Penalizing loops that deviate from the True Path.

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Abstract

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CHAOS, AND WHAT TO DO ABOUT IT

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I. INTRODUCTION

The goal of this project is to improve a variational method for finding periodic orbits introduced in refs. [1, 2]. The method evolves an initial guess in the form of a closed loop towards a true periodic orbit of a given flow $f^t(x)$ defined by:

$$\frac{dx}{dt} = v(x),\ x \in \mathbb{R}^d\,. \tag{1}$$

This is achieved by minimizing the misorientation of the unit tangent vector $\hat{t}(x)$ of the loop to the unit vector $\hat{v}(x)$ parallel to the velocity field (1). In other words minimizing the cost functional

$$\overline{F}^2 = \frac{1}{L}\oint \left(\hat{t} - \hat{v}\right)^2 dl\,, \tag{2}$$

where the integration is performed along the loop.

The improvement that will be pursued here is to replace the simple Euclidean metric δ_{ij} that is used in (2) with a metric g_{ij} that would carry information about the flow, that is minimize the functional

$$\overline{F}^2 = \frac{1}{L}\oint \left(\hat{t} - \hat{v}\right)_i\, g_{ij}\, \left(\hat{t} - \hat{v}\right)_j\, dl\,. \tag{3}$$

The motivation for the use of the variational method as well as for improving it will be given in sect. II where the problem is stated in more precise terms.

II. VARIATIONAL PRINCIPLE SEARCH OF PERIODIC ORBITS

In this project any smooth, closed curve in a $d-$dimensional space is referred to as a loop. In general a loop is not a solution of (1), in contrast to a periodic orbit, which satisfies the periodic orbit condition $f^T(x) = x$ where T the period. The unit tangent vector of the loop $\hat{t}$ will not in general be parallel to $\hat{v}$. Thus, if we could continously deform the loop in such a way that its tangent becomes parallel to the velocity field we end up with a true periodic orbit. In ref. [2] it is shown that this corresponds to minimizing (2) and that one can write down a partial differential equation (PDE) for the evolution of the loop towards a periodic orbit[4]. Numerically solving this PDE provides the periodic orbit of the system "closest"[5] to the initial loop.

The method is conceptually more complicated, harder to program and generally slower than Newton or multiple-shooting methods for the search of periodic orbits. On the other hand it has an advantage when one tries to find long or extremely unstable periodic orbits, or when one deals with hard to visualize high-dimensional systems. For multiple-shooting to converge one needs a large number of Poincaré sections in order to control local instability. Thus one needs a great deal of information about the qualitative dynamics of the flow to make a clever choice of those sections. In high-dimensional flows this is usually not the case and multiple shooting methods can easily fail to find some of the longer cycles. In the variational method described Poincaré sections play no particular role and guesses with roughly the correct topology can lead to long cycles.

The extention of the method that will be attempted here is to use a metric that penalizes variations from a true periodic orbit in the unstable eigendirections of the flow more than it does in the stable ones. The hope is that in a high-dimensional flow in which only a few of the dimensions are significant one can concetrate only on them, effectivellyconcentratethe dimensionalieffectivelyoblem and the computational load.

III. CANDIDATES FOR THE ROLE OF METRIC

A. Almost a Jacobian Matrix

The Jacobian matrix of a flow at a point x_o is defined as

$$\mathbf{J}^t_{ij} = \left. \frac{\partial f^t_i(x_o)_i}{\partial x_{0_j}} \right|_{x=x_o} . \tag{4}$$

On a periodic orbit

$$\mathbf{J}_p(x_o) = e^{\oint d\tau \mathbf{A}(\mathbf{x}(\tau))} , \tag{5}$$

where $\mathbf{A}$ the matrix of variations $A_{ij}(x) = \frac{\partial v_i}{\partial v_j}$, describes the local deformation of the neighborhood of the periodic orbit under the flow, for finite times. The eigenvalues of the Jacobian are known to be independent of the initial point x_o on the periodic orbit and provide the local measure of instability of the system.

Thus it is tempting to use $\mathbf{J}$ as our metric tensor. Yet, a loop is not a solution of the equations of the flow and we cannot calculate the Jacobian along a loop. Moreover, the

Jacobian is not in general a symmetric matrix and thus it is not diagonalizable by a unitary similarity transformation, while its eigenvectors do not form an orthonormal set.

We attemt to resolve the first difficuattempt defining the matrix

$$\mathbf{J}_L(x_o) = e^{\oint_L d\kappa A(x(\kappa))}\,, \tag{6}$$

where κ parameter with the dimensions of time (so that the argument of the exponensial be dimensionless) and texponentialion is considered along the loop.

The second difficulty is removed by the use of the metric

$$\mathbf{M}(x) = \mathbf{J}_L^T(x)\mathbf{J}_L(x) \tag{7}$$

where the superscript T denotes the transpose of a matrix. $\mathbf{M}$ is symmetric and thus diagonalizable by a unitarity transformation and possesses a complete set of orthogonal eigenvectors.

We would also like to check whether $\mathbf{J}_L$ has the property of $\mathbf{J}$ that its eigenvalues do not depend on the initial point on the loop. Treating the integral in (6) as a discrete sum over m infinitesimal steps $\Delta\kappa$ we have

$$\begin{aligned}\mathbf{J}_L(x_o) &= \lim_{m\to\infty} \exp(\sum_{n=m}^{1} \Delta\kappa\, \mathbf{A}(x_n)) \\ &= \lim_{m\to\infty} \prod_{n=m}^{1} e^{\Delta\kappa\, \mathbf{A}(x_n))}\end{aligned} \tag{8}$$

where $x_n = x(t_o + n\,\Delta\kappa)$. This establishes the group property

$$\mathbf{J}_L^{\kappa+\kappa'}(x_o) = \mathbf{J}_L^{\kappa'}(x(\kappa))\mathbf{J}_L^{\kappa}(x_o) \tag{9}$$

This is important since it is all we need to prove that the eigenvalues of $\mathbf{J_L}$ do not depend on the initial point x_o on the loop, in exactly the same way this is proved for $\mathbf{J}$ on a cycle, *cf.* Ref. [3], Paragraph 8.2.

We would also like to prove that the eigenvalues of $\mathbf{J}_L$ are invariant under a smooth conjugacy. The proof in the case of $\mathbf{J}$ for a periodic orbit depends on the chain rule for the derivatives and thus on the definition (4). Since this definition is not obeyed by $\mathbf{J}_L$ we cannot use the same arguments in the proof and we need to think [6].

APPENDIX A: PROJECT PLAN

Tentative schedule:

1. **Tue Mar 8:** Stated the problem and presented motivation.
2. **Tue Mar 15:** Will check the invariant meaning of the eigenvalues of Jacobian-like matrix along a loop.
3. **Tue Mar 29:** Will try to write down PDE using $g = J^T J$.
4. **Tue Apr 5:** Will try to implement the PDE for the Rössler system.
5. **Tue Apr 12:** Will keep trying... If successful will try to implement it for the KS system (small size-few dimensions).
6. **Tue Apr 19:** Will polish the project to high shine ... (that iskeep trying)
7. **Tue Apr 26:** Fix the last few quirks ...
8. **Tue May 2:** Project deadline

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[1] Y. Lan and P. Cvitanović. Variational method for finding periodic orbits in a general flow. *Phys. Rev. E*, 69:16217, 2004.

[2] P. Cvitanović and Y. Lan. Turbulent fields and their recurrences. In N. Antoniou, editor, *Proceedings of 10th International Workshop on Multiparticle Production: Correlations and Fluctuations in QCD*, Singapore, 2003. World Scientific.

[3] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay. *Chaos: Classical and Quantum*. `ChaosBook.org`, (Niels Bohr Institute, Copenhagen), 2004.

[4] As the form of the equation depends on the parameterization of the loop and the question of the most adequate parameterization will be addressed in this project, I prefer not to present this PDE here.

[5] Distance in the space of loops is hard to define. In practice the method converges to a loop with topology similar to the initial guess.

[6] which of course is extra price.