Lecture 8

- Chain complex C over $\mathbb{Z}[H]$ consists of

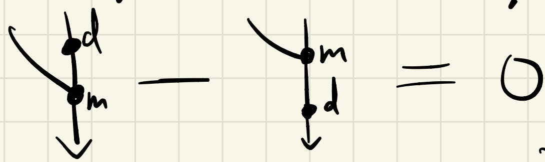
differential

1) $C \xrightarrow{d} C$

free action

2) $\mathbb{Z}[H] \otimes C \xrightarrow{m} C$

satisfying the compatibility equation

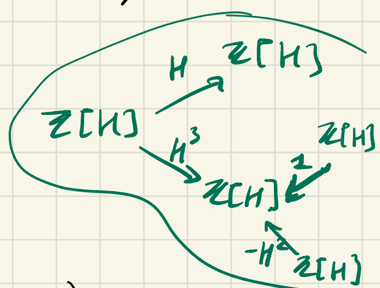


(the action is equivariant)

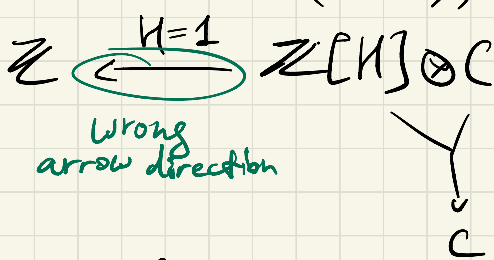
- In practice, it is a graph:

→ vertices = towers $\mathbb{Z}[H]$

→ edges labeled by elements in $\mathbb{Z}[H]$



- Want: substitute $H=0$ or $(H=1)$, but there is a problem:



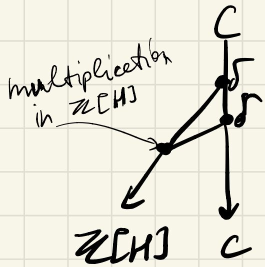
This is not a real problem though, since our modules are free.

def. Type D structure over $\mathbb{Z}[H]$

(aka twisted complex, or co-module)

is a free $\mathbb{Z}$-module C together with

with $C \xrightarrow{\delta} \mathbb{Z}[H] \otimes C$ satisfying



$= 0$

Notation: $C^{\mathbb{Z}[H]}$

In practice, its again a graph:

- vertices: $\mathbb{Z}$
- edges: labelled by elements in $\mathbb{Z}[H]$
- going twice should square to 0

$$\begin{tikzcd}
 & & \mathbb{Z}_e \\
\mathbb{Z}_a \arrow[urr, "H"] \arrow[rr, "1"] \arrow[d, "H^2"'] & & \mathbb{Z}_b \arrow[d, "H^3"] \\
\mathbb{Z}_c \arrow[rr, "H"'] & & \mathbb{Z}_d
\end{tikzcd}$$

For example: $\delta(a) = 1 \otimes b + H^2 \otimes c + H \otimes e$

- So, as algebraic structures

$\left\{\begin{matrix}\text{chain complexes} \\ \text{over } \mathbb{Z}[H]\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\text{type } D \text{ structures} \\ \text{over } \mathbb{Z}[H]\end{matrix}\right\}$

$\boxed{C^{\mathbb{Z}[H]} \boxtimes \mathbb{Z}[H]} \longleftarrow C^{\mathbb{Z}[H]}$

$d \longleftarrow \delta$

towers $\longleftarrow$ generators

'just a notation

- What do we get? Well, now we can sub $H=1$!

$$\begin{tikzcd} C \arrow[d] \arrow[dr] & \\ C \otimes \mathbb{Z}[H] \arrow[dr, "H=1"] & \\ & \mathbb{Z} \end{tikzcd} \rightsquigarrow \begin{tikzcd} C \arrow[d, "d = \delta|_{H=1}"] \\ C \end{tikzcd}$$

type D structure $\qquad$ chain complex

$$C^{\mathbb{Z}[H]} \xrightarrow[H=1]{} C^{\mathbb{Z}[H]}\Big|_{H=1}$$

graph $\bullet \xrightarrow{H} \bullet$

as a chain complex over $\mathbb{Z}[H]$

as a type D structure

$$\mathbb{Z}_a \xrightarrow{H} \mathbb{Z}_b$$

$\delta(a) = H \otimes b$

$$K \longrightarrow CBN(K)_{\mathbb{Z}[H]}$$

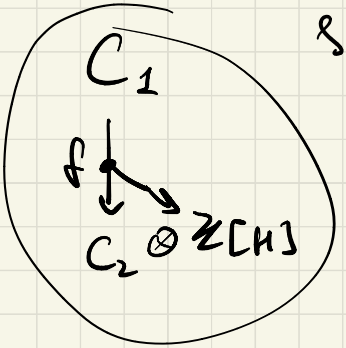
chain complex over $\mathbb{Z}[H]$, knot invariant up to $\mathbb{Z}[H]$-equivariant homotopy equivalence

$$K \longrightarrow CKh(K)^{\mathbb{Z}[H]}$$

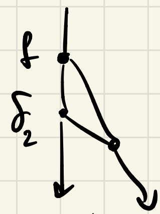
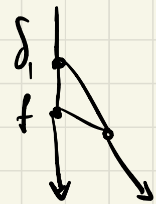
$\left(\bigcirc \rightsquigarrow V = \langle 1, x \rangle_{\mathbb{Z}} \right.$ differential is deformed $\left. \right)$

type D structure over $\mathbb{Z}[H]$, knot invariant up to $\boxed{\text{homotopy equivalence}}$?

• Type D structure <u>homomorphism</u> between $C_1^{\mathbb{Z}[H]}$ and $C_2^{\mathbb{Z}[H]}$ is a map



such that the following equation is satisfied:

$D(f) \underset{\text{def.}}{=}$  $-$  $= 0$

postcompose precompose

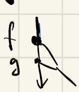
• f_1 & f_2 are <u>homotopic</u> if

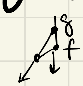
$f_1 - f_2 = D(h)$ for some $h : C_1 \longrightarrow C_2 \otimes \mathbb{Z}[H]$

$\quad\quad\quad\quad \| \;\; h\circ\delta_1 + \delta_2 \circ h$ (h need not be homo)

• $C_1^{\mathbb{Z}[H]} \xrightarrow{f} C_2^{\mathbb{Z}[H]}$ (with $\mathbb{Z}[H]$) is a <u>homotopy equivalence</u> if there is $C_1 \xleftarrow{g} C_2$ (with $\mathbb{Z}[H]$) s.t.

$f \circ g \simeq id$ & $g \circ f \simeq id$

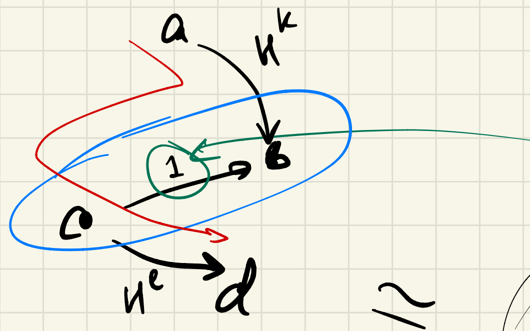




Downside of type D structures: no homology

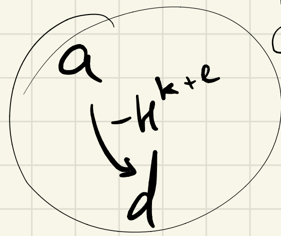
But, we can still cancel things!

1)



needs to be invertible!

$\simeq$

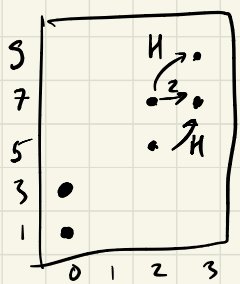


$\delta(a) = (-H^{k+l}) \otimes d$

2) More generally, this also works if a and b are isomorphic type D structures (the type D str. analogue of (C1) Lemma, see "Immersed curves in Khovanov homology", Section 2.2

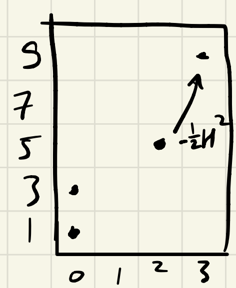
<u>Example</u> As type D structure,

• $Ckh($  $)^{\mathbb{Z}[H]} \simeq$



over $\mathbb{Z}$

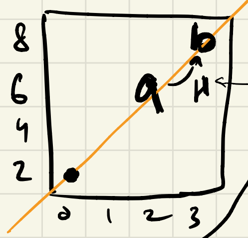
over $\mathbb{Q}$ we cancel 2



over $\mathbb{Q}$

can be canceled until

• $\widetilde{CKh}\left(\left(\ \right)\right)^{\mathbb{Z}[H]} \simeq$



can be canceled until

remark 1

grading of this arrow is computed as

• $q(a \xrightarrow{H} b) =$

$= q(b) + q(H) - q(a) =$

$= 8 + (-2) - 6 = 0$

• $h(a \xrightarrow{H} b) =$

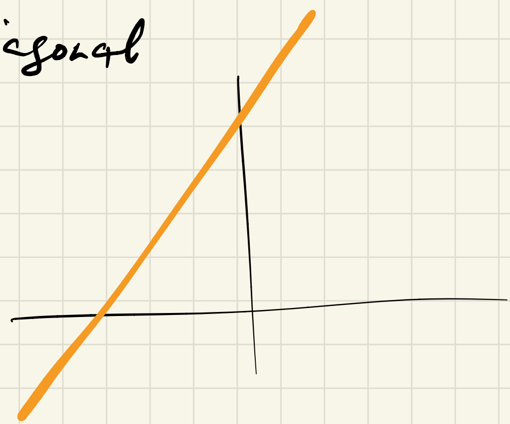
$= h(b) + h(H) - h(a) = 3 + 0 - 2 = 1$

The upshot is that $\delta: CKh(K) \to CKh(K) \otimes \mathbb{Z}[H]$ still preserves the q-grading and increases h-grading by 1

Remark 2 Lee proved that $Kh(K)$ for alternating knots is thin, i.e. is concentrated on a diagonal

$$\text{const} = q - 2h$$

(can be expressed in terms of $\sigma(K)$)



Open q. Characterize thin knots topologically / geometrically

Next:

Proposition

$$H_*\left(CKh(K)^{\mathbb{Z}[H]}\Big|_{H=1}\right) = \mathbb{Z}\oplus\mathbb{Z}$$

In other words, if we use

$m: V\otimes V \to V$ $\begin{cases} 1\otimes x \mapsto x & 1\otimes 1 \mapsto 1 \\ x\otimes 1 \mapsto x & x\otimes x \mapsto x \end{cases}$ (since $H=1$)

(do not preserve q-grading!)

$\Delta: V \to V\otimes V$ $\begin{cases} 1 \mapsto 1\otimes x + x\otimes 1 - 1\otimes 1 \\ x \mapsto x\otimes x \end{cases}$ (since $H=1$!)

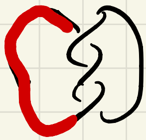
the homology of Khovanov complex $= \mathbb{Z}\oplus\mathbb{Z}$

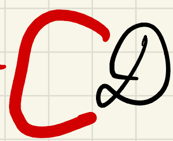
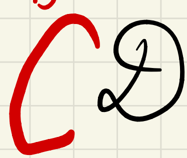
$\square$ idea work with basis $\underset{\overset{\shortparallel}{a}}{\boxed{x}}$, $\underset{\overset{\shortparallel}{b}}{\boxed{1-x}}$.

In this basis:

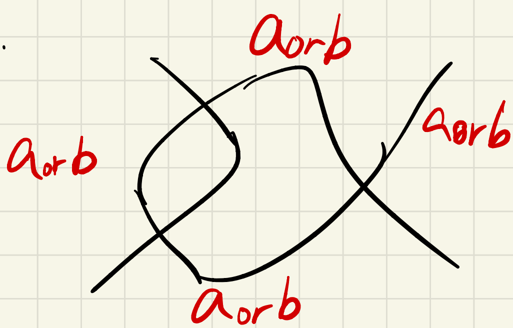
$m: \begin{cases} a\otimes a \mapsto a & a\otimes b \mapsto 0 \\ b\otimes b \mapsto b & b\otimes a \mapsto 0 \end{cases}$

$$\Delta : \begin{cases} a \mapsto a \otimes a \\ b \mapsto -b \otimes b \end{cases}$$

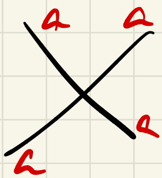
Then, if we fix an edge on a diagram a , and consider all the generators which have a on circles involving that edge, we get a subcomplex!

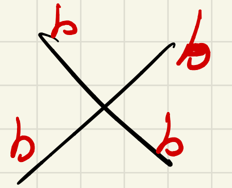
$CKh(K)^{\mathbb{Z}[H]}\Big|_{H=1} =$ a  $\oplus$ b 

Moreover, we can consider splitting according to all possible colorings of edges:



Statement is that whenever

I have 

or 

,

then those subcomplexes are contractible

And the only two that are not like this are