Reduction of continuous SO(2) symmetry of a 2-mode system using method of slices

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Abstract

We study a 4-dimensional ODE normal form that is equivariant under SO(2) transformations as a simple system that posess a continuous symmetry and can exhibit chaos. We apply the method of slices to reduce the SO(2) symmetry, and construct Poincaré return maps on the slice manifold which enables us to find the relative periodic orbits.

Porter - Knobloch System

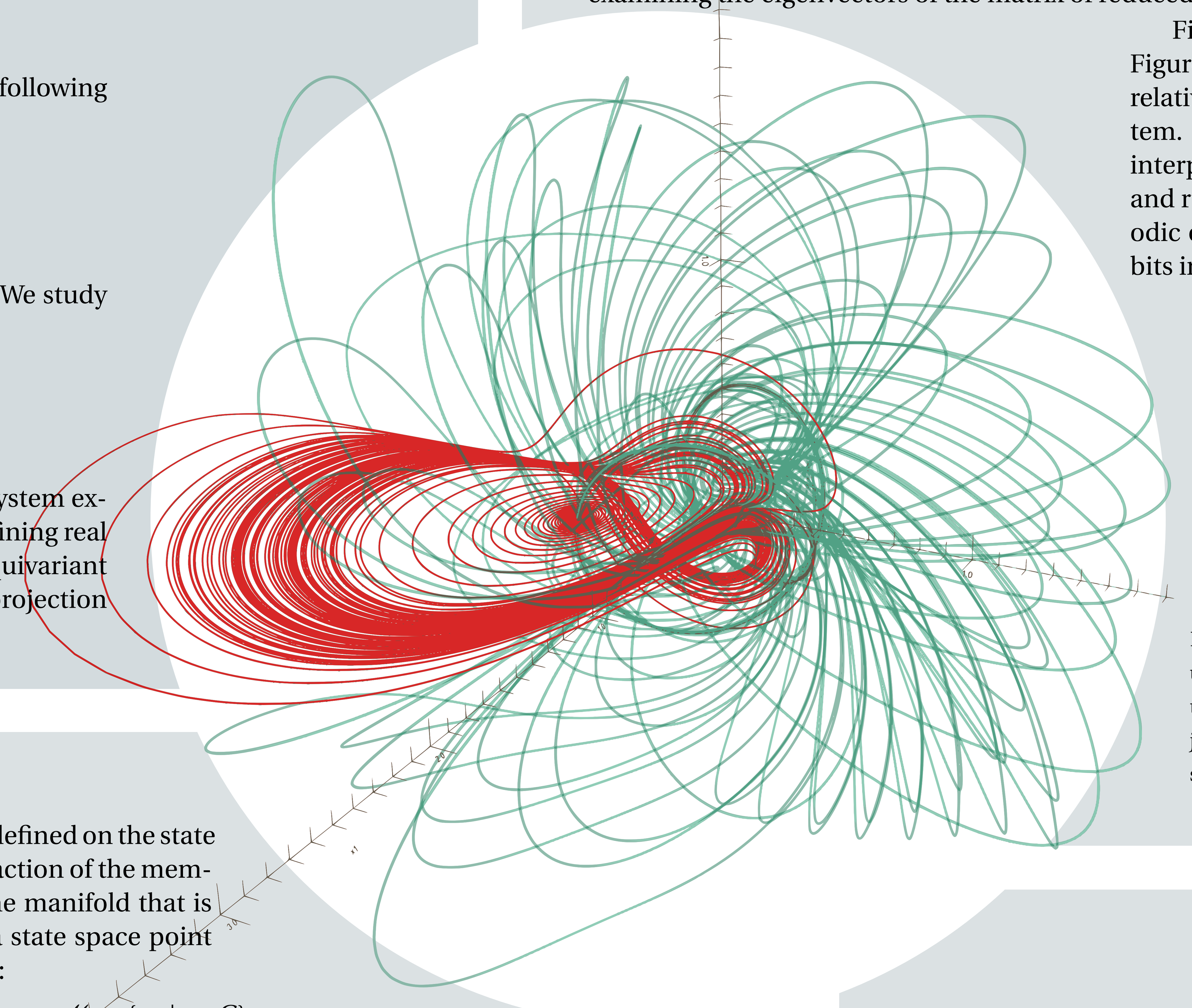
Porter and Knobloch studied bifurcations in 1:2 resonance ODE system of the following form [1]:

$$\dot z_1 = (\mu_1 - i\, e_1)\, z_1 + a_1\, z_1 |z_1|^2 + b_1\, z_1 |z_2|^2 + c_1\, \overline{z}_1\, z_2$$
$$\dot z_2 = (\mu_2 - i\, e_2)\, z_2 + a_2\, z_2 |z_1|^2 + b_2\, z_2 |z_2|^2 + c_2\, z_1^2 ,$$

which is equivariant under the U(1) transformation $(z_1, z_2) \to (e^{i\phi} z_1, e^{i2\phi} z_2)$. We study the following symplified version of this system:

$$\dot z_1 = \mu_1\, z_1 - z_1 |z_1|^2 + c_1\, \overline{z}_1\, z_2$$
$$\dot z_2 = (1 - i)\, z_2 + a_2\, z_2 |z_1|^2 + z_1^2 ,$$

with the parameter values $\mu_1 \to -2.8, a_2 \to -2.66, c_1 \to -7.75$ with which the system exhibits chaos. We represent this system in a four dimensional state space by defining real variables $x_i = Re[z_i]$ and $y_i = Im[z_i]$ and obtain a four dimensional SO(2)-equivariant system. Green curve in the main figure (in the middle) is a three dimensional projection of a typical trajectory of this system.



Method of Slices

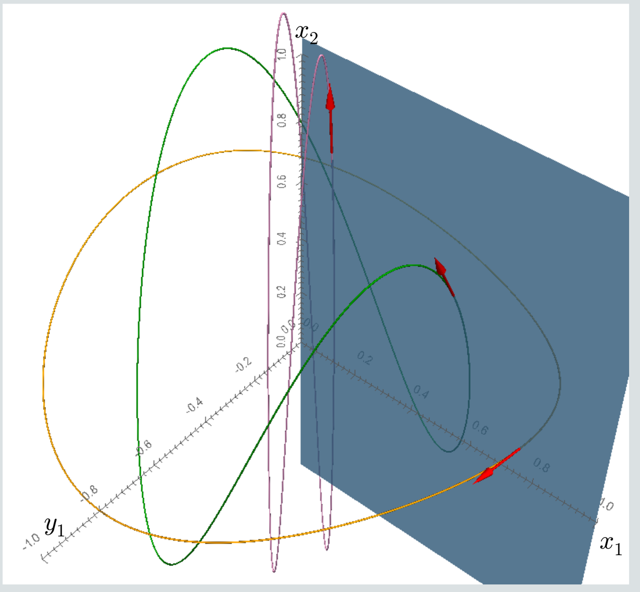


Fig 1 A 3D projection of the slice hyperplane and three group orbits of the Porter-Knobloch system. Group tangents evaluated at the points of intersection are drawn as red arrows.

Consider a dynamical system that is defined on the state space $\mathcal{M}$ and equivariant under the action of the members, g, of group of the group G. The manifold that is obtained by the action of the g on a state space point $x \in \mathcal{M}$ is called the "group orbit of x":

$$\mathcal{M}_x = \{g x | g \in G\}$$

Points on a group orbit are dynamically equivalent. The general symmetry reduction problem is finding a representation of the system, such that every group orbit is represented by a single representative point.

In the method of slices, one looks for a submanifold $\hat{\mathcal{M}}$ in the full state space in such a way that every group orbit intersects $\hat{\mathcal{M}}$ only once. Points $\hat x$ at which the group orbits pierce the slice are taken as the representatives of the group orbit, thus, on the slice, equivalent set of points $\mathcal{M}_{\hat x}$ are represented by a single point $\hat x$.

While in general, finding a good slice is a non-trivial problem, for the case at hand, a single hyperplane that includes $\hat x' = (1,0,0,0)$ and is perpendicular to the group tangent evaluated at this point, $\mathbf{T}\hat x' = t' = (0,1,0,0)$ where $\mathbf{T}$ is the $SO(2)$ Lie algebra element, can represent every group orbit that has a non-zero first mode component. This slice is illustrated in a 3D projection in Figure 1.

Computationally, the slicing problem is finding the parameters of group action as a function of time that maps the points $x(t)$ in the full state space to their representatives $\hat x(t) = g(-\theta(t)) x(t)$ on the slice. One can also obtain the dynamics within the slice hyperplane by directly integrating

$$\hat v(\hat x) = v(\hat x) - \dot\theta(\hat x)\, t(\hat x) , \qquad \dot\theta(\hat x) = \langle v(\hat x) | t' \rangle / \langle t(\hat x) | t' \rangle$$

where $v(\hat x)$ and $\hat v(\hat x)$ are velocities in symmetry equivariant and symmetry reduced state spaces; and $t(x)$ is the group tangent evaluated at the point x. For the derivation of these equations and the detailed discussion of the method of slices, see [2]. Reduced trajectory of the Porter-Knobloch system is the red curve in the main figure which is not a projection since within the slice, y_2 component of the points are zero.

Symbolic Dynamics and Relative Periodic Orbits

Relative periodic orbits (trajectories of points x_p which after a finite time T intersects the group orbit $\mathcal{M}_{x_p}$ of x_p) in the symmetry equivariant state space become periodic orbits in the symmetry reduced state space. In order to find them, we choose a Poincaré section in the reduced state space (Figure 2, 3) and construct the return map of the arclengths (Figure 4) of the points on the Poincaré section curve shown in Figure 3.

We position the Poincaré section plane such that it includes the relative equilibrium $\hat x_{TW}$ of the system and the direction towards which the nearby perturbations expand which we find by examining the eigenvectors of the matrix of reduced velocity gradients ($\hat A_{i,j} = \partial_i \hat v(\hat x_{TW})_j$).

Fixed points of the Poincaré return map in Figure 4 and its higher iterates correspond to the relative periodic orbits of the Porter-Knobloch system. We take the fixed points that we get from interpolations to the return map as initial guesses and run Newton solvers to find the relative periodic orbits of the flow. We plotted two such orbits in the reduced state space in Figure 5.

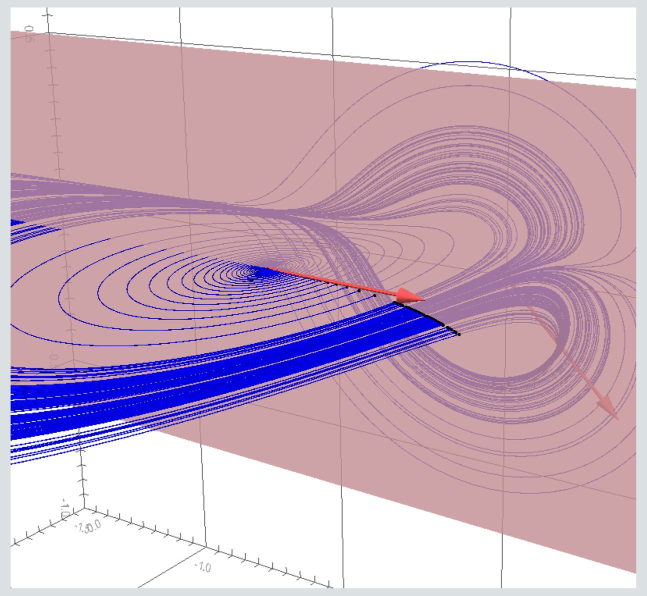


Fig 2 Symmetry reduced trajectory (blue curve) of the Porter-Knobloch system and the Poincaré section plane. Arrow points to the direction towards which the perturbations in the neighborhood of the relative equilibrium expand.

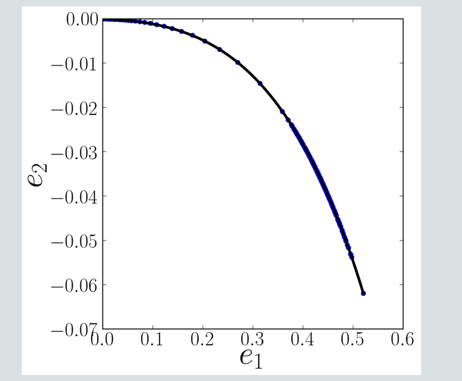


Fig 3 Poincaré section of Figure 2. Positions are relative to the relative equilibrium and projected onto the basis vectors which spans the Poincaré section plane.

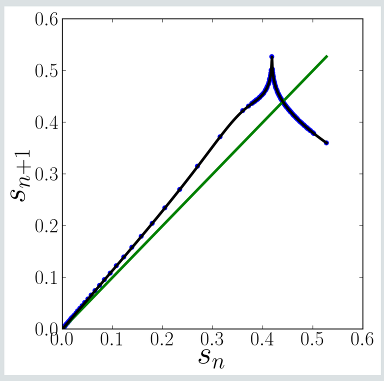


Fig 4 Return map of the arclengths along the curve in Figure 3. Fixed points this and higher order return maps lies on the relative periodic orbits of the Porter-Knobloch system.

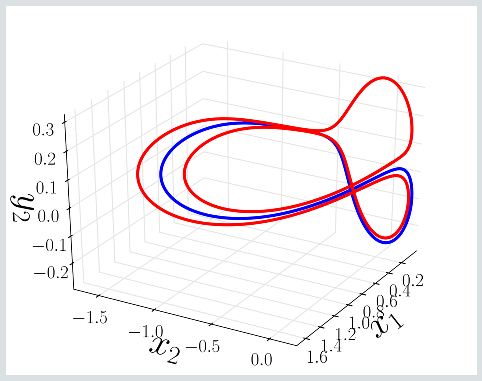


Fig 5 Relative periodic orbits $\overline{0}$ and $\overline{01}$ of the Porter-Knobloch system in the reduced state space.

Conclusions

We introduced a simple four dimensional system that exhibits chaos and is equivariant under continuous SO(2) transformations.

We showed that the method of slices can be used to reduce the SO(2) symmetry and it is particularly easy to apply to the case at hand since a single hyperplane captures all the group orbits that has a component in the first mode.

We found relative periodic orbits of the system by constructing a Poincaré return map in the reduced state space.

References

[1] J. Porter and E. Knobloch, *Dynamics in the 1:2 spatial resonance with broken reflection symmetry.* Physica D, **201**, pp 318 – 344, 2005.

[2] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay *Chaos: Classical and Quantum*, (Niels Bohr Inst., Copenhagen, 2013). ChaosBook.org