Penalizing loops that deviate from the True Path.

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Abstract

A variational method for periodic orbit searches in a general flow is developed. The method is based on penalizing the misorientation of the tangent vector of a guess-loop to the velocity field of the given flow. The loop is continuously evolved into a periodic orbit by a fictitious time flow. The main goal of the project is the definition of a natural and advantageous (i.e. informed of the flow) metric in the space of loops.

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CHAOS, AND WHAT TO DO ABOUT IT

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I. INTRODUCTION

The goal of this project is to improve the variational method for finding periodic orbits introduced in refs. [1, 2]. The method evolves an initial guess in the form of a closed loop L towards a periodic orbit p of a given flow $f^t(x)$ defined by:

$$\frac{dx}{dt} = v(x),\ x \in \mathbb{R}^{d+1} . \tag{1}$$

Given a loop parameterized by a parameter s, this is achieved by minimizing the misorientation of the tangent vector $\tilde{v}(x) = dx/ds$ of the loop to the velocity vector (1). Locally, the loop deviates from the true flow by the misalignment of the loop tangent vector

$$F(x) = v(x) - \lambda(x)\tilde{v}(x) ,\ x = x(s) \in L . \tag{2}$$

$\lambda(x)$ is an auxiliary undetermined function which compensates for the fact that not only the direction, but also the magnitudes of the two fields need to be equal on a periodic orbit. A cost functional

$$F^2 = \frac{1}{S} \oint_L \left(v(x) - \lambda(x)\, \tilde{v}(x)\right)^2 \, ds , \tag{3}$$

is an indication of the deviation of the of the guess loop L from a periodic orbit of the flow (1). Here S is a normalization factor.

The improvement that will be pursued here is to replace the simple Euclidean metric δ_{ij} that is used in (3) with a metric $g_{ij}(x)$ that would carry information about the flow, that is minimize the functional

$$F^2 = \frac{1}{S} \oint \left(v - \lambda\tilde{v}\right)_i \, g_{ij} \, \left(v - \lambda\tilde{v}\right)_j \, ds . \tag{4}$$

II. VARIATIONAL SEARCHES FOR PERIODIC ORBITS

In this project any smooth, closed curve in our $d+1-$dimensional space is referred to as a loop L. In general a loop is not a solution of (1), in contrast to a periodic orbit, which satisfies the periodic orbit condition $f^T(x) = x$, where T the period. The loop tangent vector $\tilde{v}$ is in general not parallel to $\hat{v}$. Thus, if we could continously deform the loop in such a way that its tangent becomes parallel to the velocity field everywere we would end up with a periodic orbit. In ref. [2] it is shown that this corresponds to minimizing (3) and that one can write down a partial differential equation (PDE) for the evolution of the loop towards a periodic orbit. As the form of the equation depends on the parameterization of the loop and the choice of metric, such a PDE will only be presented here for our specific choices, *cf.* sect. III C. Numerically solving this PDE provides the periodic orbit of the system "closest" to the initial loop. Distance in the space of loops is hard to define. In practice the method converges to a loop with topology "similar" to the initial guess.

The method is conceptually more complicated, harder to program and generally slower than Newton or multiple-shooting methods for the search of periodic orbits. On the other hand it has an advantage when one tries to find long or extremely unstable periodic orbits, or when one deals with hard-to-visualize high-dimensional systems. For multiple-shooting to converge one needs a large number of Poincaré sections in order to control local instability. Thus one needs a great deal of information about the qualitative dynamics of the flow to make a clever choice of those sections. In high-dimensional flows this information is usually not available and multiple shooting methods can easily fail to find the longer cycles. In the variational method described here Poincaré sections play no role and guesses with roughly the correct topology can lead to long cycles. In practice one constructs guess-loops by patching together pieces of trajectories to form a discontinuous loop, transforms in Fourier space and drops the high-frequency components to get a closed curve, *cf.* ref. [1]

The extension of the method that will be attempted here is to use a metric that penalizes variations from a true periodic orbit in the unstable eigendirections of the flow more than it does in the stable ones. The hope is that in a high-dimensional flow in which only a few of the these eigendirections are significant, one can concentrate on them, effectively reducing the dimensionality of the problem and the computational load.

III. CANDIDATES FOR THE ROLE OF METRIC

A. A Jacobian Matrix for a Loop

The Jacobian matrix $\mathbf{J}^t(x_o)$ describes the deformation of the neighborhood of a point x_o under a flow f^t, in the linear approximation,

$$\mathbf{J}^t_{ij}(x_o) = \left.\frac{\partial f^t_i(x_o)_i}{\partial x_{o_j}}\right|_{x=x_o} . \tag{5}$$

Using (5) one can show (e.g. *cf.* ref. [3], Chapter 4) that the Jacobian can be computed by means of the time-ordered product

$$\begin{aligned}\mathbf{J}^t(x_o) &= \mathbf{T}e^{\int_0^t d\tau \mathbf{A}(f^\tau(x_o))} \\ &= \lim_{m\to\infty}\prod_{n=m}^{1} e^{\Delta t\,\mathbf{A}\left(f^{n\Delta t}(x_o)\right)} ,\end{aligned} \tag{6}$$

where $\Delta t = t/m$ and $\mathbf{A}$ the matrix of variations defined by

$$A_{ij}(x) = \frac{\partial v_i}{\partial x_j} . \tag{7}$$

Along a periodic orbit

$$\mathbf{J}_p \equiv \mathbf{J}^{T_p}(x_o) = \mathbf{T}e^{\oint d\tau \mathbf{A}(f^\tau(x_o))} . \tag{8}$$

$\mathbf{J}_p$ describes the local deformation of the neighborhood of the periodic orbit under the flow. Its eigenvalues are independent of the initial point x_o on the periodic orbit, and provide the local measure of instability of the system.

Thus we would like to use $\mathbf{J}$ as our metric tensor. Yet, a loop is not a solution of the equations of the flow and we cannot calculate the Jacobian along it. Inspired by the time-ordered product (6), we *define* the Jacobian along any path as

$$\begin{aligned}\tilde{\mathbf{J}}^s(x_o) &\equiv \mathbf{T}e^{\int ds \mathbf{A}(x(s)))} \\ &\equiv \lim_{m\to\infty}\prod_{n=m}^{1} e^{\Delta s\,\mathbf{A}(x(n\,\Delta s))} ,\end{aligned} \tag{9}$$

where $s \in [s_i, s_f]$ parameterizes the path, $\Delta s = (s_f - s_i)/m$ and $\mathbf{T}$ now reminds us that the integration is ordered with respect to s. We will denote $\tilde{\mathbf{J}}(x_o)$ evaluated around a closed loop, starting at point $x_o(s)$ as $\tilde{\mathbf{J}}_L(x_o)$.

Obviously, $\tilde{\mathbf{J}}^s(x_o)$ is a solution of the differential equation

$$\frac{d\tilde{\mathbf{J}}^s(x_o)}{ds} = \mathbf{A}(x(s))\tilde{\mathbf{J}}^s(x_o). \tag{10}$$

B. Properties of $\mathbf{J}_L$

We now prove that $\tilde{\mathbf{J}}_L$ shares with $\mathbf{J}_p$ the property that its eigenvalues do not depend on the initial point on the loop. Definition (9) establishes the group property

$$\tilde{\mathbf{J}}^{s+s'}(x_o) = \tilde{\mathbf{J}}^{s'}(x(s))\tilde{\mathbf{J}}^{s}(x_o)\,. \tag{11}$$

Next we consider the eigenvalue-eigenvector equation for $\tilde{\mathbf{J}}_L$

$$\tilde{\mathbf{J}}_L(x)e_i(x) = \Lambda_{L,i}e_i(x)\,, \tag{12}$$

with $\tilde{\mathbf{J}}_L$ evaluated at a specific point $x(s)$. At a different point on the loop $x'(s')$ we have the Jacobian $\tilde{\mathbf{J}}_L(x')$. Let S_L be the "period" of the loop in s and $\Delta s = s' - s$. Using (11) we can write

$$\tilde{\mathbf{J}}^{S_L+\Delta s}(x) = \tilde{\mathbf{J}}^{S_L}(x')\tilde{\mathbf{J}}^{\Delta s}(x) = \tilde{\mathbf{J}}_L(x')\tilde{\mathbf{J}}^{\Delta s}(x)\,, \tag{13}$$

and also

$$\tilde{\mathbf{J}}^{\Delta s+S_L}(x) = \tilde{\mathbf{J}}^{\Delta s}(x)\tilde{\mathbf{J}}^{S_L}(x) = \tilde{\mathbf{J}}^{\Delta s}(x)\tilde{\mathbf{J}}_L(x)\,. \tag{14}$$

Thus

$$\tilde{\mathbf{J}}^{\Delta s}(x)\tilde{\mathbf{J}}_L(x) = \tilde{\mathbf{J}}_L(x')\tilde{\mathbf{J}}^{\Delta s}(x)\,. \tag{15}$$

Multiply (12) with $\tilde{\mathbf{J}}^{\Delta s}(x)$ and use (15) to get

$$\tilde{\mathbf{J}}_L(x')\tilde{\mathbf{J}}^{\Delta s}(x)e_i(x) = \Lambda_{L,i}\tilde{\mathbf{J}}^{\Delta s}(x)e_i(x)\,. \tag{16}$$

But this is the eigenvalue-eigenvector equation for $\tilde{\mathbf{J}}_L(x')$. Thus $\tilde{\mathbf{J}}_L(x')$ has the same eigenvalues $\Lambda_{L,i}$ as $\tilde{\mathbf{J}}_L(x)$, but corresponding eigenvectors

$$e_i(x') \equiv \tilde{\mathbf{J}}^{\Delta s}(x)e_i(x)\,, \tag{17}$$

that is, the eigenvectors are transported by the Jacobian matrix $\tilde{\mathbf{J}}^{\Delta s}(x)$. The case of the Jacobian for a periodic orbit $\mathbf{J}_p(x)$ is a special case of this result. The crucial part for the proof was the group property (11) and not the specific definition of $\tilde{\mathbf{J}}$ or the fact that it was computed along a loop and not a periodic orbit.

We would also like to prove that the eigenvalues of $\mathbf{J}_L$ are invariant under a change of variables, $y = h(x)$. For notational simplicity we temporarilly drop the index L in $\mathbf{J}_L$.

Let $\mathbf{K}^s(y_o)$ denote the Jacobian for the loop in the new variables, with $y_o = h(x_o)$. Then, according to the definition of the Jacobian on a loop

$$\frac{d\mathbf{K}^s}{ds} = \frac{\partial u}{\partial y}\mathbf{K}^s\,, \tag{18}$$

where

$$\begin{aligned} u_i &= \frac{dy_i}{dt} \\ &= \frac{\partial h_i}{\partial x_j} v_j \end{aligned} \tag{19}$$

and in we adopt the convention of summing over repeated indices. For notational compactness we define the matrix

$$H_{ij} = \frac{\partial h_i}{\partial x_j}\,. \tag{20}$$

Then, for the matrix elements of $\partial u/\partial y$ we have

$$\begin{aligned} \frac{\partial u_i}{\partial y_j} &= \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial y_j} \\ &= \frac{\partial}{\partial x_k}\left(H_{im}v_m\right)\frac{\partial x_k}{\partial y_j} \\ &= \left(\frac{\partial H_{im}}{\partial x_k}v_m + H_{im}A_{mk}\right)H^{-1}_{kj}\,, \end{aligned} \tag{21}$$

where $H^{-1}_{ij} = \partial h^{-1}_i/\partial y_j$.

To relate the eigenvalues of $\mathbf{J}$ and $\mathbf{K}$ we form the matrix

$$\mathbf{N}^s \equiv \mathbf{K}^s(y_o) - \mathbf{H}(x(s))\mathbf{J}^s(x_o)\mathbf{H}^{-1}(x_o)\,, \tag{22}$$

Differentiating with respect to s,

$$\frac{dN^s_{ij}}{ds} = \frac{dK^s_{ij}}{ds} - \frac{dH_{im}}{ds}J^s_{mk}H^{-1}_{kj} - H_{im}\frac{dJ^s_{mk}}{ds}H^{-1}_{kj}\,, \tag{23}$$

or, using (10) and (18),

$$\frac{dN^s_{ij}}{ds} = \frac{\partial u_i}{\partial y_m}K^s_{mj} - \frac{\partial H_{im}}{\partial x_n}\frac{dx_n}{ds}J_{mk}H^{-1}_{kj} - H_{im}A_{mn}J^s_{nk}H^{-1}_{kj}\,. \tag{24}$$

With the use of (21) and renaming of the dummy indices $n \leftrightarrow m$ in the last term, this reads

$$\frac{dN^s_{ij}}{ds} = \left(\frac{\partial H_{in}}{\partial x_k}v_n + H_{in}A_{nk}\right)H^{-1}_{km}(x(s))K^s_{mj} - \left(\frac{\partial H_{im}}{\partial x_n}\frac{dx_n}{ds} + H_{in}A_{nm}\right)J^s_{mk}H^{-1}_{kj}(x_o)\,. \tag{25}$$

Inserting the identity matrix $\mathbf{1} = \mathbf{H}^{-1}(x(s))\mathbf{H}(x(s))$ in the second term and noting that

$$\frac{\partial H_{in}}{\partial x_k} = \frac{\partial^2 h_i}{\partial x_k \partial x_n} = \frac{\partial H_{ik}}{\partial x_n}\,, \tag{26}$$

we get

$$\frac{dN^s_{ij}}{ds} = \left(\frac{\partial H_{ik}}{\partial x_n} v_n + H_{in}A_{nk}\right) H^{-1}_{km}(x(s))K^s_{mj} - \left(\frac{\partial H_{im}}{\partial x_n}\frac{dx_n}{ds} + H_{in}A_{nm}\right) H^{-1}_{mq}H_{ql}J^s_{lk}H^{-1}_{kj}(x_o)\,. \tag{27}$$

Were it not $\frac{dx}{ds} \neq v$ this could be factored to get $dN^s_{ij}/ds = (\ldots)_{ik}N^s_{kj}$. This would allow us to complete the proof. The proof is valid only if we are computing the Jacobian on a periodic orbit and its remaining part is given in Appendix A.

C. The variational method

We will try to motivate our variational method in terms of the path integral formulation of a random work, *cf.* ref. [4]. We consider a discretized loop consisting of N point. At each point x_i we form the difference $x_{i+1} - f^{\Delta t_i}(x)$, where $x_i + 1$ the next point on the loop and $f^{\Delta t_i}(x_i)$ the point at which the flow would transport x_i at time Δt_i. On a periodic orbit the cost function

$$F^2 = \sum_{i=0}^{N} \frac{1}{\Delta t_i}\left(x_{i+1} - f^{\Delta t_i}(x_i)\right)^2 \tag{28}$$

is obviously equal to zero, and since it is a positive definite quantity this value would be a minimum. The factor of Δt_i has been used so that the cost function has the dimensions of a diffusion constant in agreement with the stochastic integral formulation. We rewrite this quantity as

$$\begin{aligned} F^2 &= \sum_{i=0}^{N} \frac{1}{\Delta t_i}\left(x_{i+1} - x_i - f^{\Delta t_i}(x_i) - x_i\right)^2 \\ &= \sum_{i=0}^{N} \Delta t_i \left(\frac{x_{i+1} - x_i}{\Delta t_i} - \frac{f^{\Delta t_i}(x_i) - x_i}{\Delta t_i}\right)^2 \end{aligned} \tag{29}$$

and take the limit $\Delta t_i \to 0$ to get

$$\begin{aligned} F^2 &= \int_{t_i=0}^{T} dt \left(\frac{dx}{dt} - v(x)\right)^2 \\ &= \int_{t_i=0}^{T} dt \left(\frac{ds}{dt}\right)^2 \left(\tilde{v}(x) - v(x)\frac{dt}{ds}\right)^2\,, \end{aligned} \tag{30}$$

where T our guess of the period for the cycle. Thus far we have not specified the relation between the parameter s and the time t. We choose s to be such that at each point

$$\frac{|v|dt}{|\tilde{v}|ds} = \cos\theta\,, \tag{31}$$

where θ the angle from v to $\tilde{v}$

$$\cos\theta = \frac{\tilde{v}.v}{|\tilde{v}||v|} \tag{32}$$

and thus

$$\frac{dt}{ds} = \frac{\tilde{v}.v}{v^2}\,. \tag{33}$$

This allows to rewrite (30) as

$$F^2 = \int_{t_i=0}^{T} dt \left(\frac{v^2}{\tilde{v}.v}\right)^2 \left(\tilde{v}(x) - v(x)\frac{\tilde{v}.v}{v^2}\right)^2\,. \tag{34}$$

Now observe that we can write

$$\begin{aligned}\tilde{v} &= (\mathbf{1} - \frac{v\otimes v}{v^2} + \frac{v\otimes v}{v^2})\tilde{v}\\ &= \mathbf{P}^{\perp}v + v\frac{v.\tilde{v}}{v^2}\end{aligned} \tag{35}$$

where $a\otimes b$ denotes the tensor product of vectors a and b and we have introduced the matrix

$$P^{\perp} = \mathbf{1} - \frac{v\otimes v}{v^2} \tag{36}$$

which projects any vector to the plane perpedicular to v. To see this write a vector $a \in \mathbb{R}^{d+1}$ as $a = a_{\|}\hat{v} + \sum_{i=1}^{d} a_i\hat{e}_i$, where $\hat{e}_i$ unit vectors that form a basis in the complement of v and we have written the components of a as $a_{\|}, a_i$. Then

$$\begin{aligned}\mathbf{P}^{\perp}a &= a - a_{\|}\frac{v\otimes v}{v^2}\hat{v} - \sum_{i=1}^{d} a_i\frac{v\otimes v}{v^2}\hat{e}_i\\ &= a - a_{\|}\frac{v.\hat{v}}{v^2}v - \sum_{i=1}^{d} a_i\frac{v.\hat{e}_i}{v^2}v\\ &= a - a_{\|}\hat{v}\\ &= \sum_{i=1}^{d} a_i\hat{e}_i\,.\end{aligned} \tag{37}$$

which verifies our assertion.

Substituting (35) in (34) we get

$$F^2 = \int_{s_i}^{s_f} ds\, \frac{v^2}{v.\tilde{v}} \left(\mathbf{P}^{\perp}\tilde{v}\right)^2 \tag{38}$$

where we have changed the integration variable to s since it is not natural to integrate over t along a curve that is not a solution of the equations of motion. We now see the motivation for choice (31). The projection operator $\mathbf{P}^{\perp}$ appears naturally in our cost function and its role is to penalize misorientation of the fields by projecting on the directions perpendicular to the velocity vector.

For a variational method to work this functional has to be minimized monotonically towards zero, while the loop evolves towards a periodic orbit. Thus we need to write a differential equation for the evolution of each point $x(s)$ of the loop. We can think of such an equation as defining a flow in loop space with a parameter τ playing the role of the time variable and thus referred to as *fictitious time*. Therefore each point on the loop will be a function of two variables s and τ. Differentiating (38) with respect to fictitious time

$$\frac{dF^2}{d\tau} = \frac{1}{S}\int\left[\frac{\partial}{\partial\tau}\left(\frac{v^2}{v.\tilde{v}}\right)\left(\mathbf{P}^{\perp}\tilde{v}\right)^2 + 2\frac{v^2}{\tilde{v}.v}\left(\mathbf{P}^{\perp}\tilde{v}\right)^T\frac{\partial}{\partial\tau}\left(\mathbf{P}^{\perp}\tilde{v}\right)\right]ds\,. \tag{39}$$

Since there is no principle associated with the fictitious time flow other than the requirement to minimize (38), we are free to define this flow at our convenience. We use the ansatz

$$\frac{\partial}{\partial\tau}\left(\mathbf{P}^{\perp}\tilde{v}\right) = -\frac{1}{2}\mathbf{P}^{\perp}\tilde{v} + X(s,\tau)\,, \tag{40}$$

where $X(\tau)$ undermined function. Substituting in (39) yields

$$\frac{dF^2(\tau)}{d\tau} = -F^2(\tau) + \frac{1}{S}\int\left[\frac{\partial}{\partial\tau}\left(\frac{v^2}{v.\tilde{v}}\right)\left(\mathbf{P}^{\perp}\tilde{v}\right)^2 + 2\frac{v^2}{\tilde{v}.v}\left(\mathbf{P}^{\perp}\tilde{v}\right)^T X\right]\,, \tag{41}$$

and thus choosing

$$X = -\frac{\mathbf{P}^{\perp}\tilde{v}}{2}\frac{v.\tilde{v}}{v^2}\frac{\partial}{\partial\tau}\left(\frac{v^2}{v.\tilde{v}}\right) \tag{42}$$

the terms inside the integral cancel out and we get

$$\frac{dF^2(\tau)}{d\tau} = -F^2(\tau)\,, \tag{43}$$

with solution

$$F^2(\tau) = F^2(0)e^{-\tau}\,. \tag{44}$$

The functional evolves exponentially to zero, as desired.

To get a differential equation for the evolution of the loop under the fictitious time flow, we simply perform the differentiations in (40) explicitly. We have

$$\begin{aligned}
\frac{\partial P^{\perp}_{ij}}{\partial \tau} &= \frac{\partial P^{\perp}_{ij}}{\partial x_k}\frac{\partial x_k}{\partial \tau} \\
&= -\frac{\partial}{\partial x_k}\left(\frac{v_i v_j}{v^2}\right)\frac{\partial x_k}{\partial \tau} \\
&= -\left(A_{ik}\frac{v_j}{v^2} + \frac{v_i}{v^2}A_{jk} - \frac{2\, v_i v_j v_m}{v^4}A_{mk}\right)\frac{\partial x_k}{\partial \tau}\,, \qquad (45)
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\partial P^{\perp}_{ij}}{\partial \tau}\tilde{v}_j &= -\left(\frac{v_j \tilde{v}_j}{v^2}A_{ik} + \frac{v_i \tilde{v}_j}{v^2}A_{jk} - \frac{2\, v_i v_j \tilde{v}_j v_m}{v^4}A_{mk}\right)\frac{\partial x_k}{\partial \tau} \\
&= -\frac{1}{v^2}\left(v_j \tilde{v}_j\left(\delta_{im} - \frac{v_i v_m}{v^2}\right)A_{mk} + v_i \tilde{v}_j\left(\delta_{jm} - \frac{v_j v_m}{v^2}\right)A_{mk}\right)\frac{\partial x_k}{\partial \tau}\,, \qquad (46)
\end{aligned}$$

or

$$\frac{\partial \mathbf{P}^{\perp}}{\partial \tau}\tilde{v} = -\frac{1}{v^2}\left(v.\tilde{v}\,\mathbf{1} + v\otimes\tilde{v}\right)\mathbf{P}^{\perp}\mathbf{A}\frac{\partial x}{\partial \tau}\,. \qquad (47)$$

.

On the other hand

$$\mathbf{P}^{\perp}\frac{\partial \tilde{v}}{\partial \tau} = \mathbf{P}^{\perp}\frac{\partial^2 x}{\partial \tau \partial s} \qquad (48)$$

and

$$\begin{aligned}
\frac{\partial}{\partial \tau}\left(\frac{v_i v_i}{v_j \tilde{v}_j}\right) &= \frac{\partial}{\partial x_k}\left(\frac{v_i v_i}{v_j \tilde{v}_j}\right)\frac{\partial x_k}{\partial \tau} \\
&= \frac{2v_i}{v_j \tilde{v}_j}A_{ik}\frac{\partial x_k}{\partial \tau} - \frac{v_i v_i}{(v_m \tilde{v}_m)^2}\frac{\partial}{\partial x_k}(v_j \tilde{v}_j)\frac{\partial x_k}{\partial \tau} \\
&= \frac{2v_i}{v_j \tilde{v}_j}A_{ik}\frac{\partial x_k}{\partial \tau} - \frac{v_i v_i}{(v_m \tilde{v}_m)^2}A_{jk}\tilde{v}_j\frac{\partial x_k}{\partial \tau} - \frac{v_i v_i}{(v_m \tilde{v}_m)^2}v_j\frac{\partial^2 x_j}{\partial \tau \partial s}\,. \qquad (49)
\end{aligned}$$

We can write

$$\begin{aligned}
\frac{2v_i}{v_j \tilde{v}_j}A_{ik}\frac{\partial x_k}{\partial \tau} - \frac{v_i v_i}{(v_m \tilde{v}_m)^2}\tilde{v}_j A_{jk}\frac{\partial x_k}{\partial \tau} &= \frac{v^2}{(v.\tilde{v})^2}\left(2\frac{v_j v_m \tilde{v}_m}{v^2} - \tilde{v}_j\right)A_{jk}\frac{\partial x_k}{\partial \tau} \\
&= \frac{v^2}{(v.\tilde{v})^2}\left(-\left(\delta_{jm} - \frac{v_j v_m}{v^2}\right)\tilde{v}_m + \frac{v_j v_m}{v^2}\tilde{v}_m\right)A_{jk}\frac{\partial x_k}{\partial \tau} \\
&= \frac{v^2}{(v.\tilde{v})^2}\left(-P^{\perp}_{jm}\tilde{v}_m + P^{\parallel}_{jm}\tilde{v}_m\right)A_{jk}\frac{\partial x_k}{\partial \tau} \qquad (50)
\end{aligned}$$

or in matrix notation

$$\frac{\partial}{\partial \tau}\left(\frac{v^2}{v.\tilde{v}}\right) = \frac{v^2}{(v.\tilde{v})^2}\left(\left[\left(\mathbf{P}^{\parallel} - \mathbf{P}^{\perp}\right)\tilde{v}\right]^T\mathbf{A}\frac{\partial x}{\partial \tau} - v^T\frac{\partial^2 x}{\partial \tau \partial s}\right) \qquad (51)$$

$$X = -\frac{1}{2v.\tilde{v}} \left(\mathbf{P}^{\perp}\tilde{v}\right) \left(\left[\left(\mathbf{P}^{\parallel} - \mathbf{P}^{\perp}\right) \tilde{v} \right]^T \mathbf{A} \frac{\partial x}{\partial \tau} - v^T \frac{\partial^2 x}{\partial \tau \partial s} \right) \tag{52}$$

Gathering everything together in (40) we get

$$\left(\frac{1}{v^2} \left(v.\tilde{v}\, \mathbf{1} + v \otimes \tilde{v}\right) \mathbf{P}^{\perp}\mathbf{A} \right) \frac{\partial x}{\partial \tau} - \mathbf{P}^{\perp} \frac{\partial^2 x}{\partial s \partial \tau} + \frac{1}{2v.\tilde{v}} \left(\mathbf{P}^{\perp}\tilde{v}\right) \left(\left[\left(\mathbf{P}^{\parallel} - \mathbf{P}^{\perp}\right) \tilde{v} \right]^T \mathbf{A} \frac{\partial x}{\partial \tau} - v^T \frac{\partial^2 x}{\partial \tau \partial s} \right) = -\frac{1}{2}\mathbf{P}\tilde{v}\,. \tag{53}$$

This is the PDE that governs the evolution of a loop towards a periodic orbit.

IV. DISCUSSION

The result of this project needs further refinement. The form of (53) can pose some computational difficulties and a different parameterization or ansatz needs to be implemented.

Most importantly we still did not incorporate information about the dynamics. Our suggested solution is to work with the cost functional

$$F^2 = \int_{s_i}^{s_f} ds\, \frac{v^2}{v.\tilde{v}} \tilde{v}_i g_{ij} \tilde{v}_j\,, \tag{54}$$

with

$$g = \left(\tilde{\mathbf{J}}_L \mathbf{P}^{\perp}\right)^T \left(\tilde{\mathbf{J}}_L \mathbf{P}^{\perp}\right) \tag{55}$$

The variational principle that would minimize this functional still needs to be formulated.

APPENDIX A: EIGENVALUES OF $\mathbf{J}_p$ UNDER SMOOTH CONJUGACIES

From (27) with $s = t$, that is when calculating $\mathbf{N}^t(x_o)$ along a periodic orbit we get

$$\begin{aligned}
\frac{dN^t_{ij}}{dt} &= \left(\frac{\partial H_{im}}{\partial x_n} v_n + H_{in} A_{nm} \right) H^{-1}_{mq}(x(t)) K^t_{qj} - \left(\frac{\partial H_{im}}{\partial x_n} v_n + H_{in} A_{nm} \right) H^{-1}_{mq}(x(t)) H_{ql}(x(t)) J^t_{lk} H^{-1}_{kj}(x_o) \\
&= \left(\frac{\partial H_{im}}{\partial x_n} v_n + H_{in} A_{nm} \right) H^{-1}_{mq}(x(t)) \left(K^t_{qj} - H_{ql}(x(t)) J^t_{lk} H^{-1}_{kj}(x_o) \right) \\
&= \left(\frac{\partial H_{im}}{\partial x_n} v_n + H_{in} A_{nm} \right) H^{-1}_{mq}(x(t)) N_{qj}\,,
\end{aligned} \tag{A1}$$

from the definition (22) of $\mathbf{N}^s$. Thus we have a differential equation of the form $d\mathbf{N}^t/dt = \mathbf{B}(t)\mathbf{N}^t$ where $B_{ij} = \left(\frac{\partial H_{im}}{\partial x_n} v_n + H_{in} A_{nm} \right) H^{-1}_{mj}(x(t))$. The initial condition is found from (22) to be $\mathbf{N}^0 = 0$ and thus the solution will be $\mathbf{N}^t \equiv 0$ for all times. Thus

$$\mathbf{K}^t(y_o) = \mathbf{H}(x(t)) \mathbf{J}^t \mathbf{H}^{-1}(x_o)\,. \tag{A2}$$

On a periodic orbit for t equal to the period T_p we have $x(T_p) = x_o$ and thus

$$\mathbf{K}_p(y_o) = \mathbf{H}(x_o)\mathbf{J}_p\mathbf{H}(x_o)\,, \tag{A3}$$

which means that $\mathbf{K}$ and $\mathbf{J}$ are related by a similarity transformation and thus have the same eigenvalues.

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