- Office hours: please register (see announcement OR syllabus)
- S-grade: policy to be determined
- 5 April: last day to withdraw
- policy on final exam: TBD
- homework → on canvas
- online help sessions: see announcements/syllabus (if you cannot make it to my office hours)

Determinant

Recall:
- exists only for square matrices
- $\det([8]) = 8$ (1×1 matrix)
- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$
- larger-size matrices: cofactor expansion along row/column
- $\det\left(\underbrace{\begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}}_{\text{upper-triangular matrix}}\right) =$ product of diagonal elements $= \det\left(\underbrace{\begin{bmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{bmatrix}}_{\text{lower-triang.}}\right)$

- effect on $\det(\cdot)$ by row operations (in 3.2) $\rightsquigarrow$ computing $\det(\cdot)$ by row-reducing to REF

Properties

- $\det(A) \neq 0 \iff A$ invertible
- $\det(A \cdot B) = \det(A) \cdot \det(B)$

$\boxed{\text{Warning:}}$: $\det(A+B) \neq \det(A) + \det(B)$

$\det(k \cdot A) \neq k \cdot \det(A)$

instead: $\det(k \cdot A) = k^n \cdot \det(A)$

where n is the size of A

Problem from last HW

Prove that $\det(PAP^{-1}) = \det(A)$.

wrong solution:

$\det(PAP^{-1})$

$\|$

$\det(\cancel{PP^{-1}}A)$

$\|$

$\det(A)$

wrong because

$A \cdot B \neq B \cdot A$

for matrices. It was used here

correct solution

$\det(PAP^{-1})$

$\|$ using

$\det(P) \cdot \det(A) \cdot \det(P^{-1})$

$\|$ $ab = ba$ for numbers

$\det(P) \cdot \det(P^{-1}) \cdot \det(A)$

$\|$ see lecture notes

$\cancel{\det(P)} \cdot \dfrac{1}{\cancel{\det(P)}} \cdot \det(A)$

$\|$

$\det(A)$

See 3.2 # 33, 34, 35, 36
and also $\det(P^{-1}) = \dfrac{1}{\det(P)}$

You want to understand well these problems.

Start 3.3 $\{$ Crammer's rule

A method to solve lin. systems using determinants.

Example 1 (see the book)

$$\begin{cases} 3x_1 - 2x_2 = 6 \\ -5x_1 + 4x_2 = 8 \end{cases}$$

sub first col. by $\bar{b}$

$$A_1(b) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}$$

sub 2nd col. by $\bar{b}$

$$A_2(b) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}$$

$A_1(\bar{b})$ sub

$$\underbrace{\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}}_{= A} \cdot \bar{x} = \underbrace{\begin{bmatrix} 6 \\ 8 \end{bmatrix}}_{= \bar{b}}$$

sub $A_2(\bar{b})$

$\det(A) = 3\cdot 4 - (-5\cdot 2) = 12 - 10 = 2$

$\det(A_1(b)) = \det\begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix} = 24 - (-2\cdot 8) = 24 + 16 = \underline{40}$

$\det(A_2(b)) = \det\begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix} = 24 - (-5\cdot 6) = 24 + 30 = 59$

$x_1 = \dfrac{\det(A_1(b))}{\det(A)} = \dfrac{40}{2} = 20$ $\qquad x_2 = \dfrac{\det(A_2(b))}{\det(A)} = \dfrac{54}{2} = 27$

General case of $n \times n$ matrix

$$A = \begin{bmatrix} | & | & & | \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \\ | & | & & | \end{bmatrix}$$

columns

- Crammer's rule only works for invertible matrices!

Want: solve $A \cdot \bar{x} = \bar{b}$ for A invertible

To find x_i, we do:

① Replace the i-th column by $\bar{b}$, namely take

$$A_i(\bar{b}) = \begin{bmatrix} | & | & & | & & | \\ \bar{a}_1 & \bar{a}_2 & \cdots & \bar{b} & \cdots & \bar{a}_n \\ | & | & & | & & | \end{bmatrix}$$

instead of $\bar{a}_i$

② find $\det(A_i(\bar{b}))$
find $\det(A)$

note: this is not 0 since A is invertible

③ $\boxed{x_i = \dfrac{\det(A_i(\bar{b}))}{\det(A)}}$ Crammer's rule

Example 2 from the Book

$$\begin{cases} 3s\,x_1 - 2x_2 = 4 \\ -6x_1 + s\,x_2 = 1 \end{cases}$$

(s is a parameter) $\vec{b}$

Q. When does this system have a unique solution? Describe it using Crammer's rule

step 1

- solution unique

$\Updownarrow$

- no free variables + consistency

$\Updownarrow$

- $\det(A) \neq 0$

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}$$

step 2

$$\det(A) = 3s^2 - (-6)\cdot(-2) =$$

$$= 3s^2 - 12 =$$

$$= 3(s^2-4) = 3(s-2)(s+2)$$

$\det(A) \neq 0$ if and only if $s \neq 2$, $s \neq -2$

Thus solution is unique when $s \neq 2$ and $s \neq -2$.

solution to the system

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$A_1(b) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix} \qquad A_2(b) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1(b))}{\det(A)} = \frac{4s+2}{3s^2-12}$$

$$x_2 = \frac{\det(A_2(b))}{\det(A)} = \frac{3s+24}{3s^2-12}$$

Adjugate matrix

square matrix A is given

Recall that the $\underline{(i,j)\text{-cofactor}}$ is

$$\boxed{C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})}$$

the $(n-1)\times(n-1)$ matrix obtained by deleting the cross at (i,j)

Example.

(2,3) entry

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then } C_{23} = (-1)^5 \cdot \det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} = -(8-14) = 6$$

Now we record all these $C_{i,j}$ into one matrix $\underline{\underline{\text{adj}\,A}}$, adjugate A:

$$\boxed{\text{the } (i,j)\text{-entry of } \text{adj}\,A \text{ is } C_{ji}}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

number $= (-1)^{(i+j)} \det(A_{ij})$

$$\text{adj}\, A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \cdots$$

(note that it is C_{12}, not C_{21})

Formula for the inverse of a matrix:

$$\boxed{A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}\, A}$$

in more detail:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

every C_{ji} here is a det of $(n-1)\times(n-1)$ matrix

Interesting consequence (see HW)

If A is a square matrix with **integer** coefficients, and $\det(A) = 1$, then A^{-1} also has integer coefficients.

Example

(2,1) entry

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}.$$

Find A^{-1} using the adjugate formula.

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} C_{11} & C_{21} & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

$C_{11} = (-1)^{1+1} \cdot \det\begin{pmatrix} -1 & 1 \\ 4 & -2 \end{pmatrix} = -2$ $\qquad C_{12} = -\det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} = 3$

$C_{21} = (-1)^{2+1} \cdot \det\begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix} = 14$ $\qquad C_{22} = \det\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} = -7$

$C_{13} = \det\begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} = 5$ $\qquad C_{23} = -\det(\ldots) = -7$

$C_{31} = 4$ $\qquad C_{32} = 1$ $\qquad C_{33} = -3$

$$\det(A) = \det\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix} =$$

$$= 2\times(-2) + 1\times(3) + 3\times(5) =$$

det of 2x2 matrices

$$= -4 + 3 + 15 = 14$$

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} =$$

$$= \frac{1}{14}\cdot\begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix} =$$

$$= \begin{pmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{pmatrix}$$ ← answer

Determinants as scaling factors of areas and volumes

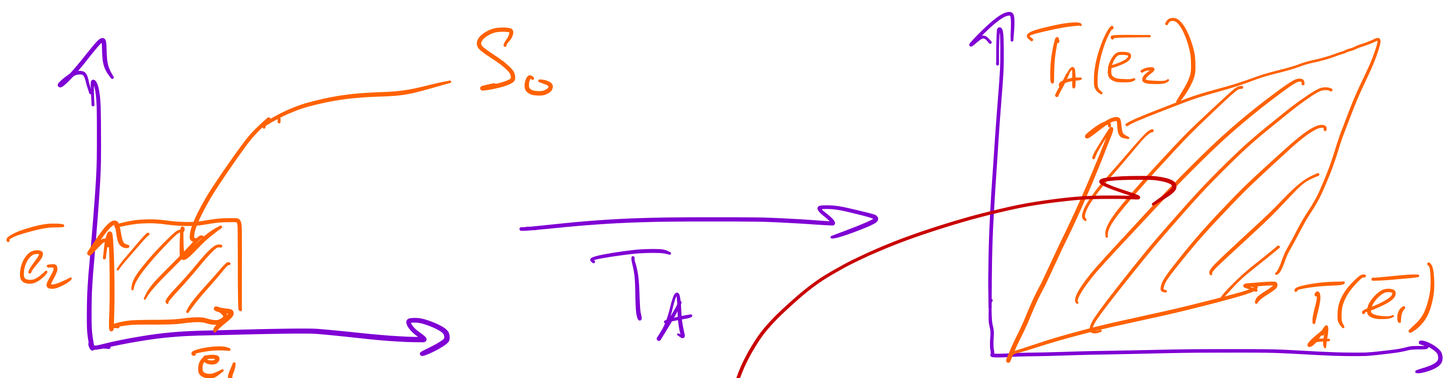
Take A to be 2×2 matrix

Then there is associated linear transformation

$$T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\bar{x} \longmapsto A\cdot\bar{x}$$

1.8 + 1.9

Effect on standard basis



square $S_0 \longmapsto$ parallelogram $T_A(S_0)$

$$\boxed{\text{Area of } T(S_0) = |\det A|}$$

(Theorem 3, 3.3)

example if $\det(A) = 0$ (A non-invertible)

area $T(S_0) = 0$

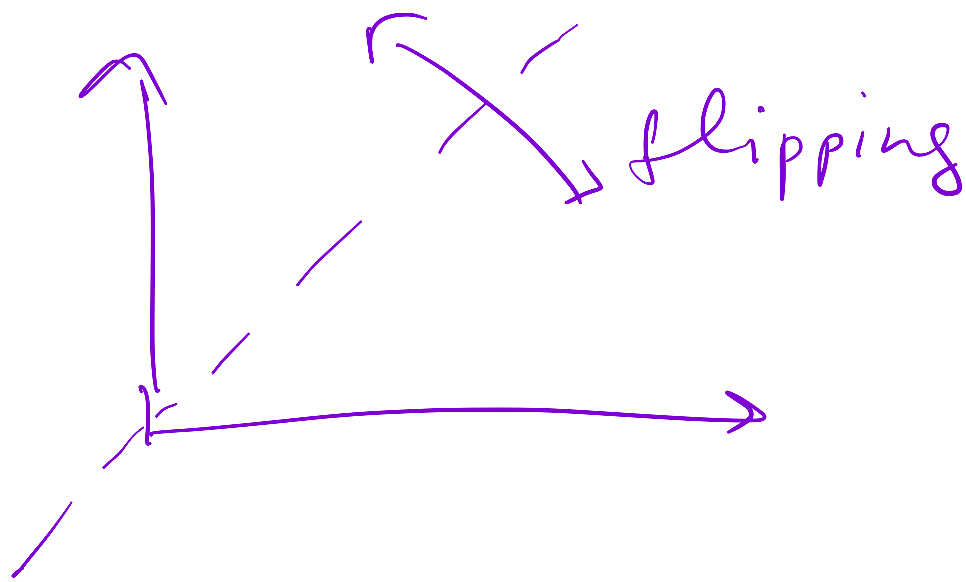
Explanation of matrix $\longleftrightarrow$ lin. transformation

example $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

sections 1.8, 1.8

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$



$\boxed{\det(A) \longleftrightarrow \text{area}}$ (still 3.3)

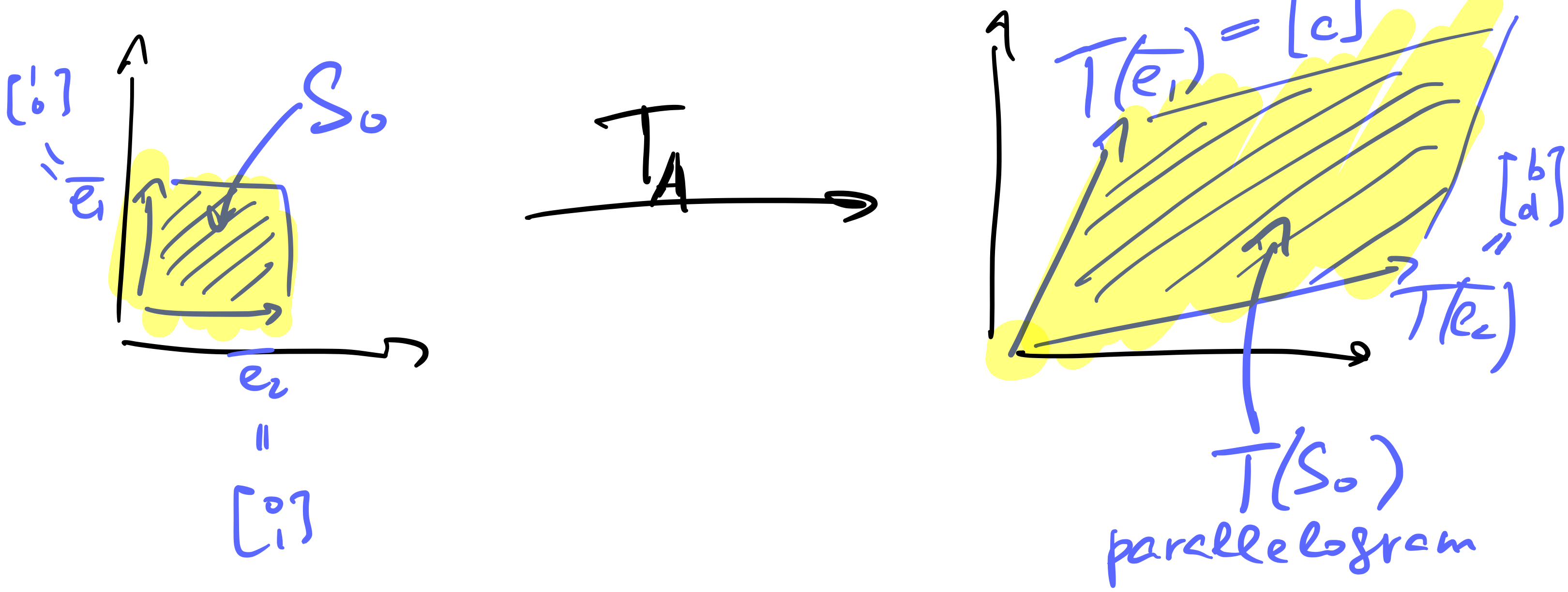
- A is 2×2, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- There is an associated lin. transformation

$$\begin{pmatrix} 1.8 \\ 1.9 \end{pmatrix} \quad T_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \bar{x} \longmapsto A\cdot\bar{x}$$

recall: $(*)$

$$T_A(\bar{e}_1) = \begin{bmatrix} a \\ c \end{bmatrix}$$
$$T_A(\bar{e}_2) = \begin{bmatrix} b \\ d \end{bmatrix}$$

- Effect on standard basis:



$$\boxed{\text{Area of } T_A(S_0) = \underbrace{|\det A|}_{\text{absolute value}}}$$

$(*)$

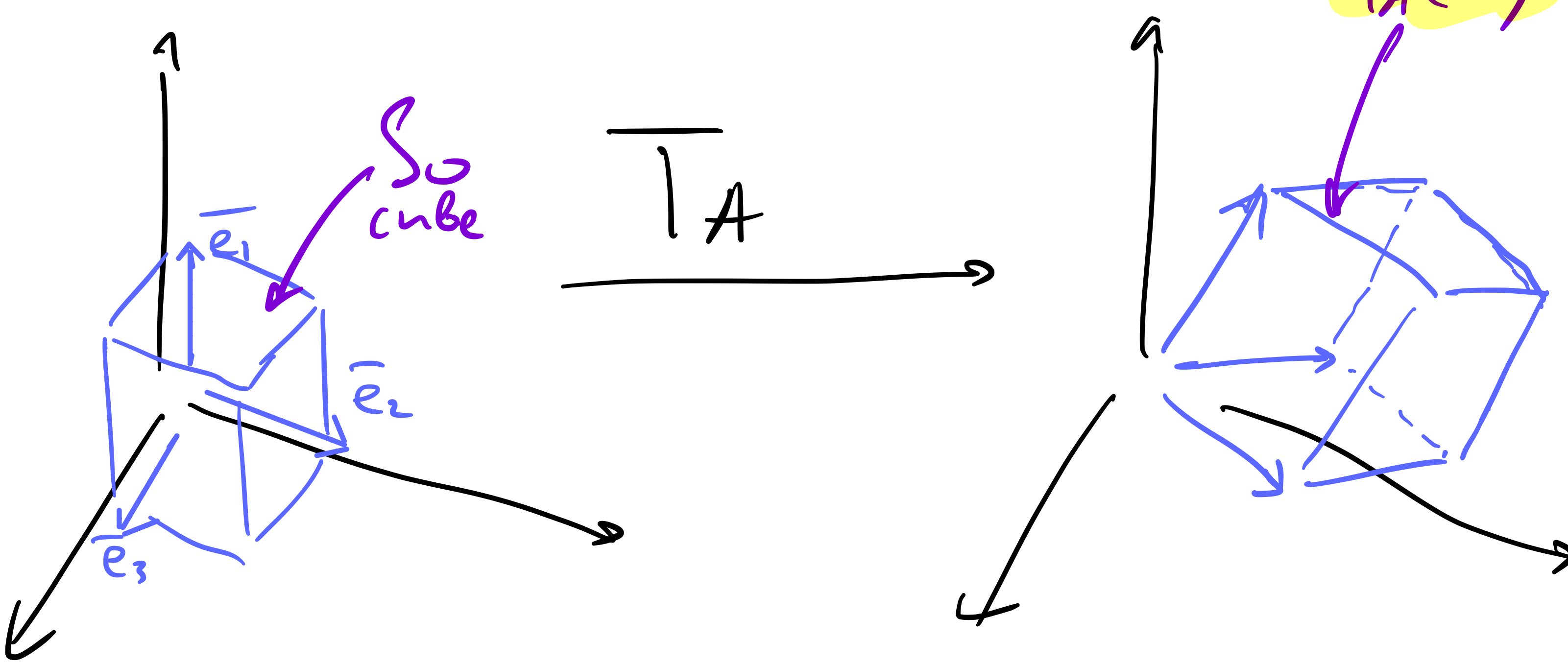
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \overset{\bar{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} a \\ c \end{bmatrix}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underset{\bar{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$\boxed{\det(A) \longleftrightarrow \text{volume}}$

- A is 3×3
- $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

  $x \longmapsto A\bar{x}$

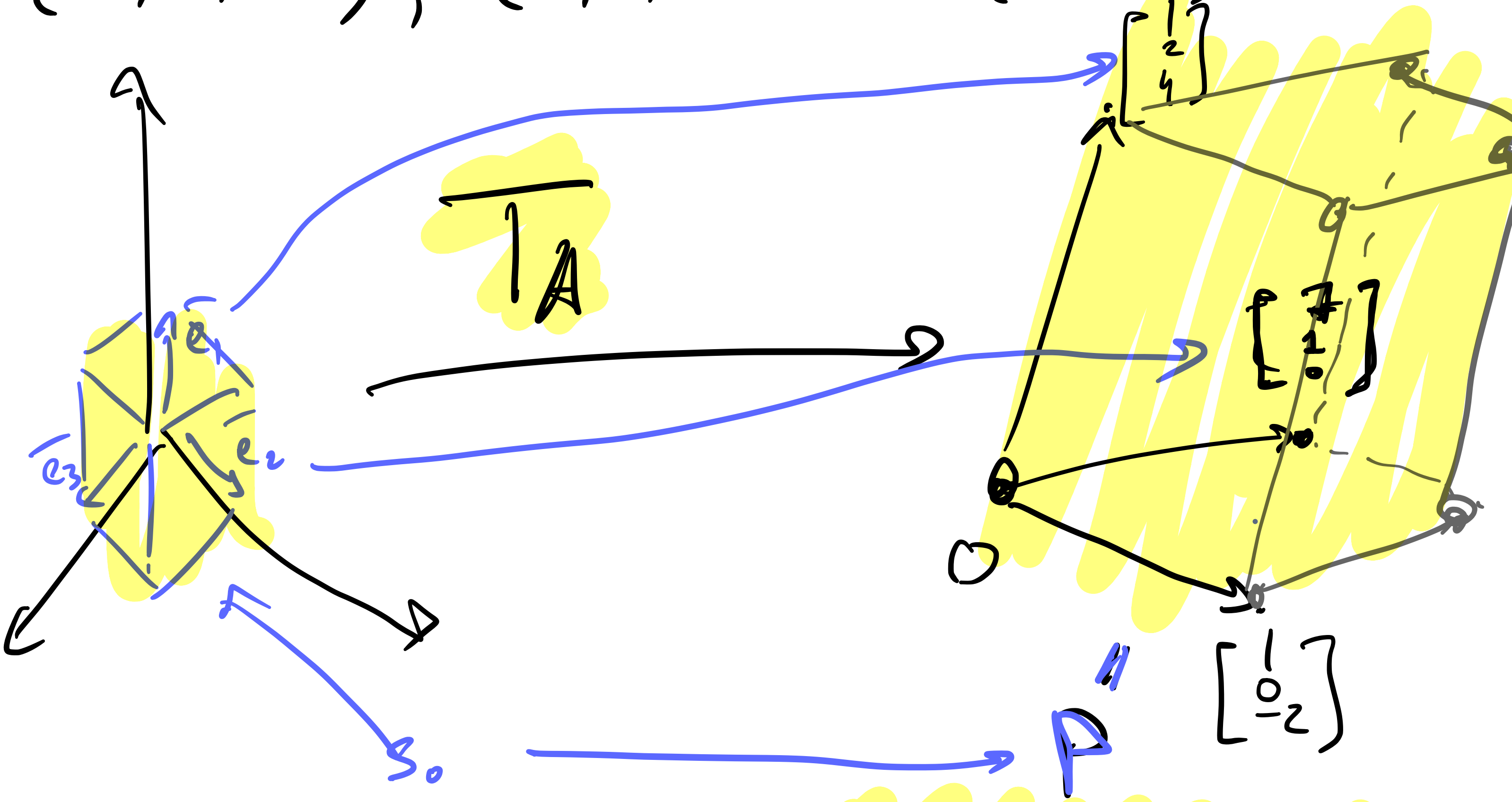
- Effect on standard basis



$$\boxed{\text{Volume of } T(S_0) = |\det A|}$$

(see the textbook for the proofs)

<u>Problem</u> find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1,0,-2)$, $(1,2,4)$ and $(7,1,0)$



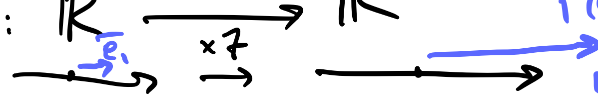
$$\boxed{P = T_A(S_0)} \text{ for } A = \begin{bmatrix} | & | & | \\ T(\bar{e}_1) & T(\bar{e}_2) & T(\bar{e}_3) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 7 & 1 \\ 2 & 1 & 0 \\ 4 & 0 & -2 \end{bmatrix}$$

$\bar{e}_1 \longmapsto \begin{bmatrix}1\\2\\4\end{bmatrix}$ $\quad \bar{e}_3 \longmapsto \begin{bmatrix}1\\0\\-2\end{bmatrix}$

$\bar{e}_2 \longmapsto \begin{bmatrix}7\\1\\0\end{bmatrix}$

<u>1×1 case</u> $\qquad A = [7]$ $\qquad$ (instead of area) length $= 7$

$T: \mathbb{R} \xrightarrow{\times 7} \mathbb{R}$ $\qquad T(\bar{e}_1) = 7$ $\qquad$ $= \det(A)$



Now

$$\text{Volume}(P) = |\det A| =$$

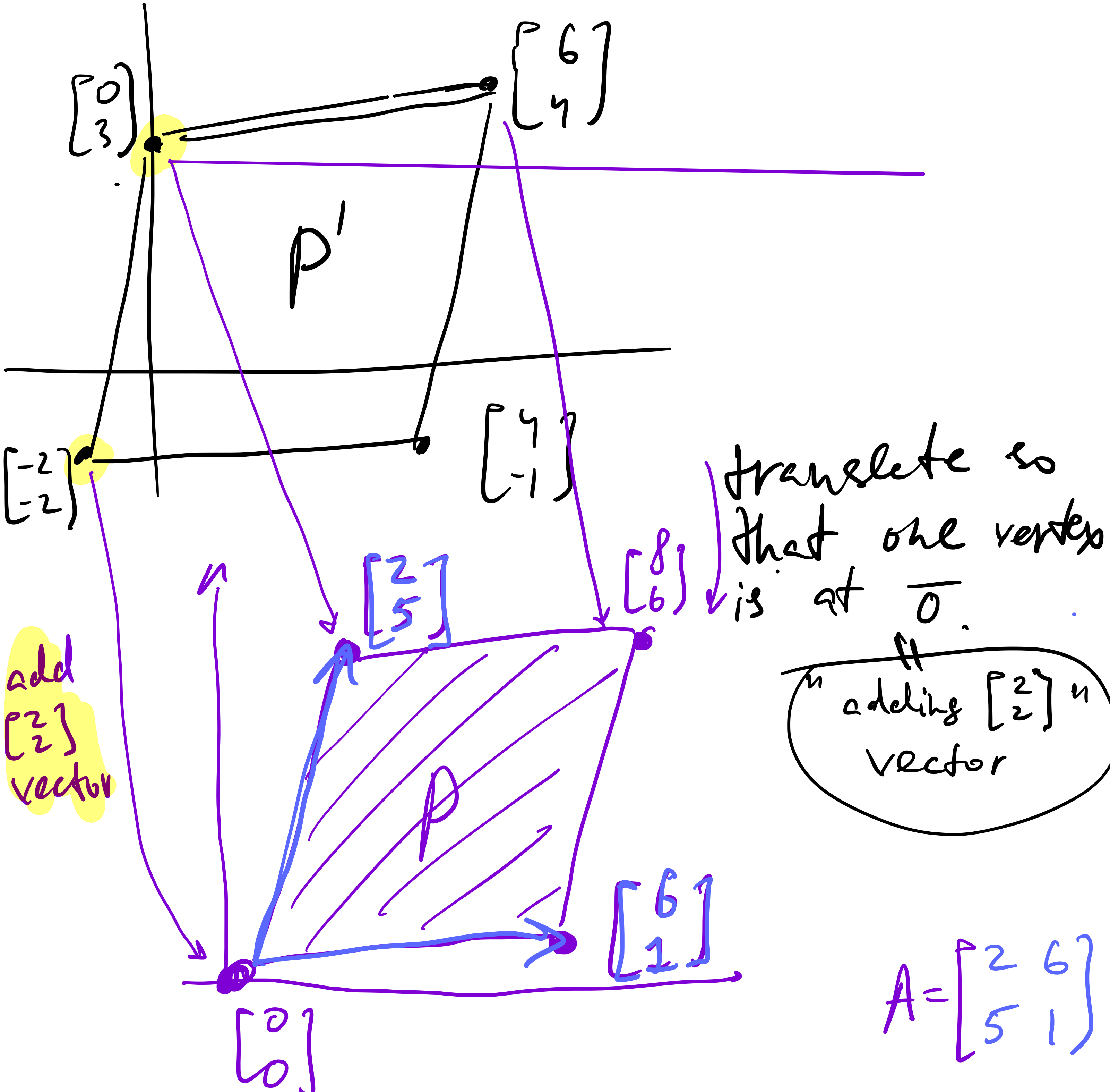
$$= \left|\det \begin{pmatrix} 1 & 7 & 1 \\ 2 & 1 & 0 \\ 4 & 0 & -2 \end{pmatrix}\begin{array}{c} + \\ - \\ + \end{array}\right| =$$

$$= \left| +1\cdot\det\begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix} - \cancel{0\cdot\det(\ldots)} - 2\cdot\det\begin{pmatrix} 1 & 7 \\ 2 & 1 \end{pmatrix}\right| =$$

$$= |-4 - 2\cdot(1-14)| = |-4+26| = 22$$

Example 4 (see the book)

Calculate the area of the parallelogram P' determined by the points $(-2,-2), (0,3), (4,-1), (6,4)$



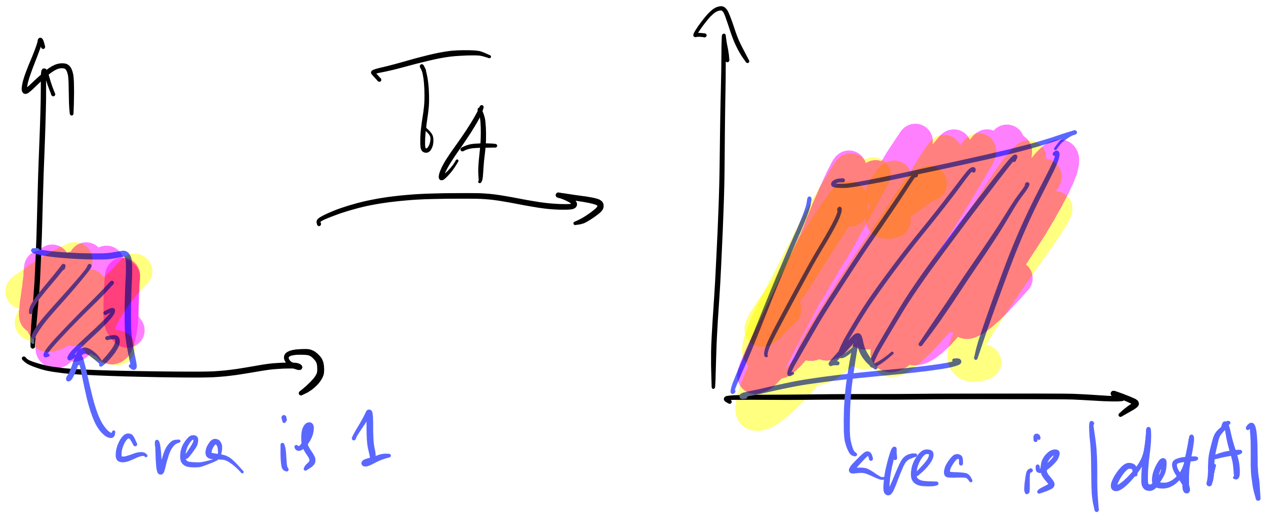
$$A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$$

$$\text{Area}(P') = \text{Area}(P) = |\det A| =$$

$$= \left|\det\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\right| = 28$$

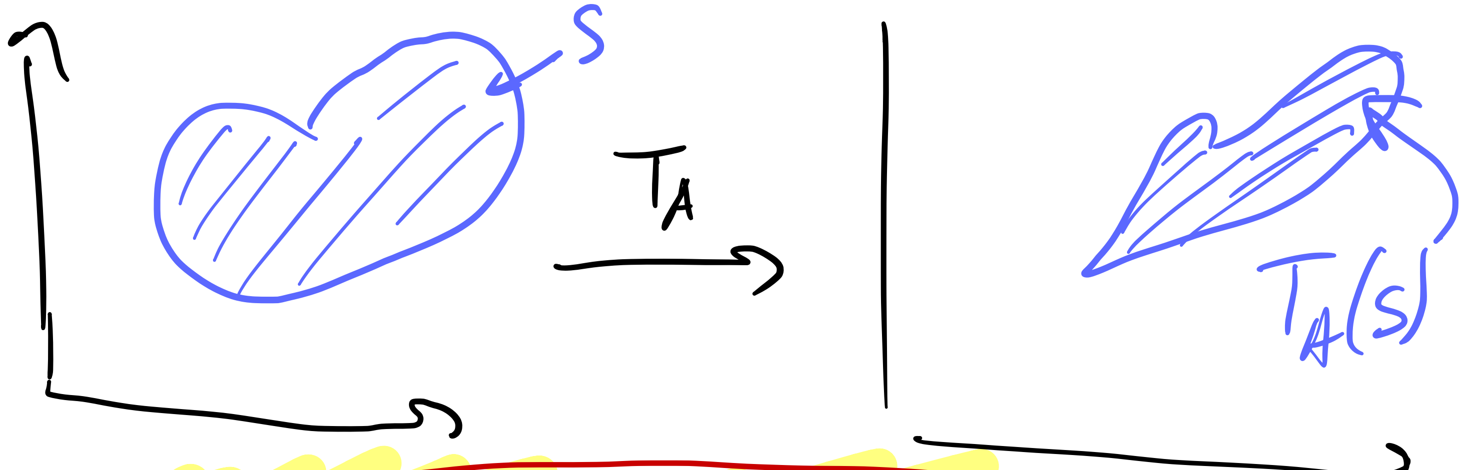
Remark det(A) is not an area/volume, but rather a <u>scaling factor</u>

<u>2x2</u>



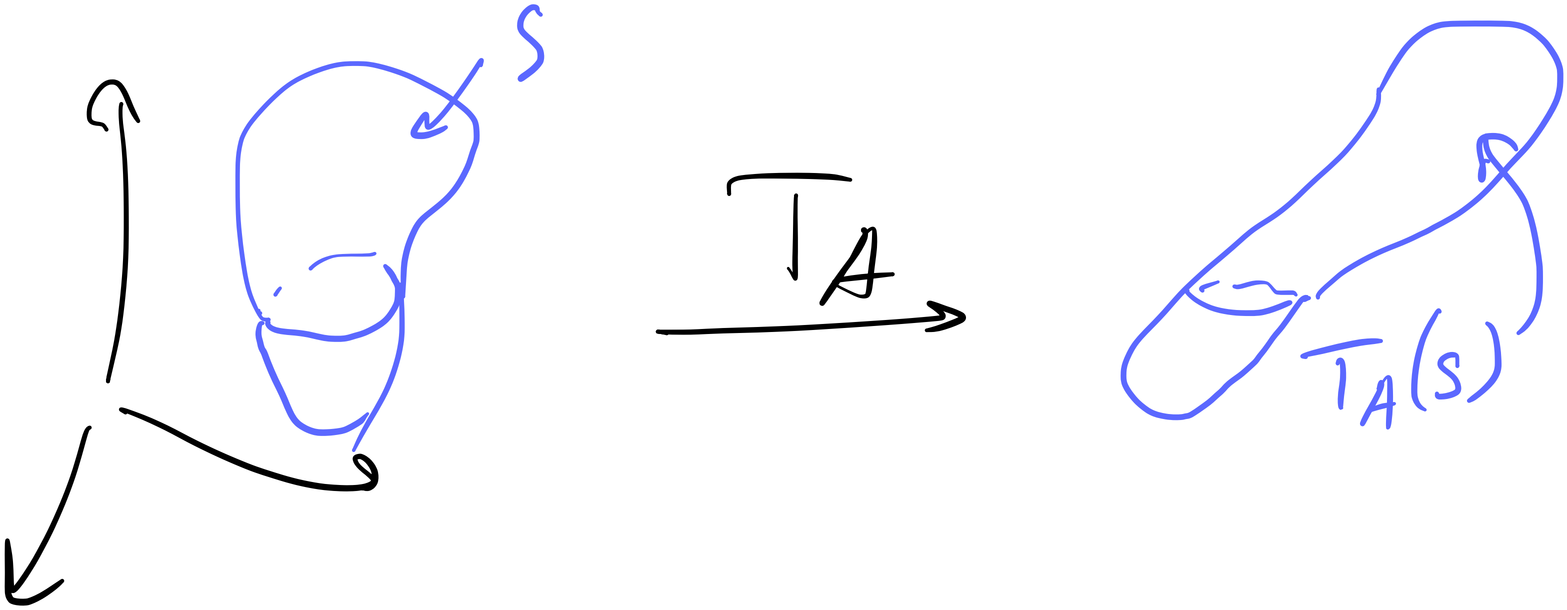
The area here is expanded (scaled) by |det A|.

The same is true for all shapes



$$\boxed{\text{Area}(T_A(S)) = |\det A| \times \text{Area}(S)}$$

The same is true for <u>3x3</u>



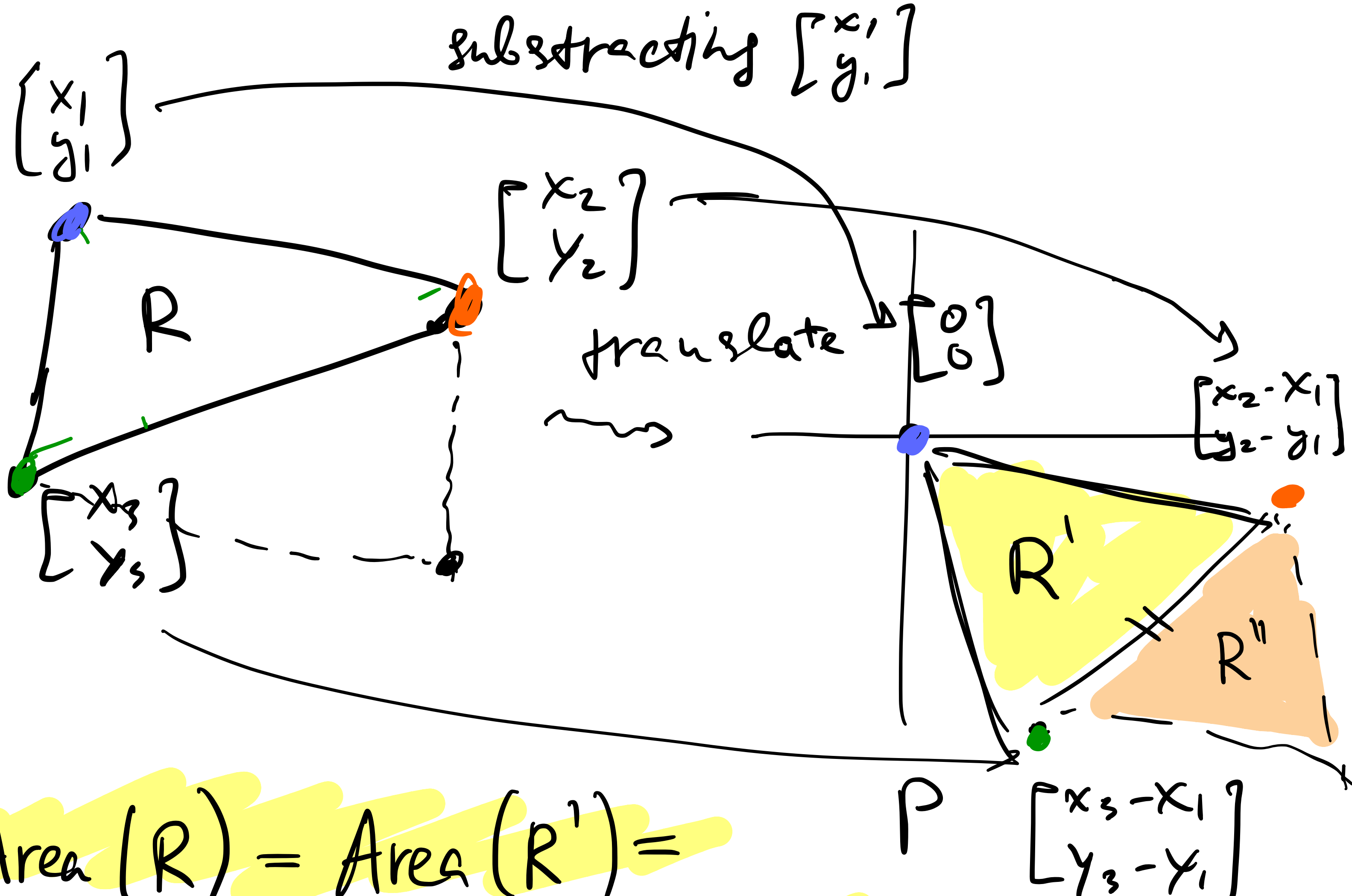
$$\boxed{\mathrm{Vol}(T_A(S)) = |\det(A)| \times \mathrm{Vol}(S)}$$

HW, 3.3 #30

R = triangle with vertices
(x_1, y_1), (x_2, y_2), (x_3, y_3)

Show

$$\boxed{\text{Area}(R) = \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|}$$



$$\text{Area}(R) = \text{Area}(R') =$$

$$= \frac{1}{2}\text{Area}(P) = \frac{1}{2}\det\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

- we set

$\hookrightarrow \text{Area}(R) = \frac{1}{2} \cdot \left| \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \right|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

- we want

$\hookrightarrow \text{Area}(R) = \frac{1}{2} \cdot \left| \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \right|$

- Thus, it remains to show

$$\left| \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \right| = \left| \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \right|$$

finish at home

4.9 Markov chains

Will be introducing them while going through important example.

Page Rank (google's first algorithm)

Internet consists of 4 websites $\rightarrow$

site 1	0.4
site 2	0.1
site 3	0.3
site 4	0.2

$$\begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix}$$

1. We describe the current state of where people are by a **state-vector** $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, example here,

where x_i = proportion of the population currently on website i =
= probability that a random person is on website i

Initial state-vector is the state vector from which the process starts

- **Important**: sum of entries of the state vector should be equal to one

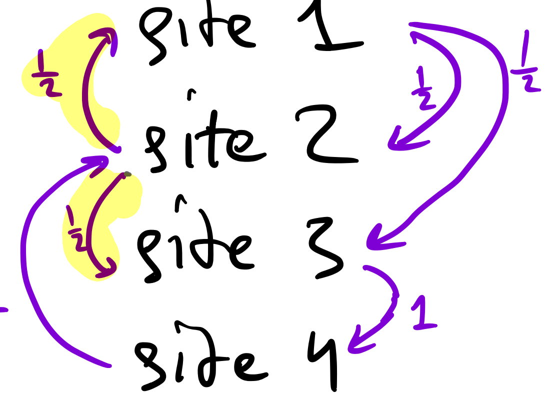
$$x_1 + x_2 + x_3 + x_4 = 1$$

(2) Now, mathematical model of people surfing the web is described by

stochastic matrix P

Its i^{th} column represents probabilities of people on website i clicking the link to other websites

- numbers = probability of clicking
- arrows = links



Ex. consider this model. The matrix is then

$$P = \begin{array}{c|cccc} & \multicolumn{4}{c}{\text{(from)}} \\ & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \frac{1}{2} & 0 & 0 \\ 2\ \text{(to)} & \frac{1}{2} & 0 & 0 & 1 \\ 3 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

remark

sum of numbers in each column is 1

③ Thus we have

- the state, recorded by a **state-vector $\overline{X}$**

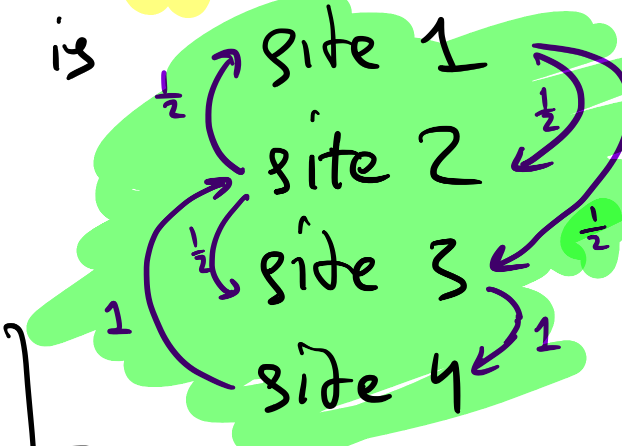
- the change in state is recorded by a **stochastic matrix P**

key relation

$$\text{state } \overline{X}_n \xrightarrow{\text{after a change}} P \cdot \overline{X}_n = \overline{X}_{n+1}$$

Ex. if initial state is $\begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix} = \overline{X}_0$

and website link graph is



then the next state will be

$$\overline{X}_1 = P \cdot \overline{X}_0 = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.05 \\ 0.4 \\ 0.25 \\ 0.3 \end{bmatrix}$$

multiplication of matrix by a vector

④ Repeating this process, we get

$\overline{X_0}$ initial state

$\overline{X_1} = P\,\overline{X_0}$ after one click

$\overline{X_2} = P \cdot \overline{X_1}$ after two clicks

$\vdots$

This sequence is called a **Markov chain** (probabilistic model!)

Having described the mathematical model for web surfing, we now pose the question:

Q. How to assign "popularity" = ranking metric for websites?

The answer is beautiful:

⑤ Critical property of Markov chain is that almost always the states $\overline{X_n}$ converge to a unique **steady-state vector**

$$\overline{X_0} = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.3 \\ 0.2 \end{bmatrix} \quad \overline{X_1} = P \cdot \overline{X_0} = \begin{bmatrix} 0.05 \\ 0.4 \\ 0.25 \\ 0.3 \end{bmatrix} \quad \overline{X_2} = P \cdot \overline{X_1} = \begin{bmatrix} 0.2 \\ 0.325 \\ 0.225 \\ 0.25 \end{bmatrix}$$

Markov chain

$$\overline{X_3} = \dots \quad \overline{X_n} = \dots \xrightarrow[\text{converge}]{} \boxed{\overline{q}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

?

($\overline{X_n}$ is very close to $\overline{q}$ for large n)

Because people constantly click on links, we can assume in our model that n is large.

key idea

Therefore, the steady-state vector $\overline{q}$ is a good approximation for how people are distributed between websites at any given moment

In other words, the if $\bar{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$,

q_i should be the rank of website i

answer to the question!

⑥ How do we find the steady-state vector $\bar{q}$?

It turns out that $\bar{q}$ satisfies

$$\boxed{P.\bar{q} = \bar{q}}$$

(thats why steady-state)

Ex. in our problem: $P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$P\bar{q} = \bar{q}$ lets try to solve for $\bar{q}$

$P\bar{q} - \bar{q} = 0$

$(P - I)\cdot\bar{q} = 0$ the usual linear system of equations!

This is the equation to solve, in order to find $\bar{q}$

$(P-I)\cdot \bar{q} = 0$ $\qquad \left(I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$

$$\left[\begin{array}{cccc|c} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \leftarrow \text{augmented matrix}$$

row-reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \text{ rref}$$

free

solution

$$\begin{cases} x_1 = \frac{2}{3} x_4 \\ x_2 = \frac{4}{3} x_4 \\ x_3 = x_4 \\ x_4 = x_4 \end{cases} = x_4 \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \\ 1 \end{bmatrix}$$

$x_i = q_i$

So, to obtain the steady-state vector, we remember that sum of entries should be 1!

$$x_1 + x_2 + x_3 + x_4 = \frac{2}{3} x_4 + \frac{4}{3} x_4 + x_4 + x_4 =$$

$$= x_4 \left(\frac{2}{3} + \frac{4}{3} + 1 + 1 \right) = x_4 \cdot 4 = 1$$

$\Rightarrow x_4 = \frac{1}{4}$, and steady state vector is

$$\bar{q} = \frac{1}{4} \cdot \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}$$

rankings for our four websites!

Theorem. If P is a stochastic matrix describing a particular Markov chain, then, if P is regular, there exists a unique steady-state vector $\bar{q}$.

• And $\bar{q}$ can be found by solving $(P-I)\bar{q} = 0$ and remembering that sum of entries $= 1$

regular above means that for some $k \geq 1$ the matrix P^k has only strictly positive entries

HW read 4.9, especially examples! problems I will assign on Canvas

Example (Markov chain)

Suppose stochastic matrix of Markov chain

$$P = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}$$

Q1 Is it regular? Yes

Q2 If yes, find the unique steady-state vector.

(Markov chain:

$\overline{x_0}$, $\overline{x_1} = P x_0$, $\overline{x_2} = P \cdot \overline{x_1}$, ...)

sum of entries of all states is 1

Recall: P is called regular if some power P^k contains strictly positive entries.

$P^1 = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$ not enough for regularity

$P^2 = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$ entries > 0 $\Rightarrow$ regular!

(Example when 0 doesn't go away is $\begin{bmatrix} 1 & \frac{1}{4} \\ 0 & \frac{3}{4} \end{bmatrix} = P$. Every power P^k will have a form $\begin{bmatrix} 1 & * \\ 0 & * \end{bmatrix}$)

example of a <u>not regular</u> matrix

Now, to find steady-state vector for P, we want to solve

$$P = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P \cdot \bar{q} = \bar{q}$$

$$P \cdot \bar{q} - \bar{q} = \bar{0}$$

$$\boxed{(P - I) \cdot \bar{q} = \bar{0}}$$

the equation to solve for $\bar{q}$

$$\begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

augmented matrix $\left[\begin{array}{cc|c} -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -1 & 0 \end{array}\right]$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \updownarrow \text{swap}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

free q_2

(1) $\begin{cases} q_1 = 2q_2 \\ q_2 = q_2 \end{cases}$

$\bar{q} = q_2 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (2)

But the sum of entries $= 1$, so

$q_1 + q_2 = 2q_2 + q_2 = 3q_2 = 1$ (3)

$\Rightarrow q_2 = \frac{1}{3}$

$$\bar{q} = q_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

steady-state vector

---

recap To calculat steady-state vector:

1. Solve $(P-I)\cdot\bar{q}=0$ for $\bar{q}$
2. write in parametric vector form
3. Require sum entries to be 1

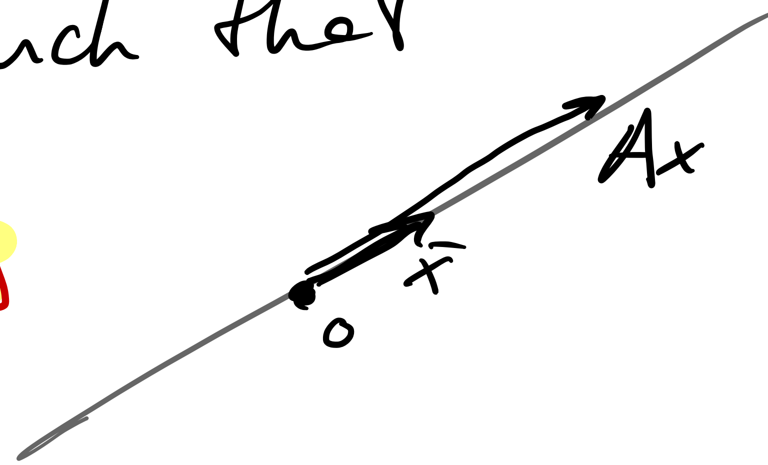
Read 4.9!

$\boxed{5.1}$ Eigenvectors & eigenvalues

Given a square $n \times n$ matrix A,

an eigenvector is a non-zero vector $\bar{x}$ in $\mathbb{R}^n$, such that

$$\boxed{A\bar{x} = \lambda \cdot \bar{x}}$$



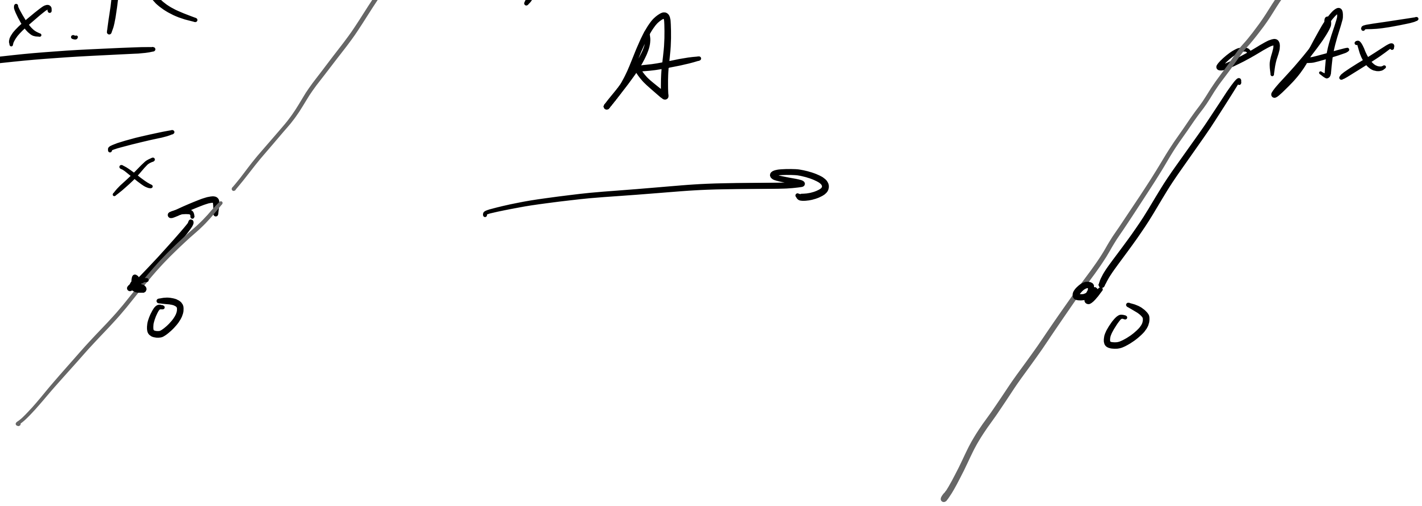
for a scalar $\boxed{\lambda}$ ← real number

eigenvalue of A, and

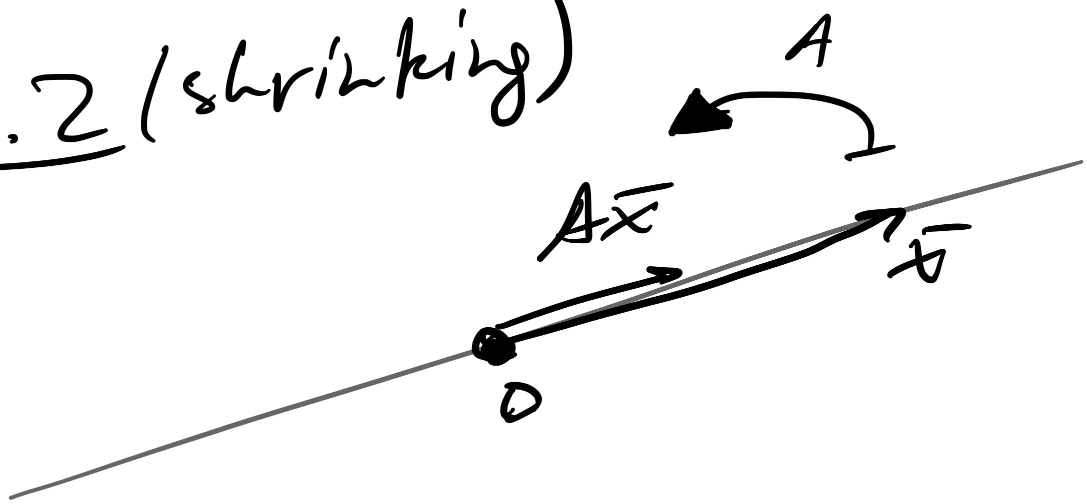
$\bar{x}$ is called an eigenvector corresponding to the eigenvalue λ

Intuitively, eigenvector is a vector that is "stretched" or "shrinked" by linear transformation A.

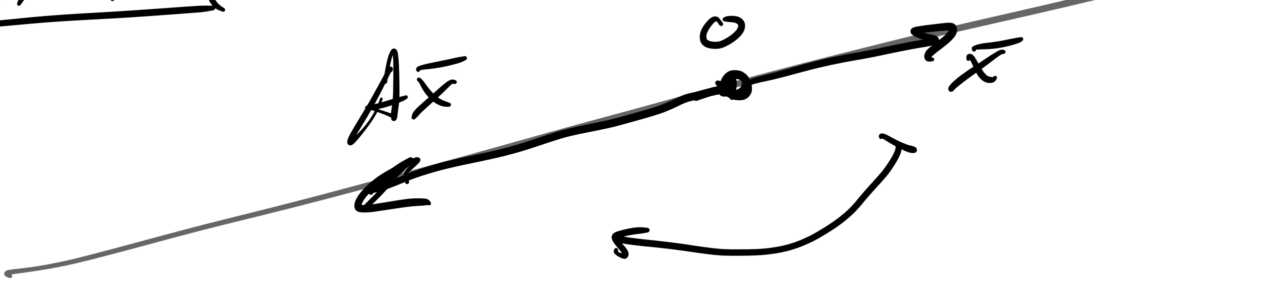
Ex. 1 (stretching)



Ex. 2 (shrinking)



Ex. 3 (stretching + flipping)



$\boxed{\begin{array}{l}\underline{\text{point}}\text{ image of } \bar{x} \text{ under } A \ (= A\cdot\bar{x}) \\ \text{is a multiple of } \bar{x}. \\ \text{That multiple is } \lambda\end{array}}$

Read 5.1 in the book, because it is an important section

Problem

① given $n \times n$ matrix A find all the eigenvalues (λ_i) of A (section 5.2)

② For each of these eigenvalues, find one (or all) eigenvectors corresponding to it (section 5.1)

Example 2 (from the book)

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\bar{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Are $\bar{u}$ & $\bar{v}$ eigenvectors of A?

$$A \cdot \bar{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix}$$

is it a multiple of $\bar{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$?

Yes!

$$A \cdot \bar{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \cdot \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4 \cdot \bar{u}$$

$$\boxed{A \cdot \bar{u} = -4\, \bar{u}}$$

$$\left(\frac{-24}{6} = -4 \quad \frac{20}{-5} = -4 \right)$$

$\Rightarrow$ $\bar{u}$ is eigenvector, with eigenvalue -4.

for $\bar{v}$:

$$A \cdot \bar{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$$

Q. Is this a multiple of $\bar{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$?

<u>No!</u> there is no λ such that

$$\begin{bmatrix} -9 \\ 11 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and so $\bar{v}$ <u>is not</u> an eigenvector for A.

- Every eigenvector has eigenvalue λ

- Every eigenvalue λ has many eigenvectors

next time: clarify.

Before next time:

repeat subspace

basis

Recall ① given $n \times n$ matrix A, to find out if a vector $\bar{v}$ is an eigenvector, check

$$A \cdot \bar{v} = \lambda \cdot \bar{v}$$

See Ex. 2 in the Book

If yes, λ is called eigenvalue.

See prev. lecture for an example.

Now, second type of basic problems:

② to find out if λ is an eigenvalue of matrix A, solve $A\bar{x} = \lambda \cdot \bar{x}$ for $\bar{x}$

Example 3 from the book

Show that $\overset{=\lambda}{7}$ is an eigenvalue of matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find also the corresponding eigenvectors.

Want: solve $\underbrace{\boxed{A \cdot \bar{x} = 7 \cdot \bar{x}}}$ for $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

eigenvector   eigenvalue

$\Updownarrow$

$$A\bar{x} - 7\bar{x} = 0$$

when $\bar{x}$ is taken out, this appears

$\Updownarrow$

$$(A - 7 \cdot I_2) \cdot \bar{x} = 0$$

$$\left(A - 7I_2 = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \right)$$

$\Updownarrow$

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

solve using augmented matrix

$$\left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right]$$

2 row-reduce

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad (*)$$

$$\rightarrow \begin{cases} x_1 - x_2 = 0 \\ x_2 \text{ free} \end{cases}$$

$\downarrow$ param. vect. form

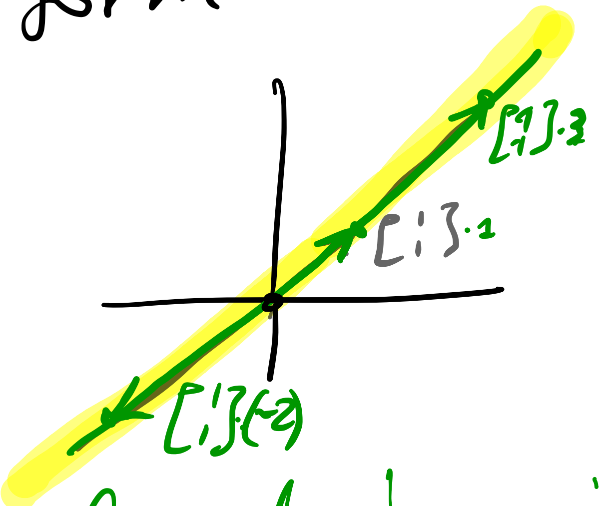
$$\boxed{\bar{x} = x_2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

So, 7 is an eigenvalue for $A=\begin{bmatrix}1&6\\5&2\end{bmatrix}$

and the corresponding eigenvectors form a line

$$\bar{x} = x_2 \cdot \begin{bmatrix}1\\1\end{bmatrix}$$

x_2 is arbitrary



Conclusion in this example we had a line worth of eigenvectors corresponding to eigenvalue $\lambda = 7$ of $\begin{bmatrix}1&6\\5&2\end{bmatrix}$.

(*) if there we have a contradiction, then 7 wouldn't be an eigenvalue.

Fact given an eigenvalue λ for matrix A, we have:

(n×n matrix)

$$\{\text{eigenvectors corresponding to eigenvalue } \lambda \text{ of } A\}$$

$\parallel$ By definition

$$\{\text{all solutions to equation } A\bar{x} = \lambda \cdot \bar{x}\}$$

see 5.1

$\parallel$

$$\{Nul(A - \lambda I)\}$$

this equation is equivalent to $(A - \lambda I)\cdot\bar{x} = 0$

this set always forms a subspace of R^n

[**Definition** given a scalar (number) λ, the eigenspace corresponding to λ is the space of all vectors $\bar{x}$ satisfying $A\bar{x} = \lambda\bar{x}$]

$\boxed{\text{The } \lambda\text{-eigenspace of } A \text{ is equal to null space of } A - \lambda I}$

Example of problems

1. Find a basis of the λ-eigenspace
2. Find the dimension of the λ-eigenspace
   $\|$
   (# of vectors in a basis

Example 4 from the book

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.
An eigenvalue of A is 2.
Find a basis for the corresponding eigenspace.

eigenspace corresp. to 2 =

$= Nul(A - 2 * I_3) =$

$= Nul\left(\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}\right)$

To find basis of Nul space one needs to write solutions in param. vector form:

$$\left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{array}\right]$$

$\Big\downarrow$ row-reduce

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free free

$$\begin{cases} x_1 - \frac{1}{2}x_2 + 3x_3 = 0 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases} \rightsquigarrow \begin{cases} x_1 = \frac{1}{2}x_2 - 3x_3 \\ x_2 = x_2 \text{ free} \\ x_3 = x_3 \text{ free} \end{cases}$$

param. vector form: $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

The vectors (yellow) above form a basis of null space

$$\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ basis of } \mathrm{Nul}(A-2I)$$

$$\parallel$$

remark dimension $= 2$.

eigenspace of A corresponding to $\lambda = 2$

answer

Theorem If $\overline{v_1}, \dots, \overline{v_r}$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$, then the set $\{\overline{v_1}, \dots, \overline{v_r}\}$ is linearly independent

Problem (Exercise 29 in 5.1)

Explain why $n \times n$ matrix can have at most n distinct eigenvalues.

HW: think about this, use the previous theorem.

If we have a matrix A, how do we find <u>all</u> of its eigenvalues? (Before λ was given to us)

$\boxed{\text{Section 5.2}}$ The characteristic equation

"λ is an eigenvalue of A"

by definition

$\Leftrightarrow$ "$A\bar{x} = \lambda\bar{x}$ has a non-zero solution in $\bar{x}$"

move $\lambda\bar{x}$ to the left

$\Leftrightarrow$ "$A\bar{x} - \lambda\bar{x} = 0$ has a non-zero solution"

factor out vector $\bar{x}$

$\Leftrightarrow$ "$(A - \lambda \cdot I_n) \cdot \bar{x} = 0$ has a non-zero sol."

$\Leftrightarrow$ "$(A - \lambda I_n)$ is not invertible"

$\Leftrightarrow$ "$\boxed{\det(A - \lambda I_n) = 0}$"

To find all the eigenvalues of $n \times n$ matrix A,

$$\boxed{\text{Solve for } \lambda \text{ the equation } \boxed{\det(A - \lambda \cdot I_n) = 0}}$$

<u>Example 1 from the book</u>

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Using , we compute:

- $A - \lambda \cdot I_2 = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} =$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

- $\det(A - \lambda I_2) = (2-\lambda)\cdot(-6-\lambda) - 3\cdot 3 =$

$= (\lambda - 2)(6 + \lambda) - 9 = \lambda^2 + 4\lambda - 12 - 9 =$

$= \lambda^2 + 4\lambda - 21$

Now, to find eigenvalues, we solve

$$\det(A-\lambda I)=0$$

$$\lambda^2+4\lambda-21=0$$

$$(\lambda+7)(\lambda-3)=0$$

$$\lambda=3 \quad \text{or} \quad \lambda=-7$$

two eigenvalues of $A=\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

As the last example shows, if A is $n\times n$, then

$\boxed{\det(A-\lambda I) \text{ is a polynomial in } \lambda \text{ of degree } n}$

This is called the characteristic polynomial of matrix A.

The equation we are solving to find eigenvalues:

$$\boxed{\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial}} = 0}$$

is called the characteristic equation of matrix A

To find eigenvalues,

1. Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda \cdot I_n)$

2. Solve $p(\lambda) = 0$ for λ.

(To do step 2 we factorize $p(\lambda)$ into products of $(\lambda - ?)$)

Example $A = \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix}$

① $p(\lambda) = \det(A - \lambda \cdot I_2) = \det\begin{pmatrix} 2-\lambda & 0 \\ 6 & 2-\lambda \end{pmatrix} =$

$= (2-\lambda)^2 - 0 = (2-\lambda)^2$

② $(2-\lambda)^{2} = 0 \rightsquigarrow \lambda = 2$ only eigenvalue.

multiplicity of $\lambda = 2$.

def
Each root of characteristic equation $p(\lambda) = 0$ has a multiplicity, which is the power of the corresp linear term.

Thus for $A = \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix}$ we have one eigenvalue $\lambda = 2$ of multiplicity 2

Remark multiplicity 2 can be though of as there are two eigenvalues, but both equal to 2.

multiplicity $\geq$ dimension of Eigenspace

Example suppose 4x4 matrix has characteristic polynomial

$p(\lambda) = \lambda^4 - 3\lambda^3 + 2\lambda^2 =$ Eigenvalues? Multiplicities?

$= \lambda^2(\lambda^2 - 3\lambda + 2) =$

$= \lambda^2(\lambda - ?)(\lambda - ?) =$

$= (\lambda - 0)^{2}(\lambda - 1)^{1}(\lambda - 2)^{1}$

So we get

Eigenvalues	
0	2
1	1
2	1

(see also Ex. 4 from the book)

Example 3 from the Book

Find the characteristic equation for

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find also eigenvalues & multiplicities.

$$A-\lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = \underline{(5-\lambda)}(3-\lambda)\underline{(5-\lambda)}(1-\lambda)$$

Eigenvalues	Multiplicities
5	2
3	1
1	1

For any <u>upper triangular</u> matrix

$$A = \begin{bmatrix} a_{11} & * & * & \cdots & * \\ 0 & a_{22} & * & & \\ 0 & 0 & \ddots & & * \\ \vdots & & & \ddots & \\ 0 & & & 0 & a_{nn} \end{bmatrix}$$

we obtain that $A - \lambda I = \begin{bmatrix} a_{11}-\lambda & & * \\ & \ddots & \\ 0 & & a_{nn}-\lambda \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = (a_{11} - \lambda)\cdot(a_{22} - \lambda)\cdots(a_{nn} - \lambda)$$

eigenvalues $= \{$solutions to $p(\lambda) = 0\}$

$$\lambda = a_{11}, \quad \lambda = a_{22}, \quad \ldots, \quad \lambda = a_{nn}$$

Thus for triangular (both upper and lower) matrices

1. eigenvalues = diagonal entries
2. multiplicities = # times those entries appear in the diagonal

We know the basics now:

given $n \times n$ matrix A, we can

1) find its eigenvalues λ_i (by solving $p(\lambda) = 0$) $\left.\right\}$ 5.2

$(\det(A - \lambda I)$

2) given an eigenvalue λ_i of A, we know how to find a basis for the corresponding λ_i-eigenspace (by finding a basis for $\mathrm{Nul}(A - \lambda_i I)$) $\left.\right\}$ 5.1

5.3 Given two $n \times n$ matrices A, B, we say that A is similar to B if there exists some invertible $n \times n$ matrix P such that

$$P^{-1} \cdot A \cdot P = B$$

(or, equivalently, $A = PBP^{-1}$)

In short

$$P^{-1} \cdot A \cdot \underset{\text{invertible}}{P} = B \implies A \text{ similar to } B$$

<u>Remark</u> $P^{-1} \cdot A \cdot P \neq A \cdot P^{-1} \cdot P = A$
!
because matrices do not commute.

<u>Remark</u> If "A similar to B"

$$\Updownarrow$$

$$P^{-1} \cdot A \cdot P = B$$

$$\Updownarrow$$

(By multiplying by P from left and by P^{-1} from the right)

$$A = P \cdot B \cdot P^{-1}$$

$$\Updownarrow \qquad P^{-1} = Q$$

$$A = Q^{-1} B Q$$

$$\Updownarrow$$

B is similar to A

therefore
"<u>A similar to B</u>" P

$$\Updownarrow$$

"<u>B similar to A</u>" $Q = P^{-1}$

- Theorem If A and B are similar, then they have the same characteristic polynomial.

- we will use similarity for:
  (1) diagonalization (5.3)
  (2) change of basis (5.4)

5.3 Diagonalization

D in this section will always be a diagonal matrix $\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$

Given $n \times n$ matrix A, a diagonalization of A is a factorization of the form:

$$\boxed{A = PDP^{-1}}$$

invertible matrix

diagonal matrix

In other words, diagonalization is the same as writing A as similar to a diagonal matrix.

Warning diagonalization may or may not exist!

Typical problem and application:

(1) find a diagonalization of A later

(2) Using the diagonalization, find a formula for the powers of A ($A^k = ?$) now

We first address ②.

If $D = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$, then

$D^2 = \begin{bmatrix} a_{11}^2 & & & 0 \\ & a_{22}^2 & & \\ & & \ddots & \\ 0 & & & a_{nn}^2 \end{bmatrix}$, $D^3 = \dots$, and in general

$$D^k = \begin{bmatrix} a_{11}^k & & & 0 \\ & a_{22}^k & & \\ & & \ddots & \\ 0 & & & a_{nn}^k \end{bmatrix}$$

If $A = PDP^{-1}$ ($\Leftrightarrow A$ is diagonalized),

then $A^k = \underbrace{(PDP^{-1}) \cdot (PDP^{-1}) \cdot \ldots \cdot (PDP^{-1})}_{k \text{ times}} =$

$$= PD\underbrace{P^{-1} \cdot P}_{I}D\underbrace{P^{-1} \cdot P}_{I}DP^{-1} \cdot \ \ldots \ \underbrace{\cdot}_{I} PDP^{-1} =$$

$$= P \cdot \underbrace{D \cdot D \cdot \ldots \cdot D}_{k \text{ times}} \cdot P^{-1} = P \cdot D^k P^{-1}$$

much easier to compute than A^k

Example 2 from the book

$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, and it is diagonilized for us

$\underbrace{A = PDP^{-1}}_{\text{given for us right now}}$, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

Find a formule for A^k.

$A = PDP^{-1} \xRightarrow[\text{before}]{\text{explained}} \underbrace{A^k = P \cdot D^k \cdot P^{-1}} =$

$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \cdot P^{-1}$

$P^{-1} = \frac{1}{\det P} \cdot \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

adjugate

(another way to do it $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right]$)

P $\quad I_2 \quad\quad P^{-1}$

$$\underline{A^k = P \cdot D^k \cdot P^{-1}} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} =$$

$$= \begin{bmatrix} 5^k & 3^k \\ -5^k & (-2)\cdot 3^k \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} =$$

$$= \boxed{\begin{bmatrix} (2\cdot 5^k - 3^k) & (5^k - 3^k) \\ (-2\cdot 5^k + 2\cdot 3^k) & (-5^k + 2\cdot 3^k) \end{bmatrix}}$$

formula for A^k

$\left(A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\right)$

---

we now start with the

problem 1 how do we find diagonalization of A?

Given $n \times n$ matrix A, an <u>eigenvector basis</u> is a collection of n linearly independent eigenvectors $\bar{v}_1, \ldots, \bar{v}_n$.

(Because we have n lin. indep. vectors in $\mathbb{R}^n$, the form a basis of $\mathbb{R}^n$)

<u>Main point</u>

Finding a diagonalization is the same thing as finding an eigenvector basis

also understand 5.1 & 5.2 !

go back and recall those things!

5.3 Given $n \times n$ matrix A

- eigenvector basis is a basis $\{\overline{v_1}, \ldots, \overline{v_n}\}$ of $\mathbb{R}^n$ such that each $\overline{v_i}$ is an eigenvector, meaning

$$A \cdot \overline{v_i} = \lambda \cdot \overline{v_i}$$

- diagonalization of A is a factorization

$$A = \underset{\text{invertible}}{P}\ \underset{\text{diagonal}}{D}\ P^{-1}$$

key point

$$\boxed{\text{finding diagonalization } A = PDP^{-1}} \iff \boxed{\text{finding an eigenvector basis for } A}$$

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \longleftrightarrow \text{eigenvector basis } \{\overline{v_1}, \ldots, \overline{v_n}\}$$

$$P = \begin{bmatrix} | & & | \\ \overline{v_1} & \cdots & \overline{v_n} \\ | & & | \end{bmatrix} \qquad \underbrace{\lambda_1 \ \ldots \ \lambda_n}_{\text{eigenvalues}}$$

in words: columns of P are the eigenvectors

- entries on the diagonal of D are the eigenvalues

Reason for: suppose $A = PDP^{-1}$, where

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad P = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}$$

Then $P\bar{e}_1 = \bar{v}_1$, $P\bar{e}_2 = \bar{v}_2$, ..., $P\bar{e}_n = \bar{v}_n$

Therefore

$$A \cdot \bar{v}_i = A \cdot P \cdot \bar{e}_i \qquad (\text{want} = \lambda_i \cdot \bar{v}_i)$$

$$= PDP^{-1}P\bar{e}_i$$

$$= PD \cdot \bar{e}_i$$

$$= P \cdot \lambda_i \cdot \bar{e}_i$$

$$= \lambda_i \cdot P \cdot e_i$$

$$= \lambda_i \cdot \bar{v}_i$$

Thus

$$A\bar{v}_i = \lambda_i \cdot \bar{v}_i,$$

and so

$\bar{v}_i$ is an eigenvector

Task learn how to find eigenvector basis using 5.1 & 5.2 techniques And then use that basis to find diagonalization.

Example 3 from the Book

Diagonalize matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

overview of the process

1. Find eigenvalues (5.2)
2. Find a basis for each eigenspace (5.1)
3. Put bases from 2 together to form an eigenvector basis, and use that to find $A = PDP^{-1}$

① $A - \lambda I = \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I)$$

characteristic polynomial

$$= (1-\lambda) \cdot \det\begin{pmatrix} -5-\lambda & -3 \\ 3 & 1-\lambda \end{pmatrix} - 3 \cdot \det\begin{pmatrix} -3 & -3 \\ 3 & 1-\lambda \end{pmatrix} + 3 \cdot \det\begin{pmatrix} -3 & -5-\lambda \\ 3 & 3 \end{pmatrix}$$

$$= \vdots = -\lambda^3 - 3\lambda^2 + 4$$

- $p(\lambda) = 0$

$$-\lambda^3 - 3\lambda^2 + 4 = 0$$

$$-(\lambda^3 + 3\lambda^2 - 4) = 0$$

idea for guessing roots of cubic polynomials when $a\lambda^3$ leading coefficient $= \pm 1$

take the last term -4, and try its divisors $\lambda = 1, -1, 2, -2$

let's check $\lambda = 1$: $\quad 1^3 + 3 \cdot 1^2 - 4 = 0$ ✓

so $\lambda = 1$ <u>is</u> a root

$$(\lambda - 1)(\lambda^2 + 4\lambda + 4) = (\lambda^3 + 3\lambda^2 - 4)$$

guess

$$(\lambda - 1)(\lambda + 2)(\lambda + 2) = \lambda^3 + 3\lambda^2 - 4$$

and so we have

eigenvalues	multiplicities
1	1
-2	2

<u>step 2</u> (finding bases of eigenspaces)

- for $\lambda = 1$ the eigenspace is

$$\text{Nul}\left(A - 1 \cdot I\right) \leftarrow$$ space of solutions of $(A - 1 \cdot I)\bar{x} = 0$

$$\|$$

$$\| \quad \mathrm{Nul}\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \leftarrow \text{augm. matrix}$$

row-reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 \text{ free} \end{cases} \rightsquigarrow \begin{cases} x_1 = x_3 \\ x_2 = -x_3 \\ x_3 = x_3 \text{ (free)} \end{cases} \rightsquigarrow$$

$$\rightsquigarrow \overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

parametric vector form

the vector forms the basis of $\mathrm{Nul}(A - 1 \cdot I)$

- for $\lambda = -2$ we have

$$\mathrm{Nul}\left(A - (-2) \cdot I\right)$$

$$\|$$

$$\mathrm{Nul}\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right]$$

$\vdots$ } row-reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 \text{ free} \\ x_3 \text{ free} \end{cases} \rightsquigarrow \begin{cases} x_1 = -x_2 - x_3 \\ x_2 = x_2 \text{ free} \\ x_3 = x_3 \text{ free} \end{cases}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

param. vector form

two vectors in a basis!

Step ③ Put eigenvectors together:

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{\lambda = 1}^{\vec{v}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\lambda = -2}^{\vec{v}_2}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\lambda = -2}^{\vec{v}_3} \right\}$$

think here vectors are always lin. indep.

we have 3 vectors in $\mathbb{R}^3$, and they are lin. independent $\Rightarrow$ we have an eigenvector basis for A

$(A \cdot \vec{v}_i = \lambda_i \cdot \vec{v}_i)$

Now, to find diagonalization $A = PDP^{-1}$, we take

$$P = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$A = PDP^{-1}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \cdot P^{-1}$$

Important if here we obtain less than 3 vectors (not enough for a basis in $\mathbb{R}^3$), then the answer is

"A is not diagonalizable"

<u>Recall</u> we are studying the problem of <u>diagonalizing matrix A</u>

(appearing according to multiplicities)

<u>overview of the process</u>

1. Find eigenvalues $\lambda_1, \dots, \lambda_n$ (5.2)
2. Find a basis for each eigenspace (5.1)
3. Put bases from (2) together $\{v_1, \dots, v_n\}$ to form an eigenvector basis, and use that to find $A = PDP^{-1}$
4. if here there is less than n vectors, then diagonalization does not exist.

$$P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

More on step ③

example $\lambda_1 = 3 \rightsquigarrow \{\overline{v}_1, \overline{v}_2\}$
$\lambda_2 = 2 \rightsquigarrow \{\overline{v}_3\}$

$P = \begin{bmatrix} | & | & | \\ \overline{v}_1 & \overline{v}_2 & \overline{v}_3 \\ | & | & | \end{bmatrix}$ or $P = \begin{bmatrix} | & | & | \\ \overline{v}_2 & \overline{v}_1 & \overline{v}_3 \\ | & | & | \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

given $n \times n$ matrix A with eigenvalues $\lambda_1 \dots \lambda_p$

Eigenspaces	Bases for eigenspaces
λ_1-eigenspace	B_1 $\left(= \{\overline{v}_1, \dots, \overline{v}_k\}\right)$ basis of λ_1-eigenspace
λ_2-eigenspace	B_2 (a set)
$\vdots$	
λ_p-eigenspace	B_p (a set)

We have seen earlier that:

> the union $B_1 \cup B_2 \cup \dots \cup B_p$
> is always linearly independent

Therefore, we have

> A is diagonalizable
> $\iff$ the set $B_1 \cup B_2 \cup \dots \cup B_p$
> has n elements (so that it is basis of $\mathbb{R}^n$)

Example 4 from the book

Diagonalize if possible $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

① $p(\lambda) = \det(A - \lambda I) =$

$$= \det \begin{pmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{pmatrix}$$

$$= \dots$$

$$= -(\lambda - 1) \cdot (\lambda + 2)^{2}.$$ Thus eigenvalues:

$\lambda = 1$: multiplicity 1

$\lambda = -2$: multiplicity 2

② $\lambda = 1$ eigenspace:

$$\mathrm{Nul}\,[A - 1 \cdot I] = \mathrm{Nul}\begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

... augmented matrix & parametric vector form ...

gives a basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \mathcal{B}_1$ for

$\lambda = -2$ eigenspace

$$\mathrm{Nul}\left[A-(-2)\cdot I\right] = \mathrm{Nul}\begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

... augmented matrix & param. vect. form...

gives a basis $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\} = B_2$ for

Rmk we see that here we have multiplicity 2 (for $\lambda=-2$), but only 1 vector in a basis of eigenspace!

③ So, collection $B_1 \cup B_2$

$$= \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

has only two vectors, while we need three! (since A is 3×3)

So thus A is not diagonalizable

Given $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_p$, consider

eigenvalues	λ_1	λ_2	...	...	...	λ_p
multiplicities	m_1	m_2	...	...	...	m_p
dimension of eigenspace	d_1	d_2	...	...	...	d_p

<u>Theorem</u> we always have

these are precisely numbers of vectors in bases of eigenspaces

$$\boxed{\begin{aligned} d_1 &\le m_1 \\ d_2 &\le m_2 \\ &\vdots \\ d_p &\le m_p \end{aligned}}$$

Moreover $m_1 + \dots + m_p \le n$

Therefore, A is diagonalizable if and only if all these inequalities are equalities.

(simply because A diagonalizable $\Leftrightarrow d_1 + d_2 + \dots + d_p = n$)

In particular

Important!

(*) $\boxed{\text{If } A \text{ has } n \text{ distinct eigenvalues, then } A \text{ is diagonalizable}}$

(Because then $m_1 = m_2 = \ldots = m_n = 1$)

Example 5 from the book

Is $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$ diagonalizable?

- A upper triangular $\Rightarrow$ $\lambda_1 = 5$, $\lambda_2 = 0$, $\lambda_3 = -2$
- 3 distinct eigenvalues $\Rightarrow$ diagonalizable! (by (*))

Remark If $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

we would not be able to use (*), because $\lambda_1 = \lambda_3 = 5$. So here one would run the usual algorithm

① ...
② ...
③ ...
④ ...

5.4

Goal: understand that for $T: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A \cdot \bar{x}$

diagonalization $A = P \cdot D \cdot P^{-1}$

$$P = \underbrace{\begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}}_{\text{Bases for eigenspaces}}, \quad D = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}_{\text{eigenvalues}}$$

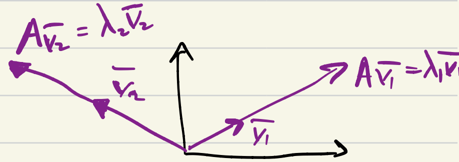
is equivalent to the fact that

Linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A \cdot \bar{x}$

? in the basis $\mathcal{B} = \{\bar{v}_1, \dots, \bar{v}_n\}$

is represented by a diagonal matrix D

Main idea



$A\bar{v}_1 = c_1\bar{v}_1 + c_2\bar{v}_2 = \lambda_1 \bar{v}_1 \Rightarrow c_1 = \lambda_1, c_2 = 0$

$A\bar{v}_2 = c_3\bar{v}_1 + c_4\bar{v}_2 = \lambda_2 \bar{v}_2 \Rightarrow c_3 = 0, c_4 = \lambda_2$

Thus: $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$!

In order to understand this better, let's step back, and introduce abstract vector spaces.

• usual vector space for was $\mathbb{R}^n$

An abstract vector space is a set where:

(a) there is an addition $(\vec{v}_1 + \vec{v}_2)$

(b) there is a "multiplication by a number" $(c \cdot \vec{v})$

such that certain axioms are satisfied (page 192)

Example 1 $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots$ all of them are abstract vector spaces.

Example 2 consider the set $\mathbb{P}_2$ of all polynomials of degree ≤ 2

(for instance $f(x) = x^2 - 3x + 7$ is an element of $\mathbb{P}_2$)

It is an abstract vector space since if f and g are in $\mathbb{P}_2$

a) $f(x) + g(x)$ is still a polynomial of deg 2 or less

b) $17 \cdot f(x)$ is a polynomial of deg 2 or less example $\quad (1+x^2) + (-3-x) = -2-x+x^2$

$$\overline{v_1} + \overline{v_2} = \overline{w}$$

This vector space $\mathbb{P}_2$ has a basis $B = \{1, x, x^2\}$ because every

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2} = a_0 \cdot (1) + a_1 \cdot (x) + a_2 \cdot (x^2)$$

$$\left(\overline{W} = c_1 \overline{V_1} + c_2 \overline{V_2} + c_3 \overline{V_3} \right)$$

Thus, using basis $\{\overset{\overline{v_1}}{1}, \overset{\overline{v_2}}{x}, \overset{\overline{v_3}}{x^2}\}$ of $\mathbb{P}_2$, we parameterize $\mathbb{P}_2$ by $\mathbb{R}^3$

$$\mathbb{P}_2 \xleftrightarrow{\text{bijection}} \mathbb{R}^3$$

$$f(x) = a_0 + a_1 x + a_2 x^2 \longleftrightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

A <u>linear transformation</u> between two abstract vector spaces

$$T: V \longrightarrow W$$

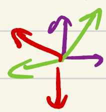
is a map satisfying

1) $T(\bar{v}_1 + \bar{v}_2) = T(\bar{v}_1) + T(\bar{v}_2)$ (preserves addition)
2) $T(c \cdot \bar{v}) = c \cdot T(\bar{v})$ (preserves scalar multiplication)

<u>Example</u> $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $\bar{x} \longmapsto A \cdot \bar{x}$ — lin. transformations given by matrices

<u>Q.</u> Can $T: V \longrightarrow W$ be always represented by a matrix?

"choosing a coordinate system"!



<u>A.</u> Yes! But first we need to <u>choose a basis</u> of V and W.

<u>remark</u> $\mathbb{R}^n$ comes with a distinguished basis $\{\bar{e}_1, \dots, \bar{e}_n\}$, but abstract vector spaces do not come with basis, and this is why we have to pick one

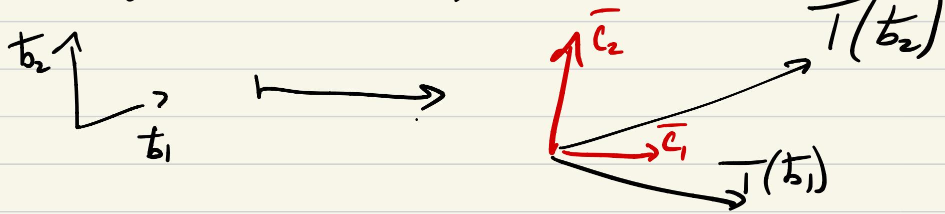
$$T: V \longrightarrow W$$

Let $B = \{\bar{b}_1 \dots \bar{b}_n\}$ be a basis for V

$C = \{\bar{c}_1, \dots, \bar{c}_n\}$ be a basis for W

Then T is determined by $\underbrace{T(\bar{b}_1)} \dots \underbrace{T(\bar{b}_n)}$

$$T: V \longrightarrow W$$



W

$$\bar{w} = x_1\bar{c}_1 + x_2\bar{c}_2 \Rightarrow [\bar{w}]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

① T is determined by $T(\bar{b}_1) \dots T(\bar{b}_n)$

② every $\bar{w}$ in W can be uniquely written $\bar{w} = x_1\bar{c}_1 + \dots + x_n\bar{c}_n$

using Both we obtain

$$[T(b_1)]_C, [T(b_2)]_C, \dots, [T(b_n)]_C$$

and we get a matrix

$$\boxed{M = \begin{bmatrix} | & | & & | \\ [T(b_1)]_C & [T(b_2)]_C & \cdots & [T(b_n)]_C \\ | & | & & | \end{bmatrix}}$$

matrix for $T: V \to W$
relative to bases $\mathcal{B}$ for V
$\mathcal{C}$ for W

$\boxed{\text{def.}}$

Example differentiation

$$\left(\frac{d}{dx}\right): \mathbb{P}_2 \longrightarrow \mathbb{P}_1$$

$$7 - x + x^2 \longmapsto -1 + 2x$$

$$a_0 + a_1 x + a_2 x^2 \longmapsto a_1 + 2a_2 x$$

observe that $\left(\frac{d}{dx}\right)$ is linear! Thus we can have a matrix M which represents differentiation on $\mathbb{P}_1$, as long as we pick basis for both

$$\left(\frac{d}{dx}\right) : \mathbb{P}_2 \longrightarrow \mathbb{P}_1$$

. choose bases : $\mathcal{B} = \{\overset{\bar{b}_1}{1}, \overset{\bar{b}_2}{x}, \overset{\bar{b}_3}{x^2}\}$ for $\mathbb{P}_2$

$C = \{\overset{\bar{c}_1}{1}, \overset{\bar{c}_2}{x}\}$ for $\mathbb{P}_1$

. now let's compute the matrix

$$T(\bar{b}_1) = \frac{d}{dx}(1) = 0 \quad = 0 \cdot \overset{\bar{c}_1}{1} + 0 \cdot \overset{\bar{c}_2}{x}$$

$$T(\bar{b}_2) = \frac{d}{dx}(x) = 1 \quad = 1 \cdot \overset{\bar{c}_1}{1} + 0 \cdot \overset{\bar{c}_2}{x}$$

$$T(\bar{b}_3) = \frac{d}{dx}(x^2) = 2x \quad = 0 \cdot \overset{\bar{c}_1}{1} + 2 \cdot \overset{\bar{c}_2}{x}$$

$$\Rightarrow M = \begin{bmatrix} | & | & | \\ [T(\bar{b}_1)]_C & [T(\bar{b}_2)]_C & [T(\bar{b}_3)]_C \\ | & | & | \end{bmatrix}$$

$$\| \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(4.1 can be helpfull)

matrix representing

$$\frac{d}{dx} : \mathbb{P}_2 \longrightarrow \mathbb{P}_1$$

with respect to bases

$$\{1, x, x^2\} \qquad \{1, x\}$$

Recall: given

- two abstract vector spaces V^3 and W^2

(assume for simplicity $\dim(V)=3$, $\dim(W)=2$)

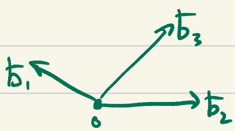
- linear transformation $T: V \to W$
- two bases, $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \bar{b}_3\}$ for V and $\mathcal{C} = \{\bar{c}_1, \bar{c}_2\}$ for W

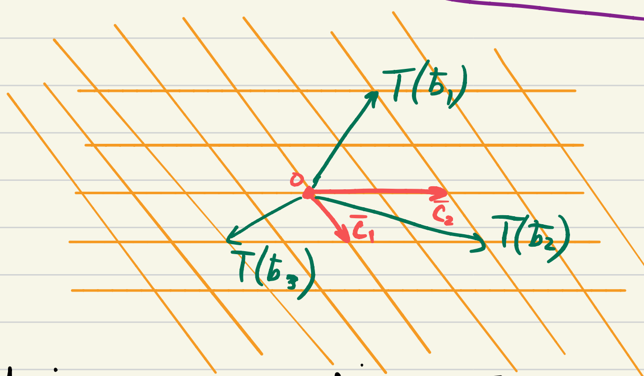
remark

$T: \mathbb{R}^n \to \mathbb{R}^n$
$x \longmapsto A x$

A is matrix for T relative to $\mathcal{B} = \{\bar{e}_1 \dots \bar{e}_n\}$ for both $\mathbb{R}^n$ and $\mathbb{R}^n$

$$T: V^3 \longrightarrow W^2$$





We can form a matrix representing $T: V \to W$ relative to bases $\mathcal{B}$ and $\mathcal{C}$. Its columns are coordinates of $T(\bar{b}_i)$ in basis $\mathcal{C}$:

$$\boxed{M = \begin{bmatrix} | & | & | \\ [T(\bar{b}_1)]_{\mathcal{C}} & [T(\bar{b}_2)]_{\mathcal{C}} & [T(\bar{b}_3)]_{\mathcal{C}} \\ | & | & | \end{bmatrix}}$$

In this example we have

$$M = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Example . $T : V^2 \longrightarrow W^2$

. bases $\{\vec{d}_1, \vec{d}_2\}$ for V, $\{\vec{b}_1, \vec{b}_2\}$ for W
$\qquad \qquad \qquad \qquad \mathcal{D} \qquad \qquad \qquad \qquad \mathcal{B}$

. $T(\vec{d}_1) = 2\vec{b}_1 - 3\vec{b}_2$, $T(\vec{d}_2) = -4\vec{b}_1 + 5\vec{b}_2$

$$M = \begin{bmatrix} | & | \\ [T(\vec{d}_1)]_{\mathcal{B}} & [T(\vec{d}_2)]_{\mathcal{B}} \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$$

(relative to bases $\mathcal{D}$ and $\mathcal{B}$)

Example . $\mathcal{E} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ standard basis in $\mathbb{R}^3$

$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\bar{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

. $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\}$ basis for V

. $T : \overset{(\mathcal{E})}{\mathbb{R}^3} \longrightarrow \overset{(\mathcal{B})}{V}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto (x_1 + x_2)\bar{b}_1 + (x_2 - x_3)\bar{b}_2$$

$$M = \begin{bmatrix} | & | & | \\ [T(\bar{e}_1)]_{\mathcal{B}} & [T(\bar{e}_2)]_{\mathcal{B}} & [T(\bar{e}_3)]_{\mathcal{B}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

(matrix for T relative to $\mathcal{E}$ and $\mathcal{B}$)

$T(\bar{e}_1) = T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1+0)\bar{b}_1 + (0-0)\bar{b}_2 = \bar{b}_1$

$T(\bar{e}_2) = \bar{b}_1 + \bar{b}_2$ $\quad T(\bar{e}_3) = -\bar{b}_2$

key point: if matrix M represents

$T: \underset{\text{basis } B}{V} \longrightarrow \underset{\text{basis } C}{W}$, then for $\bar{v}$ in V we have:

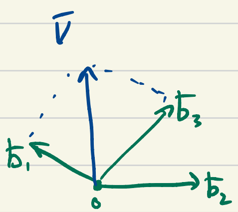
$$\boxed{M \cdot [\bar{v}]_B = [T(\bar{v})]_C} \quad (*)$$

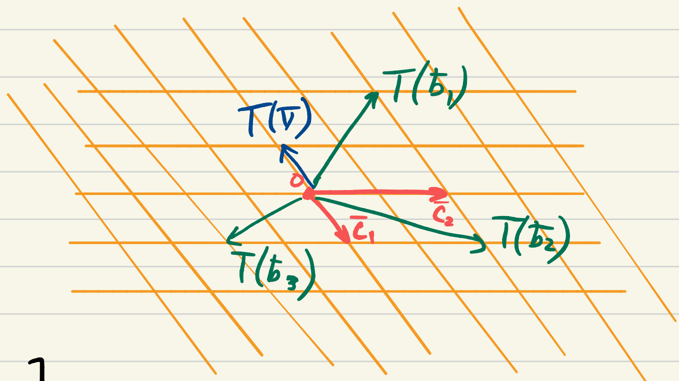
"coordinates change according to M!"

Example: suppose below $\bar{v} = \bar{b}_3 + \bar{b}_1$

$\bar{v} = \underline{1} \cdot \bar{b}_1 + \underline{0} \cdot \bar{b}_2 + \underline{1} \cdot \bar{b}_3$

$$T: V^3 \longrightarrow W^2$$





Then

- $[\bar{v}]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad (\bar{v} = 1 \cdot \bar{b}_1 + 0 \cdot \bar{b}_2 + 1 \cdot \bar{b}_3)$
- $[T(\bar{v})]_C = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\quad (= T(\bar{b}_1) + T(\bar{b}_3)$, clear from the picture$)$
- we remember $M = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Formula $(*)$ becomes

$$\boxed{\begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{= [v]_B} = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{= T(\bar{v})_C}} \quad \checkmark \text{ true}$$

Example let $T: \overset{B}{\mathbb{P}_2} \longrightarrow \overset{C}{\mathbb{P}_2}$ (differentiation)

$$a_0 + a_1 x + a_2 x^2 \mapsto a_1 + 2a_2 x$$

So find matrix $[T]_B$

$$B = \{\overset{b_1}{1}, \overset{b_2}{x}, \overset{b_3}{x^2}\}$$

$$C = B$$

$$T(b_1) = \frac{d}{dx}(1) = 0$$

$$T(b_2) = \frac{d}{dx}(x) = 1$$

$$T(b_3) = \frac{d}{dx}(x^2) = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

$$[T]_B = \begin{bmatrix} | & | & | \\ [T(b_1)]_B & [T(b_2)]_B & [T(b_3)]_B \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Special case of the story above: When $V = W$, $B = C$!

$T : {}^{B}V \longrightarrow {}^{B}V = {}^{C}W$, and suppose one choose a basis B for both spaces B. The we get a matrix M representing T in bases B.

It has a special name: "B-matrix for T"

notation: $[T]_B$

Meaning of diagonalization suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x \longmapsto A\bar{x}$,

$$A = PDP^{-1}, \quad P = \begin{bmatrix} | & & | \\ \bar{v}_1 & \cdots & \bar{v}_n \\ | & & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

eigenvectors   eigenvalues

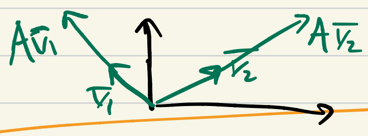
Then the key point is that

T is represented by a diagonal matrix D in basis $B = \{\bar{v}_1 \cdots \bar{v}_n\}$ $\iff$ $[T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Comment: $[T]_{\varepsilon} = A$, where ε = standard basis

reason $A(\bar{v}_i) = \lambda_i \bar{v}_i$

Because eigenvectors!



last two classes were to understand this

$$\begin{tikzcd}
\bar{x} \arrow[r, mapsto, "A\cdot"] \arrow[d, "P^{-1}\cdot"'] & A\bar{x} \\
{[\bar{x}]_B} \arrow[r, mapsto, "D\cdot"'] & {[A\bar{x}]_B} \arrow[u, "P\cdot"']
\end{tikzcd}$$

$$D = [T]_B$$

Example Suppose

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\bar{x} \longmapsto A \cdot \bar{x} \iff [T]_{\mathcal{E}} = A$$

$\mathcal{E}$ ← std basis

where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$

Find a basis B for $\mathbb{R}^2$ such that the B-matrix for T is diagonal.

Solution: Basis B will be the eigenvector basis, and so one needs to

① find eigenvalues (5.2)

② find eigenvectors (5.1)

① eigenvalues $\quad p(\lambda) = \det(A - \lambda I_2) = 0$

$$\lambda = 3, \quad \lambda = 5$$

② eigenvectors $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ for, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for

Thus if $B = \left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, then

the B-matrix for T is equal

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

matrix for T relative to Basis B for Both

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

5.6 Application of eigenvalues and eigenvectors to discrete dynamical systems

In 4.3 we studied Markov chains, processes defined by

$$\bar{X}_0 \xmapsto{P\cdot} \bar{X}_1 \xmapsto{P\cdot} \bar{X}_2 \longmapsto \cdots \quad \bar{q}$$

$$\bar{X}_1 = P\bar{X}_0, \quad \bar{X}_2 = P\cdot\bar{X}_1$$

Recall that P was a stochastic matrix.

with special property that columns sum to 1, for instance P could be $\begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix}$

Key point was that $\bar{X}_n \longrightarrow \bar{q}$ (steady state vector) as $n \to \infty$.

Question What if columns of P do not sum to 1?

Enter $\boxed{\text{discrete dynamical systems}}$

EXAMPLE 1 Denote the owl and wood rat populations at time k by $\overline{\mathbf{x}}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$, where k is the time in months, O_k is the number of owls in the region studied, and R_k is the number of rats (measured in thousands). Suppose

$$\begin{cases} O_{k+1} = (.5)O_k + (.4)R_k \\ R_{k+1} = -p \cdot O_k + (1.1)R_k \end{cases} \tag{3}$$

where p is a positive parameter to be specified. The $(.5)O_k$ in the first equation says that with no wood rats for food, only half of the owls will survive each month, while the $(1.1)R_k$ in the second equation says that with no owls as predators, the rat population will grow by 10% per month. If rats are plentiful, the $(.4)R_k$ will tend to make the owl population rise, while the negative term $-p \cdot O_k$ measures the deaths of rats due to predation by owls. ~~(In fact, 1000p is the average number of rats eaten by one owl in one month.)~~ Determine the evolution of this system when the predation parameter p is .104.

$$\overline{X}_{k+1} = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix} \overline{X}_k$$

means understand what happens long-term, as $k \to \infty$

Method

① write in matrix form

② Diagonalize A (find eigenvalues and eigenvectors)

③ The most important information is the largest eigenvalue, and its eigenvector.

$$\overline{X_k} = \begin{bmatrix} O_k \\ R_k \end{bmatrix}, \quad \underbrace{\overline{X_{k+1}} = A \cdot \overline{X_k}}_{1}, \quad A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$$

2 Diagonilize A: (using 5.2, 5.1, ...)

$\underline{\text{eigenvalues}}$, $p(\lambda) = \det(A - \lambda I) = 0$

$$\vdots$$

$$\lambda = 0.58$$
$$\lambda = 1.02$$

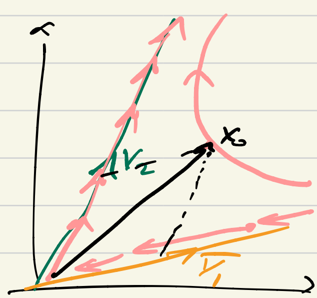
$\underline{\text{eigenvectors}}$. for $\lambda = 0.58$ eigenspace (...)

has a basis $\left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\} = V_1$

. for $\lambda = 1.02$ eigenspace (...)

has a basis $\left\{ \begin{bmatrix} 10 \\ 13 \end{bmatrix} \right\} = V_2$

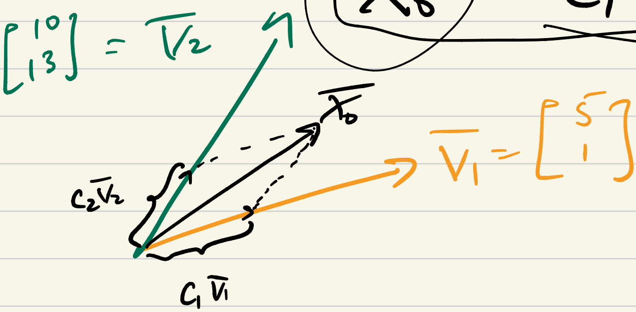
3 now, the key point:



given any initial configuration $\overline{X_0} = \begin{bmatrix} O_0 \\ R_0 \end{bmatrix}$, we can uniquely write $\overline{X_0}$ in the form

$$\boxed{\overline{X_0} = c_1 \overline{V_1} + c_2 \overline{V_2}}$$

(Because $\{\overline{V_1}, \overline{V_2}\}$ is a basis for $\mathbb{R}^2$)



Using this, we now find

main point

$$\overline{X_n} \underset{\text{by def. of evolutionary system}}{=\!=\!=} A^n \overline{X_0} = A^n(c_1\overline{V_1} + c_2\overline{V_2}) =$$

$$= c_1 (0.58)^n \overline{V_1} + c_2 \cdot (1.02)^n \overline{V_2}$$

this is because each time A is applied it stretches $\overline{V_2}$ by 1.02 and shrink $\overline{V_1}$ by 0.58

and so

$$\bar{X}_0 \longmapsto A X_0 \longmapsto A^2 X_0 \longmapsto \cdots$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ \boxed{c_1\bar{V}_1 + c_2\bar{V}_2} & \overset{A\cdot}{\longmapsto} & \boxed{c_1(0.58)\bar{V}_1 + c_2(1.02)\cdot\bar{V}_2} & \longmapsto & \boxed{c_1(0.58)^2\bar{V}_1 + c_2(1.02)^2\bar{V}_2} \longmapsto \cdots \end{array}$$

Now,

0 when $n \to \infty$

$$\bar{X}_n = \cancel{c_1 \cdot (0.58)^n \bar{V}_1} + c_2(1.02)^n \cdot \bar{V}_2$$

When n is large, $(0.58)^n \longrightarrow 0$

$\Rightarrow$ when n is large

$$\begin{bmatrix} O_n \\ R_n \end{bmatrix} = \bar{X}_n \approx c_2(1.02)^n \cdot \bar{V}_2 = c_2(1.02)^n \begin{bmatrix} 10 \\ 13 \end{bmatrix}$$

approximately

Answer over the long run, the population of owl and rats will have a ration

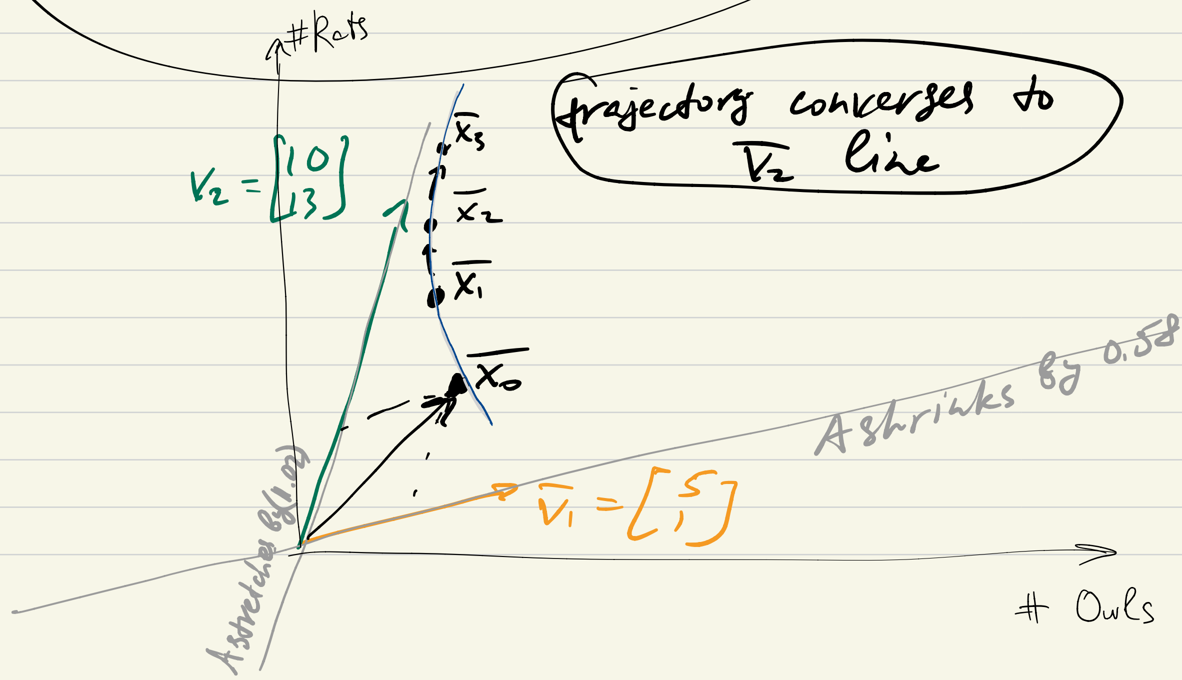
$$\boxed{O_n : R_n \approx 10 : 13}$$

$$\left(= \frac{\cancel{c_2 (1.02)^n}\, 10}{\cancel{c_2 (1.02)^n}\, 13} \right)$$

Also, both grow by 2% each month!
(due to $(1.02)^n$ exponential multiplier)

$$O_n \approx c_2 \cdot (1.02)^n \cdot 10$$

$$R_n \approx c_2 \cdot (1.02)^n \cdot 13$$



What we want is to study similar problems in general!

We will have 3 cases:

① $0 < \lambda_1 < 1 < \lambda_2$ (our case here)

② $0 < 1 < \lambda_1 < \lambda_2$

③ $0 < \lambda_1 < \lambda_2 < 1$

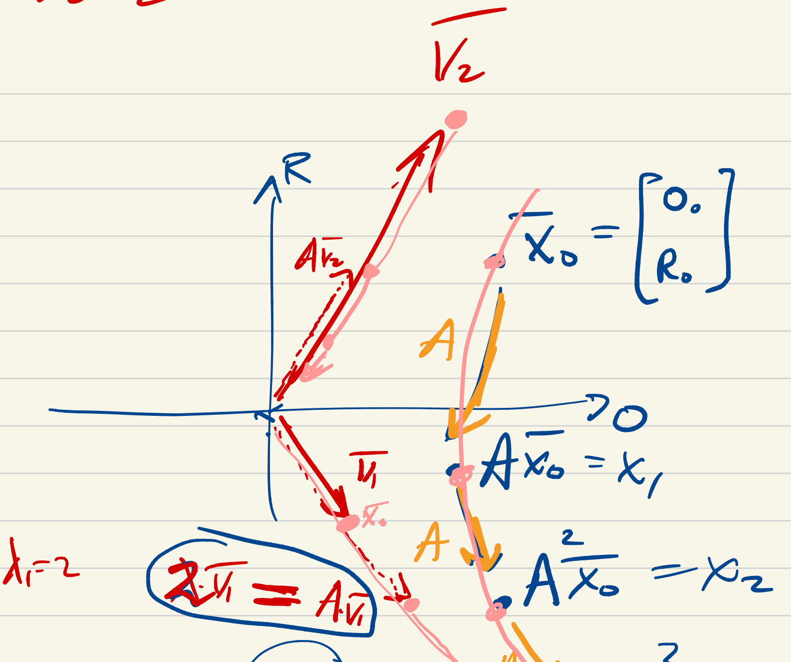
Next time we will study general pictures for all 3 cases

Read the book (5.6)!

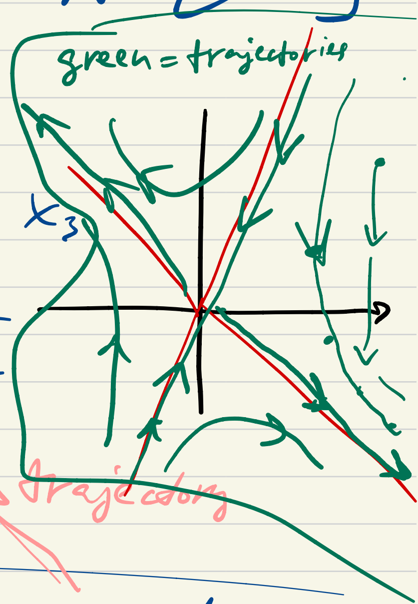
$\lambda_2 = \frac{1}{2}$

$\bar{v}_1, \bar{v}_2$ = eigenvectors

$A\bar{v}_2 = \lambda_2 \cdot \bar{v}_2 = \frac{1}{2}\bar{v}_2$



$A = \begin{bmatrix} - & \cdot \\ \cdot & \cdot \end{bmatrix}$



$A \cdot \bar{x}_0 = \bar{X}_1$

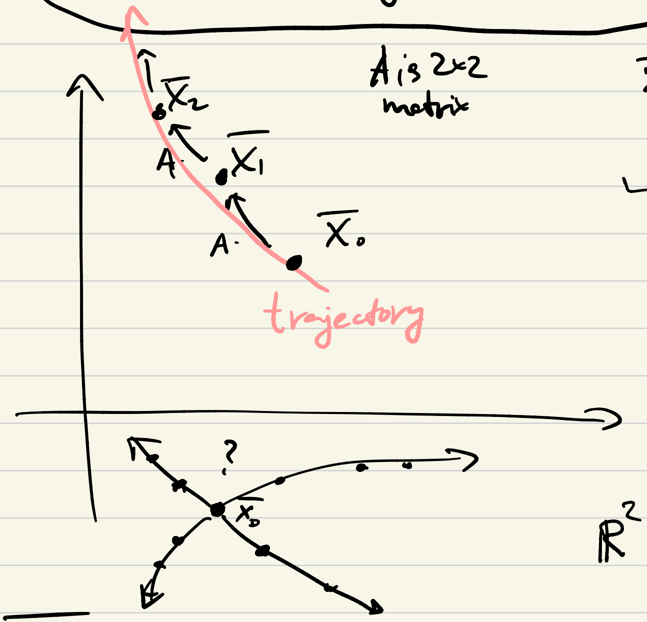
$A \cdot \bar{X}_1 = A \cdot A \cdot \bar{x}_0 = A^2 \bar{x}_0 = \bar{x}_2$

$A\bar{x}_2 = A \cdot A^2\bar{x}_0 = A^3\bar{x}_0$

Now, trajectory of discrete dyn. system is $\bar{x}_0 \longrightarrow A\bar{x}_0 \longrightarrow A^2\bar{x}_0 \longrightarrow \dots$

Q. Given initial $\bar{x}_0$, describe the trajectory.

Graphical description of discrete dynamical systems (5.6)



A is 2×2 matrix

$$\underbrace{\overline{x}_0 \xmapsto{A\cdot} \underset{\overline{x}_1}{\underset{\parallel}{A\overline{x}_0}} \xmapsto{A\cdot} \underset{\overline{x}_2}{\underset{\parallel}{A^2\overline{x}_0}} \mapsto \cdots}_{\text{discrete dynamical system}}$$

Big Question given A, describe all trajectories

Let's first understand cases of diagonal matrices $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Assumption: from now on we assume λ_i are positive and $\neq 1$

Three cases for $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$:

① **Attractor**: $0 < \lambda_1, \lambda_2 < 1$

② **Repeller**: $0 < 1 < \lambda_1, \lambda_2$

③ **Saddle**: $0 < \lambda_1 < 1 < \lambda_2$

(or $0 < \lambda_2 < 1 < \lambda_1$)

We will study these cases separately.

Example 2 from the Book (attractor)

Plot several trajectories of the dynamical system $\overline{x}_{k+1} = A \cdot \overline{x}_k$ when

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}$$

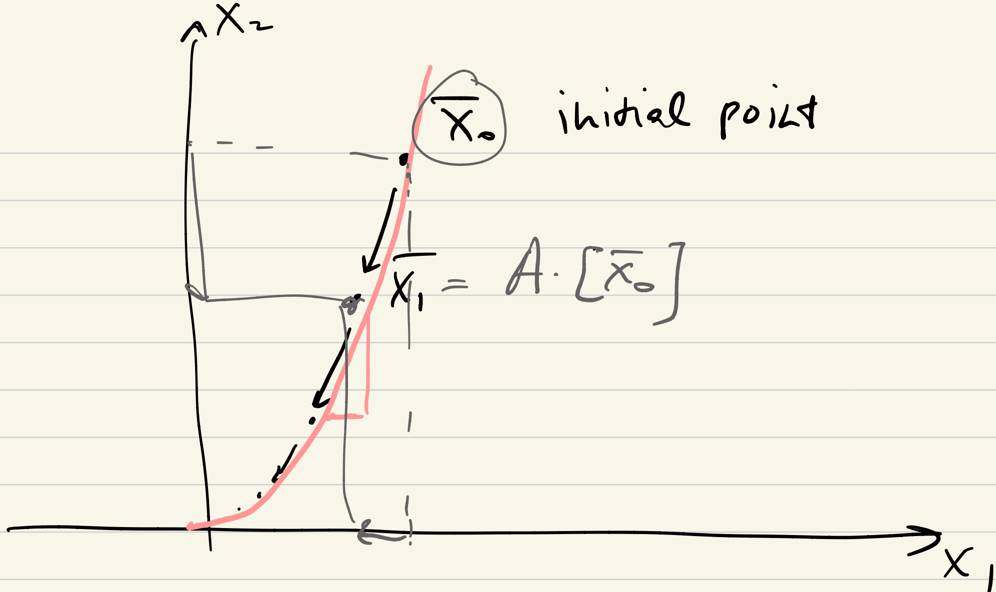
$\lambda_1 = 0.8 < 1$

$\lambda_2 = 0.64 < 1$ $\Rightarrow$ attractor

$$\overline{x}_{k+1} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix} \cdot \overline{x}_k$$ scales $\overline{e}_1$ by 0.8, e_2 by 0.64

If $\overline{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ then $\boxed{\overline{x}_k = \begin{bmatrix} (0.8)^k \cdot c_1 \\ (0.64)^k \cdot c_2 \end{bmatrix}}$

$$\left(\text{because } \underbrace{\begin{bmatrix} 0.8 & 0 \\ 0 & 0.64 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.8 c_1 \\ 0.64 \cdot c_2 \end{bmatrix} \right)$$



vertical direction ($\bar{e}_2$) drops faster (since $0.64 < 0.8$) than the horisontal one ($\bar{e}_1$) $\Rightarrow$ the trajectory is not a straight line!

The typical shape for all the trajectories of an <u>attractor</u>:

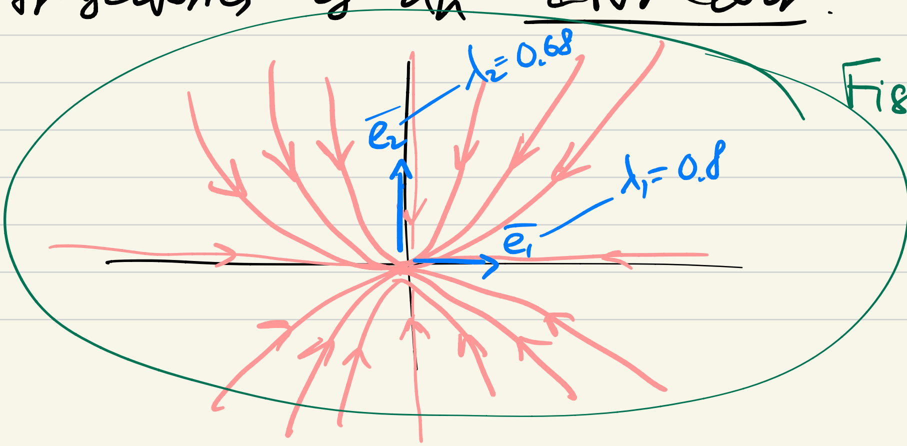


Figure 1 on p. 306

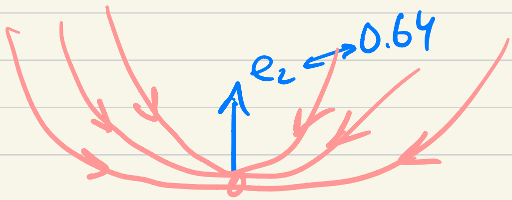
Features:

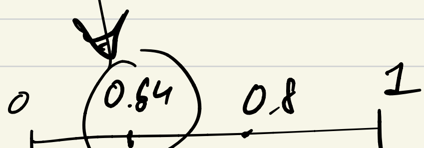
① All the points are "attracted" to the origin

② For the second eigenvalue λ_2 the $\overline{e_2}$-component disappers faster ($0.64 < 0.8$)

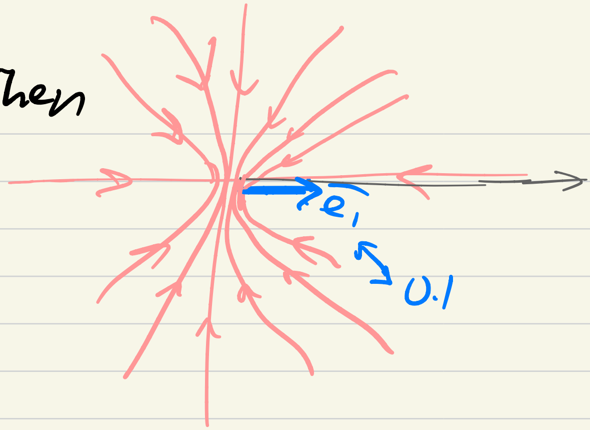
$\Rightarrow$ trajectories are "flat" near the origin

Shortcut for "which way" the trajectories are flat: trajectories are $\perp$ to the eigenvect, whose eigenvalue is further from 1 (smallest eigenvalue)





if $A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.64 \end{bmatrix}$, then



$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $\mathbb{R}^2$

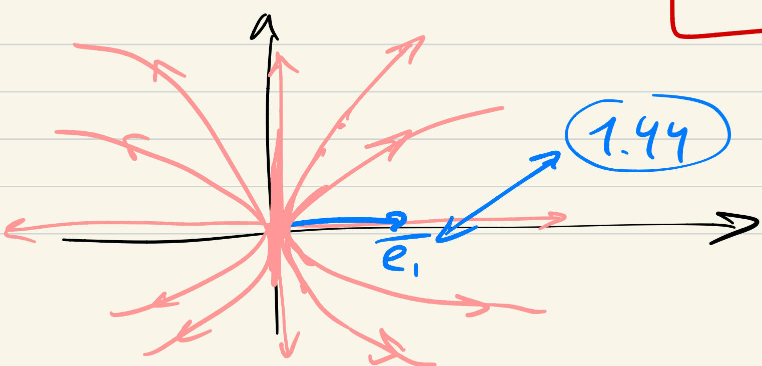
$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$ for $\mathbb{R}^n$

Example 3 in the Book (repeller)

$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$ scales $\bar{e}_1$ by 1.44

$\bar{e}_2$ by 1.2

$X_0 = c_1 \cdot \bar{e}_1 + c_2 \cdot \bar{e}_2 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \boxed{X_k = \begin{bmatrix} (1.44)^k c_1 \\ (1.2)^k c_2 \end{bmatrix}}$

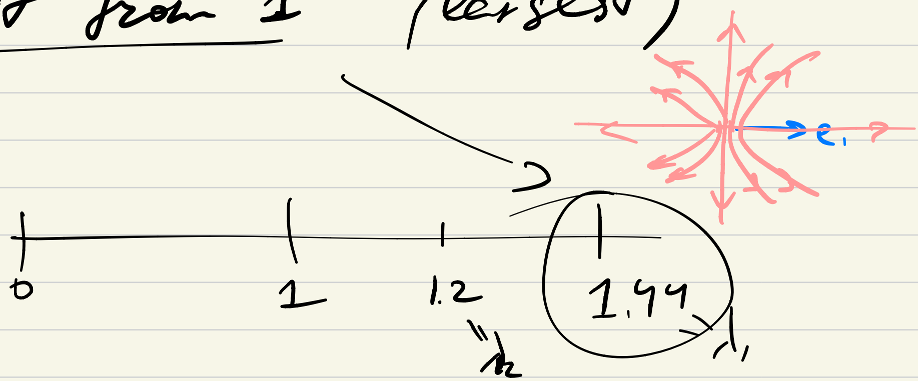


① points are "repelled" from $\bar{0} = \begin{bmatrix}0\\0\end{bmatrix}$

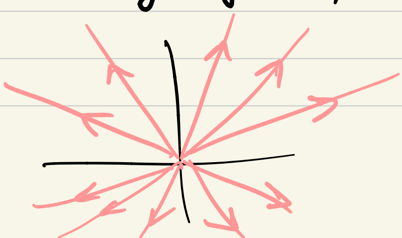
② again trajectories are "flat", because $\overline{e_1}$ scales faster than $\overline{e_2}$. $(1.44 > 1.2)$

<u>Shortcut for "which way" flat near 0</u>

trajectories are $\perp$ to the eigenvector corresponding to the eigenvalue <u>furthest from 1</u> (largest)



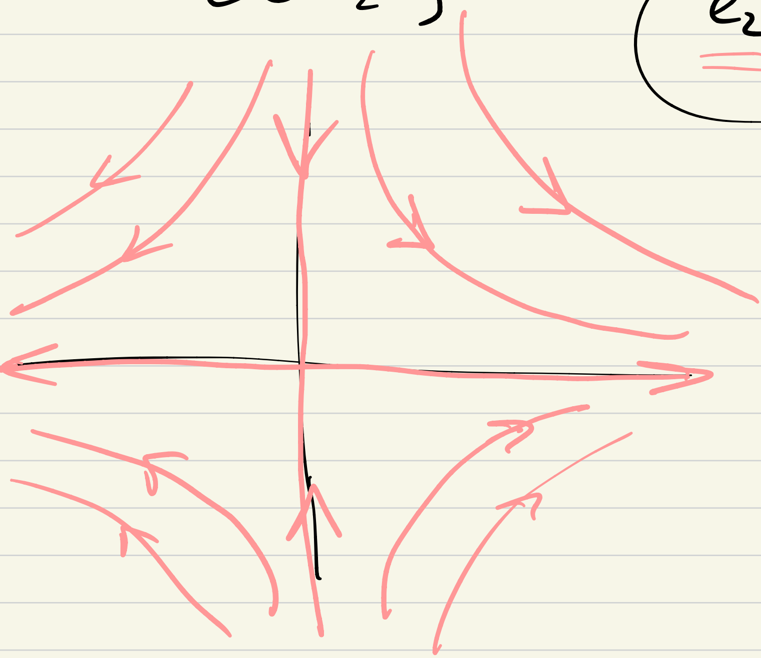
<u>Remark</u> trajectories are all straight lines only if $\lambda_1 = \lambda_2$



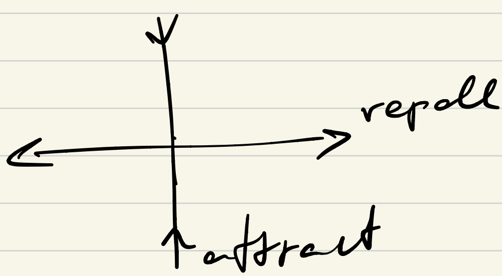
Example 4 (saddle)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$ scales e_1 by 2 $\Rightarrow$ repells along the first coordinate

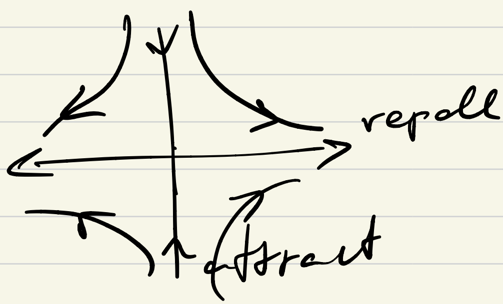
e_2 by $\frac{1}{2}$ $\Rightarrow$ attracts along second coordinate



step 1



step 2



Q. What if A is not diagonal?

A. All the same, except the role of $\overline{e_1}$ and $\overline{e_2}$ are played by eigenvectors $\overline{v_1}$ and $\overline{v_2}$

Example 5 in the book

$$A = \begin{bmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{bmatrix}$$ (should be diagonalizable!)

① eigenvalues: $p(\lambda) = \det(A - \lambda I) = 0$

$$\lambda = 2$$
$$\lambda = \frac{1}{2}$$

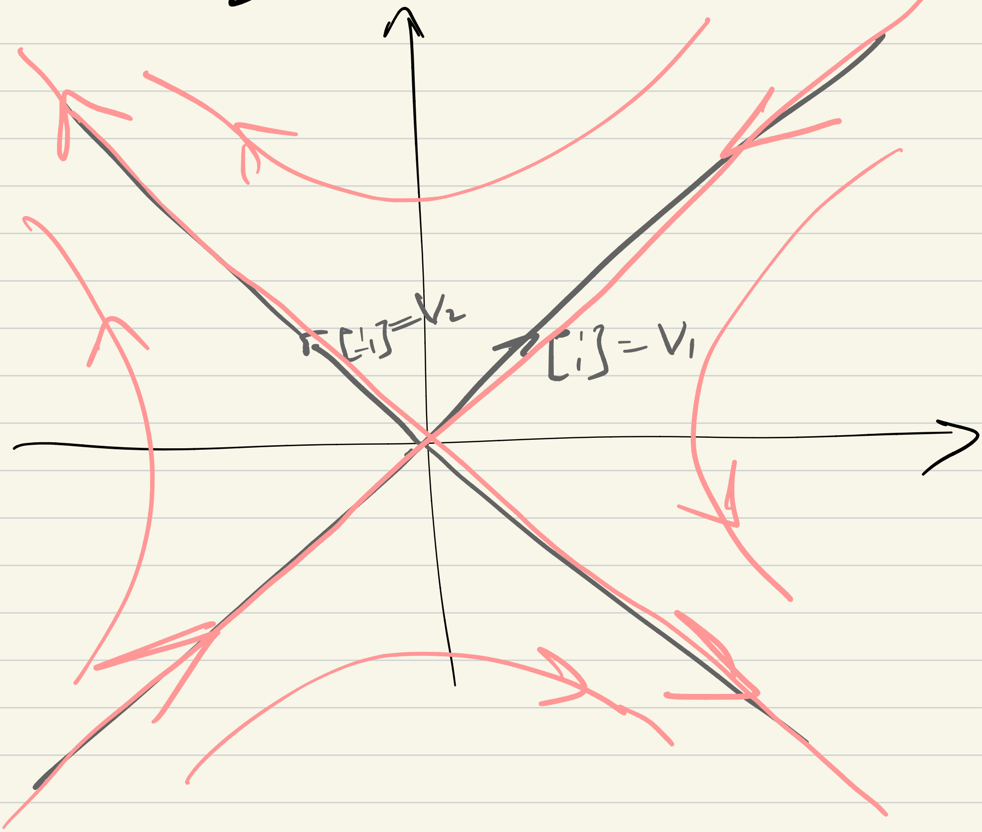
② eigenvectors: (...) you find

$\lambda_2 = 2 \rightsquigarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \overline{v_2}$

$\lambda_1 = \frac{1}{2} \rightsquigarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \overline{v_1}$

3 along $\begin{bmatrix}1\\-1\end{bmatrix}$ it repells since $\lambda=2$

along $\begin{bmatrix}1\\1\end{bmatrix}$ it attracts since $\lambda=\frac{1}{2}$



read 5.6 (excluding case of complex λ)

$$[T]_B = \begin{bmatrix} | & | \\ [T(b_1)]_B & [T(b_2)]_B \\ | & | \end{bmatrix}$$

$$\underline{T(b_1)} = A \cdot b_1 = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \underline{\begin{bmatrix} 5 \\ 0 \end{bmatrix}}$$

$[T(b_1)]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ such that

$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \underline{c_1 \cdot b_1 + c_2 \cdot b_2}$ solve for c_1 and c_2

the same for $[T(b_2)]_B$

$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ repeller $1 < \lambda_1, \lambda_2$

$A' = \begin{bmatrix} 1/\lambda_1 & 0 \\ 0 & 1/\lambda_2 \end{bmatrix}$ attractor $\lambda_1, \lambda_2 < 1$

trajectories coincide!
directions differ!