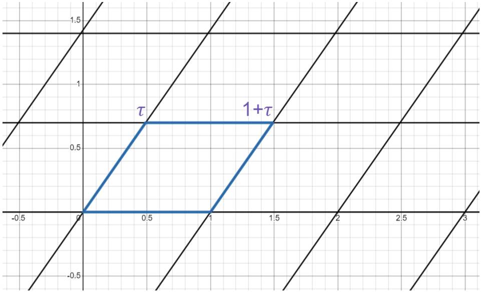
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1

Abel’s map at genus 1

Appropriate differential

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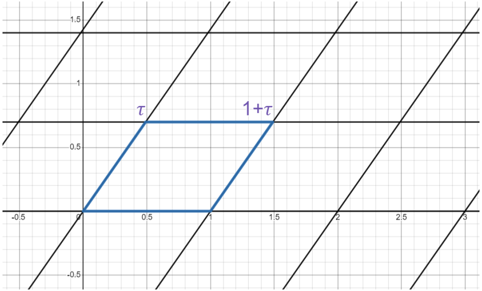

Fundamental domain and continuation at genus 1

What about higher genus?

- How do we represent the fundamental domain?
- What choice of holomorphic differentials can we make?
- What consequences does this have for Abel’s map?

Outline

Theta functions

Bertola 2006 Section 5.1

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z},\tau) := \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T\vec{z}\right), \quad \mathbf{e}(z) = \exp(2\pi i z)$$

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Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n}\mapsto-\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z}+\vec{\lambda}) = \sum_{\vec{n}\in\mathbb{Z}^g} \cancelto{1}{\mathbf{e}(\vec{n}^T\vec{\lambda})}\mathbf{e}(\ldots) = \Theta(\vec{z})$$

$$\Theta(\vec{z}+\tau\vec{\lambda}) = \begin{bmatrix}\text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry}\end{bmatrix} = \mathbf{e}\left(-\frac{1}{2}\vec{\lambda}^T\tau\lambda - \vec{\lambda}^T\vec{z}\right)\Theta(\vec{z})$$

Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus g, with period matrix τ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\to \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

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$$\begin{aligned} \theta : \mathcal{M} &\to \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

Properties:

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \mathbf{e}\left(-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right)\theta(P)$$

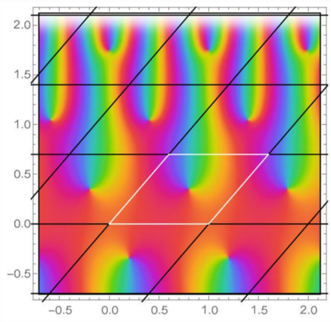
Theta function at genus 1

$$\theta(z) = \sum_{n\in\mathbb{Z}} \mathbf{e}\left(\frac{1}{2}n^2\tau + nz\right)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e}\left(-\frac{1}{2}\tau - \xi\right)\theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

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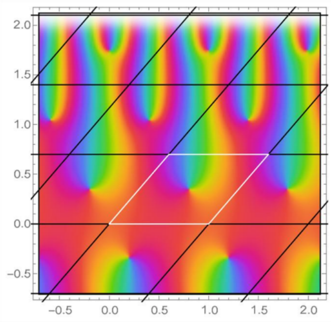

Theta function for $\tau = 0.7 + 0.6i$
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What about higher genus?

- What does the Theta function look like at higher genus?

Theta function with characteristics

Bertola 2006 Section 5.1

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e} \left(\frac{1}{2} (\vec{n} + \epsilon/2)^T \tau (\vec{n} + \epsilon/2) + (\vec{n} + \epsilon/2)^T (\vec{z} + \epsilon'/2) \right)$$

Properties:

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left(\frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta} \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

Theta function with characteristics

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Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \mathbf{e}\left(\frac{1}{8}\epsilon^T \tau \epsilon + \frac{1}{2}\epsilon^T \vec{z} + \frac{1}{4}\epsilon^T \epsilon'\right) \Theta \left(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2}\right)$$

Properties:

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau\vec{\beta}) = \mathbf{e}\left(\frac{1}{2}(\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2}\beta^T \tau \beta - \vec{\beta}\vec{z}\right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

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Odd theta functions and zeros

$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

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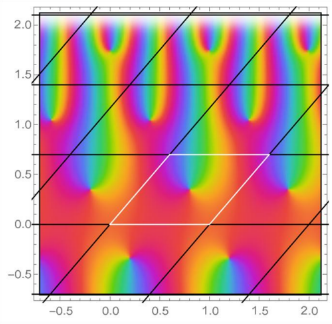
$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0$$

$$\implies \Theta \left(\frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0$$



Theta function for $\tau = 0.7 + 0.6i$
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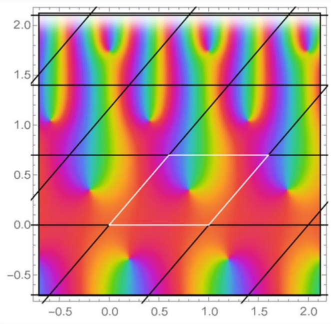

Theta function for $\tau = 0.7 + 0.6i$
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What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

Odd theta function at genus 1

Odd theta function at genus 1

We define

$$\theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$$

It has equivalent definition

$$\theta_1(z) = 2iq^{1/8} \sin(\pi z) \prod_{j>0} (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j), \quad q = \mathbf{e}(\tau), w = \mathbf{e}(z)$$

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Jacobi Triple Product:

$$f(x,y) = \prod_{m>0} (1-x^{2m})(1+x^{2m-1}y^2)(1+x^{2m-1}y^{-2})$$

$$f(x,xy) = \prod_{m>0} (1-x^{2m})(1+x^{2m+1}y^2)(1+x^{2m-3}y^{-2}) = \frac{1+x^{-1}y^{-2}}{1+xy^2} f(x,y) = x^{-1}y^{-2} f(x,y)$$

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$$f(x,y) = \sum_{n=-\infty}^{\infty} c_n(x) y^{2n} \implies f(x,y) = xy^2 f(x,xy) = \sum_n c_n(x) x^{2n+1} y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1} c_n(x) \implies c_n(x) = c_0(x) x^{n^2} \implies f(x,y) = c_0(x) \sum_n x^{n^2} y^{2n}$$

This relates the two forms of the theta function : $\prod_j (1-q^j)(1-wq^j)(1-w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2} \mathbf{e}(z)^n$

Odd theta function at genus 1

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$$f(x,xy)=\prod_{m>0}(1-x^{2m})(1+x^{2m+1}y^2)(1+x^{2m-3}y^{-2})=\frac{1+x^{-1}y^{-2}}{(1+xy^2)}f(x,y)=x^{-1}y^{-2}f(x,y)$$

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This relates the two forms of the theta function : $\prod_j(1-q^j)(1-wq^j)(1-w^{-1}q^j)\simeq\sum_n \mathbf{e}(\tau)^{n^2}\mathbf{e}(z)^n$

What about higher genus?

- Are there similar Jacobi formulas for higher genus theta functions?

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix}\text{Find function } t(z) \\ \text{with known simple zero}\end{bmatrix} \to \begin{bmatrix} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity}\end{bmatrix} \to \left(\frac{f}{g}\right) = \emptyset \to \frac{f}{g} = \text{const.}$$

Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions, so extra factors can easily cancel.

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

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- $f(z) = C \prod (z - z_i)^{n_i}$

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

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Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) =$ $C \prod \left(\theta \begin{bmatrix} a_i + 1 \\ b_i + 1 \end{bmatrix} (z) \right)^{n_i}$

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

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Genus 0:

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Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left(\theta \begin{bmatrix} a_i + 1 \\ b_i + 1 \end{bmatrix} (z) \right)^{n_i}$

Genus > 0:

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod \left(g_{P_i}(P)\right)^{n_i}$

Outline

Analogous function for Genus 0

$$\tilde{F}(z,\alpha) = \frac{(z+\alpha)}{(z)(\alpha)}$$

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$$\downarrow$$

$$\alpha\tilde{F}(z,\alpha)dz = \sum_{n=0,1} \alpha^n g^{(n)}(z)dz = dz + \alpha\frac{dz}{z}$$

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$$\downarrow$$

$$\tilde{F}(z_1,\alpha_1)\tilde{F}(z_2,\alpha_2) = \tilde{F}(z_1,\alpha_1+\alpha_2)\tilde{F}(z_2-z_1,\alpha_2) + \tilde{F}(z_2,\alpha_1+\alpha_2)\tilde{F}(z_1-z_2,\alpha_1)$$

$$g^{(1)}(z_1)g^{(1)}(z_2) = g^{(1)}(z_1)g^{(1)}(z_2-z_1) + g^{(1)}(z_2)g^{(1)}(z_1-z_2)$$

$$\frac{1}{(t-a)(t-b)} = \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)}$$

Analogous function for Genus 0

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$$\downarrow$$

$$\alpha\tilde{F}(z,\alpha)dz = \sum_{n=0,1} \alpha^n g^{(n)}(z)dz = dz + \alpha\frac{dz}{z}$$

$$\downarrow$$

$$\tilde{F}(z_1,\alpha_1)\tilde{F}(z_2,\alpha_2) = \tilde{F}(z_1,\alpha_1+\alpha_2)\tilde{F}(z_2-z_1,\alpha_2) + \tilde{F}(z_2,\alpha_1+\alpha_2)\tilde{F}(z_1-z_2,\alpha_1)$$

$$g^{(1)}(z_1)g^{(1)}(z_2) = g^{(1)}(z_1)g^{(1)}(z_2-z_1) + g^{(1)}(z_2)g^{(1)}(z_1-z_2)$$

$$\frac{1}{(t-a)(t-b)} = \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)}$$

$$\downarrow$$

use differentials to calculate multiple polylogarithms

Kronecker function

Brown and Levin 2013 Section 3.4

Definitions of the Kronecker function

The Kronecker function $F(z, \alpha, \tau)$ has equivalent definitions

1. In terms of the odd theta function
$$\frac{\theta_1'(0)\theta_1(z+\alpha)}{\theta_1(z)\theta_1(\alpha)}$$
2. In terms of a sum over exponentials
$$-2\pi i \left(\frac{\tilde{z}}{1-\tilde{z}} + \frac{1}{1-w} + \sum_{m,n>0} (\tilde{z}^m w^n - \tilde{z}^{-m} w^{-n}) q^{mn} \right), \quad \begin{pmatrix} \tilde{z} \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$
3. In terms of a sum over Eisenstein functions and series
$$\frac{1}{\alpha} \exp \left(- \sum_{j>0} \frac{(-\alpha)^j}{j} (E_j(z,\tau) - e_j(\tau)) \right)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Periodicity Properties:

$$F(z+1,\alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z,\alpha)$$

$$F(z+\tau,\alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+\tau)}{\theta_1(z+\tau)\theta_1(\alpha)} = \frac{\mathbf{e}(-z-\alpha)}{\mathbf{e}(-z)} F(z,\alpha)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

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The Fay identity

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Periodicity Properties:

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What about higher genus?

- How can we define the Kronecker function at higher genus?

Setup for derivation of the Fay identity

Matthes 2019

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

$\downarrow$ rewrite using Theta functions $\downarrow$

$$\frac{\theta_1(z_1+\alpha_1)\theta_1(z_2+\alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1)\theta_1(z_2)\theta_1(\alpha_2)} = \frac{\theta_1(z_1+\alpha_1+\alpha_2)\theta_1(z_2-z_1+\alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1+\alpha_2)\theta_1(z_2-z_1)\theta_1(\alpha_2)} + \frac{\theta_1(z_2+\alpha_1+\alpha_2)\theta_1(z_1-z_2+\alpha_1)}{\theta_1(z_2)\theta_1(\alpha_1+\alpha_2)\theta_1(z_1-z_2)\theta_1(\alpha_1)}$$

$\downarrow$ multiply common denominator and relabel $\downarrow$

$$\begin{aligned}
&\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2+\beta_1)\theta_1(\alpha_2-\beta_1)+\\
&\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0+\beta_2)\theta_1(\alpha_0-\beta_2)+\\
&\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1+\beta_0)\theta_1(\alpha_1-\beta_0) = 0
\end{aligned}$$

$\downarrow$ long process involving odd and even theta functions at genus 1 $\downarrow$

...

Setup for derivation of the Fay identity

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$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

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$$\begin{aligned}&\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2+\beta_1)\theta_1(\alpha_2-\beta_1)+\\&\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0+\beta_2)\theta_1(\alpha_0-\beta_2)+\\&\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1+\beta_0)\theta_1(\alpha_1-\beta_0)=0\end{aligned}$$

$\downarrow$ long process involving odd and even theta functions at genus 1 $\downarrow$

...

What about higher genus?

- What does the Fay identity look like at higher genus when theta functions are more complicated?

Differentials from the Kronecker function

Broedel et al. 2015 Section 3.3.3

$$\alpha F(z,\alpha)dz = \sum_{n=0}^{\infty} g^{(n)}(z) dz \alpha^n$$

$$g^{(0)}(z) = 1$$

$$g^{(1)}(z) = \pi \cot(\pi z) + 4\pi \sum_{m=1}^{\infty} \sin(2\pi m z) \sum_{n=1}^{\infty} q^{mn}$$

$$g^{(2)}(z) = -2\zeta_2 + 8\pi^2 \sum_{m=1}^{\infty} \cos(2\pi m z) \sum_{n=1}^{\infty} n q^{mn}$$

$$\vdots$$

$$\boxed{g^{(n)}(-z) = (-1)^n g^{(n)}(z)}$$

Fay identity for differentials

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

↓ decompose ↓

↓ match coefficients of $\alpha_1^m\alpha_2^n$ ↓

↓ induction ↓

$$\begin{aligned} g^{(m)}(z_1)g^{(n)}(z_2) &= (-1)^{n+1}g^{(m+n)}(z_1-z_2) \\ &+\sum_{r=0}^{m}\binom{m+r-1}{r}g^{(m+r)}(z_1)g^{(n-r)}(z_2-z_1) \\ &+\sum_{r=0}^{m}\binom{n+r-1}{r}g^{(n+r)}(z_2)g^{(m-r)}(z_1-z_2) \end{aligned}$$

$z_1 = t-x \quad ; \quad z_2 = t \implies$ repeated t dependence $\to$ repeated x dependence

Periodicity instead of holomorphicity

Broedel et al. 2015 Section 3.2.3

Elliptic version of Kronecker function

$$\Omega(z,\alpha) = \mathbf{e}\left(\alpha\frac{\Im(z)}{\Im(\tau)}\right) F(z,\alpha)$$

$$\Omega(z+1,\alpha) = \mathbf{e}\left(\alpha\frac{\Im(z+1)}{\Im(\tau)}\right) F(z+1,\alpha) = \Omega(z+1,\alpha)$$

$$\Omega(z+\tau,\alpha) = \mathbf{e}\left(\alpha\frac{\Im(z+\tau)}{\Im(\tau)}\right) F(z+1,\alpha) = \mathbf{e}(\alpha)\mathbf{e}\left(\alpha\frac{\Im(z)}{\Im(\tau)}\right)\mathbf{e}(-\alpha)F(z,\alpha) = \Omega(z,\alpha)$$

Similarly, we find

$$\alpha\Omega(z,\alpha) = \sum_{n=0}^{\infty} f^{(n)}(z)dz\alpha^n$$

for perfectly elliptic, but non-holomorphic f.

Independence of the differentials

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Independence of the differentials

Brown and Levin 2013 Lemma 8

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Let us assume that the first w differentials are not independent

$$\sum_{k \leq w} c_k f^{(k)}(z)dz = 0$$

Independence of the differentials

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$$\sum_{k\leq w} c_k f^{(k)}(z)dz = 0$$

Then, we find that

$$d\left(\sum_{k\leq w} c_k f^{(k)}(z)dz\right) = \nu \wedge \left(\sum_{k\leq w-1} c_k f^{(k)}(z)dz\right) = 0 \implies \sum_{k\leq w-1} c_k f^{(k)}(z)dz = 0$$

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Thus,

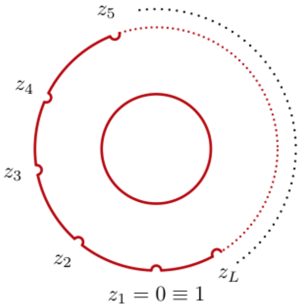
$$\sum_{k\leq w-1} c_k f^{(k)}(z)dz \neq 0 \implies \sum_{k\leq w} c_k f^{(k)}(z)dz \neq 0$$

and since $c_0 f^{(0)}(z)dz \neq 0$, all the differentials are independent by induction.

Application of properties

Properties of differentials:

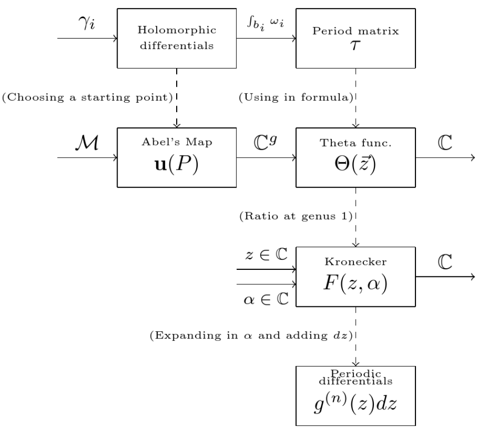
- Periodic or quasi-periodic, with particular τ
  $\rightarrow$ faithful to compact Riemann surface
- Integrability and independence
  $\rightarrow$ suitable for homotopy-invariant integrals
- Constant ($g^{(0)}$) and simple pole ($g^{(1)}$)
  $\rightarrow$ constructing elliptic polylogarithms
- Fay identity
  $\rightarrow$ rearranging dependence for integral evaluation

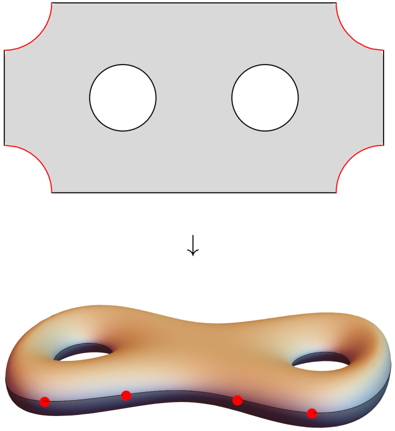


Annulus from open string
Broedel and Kaderli 2022

Outline

Big picture





Sketch of construction for genus 2 analogous to annulus

Questions gathered so far

What about higher genus?

- How do we represent the fundamental domain?
- What choice of holomorphic differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?
- Are there similar Jacobi formulas for higher genus theta functions?
- How can we define the Kronecker function at higher genus?
- What does the Fay identity look like at higher genus when theta functions are more complicated?

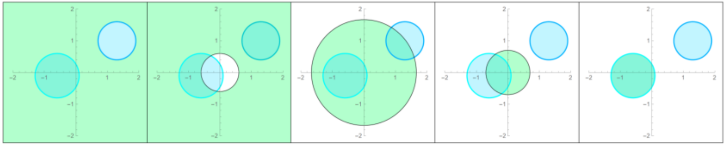
Schottky group

Choosing mutually disjoint discs $\{D_i, D_i'\}$ with interiors $\{\mathring{D}_i, \mathring{D}_i'\}$ on a Riemann sphere, we can choose mobius transformations γ_i such that the exterior of D_i is mapped to the interior of D_i'

$$\gamma_i \in \mathrm{PSL}_2(\mathbb{C}), \quad \gamma_i : z \mapsto \frac{az+b}{cz+d}$$

$$\gamma_i(\bar{C} \setminus \mathring{D}_i) = D_i'$$
$$\gamma_i(\partial D_i) = \partial D_i'$$

The transformations formed by composition of γ_i form a group called a **Schottky group**, usually denoted as Γ.



Mobius transformations mapping outside of one disc to inside of another
Chan 2022

Schottky cover

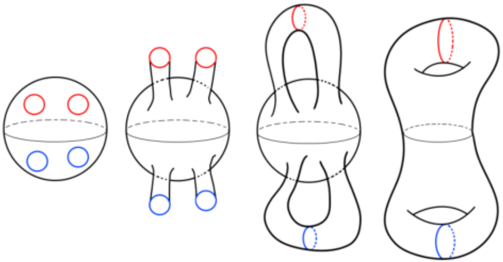
Bobenko and Klein 2011 and Chan 2022

Schottky cover

Given a Schottky group Γ with associated discs $\{D_i, D_i'\}_{i=1}^{g}$ we can define

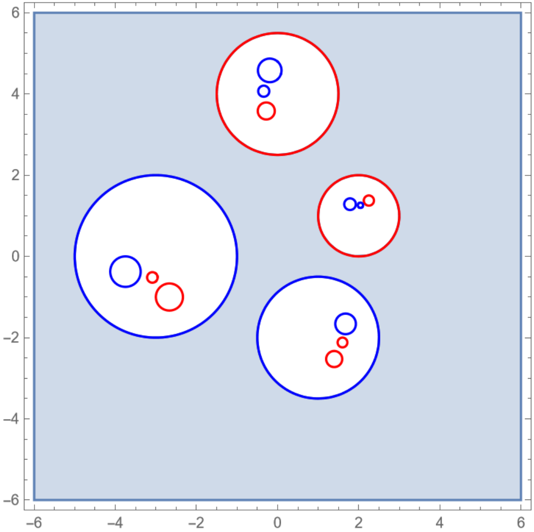
$$F := \bar{\mathbb{C}} \setminus \bigcup_i (\mathring{D}_i \cup \mathring{D}_i') \quad ; \quad \Omega := \bigcup_{\gamma \in \Gamma} \gamma(F)$$

Then, $\mathcal{M} := \Omega/\Gamma$ is a Riemann surface of genus g with fundamental domain F.



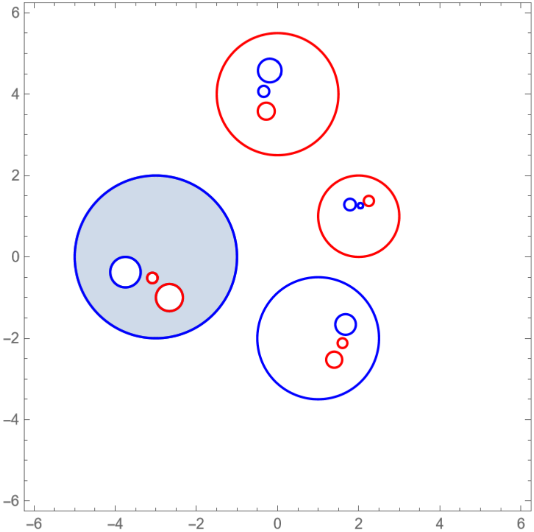
Schottky cover mapping to compact Riemann surface
Chan 2022

Schottky cover visual



Schottky cover visual for verbal explanation

Schottky cover visual



Schottky cover visual for verbal explanation

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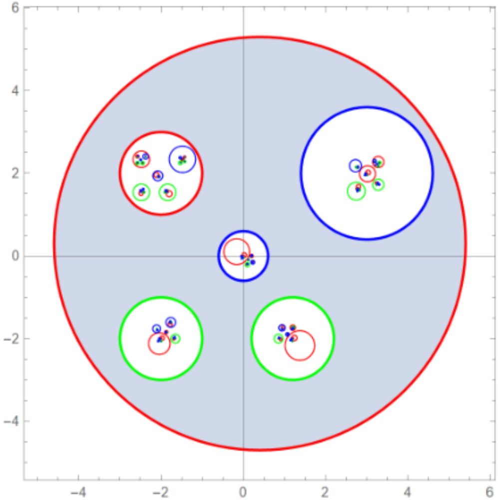
Chan 2022



Schottky cover visual for verbal explanation
Chan 2022

Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

We can define fixed points and cosets

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P_i') = P_i'$$

$$\Gamma/\Gamma_i = \{\gamma_{j_1}^{n_1} \cdots \gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

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$$\Gamma/\Gamma_i = \{\gamma_{j_1}^{n_1} \cdots \gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

And use these to define holomorphic differentials using fixed points P_i

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left(\frac{1}{z - \gamma(P_i')} - \frac{1}{z - \gamma(P_i)} \right) dz = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_i \backslash \Gamma} \left(\frac{1}{\gamma(z) - P_i'} - \frac{1}{\gamma(z) - P_i} \right) d(\gamma(z))$$

Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

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Which can then be used to define Abel's map

$$u_i[p] = \int_{p_0}^{p} \omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \ln\{p, \gamma(P_i'), p_0, \gamma(P_i)\}$$

where

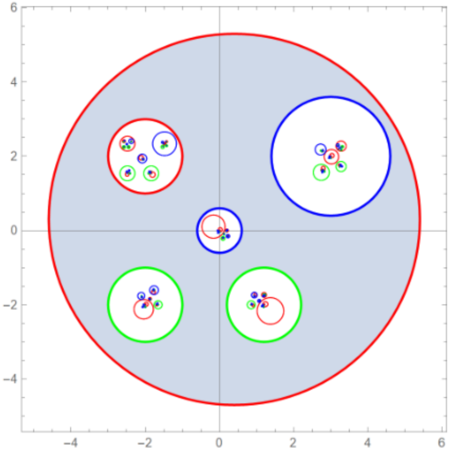
$$\{a, b, c, d\} = \frac{(a-b)(c-d)}{(a-d)(c-b)}$$

Schottky differentials with visual

Bobenko and Klein 2011 and Chan 2022

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P_i') = P_i'$$

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left(\frac{1}{z - \gamma(P_i')} - \frac{1}{z - \gamma(P_i)} \right) dz$$



Schottky cover visual for verbal explanation
Chan 2022

Attempt at a Kronecker function

Chan 2022

Focusing on three of the conditions:

1. Generalized Kronecker function should be quasi-periodic
2. Generalized Kronecker function should reduce to aforementioned genus 1 form
3. Generalized Kronecker function should satisfy integrability in a particular way

Attempt at a Kronecker function

Chan 2022

Focusing on three of the conditions:

1. Generalized Kronecker function should be quasi-periodic
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3. Generalized Kronecker function should satisfy integrability in a particular way

$$K(z, \{w_1, ..., w_g\}|\Gamma) = \sum_{\gamma\in\Gamma} \frac{\gamma'(z)}{\gamma(z)-1} w_1^{\mathrm{ord}_1 \gamma} ... w_g^{\mathrm{ord}_g \gamma}$$

Attempt at a Kronecker function

Chan 2022

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At Genus 1, for $\gamma : z \mapsto \mathbf{e}(\tau)z = qz$

$$K(z, w|\Gamma) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n z - 1} w^n = \cdots = \frac{1}{z} \left[\frac{z}{1-z} - \frac{1}{1-w} - \sum_{m,n>0} q^{mn}(z^m w^n - z^{-m} w^{-n}) \right]$$

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$$K(z, w|\Gamma) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n z - 1} w^n = \cdots = \frac{1}{z} \left[\frac{z}{1-z} - \frac{1}{1-w} - \sum_{m,n>0} q^{mn}(z^m w^n - z^{-m} w^{-n}) \right]$$

Recall

$$F(\xi, \alpha) = -2\pi i \left(\frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} q^{mn}(z^m w^n - z^{-m} w^{-n}) \right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$

Open questions

Obtaining the Kronecker function in terms of theta functions

- Choice of characters, description of theta's behavior
- Existing attempts inspiring representation with theta functions

More detail in Schottky cover description

- Matching Schottky fundamental domain with usual fundamental polygon
- Alternative choices for generalized Kronecker function

Connection to algebraic curves

- Mapping to other language of describing Riemann surfaces

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