

# Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces  
in Mathematical Physics



# Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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# Holomorphic Differentials

## Definition and existence of holomorphic differentials

Definition:  $\omega = f_\alpha dz_\alpha = f_\beta dz_\beta$  ,  $f$  holomorphic

Existence:  $\dim \mathcal{H}^1 = g$  (genus of compact Riemann surface)

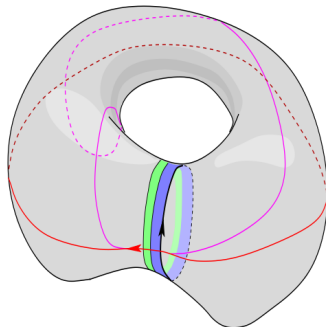
*Proof outline:*

- $\dim \mathcal{H}^1 \leq \# \text{ of a-cycles} = g$
- $\# \text{ of harmonic differentials} = \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

*Normalization & period matrix:*

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials  
Bertola 2006

# Abel's map

Bertola 2006 Section 4.2

## Formal definition of Abel's map

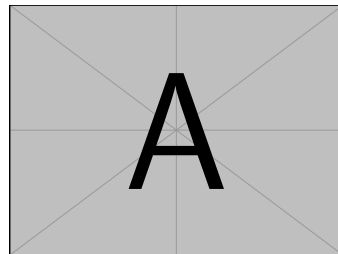
For a particular choice of a point  $P_0$  on the fundamental domain  $\mathcal{L}$ , using the normalized harmonic differentials  $\omega_i$ , we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g, \quad P \mapsto \begin{pmatrix} \int_{P_0}^P \omega_1 \\ \vdots \\ \int_{P_0}^P \omega_g \end{pmatrix}$$

*Analytic continuation beyond the fundamental domain:*

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$



Fundamental domain  
**ImageSource**

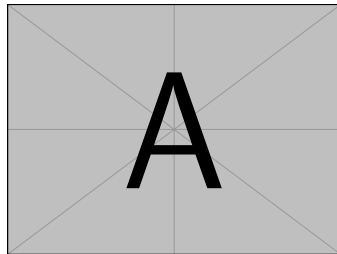
# Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1  
**ImageSource**

## What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

# Outline

1. Abel's map

2. Theta functions

3. Kronecker function

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# Theta functions

Bertola 2006 Section 5.1

## Definition of the Theta function

Given a symmetric matrix  $\tau$  with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, \tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e} \left( \frac{1}{2} \vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z} \right) \quad , \quad \mathbf{e}(z) = \exp(2\pi i z)$$

*Properties:* For  $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \stackrel{\vec{n} \mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z} + \vec{\lambda}) = \sum_{\vec{n} \in \mathbb{Z}^g} \cancel{\mathbf{e}(\vec{n}^T \vec{\lambda})} \overset{1}{\mathbf{e}(\dots)} = \Theta(\vec{z})$$

$$\Theta(\vec{z} + \tau \vec{\lambda}) = \left[ \begin{array}{c} \text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry} \end{array} \right] = \mathbf{e} \left( -\frac{1}{2} \vec{\lambda}^T \tau \vec{\lambda} - \vec{\lambda}^T \vec{z} \right) \Theta(\vec{z})$$



# Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

## Definition of Theta function on a compact Riemann surface

For a compact Riemann surface  $\mathcal{M}$  of genus  $g$ , with period matrix  $\tau$  and Abel's map  $\mathbf{u}$ , we can identify

$$\begin{aligned}\theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P))\end{aligned}$$

*Properties:*

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \mathbf{e} \left( -\frac{1}{2} \tau_{ii} - \mathbf{u}_i(P) \right) \theta(P)$$

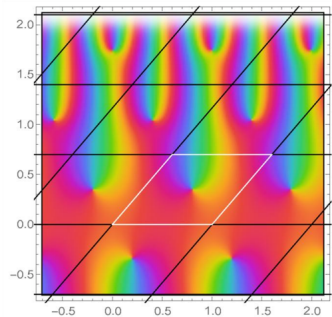
# Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \mathbf{e}\left(\frac{1}{2}n^2\tau + nz\right)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e}\left(-\frac{1}{2}\tau - \xi\right)\theta(z)$$



Theta function for  $\tau = 0.7 + 0.6i$   
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

# Theta function with characteristics

Bertola 2006 Section 5.1

## Definition of Theta function with characteristics

Consider vectors  $\epsilon, \epsilon' \in \mathbb{R}^g$ . We can then define the Theta function with characteristics  $\epsilon, \epsilon'$  as

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) := \mathbf{e} \left( \frac{1}{8} \epsilon^T \tau \epsilon + \frac{1}{2} \epsilon^T \vec{z} + \frac{1}{4} \epsilon^T \epsilon' \right) \Theta \left( \vec{z} + \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right)$$

*Properties:*

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left( \frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta}^T \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

# Odd theta functions and zeros

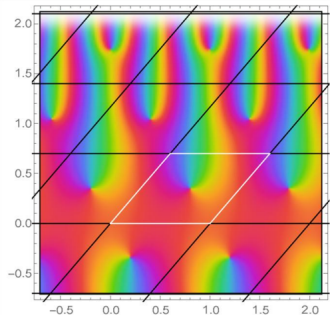
$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Rightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\Rightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0$$

$$\Rightarrow \Theta \left( \frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0$$



Theta function for  $\tau = 0.7 + 0.6i$   
Chan 2022

## What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

# Odd theta function at genus 1

## Odd theta function at genus 1

We define

$$\theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$$

It has equivalent definition

$$\theta_1(z) = 2iq^{1/8} \sin(\pi z) \prod_j (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j) \quad , \quad q = e(\tau), w = ez$$

*Jacobi Triple Product:*

$$f(x, y) = \prod_{j>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x, xy) = \prod_{j>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{1 + xy^2} f(x, y) = x^{-1}y^{-2} f(x, y)$$

# Odd theta function at genus 1

$$f(x, y) = \prod_{j>0} (1 - x^{2m}) (1 + x^{2m-1} y^2) (1 + x^{2m-1} y^{-2})$$

$$f(x, xy) = \prod_{j>0} (1 - x^{2m}) (1 + x^{2m+1} y^2) (1 + x^{2m-3} y^{-2}) = \frac{1 + x^{-1} y^{-2}}{(1 + xy^2)} f(x, y) = x^{-1} y^{-2} f(x, y)$$

$$f(x, y) = \sum_{n=-\infty}^{\infty} c_n(x) y^{2n} \implies f(x, y) = xy^2 f(x, xy) = \sum_n c_n(x) x^{2n+1} y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1} c_n(x) \implies c_n(x) = c_0(x) x^{n^2} \implies f(x, y) = c_0(x) \sum_n x^{n^2} y^{2n}$$

This relates the two forms of the theta function :  $\prod_j (1 - q^j) (1 - wq^j) (1 - w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2} \mathbf{e}z^n$

What about higher genus?

- Are there similar formulas for higher genus theta functions?

# (Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor  $(f) = \sum n_i P_i$

$$\left[ \begin{array}{c} \text{Find function } t(z) \\ \text{with known simple zero} \end{array} \right] \rightarrow \left[ \begin{array}{c} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity} \end{array} \right] \rightarrow \left( \frac{f}{g} \right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that  $\deg((f)) = \sum n_i = 0$  for meromorphic functions, so extra factors can easily cancel.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus  $> 0$ :

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod (g_{P_i}(P))^{n_i}$

Genus 1:

- Decompose  $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left( \theta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z) \right)^{n_i}$

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## Analogous function for Genus 0

$$F(z, \alpha) = \frac{(z + \alpha)}{(z)(\alpha)}$$

↓

$$\alpha F(z, \alpha) dz = \sum_{n=0,1} g^{(n)}(z) dz = dz + \frac{dz}{z}$$

↓

use differentials to calculate multiple polylogarithms

## Kronecker function at Genus 1

Elliptic version of what was shown above:

$$\frac{\theta_1(z + \alpha)}{\theta_1(z)\theta_1(\alpha)}$$

# Kronecker function

Brown and Levin 2013 Section 3.4

## Definitions of the Kronecker function

The Kronecker function  $F(z, \alpha, \tau)$  has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(z + \alpha)}{\theta_1(z)\theta_1(\alpha)}$$

2. In terms of a sum over exponentials

$$-2\pi i \left( \frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} (z^m w^n - z^{-m} w^{-n}) q^{mn} \right), \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$

3. In terms of a sum over Eisenstein functions and series

$$\frac{1}{\alpha} \exp \left( - \sum_{j>0} \frac{(-\alpha)^j}{j} (E_j(z, \tau) - e_j(\tau)) \right)$$

# Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

## What about higher genus?

- How can we define the Kronecker function at higher genus?

*Periodicity Properties:*

$$F(z+1, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z, \alpha)$$
$$F(z+\tau, \alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha+\tau)}{\theta_1(z+\tau)\theta_1(\alpha)} = \frac{e(-z-\alpha)}{e(-z)} F(z, \alpha)$$

## The Fay Relation

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

# Setup for derivation of the Fay relation

Mat19

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

↓ rewrite using Theta functions ↓

$$\frac{\theta_1(z_1 + \alpha_1)\theta_1(z_2 + \alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1)\theta_1(z_2)\theta_1(\alpha_2)} = \frac{\theta_1(z_1 + \alpha_1 + \alpha_2)\theta_1(z_2 - z_1 + \alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1 + \alpha_2)\theta_1(z_2 - z_1)\theta_1(\alpha_2)} + \frac{\theta_1(z_1 + \alpha_1 + \alpha_2)\theta_1(z_1 - z_2 + \alpha_1)}{\theta_1(z_1)\theta_1(\alpha_1 + \alpha_2)\theta_1(z_1 - z_2)\theta_1(\alpha_1)}$$

↓ multiply common denominator and relabel ↓

$$\begin{aligned} &\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2 + \beta_1)\theta_1(\alpha_2 - \beta_1) + \\ &\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0 + \beta_2)\theta_1(\alpha_0 - \beta_2) + \\ &\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1 + \beta_0)\theta_1(\alpha_1 - \beta_0) = 0 \end{aligned}$$

↓ long process involving odd and even theta functions at genus 1 ↓

...

What about higher genus?

- What does the Fay identity look like at higher genus when theta functions are more complicated?

# Differentials from the Kronecker function

$$\alpha F(z, \alpha) dz = \sum_{n=0}^{\infty} g^{(n)}(z) \alpha^n$$

$$g^{(0)}(z) = 1$$

$$g^{(1)}(z) = \pi \cot(\pi z) + 4\pi \sum_{m=1}^{\infty} \sin(2\pi m z) \sum_{n=1}^{\infty} q^{mn}$$

$$g^{(2)}(z) = -2\zeta_2 + 8\pi^2 \sum_{m=1}^{\infty} \cos(2\pi m z) \sum_{n=1}^{\infty} n q^{mn}$$

$\vdots$

# Differentials from the Kronecker function

$$\alpha F(z, \alpha) dz = \sum_{n=0}^{\infty} g^{(n)}(z) \alpha^n$$

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$$g^{(2)}(z) = -2\zeta_2 + 8\pi^2 \sum_{m=1}^{\infty} \cos(2\pi m z) \sum_{n=1}^{\infty} n q^{mn}$$

$\vdots$

$$\boxed{g^{(n)}(-z) = (-1)^n g^{(n)}(z)}$$

# Independence of the differentials

Brown and Levin (probably needs to be several slides)

## Fay relation for differentials

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

$$\left( \sum_{n=0}^{\infty} g^{(n)}(z_1) \alpha_1^n \right) \left( \sum_{n=0}^{\infty} g^{(n)}(z_2) \alpha_2^n \right) = \left( \sum_{n=0}^{\infty} g^{(n)}(z_1) (\alpha_1 + \alpha_2)^n \right) \left( \sum_{n=0}^{\infty} g^{(n)}(z_2 - z_1) \alpha_2^n \right) + \\ \left( \sum_{n=0}^{\infty} g^{(n)}(z_2) (\alpha_1 + \alpha_2)^n \right) \left( \sum_{n=0}^{\infty} g^{(n)}(z_1 - z_2) \alpha_1^n \right)$$

↓ match powers and use  $g^{(0)} = 1$

$$g^{(m)}(z_1)g^{(n)}(z_2) = (-1)^{n+1}g^{(m+n)}(z_1 - z_2) + \\ + \sum_{r=0}^n \binom{m+r-1}{m-1} g^{(n-r)}(z_2 - z_1)g^{(m+r)}(z_1) \\ + \sum_{r=0}^m \binom{n+r-1}{n-1} g^{(m-r)}(z_1 - z_2)g^{(n+r)}(z_2)$$



# Application of properties

big picture view of when these come into string theory

# Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

# Why we care about higher genus

Connection to string theory

# Questions gathered so far

## What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

# Schottky cover definition

# Schottky group example

# Differentials and theta functions




# Attempt at a Kronecker function

Chan 2022



# Open questions

# References

-  Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.
-  Brown, Francis C. S. and Andrey Levin (2013). *Multiple Elliptic Polylogarithms*.
-  Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”.