Report .1

Theta functions, Kronecker functions and bilinear relations

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1 Introduction & Background

[Cha22] [Ber10]

2 Decomposition of functions

One of the goals we have as we describe complex functions on Riemann surfaces is to find a simple way to decompose any arbitrary function.

2.1 Genus-Zero decompostion

I wrote this proof myself as an exercise. I think I am missing something in terms of how to formally treat the point at infinity; without that, my arguments that no poles $\implies$ bounded is not sound. See † below.

Theorem 1 (Decomposition for Genus-Zero)**.** *Let f be a meromorphic function on $\mathbb{C}$ (including the point at ∞). Let z_i be its zeros with multiplicity n_i and q_j be the poles with multiplicity p_j. Then, there exists a constant $C \in \mathbb{C}$ such that*

$$f(z) = C\frac{\prod_i (z - z_i)^{n_i}}{\prod_j (z - q_j)^{p_j}} \tag{.1|1}$$

This is equivalent to saying that a meromorphic function is uniquely defined, up to the scaling factor C, by the locations and multiplicity of its zeros and poles.

Proof. Consider a meromorphic function g with zeros and poles as described above. Let us define h as $h(z) = f(z)/g(z)$. We will show that $h(z)$ must be a constant function, thus showing that g is of the form desired. In order to show that $h(z)$ is constant, we can show that it is bounded and then use Liouville's Theorem [refer to reportA5? or to a source?].

At all points besides the z_i and q_j, we see that $f(z)$ and $g(z)$ have no zeros or poles, so $h(z)$ cannot have a pole at those locations.

For each zero z_i, with multiplicity n_i, we can write

$$f(z) = (z - z_i)^{n_i} \tilde{f}(z) \tag{.1|2}$$

$$g(z) = (z - z_i)^{n_i} \tilde{g}(z) \tag{.1|3}$$

for some holomorphic functions $\tilde{f}$ and $\tilde{g}$ defined on a disc around z_i that satisfy $\tilde{f} \neq 0 \neq \tilde{g}$.

Then, on that disc we have $h(z) = \frac{\tilde{f}(z)}{\tilde{g}(z)}$. Since this is a ratio of two non-zero holomorphic functions, h does not have a pole at z_i.

Similarly, for each pole q_j with multiplicity p_j, we can write

$$f(z) = (z - q_j)^{-p_j} \tilde{f}(z) \tag{.1|4}$$

$$g(z) = (z - q_j)^{-p_j} \tilde{g}(z) \tag{.1|5}$$

and conclude that h does not have poles at q_j either.

Since h has no poles, it must be bounded. † *Some step is missing here.* □

2.2 Genus-One decomposition

3 Main theorem

3.1 Preconditions

3.2 Boundary conditions

3.3 Proof

4 Outlook & open questions

Bibliography

[Ber10] Marco Bertola. Riemann surfaces and theta functions mast 661 g / mast 837. 2010.

[Cha22] Zhi Cong Chan. Towards a higher-genus generalization of the kronecker function using schottky covers. 2022.