Heegaard Floer and Khovanov theories through the lens of immersed curves I

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(joint work with Liam Watson and Claudius Zibrowius)

1. Khovanov homology

Is a knot invariant discovered by Khovanov:

$$K \rightsquigarrow \widetilde{Kh}(K)$$

oriented knot in S^3

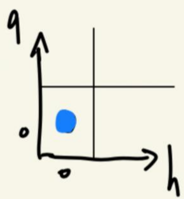
$\mathbb{Z}_q \oplus \mathbb{Z}_h$-graded vector space over a field $\mathbb{k}$

- The construction is algebraic/combinatorial
- Recovers Jones polynomial via graded Euler characteristic

Examples

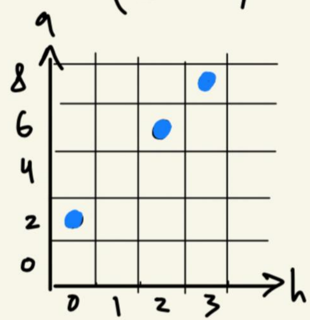
① Unknot
0_1

$\widetilde{Kh}\Big($  $\Big) = \mathbb{k}$



② Right-hand trefoil
3_1

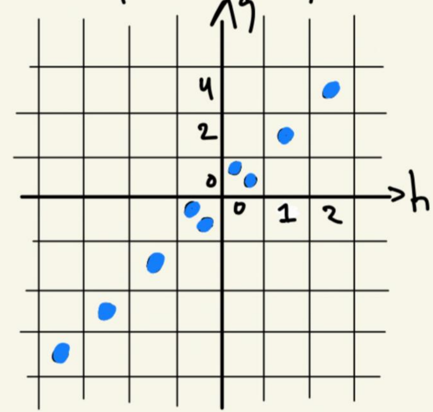
$\widetilde{Kh}\Big($  $\Big) = \mathbb{k}^3$



Jones$\Big($  $\Big) = q^1 + q^3 - q^4$

③ Stevedore knot
6_1

$\widetilde{Kh}\Big($  $\Big) = \mathbb{k}^9$

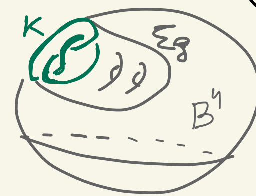


Q Why study Khovanov homology?

A1 It is powerful

- Strong knot invariant: detects the unknot (KM)

def. smooth 4-ball genus is $g_4(K) := \min\{ g \mid \partial(B^4, \Sigma_g) = (S^3, K)\}$



- * Rasmussen: $\widetilde{Kh}(K) \rightsquigarrow$ number $s(K)$, s.t.

$$\left|\frac{s(K)}{2}\right| \leq g_4(K) \leq u(K)$$

$$\Rightarrow \boxed{u(T(p,q)) = \frac{(p-1)(q-1)}{2}}$$

Milnor's conjecture

A2. It is Beautiful and mysterious.

Algebraic construction, yet deep connections to geometry

- Spectral sequences to different Floer homologies of knots

$$\widetilde{Kh}(K;\mathbb{F}_2) \rightrightarrows \widehat{HF}(-\Sigma_2(K)) \qquad (OS)$$

$$\widetilde{Kh}(K;\mathbb{Q}) \rightrightarrows \widehat{HFK}(mK) \qquad (\text{Dowlin})$$

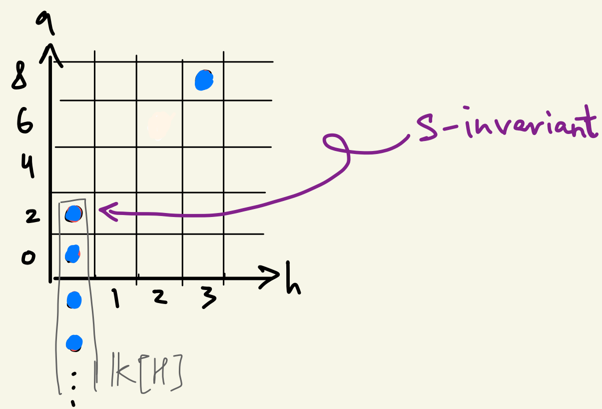
$$\widetilde{Kh}(K;\mathbb{Z}) \rightrightarrows I^{\natural}(mK) \qquad (KM)$$

- Can be redefined using Lagrangian Floer theory:
  - via Floer theory of Hilbert schemes of Milnor fibers (SSMA)
  - *via wrapped Floer theory of $S^2 \setminus 4pt$ (KWZ)

<u>Remark</u>

- A variation of Khovanov homology is $\underline{\widetilde{BN}(K) \text{ reduced Bar-Natan homology}}$ ($\mathbb{k}[H]$-module)

- $\widetilde{BN}($$) = \mathbb{k}[H] \oplus \mathbb{k}$



- $\widetilde{BN}(L)$ is a module over $\mathbb{k}[H_1, H_2, \dots, H_n]$ (KWZ)

$n = \#$ of link components

2. Heegaard Floer homology

(1) Y^3 closed oriented $\xrightarrow{(OS)}$ $\widehat{HF}(Y^3)$ graded $\Bbbk$-vector space

<u>Heegaard Floer homology</u>

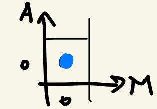
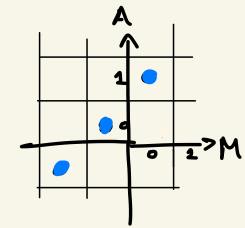
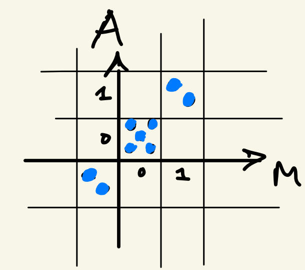
(2) K oriented knot in S^3 $\xrightarrow{(OS,\ Rasmussen)}$ $\widehat{HFK}(K)$ $\overset{A}{\mathbb{Z}} \oplus \overset{M}{\mathbb{Z}}$-graded $\Bbbk$-vector space

<u>knot Floer homology</u>

(categorifies the Alexander polynomial)

Examples

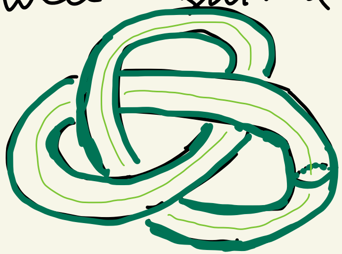
3-manifold	$\widehat{HF}$
S^3	$\mathbb{k}$
$L(p,q)$	$\mathbb{k}^p$
$\Sigma(2,3,7)$	$\mathbb{k}^3$
$\left\{ \begin{array}{l} x^2+y^3+z^7=0 \\ \lvert x\rvert^2+\lvert y\rvert^2+\lvert z\rvert^2=1 \end{array} \middle\vert \begin{array}{c} (x,y,z) \\ \cap \\ \mathbb{C}^3 \end{array} \right\}$	

Knot	$\widehat{HFK}$
$O_1 =$ 	$\mathbb{k}$ 
$3_1 =$ 	$\mathbb{k}^3$ 
$6_1 =$ 	$\mathbb{k}^9$ 

Q Why study Heegaard Floer theory?

A1 It is powerful

- Very well suited for studying surgery questions:



$S^3 \setminus \nu(K)$ $\xrightarrow{\text{glue solid torus back in}}$ $S^3_{p/q}(K)$ "p/q Dehn surgery"

Thm (KMOS)
$S^3_{p/q}(K) = S^3_{p/q}(U) \Rightarrow K = U$

[known as Property P is $\frac{p}{q} = \frac{1}{q}$ (GL)
Property R is $\frac{p}{q} = \frac{0}{1}$ (Gabai)]

- $\widehat{HFK}(K)$ detects $\to$ the 3-genus (OS)
  $\to$ being fibered (GN)
- HF detects exotic smooth structures on 4-manifolds (OS)

A2 It is Beautiful

- Brings together low-dimensional topology and symplectic topology

- Has many connections to other areas

$$\text{HF theory} \longleftrightarrow \pi_1 \text{ (L-space conjecture)}$$

$$\text{HF theory} \longleftrightarrow \text{gauge theory} \left(\begin{array}{l}\text{monopoles} \\ \text{instantons}\end{array}\right)$$

$$\text{HF theory} \longleftrightarrow \text{foliations, sutured mfld theory of Gabai}$$

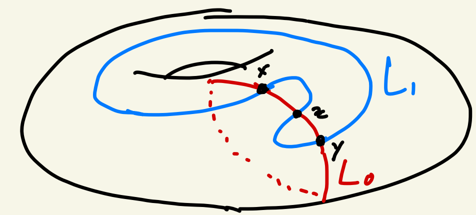
3. Lagrangian Floer homology of immersed curves on a surface

→ definition of $\widehat{HF}(L(p,q))$

→ central idea in our research

- L_0, L_1 two smoothly immersed curves in a surface Σ

Example



Lagrangian Floer chain complex

- $CF_*(L_0,L_1) = \langle L_0 \cap L_1 \rangle_{\mathbb{k}}$

- $CF_*(L_0,L_1) \circlearrowleft \partial$ counts immersed lunes (discs with two convex corners)

$$\partial(x) = \sum_{\text{lunes}} y$$

- $HF_*(L_0,L_1) = H_*(CF_*(L_0,L_1))$ Lagrangian Floer homology

<u>Theorem</u> Given some assumptions are satisfied,

(1) $\partial^2 = 0$

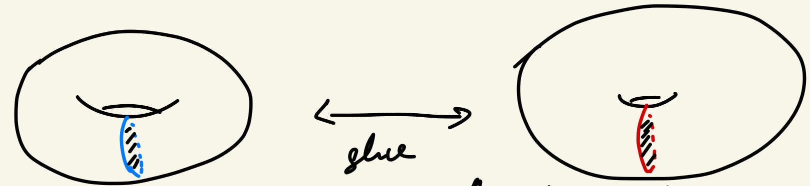
(2) $HF_*(L_0, L_1)$ is invariant with respect to isotopies

(3) If L_0 and L_1 are not homotopic, then

$\dim HF_*(L_0, L_1)$ = <u>minimal intersection number</u>

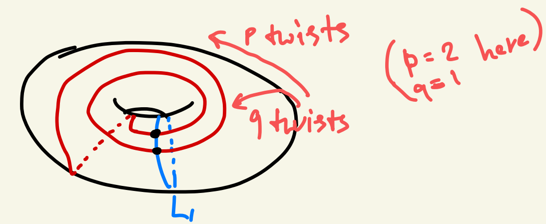
$\widehat{HF}$ for lens spaces

- $L(p,q) = S^1\times D^2 \cup_h S^1\times D^2$



"inside" solid torus "inside" solid torus

such that blue and red curves on T^2 look →

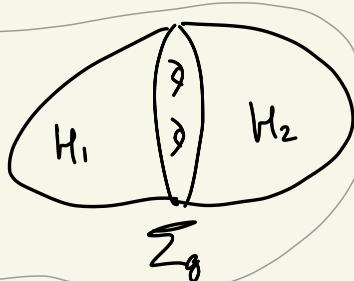


As a result:

$$\widehat{HF}(L(p,q)) = HF_*(L_0, L_1) = k^p$$

In general

$Y^3 =$



⟹ two Lagrangian submanifolds inside symplectic manifold

$$T_1^g \hookrightarrow Sym^g(\Sigma_g \setminus 1pt) \hookleftarrow T_2^g$$

⇓

$$\widehat{HF}(Y) = HF_*(T_1, T_2)$$

H_i = "inside" handlebody

Thank you!