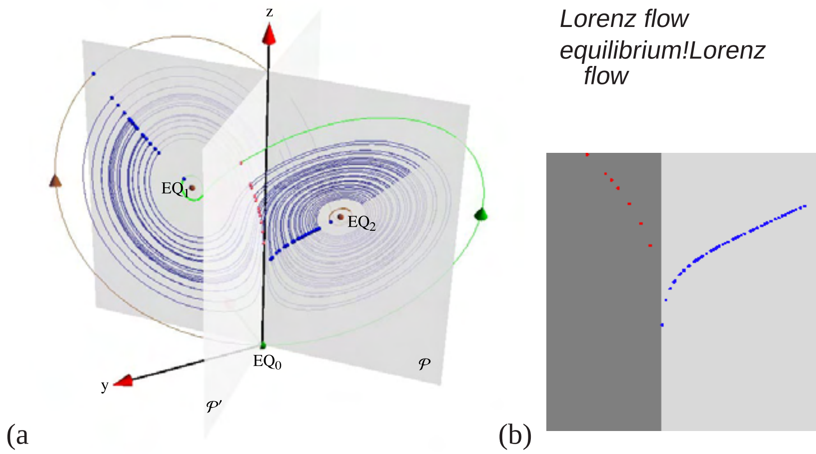

Lorenz flow
equilibrium!Lorenz
flow

Figure 3.7: (a) Lorenz flow figure 2.5 cut by $y = x$ Poincaré section plane $\mathcal{P}$ through the z axis and both $EQ_{1,2}$ equilibria. Points where flow pierces into section are marked by dots. To aid visualization of the flow near the EQ_0 equilibrium, the flow is cut by the second Poincaré section, $\mathcal{P}'$, through $y = -x$ and the z axis. (b) Poincaré sections $\mathcal{P}$ and $\mathcal{P}'$ laid side-by-side. The singular nature of these sections close to EQ_0 will be elucidated in example 4.6 and figure 10.8 (b). (E. Siminos)

ease of computation, pick linear sections (3.6) if you can. (c) If equilibria play important role in organizing a flow, pick sections that go through them (see example 3.5). (c) If you have a global discrete or continuous symmetry, pick sections left invariant by the symmetry (see example 9.7). (d) If you are solving a local problem, like finding a periodic orbit, you do not need a global section. Pick a section or a set of (multi-shooting) sections on the fly, requiring only that they are locally orthogonal to the flow(see sect. H.2.1). (e) If you have another rule of thumb dear to you, let us know.

chapter 9

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Example 3.5 ***Sections of Lorenz flow:*** *(Continued from example 2.2.) The plane $\mathcal{P}$ fixed by the $x = y$ diagonal and the z-axis depicted in figure 3.7 is a natural choice of a Poincaré section of the Lorenz flow of figure 2.5, as it contains all three equilibria, $x_{EQ0} = (0, 0, 0)$ and the (2.13) pair $EQ_{1,2}$. A section has to be supplemented with the orientation condition (3.4): here points where flow pierces* into *the section are marked by dots.*

[10] *$EQ_{1,2}$ are centers of out-spirals, and close to them the section is transverse to the flow. However, close to EQ_0 trajectories pass the z-axis either by crossing the section $\mathcal{P}$ or staying on the viewer's side. We are free to deploy as many sections as we wish: in order to capture the whole flow in this neighborhood we add the second Poincaré section, $\mathcal{P}'$, through the $y = -x$ diagonal and the z-axis. Together the two sections, figure 3.7 (b), capture the whole flow near EQ_0. In contrast to Rössler sections of figure 3.5, these appear very singular. We explain this singularity in example 4.6, and postpone construction of a Poincaré return map to example 9.7.*

(E. Siminos and J. Halcrow)

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3.2 Stroboscopic Poincaré sections

[11]

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[10] Predrag: figure 3.7 (b): generate .eps from lorenz2Poinc2D.pdf, this is the old version
[11] Predrag: not sure this is worth describing

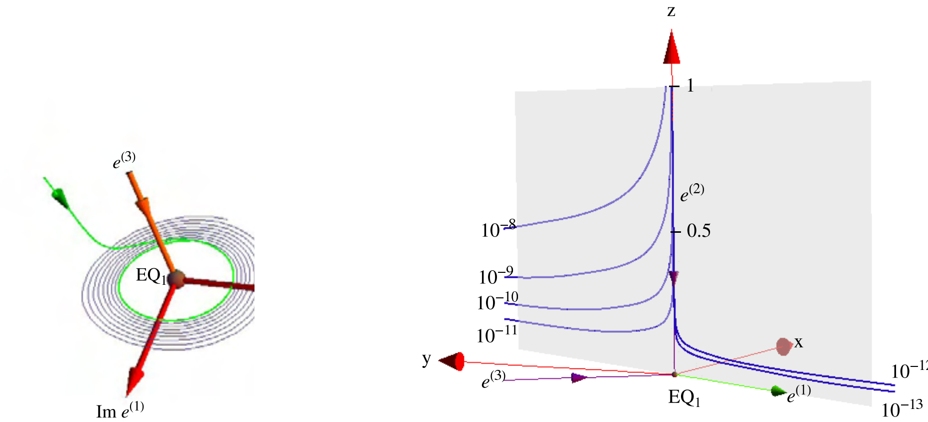

Figure 4.7: (a) A perspective view of the linearized Lorenz flow near EQ_1 equilibrium, see figure 3.7 (a). The unstable eigenplane of EQ_1 is spanned by $\mathrm{Re}\,\mathbf{e}^{(1)}$ and $\mathrm{Im}\,\mathbf{e}^{(1)}$. The stable eigenvector $\mathbf{e}^{(3)}$. (b) Lorenz flow near the EQ_0 equilibrium: unstable eigenvector $\mathbf{e}^{(1)}$, stable eigenvectors $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$. Trajectories initiated at distances $10^{-8}\cdots 10^{-12}$, 10^{-13} away from the z-axis exit finite distance from EQ_0 along the $(\mathbf{e}^{(1)},\mathbf{e}^{(2)})$ eigenvectors plane. Due to the strong $\lambda^{(1)}$ expansion, the EQ_0 equilibrium is, for all practical purposes, unreachable, and the $EQ_1 \to EQ_0$ heteroclinic connection never observed in simulations such as figure 2.5. (E. Siminos; continued in figure 10.8.)

$EQ_{1,2}$ equilibria have no symmetry, so their eigenvalues are given by the roots of a cubic equation, the secular determinant $\det(A-\lambda\mathbf{1})=0$:

$$\lambda^3+\lambda^2(\sigma+b+1)+\lambda b(\sigma+\rho)+2\sigma b(\rho-1)=0\,. \tag{4.37}$$

For $\rho>24.74$, $EQ_{1,2}$ have one stable real eigenvalue and one unstable complex conjugate pair, leading to a spiral-out instability and the strange attractor depicted in figure 2.5.

As all numerical plots of the Lorenz flow are here carried out for the Lorenz parameter choice $\sigma=10, b=8/3, \rho=28$, we note the values of these eigenvalues for future reference,

$$\begin{array}{lcl} EQ_0:\ (\lambda^{(1)},\lambda^{(2)},\lambda^{(3)}) &=& (\,11.83\,,\ -2.666,\quad -22.83\,)\\ EQ_1:\ (\mu^{(1)}\pm\, i\,\omega^{(1)},\lambda^{(3)}) &=& (\,0.094\,\pm i\,10.19,\quad -13.85\,)\,, \end{array} \tag{4.38}$$

as well as the rotation period $T_{EQ1}=2\pi/\omega^{(1)}$ about EQ_1, and the associated expansion/contraction multipliers $\Lambda^{(i)}=\exp(\mu^{(j)}T_{EQ1})$ per a spiral-out turn:

$$T_{EQ1}=0.6163\,,\qquad (\Lambda^{(1)},\Lambda^{(3)})=(\,1.060\,,1.957\times10^{-4}\,)\,. \tag{4.39}$$

We learn that the typical turnover time scale in this problem is of order $T\approx T_{EQ1}\approx 1$ (and not, let us say, 1000, or 10^{-2}). Combined with the contraction rate (4.34), this tells us that the Lorenz flow strongly contracts state space volumes, by factor of $\approx 10^{-4}$ per mean turnover time.

In the EQ_1 neighborhood the unstable manifold trajectories slowly spiral out, with very small radial per-turn expansion multiplier $\Lambda^{(1)}\simeq 1.06$, and very strong contraction multiplier $\Lambda^{(3)}\simeq 10^{-4}$ onto the unstable manifold, figure 4.7 (a). This contraction confines, for all practical purposes, the Lorenz attractor to a 2-dimensional surface evident in the section figure 3.7. [17] [18]

In the $x_{EQ0}=(0,0,0)$ equilibrium neighborhood the extremely strong $\lambda^{(3)}\simeq -23$ contraction along the $\mathbf{e}^{(3)}$ direction confines the hyperbolic dynamics near EQ_0 to

[17] Predrag: ChaosBook: develop this text from steady.tex: "For an unstable complex pair $\lambda^{(n,n+1)}$ of equilibrium EQ, let $Wmnfldu(n,n+1)EQ$ denote the orbit of a circle of infinitesimal radius in the plane about EQ spanned by $\mathbf{e}_r^{(n)}$, $\mathbf{e}_i^{(n)}$. This part of the EQ unstable manifold is 2-dimensional; its shape can be traced out by computing a set of trajectories with initial conditions $EQ+\epsilon(\mathbf{e}_r^{(n)}\cos\theta+\mathbf{e}_i^{(n)}\sin\theta)$ for a set of values of θ.

[18] Predrag: remember to delete halcrow/figs/hyperb.* ??

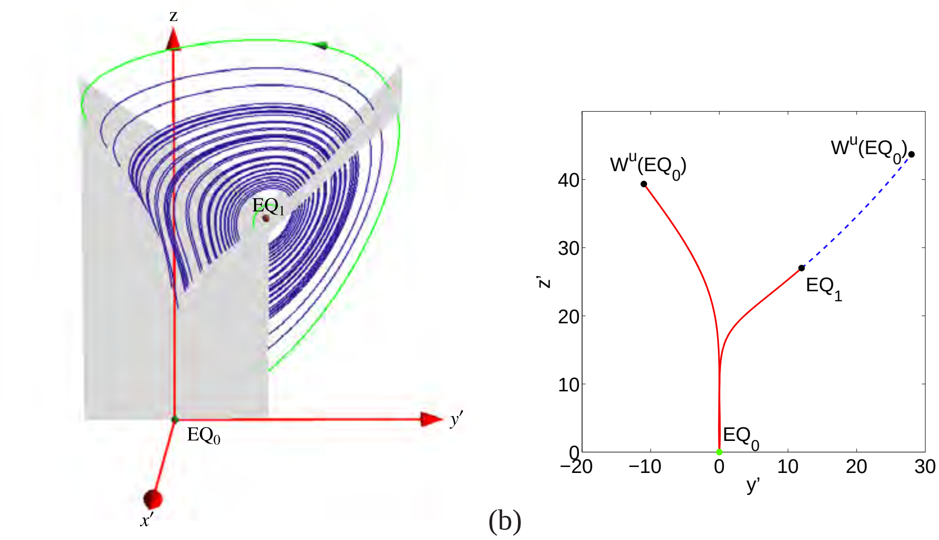

Figure 10.8: (a) A Poincaré section of the Loren flow in the doubled-polar angle representation, fig ure 10.8, given by the $[y', z]$ plane that contains th z-axis and the equilibrium EQ_1. x' axis points to ward the viewer. (b) The Poincaré section of th Lorenz flow by the section plane (a); compare wit figure 3.7. Crossings *into* the section are marke red (solid) and crossings *out of* the section a marked blue (dotted). Outermost points of bot in- and out-sections are given by the EQ_0 unstabl manifold $W^u(EQ_0)$ intersections. (E. Siminos)

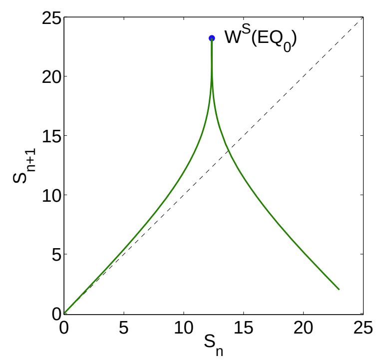

Figure 10.9: The Poincaré return map $s_{n+1} = P(s_n)$ parameterized by Euclidean arclength s measured along the EQ_1 unstable manifold, from x_{EQ1} to $W^u(EQ_0)$ section point, uppermost right point of the blue segment in figure 10.8 (b). The critical point (the "crease") of the map is given by the section of the heteroclinic orbit $W^s(EQ_0)$ that descends all the way to EQ_0, in infinite time and with infinite slope. (E. Siminos)

But there is trouble in paradise. By a fluke, the Lorenz attractor, the first flow to popularize strange attractors, turns out to be topologically one of the simplest strange attractors. But it is not "uniformly hyperbolic." The flow near EQ_1 is barely unstable, while the flow near EQ_0 is arbitrarily unstable. So binary symbolic dynamics enumeration of cycles mixes cycles of vastly different stabilities, and is not very useful - presumably the practical way to compute averages is by stability ordering. [15]

If a flow is locally unstable but globally bounded, any open ball of initial points will be stretched out and then folded back.

At this juncture we show how this works on the simplest example: unimodal mappings of the interval. The erudite reader should skim through this chapter and then take a more demanding path, via the Smale horseshoes of chapter 11. Unimodal maps are easier, but physically less motivated. The Smale horseshoes are the high road, more complicated, but the right tool to generalize what we learned from the 3-disk dynamics, and begin analysis of general dynamical systems. It is up to you - unimodal maps suffice to get quickly to the heart of this treatise.

[15] Predrag: link this to the stability ordering section