CHEBFUN GUIDE 1: GETTING STARTED WITH CHEBFUN

Lloyd N. Trefethen, October 2009, latest revision May 2014

1.1 What is a chebfun?

A chebfun is a function of one variable defined on an interval $[a, b]$. The syntax for chebfuns is almost exactly the same as the usual Matlab syntax for vectors, with the familiar Matlab commands for vectors overloaded in natural ways. Thus, for example, whereas `sum(f)` returns the sum of the entries when `f` is a vector, it returns a definite integral when `f` is a chebfun.

Chebfun with a capital C is the name of the software system.

The aim of Chebfun is to "feel symbolic but run at the speed of numerics". More precisely our vision is to achieve for functions what floating-point arithmetic achieves for numbers: rapid computation in which each successive operation is carried out exactly apart from a rounding error that is very small in relative terms [Trefethen 2007].

The implementation of Chebfun is based on the mathematical fact that smooth functions can be represented very efficiently by polynomial interpolation in Chebyshev points, or equivalently, thanks to the Fast Fourier Transform, by expansions in Chebyshev polynomials. For a simple function, 20 or 30 points often suffice, but the process is stable and effective even for functions complicated enough to require 1000 or 1,000,000 points. Chebfun makes use of adaptive procedures that aim to find the right number of points automatically so as to represent each function to roughly machine precision, that is, about 15 digits of relative accuracy. (Originally Chebfun stored function values at Chebyshev points; in Version 5 it switched to storing Chebyshev expansion coefficients.)

The mathematical foundations of Chebfun are for the most part well established by results scattered throughout the 20th century. A key early figure, for example, was Bernstein in the 1910s. Much of the relevant material can be found collected in the Chebfun-based book [Trefethen 2013].

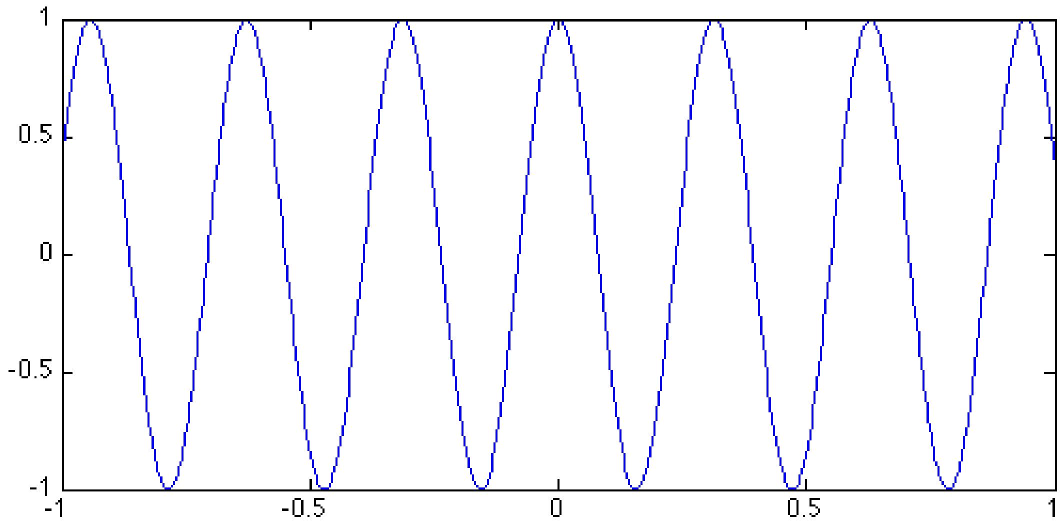
Chebfun was originally created by Zachary Battles and Nick Trefethen at Oxford during 2002-2005 [Battles & Trefethen 2004]. Battles left the project in 2005, and soon four new members were added to the team: Ricardo Pachon (from 2006), Rodrigo Platte (from 2007), and Toby Driscoll and Nick Hale (from 2008). In 2009, Asgeir Birkisson and Mark Richardson also became involved, and other contributors included Pedro Gonnet, Joris Van Deun, and Georges Klein. Nick Hale served as Director of the project during 2010-2014. The Chebfun Version 5 rewrite was directed by Nick Hale during 2013-2014, and the team included Anthony Austin, Asgeir Birkisson, Toby Driscoll, Hrothgar, Mohsin Javed, Hadrien Montanelli, Nick Trefethen, Grady Wright, and Kuan Xu. Further information about Chebfun history is available at the Chebfun web site.

This Guide is based on Chebfun Version 5, released in 2014. Chebfun is available at http://www.maths.ox.ac.uk/chebfun/ (http://www.maths.ox.ac.uk/chebfun/), and an earlier version of the Guide for Version 4 can found there.

1.2 Constructing simple chebfuns

The `chebfun` command constructs a chebfun from a specification such as a string or an anonymous function. If you don't specify an interval, then the default interval $[-1, 1]$ is used. For example, the following command makes a chebfun corresponding to $\cos(20x)$ on $[-1, 1]$ and plots it.

```
f = chebfun('cos(20*x)');
plot(f)
```



From this little experiment, you cannot see that `f` is represented by a polynomial. One way to see this is to find the length of `f`:

```
length(f)
```

```
ans =
    59
```

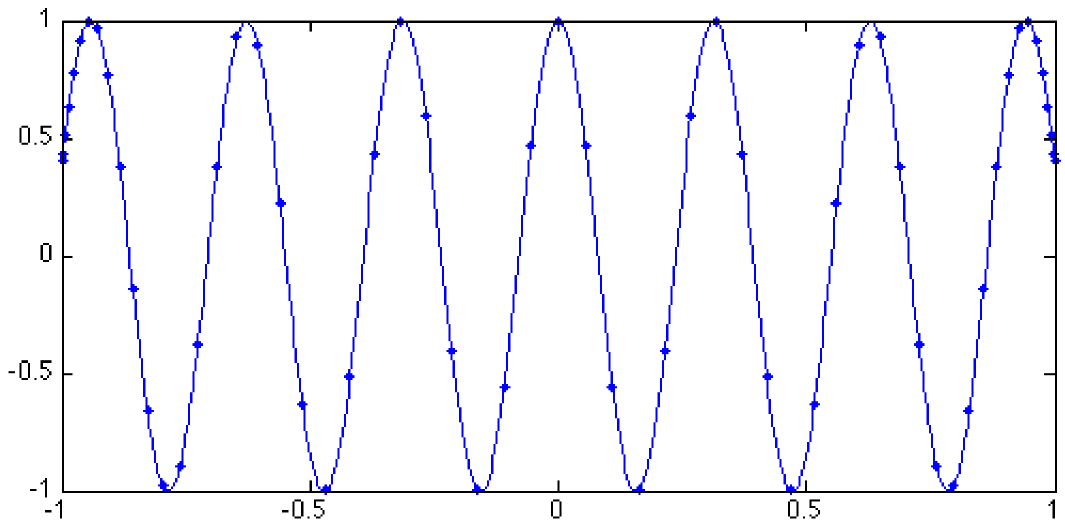
Another is to remove the semicolon that suppresses output:

```
f
```

```
f =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]       59       0.41     0.41
Epslevel = 2.183626e-15.  Vscale = 1.
```

These results tell us that `f` is represented by a polynomial interpolant through 59 Chebyshev points, i.e., a polynomial of degree 58. These numbers have been determined by an adaptive process. We can see the data points by plotting `f` with the `'.-'` option:

```
plot(f,'.-')
```



The formula for $N+1$ Chebyshev points in $[-1,1]$ is

$$x(j) = -\cos(j\pi/N), \quad j = 0:N,$$

and in the figure we can see that the points are clustered accordingly near 1 and -1. Note that in the middle of the grid, there are about 5 points per wavelength, which is evidently what it takes to represent this cosine to 15 digits of accuracy. For intervals other than $[-1,1]$, appropriate Chebyshev points are obtained by a linear scaling.

The curve between the data points is the polynomial interpolant, which is evaluated by the barycentric formula introduced by Salzer [Berrut & Trefethen 2004, Salzer 1972]. This method of evaluating polynomial interpolants is stable and efficient even if the degree is in the millions [Higham 2004].

What is the integral of f from -1 to 1? Here it is:

```
sum(f)
```

```
ans =
   0.091294525072763
```

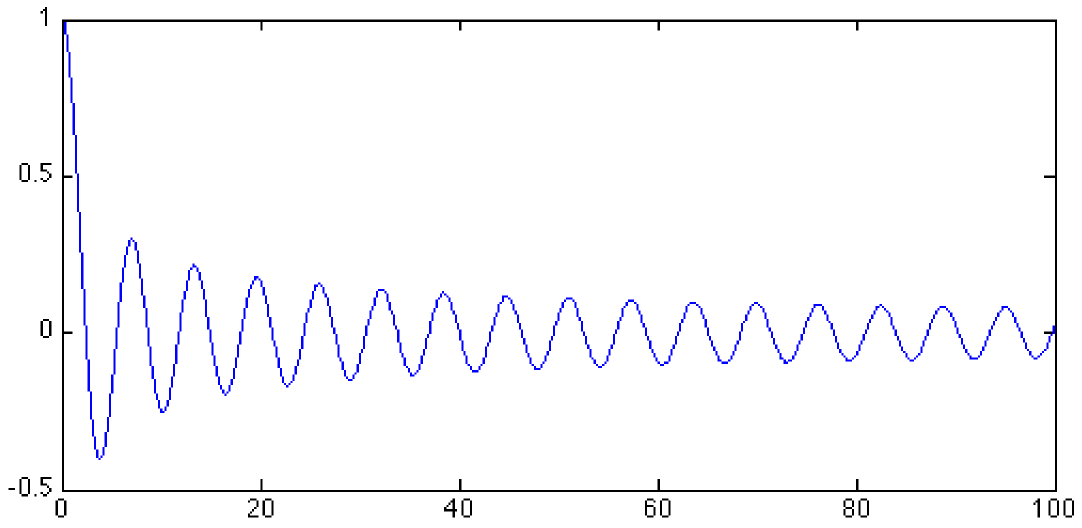
This number was computed by integrating the polynomial (Clenshaw-Curtis quadrature -- see Section 2.1), and it is interesting to compare it to the exact answer from calculus:

```
exact = sin(20)/10
```

```
exact =
   0.091294525072763
```

Here is another example, now with the chebfun defined by an anonymous function instead of a string. In this case the interval is specified as $[0,100]$.

```
g = chebfun(@(t) besselj(0,t),[0,100]);
plot(g), ylim([-.5 1])
```



The function looks complicated, but it is actually a polynomial of surprisingly small degree:

```
length(g)
```

```
ans =
    89
```

Is it accurate? Well, here are three random points in $[0, 100]$:

```
format long
x = 100*rand(3,1)
```

```
x =
  91.337585613901936
  63.235924622540949
   9.754040499940952
```

Let's compare the chebfun to the true Bessel function at these points:

```
exact = besselj(0,x);
error = g(x) - exact;
[g(x) exact error]
```

```
ans =
  -0.070938332803690  -0.070938332803690  -0.000000000000000
   0.093053890749966   0.093053890749966                   0
  -0.227754955147806  -0.227754955147806  -0.000000000000000
```

If you want to know the first 5 zeros of the Bessel function, here they are:

```
r = roots(g); r = r(1:5)
```

```
r =
   2.404825557695767
   5.520078110286297
   8.653727912911013
  11.791534439014285
  14.930917708487780
```

Notice that we have just done something nontrivial and potentially useful. How else would you find zeros of the Bessel function so readily? As always with numerical computation, we cannot expect the answers to be exactly correct, but they will usually be very close. In fact, these computed zeros are accurate to close to machine precision:

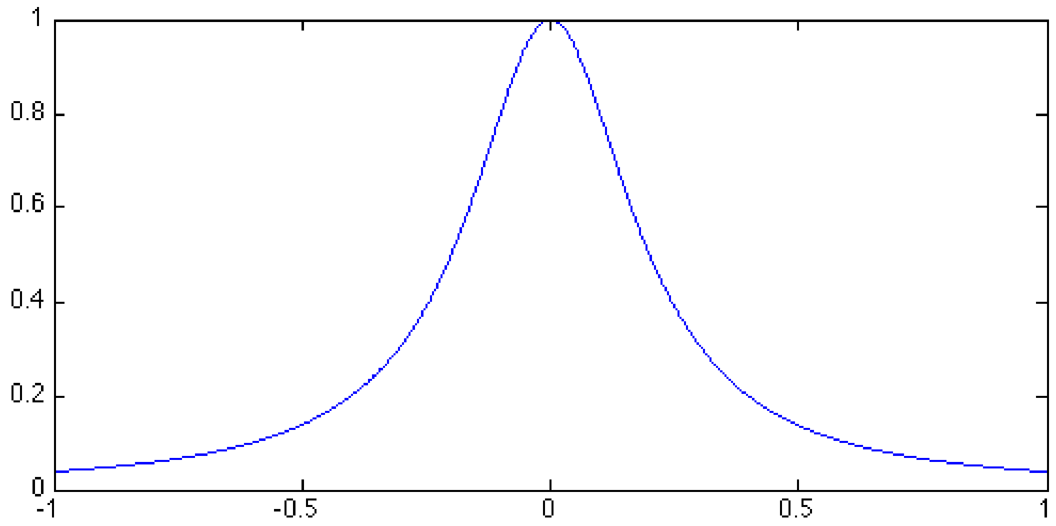
```
besselj(0,r)
```

```
ans =
   1.0e-14 *
   0.314549201444184
  -0.455365848063064
   0.010653221425832
   0.071369727817872
   0.161773287356131
```

Most often we get a chebfun by operating on other chebfuns. For example, here is a sequence that uses plus, times, divide, and power operations on an initial chebfun `x` to produce a famous function of Runge:

```
x = chebfun('x');
f = 1./(1+25*x.^2);
length(f)
clf, plot(f)
```

```
ans =
   183
```



1.3 Operations on chebfuns

There are more than 200 commands that can be applied to a chebfun. For a list of many of them you can type `methods` :

```
methods chebfun
```

```
Methods for class chebfun:

abs              cot              issing           rank
acos             cotd             iszero           rdivide
acosd            coth             join             real
acosh            cov              jump             reallog
acot             csc              ldivide          rem
acotd            cscd             le               remez
acoth            csch             legpoly          repmat
acsc             ctranspose       length           residue
acscd            cumprod          log              restrict
acsch            cumsum           log10            roots
airy             cylinder         log1p            round
all              diff             log2             sec
and              dirac            logical          secd
angle            disp             loglog           sech
any              display          lt               semilogx
arcLength        domain           lu               semilogy
area             ellipj           mat2cell         sign
asec             ellipke          max              simplify
asecd            end              mean             sin
asech            epslevel         merge            sinc
asin             eq               mesh             sind
asind            erf              min              sinh
asinh            erfc             minandmax        size
atan             erfcinv          minus            sound
atan2            erfcx            mldivide         spy
atan2d           erfinv           mod              sqrt
```

```
atand            exp              mrdivide         std
atanh            expm1            mtimes           subsasgn
besselh          feval            ne               subspace
besseli          fill             newDomain        subsref
besselj          find             nextpow2         sum
besselk          fix              norm             surf
bessely          fliplr           normal           surface
bvp4c            flipud           normest          surfc
bvp5c            floor            not              svd
cat              fred             null             tan
ceil             ge               num2cell         tand
cf               get              or               tanh
cheb2cell        gt               orth             times
cheb2quasi       heaviside        overlap          transpose
chebellipseplot  horzcat          pde15s           uminus
chebfun          hscale           permute          unwrap
chebpade         hypot            pinv             uplus
chebpoly         imag             plot             vander
chebpolyplot     innerProduct     plot3            var
chebtune         integral         plus             vertcat
complex          inv              poly             volt
compose          isdelta          polyfit          vscale
cond             isempty          pow2             waterfall
conj             isequal          power            why
conv             isfinite         prod             xor
cos              isinf            qr
cosd             isnan            quantumstates
cosh             isreal           quasi2cheb

Static methods:

constructor      ode113           pchip
interp1          ode15s           spline
lagrange         ode45
```

To find out what a command does, you can use `help` .

```
help chebfun/sum
```

```
 SUM   Definite integral of a CHEBFUN.
    SUM(F) is the integral of a column CHEBFUN F over its domain of definition.

    SUM(F, A, B), where A and B are scalars, integrates a column CHEBFUN F over
    [A, B], which must be a subdomain of F.domain:

                          B
                          /
                SUM(F) =  | F(t) dt.
                          /
                         A

    SUM(F, A, B), where A and B are CHEBFUN objects, returns a CHEBFUN S which
    satisfies

                        B(s)
                        /
                S(s) =  | F(t) dt.
                        /
                      A(s)

    SUM(F, DIM), where DIM is one of 1, 2, sums F over the dimension DIM. If F
    is a column CHEBFUN and DIM = 1 or if F is a row CHEBFUN and DIM = 2 then
    this integrates in the continuous dimension of F, as described above.
    Otherwise, SUM(F, DIM) sums across the columns (rows) of the column (row)
    CHEBFUN F.

  See also CUMSUM, DIFF.
```

Most of the commands in the list exist in ordinary Matlab; some exceptions are `domain`, `restrict`, `chebpoly`, and `remez`. We have already seen `length` and `sum` in action. In fact we have already seen `subsref` too, since that is the Matlab command for (among other things) evaluating arguments in parentheses. Here is another example of its use:

```
f(0.5)
```

```
ans =
   0.137931034482759
```

Here for comparison is the true result:

```
1/(1+25/4)
```

```
ans =
   0.137931034482759
```

In this Runge function example, we have also implicitly seen `times`, `plus`, `power`, and `rdivide`, all of which have been overloaded from their usual Matlab uses to apply to chebfuns.

In the next part of this tour we shall explore many of these commands systematically. First, however, we should see that chebfuns are not restricted to smooth functions.

1.4 Piecewise smooth chebfuns

Many functions of interest are not smooth but piecewise smooth. In this case a chebfun may consist of a concatenation of smooth pieces, each with its own polynomial representation. Each of the smooth pieces is called a "fun". This enhancement of Chebfun was developed initially by Ricardo Pachon during 2006-2007, then also by Rodrigo Platte starting in 2007 [Pachon, Platte and Trefethen 2009]. Essentially funs are the "classic chebfuns" for smooth functions on $[-1, 1]$ originally implemented by Zachary Battles in Chebfun Version 1.

Later we shall describe the options in greater detail, but for the moment let us see some examples. One way to get a piecewise smooth function is directly from the constructor, taking advantage of its capability of automatic edge detection. For example, in the default "splitting off" mode a function with a jump in its derivative produces a warning message,

```
f = chebfun('abs(x-.3)');
```

```
Warning: Function not resolved using 65537 pts. Have you tried 'splitting on'?
```

The same function can be successfully captured with splitting on:

```
f = chebfun('abs(x-.3)','splitting','on');
```

The `length` command reveals that `f` is defined by four data points, two for each linear interval:

```
length(f)
```

```
ans =
     4
```

We can see the structure of `f` in more detail by typing `f` without a semicolon:

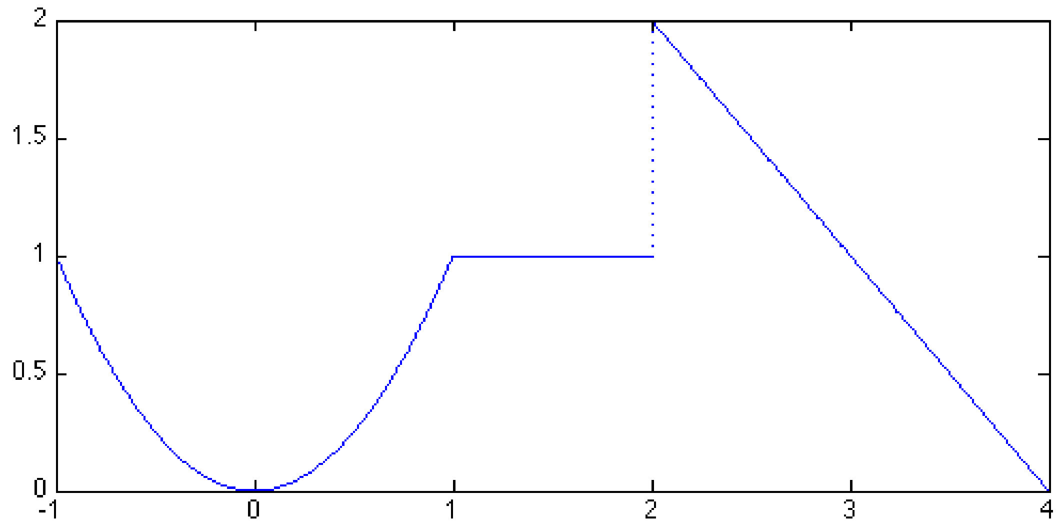
```
f
```

```
f =
   chebfun column (2 smooth pieces)
       interval       length   endpoint values
[      -1,     0.3]        2       1.3        0
[     0.3,       1]        2         0      0.7
Epslevel = 6.492624e-16.  Vscale = 1.300000e+00.  Total length = 4.
```

This output confirms that f consists of two funs, each defined by two points and two corresponding function values. The functions live on intervals defined by breakpoints at -1, 1, and a number very close to 0.3. The `Vscale` field is related to the maximum absolute value of `f` and `Epslevel` gives some information (to be discussed later) about its relative accuracy.

Another way to make a piecewise smooth chebfun is to construct it explicitly from various pieces. For example, the following command specifies three functions x^2, 1, and $4 - x$, together with a vector of endpoints indicating that the first function applies on $[-1, 1]$, the second on $[1, 2]$, and the third on $[2, 4]$:

```
f = chebfun({@(x) x.^2, @(x) 1+0*x, @(x) 4-x},[-1 1 2 4]);
plot(f)
```



We expect `f` to consist of three pieces of lengths 3, 1, and 2, and this is indeed the case:

```
f
```

```
f =
   chebfun column (3 smooth pieces)
       interval       length   endpoint values
[      -1,       1]        3         1        1
[       1,       2]        1         1        1
[       2,       4]        2         2        0
Epslevel = 6.492624e-16.  Vscale = 2.000000e+00.  Total length = 6.
```

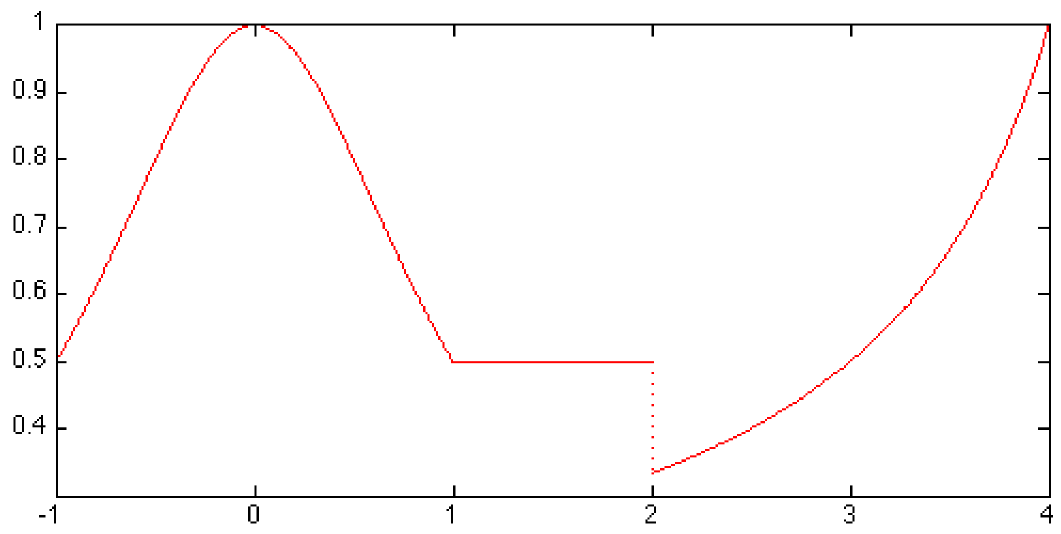
Our eyes see pieces, but to Chebfun, `f` is just another function. For example, here is its integral.

```
sum(f)
```

```
ans =
   3.666666666666667
```

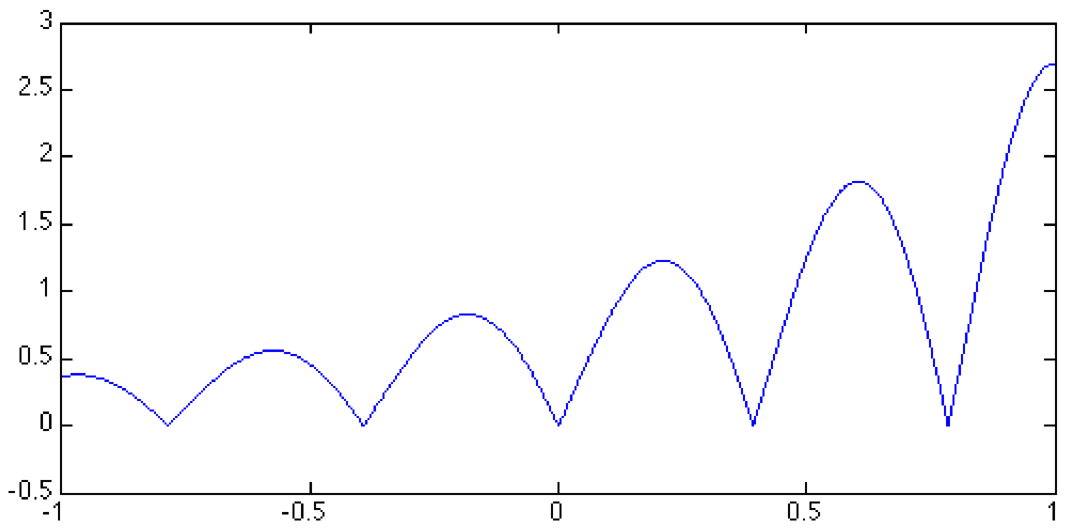
Here is an algebraic transformation of `f`, which we plot in another color for variety.

```
plot(1./(1+f),'r')
```



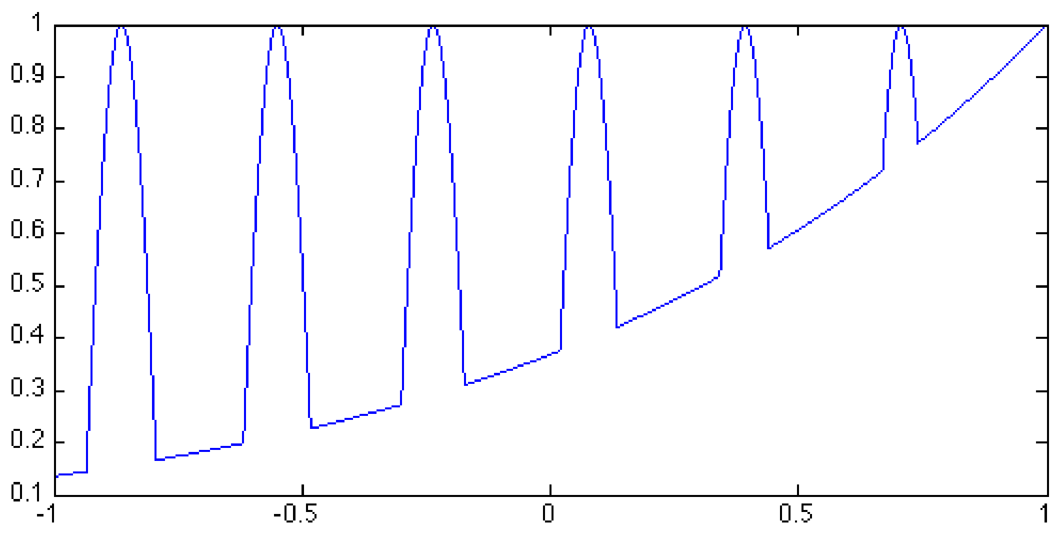
Some Chebfun commands naturally introduce breakpoints in a chebfun. For example, the `abs` command first finds zeros of a function and introduces breakpoints there. Here is a chebfun consisting of 6 funs:

```
f = abs(exp(x).*sin(8*x));
plot(f)
```



And here is an example where breakpoints are introduced by the `max` command, leading to a chebfun with 13 pieces:

```
f = sin(20*x);
g = exp(x-1);
h = max(f,g);
plot(h)
```



As always, h may look complicated to a human, but to Chebfun it is just a function. Here are its mean, standard deviation, minimum, and maximum:

```
mean(h)
```

```
ans =
   0.578242020778010
```

```
std(h)
```

```
ans =
   0.280937455806246
```

```
min(h)
```

```
ans =
   0.135335283236613
```

```
max(h)
```

```
ans =
   1.000000000000001
```

A final note about piecewise smooth chebfuns is that the automatic edge detection or "splitting" feature, when it is turned on, may subdivide functions even though they do not have clean point singularities, and this may be desirable or undesirable depending on the application. For example, considering $\sin(x)$ over $[0, 1000]$ with splitting on, we end up with a chebfun with many pieces:

```
tic, f = chebfun('sin(x)',[0 1000*pi],'splitting','on'); toc
disp(f)
```

```
Elapsed time is 0.762380 seconds.
ans =
f =
   chebfun column (32 smooth pieces)
       interval       length   endpoint values
[       0,      98]       87         0    -0.71
[      98,   2e+02]       88     -0.71        1
[   2e+02, 2.9e+02]       88         1    -0.71
[ 2.9e+02, 3.9e+02]       87     -0.71        0
[ 3.9e+02, 4.9e+02]       87         0     0.71
[ 4.9e+02, 5.9e+02]       87      0.71       -1
[ 5.9e+02, 6.9e+02]       86        -1     0.71
[ 6.9e+02, 7.9e+02]       85      0.71        0
[ 7.9e+02, 8.8e+02]       86         0    -0.71
[ 8.8e+02, 9.8e+02]       86     -0.71        1
[ 9.8e+02, 1.1e+03]       86         1    -0.71
[ 1.1e+03, 1.2e+03]       86     -0.71        0
[ 1.2e+03, 1.3e+03]       85         0     0.71
[ 1.3e+03, 1.4e+03]       86      0.71       -1
[ 1.4e+03, 1.5e+03]       86        -1     0.71
[ 1.5e+03, 1.6e+03]       85      0.71        0
[ 1.6e+03, 1.7e+03]       86         0    -0.71
[ 1.7e+03, 1.8e+03]       86     -0.71        1
[ 1.8e+03, 1.9e+03]       86         1    -0.71
[ 1.9e+03,   2e+03]       85     -0.71        0
[   2e+03, 2.1e+03]       85         0     0.71
[ 2.1e+03, 2.2e+03]       86      0.71       -1
[ 2.2e+03, 2.3e+03]       86        -1     0.71
[ 2.3e+03, 2.4e+03]       87      0.71        0
[ 2.4e+03, 2.5e+03]       85         0    -0.71
[ 2.5e+03, 2.6e+03]       86     -0.71        1
[ 2.6e+03, 2.7e+03]       84         1    -0.71
[ 2.7e+03, 2.7e+03]       85     -0.71        0
[ 2.7e+03, 2.8e+03]       85         0     0.71
[ 2.8e+03, 2.9e+03]       84      0.71       -1
[ 2.9e+03,   3e+03]       86        -1     0.71
[   3e+03, 3.1e+03]       85      0.71        0
Epslevel = 3.487391e-13.  Vscale = 1.000000e+00.  Total length = 2748.
```

In this case it is more efficient -- and more interesting mathematically -- to omit the splitting and construct one global chebfun:

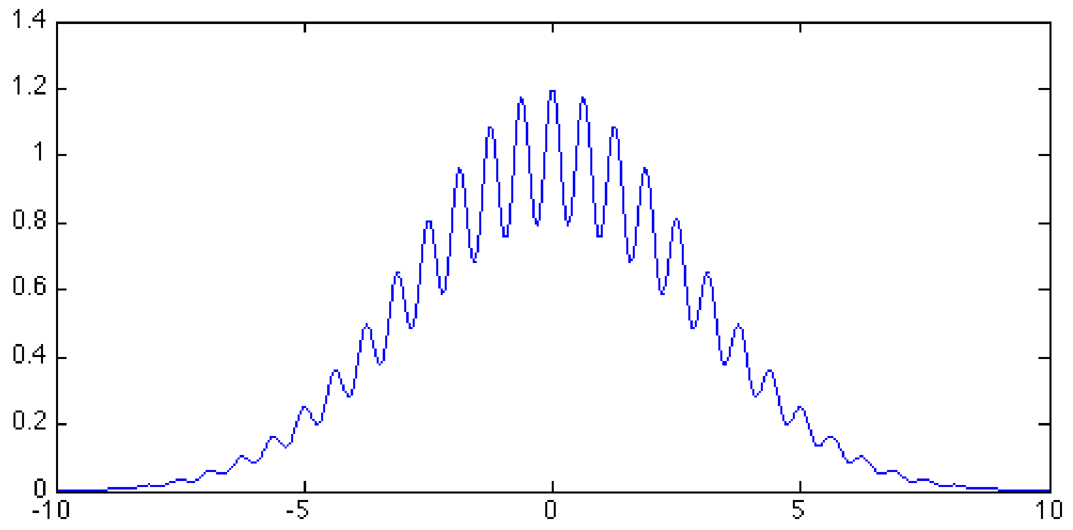
```
tic, f2 = chebfun('sin(x)',[0 1000*pi]); toc
disp(f2)
```

```
Elapsed time is 0.024330 seconds.
ans =
f2 =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[       0, 3.1e+03]     1684         0        0
Epslevel = 3.487867e-13.  Vscale = 9.999862e-01.
```

1.5 Infinite intervals and infinite function values

A major change from Chebfun Version 2 to Version 3 was the generalization of chebfuns to allow certain functions on infinite intervals or which diverge to infinity; the initial credit for these innovations belongs to Nick Hale, Rodrigo Platte, and Mark Richardson. For example, here is a function on the whole real axis,

```
f = chebfun('exp(-x.^2/16).*(1+.2*cos(10*x))',[-inf,inf]);
plot(f)
```



and here is its integral:

```
sum(f)
```

```
Warning: Result may not be accurate as the function decays slowly at infinity.
ans =
   7.089817931976723
```

Here's the integral of a function on [1,inf]:

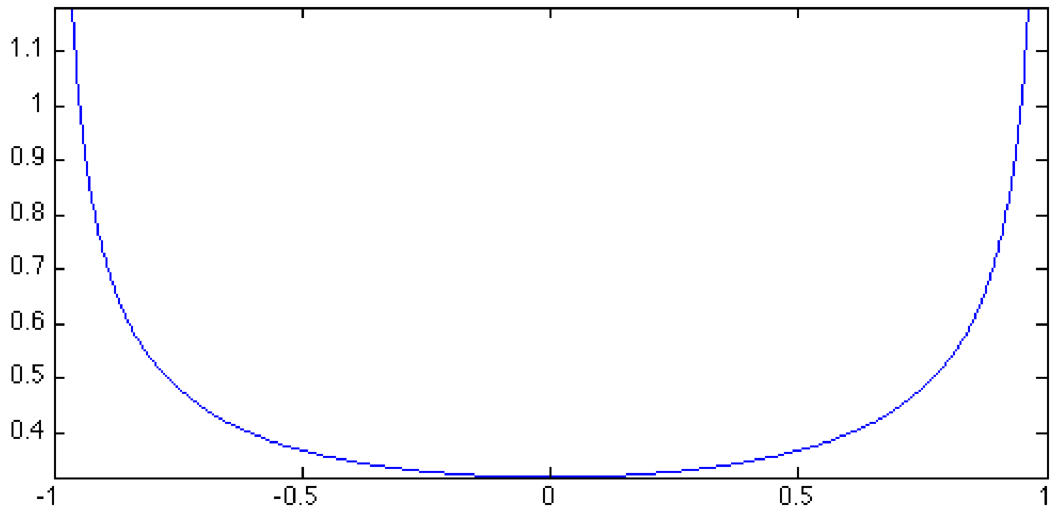
```
sum(chebfun('1./x.^4',[1 inf]))
```

```
ans =
   0.333333333254510
```

Notice that several digits of accuracy have been lost here. Be careful! -- operations involving infinities in Chebfun are not always as accurate and robust as their finite counterparts.

Here is an example of a function that diverges to infinity, which we can capture by including the flag `'blowup 2'` (try help chebfun for details):

```
h = chebfun('(1/pi)./sqrt(1-x.^2)','blowup',2);
plot(h)
```



In this case the integral comes out just right:

```
sum(h)
```

```
ans =
   1.000000000000000
```

For more on the treatment of infinities in Chebfun, see Chapter 9.

1.6 Rows, columns, and quasimatrices

Matlab doesn't only deal with column vectors: there are also row vectors and matrices. The same is true of Chebfun. The chebfuns shown so far have all been in column orientation, which is the default, but one can also take the transpose, compute inner products, and so on:

```
x = chebfun(@(x) x)
```

```
x =
   chebfun column (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        2        -1         1
Epslevel = 6.492624e-16.  Vscale = 1.
```

```
x'
```

```
ans =
   chebfun row (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        2        -1        1
Epslevel = 6.492624e-16.  Vscale = 1.
```

```
x'*x
```

```
ans =
   0.666666666666667
```

One can also make matrices whose columns are chebfuns or whose rows are chebfuns, like this:

```
A = [1 x x.^2]
```

```
A =
   chebfun column1 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        3         1        1
Epslevel = 2.220446e-15.  Vscale = 1.
   chebfun column2 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        3        -1        1
Epslevel = 6.492624e-16.  Vscale = 1.
   chebfun column3 (1 smooth piece)
       interval       length   endpoint values
[      -1,       1]        3         1        1
Epslevel = 1.298525e-15.  Vscale = 1.000000e+00.
(P.S. I am an array-valued CHEBFUN!)
```

```
A'*A
```

```
ans =
   2.000000000000000   0.000000000000000   0.666666666666667
   0.000000000000000   0.666666666666667   0.000000000000000
   0.666666666666667   0.000000000000000   0.400000000000000
```

These are called *quasimatrices*, and they are discussed in Chapter 6.

1.7 How this Guide is produced

This guide is produced in Matlab using the "publish" command with a style sheet somewhat different from the usual; the output of publish is then processed by Markdown. To publish a chapter for yourself, make sure the chebfun guide directory is in your path and then type, for example, `open(publish('guide1'))`. The formatting may not be exactly right but it should certainly be legible.

1.8 References

[Battles & Trefethen 2004] Z. Battles and L. N. Trefethen, "An extension of Matlab to continuous functions and operators", *SIAM Journal on Scientific Computing* 25 (2004), 1743-1770.

[Berrut & Trefethen 2005] J.-P. Berrut and L. N. Trefethen, "Barycentric Lagrange interpolation", *SIAM Review 46* (2004), 501-517.

[Higham 2004] N. J. Higham, "The numerical stability of barycentric Lagrange interpolation", *IMA Journal of Numerical Analysis* 24 (2004), 547-556.

[Pachon, Platte & Trefethen 2009] R. Pachon, R. B. Platte and L. N. Trefethen, "Piecewise smooth chebfuns", *IMA J. Numer. Anal.*, 2009.

[Salzer 1972] H. E. Salzer, "Lagrangian interpolation at the Chebyshev points cos(nu pi/n), nu = 0(1)n; some unnoted advantages", *Computer Journal* 15 (1972), 156-159.

[Trefethen 2007] L. N. Trefethen, "Computing numerically with functions instead of numbers", *Mathematics in Computer Science* 1 (2007), 9-19.

[Trefethen 2013] L. N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, 2013.