ETH zürich

D-PHYS

Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics

Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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Holomorphic Differentials

Existence of holomorphic differentials

The dimension of the space of holomorphic differentials is $\dim \mathcal{H}^1 = g$, the genus of the compact Riemann surface.

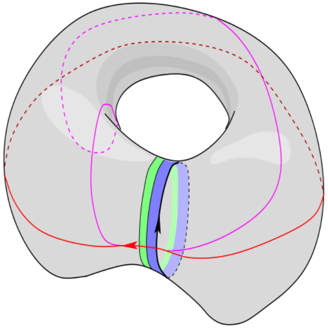
Proof outline:

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



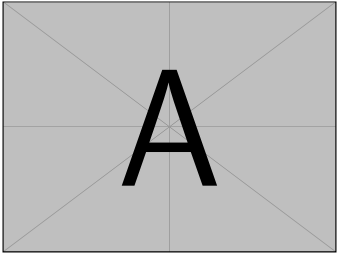
Regions used to define harmonic differentials
Ber06

Abel's map

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials ω_i, we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g, \quad P \qquad \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$



Fundamental domain
ImageSource

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

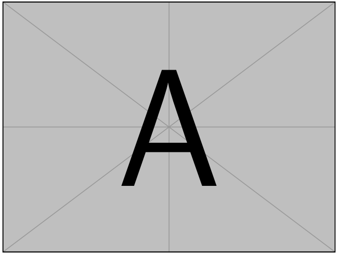
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1
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What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

Outline

Theta functions

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp\left(2\pi i \left[\frac{1}{2}\vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z}\right]\right)$$

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$$\Theta(\vec{z}+\tau\vec{\lambda}) = \left[\begin{array}{c}\text{shift } \vec{n}\\ \text{use } \tau \text{ symmetry}\end{array}\right] = \exp\left(2\pi i\left[-\frac{1}{2}\vec{\lambda}^T\tau\lambda - \vec{\lambda}^T\vec{z}\right]\right)\Theta(\vec{z})$$

Theta function on a compact Riemann surface

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus g, with period matrix τ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

Properties:

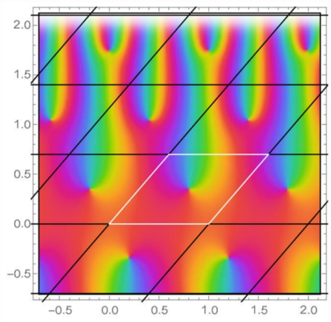
$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \exp\left(2\pi i \left[-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right]\right)\theta(P)$$

Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \exp(2\pi i[\frac{1}{2}n^2\tau + nz])$$

$$\theta(z) = \theta(-z)$$
$$\theta(z+1) = \theta(z)$$
$$\theta(z+\tau) = \theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Cha22

What about higher genus?

- What does the Theta function look like at higher genus?

Theta function with characteristics

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \exp\left(2\pi i \left[\frac{1}{8}\epsilon^T \tau \epsilon + \frac{1}{2}\epsilon^T \vec{z} + \frac{1}{4}\epsilon^T \epsilon'\right]\right) \Theta(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2})$$

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Properties:

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau\vec{\beta}) = \exp\left(2\pi i \left[\frac{1}{2}(\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2}\beta^T \tau \beta - \vec{\beta}\vec{z}\right]\right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

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$$\Theta\begin{bmatrix}\epsilon + 2\eta\\ \epsilon' + 2\eta'\end{bmatrix}(\vec{z}) = \exp(\pi i\epsilon^T\eta')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

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$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

Odd theta functions and zeros

(Application) Decomposing meromorphic functions

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix}\text{Find function } t(z) \\ \text{such that } t(0)=0\end{bmatrix} \to \begin{bmatrix} g(z) = \prod t(P-P_i)^{n_i} \\ \text{respecting possible periodicity}\end{bmatrix} \to \left(\frac{f}{g}\right) = \emptyset \to \frac{f}{g} = \text{const.}$$

At genus 0:

At genus 1

At higher genus:

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kronecker

Outline

motivation

References