M722: Modern techniques in knot theory

- in-depth questions
  clarifications ⟶ after classes
  discussion (instead of office hours)

- questions/comments during the class are highly encouraged, this course is for you

- grades: A ⟶ 60% of problems

  B ⟶ 40% of problems

  C ⟶ 20% of problems

- problems will be given throughout the course
  (algebra heavy)

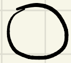
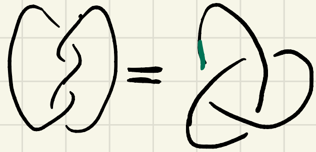
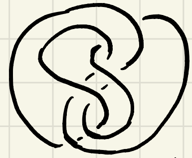
- grading problems: all at the end

- will start with slides, always let me know when I am going too fast

- video-lectures & notes will be posted.

Knot theory

a knot is a smooth (or PL) embedding $S^1 \hookrightarrow S^3$, considered up to isotopy (homotopy through smooth embeddings)

Examples:  unknot  right-handed trefoil  Stevedore knot ...

× left-handed trefoil → 

Goals of knot theory:

(1) Distinguish different knots (knot invariants)

Recognize the same knots (Reidemeister moves)

(holy grail: classify knots, i.e. find a complete computable & comparable invariant)

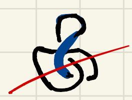
(2) Study all knots as a family (knot concordance group)

(3) Use knots to study 3 mnfds (double-branched covers, surgery)

and 4 mnfds (handle attachments, Kirby calculus)

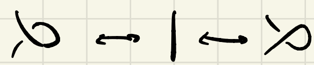
⋮

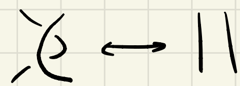
We focus on knot invariants, but before that:

Diagram of a knot is projection of a knot onto a plane with no triple points.  ✓ 

To recognize the same knots we mainly use:

Theorem (Reidemeister) Two diagrams D_1, D_2 of the same knot K are related by a sequence of moves $D_1 \longrightarrow \longrightarrow \longrightarrow \longrightarrow \longrightarrow D_2$ where each move is one of three:

RI 

RII 

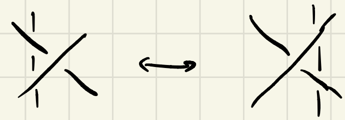
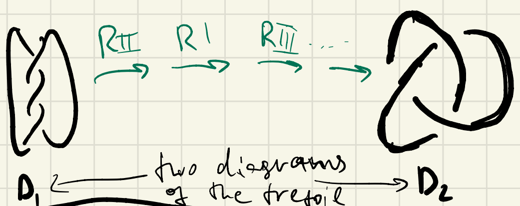
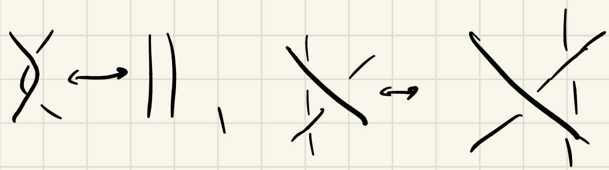
RIII 

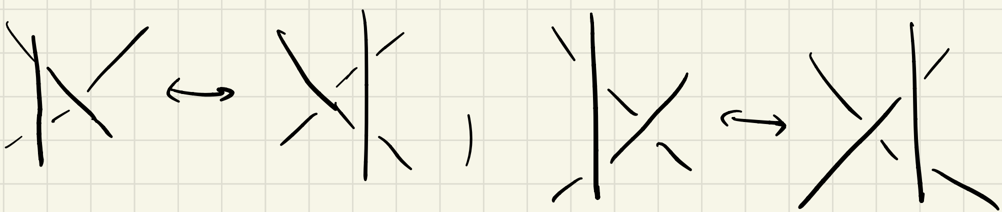
illustration:



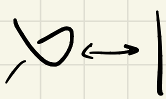
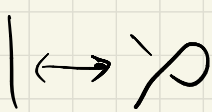
See "Lickorish, knot theory" for the proof if interested

Problem 1

moves 



follow from the RI, RII, RIII

Q. excluding  and leaving only  in RI will or not

break the statement of the theorem?
(probably not)

Knot invariants

Suggestive/easy to define, hard to compute:

$\boxed{A}$

- minimal crossing number
- unknotting number
- Seifert genus
- $\pi_1(S^3 \setminus K)$
- representations of $\pi_1(S^3 \setminus K)$

Hard to discover
easier, to compute:
strong

$\boxed{B}$

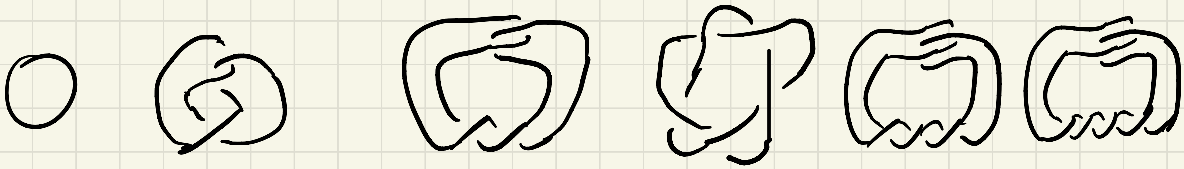
- Alexander poly
- Jones poly
- HOMFLY-pt poly
- knot Floer homology
- Khovanov homology

Big part of knot theory is discovering B, relating them to A, and then computing A

We will focus on Jones poly and Khovanov homology, but first let me cover the A type invariants

Minimal crossing number $c(K)$ (over all possible diagrams)

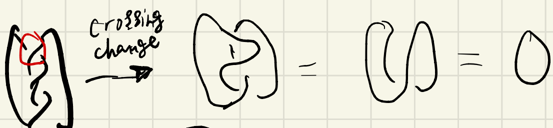
This invariant is traditionally used to tabulate knots, see knot_atlas & knot_info databases in the web.



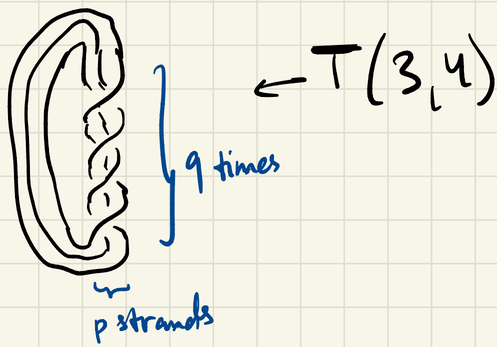
0_1 | 3_1 (trefoil) | 4_1 (figure eight knot) | 5_1 ((2,5)-torus knot) | 5_2 | 6_1 (Stevedore knot)

Unknotting number $u(K)$ number of strands collisions $\times \leftrightarrow \times$ (in 3D) needed to untie the knot

$u($  $) = 1$, since



(p,q)-torus knot is



(p&q are co-prime)

P2 $T(p,q) = T(q,p)$

P3 $u(T(p,q)) \leq \frac{(p-1)(q-1)}{2}$

Theorem (Kronheimer-Mrowka)

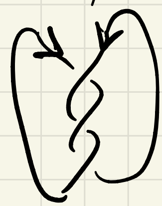
$$u(T(p,q)) = \frac{(p-1)(q-1)}{2}$$ (aka Milnor conjecture)

(One of our goals is to prove this without using gauge theory)

Orientated knot

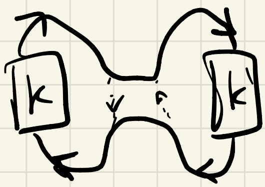
it so happens!

$=$



Rmk sometimes reversing orientation changes K!

Connected sum of oriented knots

$K \# K' =$ 

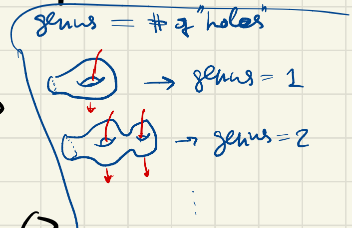
Pf The connected sum operation is well-defined.

Conjectures

1) $u(K \# K') = u(K) + u(K')$

2) $c(K \# K') = c(K) + c(K')$

Knot that is <u>not</u> a connected sum of two non-triv knots is called <u>prime</u>. Non-prime knots are called <u>composite</u>.



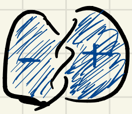
<u>Prop 1</u> $K \# K' = O \Rightarrow$

$\Rightarrow$ Both $K = O$ and $K' = O$

The proof is based on <u>Seifert genus</u>:

$$g_3(K) \overset{\text{def.}}{=} \left\{ \min g(\Sigma) \;\middle|\; \begin{array}{l} \text{oriented } \Sigma \hookrightarrow S^3 \setminus K, \\ \partial \Sigma = K \end{array} \right\}$$

(Seifert surface)

-  $\longrightarrow g_3(RHT) = 1$. Existence is not hard
- $g_3(K) = 0 \iff K = U$ (easy) 
- $g_3(K \# K') = g_3(K) + g_3(K')$ (harder, see Lickorish)

<u>Prop 2</u> There is a <u>unique</u> decomposition of a knot into connected sum of prime knots $\left(K = \underbrace{P_1 — P_2 — P_3 — P_4}_{P_1 \# P_2 \# P_3 \# P_4} \right)$

Thus in the Rolfsen Table of knots only prime knots are listed, and

also up to mirroring (changing all crossings $\times \leftrightarrow \times$ in the diagram)
and up to orientation reversing

$\pi_1(K)$ is a good invariant, But
very hard to compare $\pi_1(K)$ to $\pi_1(K')$

(There is a sense in which it is a complete invariant, see "peripheral subgroup" page on wiki.)

Much Better invariant is representations:

$$\boxed{\hom(\pi_1(K), G)/\text{conjugation}}$$

Before Alexander & Jones poly came, taking G to be some finite group produced the most computable/strong knot invariants