CHEBFUN GUIDE 5: COMPLEX CHEBFUNS

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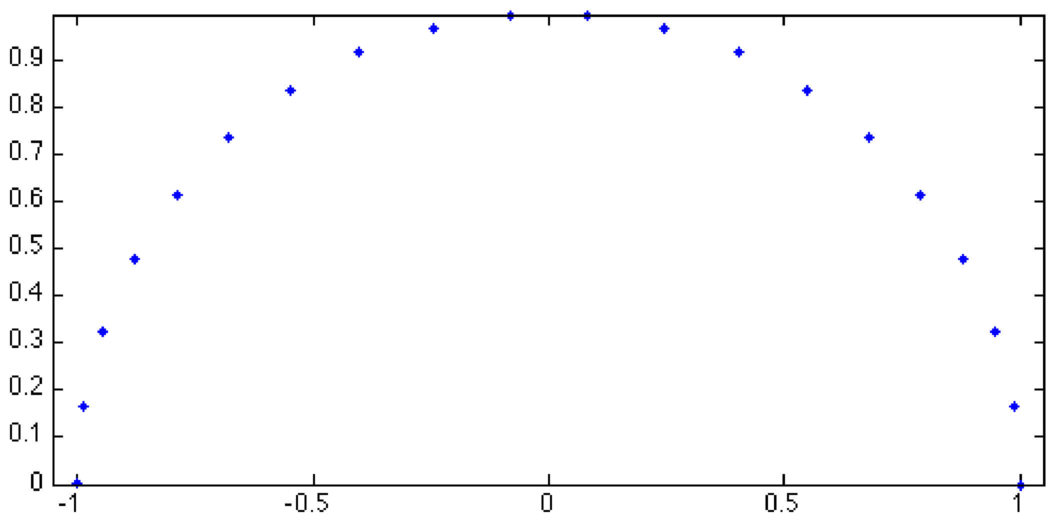
5.1 Complex functions of a real variable

One of the attractive features of Matlab is that it handles complex arithmetic well. For example, here are 20 points on the upper half of the unit circle in the complex plane:

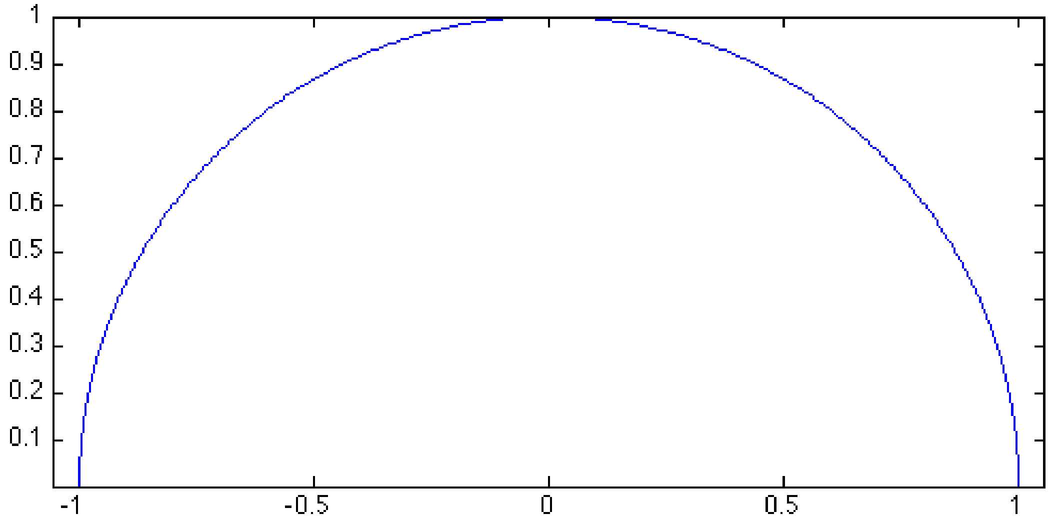
```
s = linspace(0,pi,20);
f = exp(1i*s);
plot(f,'.')
axis equal
```



In Matlab, both the variables `i` and `j` are initialized as i, the square root of -1, but this code uses `1i` instead (just as one might write, for example, `3+2i` or `2.2-1.1i`). Writing the imaginary unit in this fashion is a common trick among Matlab programmers, for it avoids the risk of surprises caused by `i` or `j` having been overwritten by other values. The `axis equal` command ensures that the real and imaginary axes are scaled equally.

Here is a Chebfun analogue.

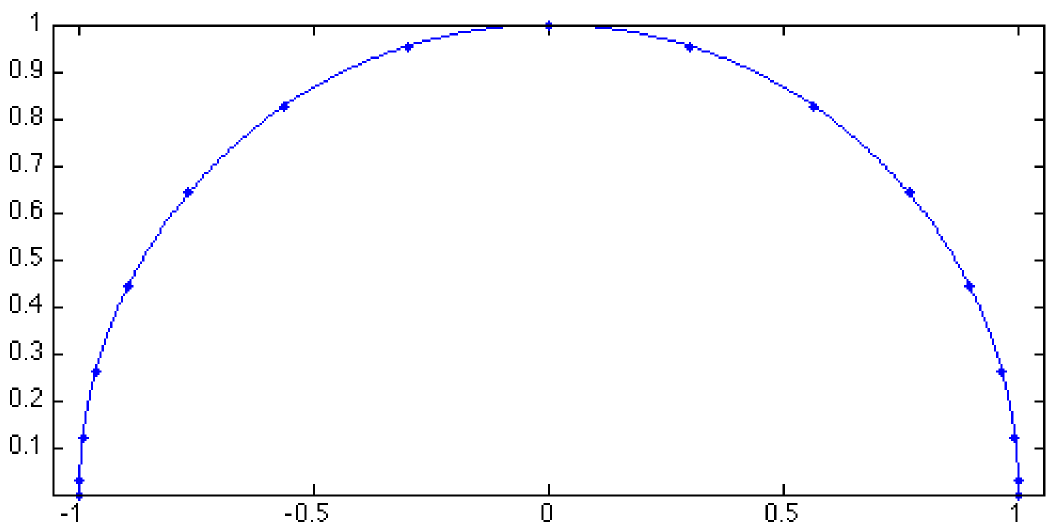
```
s = chebfun(@(s) s,[0 pi]);
f = exp(1i*s);
plot(f)
axis equal
```



The Chebfun semicircle is represented by a polynomial of low degree:

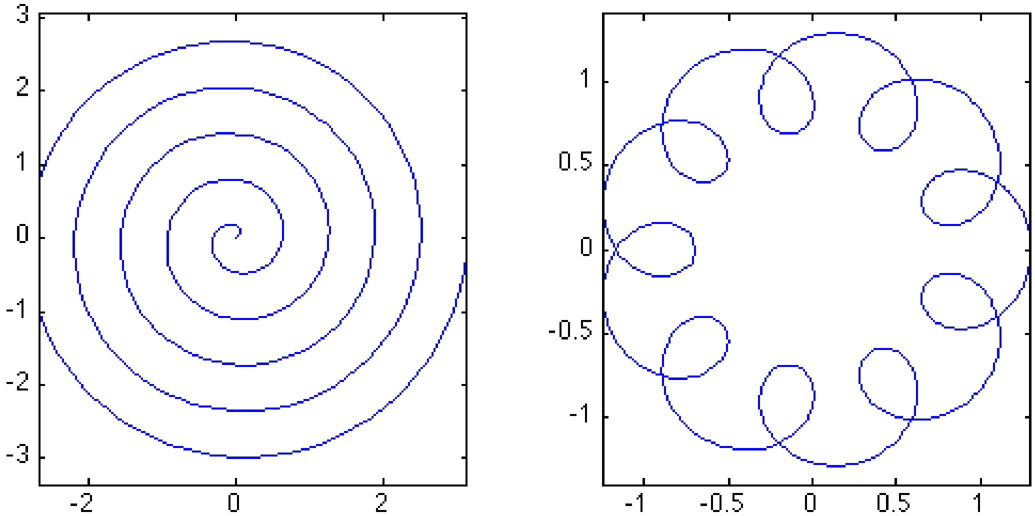
```
length(f)
plot(f,'.-')
axis equal
```

ans =
17

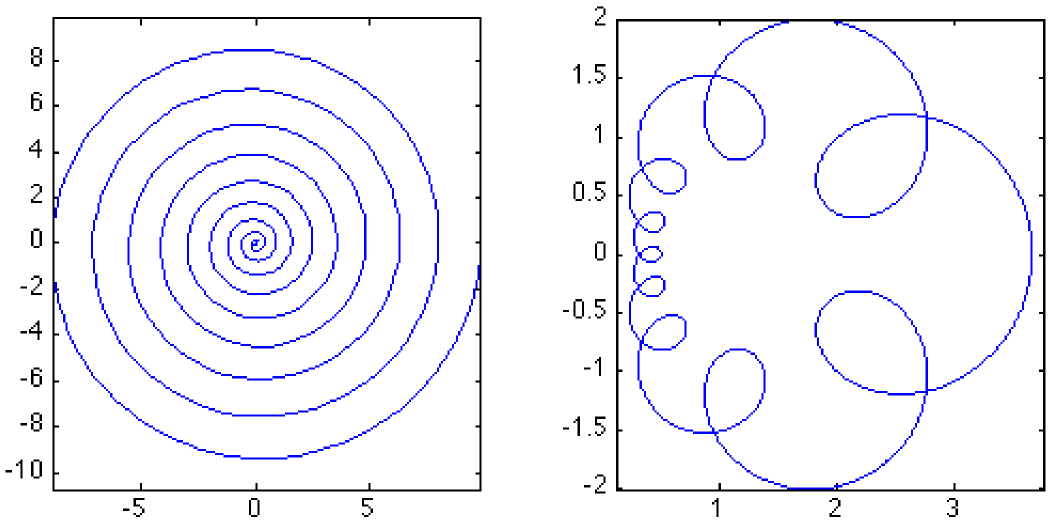


We can have fun with variations on the theme:

```
subplot(1,2,1), g = s.*exp(10i*s); plot(g), axis equal
subplot(1,2,2), h = exp(2i*s)+.3*exp(20i*s); plot(h), axis equal
```




```
subplot(1,2,1), plot(g.^2), axis equal
subplot(1,2,2), plot(exp(h)), axis equal
```



Such plots make pretty pictures, but as always with Chebfun, the underlying operations involve precise mathematics carried out to many digits of accuracy. For example, the integral of `g` is $-\pi i/10$,

```
sum(g)
```

```
ans =
 -0.000000000000002 - 0.314159265358979i
```

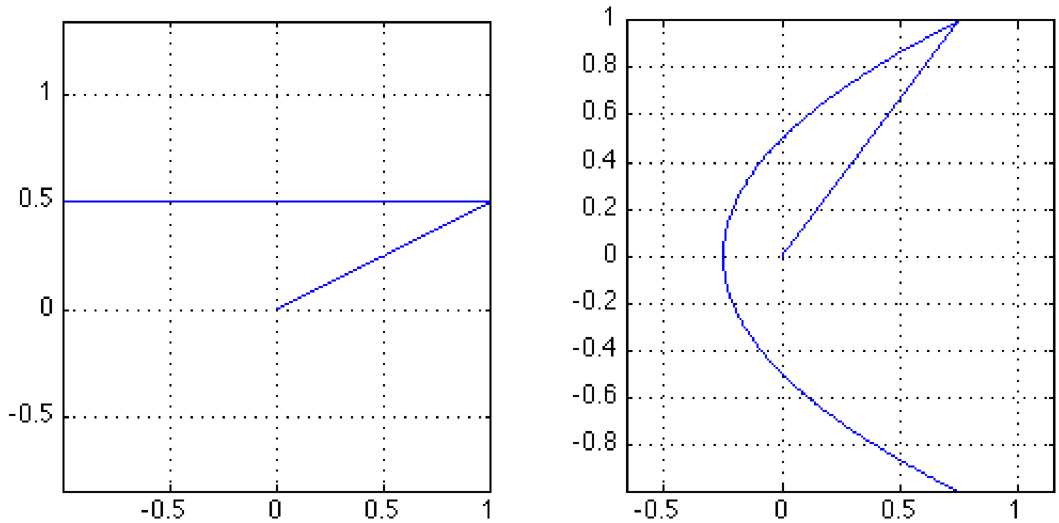
and the integral of `h` is zero:

```
sum(h)
```

```
ans =
     -6.999851837738844e-16 + 1.291664876099699e-16i
```

Piecewise smooth complex chebfuns are also possible. For example, the following starts from a chebfun `z` defined as $(1+0.5i)s$ for s on the interval $[0,1]$ and $1+0.5i-2(s-1)$ for s on the interval $[1,2]$.

```
z = chebfun({@(s) (1+.5i)*s, @(s) 1+.5i-2*(s-1)},[0 1 2]);
subplot(1,2,1), plot(z), axis equal, grid on
subplot(1,2,2), plot(z.^2), axis equal, grid on
```



Actually, this way of constructing a piecewise chebfun is rather clumsy. An easier method is to use the `join` command, in which a construction like `join(f,g,h)` constructs a single chebfun with the same values as `f`, `g`, and `h`, but on a domain concatenated together. Thus if the domains of `f`, `g`, `h` are $[a,b]$, $[c,d]$, and $[e,d]$, then `chebjoin(f,g,h)` has three pieces with domains $[a,b]$, $[b,b+(d-c)]$, $[b+(d-c),b+(d-c)+(d-e)]$. Using this trick, we can construct the chebfun `z` above in the following alternative manner:

```
s = chebfun(@(s) s,[0 1]);
zz = join((1+.5i)*s, 1+.5i-2*s);
norm(z-zz)
```

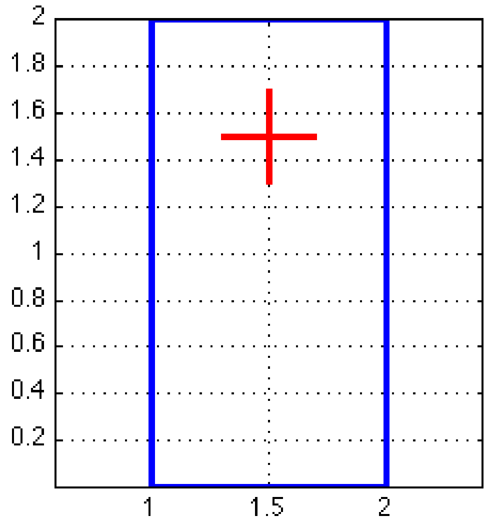
```
ans =
     1.281975124255709e-16
```

5.2 Analytic functions and conformal maps

A function is *analytic* if it is differentiable in the complex sense, or equivalently, if it has a convergent Taylor series near each point. Analytic functions do interesting things in the complex plane. In particular, away from points where the derivative is zero, they are *conformal maps*, which means that though they may scale and rotate an infinitesimal region, they preserve angles between intersecting curves.

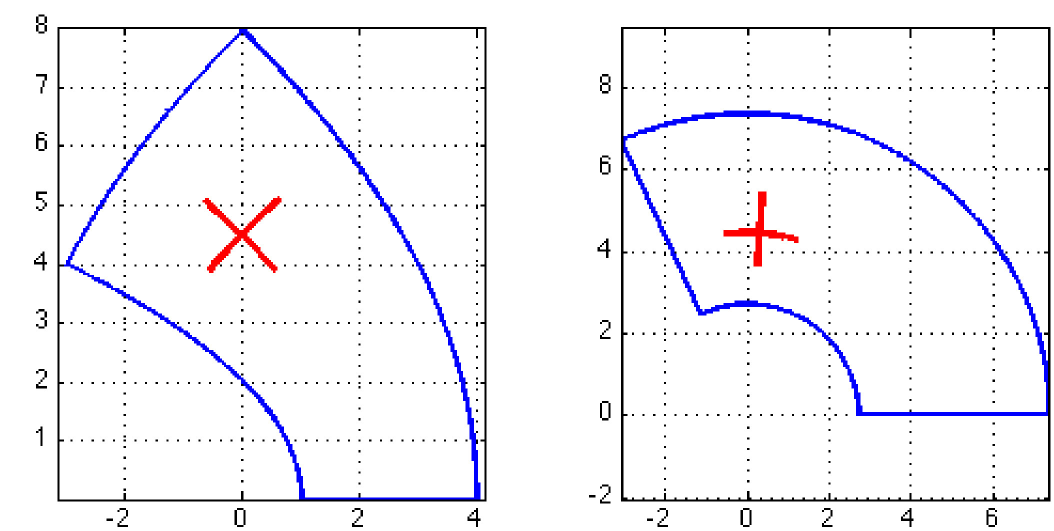
For example, suppose we define `R` to be a chebfun corresponding to the four sides of a rectangle and we define `X` to be another chebfun corresponding to a cross inside `R`.

```
s = chebfun('s',[0 1]);
R = join(1+s, 2+2i*s, 2+2i-s, 1+2i-2i*s);
LW = 'linewidth'; lw1 = 2; lw2 = 3;
clf, subplot(1,2,1), plot(R,LW,lw2), grid on, axis equal
X = join(1.3+1.5i+.4*s, 1.5+1.3i+.4i*s);
hold on, plot(X,'r',LW,lw2)
```



Here we see what happens to `R` and `X` under the maps z^2 and $\exp(z)$:

```
clf
subplot(1,2,1), plot(R.^2,LW,lw1), grid on, axis equal
hold on, plot(X.^2,'r',LW,lw2)
subplot(1,2,2), plot(exp(R),LW,lw1), grid on, axis equal
hold on, plot(exp(X),'r',LW,lw2)
```

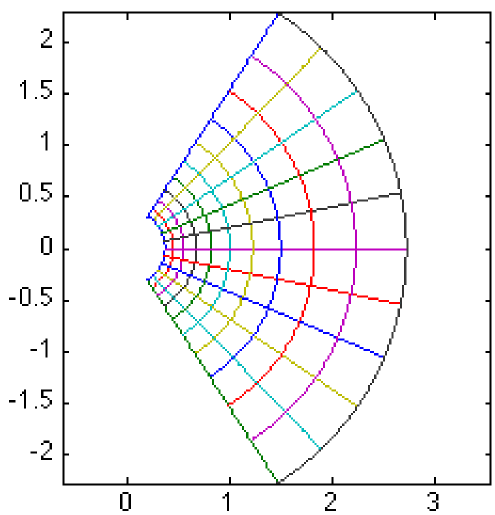


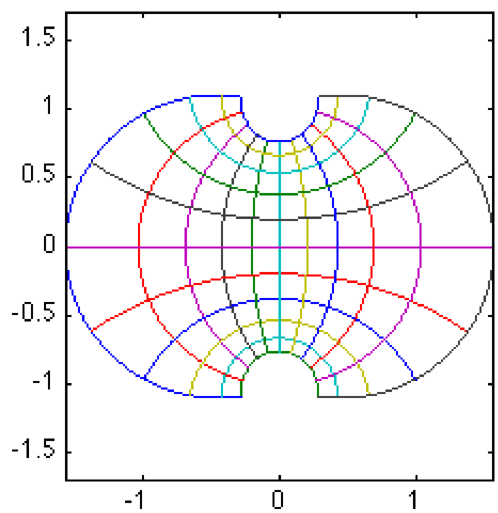
We can take the same idea further and construct a grid in the complex plane, each segment of which is a piece of a chebfun that happens to be linear. In this case we accumulate the various pieces as successive columns of a quasimatrix, i.e., a "matrix" whose columns are chebfuns.

```
x = chebfun(@(x) x);
S = chebfun;                     % make an empty chebfun
for d = -1:.2:1
  S = [S d+1i*x 1i*d+x];         % add 2 more lines to the collection
end
clf,
% subplot(1,2,1), plot(S), axis equal
```

Here are the exponential and tangent of the grid:

```
subplot(1,2,1), plot(exp(S)), axis equal
subplot(1,2,2), plot(tan(S)), axis equal
```



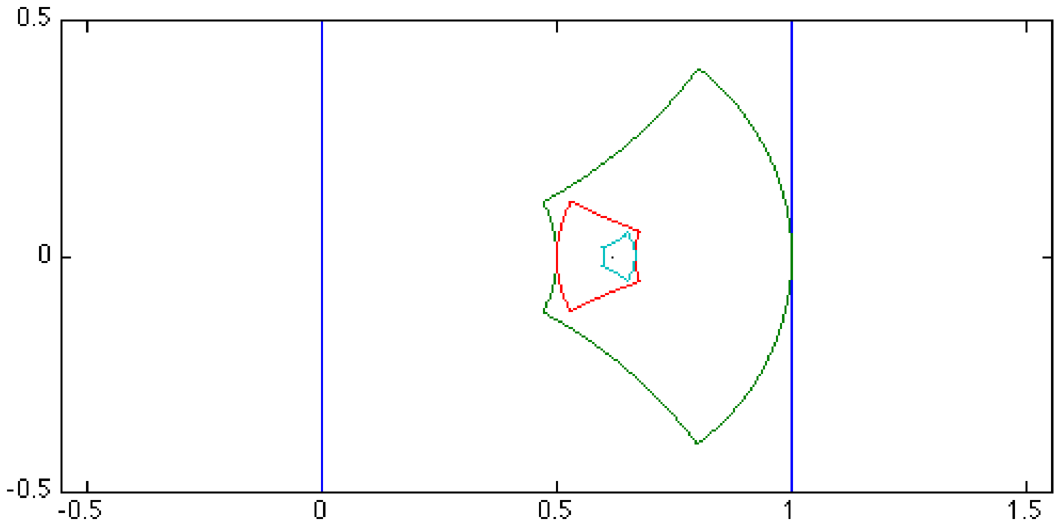


Here is a sequence that puts all three images together on a single scale:

```
  clf
% plot(S), hold on
% plot(1.6+exp(S))
% plot(6.6+tan(S))
% axis equal, axis off
```

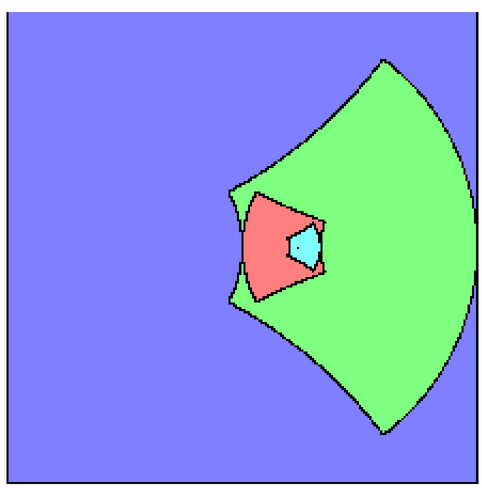
A particularly interesting set of conformal maps are the *Moebius transformations*, the rational functions of the form $(az + b)/(cz + d)$ for constants a, b, c, d. For example, here is a square and its image under the map $w = 1/(1 + z)$, and the image of the image under the same map, and the image of the image of the image. We also plot the limit point given by the equation $z = 1/(1 + z)$, i.e., $z = (sqrt(5) - 1)/2$.

```
moebius = @(z) 1./(1+z);
s = chebfun(@(s) s,[0 1]);
S = join(-.5i+s, 1-.5i+1i*s, 1+.5i-s, .5i-1i*s);
clf
for j = 1:3
  S = [S moebius(S(:,j))];
end
plot(S)
hold on, axis equal
plot((sqrt(5)-1)/2,0,'.k','markersize',4)
```



Here's a prettier version of the same image using the Chebfun `fill` command.

```
S = join(-.5i+s, 1-.5i+1i*s, 1+.5i-s, .5i-1i*s);
clf
fill(real(S),imag(S),[.5 .5 1]), axis equal, hold on
S = moebius(S); fill(real(S),imag(S),[.5 1 .5])
S = moebius(S); fill(real(S),imag(S),[1 .5 .5])
S = moebius(S); fill(real(S),imag(S),[.5 1 1 ])
plot((sqrt(5)-1)/2,0,'.k','markersize',4)
axis off
```



5.3 Contour integrals

If s is a real parameter and $z(s)$ is a complex function of s, then we can define a contour integral in the complex plane like this:

$$\int f(z(s))z'(s)ds.$$

The contour in question is the curve described by $z(s)$ as s varies over its range.

For example, in the example at the end of Section 5.1 the contour consists of two straight segments that begin at 0 and end at $-1 + .5i$. We can compute the integral of $\exp(-z^2)$ over the contour like this:

```
f = exp(-z.^2);
I = sum(f.*diff(z))
```

```
I =
 -0.842544559526136 + 0.166587147924074i
```

Notice how easily the contour integral is realized in Chebfun, even over a contour consisting of several pieces. This particular integral is related to the complex error function [Weideman 1994].

According to *Cauchy's theorem*, the integral of an analytic function around a closed curve is zero, or equivalently, the integral between two points z_1 and z_2 is path-independent. To verify this, we can compute the same integral over the straight segment going directly from 0 to $-1 + 0.5i$:

```
w = chebfun('(-1+.5i)*s',[0 1]);
f = exp(-w.^2);
I2 = sum(f.*diff(w))
```

```
I2 =
 -0.842544559526136 + 0.166587147924074i
```

A *meromorphic function* is a function that is analytic in a region of interest in the complex plane apart from possible poles. According to the *Cauchy integral formula*, $1/2\pi i$ times the integral of a meromorphic function f around a closed contour is equal to the sum of the residues of f associated with any poles it may have in the enclosed region. The *residue* of f at a point z_0 is the coefficient of the degree -1 term in its Laurent expansion at z_0. For example, the function $\exp(z)/z^3$ has Laurent series $z^{-3} + z^{-2} + (1/2)z^{-1} + (1/6)z^0 + \ldots$ at the origin, and so its residue there is $1/2$. We can confirm this by computing the contour integral around a circle:

```
z = chebfun('exp(1i*s)',[0 2*pi]);
f = exp(z)./z.^3;
I = sum(f.*diff(z))/(2i*pi)
```

```
I =
  0.500000000000000 - 0.000000000000000i
```

Notice that we have just computed the degree 2 Taylor coefficient of $\exp(z)$.

When Chebfun integrates around a circular contour like this, it does not normally take advantage of the fact that the integrand is periodic. That would be Fourier analysis as opposed to Chebyshev analysis, and a "Fourtech" approach would be more efficient for such problems [Davis 1959]. (This can be achieved by calling the Chebfun constructor with the `periodic` flag; an example will be added.) Chebyshev analysis is more flexible, however, since it does not require periodicity, and the loss in efficiency is only about a factor of $\pi/2$ [Hale & Trefethen 2008].

The contour does not have to have radius 1, or be centered at the origin:

```
z = chebfun('1+2*exp(1i*s)',[0 2*pi]);
f = exp(z)./z.^3;
I2 = sum(f.*diff(z))/(2i*pi)
```

```
I2 =
  0.500000000000000 - 0.000000000000000i
```

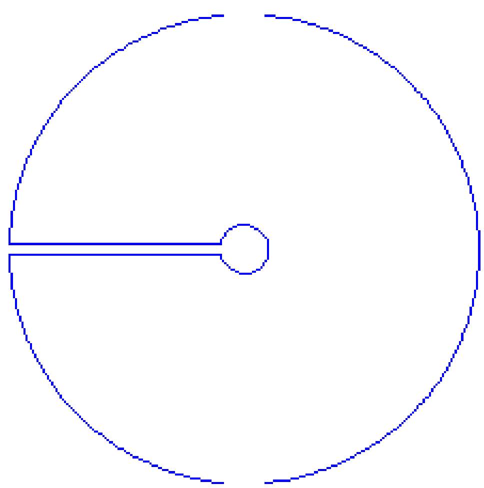
Nor does the contour have to be smooth. Here let us compute the same result by integration over a square:

```
s = chebfun('s',[-1 1]);
z = join(1+1i*s, 1i-s, -1-1i*s, -1i+s);
f = exp(z)./z.^3;
I3 = sum(f.*diff(z))/(2i*pi)
```

```
I3 =
   0.500000000000000
```

In Chebfun one can also construct more interesting contours of the kind that appear in complex variables texts. Here is an example:

```
  c = [-2+.05i, -.2+.05i, -.2-.05i, -2-.05i];    % 4 corners
  s = chebfun('s',[0 1]);
  z = join(c(1)+s*(c(2)-c(1)), c(2)*c(3).^s./c(2).^s, ...
       c(3)+s*(c(4)-c(3)), c(4)*c(1).^s./c(4).^s);
  clf, plot(z), axis equal, axis off
```



The integral of $f(z) = \log(z)\tanh(z)$ around this contour will be equal to $2\pi i$ times the sum of the residues at the poles of f at $\pm\pi i/2$.

```
f = log(z).*tanh(z);
I = sum(f.*diff(z))
Iexact = 4i*pi*log(pi/2)
```

```
I =
  0.000000000000005 + 5.674755637702225i
Iexact =
  0.000000000000000 + 5.674755637702224i
```

5.4 Cauchy integrals and locating zeros and poles

Here are some further examples of computations with Cauchy integrals. The Bernoulli number B_k is $k!$ times the kth Taylor coefficient of $z/((exp(z) - 1)$. Here is B_{10} compared with its exact value $5/66$.

```
k = 10;
z = chebfun('5*exp(1i*s)',[0 2*pi]);
f = z./((exp(z)-1));
B10 = factorial(k)*sum((f./z.^(k+1)).*diff(z))/(2i*pi)
exact = 5/66
```

```
B10 =
  0.075757575757576 + 0.000000000000001i
exact =
   0.075757575757576
```

Notice that we have taken z to be a circle of radius 5. If the radius is 1, the accuracy is a good deal lower:

```
z = chebfun('exp(1i*s)',[0 2*pi]);
f = z./((exp(z)-1));
B10 = factorial(k)*sum((f./z.^(k+1)).*diff(z))/(2i*pi)
```

```
B10 =
  0.075757575601521 - 0.000000000169988i
```

This problem of numerical instability would arise no matter how one calculated the integral over the unit circle; it is not the fault of Chebfun. For a study of how to pick the optimal radius, see [Bornemann 2009].

Another use of Cauchy integrals is to count zeros or poles of functions in specified regions. According to the *principle of the argument*, the number of zeros minus the number of poles of f in a region is

$$N = \frac{1}{2\pi i} \int \frac{f'(z)dz}{f(z)}$$

where the integral is taken over the boundary. Since $f' = df/dz = (df/ds)(ds/dz)$, we can rewrite this as

$$N = \frac{1}{2\pi i} \int \frac{df}{fds} ds.$$

For example, the function $f(z) = \sin(z)^3 + \cos(z)^3$ clearly has no poles; how many zeros does it have in the disk about 0 of radius 2? The following calculation shows that the answer is 3:

```
z = chebfun('2*exp(1i*s)',[0 2*pi]);
f = sin(z).^3 + cos(z).^3;
N = sum((diff(f)./f))/(2i*pi)
```

```
N =
  3.000000000000001 - 0.000000000000001i
```

What is really going on here is a calculation of the change of the argument of f as the boundary is traversed. Another way to find that number is with the Chebfun overloads of the Matlab commands `angle` and `unwrap`:

```
anglef = unwrap(angle(f));
N = (anglef(end)-anglef(0))/(2*pi)
```

```
N =
   2.999999999999858
```

Variations on this idea enable one to locate zeros and poles as well as count them. For example, we can locate a single zero with the formula

$$r = \frac{1}{2\pi i} \int z(df/ds)/f\,ds$$

[McCune 1966]. Here is the zero of the function above in the unit disk:

```
z = chebfun('exp(1i*s)',[0 2*pi]);
f = sin(z).^3 + cos(z).^3;
r = sum(z.*(diff(f)./f))/(2i*pi)
```

```
r =
 -0.785398163397448 + 0.000000000000000i
```

We can check the result by a more ordinary Chebfun calculation:

```
x = chebfun('x');
f = sin(x).^3 + cos(x).^3;
r = roots(f)
```

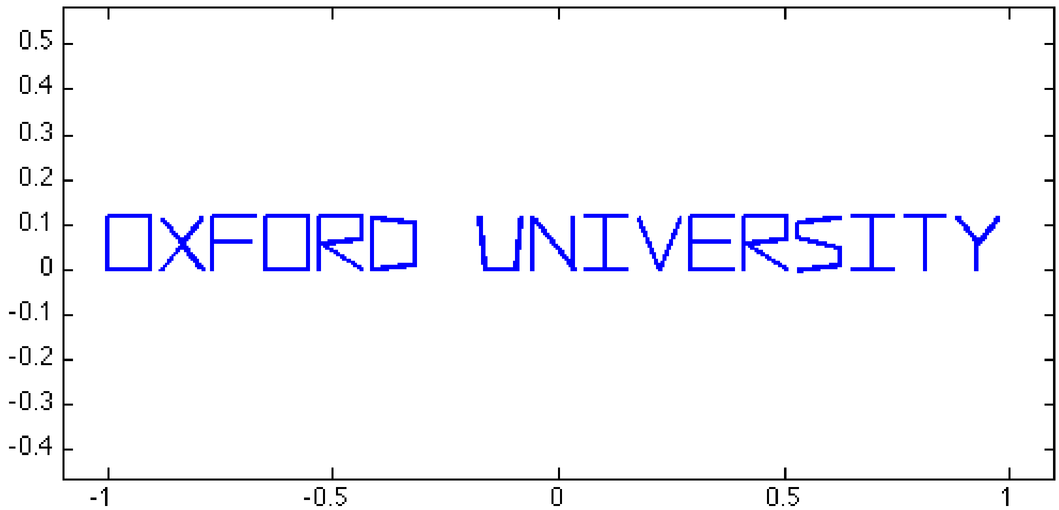
```
r =
  -0.785398163397449
```

To find multiple zeros via Cauchy integrals, and for many other generalizations of the ideas in this chapter, see [Austin, Kravanja & Trefethen 2013].

5.5 Alphabet soup

The Chebfun command `scribble` returns a piecewise linear complex chebfun corresponding to a word spelled out in capital letters. For example:

```
f = scribble('Oxford University');
LW = 'linewidth'; lw = 2;
plot(f,LW,lw), xlim(1.1*[-1 1]), axis equal
```



This chebfun happens to have 67 pieces:

```
length(domain(f))-1
```

```
ans =
    67
```

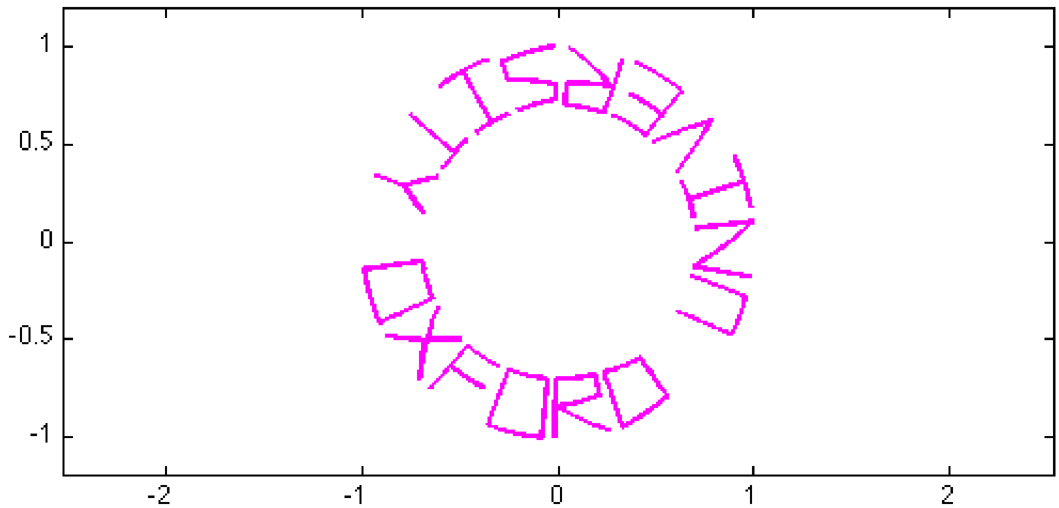
Though its applications are unlikely to be mathematical, f is a precisely defined mathematical object just like any other chebfun. If we wish, we can compute with it:

```
f(0), norm(f)
```

```
ans =
   0.129411764705882
ans =
   0.847576500999202
```

Perhaps more interesting is that we can apply functions to it and learn something in the process:

```
plot(exp(3i*f),'m',LW,lw), ylim(1.2*[-1 1]), axis equal
```



Does putting a box around enhance the image? (We do this by adding a second column of a Chebfun quasimatrix -- see Chapter 6.)

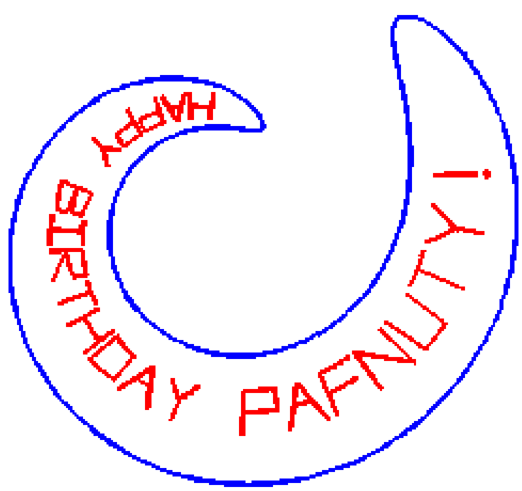
```
%L = f.ends(end);
%s = chebfun(@(x) 2*x+2,[-1 -0.5]);
%box = join(-1.1-.05i+2.2*s,1.1-.05i+.22i*s,1.1+.17i-2.2*s,-1.1+.17i-.22i*s);
%f = [f box];
%plot(f,LW,lw), xlim(1.2*[-1 1]), axis equal
```

```
clf, plot(exp((1+.2i)*f),LW,lw), axis equal, axis off
```

```
plot(tan(f),LW,lw), axis equal, axis off
```

Next May 16, you might wish to write a greeting card for Pafnuty Lvovich Chebyshev, accurate as always to 15 digits:

```
f = scribble('Happy Birthday Pafnuty!');
L = f.ends(end);
g = @(z) exp(-2.2i+(2.5i+.4)*z);
clf, plot(g(f),'r',LW,lw), axis equal, axis off
circle = 1.12*chebfun(@(x) exp(2i*pi*x/L),[0 L]);
ellipse = 1.2*(circle + 1./circle)/2 + 1i*mean(imag(f));
hold on, plot(g(ellipse),'b',LW,lw)
axis auto equal off
```



You can find an example "Birthday cards and analytic function" in the Complex Variables section of the Chebfun Examples collection, and further related explorations in the Geometry section.

5.6 References

[Austin, Kravanja & Trefethen 2013] A. P. Austin, P. Kravanja and L. N. Trefethen, "Numerical algorithms based on analytic function values at roots of unity," *SIAM Journal on Numerical Analysis*, to appear.

[Bornemann 2009] F. Bornemann, "Accuracy and stability of computing high-order derivatives of analytic functions by Cauchy integrals", *Foundations of Computational Mathematics*, 11 (2011), 1-63.

[Davis 1959] P. J. Davis, "On the numerical integration of periodic analytic functions", in R. E. Langer, ed., *On Numerical Integration*, Math. Res. Ctr., U. of Wisconsin, 1959, pp. 45-59.

[Hale & Trefethen 2008] N. Hale and L. N. Trefethen, "New quadrature formulas from conformal maps", *SIAM Journal on Numerical Analysis*, 46 (2008), 930-948.

[McCune 1966] J. E. McCune, "Exact inversion of dispersion relations", *Physics of Fluids*, 9 (1966), 2082-2084.

[Weideman 1994] J. A. C. Weideman, "Computation of the complex error function", *SIAM Journal on Numerical Analysis*, 31 (1994), 1497-1518.