Overview. I study invariants of 3-manifolds and knots through symplectic geometry. Consider a 3-manifold Y, and a knot K inside it (possibly empty). Consider a decomposition

$$(*) \qquad (Y, K) = (Y_1, T_1) \cup_{(\Sigma, 2k)} (Y_2, T_2)$$

where T_1 and T_2 are $2k$-ended tangles, and $(\Sigma, 2k)$ is a $2k$-punctured surface; see Figure 1 for an example of decomposition of the trefoil knot $T(2, 3)$ along a four-puncture sphere $(S^2, 4\text{pt})$. Frequently, decomposition $(*)$ needs to satisfy special properties, for example being a Heegaard splitting for Y. Next, the main idea is to associate to the $2k$-punctured surface a certain symplectic manifold $\mathcal{M}(\Sigma, 2k)$, and to associate to the two parts (Y_i, T_i) of decomposition $(*)$ two Lagrangian submanifolds L_i inside the symplectic manifold:

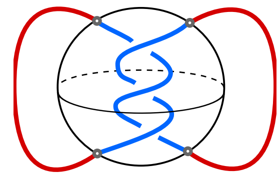


Figure 1

$$L_1 \to \mathcal{M}(\Sigma, 2k) \leftarrow L_2$$

Then sometimes, in favorable circumstances, the *Lagrangian Floer homology* $HF(L_1, L_2)$ (see the next page for the definition) is in fact a topological invariant of (Y, K), i.e. does not depend on decomposition $(*)$ and other choices. This method of constructing invariants was pioneered by Ozsváth and Szabó: they used the symplectic manifold $\mathcal{M}(\Sigma_g) = Sym^g(\Sigma_g)$ to define a 3-manifold invariant called *Heegaard Floer homology* [8]. They, and independently Rasmussen, also extended the construction to knots, resulting in an invariant called *knot Floer homology* [7, 9]. These invariants are extraordinarily powerful, and much of the contemporary research focuses on studying these and other similarly constructed symplectic geometric invariants of (Y, K).

Immersed curves and Khovanov homology. Currently, my research is centered around *Khovanov homology*, a homological knot invariant discovered by Khovanov [5], taking the form of a bigraded vector space $Kh^{h,q}(K)$. It is defined algebraically, and so it is natural to ask whether the strategy above applies to this invariant. In a joint work with Liam Watson and Claudius Zibrowius [6], we obtained the following symplectic geometric interpretation of Khovanov homology.

Theorem 1. *For any 4-ended tangle (D^3, T) there exist three immersed curves*[1]

$$\widetilde{BN}(T), \widetilde{Kh}(T), Kh(T) \looparrowright (S^2, 4pt) = \partial(D^3, T)$$

whose homotopy classes are tangle invariants of T. Moreover, let $(S^3, K) = (D^3, T_1) \cup_{(S^2,4pt)} (D^3, T_2)$ be a decomposition of a knot $K \subset S^3$ into two 4-ended tangles. Then reduced $(\widetilde{Kh})$ and full (Kh) Khovanov homology of the knot K, as well as reduced Bar-Natan's deformation $(\widetilde{BN})$ of Khovanov homology, are isomorphic to Lagrangian Floer homology:

$$\widetilde{BN}(K) \cong HF(\widetilde{BN}(mT_1), \widetilde{BN}(T_2))$$
$$\widetilde{Kh}(K) \cong HF(\widetilde{BN}(mT_1), \widetilde{Kh}(T_2))$$
$$Kh(K) \cong HF(\widetilde{BN}(mT_1), Kh(T_2))$$

where mT_1 is the mirror tangle.

For the depicted in Figure 1 decomposition of the trefoil $T_1 \cup_{(S^2,4\text{pt})} T_2$, the Lagrangian Floer intersection picture resulting in reduced Khovanov homology is illustrated in Figure 2. This theorem is central in our research, allowing us to study Khovanov homology from a new angle. To highlight one implication, we proved that Rasmussen's s-invariant is preserved under mutation. Further applications are currently being developed.

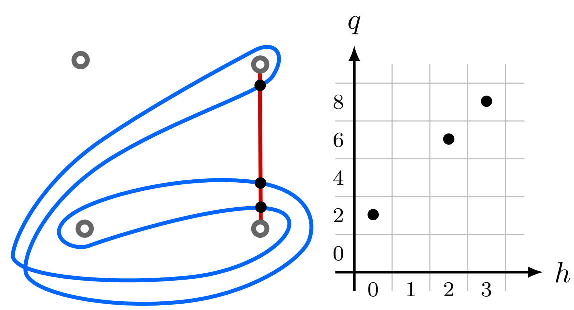
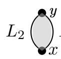


Figure 2. $HF(\widetilde{BN}(\mathrm{m}T_1), \widetilde{Kh}(T_2)) \cong \widetilde{Kh}(T(2,3)) \cong \mathbb{F}^3$

[1]In this context, immersed curves always come with a choice of a local system. For the purposes of illustration it is not important, and so we sweep this detail under the rug.

Lagrangian Floer theory and the Fukaya category. Omitting a large amount of details and certain technical conditions [2], Lagrangian Floer homology $HF(L_1, L_2)$ is a homological invariant of two Lagrangian submanifolds inside a symplectic manifold $L_1, L_2 \hookrightarrow (M, \omega)$, which is invariant under Hamiltonian isotopies of each Lagrangian. The underlying chain complex is generated by intersections: $CF(L_1, L_2) = \langle L_1 \cap L_2 \rangle_{\mathbb{F}}$ (transversality can be achieved by a small generic Hamiltonian perturbation). After fixing a generic compatible with ω almost complex structure on M, the differential $\partial : CF(L_1, L_2) \to CF(L_1, L_2)$ is defined by counting rigid pseudo-holomorphic discs between the intersection points, with Lagrangian boundary conditions on L_1 and L_2:



contributes ± 1 into coefficient c_{xy} in $\partial(x) = \sum_y c_{xy} \cdot y$

In the relevant for Theorem 1 case $\dim(M) = 2$, Lagrangians are curves on a surface, and counting rigid pseudo-holomorphic discs is equivalent to counting immersed discs with convex angles at intersections. As a result, the construction is easily generalized to *immersed* curves, and the dimension of Lagrangian Floer homology is almost always equal to the minimal intersection number of curves: $\dim HF(L_1, L_2) = \min \#(L_1 \cap L_2)$. An example of minimal intersection number 3 is depicted in Figure 2.

Another important symplectic geometric invariant, instrumental in the proof of Theorem 1, is the *Fukaya category* of a symplectic manifold $\mathcal{F}(M)$ [3, 4, 10] (see [1] for a survey). It is a unified structure, which captures how all Lagrangians intersect with each other. The objects in the Fukaya category are all Lagrangians $L_i \to M$, and morphism spaces are Lagrangian Floer complexes $CF(L_i, L_j)$. The composition in this category is defined by counting pseudo-holomorphic triangles:


$: CF(L_i, L_j) \otimes CF(L_j, L_k) \to CF(L_i, L_k)$

The composition is not associative on the nose, but is associative up to homotopy, which is given by counting pseudo-holomorphic rectangles. As such, $\mathcal{F}(M)$ is not a regular category, but rather is an A_∞ category, where higher operations are defined by counting pseudo-holomorphic polygons.

In the case $\dim(M) = 2$, where Lagrangians are curves on the surface M, the Fukaya category is similar to a curve complex, only it captures more information: minimal intersection numbers between the curves, and also all the immersed convex polygons with boundary on multiple curves.

References

[1] D. Auroux. A beginner's introduction to Fukaya categories. In *Contact and symplectic topology*, volume 26 of *Bolyai Soc. Math. Stud.*, pages 85–136. János Bolyai Math. Soc., Budapest, 2014. (ArXiv: 1301.7056).

[2] A. Floer. Morse theory for Lagrangian intersections. *J. Differential Geom.*, 28(3):513–547, 1988.

[3] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[4] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[5] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000. (ArXiv: math/9908171).

[6] A. Kotelskiy, L. Watson, and C. Zibrowius. Immersed curves in Khovanov homology. 2019. ArXiv preprint 1910.14584.

[7] P. S. Ozsváth and Z. Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004. (ArXiv: math/0209056).

[8] P. S. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004. (ArXiv: math/0101206).

[9] J. Rasmussen. *Floer homology and knot complements*. ProQuest LLC, Ann Arbor, MI, 2003. Thesis (Ph.D.)–Harvard University, (ArXiv: math/0306378).

[10] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.