

**Overview.** I study invariants of 3-manifolds and knots through symplectic geometry. Consider a 3-manifold  $Y$ , and a knot  $K$  inside it (possibly empty). Consider a decomposition

$$(*) \quad (Y, K) = (Y_1, T_1) \cup_{(\Sigma, 2k)} (Y_2, T_2)$$

where  $T_1$  and  $T_2$  are  $2k$ -ended tangles, and  $(\Sigma, 2k)$  is a  $2k$ -punctured surface; see Figure 1 for an example of decomposition of the trefoil knot along a four-puncture sphere ( $S^2, 4pt$ ). Frequently, decomposition  $(*)$  needs to satisfy special properties, for example being a Heegaard splitting for  $Y$ . Next, the main idea is to associate to the  $2k$ -punctured surface a certain symplectic manifold  $\mathcal{M}(\Sigma, 2k)$ , and to associate to the two parts  $(Y_i, T_i)$  of decomposition  $(*)$  two Lagrangian submanifolds  $L_i$  inside the symplectic manifold:

$$L_1 \rightarrow \mathcal{M}(\Sigma, 2k) \leftarrow L_2$$

Then sometimes, in favorable circumstances, the *Lagrangian Floer homology*  $HF(L_1, L_2)$  (see the next page for the definition) is in fact a topological invariant of  $(Y, K)$ , i.e. does not depend on decomposition  $(*)$  and other choices. This method of constructing invariants was pioneered by Ozsváth and Szabó: they used the symplectic manifold  $\mathcal{M}(\Sigma_g) = \text{Sym}^g(\Sigma_g)$  to define a 3-manifold invariant called *Heegaard Floer homology* [8]. They, and independently Rasmussen, also extended the construction to knots, resulting in an invariant called *knot Floer homology* [7, 9]. These invariants are extraordinarily powerful, and much of the contemporary research focuses on studying these and other similarly constructed symplectic geometric invariants of  $(Y, K)$ .

**Immersed curves and Khovanov homology.** Currently, my research is centered around *Khovanov homology*, a homological knot invariant discovered by Khovanov [5], taking the form of a bigraded vector space  $Kh^{h,q}(K)$ . It is defined algebraically, and so it is natural to ask whether the strategy above applies to this invariant. In a joint work with Liam Watson and Claudius Zibrowius [6], we obtained the following symplectic geometric interpretation of Khovanov homology.

**Theorem 1.** *For any 4-ended tangle  $(D^3, T)$  there exist three immersed curves<sup>1</sup>*

$$\widetilde{BN}(T), \widetilde{Kh}(T), Kh(T) \looparrowright (S^2, 4pt) = \partial(D^3, T)$$

whose homotopy classes are tangle invariants of  $T$ . Moreover, let  $(S^3, K) = (D^3, T_1) \cup_{(S^2, 4pt)} (D^3, T_2)$  be a decomposition of a knot  $K \subset S^3$  into two 4-ended tangles. Then reduced  $(\widetilde{Kh})$  and full  $(Kh)$  Khovanov homology of the knot  $K$ , as well as reduced Bar-Natan's deformation  $(\widetilde{BN})$  of Khovanov homology, are isomorphic to Lagrangian Floer homology:

$$\widetilde{BN}(K) \cong HF(\widetilde{BN}(mT_1), \widetilde{BN}(T_2))$$

$$\widetilde{Kh}(K) \cong HF(\widetilde{BN}(mT_1), \widetilde{Kh}(T_2))$$

$$Kh(K) \cong HF(\widetilde{BN}(mT_1), Kh(T_2))$$

where  $mT_1$  is the mirror tangle.

For the depicted in Figure 1 decomposition of the trefoil  $T_1 \cup_{(S^2, 4pt)} T_2$ , the Lagrangian Floer intersection picture resulting in reduced Khovanov homology is illustrated in Figure 2. This theorem is central in our research, allowing us to study Khovanov homology from a new angle. To highlight one implication, we proved that Rasmussen's  $s$ -invariant is preserved under mutation. Further applications are currently being developed.

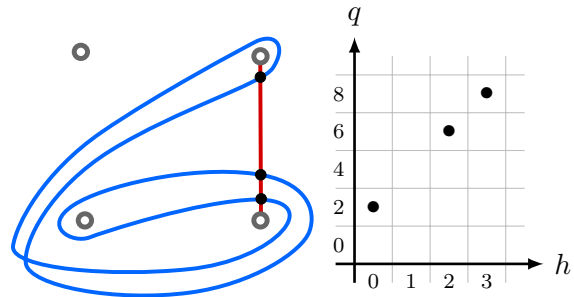


FIGURE 2.  $HF(\widetilde{BN}(mT_1), \widetilde{Kh}(T_2)) \cong \widetilde{Kh}(T(2, 3)) \cong \mathbb{F}^3$

<sup>1</sup>In this context, immersed curves always come with a choice of a local system. For the purposes of illustration it is not important, and so we sweep this detail under the rug.

**Lagrangian Floer theory and the Fukaya category.** Omitting a large amount of details and certain technical conditions [2], Lagrangian Floer homology  $HF(L_1, L_2)$  is a homological invariant of two Lagrangian submanifolds inside a symplectic manifold  $L_1, L_2 \hookrightarrow (M, \omega)$ , which is invariant under Hamiltonian isotopies of each Lagrangian. The underlying chain complex is generated by intersections:  $CF(L_1, L_2) = \langle L_1 \cap L_2 \rangle_{\mathbb{F}}$  (transversality can be achieved by a small generic Hamiltonian perturbation). After fixing a generic compatible with  $\omega$  almost complex structure on  $M$ , the differential  $\partial : CF(L_1, L_2) \rightarrow CF(L_1, L_2)$  is defined by counting rigid pseudo-holomorphic discs between the intersection points, with Lagrangian boundary conditions on  $L_1$  and  $L_2$ :

$$L_2 \begin{array}{c} \circlearrowleft^y \\ \circlearrowright^x \end{array} L_1 \text{ contributes } \pm 1 \text{ into coefficient } c_{xy} \text{ in } \partial(x) = \sum_y c_{xy} \cdot y$$

In the relevant for Theorem 1 case  $\dim(M) = 2$ , Lagrangians are curves on a surface, and counting rigid pseudo-holomorphic discs is equivalent to counting immersed discs with convex angles at intersections. As a result, the construction is easily generalized to *immersed* curves, and the dimension of Lagrangian Floer homology is almost always equal to the minimal intersection number of curves:  $\dim HF(L_1, L_2) = \min \#(L_1 \cap L_2)$ . An example of minimal intersection number 3 is depicted in Figure 2.

Another important symplectic geometric invariant, instrumental in the proof of Theorem 1, is the *Fukaya category* of a symplectic manifold  $\mathcal{F}(M)$  [3, 4, 10] (see [1] for a survey). It is a unified structure, which captures how all Lagrangians intersect with each other. The objects in the Fukaya category are all Lagrangians  $L_i \rightarrow M$ , and morphism spaces are Lagrangian Floer complexes  $CF(L_i, L_j)$ . The composition in this category is defined by counting pseudo-holomorphic triangles:

$$\triangle: CF(L_i, L_j) \otimes CF(L_j, L_k) \rightarrow CF(L_i, L_k)$$

The composition is not associative on the nose, but is associative up to homotopy, which is given by counting pseudo-holomorphic rectangles. As such,  $\mathcal{F}(M)$  is not a regular category, but rather is an  $A_\infty$  category, where higher operations are defined by counting pseudo-holomorphic polygons.

In the case  $\dim(M) = 2$ , where Lagrangians are curves on the surface  $M$ , the Fukaya category is similar to a curve complex, only it captures more information: minimal intersection numbers between the curves, and also all the immersed convex polygons with boundary on multiple curves.

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