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AN ALGORITHM FOR COMPUTING FEKETE POINTS IN THE TRIANGLE*

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Abstract. On the line and its tensor products, Fekete points are known to be the Gauss–Lobatto quadrature points. But unlike high-order quadrature, Fekete points generalize to non-tensor-product domains such as the triangle. Thus Fekete points might serve as an alternative to the Gauss–Lobatto points for certain applications. In this work we present a new algorithm to compute Fekete points and give results up to degree 19 for the triangle. For degree $d > 10$ these points have the smallest Lebesgue constant currently known. The computations validate a conjecture of Bos [*J. Approx. Theory*, 64 (1991), pp. 271–280] that Fekete points along the boundary of the triangle are the one-dimensional Gauss–Lobatto points.

Key words. triangle, Lebesgue constant, Fekete points, multivariate approximation

AMS subject classifications. 65M60, 65M70, 41A10

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1. Introduction.

The Gauss–Lobatto quadrature points are commonly used in numerical methods which rely on both accurate high-order polynomial interpolation and quadrature properties. But Gauss–Lobatto quadrature points are only known for tensor-product domains such as the line and square, making it unclear how to extend a Gauss–Lobatto numerical method to non-tensor-product domains like the triangle. Since Fekete points are known to be the Gauss–Lobatto points on the line [10] and in the d-dimensional cube [7], Fekete points are one possible generalization of Gauss–Lobatto points for the triangle. At present there is no known analytic formula for the location of the Fekete points in the triangle; thus it is of interest to compute Fekete points numerically so that their interpolation and quadrature properties can be studied.

In this work, we present a new algorithm to compute Fekete points. The algorithm is applied to the triangle, where for degree $d > 10$, it improves the Fekete points computed in [1] and produces points with the best known Lebesgue constant.

2. Fekete points.

Fekete points are closely related to optimal interpolation points; therefore we first describe some well-known facts about interpolation.

Let $\mathcal{P}_{\mathcal{N}}$ be a finite dimensional vector space made up of polynomials and let Ω be some domain such as the square or triangle. There are two polynomial spaces we will consider in this paper. The space used when working with quadrilaterals is the span of $\{x^m y^n,\ m, n \leq d\}$, which we call a *diamond truncation* because it has that shape when displayed using Pascal's triangle of the monomials in x and y (see [3, p. 157]). For triangles, the usual choice is a *triangular truncation* of polynomials, spanned by the monomials $\{x^m y^n,\ m + n \leq d\}$.

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Setting $N = \dim \mathcal{P}_\mathcal{N}$, let us now take N points, $\{z_i\}$, which we assume are solvable in $\mathcal{P}_\mathcal{N}$. That is, given an arbitrary function f, there is a unique function $g \in \mathcal{P}_\mathcal{N}$ where $g(z_i) = f(z_i)$. We will denote the interpolating polynomial g by

$$g(z) = (I_N f)(z).$$

The lemma of Lebesgue shows how well $I_N(f)$ approximates f. Take a function $h \in \mathcal{P}_\mathcal{N}$ which best approximates f in the max norm,

$$\|f\| = \max_{z\in\Omega} |f(z)|.$$

This function is not necessarily $I_N(f)$, but $h = I_N(h)$. Thus

$$\begin{aligned} \|f - I_N(f)\| &= \|f - h + I_N(h) - I_N(f)\| \\ &\le \|f - h\| + \|I_N\|\,\|h - f\| \\ &\le (1 + \|I_N\|)\,\|f - h\|, \end{aligned}$$

where the operator max-norm is defined in the usual way:

$$\|I_N\| = \max_{\|f\|=1} \|I_N(f)\|,$$

and $\|I_N\|$ is known as the Lebesgue constant. If our functional space $\mathcal{P}_\mathcal{N}$ is a good choice for approximating f, then $I_N(f)$ will be a good approximation to f if there are reasonable bounds on the Lebesgue constant.

Choices such as equally spaced (in the square or triangle) or a product Gaussian grid (restricted to the triangle) lead to disasters. In these cases it is well known that the Lebesgue constant grows exponentially with the degree. If the points are not chosen very carefully, the interpolating polynomial will have wild oscillations between the collocation points (see, for example, [8, p. 171]).

For interpolation we thus would like to find a set of points with the smallest possible Lebesgue constant. The Lebesgue constant is a function of the interpolation points, the domain Ω, and the functional space $\mathcal{P}_\mathcal{N}$. For a given Ω and $\mathcal{P}_\mathcal{N}$, Lebesgue points are the points with minimum Lebesgue constant, and thus (in this norm) are the optimal interpolation points. Almost nothing seems to be known about Lebesgue points in more than one dimension. Nor are we aware of a feasible method for computing them numerically.

Fekete points are a tractable alternative to Lebesgue points. There are some theoretical results for Fekete points in more than one dimension and numerical evidence shows they are close to optimal [1]. To define Fekete points for a domain Ω, we first pick a basis $\{g_i, i = 1, \ldots, N\}$ for $\mathcal{P}_\mathcal{N}$. Then let $V(z_1, z_2, \ldots, z_N)$ be the generalized Vandermonde matrix defined at the points $\{z_i \in \Omega, i = 1, \ldots, N\}$. That is, V is an $N \times N$ matrix whose elements are $V_{ij} = g_j(z_i)$. Fekete points are a set of points $\{z_i, i = 1, \ldots, N\}$ which maximize (for a fixed basis) the determinant of V:

$$\max_{\{z_i\}} |V(z_1, z_2, \ldots, z_N)|.$$

Fekete points are independent of our choice of basis, since any change of basis only multiplies the determinant by a constant independent of the points. The matrix V can be thought of as the inverse transform matrix. V maps spectral coefficients of a

function into that function's grid point values at z_i and maximizing $|V|$ makes this matrix far from singular.

In the square, with $\mathcal{P}_\mathcal{N}$ taken to be a diamond truncation of polynomials, Bos has recently proven that the Fekete points are unique and given by the tensor product of Gauss–Lobatto points [7]. This was proven for the interval $[-1, 1]$ in [10]. In one dimension, Fekete points are also a minimum energy configuration of point charges in the interval (attributed to Stieltjes in [15, p. 140]). However, in higher dimensions, all of these similarities break down [4]. For general domains, Fekete points are not Gaussian-like quadrature points, nor will they be minimum energy electrostatic points (the latter points are sometimes called elliptic Fekete points). For the electrostatic problem, the limiting distribution as $N \to \infty$ is known. For Fekete points, this limiting distribution is conjectured to be a certain extremal measure [4], but this is an unsolved problem. However, Hesthaven [11] has shown that a generalized form of the electrostatic problem with Gauss–Lobatto points imposed on the boundary will produce points with better Lebesgue constant than Fekete points for degree $d < 9$.

There are some analytic results for Fekete points in a triangle. If we assume the Vandermonde matrix is nonsingular, then there is a maximum number of points which can lie on the boundary. Bos [6] conjectured that in the triangle (and some other domains) Fekete points will have the maximum number of points on the boundary. Under this assumption, he proved that the boundary points will be the one-dimensional Gauss–Lobatto points. We have verified this conjecture numerically. This result has an important application: Fekete point triangular elements naturally conform with standard quadrilateral spectral elements, since this method relies on a tensor product of Gauss–Lobatto points within each quadrilateral [13], [16].

3. Fekete point cardinal functions. Let us call the determinant of the generalized Vandermonde matrix v:

$$v = \max_{\{\zeta_i\} \in \Omega} |V(\zeta_1, \zeta_2, \ldots, \zeta_N)|,$$

and let $\{z_i, i = 1, \ldots, N\}$ be a set of Fekete points which achieve this maximum. We can now write an expression for the cardinal functions (Lagrange interpolation functions) defined at these points. If $v \neq 0$, the cardinal functions $\phi_j(z) \in \mathcal{P}_\mathcal{N}$ are uniquely defined by

$$\phi_j(z_i) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. They are given by

$$\phi_j(z) = \frac{|V(\zeta_1, \zeta_2, \ldots, \zeta_N)|}{v},$$

evaluated at $\zeta_j = z$ and $\zeta_i = z_i$ for $i \neq j$. This is because at $z = z_i$ for $i \neq j$, the i'th and j'th rows of the Vandermonde matrix are equal and $|V| = 0$. When $z = z_j$, we have that the determinant of V is at its maximum and thus $|V| = v$. This expression also leads to a bound on the cardinal functions

$$|\phi_i(z)| \leq 1 \qquad \forall z \in \Omega.$$

Thus unlike general optimal interpolation points, Fekete points generate cardinal functions which achieve their maximum in Ω at their associated Fekete point. This property gives a bound on the Lebesgue constant. Writing $I_N(f)$ in terms of cardinal

functions,

$$I_N(f) = \sum_{i=1}^{N} f(z_i)\phi_i(z),$$

we see that in the max norm

$$\|I_N\| = \max_{\|f\|=1} \|I_N(f)\| = \max_{z\in\Omega} \sum_{i=1}^{N} |\phi_i(z)| \le N.$$

Combining this bound with the lemma of Lebesgue allows for simple and well-known proofs of uniform and exponential convergence of interpolation, differentiation, and quadrature when applied to an analytic function f. Note the numerical results presented later suggest the bound for the Lebesgue constant in the triangle is $C\sqrt{N}$. In the univariate case, the bound is well known to be logarithmic in N.

4. Computing Fekete points. We only know of a few works in which Fekete points are computed in more than one dimension. The earliest is due to Bos [5], who derived Fekete points for the triangle up to degree $d = 3$ and derived some approximate solutions up to degree 7. More recent work has been computational. Chen and Babuška [1] improved and extended Bos's results to degree 13. They also computed optimal L^2-norm interpolation points and showed that these points have a lower Lebesgue constant than their approximate Fekete points. Their algorithm is not described, but the minimization seems not to be effective for $d > 10$ since we will present Fekete points with smaller Lebesgue constant than both their Fekete and minimal L^2-norm points. Thus we suspect an improved algorithm could compute optimal L^2-norm points with even smaller Lebesgue constants.

Hesthaven [11] computed a different set of near-optimal interpolation points for the triangle. He uses a second-order time evolution method to minimize an electrostatic energy function. The resulting solution is further refined with a modified Powell's hybrid method. For $d < 9$ these points are quite good, with a smaller Lebesgue constant than Fekete points and sometimes even the optimal L^2-norm points. This fails to be true for larger d, and for $d > 13$ these points become significantly worse than Fekete points. All of these results are summarized in Table 5.1.

We note that Fekete points have also been computed on the sphere using the equal area resolution triangular truncation of spherical harmonics [14]. They use an exchange algorithm which is an iterative sequence of independent minimizations for each point.

4.1. Steepest ascent algorithm. The algorithm we use to maximize the determinant of the Vandermonde matrix is the steepest ascent method. We implement this with a simple gradient flow algorithm to solve the system of ODEs

$$\frac{\partial z_i}{\partial t} = \frac{\partial |V|}{\partial z_i}.$$

The points are evolved by moving them in the direction of steepest accent until an equilibrium solution is reached. The only constraint is that the points are not allowed to leave the triangle. We perform these computations in the right triangle. The algorithm will terminate at a local maximum of $|V|$, and we call any such extremal point a solution.

In practice we can find several different solutions and we evaluate these solutions with a variety of norms. At present we have no way of knowing if any of our computed solutions represent a global maximum and thus represent true Fekete points; hence we refer to these local maximums as approximate Fekete points. When we find several sets of approximate Fekete points with similar values for $|V|$, we present the additional solutions if they have dramatically different quadrature properties or Lebesgue constants.

The algorithm requires us to compute $\nabla|V|$, which has a simple expression if we write V in terms of a cardinal function basis. To show this, we first note that the partial derivative of the a determinant of a matrix with respect to an entry V_{ij} is given by

$$\frac{\partial|V|}{\partial V_{ij}} = -1^{i+j}|A_{ij}|,$$

where A_{ij} is the ij minor of V (the matrix formed by removing the ith and jth rows of V). When V is computed using a cardinal function basis, V is the identity matrix. Thus the derivative of $|V|$ with respect to any matrix element is only nonzero for elements along the diagonal. When V is written with a cardinal function basis, the ith diagonal element is the ith cardinal function evaluated at the grid points z_i and the determinant of the minor $|A_{ii}| = 1$. Thus our system of ODEs reduces to

$$\frac{\partial z_i}{\partial t} = \frac{\partial \phi_i}{\partial z_i}.$$

This algorithm has a very simple geometric interpretation. We would like the cardinal functions ϕ_i to look like delta functions at z_i. The maximum should be achieved at the grid point z_i. The steepest ascent algorithm simply moves each point towards the maximum of its associated cardinal function. The iterative nature of the algorithm comes into play because the cardinal functions change with every change in the grid points, and thus we must recompute our basis functions at each iteration. The iteration is terminated when

$$\left|\frac{\partial \phi_i}{\partial z_i}\right| \leq 10^{-12}$$

for all z_i in the interior of the triangle.

The only remaining computational issue is in evaluating the cardinal functions and their derivatives. This is done by first computing the spectral coefficients of the cardinal functions with respect to a known basis. If the Vandermonde matrix is written using this known basis, then the coefficients of the expansion of the cardinal functions in terms of this basis are given by the rows of V^{-1}. Thus we must invert V (through Gaussian elimination) at each iteration. Since we must compute this inverse numerically, it is important that V be well conditioned. For this we have found that the Dubiner basis functions [9] in the right triangle are far superior to Legendre polynomials. Using monomials for the basis functions is practically impossible. For our largest case, degree 19, the MATLAB reported condition number of V is less than 50 when computed with the Dubiner polynomials.

A potential drawback of the steepest ascent method is its slow rate of convergence. At present, a degree 19 maximization running in MATLAB on a Sun workstation takes several hours to converge to 12 digits of accuracy. The convergence can be greatly accelerated by using a combination of the steepest ascent method and Newton's method.

When the steepest ascent method has converged close enough to a solution, we switch from the steepest ascent method and instead use Newton's method. Details of this algorithm are given in Appendix I.

4.2. Initial conditions. The method is extremely sensitive to the initial condition for the grid points. With different initial conditions, we can find different local maximums of $|V|$. This makes it important to initialize the algorithm with a good initial guess. We have tried several possibilities, the best of which turns out to be points distributed to generate a density which approximates the extremal measure $g(x, y)$ for the triangle given in [2]. For the right triangle $x \geq 0$, $y \geq 0$, and $x + y \leq 1$, the extremal measure is

$$g(x, y) = \frac{1}{\sqrt{xy(1 - x - y)}}.$$

We are motivated to use this extremal measure because it is conjectured that $g(x, y) = f(x, y)$, where $f(x, y)$ is the limiting distribution of the Fekete points (as the degree approaches infinity) [4]. The limit $f(x, y)$ is known to exist and to satisfy [12]

$$g(x, y) \leq C f(x, y).$$

To distribute a finite set of points to approximate a given density $g(x, y)$ is not trivial. Our initial approximation, which has much room for improvement, is as follows. We first assume the points, at least topologically, form a nested family of triangles. Thus we compute a nested family of triangular shells which have a "mass" proportional to the number of points we have decided to place in that triangular shell. If there are k points to be placed in a given shell, we then chop that shell into k pieces, all with the same mass, and place one point in the center of each piece. This procedure produces points which respect the D_3 symmetry of the triangle, and the iterative method maintains this symmetry. Points in the triangle with such symmetry have orbits of either 1, 3, or 6 points. For a given set of points, there will be a variety of symmetry configurations which can be generated by altering the number of points within each shell, the number of shells, and the distribution of the points within a shell.

The algorithm does not place points along the edge of the triangle, although it does place a total of $3d$ points in the outermost triangular shell. In all cases we have computed, the iterative scheme quickly moves these points to the boundary of the triangle, and then aligns them to be the Gauss–Lobatto points, as conjectured. This remains true even when the initial distribution is not given D_3 symmetry. For $d > 7$, asymmetric initial data can lead to asymmetric solutions, but the boundary points are still the Gauss–Lobatto points.

5. Numerical results. We now present our computed approximate Fekete points. We use two metrics to evaluate the quality of these points in the triangle. The first metric is the Lebesgue constant, $\|I_N\|$. The cardinal functions needed to compute the Lebesgue constant are computed as described in section 4.1, and the maximum overall points in the triangle are approximated by the maximum over a grid of 2485 equally spaced points. These numbers are given for several different point distributions in Table 5.1. For comparison, we also list the Lebesgue constant for the Fekete points (Gauss–Lobatto points) in the square. Note that the actual Lebesgue constant $\|I_N\|$ is better than the estimate given in section 3. The values

TABLE 5.1

Lebesgue constants for point distributions in the triangle using the triangular polynomial truncation (unless noted otherwise). Key: "Fekete": approximate Fekete points from this paper. "CB Fekete": Chen and Babuška approximate Fekete points. "CB L^2": Chen and Babuška optimal L^2-norm points. "H": Hesthaven electrostatic points. "GL": Fekete points (Gauss–Lobatto points) in the square with the diamond polynomial truncation. For the "Fekete" points only, † denotes the presence of negative quadrature weights and ‡ denotes an asymmetric solution.

Degree	Fekete	CB Fekete	CB L^2	H	GL
6	4.17	4.17	3.79	4.07	3.50
7	4.91	4.93	4.39	4.78	3.89
8	9.43, 5.90†	5.90	5.09	5.88	4.18
9	6.80	6.80	5.92	6.92	4.50
10	8.11, 7.75†	8.00	7.09	8.40	4.71
11	8.76, 7.89†	9.52	8.34	10.09	5.02
12	9.60, 8.03†	11.08	10.08	12.52	5.20
13	9.21	13.24	12.05	15.34	5.49
14	10.8, 9.72†			22.18	5.62
15	9.97			29.69	5.91
16	12.1			41.73	6.08
17	13.3†				6.29
18	13.5				6.41
19	14.2‡				6.63

for the triangle can be seen to grow at a rate close to d. In the square, the Lebesgue constant is significantly better.

Our second metric is whether or not all the quadrature weights for a given point distribution are positive. These weights are computed by solving the linear system

$$\sum_{j=1}^{N} w_j g_i(z_j) = \int g_i \, dA$$

for a basis $\{g_i, i = 1, \ldots, N\}$ of $\mathcal{P}_\mathcal{N}$. This requires inverting the Vandermonde matrix, so again it is important that this matrix be well conditioned. Negative weights call into question the usefulness of the points for numerical integration. Thus in Table 5.1, we list two approximate Fekete grids for a given degree if the grid with the lowest Lebesgue constant is nonpositive. In all cases, when negative weights are present, they are limited to points on the edge of the triangle. One could modify the weights to make them positive, but any change in the quadrature formula would mean it no longer integrates all of $\mathcal{P}_\mathcal{N}$ exactly.

Finally we present several computed approximate Fekete points in Figures 6.1 and 6.2. Next to each point distribution is a plot of the cardinal function associated with the leftmost corner point. This cardinal function is chosen because it is associated with the smallest (and sometimes negative as indicated in Table 5.1) quadrature weight. The cardinal function values are plotted along a line from the corner point through the center of the triangle. Experience has shown that the largest amplitude oscillations of the cardinal function will be along this line. The grids plotted in Figure 6.1 all have positive quadrature, while the grids in Figure 6.2 have non-positive quadrature and the cardinal functions are more erratic.

6. Conclusions. We have presented an algorithm to compute approximate Fekete points in the triangle. This algorithm has room for improvement but has calculated grids with small Lebesgue constant and positive quadrature for up to degree 19. For degree $d > 10$, these points have the best known Lebesgue constants.

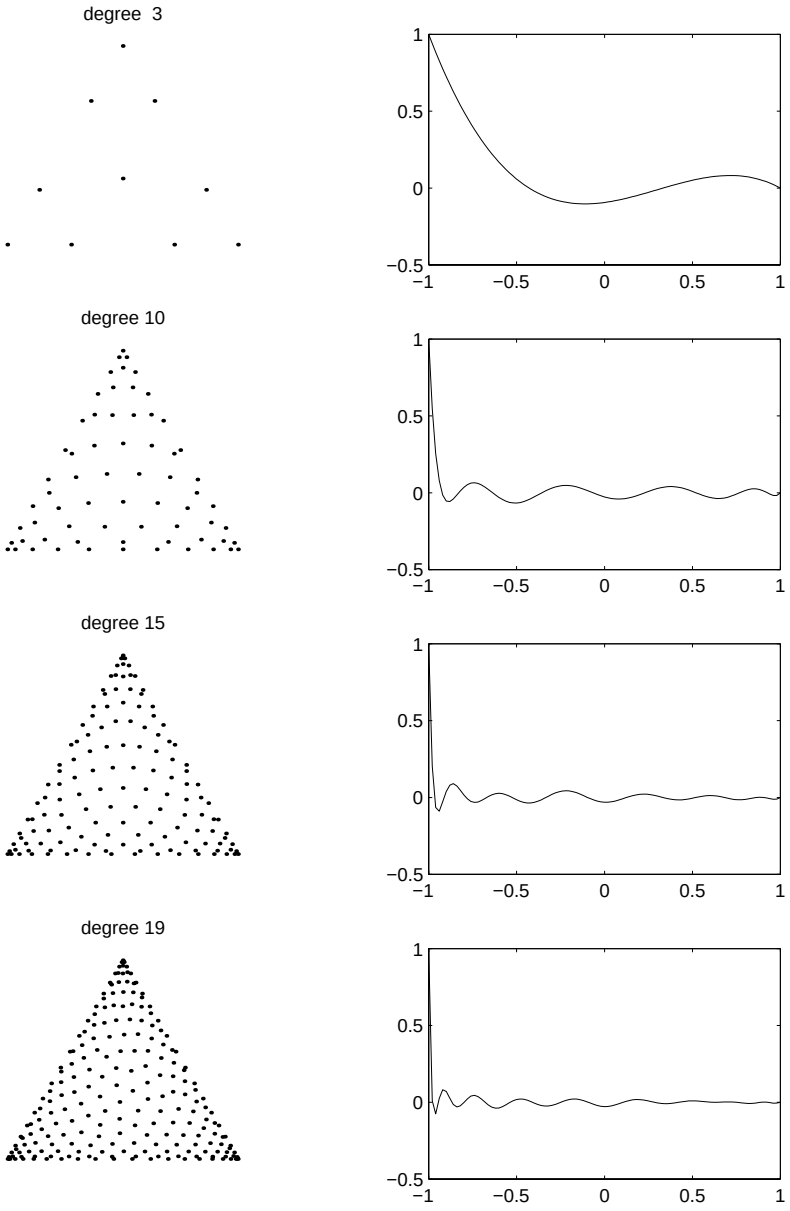

FIG. 6.1. *Approximate Fekete grids with positive quadrature. Adjacent to each grid is the cardinal function associated with the left most corner point, plotted along the line through the center of the triangle.*

Our numerical computation of Fekete points verifies a conjecture due to Bos [6] that the boundary Fekete points will be the one-dimensional Gauss–Lobatto points. This allows for easy coupling of Fekete point triangular spectral methods with the standard quadrilateral spectral element method. This latter method is also a Fekete point method, since the tensor product of Gauss–Lobatto points are the Fekete points for the square.

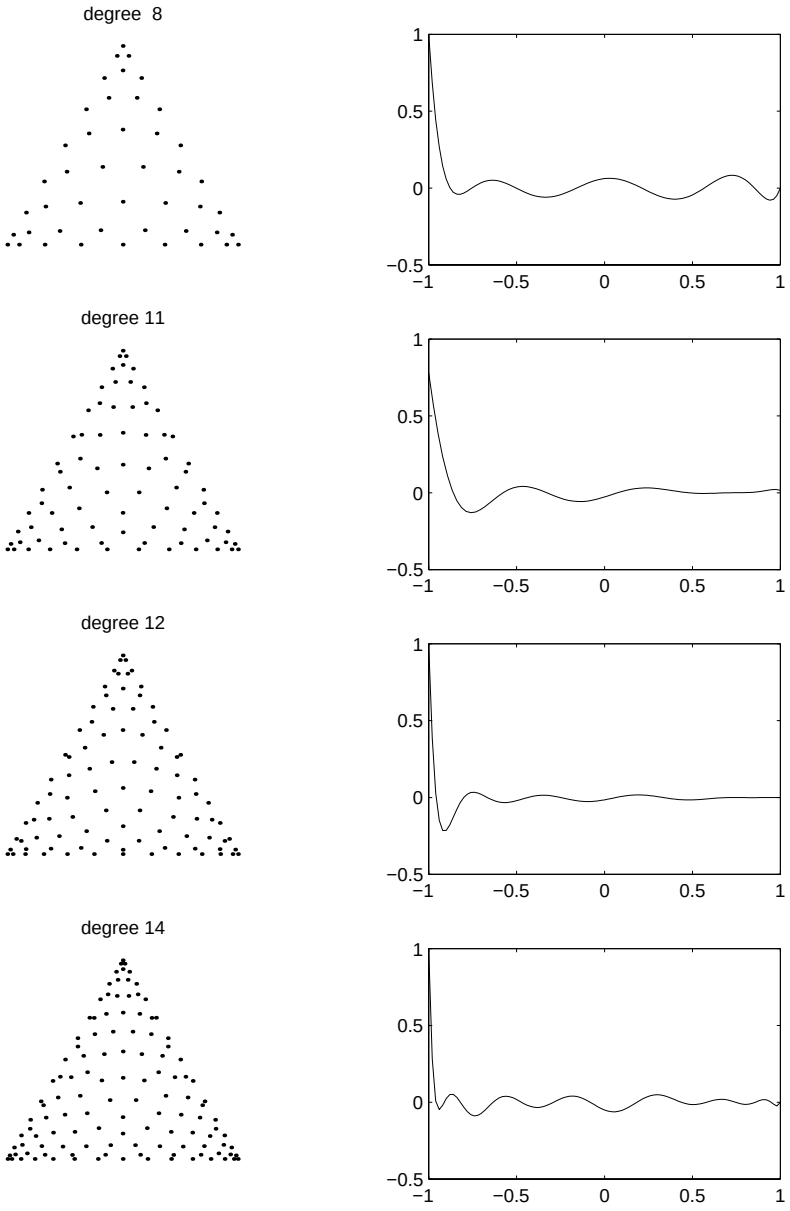

FIG. 6.2. *Approximate Fekete grids with nonpositive quadrature. Adjacent to each grid is the cardinal function associated with the left most corner point, plotted along the line through the center of the triangle.*

Appendix I. Newton acceleration. Newton's method can be used to greatly accelerate the convergence of the steepest accent algorithm presented in section 4.1. When the steepest accent method is close enough to a local maximum, one can fix the points along the boundary of the triangle and then use Newton's method to solve

the system

$$\frac{\partial |V|}{\partial z_i} = 0$$

for all z_i in the interior of the triangle. In practice, we find that Newton's method will converge if the initial conditions satisfy

$$|\partial_{z_i}|V|| < \frac{|V|}{10}$$

for all $i \le k$.

Newton's method requires forming and inverting the Hessian of $|V|$. Remarkably, the Hessian has a very simple form when $|V|$ is written in the cardinal function basis and the Hessian is evaluated at the points $\{z_i\}$ used to define the cardinal functions. The derivation is straightforward but tedious, so we only give the results. Let $z = (x, y)$ be the usual Cartesian coordinates of a point z in the triangle, and let ∂_x and ∂_y be differentiation with respect to x and y, respectively. The Vandermonde matrix is a function of the N points $\{z_i\}$, and we denote differentiation with respect to the coordinates of these points by ∂_{x_i} and ∂_{y_i}. The second derivative of $|V|$ evaluated at the points $\{z_i\}$ is given by

$$\partial_{a_i}\partial_{b_i}|V| = \partial_a\partial_b\phi_i(z_i)$$

and for $i \neq j$

$$\partial_{a_i}\partial_{b_j}|V| = \partial_a\phi_i(z_i)\partial_b\phi_j(z_j) - \partial_b\phi_i(z_j)\partial_a\phi_j(z_i),$$

where a and b can each be either x or y.

Now let k be the number of points inside the triangle and arrange the points so that $z_1, z_2, \ldots, z_k$ are all inside the triangle. Newton's method can then be written

$$\frac{\partial}{\partial t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix} = \begin{pmatrix} \partial_{x_i}\partial_{x_j}|V| & \partial_{x_i}\partial_{y_j}|V| \\ \partial_{y_i}\partial_{x_j}|V| & \partial_{y_i}\partial_{y_j}|V| \end{pmatrix}^1 \begin{pmatrix} \partial_{x_1}|V| \\ \partial_{x_2}|V| \\ \vdots \\ \partial_{x_k}|V| \\ \partial_{y_1}|V| \\ \partial_{y_2}|V| \\ \vdots \\ \partial_{y_k}|V| \end{pmatrix},$$

where the $2k \times 2k$ Hessian matrix is written in terms of four $k \times k$ blocks.

Appendix II. Tables of approximate Fekete points. We now give the coordinates for selected approximate Fekete points. We list the grids which by symmetry must have a point in the center of the triangle, since it can be seen in Table 5.1 that these grids tend to have the best Lebesgue constants. For degree 12, we give both the positive and nonpositive quadrature grids mentioned in Table 5.1. Coordinates for all grids are available via email from the first author. We list the first two barycentric coordinates of each point (equivalent to the x and y coordinates after an equilateral triangle is linearly mapped to the right triangle with unit length base and height) along with the quadrature weight `w`. The third barycentric coordinate is defined such that the sum of all three coordinates is one. Only one point is listed from points possessing symmetries of the triangle. Such a point represents `orbit`=1,3, or 6 total points via permutations of its three barycentric coordinates.

```
d= 3:
orbit=1    0.3333333333   0.3333333333  w= 0.9000000000
orbit=3    0.0000000000   0.0000000000  w= 0.0333333333
orbit=6    0.0000000000   0.2763932023  w= 0.1666666667
d= 6:
orbit=1    0.3333333333   0.3333333333  w= 0.2178563571
orbit=3    0.1063354684   0.1063354684  w= 0.1104193374
orbit=3    0.0000000000   0.5000000002  w= 0.0358939762
orbit=3    0.0000000000   0.0000000000  w= 0.0004021278
orbit=6    0.1171809171   0.3162697959  w= 0.1771348660
orbit=6    0.0000000000   0.2655651402  w= 0.0272344079
orbit=6    0.0000000000   0.0848854223  w= 0.0192969460
d= 9:
orbit=1    0.3333333333   0.3333333333  w= 0.1096011288
orbit=3    0.1704318201   0.1704318201  w= 0.0767491008
orbit=3    0.0600824712   0.4699587644  w= 0.0646677819
orbit=3    0.0489345696   0.0489345696  w= 0.0276211659
orbit=3    0.0000000000   0.0000000000  w= 0.0013925011
orbit=6    0.1784337588   0.3252434900  w= 0.0933486453
orbit=6    0.0588564879   0.3010242110  w= 0.0619010169
orbit=6    0.0551758079   0.1543901944  w= 0.0437466450
orbit=6    0.0000000000   0.4173602935  w= 0.0114553907
orbit=6    0.0000000000   0.2610371960  w= 0.0093115568
orbit=6    0.0000000000   0.1306129092  w= 0.0078421987
orbit=6    0.0000000000   0.0402330070  w= 0.0022457501
d=12:
orbit=1    0.3333333333   0.3333333333  w= 0.0626245179
orbit=3    0.1988883477   0.4005558262  w= 0.0571359417
orbit=3    0.2618405201   0.2618405201  w= 0.0545982307
orbit=3    0.0807386775   0.0807386775  w= 0.0172630326
orbit=3    0.0336975736   0.0336975736  w= 0.0142519606
orbit=3    0.0000000000   0.5000000000  w= 0.0030868485
orbit=3    0.0000000000   0.0000000000  w= 0.0004270742
orbit=6    0.1089969290   0.3837518758  w= 0.0455876390
orbit=6    0.1590834479   0.2454317980  w= 0.0496701966
orbit=6    0.0887037176   0.1697134458  w= 0.0387998322
orbit=6    0.0302317829   0.4071849276  w= 0.0335323983
orbit=6    0.0748751152   0.2874821712  w= 0.0268431561
orbit=6    0.0250122615   0.2489279690  w= 0.0237377452
orbit=6    0.0262645218   0.1206826354  w= 0.0177255972
orbit=6    0.0000000000   0.3753565349  w= 0.0043097313
orbit=6    0.0000000000   0.2585450895  w= 0.0028258057
orbit=6    0.0000000000   0.1569057655  w= 0.0030994935
orbit=6    0.0000000000   0.0768262177  w= 0.0023829062
orbit=6    0.0000000000   0.0233450767  w= 0.0009998683
d=12:    (some negative quadrature weights)
orbit=1    0.3333333333   0.3333333333  w= 0.0485965670
orbit=3    0.2201371125   0.3169406831  w= 0.0602711576
orbit=3    0.2201371125   0.4629222044  w= 0.0602711576
```

```
orbit=3    0.1877171129   0.1877171129  w= 0.0476929767
orbit=3    0.1403402144   0.4298298928  w= 0.0453940802
orbit=3    0.0833252778   0.0833252778  w= 0.0258019417
orbit=3    0.0664674598   0.0252297247  w= 0.0122004614
orbit=3    0.0218884020   0.4890557990  w= 0.0230003812
orbit=3    0.0252297247   0.0664674598  w= 0.0122004614
orbit=3    0.0000000000   0.5000000000  w= 0.0018106475
orbit=3    0.0000000000   0.0000000000  w=-0.0006601747
orbit=6    0.1157463404   0.2842319093  w= 0.0455413513
orbit=6    0.0672850606   0.3971764400  w= 0.0334182802
orbit=6    0.0909839531   0.1779000668  w= 0.0324896773
orbit=6    0.0318311633   0.3025963402  w= 0.0299402736
orbit=6    0.0273518579   0.1733665506  w= 0.0233477738
orbit=6    0.0000000000   0.3753565349  w= 0.0065962854
orbit=6    0.0000000000   0.2585450895  w= 0.0021485117
orbit=6    0.0000000000   0.1569057655  w= 0.0034785755
orbit=6    0.0000000000   0.0768262177  w= 0.0013990566
orbit=6    0.0000000000   0.0233450767  w= 0.0028825748
d=15:
orbit=1    0.3333333333   0.3333333333  w= 0.0459710878
orbit=3    0.2379370518   0.3270403780  w= 0.0346650571
orbit=3    0.3270403780   0.2379370518  w= 0.0346650571
orbit=3    0.1586078048   0.4206960976  w= 0.0384470625
orbit=3    0.2260541354   0.2260541354  w= 0.0386013566
orbit=3    0.1186657611   0.1186657611  w= 0.0224308157
orbit=3    0.0477095725   0.4761452137  w= 0.0243531004
orbit=3    0.0531173538   0.0531173538  w= 0.0094392654
orbit=3    0.0219495841   0.0219495841  w= 0.0061105652
orbit=3    0.0000000000   0.0000000000  w= 0.0001283162
orbit=6    0.1585345951   0.3013819154  w= 0.0305412307
orbit=6    0.0972525649   0.3853507643  w= 0.0262101254
orbit=6    0.0875150140   0.2749910734  w= 0.0265367617
orbit=6    0.1339547708   0.1975591066  w= 0.0269859772
orbit=6    0.0475622627   0.3524012205  w= 0.0172635676
orbit=6    0.0596194677   0.1978887556  w= 0.0188795851
orbit=6    0.0534939782   0.1162464503  w= 0.0158224870
orbit=6    0.0157189888   0.4176001732  w= 0.0127170850
orbit=6    0.0196887324   0.2844332752  w= 0.0164489660
orbit=6    0.0180698489   0.1759511193  w= 0.0120018620
orbit=6    0.0171941515   0.0816639421  w= 0.0072268907
orbit=6    0.0000000000   0.4493368632  w= 0.0023599161
orbit=6    0.0000000000   0.3500847655  w= 0.0017624674
orbit=6    0.0000000000   0.2569702891  w= 0.0018648017
orbit=6    0.0000000000   0.1738056486  w= 0.0012975716
orbit=6    0.0000000000   0.1039958541  w= 0.0018506035
orbit=6    0.0000000000   0.0503997335  w= 0.0009919379
orbit=6    0.0000000000   0.0152159769  w= 0.0004893506
d=18:
orbit=1    0.3333333333   0.3333333333  w= 0.0326079297
```

```
orbit=3    0.2515553103   0.3292984162  w= 0.0255331366
orbit=3    0.3292984162   0.2515553103  w= 0.0255331366
orbit=3    0.1801930996   0.4099034502  w= 0.0288093886
orbit=3    0.2438647767   0.2438647767  w= 0.0279490452
orbit=3    0.1512564554   0.1512564554  w= 0.0174438045
orbit=3    0.0810689493   0.4594655253  w= 0.0203594338
orbit=3    0.0832757649   0.0832757649  w= 0.0113349170
orbit=3    0.0369065587   0.0369065587  w= 0.0046614185
orbit=3    0.0149574850   0.0149574850  w= 0.0030346239
orbit=3    0.0000000000   0.5000000000  w= 0.0012508731
orbit=3    0.0000000000   0.0000000000  w= 0.0000782945
orbit=6    0.1821465920   0.3095465041  w= 0.0235716330
orbit=6    0.1246901255   0.3789288931  w= 0.0206304700
orbit=6    0.1179441386   0.2868915642  w= 0.0204028340
orbit=6    0.1639418454   0.2204868669  w= 0.0215105697
orbit=6    0.0742549663   0.3532533654  w= 0.0183482070
orbit=6    0.0937816771   0.2191980979  w= 0.0174161032
orbit=6    0.0890951387   0.1446273457  w= 0.0155972434
orbit=6    0.0409065243   0.4360543636  w= 0.0119269616
orbit=6    0.0488675890   0.2795984854  w= 0.0147074804
orbit=6    0.0460342127   0.2034211147  w= 0.0116182830
orbit=6    0.0420687187   0.1359040280  w= 0.0087639138
orbit=6    0.0116377940   0.4336892286  w= 0.0098563528
orbit=6    0.0299062187   0.3585587824  w= 0.0096342355
orbit=6    0.0132313129   0.2968103667  w= 0.0086477936
orbit=6    0.0136098469   0.2050279257  w= 0.0083868302
orbit=6    0.0124869684   0.1232146223  w= 0.0062576643
orbit=6    0.0365197797   0.0805854893  w= 0.0077839825
orbit=6    0.0118637765   0.0554881302  w= 0.0031415239
orbit=6    0.0000000000   0.4154069883  w= 0.0006513246
orbit=6    0.0000000000   0.3332475761  w= 0.0021137942
orbit=6    0.0000000000   0.2558853572  w= 0.0004393452
orbit=6    0.0000000000   0.1855459314  w= 0.0013662119
orbit=6    0.0000000000   0.1242528987  w= 0.0003331251
orbit=6    0.0000000000   0.0737697111  w= 0.0011613225
orbit=6    0.0000000000   0.0355492359  w= 0.0004342867
orbit=6    0.0000000000   0.0106941169  w= 0.0002031499
```

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