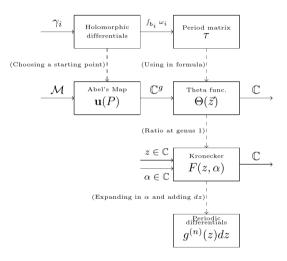


Theta Functions, Kronecker Functions, and Bilinear Relations

Artyom Lisitsyn Riemann Surfaces in Mathematical Physics

Diagram of plan



Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus



Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

Holomorphic differentials

Definition and existence of holomorphic differentials

Definition: $\omega = f_{\alpha}dz_{\alpha} = f_{\beta}dz_{\beta}$, f holomorphic

Existence: $\dim \mathcal{H}^1 = q$ (genus of compact Riemann surface)

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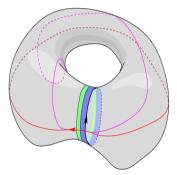
Proof outline:

- dim \mathcal{H}^1 < # of a-cycles = q
- # of harmonic differentials = $\dim H \ge 2g$
- $h = fdz + gd\bar{z} \implies \dim H = 2\dim \mathcal{H}^1$
- $q < \dim \mathcal{H}^1 < q \implies \dim \mathcal{H}^1 = q$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = au_{ij}$$



Regions used to define harmonic differentials Bertola 2006

Abel's map

Bertola 2006 Section 4.2

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain \mathcal{L} , using the normalized harmonic differentials ω_i , we have Abel's map

$$\mathbf{u}: \mathcal{L} \to \mathbb{C}^g, \quad P \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$



Genus 3 surface

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P+a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$
$$\mathbf{u}(P+b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

FTH zürich

D.PHYS

Abel's map

Bertola 2006 Section 4.2

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain L, using the normalized harmonic differentials ω_i , we have Abel's map

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Unfolding Genus 3 Surface

Analytic continuation beyond the fundamental domain:

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FTH zürich

D.PHVS

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Bertola 2006 Section 4.2

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Analytic continuation beyond the fundamental domain:



Genus 3 fundamental domain

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$$\mathbf{u}(P+b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

D.PHVS

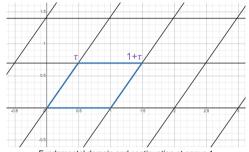
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1

Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1

What about higher genus?

- How do we represent the fundamental domain?
- What choice of holomorphic differentials can we make?
- What consequences does this have for Abel's map?

Outline

1. Abel's map

2. Theta functions

3. Kronecker function

4. Striving for higher genus

Theta functions

Bertola 2006 Section 5.1

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z}, au) := \sum_{\vec{n} \in \mathbb{Z}^q} \mathbf{e} \left(\frac{1}{2} \vec{n}^T au \vec{n} + \vec{n}^T \vec{z} \right), \quad \mathbf{e}(z) = \exp(2\pi i z)$$

Theta functions

Bertola 2006 Section 5.1

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z},\tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e} \left(\frac{1}{2} \vec{n}^T \tau \vec{n} + \vec{n}^T \vec{z} \right), \quad \mathbf{e}(z) = \exp(2\pi i z)$$

Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\begin{split} \Theta(-\vec{z}) &\overset{\vec{n}\mapsto -\vec{n}}{=} \Theta(\vec{z}) \\ \Theta(\vec{z}+\vec{\lambda}) &= \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}(\vec{n}^T\vec{\lambda}) \mathbf{e}^{(1)}(\ldots) = \Theta(\vec{z}) \\ \Theta(\vec{z}+\tau\vec{\lambda}) &= \begin{bmatrix} \mathrm{shift} \ \vec{n} \\ \mathrm{use} \ \tau \ \mathrm{symmetry} \end{bmatrix} = \mathbf{e} \left(-\frac{1}{2}\vec{\lambda}^T \tau \lambda - \vec{\lambda}^T \vec{z}\right) \Theta(\vec{z}) \end{split}$$

Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal M$ of genus g, with period matrix τ and Abel's map $\mathbf u$, we can identify

$$\theta: \mathcal{M} \to \mathbb{C}$$

$$P \mapsto \Theta(\mathbf{u}(P))$$

Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface \mathcal{M} of genus g, with period matrix τ and Abel's map \mathbf{u} , we can identify

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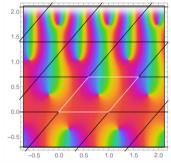
Properties:

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P+b_i) = \mathbf{e}\left(-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right)\theta(P)$$

Theta function at genus 1

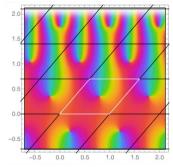
$$\begin{split} \theta(z) &= \sum_{n \in \mathbb{Z}} \mathbf{e} \left(\frac{1}{2} n^2 \tau + nz \right) \\ \theta(z) &= \theta(-z) \\ \theta(z+1) &= \theta(z) \\ \theta(z+\tau) &= \mathbf{e} \left(-\frac{1}{2} \tau - z \right) \theta(z) \end{split}$$



Theta function for $\tau = 0.7 + 0.6i$ Chan 2022

Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \mathbf{e} \left(\frac{1}{2} n^2 \tau + nz \right)$$
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Theta function for $\tau = 0.7 + 0.6i$ Chan 2022

What about higher genus?

• What does the Theta function look like at higher genus?

Theta function with characteristics

Bertola 2006 Section 5.1

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix}(\vec{z},\tau) := \sum_{\vec{n} \in \mathbb{Z}^g} \mathbf{e} \left(\frac{1}{2} (\vec{n} + \epsilon/2)^T \tau (\vec{n} + \epsilon/2) + (\vec{n} + \epsilon/2)^T (\vec{z} + \epsilon'/2) \right)$$

Properties:

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left(\frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta} \vec{z} \right) \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \alpha, \beta \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

Theta function with characteristics

Bertola 2006 Section 5.1

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta\begin{bmatrix}\epsilon\\\epsilon'\end{bmatrix}(\vec{z}) := \mathbf{e}\left(\frac{1}{8}\epsilon^T\tau\epsilon + \frac{1}{2}\epsilon^T\vec{z} + \frac{1}{4}\epsilon^T\epsilon'\right)\Theta\left(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2}\right)$$

Properties:

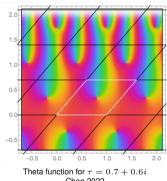
$$\Theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z} + \vec{\alpha} + \tau \vec{\beta}) = \mathbf{e} \left(\frac{1}{2} (\epsilon^T \vec{\alpha} - \vec{\beta}^T \epsilon') - \frac{1}{2} \beta^T \tau \beta - \vec{\beta} \vec{z} \right) \Theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\Theta\begin{bmatrix} \epsilon + 2\eta \\ \epsilon' + 2\eta' \end{bmatrix} (\vec{z}) = \exp(\pi i \epsilon^T \eta') \Theta\begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}), \quad \eta, \eta' \in \mathbb{Z}^g$$

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Odd theta functions and zeros

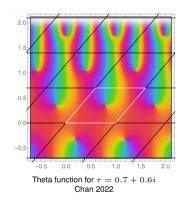
$$\begin{split} \epsilon, \epsilon' &\in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd} \\ \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) &= \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \\ &\Longrightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) &= -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) \\ &\Longrightarrow \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) &= \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) &= 0 \\ &\Longrightarrow \Theta \left(\frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) &= 0 \end{split}$$



Chan 2022

Odd theta functions and zeros

$$\begin{split} & \epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd} \\ & \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) \\ & \Longrightarrow \ \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) \\ & \Longrightarrow \ \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0 \\ & \Longrightarrow \ \Theta \left(\frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0 \end{split}$$



What about higher genus?

 Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

Odd theta function at genus 1

Odd theta function at genus 1

We define

$$\theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}(z)$$

It has equivalent definition

$$\theta_1(z) = 2iq^{1/8}\sin(\pi z)\prod_{j>0}(1-q^j)(1-wq^j)(1-w^{-1}q^j), \quad q = \mathbf{e}(\tau), w = \mathbf{e}(z)$$

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix} \text{Find function } t(P,P') \\ \text{with simple zero at } P = P' \end{bmatrix} \rightarrow \begin{bmatrix} g(P) = \prod t(P,P_i)^{n_i} \\ \text{respecting possible periodicity} \end{bmatrix} \rightarrow \left(\frac{f}{g}\right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

Recall that $deg((f)) = \sum n_i = 0$ for meromorphic functions, so extra factors can easily cancel.

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

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Genus 0:

•
$$f(z) = C \prod (z - z_i)^{n_i}$$

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

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Genus 0:

• $f(z) = C \prod (z - z_i)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- \bullet f(z) = $C \prod \left(\theta \begin{bmatrix} a_i + 1 \\ b_i + 1 \end{bmatrix} (z)\right)^{n_i}$

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix} \text{Find function } t(P,P') \\ \text{with simple zero at } P = P' \end{bmatrix} \rightarrow \begin{bmatrix} g(P) = \prod t(P,P_i)^{n_i} \\ \text{respecting possible periodicity} \end{bmatrix} \rightarrow \left(\frac{f}{g}\right) = \emptyset \rightarrow \frac{f}{g} = \text{const.}$$

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Genus 0:

• $f(z) = C \prod (z - z_i)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- \bullet f(z) = $C \prod \left(\theta \begin{bmatrix} a_i + 1 \\ b_i + 1 \end{bmatrix}(z)\right)^{n_i}$

Genus > 0:

- \bullet $\Theta(\xi) = 0$
- \bullet $q_{P'}: P \mapsto$ $\Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod (q_{P_i}(P))^{n_i}$

Outline

1. Abel's map

2. Theta functions

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4. Striving for higher genus

$$\tilde{F}(z,\alpha) = \frac{(z+\alpha)}{(z)(\alpha)}$$

$$\tilde{F}(z,\alpha) = \frac{(z+\alpha)}{(z)(\alpha)}$$

$$\downarrow$$

$$\alpha \tilde{F}(z,\alpha) dz = \sum_{z=1}^{\infty} \alpha^n g^{(n)}(z) dz = dz + \alpha \frac{dz}{z}$$

$$\tilde{F}(z,\alpha) = \frac{(z+\alpha)}{(z)(\alpha)}$$

$$\downarrow$$

$$\alpha \tilde{F}(z,\alpha) dz = \sum_{n=0,1} \alpha^n g^{(n)}(z) dz = dz + \alpha \frac{dz}{z}$$

$$\downarrow$$

$$\tilde{F}(z_1,\alpha_1) \tilde{F}(z_2,\alpha_2) = \tilde{F}(z_1,\alpha_1+\alpha_2) \tilde{F}(z_2-z_1,\alpha_2) + \tilde{F}(z_2,\alpha_1+\alpha_2) \tilde{F}(z_1-z_2,\alpha_1)$$

$$g^{(1)}(z_1) g^{(1)}(z_2) = g^{(1)}(z_1) g^{(1)}(z_2-z_1) + g^{(1)}(z_2) g^{(1)}(z_1-z_2)$$

$$\frac{1}{(t-a)(t-b)} = \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)}$$

$$\tilde{F}(z,\alpha) = \frac{(z+\alpha)}{(z)(\alpha)}$$

$$\downarrow$$

$$\alpha \tilde{F}(z,\alpha) dz = \sum_{n=0,1} \alpha^n g^{(n)}(z) dz = dz + \alpha \frac{dz}{z}$$

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$$g^{(1)}(z_1) g^{(1)}(z_2) = g^{(1)}(z_1) g^{(1)}(z_2-z_1) + g^{(1)}(z_2) g^{(1)}(z_1-z_2)$$

$$\frac{1}{(t-a)(t-b)} = \frac{1}{(t-a)(a-b)} + \frac{1}{(t-b)(b-a)}$$

$$\downarrow$$

use differentials to calculate multiple polylogarithms

Kronecker function

Brown and Levin 2013 Section 3.4

Definitions of the Kronecker function

The Kronecker function $F(z, \alpha, \tau)$ has equivalent definitions

1. In terms of the odd theta function

$$\frac{\theta_1'(0)\theta_1(z+\alpha)}{\theta_1(z)\theta_1(\alpha)}$$

2. In terms of a sum over exponentials

$$-2\pi i \left(\frac{\tilde{z}}{1-\tilde{z}} + \frac{1}{1-w} + \sum_{m,n>0} (\tilde{z}^m w^n - \tilde{z}^{-m} w^{-n}) q^{mn} \right), \quad \begin{pmatrix} \tilde{z} \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$

In terms of a sum over Eisenstein functions and series

$$\frac{1}{\alpha} \exp \left(-\sum_{j>0} \frac{(-\alpha)^j}{j} (E_j(z,\tau) - e_j(\tau)) \right)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Periodicity Properties:

$$F(z+1,\alpha) = \frac{\theta'_1(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z,\alpha)$$
$$F(z+\tau,\alpha) = \frac{\theta'_1(0)\theta_1(z+\alpha+\tau)}{\theta_1(z+\tau)\theta_1(\alpha)} = \frac{\mathbf{e}(-z-\alpha)}{\mathbf{e}(-z)}F(z,\alpha)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Periodicity Properties:

$$F(z+1,\alpha) = \frac{\theta'_1(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z,\alpha)$$

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The Fay identity

$$F(z_1, \alpha_1)F(z_2, \alpha_2) = F(z_1, \alpha_1 + \alpha_2)F(z_2 - z_1, \alpha_2) + F(z_2, \alpha_1 + \alpha_2)F(z_1 - z_2, \alpha_1)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

Periodicity Properties:

$$F(z+1,\alpha) = \frac{\theta'_1(0)\theta_1(z+\alpha+1)}{\theta_1(z+1)\theta_1(\alpha)} = F(z,\alpha)$$
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What about higher genus?

• How can we define the Kronecker function at higher genus?

Setup for derivation of the Fav identity

Matthes 2019

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

$$\downarrow \text{ rewrite using Theta functions } \downarrow$$

$$\frac{\theta_1(z_1+\alpha_1)\theta_1(z_2+\alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1)\theta_1(z_2)\theta_1(\alpha_2)} = \frac{\theta_1(z_1+\alpha_1+\alpha_2)\theta_1(z_2-z_1+\alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1+\alpha_2)\theta_1(z_2-z_1)\theta_1(\alpha_2)} + \frac{\theta_1(z_2+\alpha_1+\alpha_2)\theta_1(z_1-z_2+\alpha_1)}{\theta_1(z_2)\theta_1(\alpha_1+\alpha_2)\theta_1(z_1-z_2)\theta_1(\alpha_1)}$$

$$\downarrow \text{ multiply common denominator and relabel } \downarrow$$

$$\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2+\beta_1)\theta_1(\alpha_2-\beta_1) +$$

$$\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0+\beta_2)\theta_1(\alpha_0-\beta_2) +$$

$$\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1+\beta_0)\theta_1(\alpha_1-\beta_0) = 0$$

 \downarrow long process involving odd and even theta functions at genus 1 \downarrow

Setup for derivation of the Fay identity

Matthes 2019

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1)$$

$$\downarrow \text{ rewrite using Theta functions } \downarrow$$

$$\frac{\theta_1(z_1+\alpha_1)\theta_1(z_2+\alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1)\theta_1(z_2)\theta_1(\alpha_2)} = \frac{\theta_1(z_1+\alpha_1+\alpha_2)\theta_1(z_2-z_1+\alpha_2)}{\theta_1(z_1)\theta_1(\alpha_1+\alpha_2)\theta_1(z_2-z_1)\theta_1(\alpha_2)} + \frac{\theta_1(z_2+\alpha_1+\alpha_2)\theta_1(z_1-z_2+\alpha_1)}{\theta_1(z_2)\theta_1(\alpha_1+\alpha_2)\theta_1(z_1-z_2)\theta_1(\alpha_1)}$$

$$\downarrow \text{ multiply common denominator and relabel } \downarrow$$

$$\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2+\beta_1)\theta_1(\alpha_2-\beta_1) + \theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0+\beta_2)\theta_1(\alpha_0-\beta_2) + \theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1+\beta_0)\theta_1(\alpha_1-\beta_0) = 0$$

 \downarrow long process involving odd and even theta functions at genus 1 \downarrow

...

What about higher genus?

• What does the Fay identity look like at higher genus when theta functions are more complicated?

Differentials from the Kronecker function

Broedel et al. 2015 Section 3.3.3

$$\alpha F(z,\alpha)dz = \sum_{n=0}^{\infty} g^{(n)}(z)dz\alpha^{n}$$

$$g^{(0)}(z) = 1$$

$$g^{(1)}(z) = \pi \cot(\pi z) + 4\pi \sum_{m=1}^{\infty} \sin(2\pi m z) \sum_{n=1}^{\infty} q^{mn}$$

$$g^{(2)}(z) = -2\zeta_{2} + 8\pi^{2} \sum_{m=1}^{\infty} \cos(2\pi m z) \sum_{n=1}^{\infty} nq^{mn}$$

$$\vdots$$

$$g^{(n)}(-z) = (-1)^{n} g^{(n)}(z)$$

Fay identity for differentials

$$\begin{split} F(z_1,\alpha_1)F(z_2,\alpha_2) &= F(z_1,\alpha_1+\alpha_2)F(z_2-z_1,\alpha_2) + F(z_2,\alpha_1+\alpha_2)F(z_1-z_2,\alpha_1) \\ &\quad \quad \downarrow \text{ decompose } \downarrow \\ &\quad \quad \downarrow \text{ match coefficients of } \alpha_1^m \alpha_2^n \downarrow \\ &\quad \quad \downarrow \text{ induction } \downarrow \\ &\quad \quad g^{(m)}(z_1)g^{(n)}(z_2) = (-1)^{n+1}g^{(m+n)}(z_1-z_2) \\ &\quad \quad + \sum_{r=0}^m \binom{m+r-1}{r}g^{(m+r)}(z_1)g^{(n-r)}(z_2-z_1) \\ &\quad \quad + \sum_{r=0}^m \binom{n+r-1}{r}g^{(n+r)}(z_2)g^{(m-r)}(z_1-z_2) \end{split}$$

 $z_1 = t - x$; $z_2 = t \implies$ repeated t dependence \rightarrow repeated x dependence

2023

Periodicity instead of holomorphicity

Broedel et al. 2015 Section 3.2.3

Elliptic version of Kronecker function

$$\Omega(z,\alpha) = \mathbf{e}\left(\alpha \frac{\Im(z)}{\Im(\tau)}\right) F(z,\alpha)$$

$$\Omega(z+1,\alpha) = \mathbf{e}\left(\alpha \frac{\Im(z+1)}{\Im(\tau)}\right) F(z+1,\alpha) = \Omega(z+1,\alpha)$$

 $\Omega(z+\tau,\alpha) = \mathbf{e}\left(\alpha \frac{\Im(z+\tau)}{\Im(\tau)}\right) F(z+\tau,\alpha) = \mathbf{e}(\alpha) \mathbf{e}\left(\alpha \frac{\Im(z)}{\Im(\tau)}\right) \mathbf{e}(-\alpha) F(z,\alpha) = \Omega(z,\alpha)$

Similarly, we find

$$\alpha\Omega(z,\alpha) = \sum_{n=0}^{\infty} f^{(n)}(z)dz\alpha^n$$

for perfectly elliptic, but non-holomorphic f.

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Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Let us assume that the first w differentials are not independent

$$\sum_{k \le w} c_k f^{(k)}(z) dz = 0$$

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

Let us assume that the first w differentials are not independent

$$\sum_{k \le w} c_k f^{(k)}(z) dz = 0$$

Then, we find that

$$d\left(\sum_{k \le w} c_k f^{(k)}(z) dz\right) = \nu \wedge \left(\sum_{k \le w - 1} c_{k+1} f^{(k)}(z) dz\right) = 0 \implies \sum_{k \le w - 1} c_k f^{(k)}(z) dz = 0$$

Brown and Levin 2013 Lemma 8

$$d(f^{(k+1)}(z)dz) = \nu \wedge (f^{(k)}(z)dz), \quad \nu = 2\pi i \cdot d\left(\frac{\Im(z)}{\Im(\tau)}\right)$$

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Thus.

$$\sum_{k \le w-1} c_k f^{(k)}(z) dz \neq 0 \implies \sum_{k \le w} c_k f^{(k)}(z) dz \neq 0$$

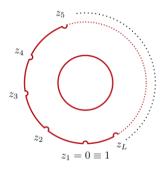
and since $c_0 f^{(0)}(z) dz \neq 0$, all the differentials are independent by induction.

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Application of properties

Properties of differentials:

- Periodic or quasi-periodic, with particular au \to faithful to compact Riemann surface
- Constant $(g^{(0)})$ and simple pole $(g^{(1)})$ \rightarrow constructing elliptic polylogarithms
- Fay identity
 rearranging dependence for integral evaluation



Annulus from open string Broedel and Kaderli 2022

Outline

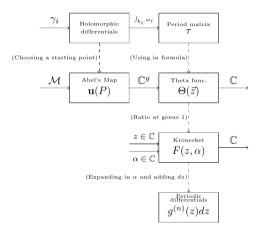
1. Abel's map

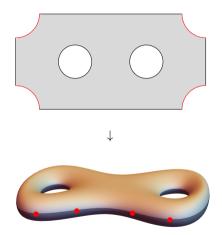
2. Theta functions

3. Kronecker function

4. Striving for higher genus

Big picture





Sketch of construction for genus 2 analogous to annulus

Questions gathered so far

What about higher genus?

- How do we represent the fundamental domain?
- What choice of holomorphic differentials can we make?
- What consequences does this have for Abel's map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?
- How can we define the Kronecker function at higher genus?
- What does the Fay identity look like at higher genus when theta functions are more complicated?

Schottky group

Robenko and Klein 2011 and Chan 2022

Schottky group

Choosing mutually disjoint discs $\{D_i, D_i'\}$ with interiors $\{\mathring{D}_i, \mathring{D}_i'\}$ on a Riemann sphere, we can choose mobius transformations γ_i such that the exterior of D_i is mapped to the interior of D_i'

$$\gamma_i \in \mathsf{PSL}_2(\mathbb{C}), \quad \gamma_i : z \mapsto \frac{az+b}{cz+d}$$

$$\gamma_i(\bar{C} \setminus \mathring{D}_i) = D_i'$$

$$\gamma_i(\partial D_i) = \partial D_i'$$

The transformations formed by composition of γ_i form a group called a **Schottky group**, usually denoted as Γ .



Mobius transformations mapping outside of one disc to inside of another Chan 2022

Schottky cover

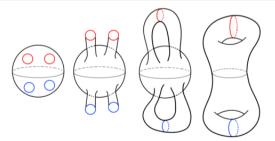
Bobenko and Klein 2011 and Chan 2022

Schottky cover

Given a Schottky group Γ with associated discs $\{D_i,D_i'\}_{i=1}^g$ we can define

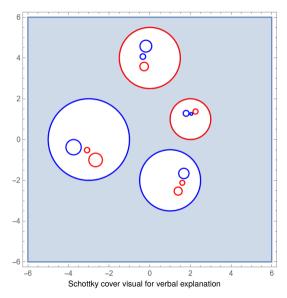
$$F := \bar{\mathbb{C}} \setminus \bigcup_{i} (\mathring{D}_{i} \cup \mathring{D}'_{i}) \quad ; \quad \Omega := \bigcup_{\gamma \in \Gamma} \gamma(F)$$

Then, $\mathcal{M} := \Omega/\Gamma$ is a Riemann surface of genus g with fundamental domain F.

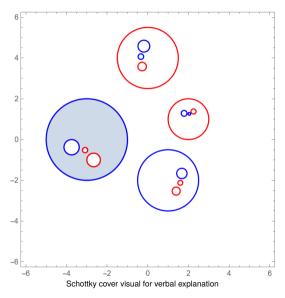


Schottky cover mapping to compact Riemann surface Chan 2022

Schottky cover visual

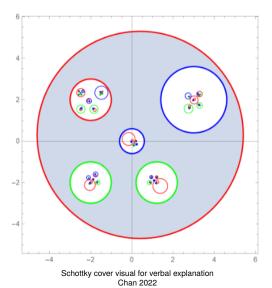


Schottky cover visual



Schottky cover visual

Chan 2022



Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

We can define fixed points and cosets

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P_i') = P_i'$$
$$\Gamma/\Gamma_i = \{\gamma_{j_1}^{n_1} \cdots \gamma_{j_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

Differentials and Abel's map

Bobenko and Klein 2011 and Chan 2022

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$$\Gamma/\Gamma_i = \{\gamma_{i_1}^{n_1} \cdots \gamma_{i_k}^{n_k} : \gamma_{j_k} \neq \gamma_i\}$$

And use these to define holomorphic differentials using fixed points P_i

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left(\frac{1}{z - \gamma(P_i')} - \frac{1}{z - \gamma(P_i)} \right) dz = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_i \setminus \Gamma} \left(\frac{1}{\gamma(z) - P_i'} - \frac{1}{\gamma(z) - P_i} \right) d(\gamma(z))$$

Differentials and Abel's map

Robenko and Klein 2011 and Chan 2022

We can define fixed points and cosets

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P_i') = P_i'$$
$$\Gamma/\Gamma_i = \{\gamma_{i_1}^{n_1} \cdots \gamma_{i_k}^{n_k} : \gamma_{i_k} \neq \gamma_i\}$$

And use these to define holomorphic differentials using fixed points P_i

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left(\frac{1}{z - \gamma(P_i')} - \frac{1}{z - \gamma(P_i)} \right) dz = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_i \setminus \Gamma} \left(\frac{1}{\gamma(z) - P_i'} - \frac{1}{\gamma(z) - P_i} \right) d(\gamma(z))$$

Which can then be used to define Abel's map

$$u_i[p] = \int_{p_0}^p \omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \ln\{p, \gamma(P_i'), p_0, \gamma(P_i)\}$$

where

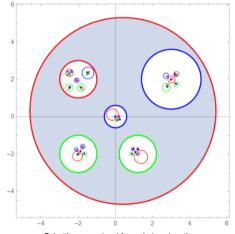
$${a,b,c,d} = \frac{(a-b)(c-d)}{(a-d)(c-b)}$$

Schottky differentials with visual

Bobenko and Klein 2011 and Chan 2022

$$\gamma_i(P_i) = P_i, \quad \gamma_i(P_i') = P_i'$$

$$\omega_i = \frac{1}{2\pi i} \sum_{\gamma \in \Gamma/\Gamma_i} \left(\frac{1}{z - \gamma(P_i')} - \frac{1}{z - \gamma(P_i)} \right) dz$$



Schottky cover visual for verbal explanation Chan 2022

Chan 2022

Focusing on three of the conditions:

- 1. Generalized Kronecker function should be quasi-periodic
- 2. Generalized Kronecker function should reduce to aforementioned genus 1 form
- 3. Generalized Kronecker function should satisfy integrability in a particular way

Chan 2022

Focusing on three of the conditions:

- 1. Generalized Kronecker function should be quasi-periodic
- 2. Generalized Kronecker function should reduce to aforementioned genus 1 form
- 3. Generalized Kronecker function should satisfy integrability in a particular way

$$K(z, \{w_1, ..., w_g\} | \Gamma) = \sum_{\gamma \in \Gamma} \frac{\gamma'(z)}{\gamma(z) - 1} w_1^{\operatorname{ord}_1 \gamma} ... w_g^{\operatorname{ord}_g \gamma}$$

Chan 2022

Focusing on three of the conditions:

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At Genus 1, for $\gamma: z \mapsto \mathbf{e}(\tau)z = az$

$$K(z, w|\Gamma) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n z - 1} w^n = \dots = \frac{1}{z} \left[\frac{z}{1 - z} - \frac{1}{1 - w} - \sum_{m, n > 0} q^{mn} (z^m w^n - z^{-m} w^{-n}) \right]$$

Chan 2022

Focusing on three of the conditions:

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At Genus 1, for $\gamma: z \mapsto \mathbf{e}(\tau)z = qz$

$$K(z, w|\Gamma) = \sum_{n \in \mathbb{Z}} \frac{q^n}{q^n z - 1} w^n = \dots = \frac{1}{z} \left[\frac{z}{1 - z} - \frac{1}{1 - w} - \sum_{m, n > 0} q^{mn} (z^m w^n - z^{-m} w^{-n}) \right]$$

Recall

$$F(\xi,\alpha) = -2\pi i \left(\frac{z}{1-z} + \frac{1}{1-w} + \sum_{m,n>0} q^{mn} (z^m w^n - z^{-m} w^{-n}) \right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e} \begin{pmatrix} z \\ \alpha \\ \tau \end{pmatrix}$$

Open questions

Obtaining the Kronecker function in terms of theta functions

- Choice of characters, description of theta's behavior
- Existing attempts inspiring representation with theta functions

More detail in Schottky cover description

- Matching Schottky fundamental domain with usual fundamental polygon
- Alternative choices for generalized Kronecker function

Connection to algebraic curves

Mapping to other language of describing Riemann surfaces

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Odd theta function at genus 1

Jacobi Triple Product:

$$f(x,y) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x,xy) = \prod_{m>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{(1 + xy^2)}f(x,y) = x^{-1}y^{-2}f(x,y)$$

$$f(x,y) = \sum_{n=-\infty}^{\infty} c_n(x)y^{2n} \implies f(x,y) = xy^2f(x,xy) = \sum_n c_n(x)x^{2n+1}y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1}c_n(x) \implies c_n(x) = c_0(x)x^{n^2} \implies f(x,y) = c_0(x)\sum_{n=0}^{\infty} x^{n^2}y^{2n}$$

This relates the two forms of the theta function : $\prod_j (1-q^j)(1-wq^j)(1-w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2} \mathbf{e}(z)^n$

What about higher genus?

Are there similar Jacobi formulas for higher genus theta functions?