Knot homologies through the lens of immersed curves

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(based on joint works with L. Watson & C. Zibrowius)

$\boxed{\text{Overview}}$

- Homology theories that can be studied via immersed curves

$\boxed{{}_{R}HFK(K)}$ knot Floer homology over $R = \frac{k[u,v]}{uv=0}$

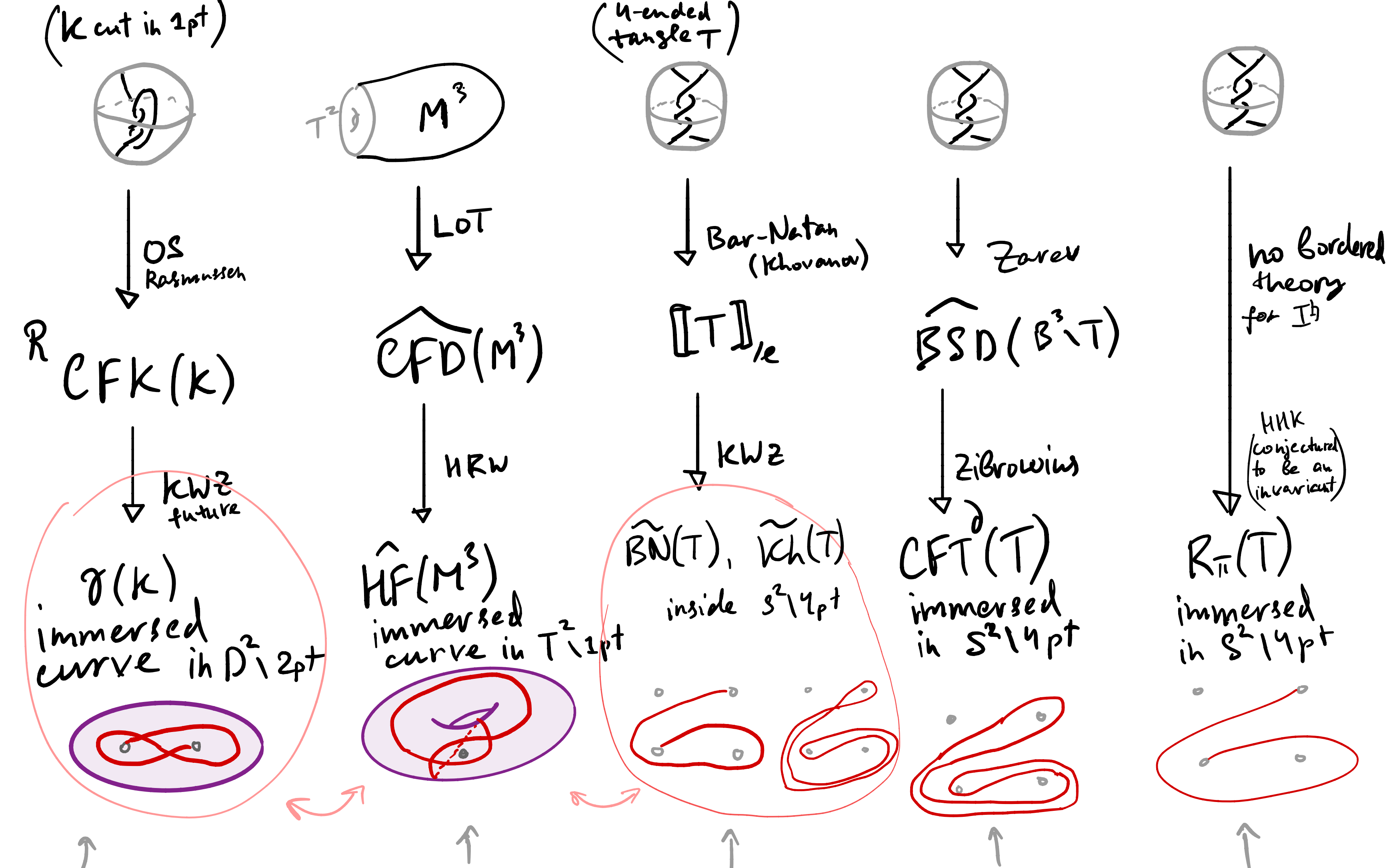
$\boxed{\widehat{HF}(Y^3)}$ Heegard-Floer homology

$\boxed{\widetilde{BN}(K), \widetilde{Kh}(K)}$ reduced Bar-Natan & Khovanov homologies

$\boxed{\widehat{HFK}(K)}$ knot Floer homology

$\boxed{I^{\natural}(K)}$ singular Instanton homology

- their bordered variants & immersed curves


(K cut in 1pt)
T^2
M^3
(4-ended tangle T)
OS
Rasmussen
LOT
Bar-Natan
(Khovanov)
Zarev
no bordered theory for $I^\natural$
${}^{R}CFK(K)$
$\widehat{CFD}(M^3)$
$[\![T]\!]_{/e}$
$\widehat{BSD}(B^3 \setminus T)$
KWZ
future
HRW
KWZ
Zibrowius
HHK
(conjectured to be an invariant)
$\gamma(K)$
immersed curve in $D^2 \setminus 2pt$
$\widehat{HF}(M^3)$
immersed curve in $T^2 \setminus 1pt$
$\widetilde{BN}(T)$, $\widetilde{Kh}(T)$
inside $S^2 \setminus 4pt$
$CFT^\partial(T)$
immersed in $S^2 \setminus 4pt$
$R_\pi(T)$
immersed in $S^2 \setminus 4pt$

used to simplify the process $CFK^-(K) \underset{LOT}{\rightsquigarrow} \widehat{CFD}(S^3 \setminus K)$

gives a criterion for when $Y^3 = M \cup_{T^2} M'$ is an L-space

used to prove mutation invariance of s-invariant and $\widetilde{BN}(K;\mathbb{F}_2)$

used to prove mutation invariance of δ-graded $\widehat{HFK}(K)$

used to obtain explicit descriptions of generators in $CI^{\natural}(K)$

- All these curve invariants satisfy the corresponding <u>gluing theorems</u> (except for the instanton curve)

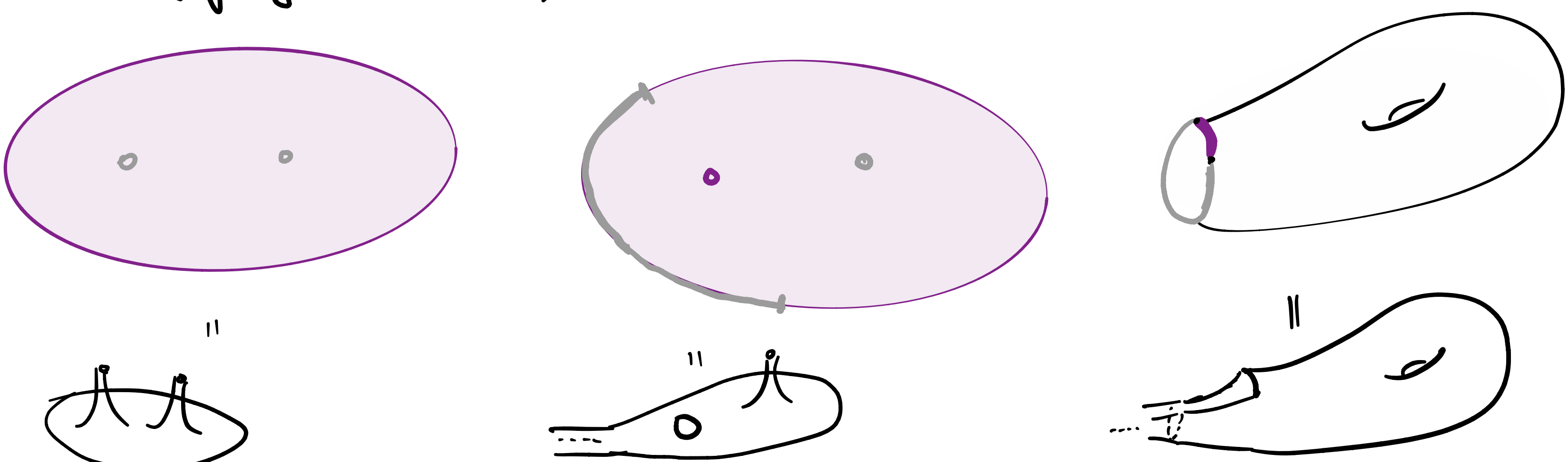
① When do immersed curves arise?

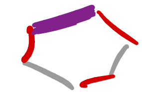
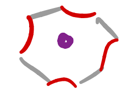
• Consider a <u>marked surface</u> ($\Leftrightarrow$ sutured surface $\Leftrightarrow$ punctured surface)

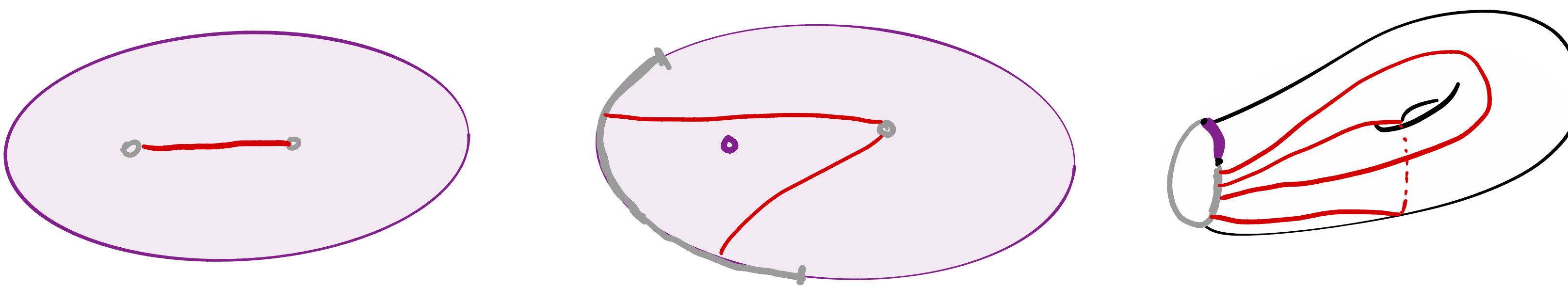
$$\left(\Sigma,\ \partial\Sigma = B \sqcup P\right)$$

actual boundary ↗ B; ↖ "punctured" part of the boundary P

satisfying $\quad B \neq \emptyset \neq P$

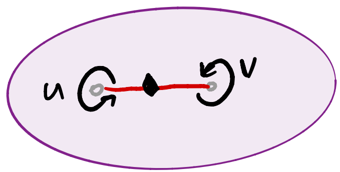


- Then we can consider a <u>full arc-system</u>, i.e. disjoint arcs $\{a_1, \dots, a_n\}$ with endpoint on P, cutting Σ into
  - discs with one B-component on the boundary 
  - annuli with one B-component in the center 

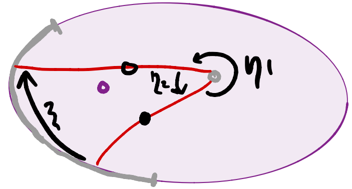


$\bullet \ \{(\Sigma, \partial\Sigma = B \sqcup P), \{a_1 \dots a_n\}\} \rightsquigarrow$ $\boxed{\text{quiver Algebra } \mathcal{A}(\Sigma) \text{ over any field } k}$

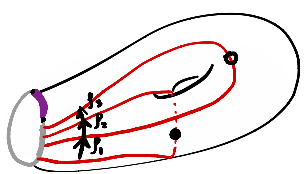
- vertices = arcs
- edges = chords along P
- relations = non-consecutive chords



$$\longrightarrow \quad \mathcal{A} = k\left[\begin{tikzcd} \bullet \arrow[loop left, "u"] \arrow[loop right, "v"] \end{tikzcd}\right] \Big/ \begin{matrix} uv = 0 \\ vu = 0 \end{matrix} = \frac{k[u,v]}{uv=0}$$



$$\longrightarrow \quad \mathcal{A} = k\left[\begin{tikzcd} \bullet \arrow[r, "\eta_1"] \arrow[r, bend right, "\xi"'] & \circ \arrow[l, bend right, "\eta_2"'] \end{tikzcd}\right] \Big/ \begin{matrix} \xi\eta_2 = 0 \\ \eta_2\xi = 0 \end{matrix}$$



$$\longrightarrow \quad \mathcal{A} = k\left[\begin{tikzcd} \bullet \arrow[r, bend left, "\rho_3"] \arrow[r, bend right, "\rho_1"'] & \circ \arrow[l, "\rho_2"'] \end{tikzcd}\right] \Big/ \begin{matrix} \rho_3\rho_2 = 0 \\ \rho_2\rho_1 = 0 \end{matrix}$$

• $\mathcal{A}$ = wrapped Fukaya category based on $\{a_1, \dots, a_n\}$ (with stops B)

Rmk changing $P \longleftrightarrow B$ changes $\mathcal{A}$ to its Koszul dual

Theorem (HKK, HRW, KWZ)

HKK: · representation theoretic proof

HRW: · algorithm · compact Fuk

KWZ: · algorithm · full Fuk

$$\frac{\left\{\begin{array}{l}\text{type D structures} \\ \text{over } \mathcal{A}(\Sigma)_{k} \quad (\leftarrow \text{field})\end{array}\right\}}{\text{homotopy equivalence}} \xleftrightarrow{\text{Bijection}} \frac{\left\{\begin{array}{l}\text{collections of immersed} \\ \text{curves with local systems} \\ \text{in } \Sigma\text{, endpoints allowed on } P\end{array}\right\}}{\text{homotopy of curves} + \text{equivalence of local systems}}$$

Thus, every time a bordered invariant (in low-dim topology) is a type D structure over $\mathcal{A}(\Sigma)$, one has an immersed curve invariant.

② Immersed curve $\gamma(K)$ coming from knot Floer invariant

knot Floer invariant

rmk: over $k[u,v]$ in general, but we specialize to $R = \frac{k[u,v]}{uv=0}$

$$\begin{tikzcd}
\text{knot } K \arrow[rr, mapsto, "\text{OS, Rasmussen}"] \arrow[dr, mapsto] & & {}_{R}CFK(K) \arrow[r, mapsto, "H_*(-)"] & {}_{R}HFK(K) \\
& {}^{R}CFK(K) \arrow[ur, mapsto, "(R \boxtimes -)"] & &
\end{tikzcd}$$

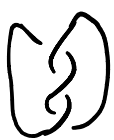
${}_{R}CFK(K)$: freely generated

${}_{R}HFK(K)$: R-module

${}^{R}CFK(K)$: type D structure

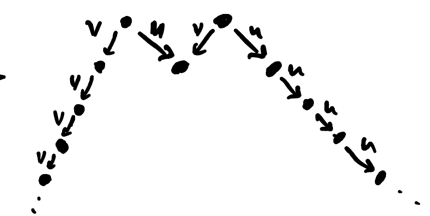
(~ quiver with coefficients on edges, s.t. "$d^2=0$")

Example



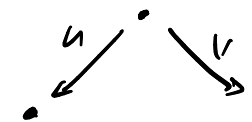
trefoil

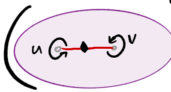
$$\begin{tikzcd}
\text{trefoil} \arrow[r, mapsto] \arrow[d, mapsto] & {[R \xleftarrow{u} R \xrightarrow{v} R]} \arrow[r, mapsto] & {} \\
{[\bullet \xleftarrow{u} \bullet \xrightarrow{v} \bullet]} \arrow[ur, mapsto] & &
\end{tikzcd}$$



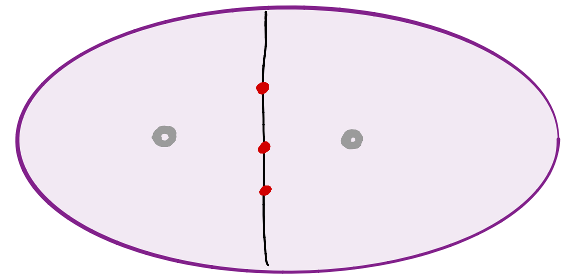
<u>Immersed curve $\gamma(K)$</u>

- ${}^{R}CFK$ = quiver with coeffs. in R s.t. "$d^2=0$"

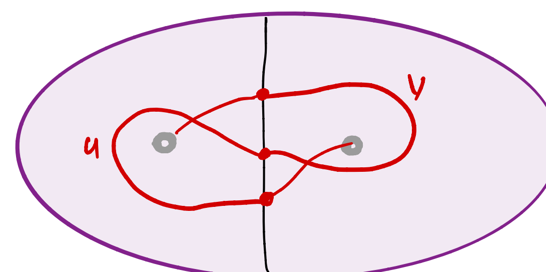


- recall $R = \mathcal{A}\Big($  $\Big)$

- put vertices of ${}^{R}CFK$ on the vertical arc ("dual" arc to the red here)



- record edges by segments around the punctures, and join the valence-1 vertices with punctures

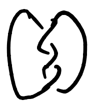


<u>Important</u> In general one gets a train-track, but the arrow-sliding algo [HRW] allows to modify it to get a set of immersed curves with local systems

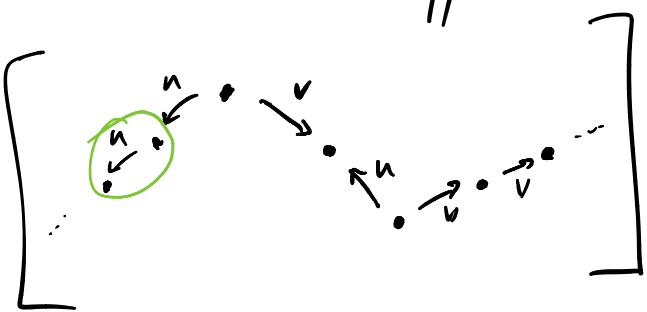
rmk Train-track moves correspond to homotopy equivalences of type D structures

Glueing theorem $\quad {}_{R}HFK(mK \# K') \cong HW(\gamma(K), \gamma(K'))$

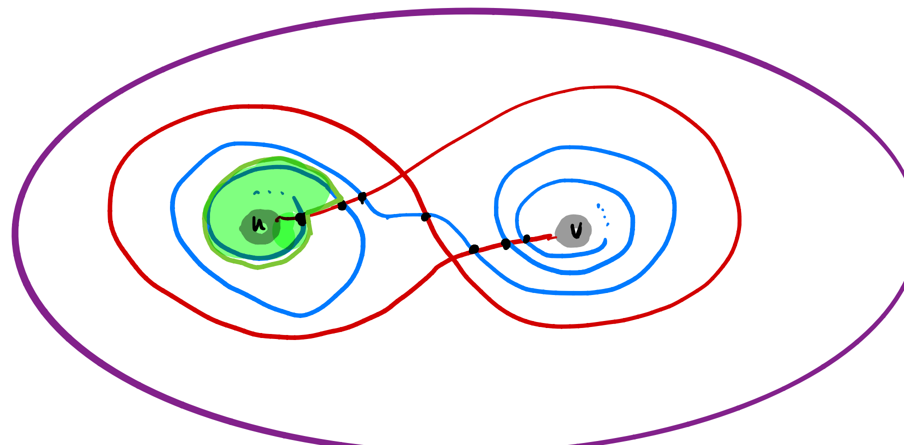
wrapped Floer homology

Example $\quad {}_{R}HFK\Big($  $=$ O $\#$  $\Big)$

$\Vert$



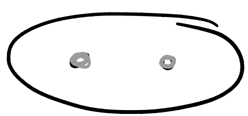
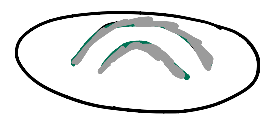
$\cong$



Note the actions U and V correspond to some of index-2 discs covering the punctures (work in progress)

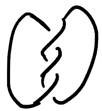
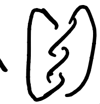
Application: geometric interpretation of LOT-theorem

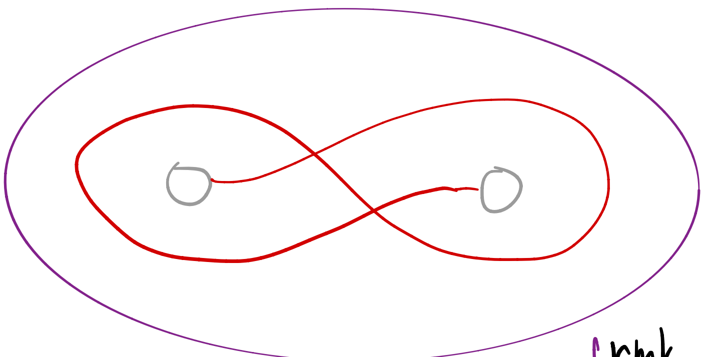
$$\begin{tikzcd}
CFK(S^3, K) \arrow[rr, "\text{stable and unstable chains}"'] \arrow[d, leftrightarrow] & & \widehat{CFD}(S^3 \setminus \nu(K)) \arrow[d, leftrightarrow] \\
\gamma(K) \arrow[rr, "\boxed{\text{adding a handle}}"'] & & \widehat{HF}(M) \\
\cap & & \cap
\end{tikzcd}$$

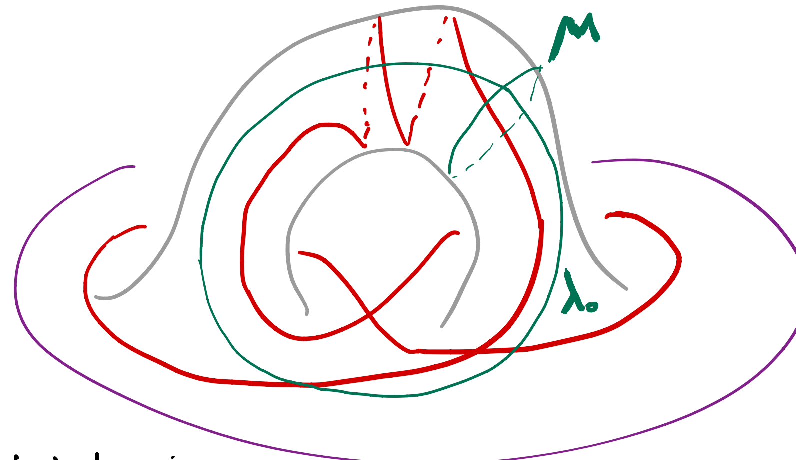
twice-punctured disc   once-punctured torus

Theorem curve $\gamma(K)$ becomes $\widehat{HF}(M)$ after adding a handle.

Example

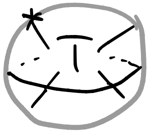
$\gamma\Big($  $\Big) \longmapsto \widehat{HF}\Big(S^3 \setminus$  $\Big)$





rmk add 2τ twists in the handle so that the red curve is homologous to Seifert longitude λ_0

③ Khovanov theoretic curve invariants

 $\xrightarrow{\text{BN}}$ $[\![T]\!]_{/\ell}$ $\xrightarrow{\text{KWZ}}$ $\mathcal{D}(T)^{\mathcal{B}}$ $\longmapsto$ $\widetilde{\mathrm{BN}}(T)$

pointed 4-ended tangle

uses cube of resolutions

complex over cobordism category $\mathrm{Cob}_{/\ell}$

new basis in $\mathrm{Cob}_{/\ell}$ uses $*$

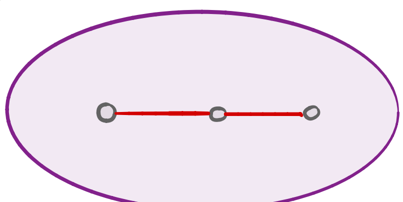
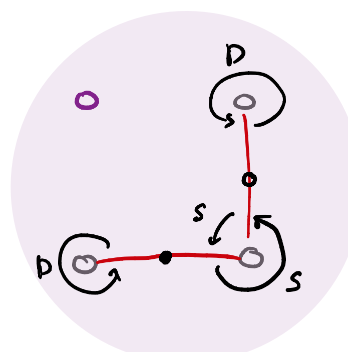
type D str.

curves in $S^2 \setminus 4\,\mathrm{pt}$

mapping cone on $H = D - S^2$

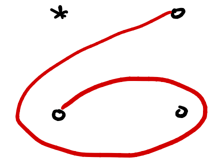
$$\left[\mathcal{D}(T) \xrightarrow{H\cdot \mathrm{id}} \mathcal{D}(T)\right]^{\mathcal{B}} \longmapsto \widetilde{\mathrm{Kh}}(T)$$

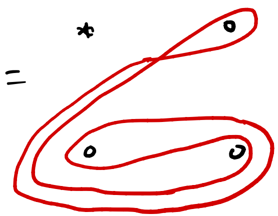
rmk unreduced invariant $\mathrm{Kh}(T)$ also exists

$\mathcal{B} = \mathcal{A}\Big($  $=$  $\Big) = \left[D \circlearrowleft \bullet \underset{S}{\overset{S}{\rightleftarrows}} \circ \circlearrowright D \right] \Big/ DS = SD = 0$

$\mathrm{Cob}_{/\ell}(\Theta, \ominus)$

Example

$\widetilde{BN}\Big($  $\Big) =$ 

$\widetilde{Kh}\Big($  $\Big) =$ 

for rational tangles $\widetilde{BN}(T)$ is a push-offs of one component to the boundary

$\widetilde{BN}(T)$ has precisely one arc component

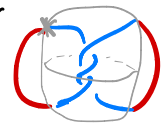
$\widetilde{Kh}(T)$ consists of "figure eights" but relationship $\widetilde{BN} \longleftrightarrow \widetilde{Kh}$ is complicated

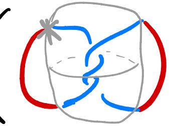
Gluing theorem

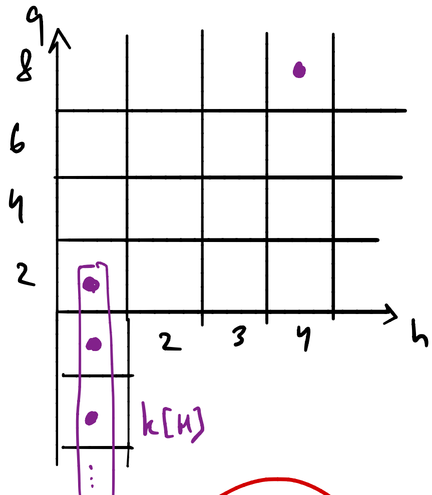
$$\widetilde{BN}(T \cup T') \cong HW\big(\widetilde{BN}(mT), \widetilde{BN}(T')\big)$$

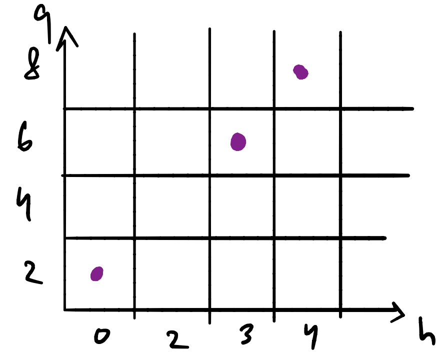
$$\widetilde{Kh}(T \cup T') \cong HW\big(\widetilde{BN}(mT), \widetilde{Kh}(T')\big) \cong HW\big(\widetilde{Kh}(mT), \widetilde{BN}(T')\big)$$

Example

$\widetilde{BN}\Big($  $\Big) = k[H] \oplus k$

$\widetilde{Kh}\Big($  $\Big) = k^3$



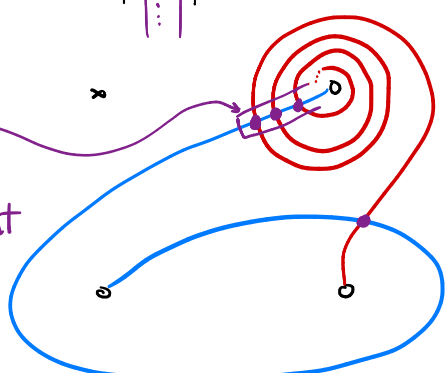


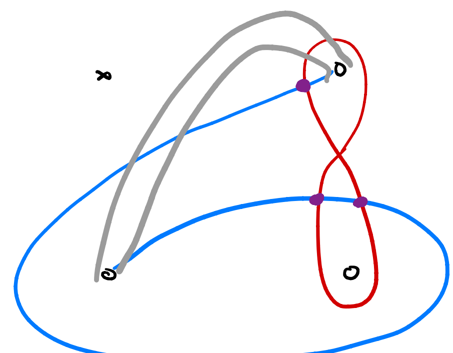
rmk

H-action is again governed by index-2 discs

rmk

This is the s-invariant





- $\widetilde{BN}(T)$ and $\widetilde{Kh}(T)$ are first examples where non-trivial local system show up

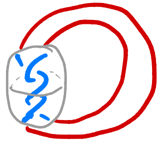
<u>Theorem</u> mutation only changes local systems on $\widetilde{BN}(T)$ by $\cdot(\pm 1)$
(uses a new idea in Cob$_{/\ell}$: "moving the basepoint using 4Tu relation")

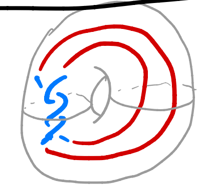
<u>Corollary</u> Rasmussen's invariant $s^k(K)$, and $\widetilde{BN}$, $\widetilde{Kh}$, Kh over $\mathbb{F}_2$ are preserved by mutation

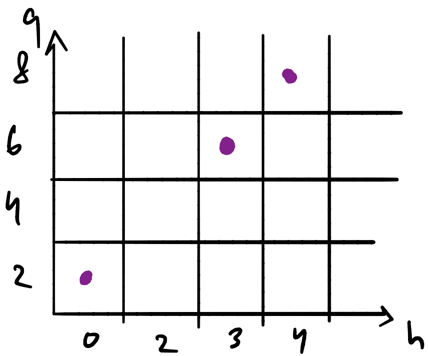
- $\widetilde{BN}(T)$ computable, python package written [ZC], C++ under way

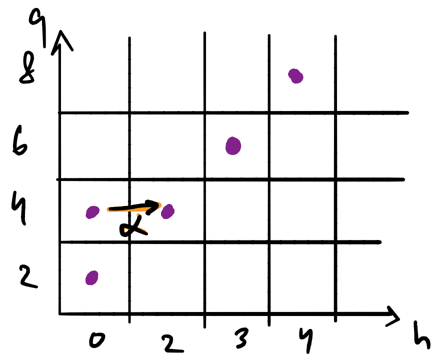
- Analyzing $\widetilde{BN}(T)$ for different tangles allows to make new conjectures about the structure of $[\![T]\!]_{/\ell}$

Recovering annular sutured Khovanov homology

$\widetilde{Kh}\Big($  $\Big) = k^3$

$\widetilde{SKh}^+\Big($  $\Big) = k^5$





Just as

$$\boxed{\widehat{HF} \xrightarrow[\text{basepoint}]{\text{adding}} \widehat{HFK}}\,,$$

we have

$$\boxed{\widetilde{Kh} \xrightarrow[\text{basepoint}]{\text{adding}} \widetilde{SKh}^+}$$

(work in progress)