

Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics



Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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Holomorphic Differentials

Existence of holomorphic differentials

The dimension of the space of holomorphic differentials is $\dim \mathcal{H}^1 = g$, the genus of the compact Riemann surface.

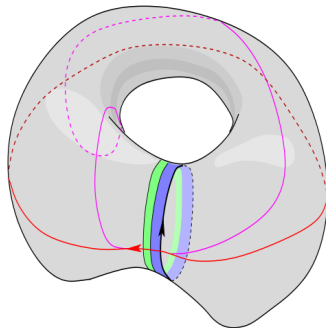
Proof outline:

- $\dim \mathcal{H}^1 \leq \# \text{ of a-cycles} = g$
- $\# \text{ of harmonic differentials} = \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

Abel's map

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain \mathcal{L} , using the normalized harmonic differentials ω_i , we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g$$
$$P \mapsto \begin{pmatrix} \int_{P_0}^P \omega_1 \\ \vdots \\ \int_{P_0}^P \omega_g \end{pmatrix}$$

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$
$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

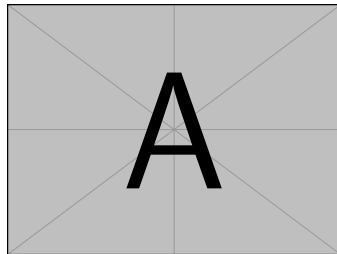
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1

ImageSource

What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

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Theta functions

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp \left(2\pi i \left[\frac{1}{2} \vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z} \right] \right)$$

Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \stackrel{\vec{n} \mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z} + \vec{\lambda}) = \sum_{\vec{n} \in \mathbb{Z}^g} \exp(2\pi i \vec{n}^T \vec{\lambda}) \exp(\dots) = \Theta(\vec{z})$$

$$\Theta(\vec{z} + \tau \vec{\lambda}) = \left[\begin{array}{c} \text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry} \end{array} \right] = \exp \left(2\pi i \left[-\frac{1}{2} \vec{\lambda}^T \tau \vec{\lambda} - \vec{\lambda}^T \vec{z} \right] \right) \Theta(\vec{z})$$

Theta function on a compact Riemann surface

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface \mathcal{M} of genus g , with period matrix τ and Abel's map \mathbf{u} , we can identify

$$\begin{aligned}\theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P))\end{aligned}$$

Properties:

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \exp\left(2\pi i \left[-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right]\right) \theta(P)$$

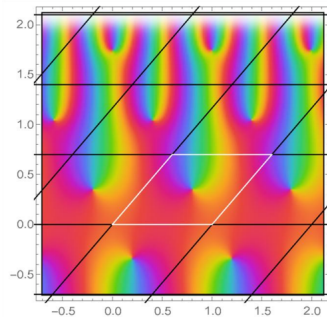
Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \exp(2\pi i [\frac{1}{2} n^2 \tau + nz])$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

(Application) Decomposing meromorphic functions and differentials

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\left[\begin{array}{l} \text{Find function } t(z) \\ \text{such that } t(0) = 0 \end{array} \right] \rightarrow \left[\begin{array}{l} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity} \end{array} \right] \rightarrow \left(\frac{f}{g} \right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

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kronecker

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motivation

References

-  Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.
-  Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”. [In.](#)