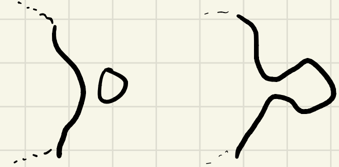
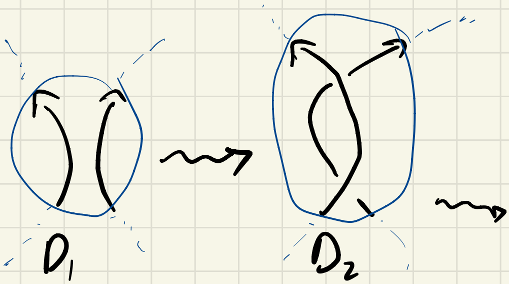


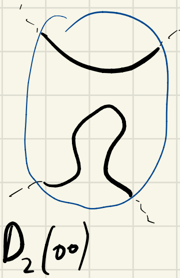
$$J_{D_2} = \Big(\cancel{J_{D_1}(q+q^{-1}) + J_{D_1}\cdot(-q)^2} \Big)\cdot \underbrace{q^1}_{\text{renormalization}} = J_{D_1}$$

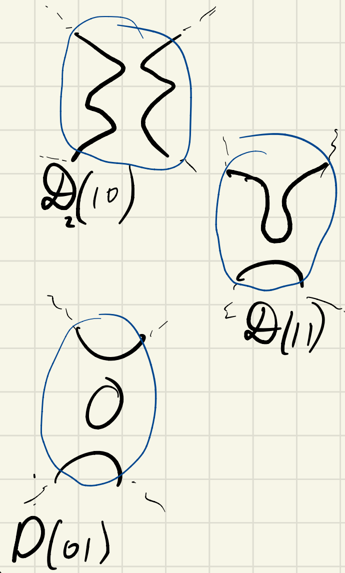
since we have 1 extra positive crossing

$n_+(D_2) - n_+(D_1) = 1$

$\boxed{R2}$







- Denote by $\langle D \rangle$ the Jones poly <u>without</u> renormalization $(-1)^{n_-} q^{n_+-2n_-}$
- $r(D)$ = renormalization for diagram D

$$J_{D_2} = \Big(\langle D_2(00)\rangle + \langle D_2(10)\rangle + \langle D_2(11)\rangle + \langle D_2(01)\rangle \Big) r(D_2)$$

$$= \Big(\cancel{\langle D_2(00)\rangle + \langle D_2(00)\rangle\cdot q^2 + \langle D_2(00)\rangle(q+q^{-1})\cdot(-q)} +$$

$$+ \langle D_1\rangle\cdot(-q) \Big)\, r(D_1)\, \frac{r(D_2)}{r(D_1)} =$$

$$= \langle D_1 \rangle \cdot r(\mathcal{D}_1) \cdot (-q) \cdot (-1)^1 q^{1-2} = \langle \mathcal{D}_1 \rangle \cdot r(\mathcal{D}_1) =$$

$$= J_{\mathcal{D}_1}, \quad \text{q.e.d.}$$

P6 prove invariance w.r.t. R3 move

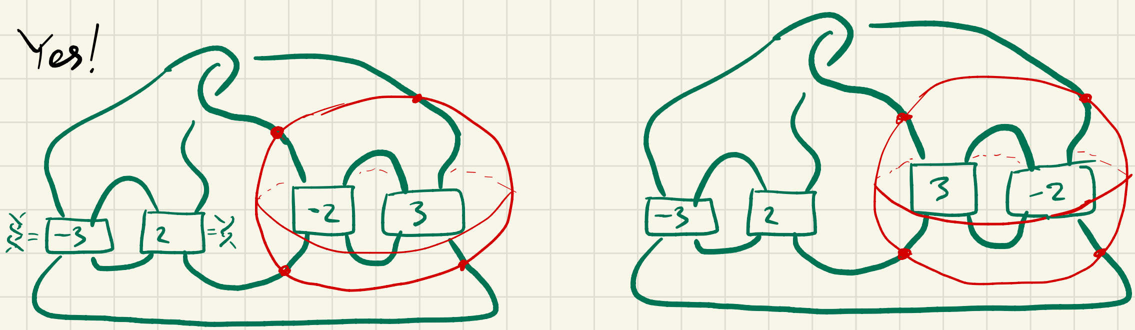
(hint:
$$\begin{tikzcd} \mathcal{D}_1 \arrow[r] \arrow[d, phantom] & 8 \text{ terms} \arrow[d, "\text{cancel}"'] & 8 \text{ terms} \arrow[d, "\text{cancel}"] & \mathcal{D}_2 \arrow[l] \\ & 5 \text{ terms} \arrow[r, phantom, "="] & 5 \text{ terms} & \end{tikzcd}$$
)

∎

Conjecture Jones poly detects the unknot,
i.e. only $\bigcirc$ has Jones poly $= 1$

Q. Are there knots $K_1 \neq K_2$ with same J?

A. Yes!



Kinoshita-Terasaka knot

Conway mutation

Conway knot

180°

<u>Fact</u> the knots are different, since $g_3(KT)=2$ & $g_3(\text{Conway})=3$ (e.g. use $\widehat{HFK}$ to see this)

<u>P7</u> Jones poly is unchanged by mutation.

(and so $J_{KT} = J_{\text{Conway}}$)

Khovanov homology

Is a link invariant in the form of a $\boxed{\text{bigraded abelian group}}$

<u>unreduced version</u>: $Kh^{h,q}(\text{Link}) \overset{\chi_h}{\rightsquigarrow} \sum_{\substack{h\in\mathbb{Z}\\ j\in\mathbb{Z}}} \mathrm{rk}\, Kh^{h,j}(L)\cdot(-1)^h\cdot q^j = J(L)(q+q^{-1})$

<u>reduced version</u>: $\widetilde{Kh}^{q,h}(\text{Knot}) \overset{\chi_h}{\rightsquigarrow} J(K)$

We start with the unreduced:

$\mathcal{D}_L \rightsquigarrow$ cube of resolutions $\rightsquigarrow$ chain complex $CKh(\mathcal{D}_L)$

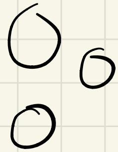
diagram

n crossings

<u>vertices</u> $\bigsqcup_{v\in\{0,1\}^n} \mathcal{D}_L(v) \xmapsto{1}$ free ab. group

<u>edges</u> $\xmapsto{2}$ differential

1 Generators of $CKh(L)$

$\mathcal{D}(v)$  $\longmapsto V^{\otimes|\mathcal{D}(v)|}$ $\left(= V^{2} \otimes V^{2} \otimes V^{2}\right)$ (if 3 circles)

8 dimensional

- $V = \langle x_+, x_- \rangle_{\mathbb{Z}}$ free ab. group

- q-grading: $|v| + q(\underline{x} = x_+ \otimes x_- \otimes x_- \otimes \ldots x_+)$

quantum

\# of 1's in the smoothing

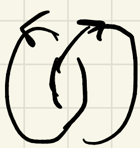
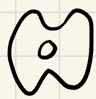
q-local

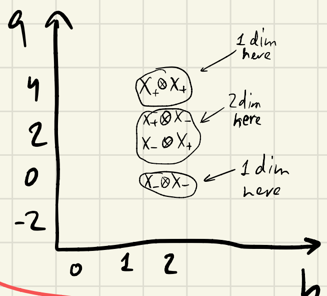
determined by $q(x_+) = 1$, $q(x_-) = -1$

eventually will be shifted ("renormalization")

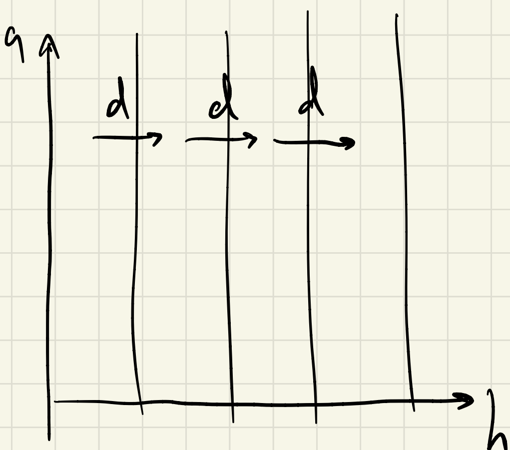
- h-grading: $|v|$ $\left(\text{e.g. } q(x_+ \otimes x_- \otimes x_+) = = q(x_+) + q(x_-) + q(x_+) = +1\right)$

homological

Example:  $\mathcal{D}$, $\mathcal{D}(11) =$  $\mapsto V^{2} \otimes V^{2}$



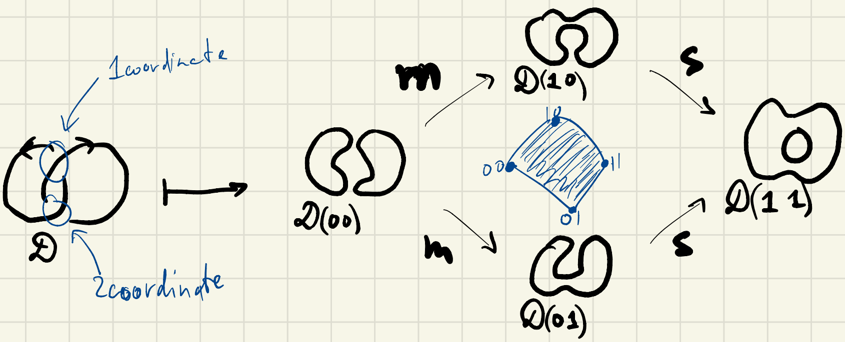
② Differential $CKh^{q,h}(L) \xrightarrow{d} CKh^{q,h+1}(L)$



$$\boxed{CKh(\mathcal{D}) = \bigoplus_{v \in \{0,1\}^n} V^{\otimes|\mathcal{D}(v)|}}$$

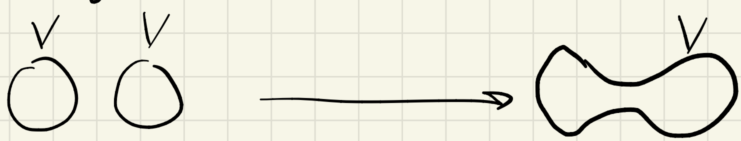
def. of Khovanov complex

Each edge in the cube corresponds to either **merge** or **split** of circles:



So, it is enough to define maps locally:

• m :

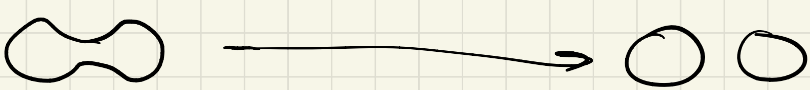


$$V\otimes V \quad \begin{cases} X_+\otimes X_+ \longmapsto X_+ \\ X_+\otimes X_- \longmapsto X_- \\ X_-\otimes X_+ \longmapsto X_- \\ X_-\otimes X_- \longmapsto 0 \end{cases}$$

$1\otimes 1 \mapsto 1$
$1\otimes x \mapsto x$
$x\otimes 1 \mapsto x$
$x\otimes x \mapsto 0$

(if we denote $X_- = x$, $X_+ = 1$, then m is equal to multiplication in $\mathbb{Z}[x]/(x^2)$)

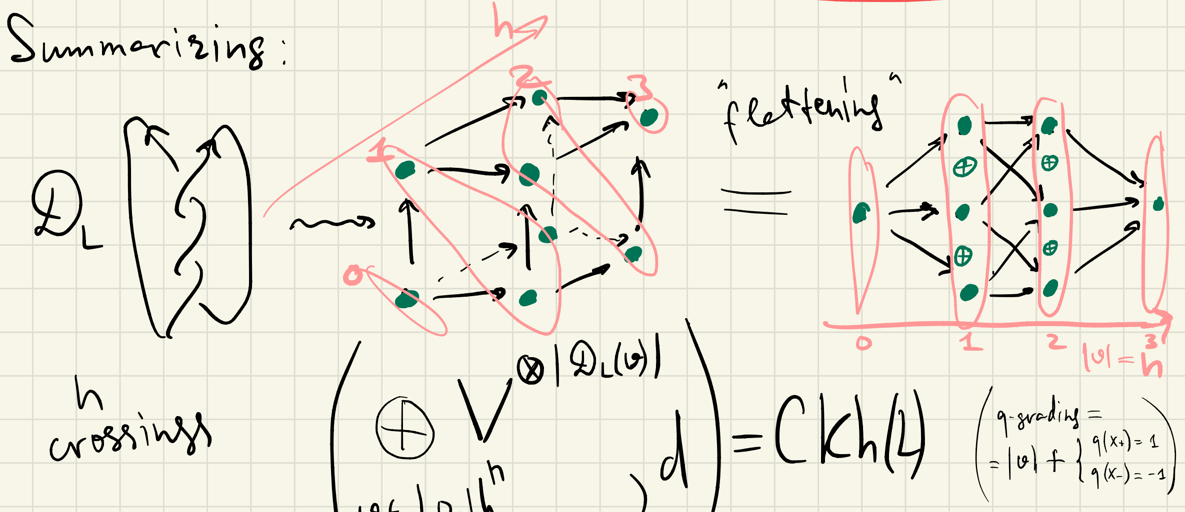
• Δ :



$$X_+ \longmapsto X_+\otimes X_- + X_-\otimes X_+$$
$$X_- \longmapsto X_-\otimes X_-$$

Explanation of m & Δ the only thing which preserves quantum grading & symmetric w.r.t. circles

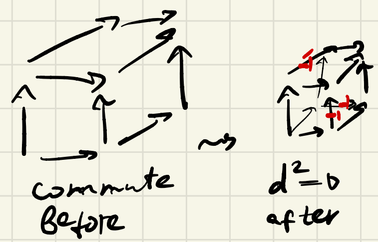
Summarizing:



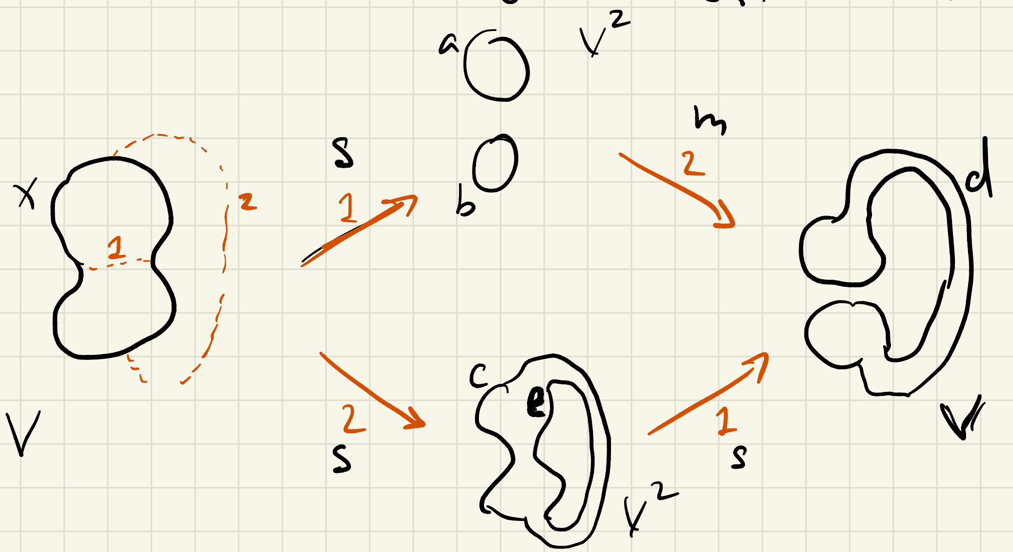
$$\left(\bigoplus_{v \in \{0,1\}^n} V^{\otimes |\mathcal{D}_L(v)|}, d \right) = Ckh(L)$$

$\left(\begin{array}{l} q\text{-grading} = \\ = |v| + \begin{cases} q(x_+) = 1 \\ q(x_-) = -1 \end{cases} \end{array} \right)$

Lemma $d^2 = 0$



☐ check the commutativity:



$$\begin{tikzcd}[column sep=small]
 & & \begin{pmatrix} a_+\otimes b_- \\ + \\ b_+\otimes a_- \end{pmatrix} \arrow[ddrr, mapsto, "f_2"] & & \\
 & & a_-\otimes b_- \arrow[dr, mapsto] & & \\
X_+ \arrow[uurr, mapsto, "f_1"] \arrow[ddrr, mapsto, "f_3"'] & X_- \arrow[ur, mapsto] \arrow[dr, mapsto] & & 0 & 2d_- \\
 & & c_-\otimes e_- \arrow[ur, mapsto] & & \\
 & & \begin{pmatrix} c_+\otimes e_- \\ + \\ e_+\otimes c_- \end{pmatrix} \arrow[uurr, mapsto, "f_4"'] & &
\end{tikzcd}$$

Problem: $d^2 = (f_2 \circ f_1 + f_4 \circ f_3)(x_+) = 4d_-$!

instead we want this sign $\quad f_2 \circ f_1 - f_4 \circ f_3 = 0$

So we introduce extra signs into differential:

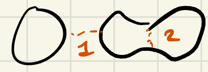
before: $$V^{\otimes \mathcal{D}(v)} \xrightarrow[F_{0110\ldots * \ldots 011}]{} V^{\otimes \mathcal{D}(v')}$$

where v differs from v'

$* = 0 \qquad * = 1$

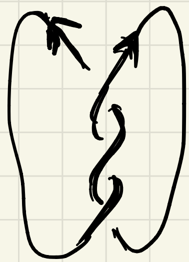
after: $$V^{\otimes \mathcal{D}(v)} \xrightarrow[(-1)^{\varepsilon}\cdot F_{0110\ldots \boxed{*} \ldots 011}]{} V^{\otimes \mathcal{D}(v')}$$

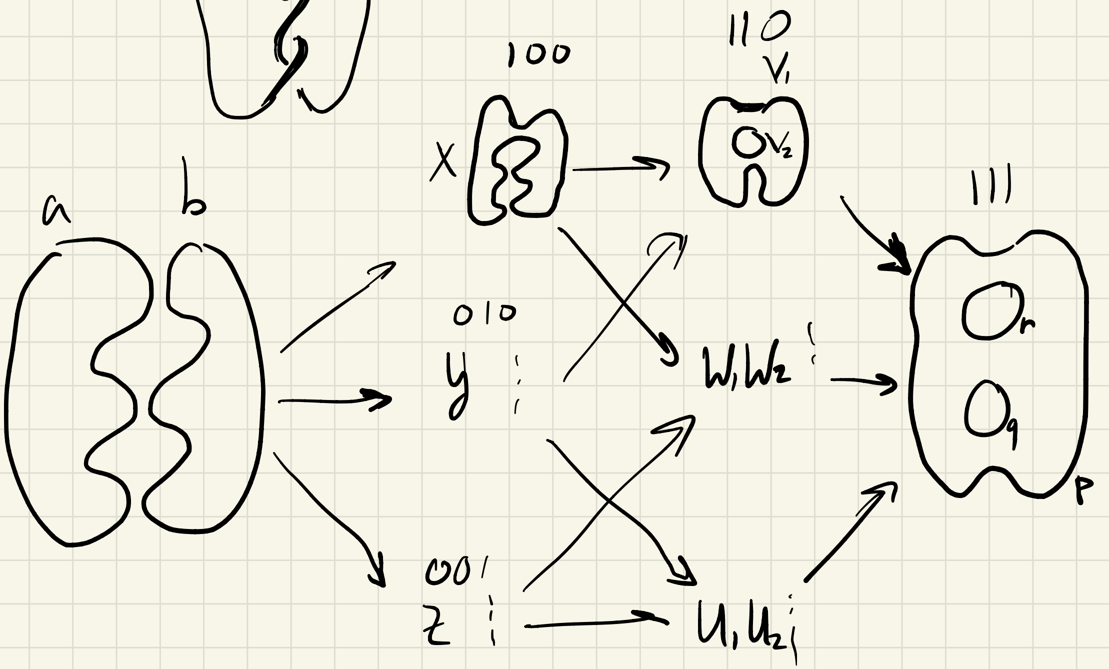
$\varepsilon = \#$ of 1's before $\boxed{*}$

The other cases are similar (like , etc)

∎

Computation for trefoil

$D =$  RHT



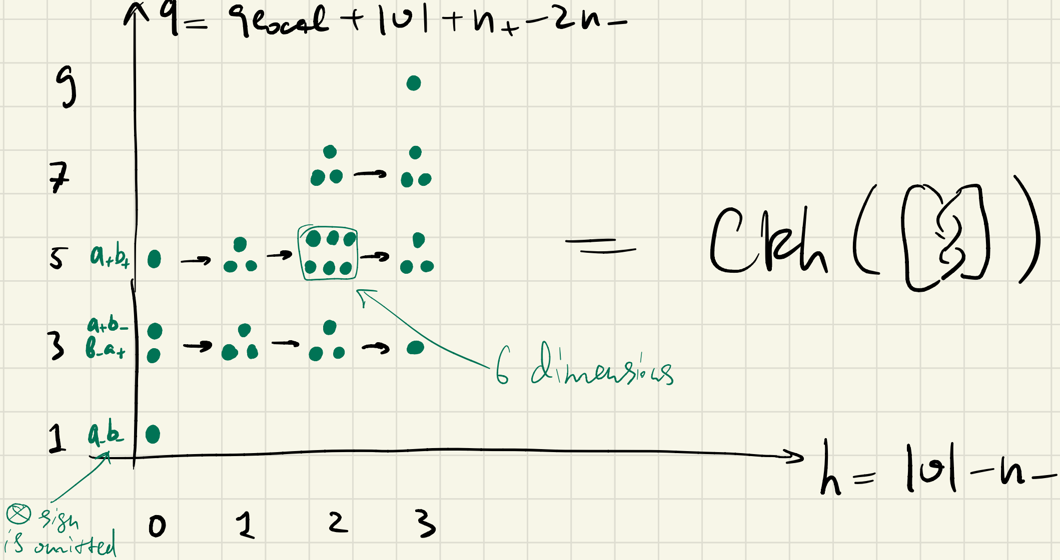
CKh splits according to global q-grading.

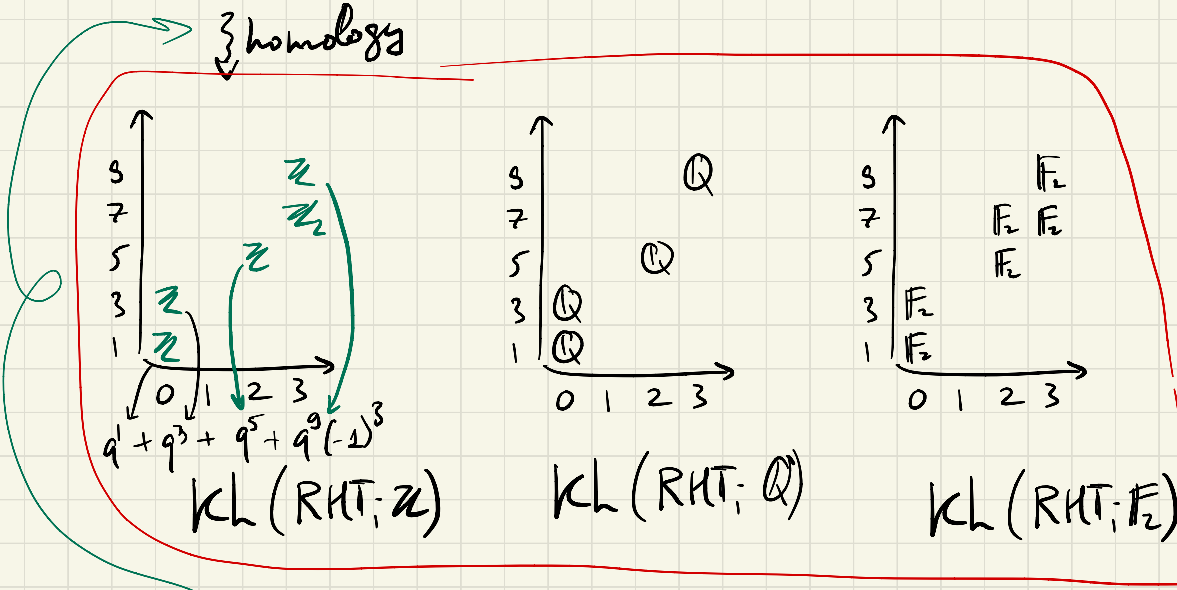
Rmk shifting bigrading:

$$h = |v| \longmapsto |v| - n_-$$

$$q = q_{local} + |v| \longmapsto q_{local} + |v| + n_+ - 2n_-$$

$q(x_+) = 1$, $q(x_-) = -1$





P8 Do this computation in detail.

- $\chi_q\big(CKh(RHT)\big) = q^1 + q^3 + q^5 - q^9 = J(RHT)\cdot(q+q^{-1})$

$$J(RHT) = q^2 + q^4 - q^8$$