ETH zürich

D-PHYS

Theta Functions, Kronecker Functions, and Bilinear Relations

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Riemann Surfaces
in Mathematical Physics

Outline

1. Abel's map
2. Theta functions
3. Kronecker function
4. Striving for higher genus

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Holomorphic Differentials

Definition and existence of holomorphic differentials

Definition: $\omega = f_\alpha dz_\alpha = f_\beta dz_\beta \quad , \quad f$ holomorphic

Existence: $\dim \mathcal{H}^1 = g$ (genus of compact Riemann surface)

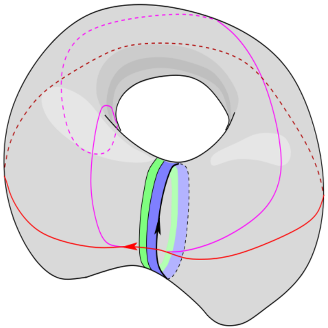
Proof outline:

- $\dim \mathcal{H}^1 \leq$ # of a-cycles $= g$
- # of harmonic differentials $= \dim H \geq 2g$
- $h = f dz + g d\bar{z} \implies \dim H = 2 \dim \mathcal{H}^1$
- $g \leq \dim \mathcal{H}^1 \leq g \implies \dim \mathcal{H}^1 = g$

Normalization & period matrix:

$$\int_{a_i} \omega_j = \delta_{ij}$$

$$\int_{b_i} \omega_j = \tau_{ij}$$



Regions used to define harmonic differentials
Bertola 2006

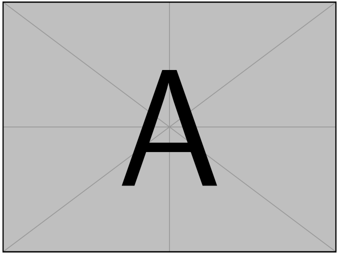
Abel's map

Bertola 2006 Section 4.2

Formal definition of Abel's map

For a particular choice of a point P_0 on the fundamental domain $\mathcal{L}$, using the normalized harmonic differentials ω_i, we have Abel's map

$$\mathbf{u} : \mathcal{L} \mapsto \mathbb{C}^g, \quad P \qquad \mapsto \begin{pmatrix} \int_{P_0}^{P} \omega_1 \\ \vdots \\ \int_{P_0}^{P} \omega_g \end{pmatrix}$$



Fundamental domain
ImageSource

Analytic continuation beyond the fundamental domain:

$$\mathbf{u}(P + a_i) = \mathbf{u}(P) + \begin{pmatrix} \int_{a_i} \omega_1 \\ \vdots \end{pmatrix} = \mathbf{u}(P) + \begin{pmatrix} \delta_{i1} \\ \vdots \end{pmatrix}$$

$$\mathbf{u}(P + b_i) = \mathbf{u}(P) + \begin{pmatrix} \tau_{i1} \\ \vdots \end{pmatrix}$$

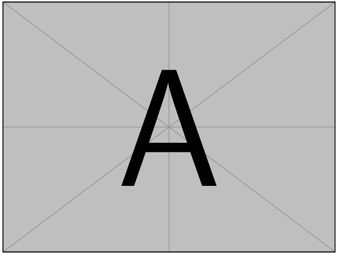
Abel's map at genus 1

Appropriate differential

$$\omega = dz$$

Abel's map

$$\mathbf{u}(z) = \int_0^z \omega = z$$



Fundamental domain and continuation at genus 1
ImageSource

What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel's map?

Outline

Theta functions

Bertola 2006 Section 5.1

Definition of the Theta function

Given a symmetric matrix τ with positive definite imaginary part, the Theta function is

$$\Theta(\vec{z},\tau) := \sum_{\vec{n}\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2}\vec{n}^T\tau\vec{n} + \vec{n}^T\vec{z}\right) \quad , \quad \mathbf{e}(z) = \exp(2\pi i z)$$

Properties: For $\vec{\lambda} \in \mathbb{Z}^g$

$$\Theta(-\vec{z}) \overset{\vec{n}\mapsto -\vec{n}}{=} \Theta(\vec{z})$$

$$\Theta(\vec{z}+\vec{\lambda}) = \sum_{\vec{n}\in\mathbb{Z}^g} \cancelto{1}{\mathbf{e}(\vec{n}^T\vec{\lambda})}\mathbf{e}(\ldots) = \Theta(\vec{z})$$

$$\Theta(\vec{z}+\tau\vec{\lambda}) = \begin{bmatrix}\text{shift } \vec{n} \\ \text{use } \tau \text{ symmetry}\end{bmatrix} = \mathbf{e}\left(-\frac{1}{2}\vec{\lambda}^T\tau\lambda - \vec{\lambda}^T\vec{z}\right)\Theta(\vec{z})$$

Theta function on a compact Riemann surface

Bertola 2006 Section 5.2

Definition of Theta function on a compact Riemann surface

For a compact Riemann surface $\mathcal{M}$ of genus g, with period matrix τ and Abel's map $\mathbf{u}$, we can identify

$$\begin{aligned} \theta : \mathcal{M} &\mapsto \mathbb{C} \\ P &\mapsto \Theta(\mathbf{u}(P)) \end{aligned}$$

Properties:

$$\theta(P + a_i) = \theta(P)$$

$$\theta(P + b_i) = \mathbf{e}\left(-\frac{1}{2}\tau_{ii} - \mathbf{u}_i(P)\right)\theta(P)$$

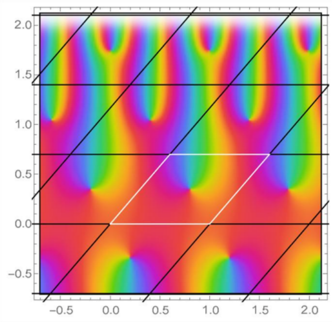
Theta function at genus 1

$$\theta(z) = \sum_{n \in \mathbb{Z}} \mathbf{e}(\frac{1}{2}n^2\tau + nz)$$

$$\theta(z) = \theta(-z)$$

$$\theta(z+1) = \theta(z)$$

$$\theta(z+\tau) = \mathbf{e}(-\frac{1}{2}\tau - \xi)\theta(z)$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- What does the Theta function look like at higher genus?

Theta function with characteristics

Bertola 2006 Section 5.1

Definition of Theta function with characteristics

Consider vectors $\epsilon, \epsilon' \in \mathbb{R}^g$. We can then define the Theta function with characteristics ϵ, ϵ' as

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) := \mathbf{e}\left(\frac{1}{8}\epsilon^T\tau\epsilon + \frac{1}{2}\epsilon^T\vec{z} + \frac{1}{4}\epsilon^T\epsilon'\right)\Theta\left(\vec{z} + \frac{\epsilon'}{2} + \frac{\tau\epsilon}{2}\right)$$

Properties:

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z} + \vec{\alpha} + \tau\vec{\beta}) = \mathbf{e}\left(\frac{1}{2}(\epsilon^T\vec{\alpha} - \vec{\beta}^T\epsilon') - \frac{1}{2}\beta^T\tau\beta - \vec{\beta}\vec{z}\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z})$$

$$\Theta\begin{bmatrix}\epsilon + 2\eta\\ \epsilon' + 2\eta'\end{bmatrix}(\vec{z}) = \exp(\pi i\epsilon^T\eta')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \quad , \quad \eta, \eta' \in \mathbb{Z}^g$$

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(-\vec{z}) = \exp(\pi i\epsilon^T\epsilon')\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\vec{z}) \quad , \quad \epsilon, \epsilon' \in \mathbb{Z}^g$$

Odd theta functions and zeros

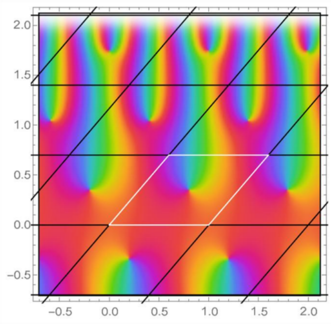
$$\epsilon, \epsilon' \in \mathbb{Z}^g, \quad \epsilon^T \epsilon' \text{ is odd}$$

$$\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z}) = \exp(\pi i \epsilon^T \epsilon') \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{z}) = -\Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (-\vec{z})$$

$$\implies \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (0) = \Theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\vec{\alpha} + \tau \vec{\beta}) = 0$$

$$\implies \Theta \left(\frac{\epsilon'}{2} + \frac{\tau \epsilon}{2} \right) = 0$$



Theta function for $\tau = 0.7 + 0.6i$
Chan 2022

What about higher genus?

- Which of the zeros located by odd characteristics are actually reached by Abel's map on compact Riemann surfaces of higher genus?

Odd theta function at genus 1

Odd theta function at genus 1

We define

$$\theta_1(z) = -\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} (z)$$

It has equivalent definition

$$\theta_1(z) = 2iq^{1/8} \sin(\pi z) \prod_j (1 - q^j)(1 - wq^j)(1 - w^{-1}q^j) \quad , \quad q = \mathbf{e}(\tau), w = \mathbf{e}z$$

Jacobi Triple Product:

$$f(x, y) = \prod_{j>0} (1 - x^{2m})(1 + x^{2m-1}y^2)(1 + x^{2m-1}y^{-2})$$

$$f(x, xy) = \prod_{j>0} (1 - x^{2m})(1 + x^{2m+1}y^2)(1 + x^{2m-3}y^{-2}) = \frac{1 + x^{-1}y^{-2}}{1 + xy^2} f(x, y) = x^{-1}y^{-2} f(x, y)$$

Odd theta function at genus 1

$$f(x,y) = \prod_{j>0}(1-x^{2m})(1+x^{2m-1}y^2)(1+x^{2m-1}y^{-2})$$

$$f(x,xy) = \prod_{j>0}(1-x^{2m})(1+x^{2m+1}y^2)(1+x^{2m-3}y^{-2}) = \frac{1+x^{-1}y^{-2}}{(1+xy^2)}f(x,y) = x^{-1}y^{-2}f(x,y)$$

$$f(x,y) = \sum_{n=-\infty}^{\infty} c_n(x)y^{2n} \implies f(x,y) = xy^2 f(x,xy) = \sum_n c_n(x)x^{2n+1}y^{2n+2}$$

$$\implies c_{n+1}(x) = x^{2n+1}c_n(x) \implies c_n(x) = c_0(x)x^{n^2} \implies f(x,y) = c_0(x)\sum_n x^{n^2}y^{2n}$$

This relates the two forms of the theta function : $\prod_j(1-q^j)(1-wq^j)(1-w^{-1}q^j) \simeq \sum_n \mathbf{e}(\tau)^{n^2}\mathbf{e}z^n$

What about higher genus?

- Are there similar formulas for higher genus theta functions that make them easier to work with in some cases?

(Application) Decomposing meromorphic functions

Chan 2022 Section 3.4 & Bertola 2006 Chapter 6

Rough outline of how to reproduce a function with divisor $(f) = \sum n_i P_i$

$$\begin{bmatrix}\text{Find function } t(z) \\ \text{with known simple zero}\end{bmatrix} \to \begin{bmatrix} g(z) = \prod t(P - P_i)^{n_i} \\ \text{respecting possible periodicity}\end{bmatrix} \to \left(\frac{f}{g}\right) = \emptyset \to \frac{f}{g} = \text{const.}$$

Recall that $\deg((f)) = \sum n_i = 0$ for meromorphic functions, so extra factors can easily cancel.

Genus 0:

- $f(z) = C \prod (z - z_i)^{n_i}$

Genus > 0:

- $\Theta(\xi) = 0$
- $g_{P'} : P \mapsto \Theta(\mathbf{u}(P) - \mathbf{u}(P') + \xi)$
- $f(P) = C \prod \left(g_{P_i}(P)\right)^{n_i}$

Genus 1:

- Decompose $z_i = \frac{b_i}{2} + \tau \frac{a_i}{2}$
- $f(z) = C \prod \left(\theta \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z) \right)^{n_i}$

Outline

Kronecker function

Brown and Levin 2013 Section 3.4

Definitions of the Kronecker function

The Kronecker function $F(\xi, \eta, \tau)$ has equivalent definitions

1. In terms of the odd theta function
$$\frac{\theta_1'(0)\theta_1(\xi+\eta)}{\theta_1(\xi)\theta_1(\eta)}$$

2. In terms of a sum over exponentials
$$-2\pi i\left(\frac{z}{1-z}+\frac{1}{1-w}+\sum_{m,n>0}(z^m w^n - z^{-m}w^{-n})q^{mn}\right) \quad , \quad \begin{pmatrix} z \\ w \\ q \end{pmatrix} = \mathbf{e}\begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix}$$

3. In terms of a sum over Eisenstein functions and series
$$\frac{1}{\eta}\exp\left(-\sum_{j>0}\frac{(-\eta)^j}{j}(E_j(\xi,\tau)-e_j(\tau))\right)$$

Properties of the Kronecker function

Brown and Levin 2013 Section 3.4

What about higher genus?

- How can we define the Kronecker function at higher genus?

Periodicity Properties:

$$F(\xi+1,\eta)=\frac{\theta_1'(0)\theta_1(\xi+\eta+1)}{\theta_1(\xi+1)\theta_1(\eta)}=F(\xi,\eta)$$

$$F(\xi+\tau,\eta)=\frac{\theta_1'(0)\theta_1(\xi+\eta+\tau)}{\theta_1(\xi+\tau)\theta_1(\eta)}=\frac{\mathbf{e}(-\xi-\eta)}{\mathbf{e}(-\xi)}F(\xi,\eta)$$

The Fay Relation

$$F(\xi_1,\eta_1)F(\xi_2,\eta_2)=F(\xi_1,\eta_1+\eta_2)F(\xi_2-\xi_1,\eta_2)+F(\xi_2,\eta_1+\eta_2)F(\xi_1-\xi_2,\eta_1)$$

Setup for derivation of the Fay relation

Mat19

$$F(\xi_1,\eta_1)F(\xi_2,\eta_2) = F(\xi_1,\eta_1+\eta_2)F(\xi_2-\xi_1,\eta_2) + F(\xi_2,\eta_1+\eta_2)F(\xi_1-\xi_2,\eta_1)$$

$\downarrow$ rewrite using Theta functions $\downarrow$

$$\frac{\theta_1(\xi_1+\eta_1)\theta_1(\xi_2+\eta_2)}{\theta_1(\xi_1)\theta_1(\eta_1)\theta_1(\xi_2)\theta_1(\eta_2)} = \frac{\theta_1(\xi_1+\eta_1+\eta_2)\theta_1(\xi_2-\xi_1+\eta_2)}{\theta_1(\xi_1)\theta_1(\eta_1+\eta_2)\theta_1(\xi_2-\xi_1)\theta_1(\eta_2)} + \frac{\theta_1(\xi_1+\eta_1+\eta_2)\theta_1(\xi_1-\xi_2+\eta_1)}{\theta_1(\xi_1)\theta_1(\eta_1+\eta_2)\theta_1(\xi_1-\xi_2)\theta_1(\eta_1)}$$

$\downarrow$ multiply common denominator and relabel $\downarrow$

$$\begin{aligned}&\theta_1(\alpha_0)\theta_1(\beta_0)\theta_1(\alpha_2+\beta_1)\theta_1(\alpha_2-\beta_1)+\\ &\theta_1(\alpha_1)\theta_1(\beta_1)\theta_1(\alpha_0+\beta_2)\theta_1(\alpha_0-\beta_2)+\\ &\theta_1(\alpha_2)\theta_1(\beta_2)\theta_1(\alpha_1+\beta_0)\theta_1(\alpha_1-\beta_0)=0\end{aligned}$$

$\downarrow$ long process involving odd and even theta functions at genus 1 $\downarrow$

...

What about higher genus?

- What does the Fay identity look like at higher genus when theta functions are more complicated?

Differentials from the Kronecker function

$$F(z,\alpha)dz = \frac{1}{\alpha}\sum_{n=0}^{\infty} g^{(n)}(z)\alpha^n$$

Examples of differentials

[1412.5535] eqn 3.31 etc.

Independence of the differentials

Brown and Levin (probably needs to be several slides)

Fay relation for differentials

one line derivation one line statement

Application of properties

big picture view of when these come into string theory

Outline

Why we care about higher genus

Connection to string theory

Questions gathered so far

What about higher genus?

- How do we represent the fundamental domain?
- What choice of differentials can we make?
- What consequences does this have for Abel’s map?
- What does the Theta function look like at higher genus?
- Which of the zeros located by odd characteristics are actually reached by Abel’s map on compact Riemann surfaces of higher genus?

Schottky cover definition

Schottky group example

Differentials and theta functions

Attempt at a Kronecker function

Chan 2022

Open questions

References

Bertola, Marco (2006). *Riemann Surfaces and Theta Functions*.

Brown, Francis C. S. and Andrey Levin (2013). *Multiple Elliptic Polylogarithms*.

Chan, Zhi Cong (2022). “Towards a Higher-Genus Generalization of the Kronecker Function Using Schottky Covers”.