1.1 **Function** is a rule that assigns to every real number x a number $f(x)$.

ex. $f(x) = x$ $f(x) = x^2$ $f(x) = \sqrt{x}$

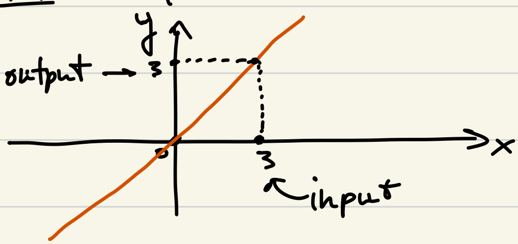
$f(x) = x$	$f(x) = x^2$	$f(x) = \sqrt{x}$
input: x	input: x	input: x
output: x	output: x^2	output: $\sqrt{x}$
$f(4) = 4$	$f(4) = 16$	$f(4) = 2$

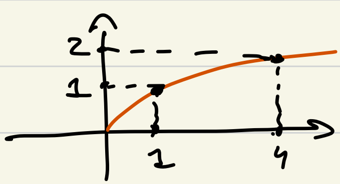
$$\boxed{f(\text{input}) = \text{output}}$$

Domain of a function $f(x)$ is the set of x-values where $f(x)$ is defined

Range of a function $f(x)$ is the set of y-values $f(x)$ takes

ex. $f(x) = x$ $f(x) = \sqrt{x}$





Domain: all numbers $(-\infty, +\infty)$
Range: all numbers $(-\infty, +\infty)$

D: $[0, +\infty)$ (non-negative numbers
Because $\sqrt{\ }$ of negative

numbers doesn't exist!

$\sqrt{-4}$ doesn't exist $\Rightarrow$ $\sqrt{x}$ is defined only for $x \geq 0$

that is <u>Domain: $[0,+\infty)$</u>

Range: $[0,+\infty)$ (because $\sqrt{x}$ is always positive!)

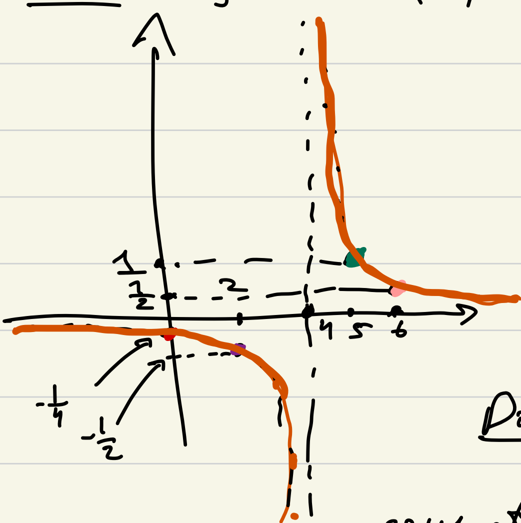
<u>Remark</u> • $\boxed{x^2=4}$ has two roots! $\quad x^2-4=0 \quad (x-2)(x+2)=0 \quad x=2,\ x=-2$

$\boxed{\text{But}}$ • $\boxed{\sqrt{4}=2}$, and -2 is not included

← convention that $\sqrt{x}$ is always ≥ 0

def. $\sqrt{x}$ means a number $a \geq 0$ such that $a^2 = x$

<u>ex.</u> $f(x) = \dfrac{1}{x-4}$



$f(0) = \dfrac{1}{-4}$

$f(2) = \dfrac{1}{-2}$

$f(4) = \dfrac{1}{0}$ undefined!

$f(5) = \dfrac{1}{5-4} = 1$

$f(6) = \dfrac{1}{2}$

<u>Domain</u> all numbers except 4
$(-\infty, 4) \cup (4, +\infty)$

← assymptote

<u>Range</u> all numbers except 0
$(-\infty, 0) \cup (0, +\infty)$

rmk

$f(x) = x+2$
$f(3) = 5$ ✓

Domain: all the numbers that you can input in $f(x)$

$f(7)=9$ ✓ $D: (-\infty, +\infty)$ (all numbers)

Domain = all possible inputs
Range = all possible outputs

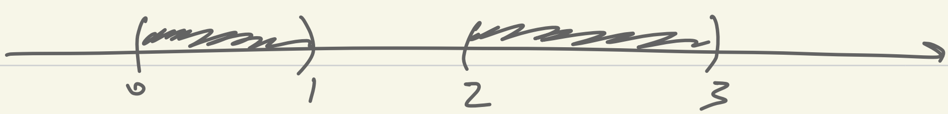
interval notation

- (a,b) = all numbers r such that $a<r<b$

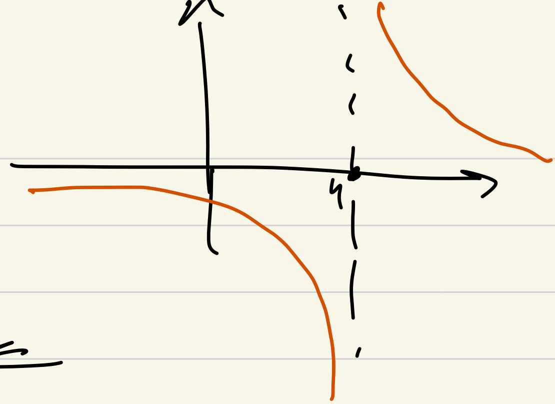
- $[a,b]$ = the same, but includes a & b, that is all numbers r such that $a \le r \le b$

- $(a,b]$ = only b is included, that is all numbers r such that $a<r\le b$

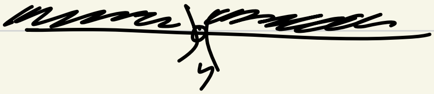
- unions: $(0,1) \cup (2,3)$



$f(x) = \dfrac{1}{x-4}$



Domain:



$(-\infty, 4) \cup (4, +\infty)$

Range:

$(-\infty, 0) \cup (0, +\infty)$

Q. Given a function $f(x) =$ formula, how do we find its domain?

Step 1 identify bad x-values for $f(x)$, that is those numbers where $f(x)$ is undefined

a. division by zero

b. negatives inside square root

c. negatives or zero in logarithm

($\log_{10} x$ is defined only if $x > 0$)

sources of bad values

Step 2 Domain is all the good values, that is everything except bad values (throw out bad values) ex. $f(x) = \underbrace{2 - x^2}_{\text{no bad } x\text{-values}} + \underbrace{\frac{1}{x}}_{\text{a. bad value } x=0} - \underbrace{\sqrt{4-x}}_{\text{b. } 4-x<0}$

b. $4-x<0$

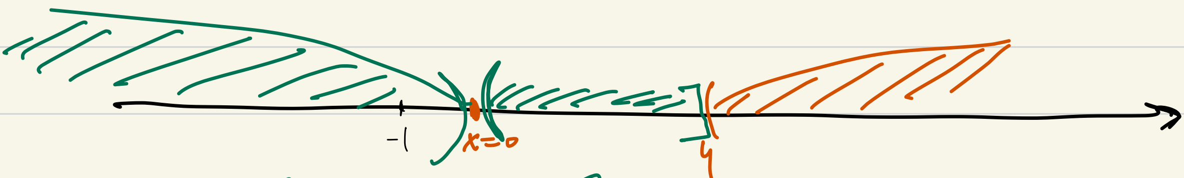
$4<x$

bad values because $4-x<0$ for those x

Domain = ?

step 1 $\boxed{\begin{array}{l}\text{bad values} \\ x=0 \\ x>4\end{array}}$ →

step 2 all good values:



bad values (where f(x) is undefined)

good values

$(-\infty, 0) \cup (0, 4]$ ← Domain

this type of bracket tells that 4 is included

this type of bracket tells that 0 is excluded

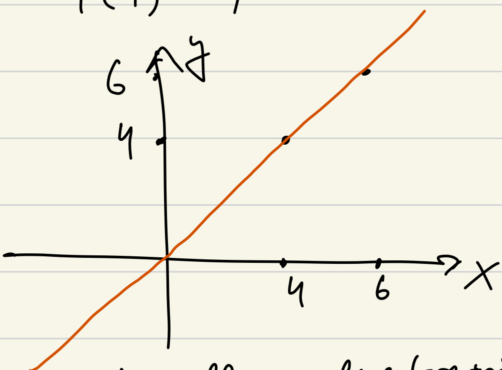
1.1. Function is a rule that assigns to every number x a number $f(x)$

<u>ex.</u> $f(x) = x$

input $= x$

output $= x$

$f(4) = 4$



Domain: all numbers $(-\infty, +\infty)$

Range: all numbers $(-\infty, +\infty)$

$f(x) = x^2$

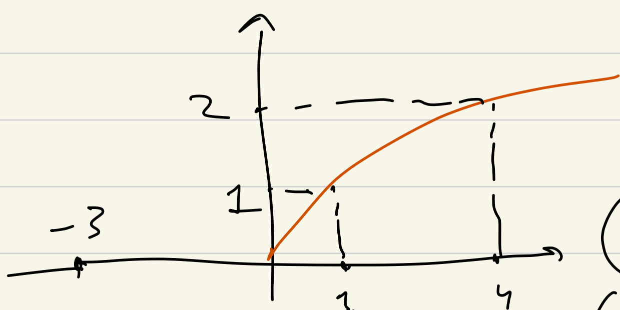
$f(4) = 16$

Domain: all numbers

Range: $y \geq 0$ (because $x^2 \geq 0$)

$f(x) = \sqrt{x}$ (number $a \geq 0$ such that $a^2 = x$)

$\sqrt{4} = 2$



Range: $y \geq 0$

($\sqrt{x}$ always ≥ 0)

Domain: $x \geq 0$ (Because $\sqrt{x}$ is defined only for $x \geq 0$)

Remark Two important facts:

1) $\sqrt{\cdot}$ is always non-negative (convention)

$\boxed{\sqrt{4}=2}$, not -2!

2) equation $x^2=4$ has two roots

$$x^2-4=0$$
$$(x-2)(x+2)=0$$
$$\underbrace{x=2,\ x=-2}$$

3) related fact $\sqrt{x^2}=|x|$ (not x!)

think about it

Domain of a function $f(x)$ = all possible inputs

(the set of x-values where $f(x)$ is defined)

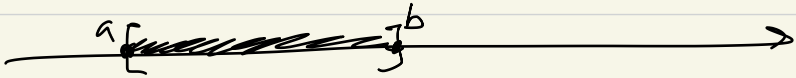
Range of a function $f(x)$ = all possible outputs

(the set of y-values that $f(x)$ takes)

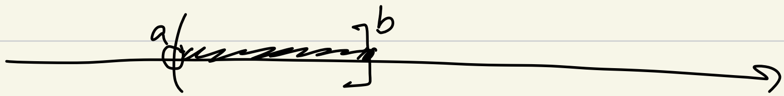
Interval notation

- (a,b) = all numbers r such that $a<r<b$

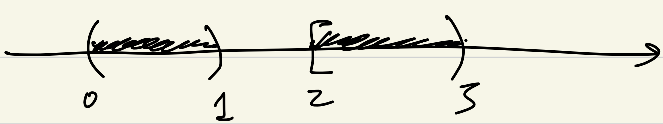
- $[a,b]$ = all numbers r such that $a\leq r\leq b$
  (endpoints included!)



- $(a,b]$ = all numbers r such that $a<r\leq b$
  (only b is included!)



- Unions: $(0,1)\cup[2,3)$ = union of intervals

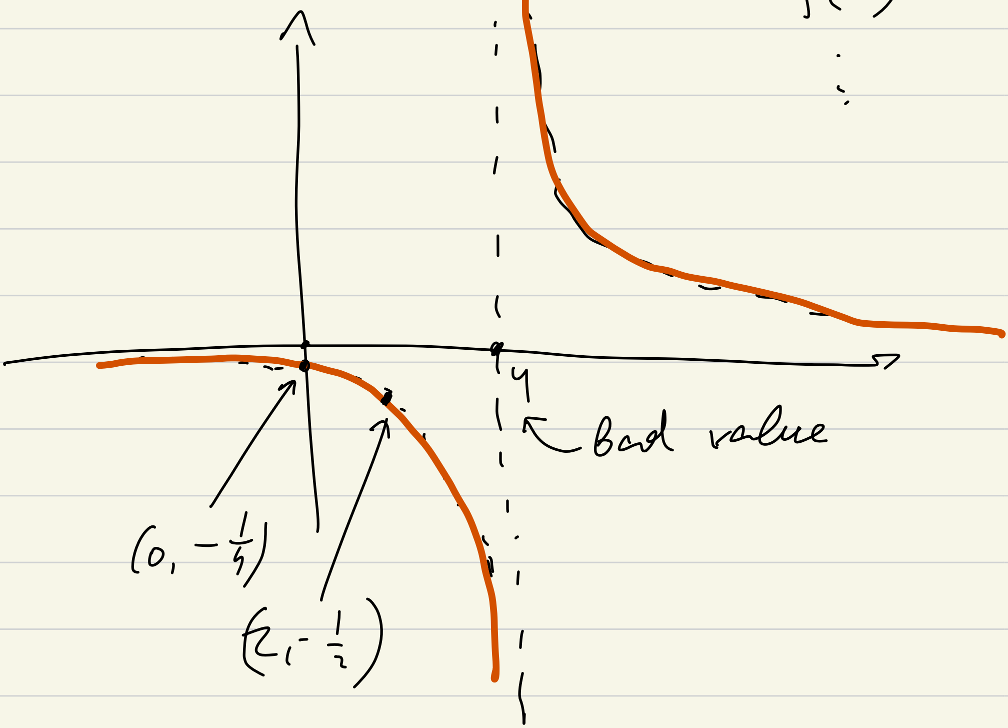


ex. $f(x) = \dfrac{1}{x-4}$

$f(0) = \dfrac{1}{-4}$

$f(2) = -\dfrac{1}{2}$

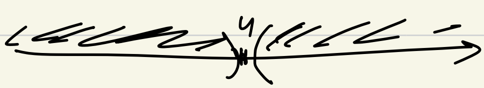
$\vdots$



Domain: $\dfrac{1}{x-4}$ is defined everywhere except $x = 4$. (we cannot divide by 0)

Thus

Domain $(-\infty, 4) \cup (4, +\infty)$



Range $\dfrac{1}{x-4}$ attains all numbers except 0

Thus $(-\infty, 0) \cup (0, +\infty)$

Q. How to find domain of $f(x)$?

Step 1 Identify bad x-values

(that is those x-values where $f(x)$ is undefined)

a. Division by zero $\left(\frac{1}{a} \Rightarrow a \neq 0\right)$

b. Negatives in square roots $\left(\sqrt{a} \Rightarrow a \geq 0\right)$

c. Negatives or zero in logarithms $\left(\log_{10} a \Rightarrow a > 0\right)$

"therefore"

Step 2 Domain is the complement of bad values.

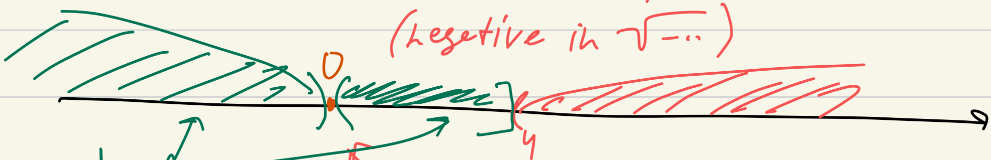
ex. $f(x) = 2 - x^2 + \frac{1}{x} + \sqrt{4-x}$

Domain?

Step 1 Bad values: $x=0$ (division by 0)

$4 - x < 0$, or $(4 < x)$

(negative in $\sqrt{-\ldots}$)



good values

"

Domain

$(-\infty, 0) \cup (0, 4]$

bad values

ex. $f(x) = \underbrace{\frac{1}{x-3}}_{} + \underbrace{\sqrt{x}}_{} + \underbrace{\sqrt{6-x}}_{}$

bad values: $\div$ by 0 when $x=3$ — neg. inside $\sqrt{\cdot}$ when $x<0$ — neg. in $\sqrt{\cdot}$ when $6-x<0$, $6<x$

bad values

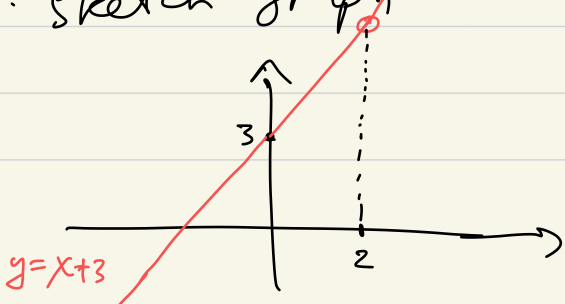
good x-values

Domain $= [0,3) \cup (3,6]$

ex. $f(x) = \dfrac{x^2+x-6}{x-2}$

- find domain : $(-\infty,2)\cup(2,\infty)$
- sketch graph

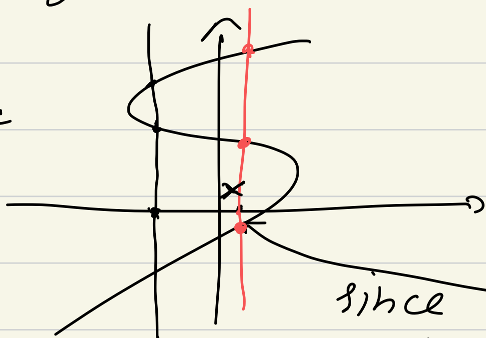
$= \dfrac{\cancel{(x-2)}(x+3)}{\cancel{x-2}} = x+3$



1.2 functions to know

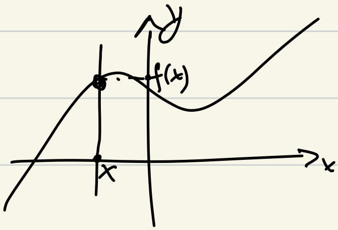
Q. I a curve is drawn on a plane, how do we know if this curve can be a graph of some function $f(x)$?

ex.



cannot be a graph of a function, since there is such x, that there is more then one "$f(x)$"

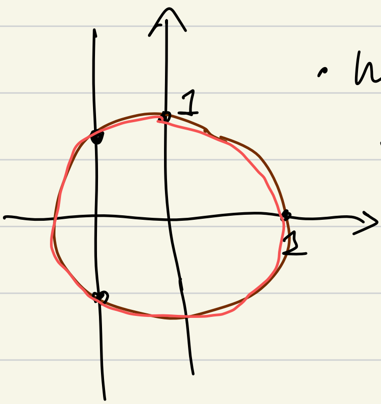
ex.



can be a graph of a functions, because for every x there is at most one $f(x)$

__Vertical line test__ curve can be a graph of function $f(x)$ only if every vertical line intersects the curve at most once

__ex.__



- not a function!
- __can__ be represented by an equation

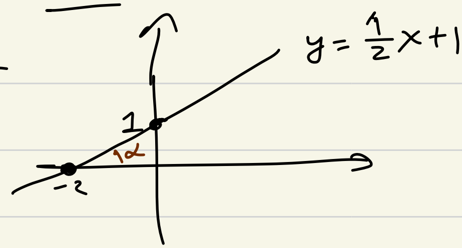
$$\underbrace{x^2+y^2-1}_{f(x,y)}=0$$

__functions__	__equations__
$y=f(x)$	$f(x,y)=0$

difference between equations and functions

Function to know

ex.



① linear functions

$$y = m\cdot x + b$$

$\tan(\alpha)$ = slope; y-intercept

Domain: $(-\infty, \infty)$

<u>Remark</u> all symbols can be called differently

<u>ex.</u> $y=f(x)$, but also could be $a=f(h)$

② (power functions) $y=x^n$, n is a positive integer $(1,2,3,4,\dots)$

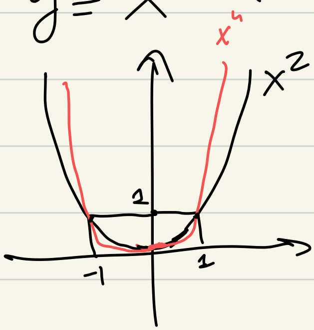
(<u>$2k=n$ even</u>)

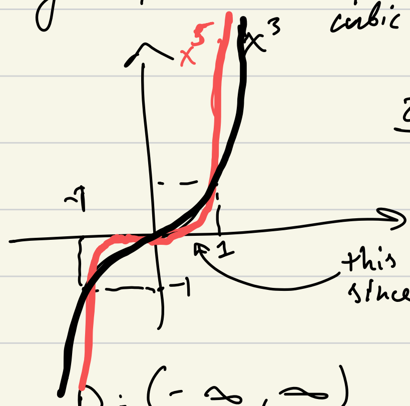
$$y=x^{2k} \quad (x^2, x^4, x^6, \dots)$$

(<u>$2k+1=n$ odd</u>)

$$y=x^{2k+1} \quad (x^3, x^5, x^7, \dots)$$

cubic (for x^3)





<u>$2^5 > 2^3$</u>

this behaviour since $\left(\frac{1}{2}\right)^5 < \left(\frac{1}{2}\right)^3$

D: $(-\infty, \infty)$

R: $[0, \infty)$

Q. $0 = x^2$
$0 = 0^2 \Rightarrow 0$ is in R

D: $(-\infty, \infty)$

R: $(-\infty, \infty)$

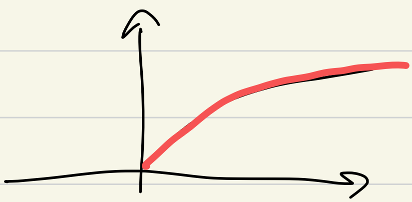
③ **polynomials:** sums of power functions with coefficients

ex. $y = 6x^3 + 7x^2 - 5x + 3$

D: $(-\infty, \infty)$ graph: more complicated

④ **roots** $y = \sqrt[n]{x} = x^{\frac{1}{n}}$ (number a such that $a^n = x$)

n even

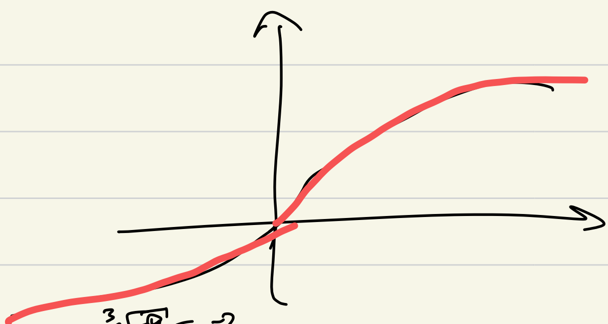


$\sqrt{-8}$ doesn't exist!

D: $[0, +\infty)$

R: $[0, +\infty)$

n odd



$\sqrt[3]{-8} = -2$

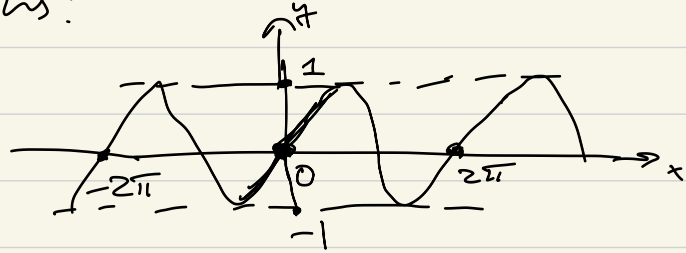
D: $(-\infty, +\infty)$

R: $(-\infty, +\infty)$

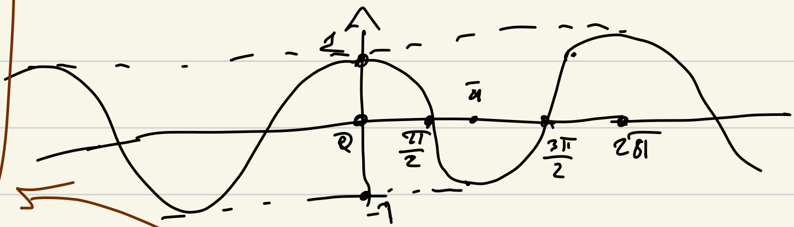
(Range = all possible outputs
Domain = all possible inputs)

⑤ trig functions:

- $\sin(x)$



- $\cos(x)$

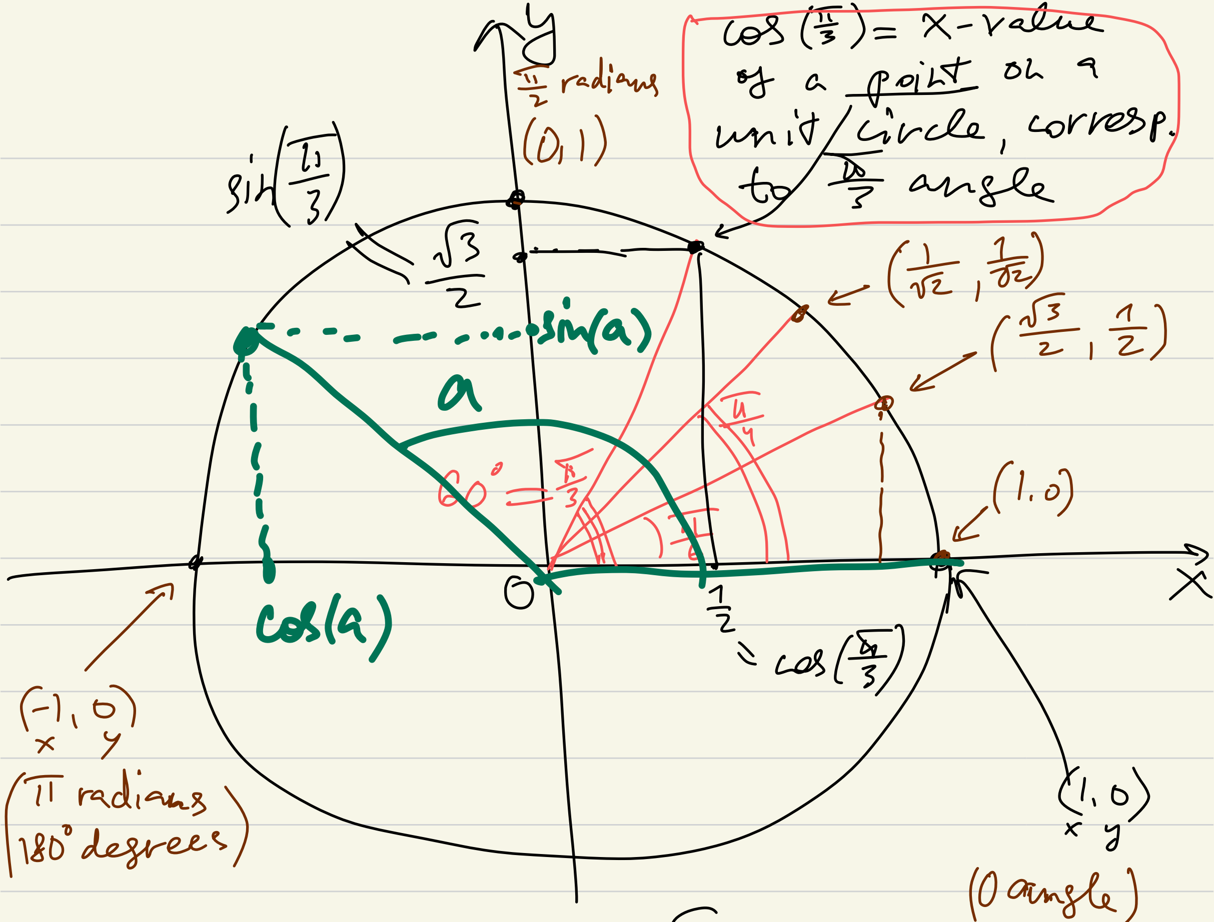


- $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$
- $\sec(x) = \dfrac{1}{\cos(x)}$
- $\csc(x) = \dfrac{1}{\sin(x)}$
- $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$

HOW TO DEFINE?

↓

UNIT CIRCLE



$$\underset{\text{x-value}}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{2}, \quad \underset{\text{y-value}}{\sin\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{2}$$

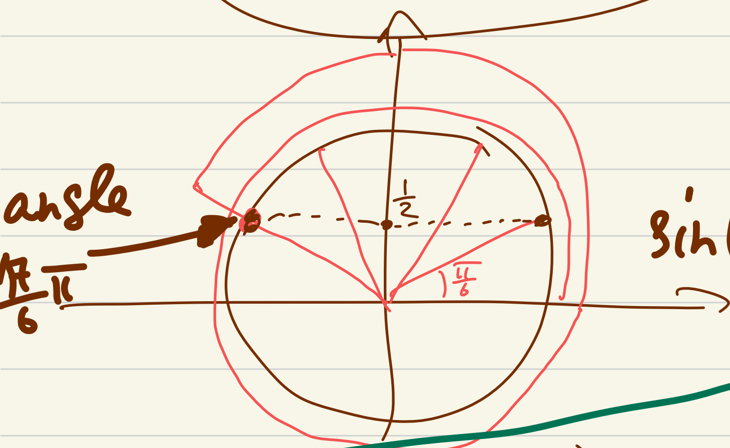
<u>need to remember</u>

- cos(–) and sin(–) of memorize $\boxed{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}}$
- be able to deduce cos(–) and sin(–) of multiples of these

ex. $\cos\left(-\frac{40}{3}\pi\right) = ?$

ex. $\sin\left(\frac{17}{6}\cdot\pi\right)=\frac{1}{2}$

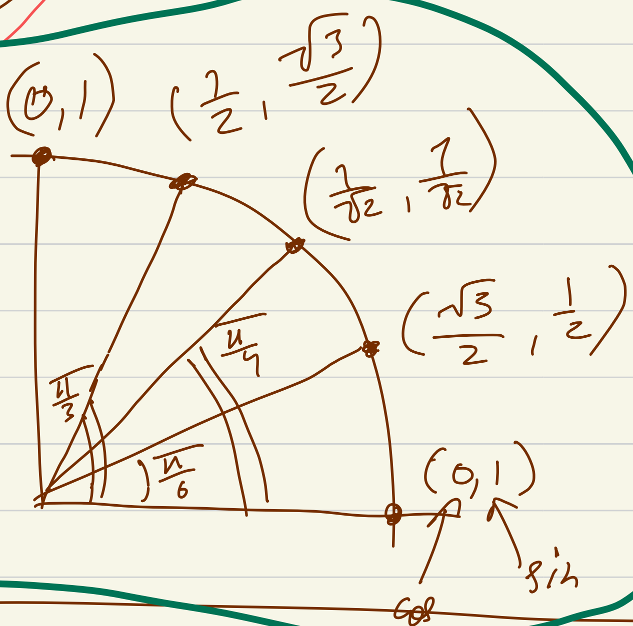
$\frac{17}{6}\pi = 2\frac{5}{6}\pi =$
$= \boxed{2\pi + 5\cdot\frac{\pi}{6}}$
(π = half circle)
(2π = full circle)



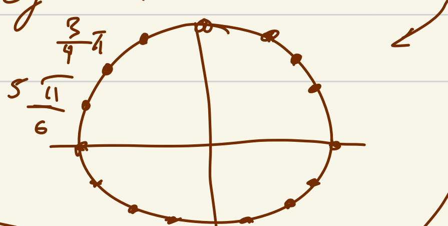
$\sin\left(\frac{17}{6}\pi\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

know
(1) By hard

$\sin(0)=0,\ \cos(0)=1$
$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2},\ \cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$
$\sin\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$
$\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2},\ \cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$
$\sin\left(\frac{\pi}{2}\right)=1,\ \cos\left(\frac{\pi}{2}\right)=0$



(2) Be able to deduce sin and cos of all these



ex. $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$

$\cos(\pi + x) = -\cos(x)$

$\vdots$

need to remember / know / be able to deduce

ex.

$$\frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin(x)} = \frac{\sin x}{\sin x} = 1$$

---

(6) Piecewise functions way to use more than one rule to define $f(x)$.

ex. $f(x) = \begin{cases} x, & \underline{x \geq 0} \\ -x, & \underline{x < 0} \end{cases}$

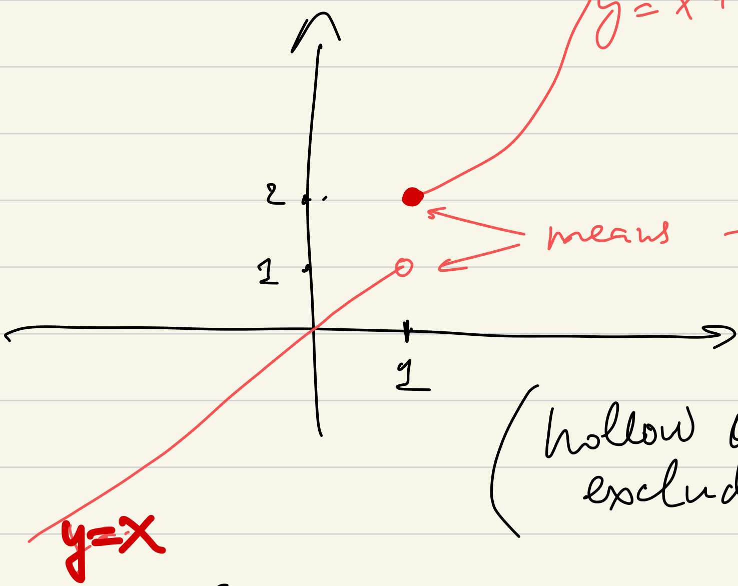
domains where the rule changes

" $|x|$ absolute value

ex. $f(x) = \begin{cases} x^2+1, & x \geq 1 \\ x, & x < 1 \end{cases}$

use this rule

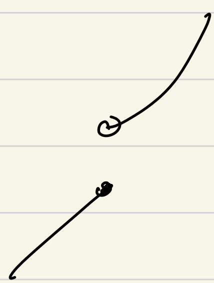
$f(1) = 1^2+1 = 2$



means $f(1) = 1^2+1 = 2$

(hollow circle excludes the end)

$\begin{cases} x^2+1, & x > 1 \\ x, & x \leq 1 \end{cases}$

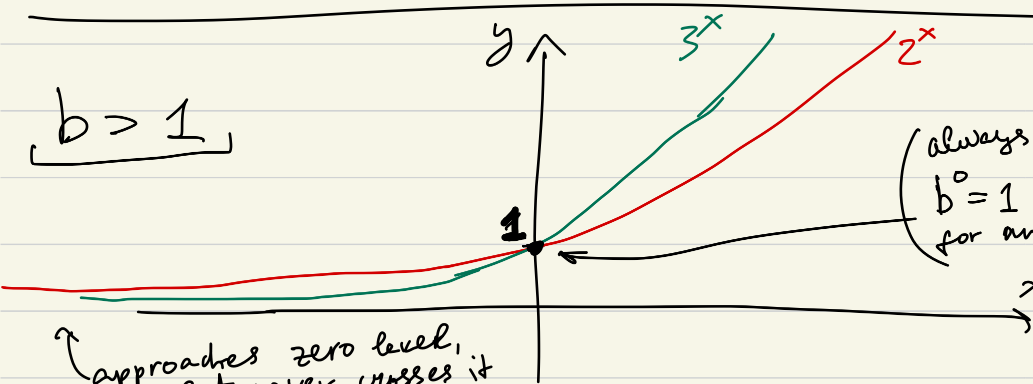


[1.4] (power functions: $y = x^b$)

· Exponential functions $\quad \boxed{y = b^x}$

ex. $y = 2^x$ $\qquad$ Base $\rightarrow$ $b > 0$

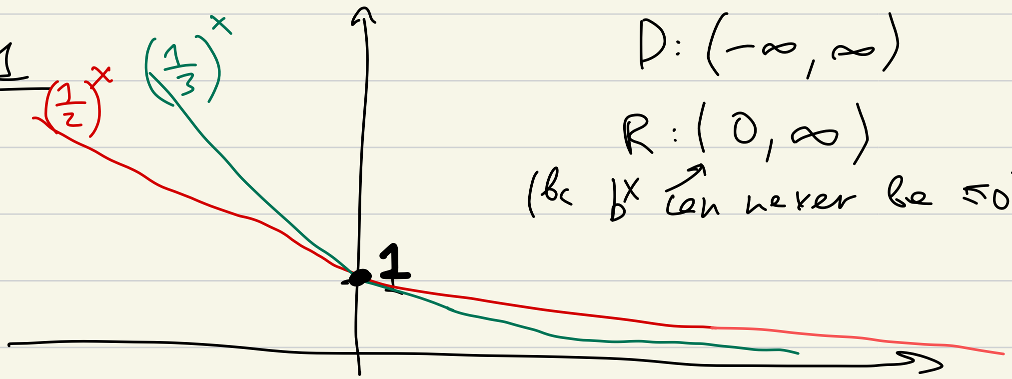
$b > 1$



(always have $b^0 = 1$ for any b)

approaches zero level, but never crosses it

$b < 1$



D: $(-\infty, \infty)$

R: $(0, \infty)$

(bc b^x can never be ≤ 0)

properties

1. $b^x \cdot b^y = b^{x+y}$

2. $(b^x)^y = b^{x \cdot y}$

3. $(ab)^x = a^x \cdot b^x$

4. $b^{-x} = \frac{1}{b^x}$

5. $\frac{b^x}{b^y} \underset{\text{by 4.}}{=} b^x b^{-y} \underset{\text{by 1.}}{=} b^{x-y}$

<u>Euler's number</u> $= \boxed{e \approx 2.718\ldots}$

(it is defined by the property $(e^x)' = e^x$)

↑ derivative

- $2 < e < 3$

$y = e^x$ will be a frequently used function for us

<u>ex.</u> $f(x) = \dfrac{1}{e^x}$ Domain $= ?$ $\left(\dfrac{1}{e^x} = e^{-x} \right.$ graph of e^{-x} is graph of e^x flipped about y-axis $\left. \right)$

$e^x = 0$ no solutions, therefore no bad value, so $D: (-\infty, \infty)$

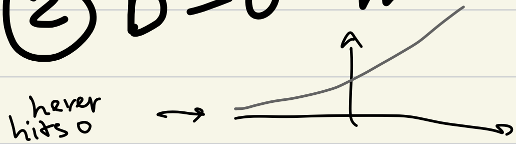
<u>ex.</u> $f(x) = \dfrac{1}{1-e^x}$, domain $= ?$

$1 - e^x = 0$

$1 = e^x$ so $x = 0$, so $x=0$ bad value, So <u>$D: (-\infty, 0) \cup (0, \infty)$</u>

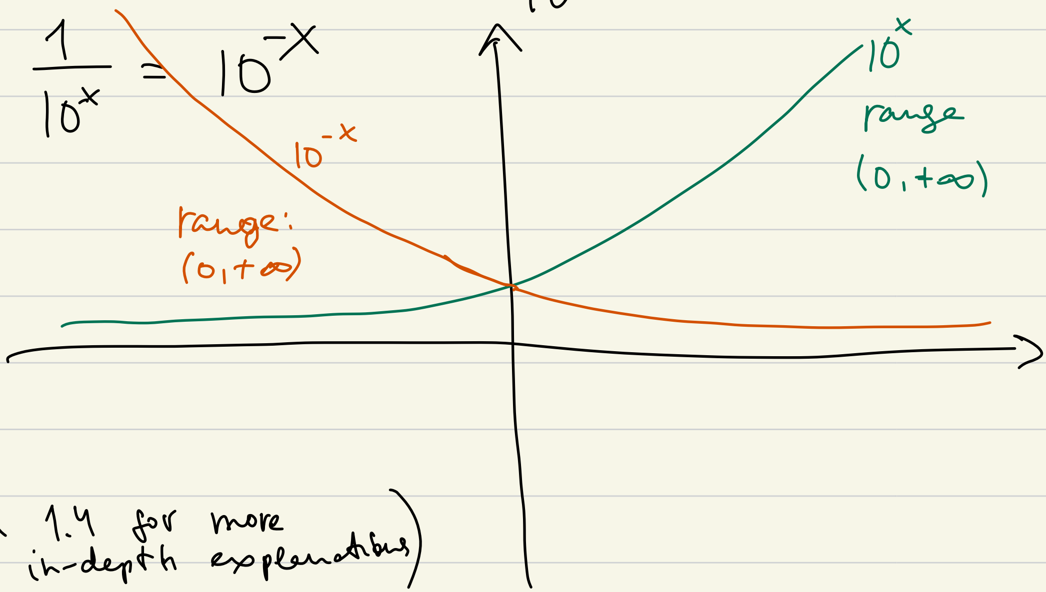
rmk ① $b^x = 1$ equivalent to $x = 0$

② $b^x = 0$ has no solutions!



ex. range of $y = \frac{1}{10^x}$?

$y = \frac{1}{10^x} = 10^{-x}$



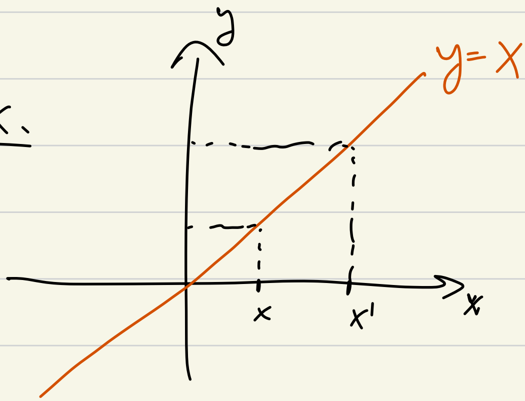
(read 1.4 for more in-depth explanations)

$\boxed{1.5}$ Inverse functions

One-to-one function is

a function that never takes the same value twice

ex.

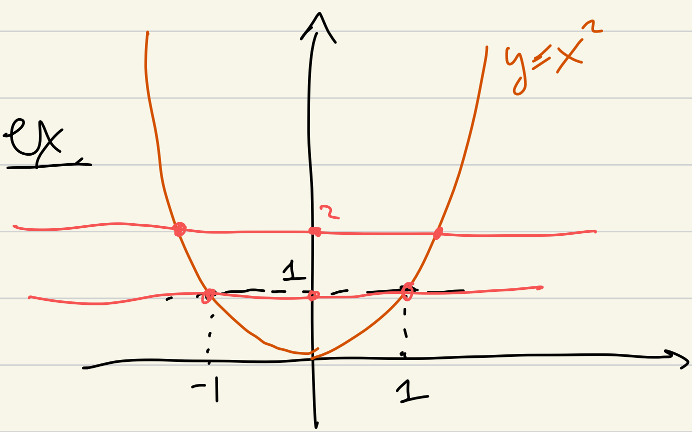


one-to-one because

if $x \neq x'$, then
$f(x) \neq f(x')$

In other words, any horizontal line hits the graph at at most one point

ex.



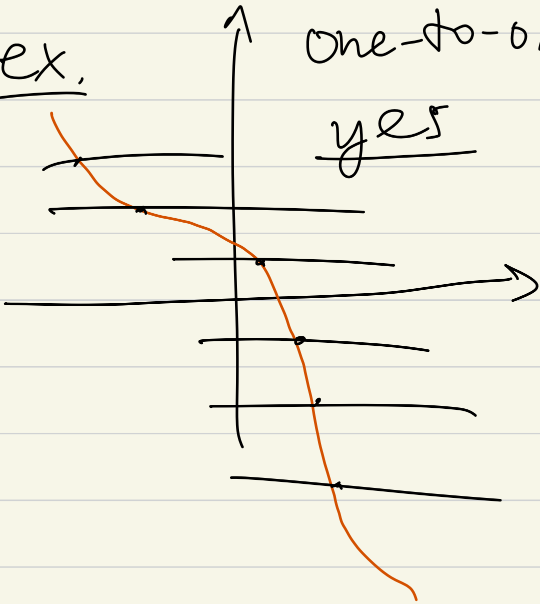
not one-to-one

$f(1) = f(-1) = 1$,
and so the same value is attained twice.

In other words, there are horizontal

lines that hit the graph at two or more points.

ex.

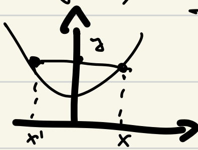


Horizontal line test

A function is one-to-one if every horizontal line hits the graph at one or zero points.

Inverse function. Given one-to-one function $f(x)$, inverse function $f^{-1}(x)$ is defined by the rule that if $f(x)=y$, then $f^{-1}(y)=x$

NOTE for inverse to exist $f(x)$ must be one-to-one

(Why?  $f^{-1}(y) = x$? $= x'$? cannot decide)

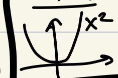
ex. if $f(x)$ is one-to-one and $f(2)=8$, then $f^{-1}(8)=2$

ex. if $f(x)=\sqrt{x}$ $\left(f(1)=1,\ f(4)=2,\ f(9)=3 \ldots \right)$

then $f^{-1}(x)=x^2$ $\left(f^{-1}(1)=1,\ f^{-1}(2)=4,\ f^{-1}(3)=9 \right)$

(because $(\sqrt{x})^2 = x$)

ex. $f(x)=x^2$, what is $f^{-1}(x)=?$



It doesn't exist! Since x^2 is not one-to-one

In other words, inverse function is a function which "reverts" the effect

$f(x)=x^5$, $f(2)=32$ $\qquad$ $f^{-1}(x)=\sqrt[5]{x}$, $f^{-1}(32)=2$

How to find inverse function

1. Write your function as $y=f(x)$
2. Change all y's to x's and all x's to y's
3. Solve for new y

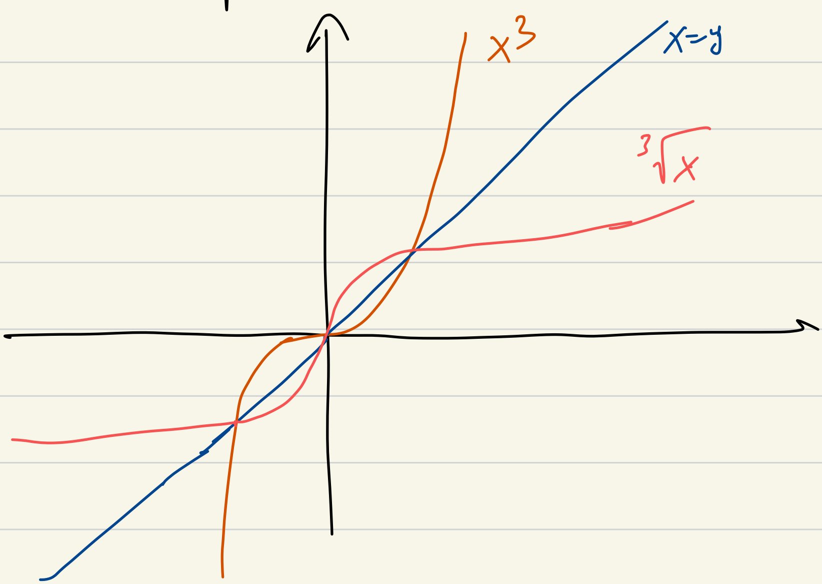
ex. $f(x)=x^3$, find $f^{-1}(x)=?$

1. $y=x^3$

2. $x=y^3$

3. $\sqrt[3]{x}=y$ $\qquad f^{-1}(x)=\sqrt[3]{x}$

What happens to the graph?
It flips about $x=y$ line!



This always happens!

Graph of $f^{-1}(x)$ is the same as for $f(x)$ but flipped about $x=y$ line

Important note:

1) Domain of $f^{-1}(x)$ = Range of $f(x)$
Range of $f^{-1}(x)$ = Domain of $f(x)$

Warnings

1) To find $f^{-1}(x)$, the function $f(x)$ has to be one-to-one.

Otherwise $f^{-1}(x)$ doesn't exist

2) $f^{-1}(x)$ is <u>not</u> $\frac{1}{f(x)}$

<u>Example</u> $f(x) = 1 + \sqrt{2+3x}$. Find the inverse, if it exists.

<u>Q.</u> Does $f^{-1}(x)$ exist? Yes, if $f(x)$ is one-to-one. How do we check if $f(x)$ is one-to-one? You just try to find $f^{-1}(x)$, and if you cannot, that means that $f(x)$ was not one-to-one from the start.

1. $y = 1 + \sqrt{2+3x}$

2. $x = 1 + \sqrt{2+3y}$

3. Solve for y.

$$x-1 = \sqrt{2+3y}$$

$$(x-1)^2 = 2+3y$$

$$(x-1)^2 - 2 = 3y$$

$$\frac{x^2-2x+\cancel{1}^{-1}\cancel{-2}}{3} = \frac{(x-1)^2-2}{3} = y$$

$$\boxed{\text{Answer} \quad f^{-1}(x) = \frac{x^2-2x-1}{3}}$$

ex. $f(x) = x^2$. Find inverse if it exists.

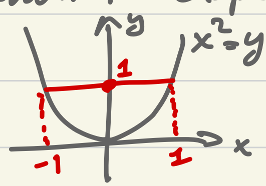
1. $y = x^2$
2. $x = y^2$
3. $\sqrt{x} = \sqrt{y^2}$

$$\underbrace{\sqrt{x} = |y|}$$

there is no single solution for y

so f^{-1} doesn't exist

another explanation



is clearly not one-to-one, and so f^{-1} doesn't exist!

Remark

$\sqrt{x^2} \neq x$!

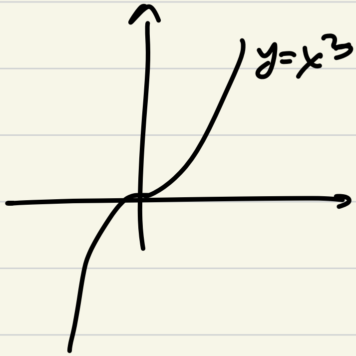
Instead $\sqrt{x^2} = |x|$!

example → $\sqrt{(-3)^2} = 3 = |-3|$

$(-3)^2 = 9$

definition $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

ex. $y = x^3$. Find an inverse if it exists



1. $y = x^3$
2. $x = y^3$
3. $\sqrt[3]{x} = y$

and so $f^{-1} = \sqrt[3]{x}$

one-to-one so f^{-1} exists!

definition $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ ("ignoring the minus sign")

Remark $\sqrt[3]{x^3} = x$

$\sqrt[2]{x^2} = |x|$ important points!

Because $\sqrt[2]{\cdot}$ has to be positive, and

so $\sqrt[2]{(-5)^2} = 5$

$\neq -5$

More generally: • $\sqrt[2k]{x^{2k}} = |x|$

• $\sqrt[2k+1]{x^{2k+1}} = x$. Therefore

• $y = x^{2k}$ do not have inverses

(k integer)

• $y = x^{2k+1}$ do have inverses.

Suppose $\sqrt[2]{x^2} = x$. Then

$\sqrt[2]{(-5)^2} = \sqrt{25} = -5$

cannot be true since $\sqrt[2]{\cdot} \geq 0$

ex.

$$x^2 = 4 \rightarrow \sqrt{x^2} = 2 \rightarrow |x| = 2$$

$$\downarrow$$

$x = 2$

$x = -2$

1.4. Logarithms

What is a logarithm?

Given exponential function $y = b^x$, $b>0$ constant, $b \neq 1$

then we call the inverse function of b^x

the logarithm of base b:

$$\boxed{\begin{aligned} f(x) &= b^x \\ f^{-1}(x) &= \log_b(x) \quad \text{logarithm} \end{aligned}}$$

remark. $a = \log_b(C)$ means that $b^a = C$

(by definition)

· In other words, $\log_b(c)$ is such a number a, that $b^a = c$

Fact $\log_b(x)$ is defined only if $x>0$, $b>0$, $b \neq 1$.

So, domain of $y = \log_b(x)$ is $x>0$, or $(0, \infty)$

ex. · $b=3$, $f(x) = 3^x$. Then $f^{-1}(x) = \log_3 x$

· $\log_3(9) = 2$ (because $3^2 = 9$)

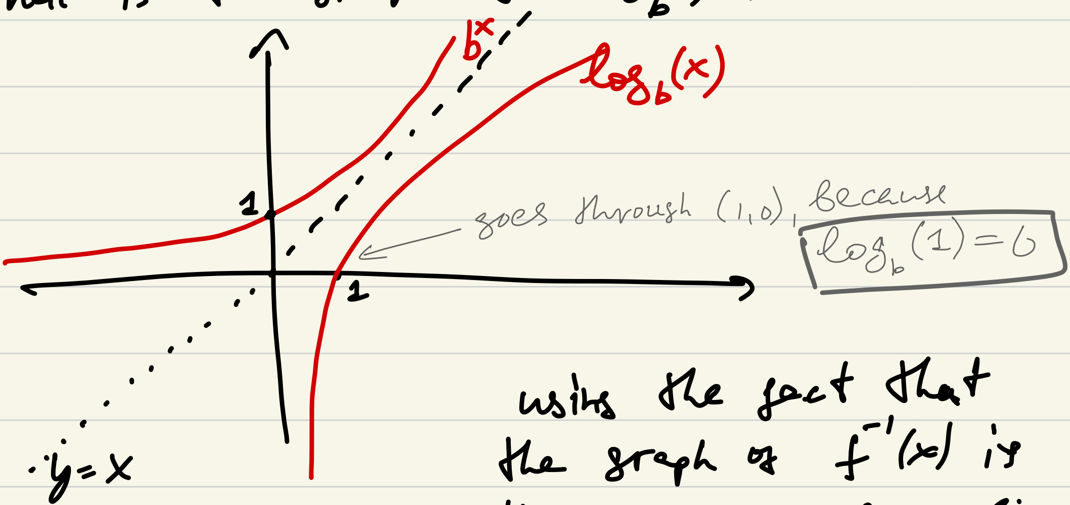
ex. $b = e$, then $f(x) = e^x$

$f^{-1}(x) = \log_e x$ ← has a special name and notation

Euler's number

. $\log_e x$ is called <u>natural logarithm</u>, and is denoted by $\boxed{\ln(x)} = \log_e(x)$

Q. What is the graph of $\log_b(x)$?



using the fact that the graph of $f^{-1}(x)$ is the graph of $f(x)$ flipped about $y=x$ line.

<u>Properties of logarithms</u> ("dual to properties of exponentials")

1. $b^{\log_b(x)} = x$ and $\log_b(b^x) = x$

(this is because b^x and $\log_b(x)$ are inverses of each other)

2. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$ (compare to $b^{x+y} = b^x \cdot b^y$)

3. $\log_b(x^p) = p \cdot \log_b(x)$ ← important property

4. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

. have a "cheat-sheet" with formules

ex. $e^{3 \cdot \ln(2)} = ?$

1way $e^{3 \cdot \ln(2)} = e^{\ln(2) \cdot 3} = \left(e^{\ln(2)}\right)^3 = 2^3 = 8$

↑ property of exponentials

2way $e^{3 \cdot \ln(2)} = e^{\ln(2^3)} = 2^3 = 8$

↑ property 3 of logarithms

($\ln(x) = \log_e(x)$, we will use this a lot)
↑ base e

Problem $f(x) = e^{3x-1}$ 1) Inverse?
2) Range?

1) $y = e^{3x-1}$

$x = e^{3y-1}$

(now we solve for y

$\ln(\cdot)$ "cancel" each other

$$\ln(x) = \ln\left(e^{3y-1}\right)$$

$$\ln(x) = 3y-1$$

$$\ln(x)+1 = 3y$$

$$\boxed{\frac{\ln(x)+1}{3} = y} \qquad \boxed{f^{-1}(x) = \frac{\ln(x)+1}{3}}$$

2) Range of $f(x) = e^{3x-1}$?

fact:

Range of $f(x)$ = Domain of $f^{-1}(x)$

And so, we just have to find Domain of $\frac{\ln(x)+1}{3} = f^{-1}(x)$

Domain:

1) Bad values: $x \leq 0$ b.c. $\ln(x)$

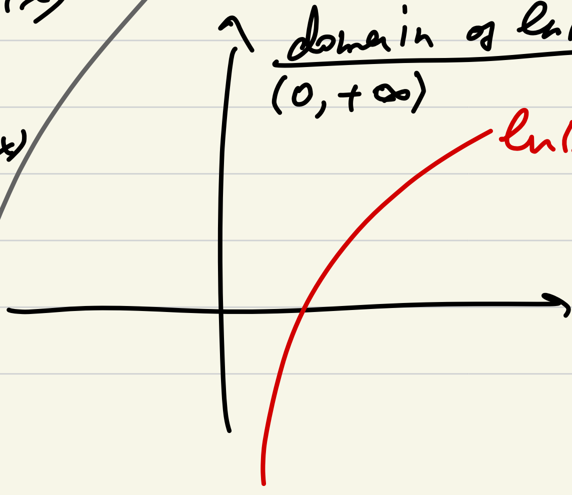
2) domain = good values

$$\boxed{(0, +\infty)}$$

remark:

domain of $\ln(x)$

$(0, +\infty)$



ex. $x = 3^y$ solve for y

$$\log_3(x) = \log_3(3^y)$$

$$\log_3(x) = y$$

Q. Is $\log_7(0)$ defined?

A. No, since $7^a = 0$ doesn't have solutions.

read 1.4

We need to be able to use the properties of log and exp. functions to simplify expressions.

Example
$$\frac{1}{27}\left(\frac{\sqrt[3]{9}}{e^{\frac{1}{2}\ln\frac{1}{9}}}\right)^6 =$$

$$e^{\frac{1}{2}\ln\frac{1}{9}} = e^{\ln\left(\frac{1}{9}\right)\cdot\frac{1}{2}} =$$
$$= \left(e^{\ln\frac{1}{9}}\right)^{\frac{1}{2}} = \left(\frac{1}{9}\right)^{\frac{1}{2}} =$$
$$= \sqrt[2]{\frac{1}{9}} = \frac{\sqrt[2]{1}}{\sqrt{9}} = \frac{1}{3}$$

$$= \frac{1}{27}\left(\frac{9^{\frac{1}{3}}}{\frac{1}{3}}\right)^6 =$$

$$= \frac{1}{27}\frac{\left(9^{\frac{1}{3}}\right)^6}{\left(\frac{1}{3}\right)^6} = \frac{1}{27}\frac{9^{\frac{1}{3}\cdot 6}}{\frac{1}{3^6}} = \frac{1}{27}9^2\cdot 3^6 =$$

$$= \frac{1}{3^3}\cdot 3^6\cdot 9^2 = 3^{6-3}\cdot 9^2 = 3^3\cdot 9^2 = 27\cdot 81 =$$

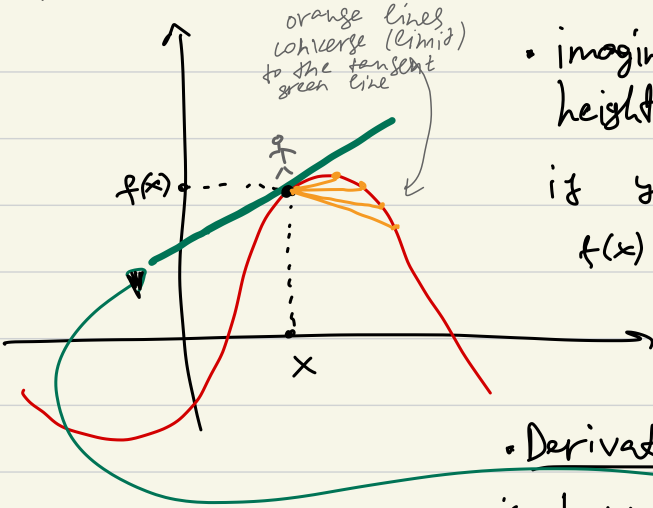
$$= 30\cdot 81 - 3\cdot 81 = 2430 - 243 = \boxed{2187}$$

2.2 Limits

Q. Why do we need a limit?

A. To define derivative. And so let me briefly say what derivative is, as a motivation:

Limit definition of derivative of f(x)



- imagine f(x) is modeling the height of a mountain, that is, if you stand at x then f(x) gives the hight

- <u>Derivative</u> of f at point x is how steep (slope) the mountain is at (x, f(x))

notation: $f'(x)$

- $f'(x)$ is the slope of green line = tangent line

<u>Q.</u> How to find the green line?

<u>Great idea</u> define the green (tangent) line as a limit of orange chords

Limits

<u>Main definition</u> (two-sided limit, "two-sided" is omitted frequently)

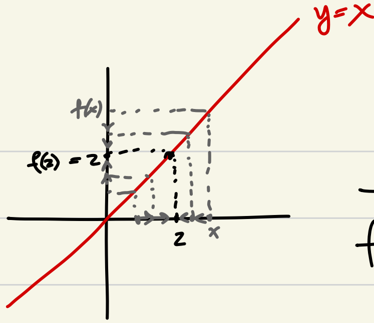
Given a function f(x) and a point=number a, the y-value that f(x) approaches when x-value approaches a is called the limit of f(x) at x=a:

$$\lim_{x \to a} f(x)$$

ex 1

$f(x) = x$

$\lim_{x \to 2} f(x) = 2$



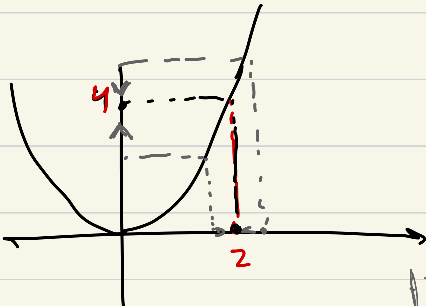
x	1.9	1.99	2.01	2.1
$f(x)$	1.9	1.99	2.01	2.1

2

$\lim_{x\to 2} f(x) = 2 = f(2)$

ex 2 $f(x) = x^2$

$\lim_{x \to 2} f(x) = 4$



x	1.9	1.99	2.01	2.1
$f(x)$	3.61	3.9601	4.0401	4.41

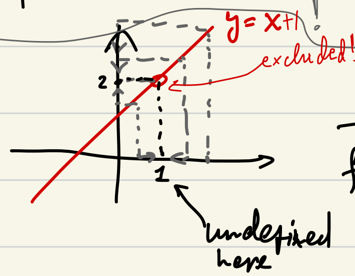
2

4

the closer x is to 2, the closer $f(x)$ is to 4

ex. 3 $f(x) = \dfrac{x^2-1}{x-1}$

$\lim_{x \to 1} f(x) = 2$



x	0.9	0.99	1.01	1.1
$f(x)$	1.9	1.99	2.01	2.1

1

2

$\boxed{\lim_{x\to 1} f(x) = 2 \neq f(1) \text{ undefined!}}$

important

$\dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$ except not defined at $x=1$

Domain: $(-\infty, 1) \cup (1, \infty)$

Rmk 1 $f(x) = \dfrac{x^2-1}{x-1} = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$ (Because $\div 0$)

Rmk 2 $f(1)$ is undefined! But, nevertheless, the limit exists: $\lim_{x\to 1} f(x) = 2$

(read 2.2 for more explanations)

Rmk 3

limit has input: $f(x)$ and a point a

output: $\lim_{x \to a} f(x)$

Rmk 4

Ex. 3 showed that the value $f(a)$ and the limit $\lim_{x \to a} f(x)$ are different concepts!

one-sided limits

1) If we restrict our x approaching a to also be $x > a$ then we write $\lim_{x \to a^+} f(x)$, and we call it the limit from the right

2) if we restrict to $x < a$ the we write $\lim_{x \to a^-} f(x)$, and we call it the limit from the left

☆ these two limits can be different!

Ex. 4 $f(x) = \begin{cases} x^2, & x < 0 \\ x+1, & x \geq 0 \end{cases}$

(piecewise function)

(rmk $f(0) = 1$ since we use $x+1$ rule for $x \geq 0$)

- $\lim_{x \to 0^-} f(x) = 0$ (Because we use "x^2" rule for $x<0$)
- $\lim_{x \to 0^+} f(x) = 1$ (because we use "$x+1$" rule for $x > 0$)

explanation via table:

x	-0.1	-0.01	0.01	0.1
$f(x)$	0.01	0.0001	1.01	1.1

left limit $\to 0$ $\quad$ $1 \leftarrow$ right limit

x^2 rule $\qquad$ $x+1$ rule

Rmk 1 last example showed that one-sided limits may be different.

Rmk 2 One-sided limits (almost) always exist!

see $\sin\left(\frac{\pi}{x}\right)$ example in the book

Fact regular (two-sided) limit exists if and only if one-sided limits are equal

(so if $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)$ is undefined)

For example, here, $\lim\limits_{x\to 0} f(x)$ is undefined!

(intuition)

☆ limit is value when you approach a, not what's at a

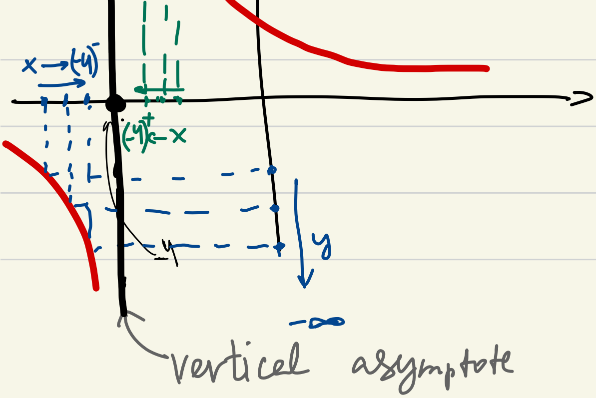
(still 2.2)

Infinite limits

If in one-sided limit the y-values $f(x)$ are getting arbitrarily large as x approaches a, then we write $\lim\limits_{x\to a^-} f(x) = \infty$ (or $\lim\limits_{x\to a^+} = \infty$)

If in one-sided limit the y-values $f(x)$ are getting arbitrarily small (like $-10000\ldots$), then we write $\lim\limits_{x\to a^-} f(x) = -\infty$ (or $\lim\limits_{x\to a^+} f(x) = -\infty$)

Ex. $f(x) = \dfrac{1}{x+4}$



- $\lim\limits_{x\to(-4)^+} f(x) = \infty$

- $\lim\limits_{x\to(-4)^-} f(x) = -\infty$

$\xrightarrow{x} -4 \xleftarrow{x}$

x	-4.1	-4.01	-3.99	-3.9
$f(x)$	-10	-100	100	10

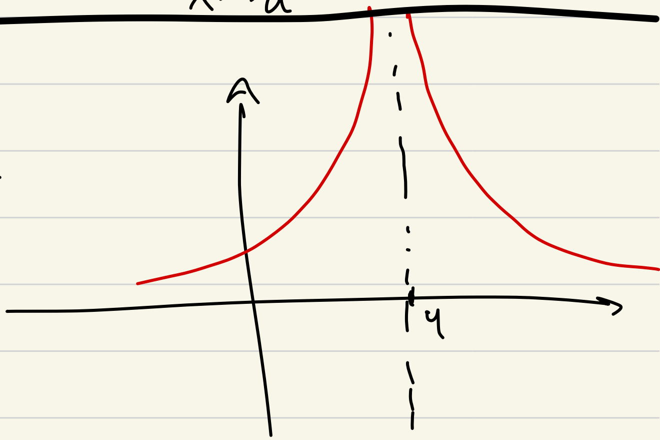
$\infty \xleftarrow{y}$

$\xrightarrow{y} -\infty$

remark here ↑ one-sided limits are not equal, and so $\lim_{x \to (-4)} f(x) = \text{DNE}$ (does not exist)!

definition vertical line $x = a$ is called a vertical asymptote if $\lim_{x \to a^{+ \text{ or } -}} f(x) = \pm \infty$

ex. $f(x) = \dfrac{1}{(x-4)^2}$



- $\lim_{x \to 4^+} f(x) = \infty$

$$\parallel$$

- $\lim_{x \to 4^-} f(x) = \infty$

Because these are equal, we can also say that the (double-sided) usual limit is $\lim_{x \to 4} f(x) = \infty$ (does exist)

2.3. How to comute limits

Limit laws

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, and are equal $\overset{\shortparallel}{C}$ and $\overset{\shortparallel}{D}$. Then:

then 1) $\lim_{x\to a}\left(f(x)+g(x)\right) = C+D$

2) $\lim_{x\to a}\left(f(x)-g(x)\right) = C-D$

3) $\lim_{x\to a}\left(\underset{\substack{\uparrow\\ \text{constant number}}}{k}\cdot f(x)\right) = k\cdot C$

4) $\lim_{x\to a}\left(f(x)\cdot g(x)\right) = C\cdot D$

5) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{C}{D}$ if $D\neq 0$

Ex. $\lim_{x\to 2}(x^2) \overset{\substack{\text{we know}\\ \downarrow}}{=} 4$. Using 3) we can deduce

$\lim_{x\to 2}(17\cdot x^2) =$ (by the rule 3))

$$17 \cdot \lim_{x \to 2} x^2 = 17 \cdot 4 = 68$$

Remark: in other words (if limits of $f(x)$ & $g(x)$ exist)

1) $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2) $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3) $\lim_{x \to a} (k \cdot f(x)) = k \cdot \lim_{x \to a} f(x)$

constant number

4) $\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$

5) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

These limit laws imply the following very useful ~~tool~~ to compute limits:

Important shortcut

If $f(x) =$ polynomial or rational function
(↑fraction of polynomials)

or any other continuous function,
(↑to be defined later, include e^x, $\sqrt[k]{\cdot}$, sin & cos, ...)

then $\boxed{\lim_{x \to a} f(x) = f(a)}$

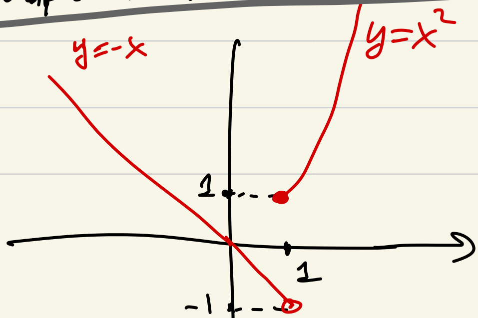
Note: if $f(x)$ is piecewise, then this doesn't work!

Ex. $f(x) = \dfrac{x^2 - 3x}{x^3 + 7}$ ← rational function

$$\lim_{x \to -1} f(x) \underset{\text{by the shortcut}}{=} f(-1) = \frac{(-1)^2 - 3\cdot(-1)}{(-1)^3 + 7} = \frac{4}{6} = \frac{2}{3}$$

So, if $f(x)$ is "nice", then limits are computed by substitution

Ex. $f(x) = \begin{cases} x^2, & x \geq 1 \\ -x, & x < 1 \end{cases}$



$\lim_{x \to 1} f(x) = \text{DNE}$ because $\lim_{x \to 1^+} f(x) = 1 \neq \lim_{x \to 1^-} f(x) = -1$

In particular,

$\lim_{x \to 1} f(x) \neq f(1) = 1$,

so this illustrates that the shortcut doesn't work for piecewise functions.

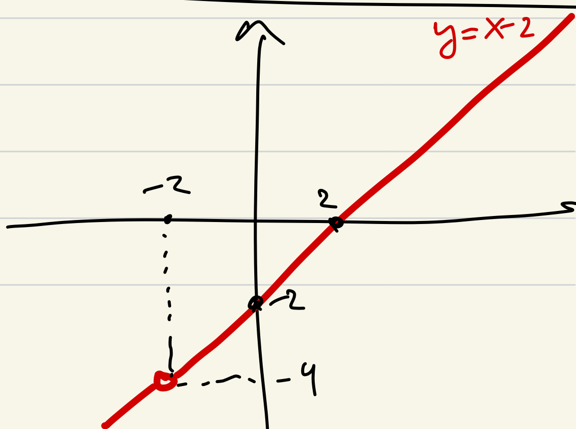
Ex. $f(x) = \dfrac{x^2 - 4}{x + 2}$.

Q1 $\lim_{x \to -2} f(x) = ?$

Q.2 $\lim_{x \to 2} f(x) = ?$

$$\frac{x^2-4}{x+2} = \frac{(x-2)\cancel{(x+2)}}{\cancel{(x+2)}} =$$

$$= \begin{cases} x-2 & \text{if } x \neq -2 \\ \text{DNE} & \text{if } x = -2 \end{cases}$$



A1 $\lim_{x \to -2} f(x) = -4 \neq f(-2)$ which is undefined!

This illustrates the the shortcut doesn't work for those x-values where the denominator becomes 0.

A2 $\lim_{x \to 2} f(x) = f(2) = 0$

the shortcut works since denominator is not 0.

2.3 How to algebraically compute limits?

Q. $\lim_{x \to a} f(x) = ?$

- number
- $+\infty$
- $-\infty$
- DNE (does not exis)

possible answers

ALGORITHM to find the limit

1. Is $f(x)$ piecewise?

NO, $f(x)$ is defined by one rule

2. Compute $f(a)$ by plugging in

If $f(a)$ = number, then that's the limit

3. If you see $\frac{0}{0}$, $\frac{\infty}{\infty}$ or $\infty - \infty$ when computing $f(a)$, then you need to simplify algebraically, then plug in one more time.

see later

(if you see square roots, then rationalize)

see later

4. If you see a non-zero number over zero (say $\frac{c^{\neq 0}}{0}$), then the limit is $\infty, -\infty$, or DNE.

Thus do this

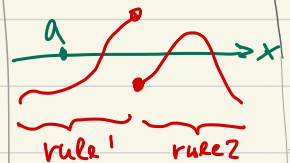
check one-sided limits to see if they agree.

YES, $f(x)$ is defined by two or more rules

2. Is a the endpoint of intervals defining rules for piecewise f?

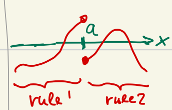
No

Just compute the limit using the rule 1



Yes

Analyze one-sided limits from left and right



Examples of computing limits for $f(x)$ that are not piecewise

1) $\lim_{x \to 2} x^2 = ?$ Answer: 4

Piecewise? No. So we plug a into $f(x)$:

$$f(2) = 2^2 = 4$$

2) $\lim_{x \to 1} x^3 - 6x^2 + 7x^2 + 5 = ?$ 7

Piecewise? No. So plug in:

$$f(1) = 1^3 - 6 \cdot 1^2 + 7 \cdot 1^2 + 5 = 1 - 6 + 7 + 5 = 7$$

3) $\lim_{x \to 1} \frac{x^2 - 1}{x + 1} = ?$ 0 Piecewise? No. So

plug in: $f(1) = \frac{1^2 - 1}{1 + 1} = \frac{0}{2} = 0$

4) $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = \lim_{x\to 1} x+1 = 2$

rmk1
cancellation of $(x-1)$'s is possible because we compute limits!

1. Piecewise? No. So we plug in
2. $f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$
3. Simplify algebraically:

rmk2
$\frac{(x-1)(x+1)}{x-1} \neq x+1$

So if you write $\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1$ it will be a mistake

$\lim_{x\to 1} x+1 = ?\ 2$

$f(1) = 1+1 = 2$

rmk3
$\lim_{x\to 1^+} \frac{x^2-1}{x-1} = 2 = \lim_{x\to 1^-} \frac{x^2-1}{x-1}$

Therefore $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$

rmk4

undefined at $x=1$
$D: (-\infty, 1) \cup (1, \infty)$

defined everywhere
$D: (-\infty, +\infty)$

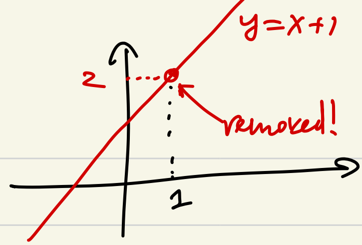
$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} \neq x+1$ (because they have different domains)

cannot do this because $x=1$
here there is division by 0

Important

the right statement!

Fact 1 $\quad \dfrac{x^2-1}{x-1} = \begin{cases} x+1, & x \neq 1 \\ DNE, & x = 1 \end{cases}$



Fact 2 Fact 1 implies $\dfrac{x^2-1}{x-1} \neq x+1$

(Because this does not have value at 1, but this does)

Fact 3 Even though $\dfrac{x^2-1}{x-1} \neq x+1$, we still have

$$\boxed{\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1}$$

(Because in the limit $x \to 1$ x-values approach 1, but are not equal to 1)

(In such cases  , where we have one point removed, we say that the function has <u>removable discontinuity</u>.)

Next example:

5) $\displaystyle \lim_{x\to -2} \frac{x+2}{x^2+x-2} = \lim_{x\to -2} \frac{\cancel{x+2}}{\cancel{(x+2)}(x-1)} =$

can cancel Because we are in the limits

$f(-2) = \dfrac{-2+2}{4-2-2} = \dfrac{0}{0}$

$\displaystyle = \lim_{x\to -2} \frac{1}{x-1} =$

Thus we need to simplify

$= f(-2) = -\dfrac{1}{3}$

6) $\lim\limits_{x \to 2} \frac{x}{(x-2)^3} = ?$ DNE!

$f(2) = \frac{2}{0} \longrightarrow$ step (4) $\rightarrow +\infty$, $\rightarrow -\infty$, $\rightarrow$ DNE

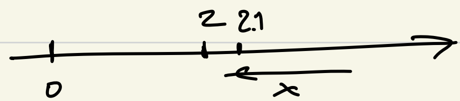
We have to check one-sided limits:

right-sided

$$\lim_{x \to 2^+} \frac{x}{(x-2)^3} = +\infty \quad \cancel{-\infty}$$

(it exists because one-sided limits always exist)

$$\frac{2.1}{(2.1-2)^3} = \frac{2.1}{(0.1)^3} > 0$$



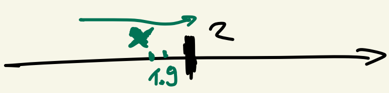
therefore we know that $\frac{x}{(x-2)^3}$ is positive when $x \to 2^+$ and thus the limit has to be $+\infty$

left-sided

$$\lim_{x \to 2^-} \frac{x}{(x-2)^3} = \cancel{+\infty} \quad -\infty$$

(it exists because one-sided limits always exixst)

$$\frac{1.9}{(1.9-2)^3} = \frac{1.9}{(-0.1)^3} = \frac{1.9}{-0.001} < 0,$$ so $\frac{x}{(x-2)^3} < 0$ when $x \to 2^-$

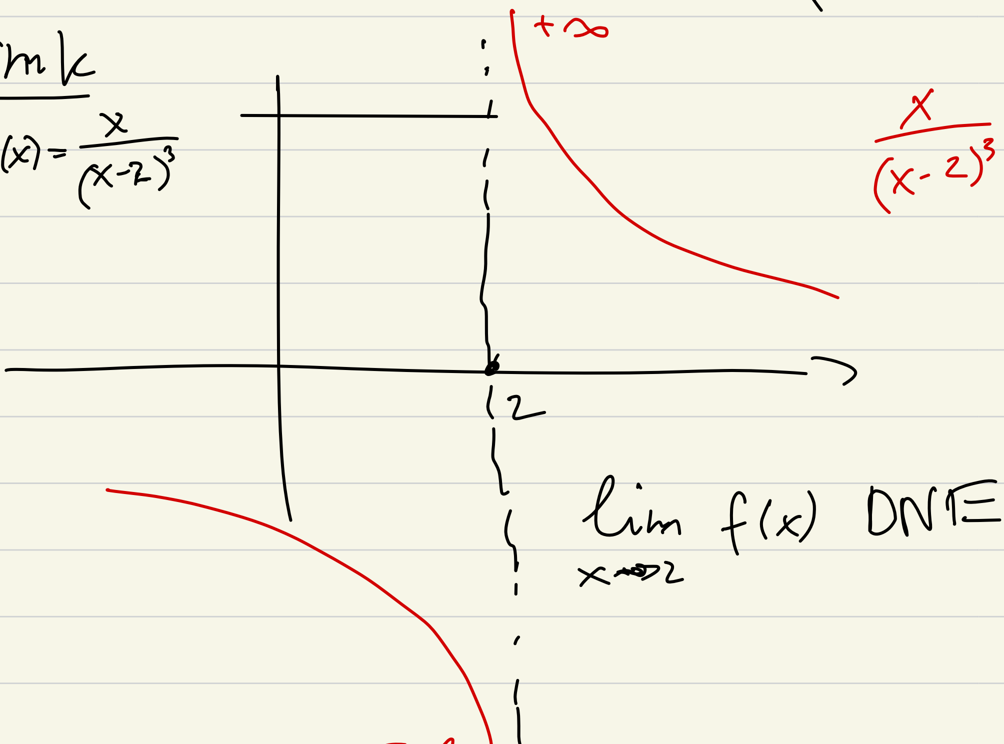


$$\lim_{x \to 2^+} \frac{x}{(x-2)^3} = +\infty \quad , \quad \lim_{x \to 2^-} \frac{x}{(x-2)^3} = -\infty$$

so $\displaystyle \lim_{x \to 2} \frac{x}{(x-2)^3}$ DNE (because one-sided limits do not agree)

rmk

$f(x) = \dfrac{x}{(x-2)^3}$



$\displaystyle \lim_{x \to 2} f(x)$ DNE

$$f(x) = \frac{1}{(x-2)^2}$$

both one-sided limits are $+\infty$, so

$$\lim_{x \to 2} f(x) = +\infty$$

2.3 How to algebraically compute limits?

Q. $\lim\limits_{x \to a} f(x) = ?$

- number
- $+\infty$
- $-\infty$
- DNE (does not exis)

possible answers

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(if you see square roots, then rationalize)

see later

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Thus do this

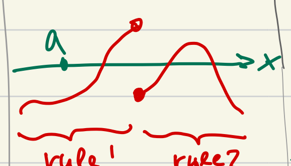
check one-sided limits to see if they agree.

YES, $f(x)$ is defined by two or more rules

2. Is a the endpoint of intervals defining rules for piecewise f?

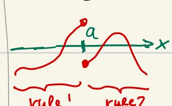
No

Just compute the limit using the rule 1



Yes

Analyze one-sided limits from left and right



Continueing examples

1) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2+x} \right) =$

$f(0) = \frac{1}{0} - \frac{1}{0} = \infty - \infty$, so need to simplisy.

$= \lim_{x \to 0} \left(\frac{x+1}{x(x+1)} - \frac{1}{x(x+1)} \right) =$

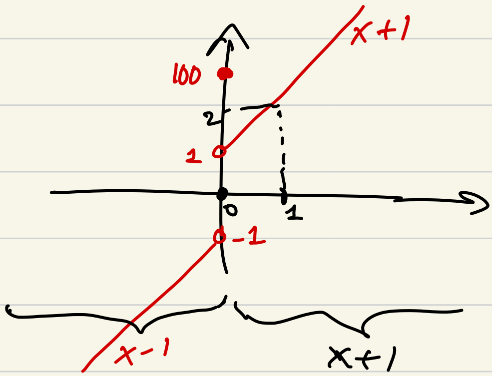
$= \lim_{x \to 0} \left(\frac{x\cancel{+1}\cancel{-1}}{x(x+1)} \right) = \lim_{x \to 0} \left(\frac{\cancel{x}}{\cancel{x}(x+1)} \right) =$ (can cancel x's because we are in limit)

$= \lim_{x \to 0} \frac{1}{x+1} = 1$

$f(0) = \frac{1}{0+1} = 1$

2) $\lim_{x \to 2} \frac{x}{(x-2)^2} = ?$ HW, hint: do it similarly to the example $\frac{x}{(x-2)^3}$ in the previous lecture.

Now, let's study limits of piecewise functions.



3) $f(x) = \begin{cases} x-1, & x<0 \\ 100, & x=0 \\ x+1, & x>0 \end{cases}$

the rule for $x>0$

• $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} (x+1) = 2$

Q. Is 1 the endpoint of defining intervales?

No

So, we just use $x+1$ rule to find the limit.

• $\lim\limits_{x\to 0} f(x) =$

Q. Is 0 the endpoint of defining intervals?

Yes!

So we analyze one sided limits:

$\lim\limits_{x\to 0^-} f(x) =$ (using rule $x-1$) $= 0-1 = -1$

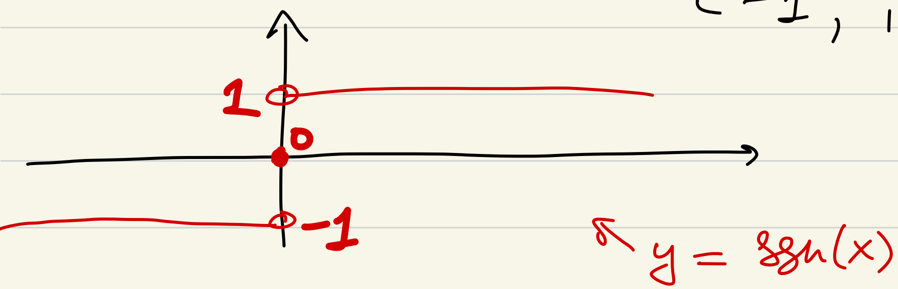
$\lim_{x \to 0^+} f(x) =$ (using rule $x+1$) $= 0 + 1 = +1$

(<u>remark</u> $f(0) = 100$ does not matter)

- One-sided limits are not equal, and so the twosided limit $\lim_{x \to 0} f(x)$ <u>DNE</u>.

<u>Definition</u> "sign"

$$\mathrm{sgn}(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$



- sgn(x) is piecewise.

- If function f(x) involves sgn(x) in some way, f(x) is piecewise.

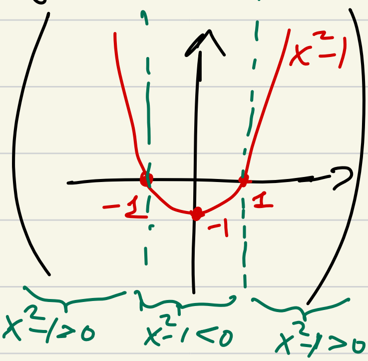
4) $\lim\limits_{x \to 1} \operatorname{sgn}(x^2-1) = ?$

Q1 Piecewise? Yes! Q2 Is $x=1$ on the boundary of defining intervals? Yes! Because rules are $\nearrow x^2-1>0$, $\rightarrow x^2-1=0$, $\searrow x^2-1<0$ and substituting 1 into x^2-1 gives 0.

Let's understand better:

Defining intervals for this function?

$$\operatorname{sgn}(x^2-1) = \begin{cases} +1, & x^2-1>0 \\ 0, & x^2-1=0 \\ -1, & x^2-1<0 \end{cases}$$

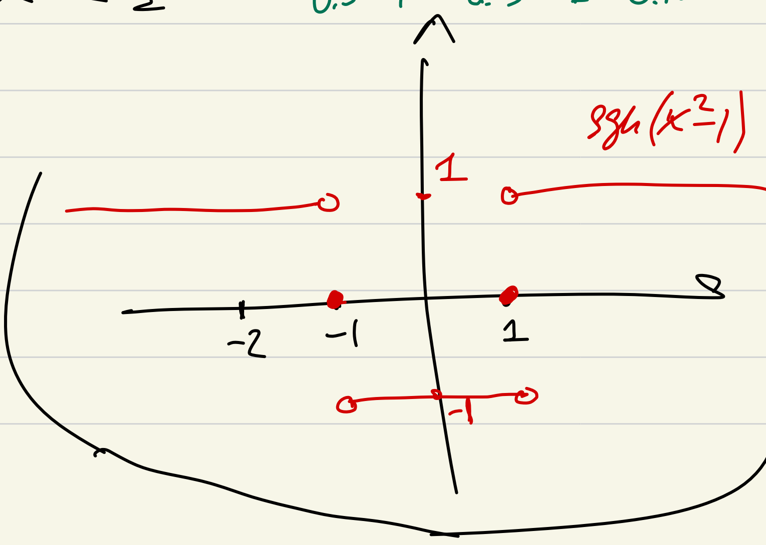


$$= \begin{cases} +1, & x>1 \text{ or } x<-1 \\ 0, & x=1 \text{ or } x=-1 \\ -1, & -1<x<1 \end{cases}$$

$x^2-1<0$ for $-1<x<1$

$0.5^2-1 = 0.25-1 = -0.75$

So the graph of $\operatorname{sgn}(x^2-1)$ is

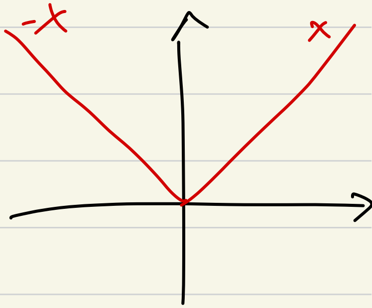


$$\lim_{x \to 1} \operatorname{sgn}(x^2-1) = DNE$$

because $\lim_{x \to 1^+} f(x) = 1 \neq -1 = \lim_{x \to 1^-} f(x)$!

$$\lim_{x \to -2} \operatorname{sgn}(x^2-1) = \operatorname{sgn}((-2)^2-1) = \operatorname{sgn}(3) = +1$$

-2 not on the Boundary!



<u>Definition</u> $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

ex: $|-4| = -(-4) = 4$

- $|x|$ piecewise!

- any function $f(x)$ that involves $|x|$ is also piecewise.

<u>ex.</u> $$\lim_{x \to 0.5} \frac{2x-1}{|2x^3-x^2|} = ?$$

Q1

Piecewise? Yes, Because we see $|\,|$.

Q2 Is 0.5 on the Boundary of defining regions?

$$\frac{2x-1}{\pm|2x^3-x^2|} = \begin{cases} \dfrac{2x-1}{+(2x^3-x^2)} & \text{if } (2x^3-x^2) \geq 0 \\ \dfrac{2x-1}{-(2x^3-x^2)} & \text{if } (2x^3-x^2) < 0 \end{cases}$$

.point $x=a$ is on the boundary if $2a^3-a^2=0$

key

Q.2 Is 0.5 on the boundary of regions with diff. rules?

$2x^3-x^2 \geq 0$ $\quad$ $2x^3-x^2 < 0$

Check algebraically by substituting 0.5.

$2\cdot(0.5)^3-(0.5)^2 = 0$, so

0.5 is on the boundary of regions.

and so we have to check one-sided limits

$$\lim_{x \to 0.5^+} \frac{2x-1}{|2x^3-x^2|} =$$

to check the sign

close to 0.5^+

sub **0.6** to check the sign

$2(0.6)^3-(0.6)^2 > 0$

(which rule to chose, + or - ? Choose + Because)

$$= \lim_{x\to 0.5^+} \frac{2x-1}{+(2x^3-x^2)} = \frac{2\cdot 0.5-1}{2\cdot(0.5)^3-(0.5)^2} = \frac{0}{0} \text{ problem!}$$

So we simplify

$$\lim_{x\to 0.5^+} \frac{2x-1}{2x^3-x^2} = \lim_{x\to 0.5^+} \frac{\cancel{2x-1}}{\cancel{(2x-1)}x^2} =$$

$$= \lim_{x\to 0.5^+} \frac{1}{x^2} = \frac{1}{(0.5)^2} = \frac{1}{0.25} = 4$$

$$\lim_{x\to 0.5^-} \frac{2x-1}{|2x^3-x^2|} = \lim_{x\to 0.5^-} \frac{2x-1}{-(2x^3-x^2)} \overset{\text{simplify}}{=} \lim_{x\to 0.5^-} \frac{1}{-x^2} = -4$$

$$2(0.4)^3-(0.4)^2 < 0$$

(plug 0.4 Because its close to 0.5^-)

$-4 \neq 4$, So $\lim_{x\to 0.5} \frac{2x-1}{|2x^3-x^2|}$ DNE ex. $\lim_{x\to 1} \frac{2x-1}{(2x^3-x^2)} = \lim_{x\to 1} \frac{2x-1}{+(2x^3-x^2)} =$

$= \frac{2\cdot 1-1}{2\cdot 1-1} = 1$

$2\cdot 1^3 - 1^2 = 1 > 0$

so 1 not on the boundary!

See Examples 7 and 8 on page 100 of the book.

Read the whole 2.3 for your benefit.

example similar to problems from the previous lecture:

$$\lim_{x\to 2}\frac{x^2-5x+6}{|x-2|}$$

intervals $\nearrow$ $x-2 \geq 0$

intervals $\searrow$ $x-2<0$

Piecewise? Yes.

Is $x=2$ on the boundary of intervals?

Yes, because $x-2$ changes from <0 to >0 exactly at $x=2$.

Therefore we analyze one-sided limits:

$2 \leftarrow x$

1) $$\lim_{x\to 2^+}\frac{x^2-5x+6}{|x-2|}=$$

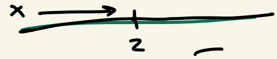
$x\to 2^+$ means that $x>2$, and $2.1-2=0.1>0$, and so so $x-2>0$, and so $|x-2|=+(x-2)$

since $x-2>0$

$$=\lim_{x\to 2^+}\frac{x^2-5x+6}{+(x-2)}=$$

$\to f(2)=\frac{0}{0}$, so we simplify

$$=\lim_{x\to 2^+}\frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}=\lim_{x\to 2^+}(x-3)=2-3=-1$$



$x \to 2^-$ means that $x < 2$, and $1.9 - 2 = -0.1 < 0$, and so

so $x - 2 < 0$, and

so $|x-2| = -(x-2)$

2) $$\lim_{x \to 2^-} \frac{x^2-5x+6}{|x-2|} =$$

since $x-2<0$

$$= \lim_{x \to 2^-} \frac{x^2-5x+6}{-(x-2)} =$$

$$= \lim_{x \to 2^-} \frac{\cancel{(x-2)}(x-3)}{-\cancel{(x-2)}} = \lim_{x \to 2^-} -(x-3) = -(2-3) = +1$$

Thus one sided limits are <u>not</u> equal, and so $\boxed{\lim_{x \to 2} \frac{x^2-5x+6}{|x-2|} \text{ DNE}}$

<u>Remark</u> When computing limits $\lim_{x\to a^+ \text{or} -} f(x)$, you substitute a and compute $f(a)$

- if you see a <u>number</u>, then thats the limit;
- if you see $\frac{c}{0}$ where $c \neq 0$, then the limit is + or − ∞, and so you have to check the sign
- if you see $\frac{0}{0}$, or $\frac{\infty}{\infty}$ or $\infty - \infty$, then you need simplify further

---

<u>Q.</u> What to do if you need to simplify, but there are square roots?

A. "multiply by a conjugate".

<u>Ex.</u> $\lim_{x\to 0} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{x} \right) = ?$

<u>rmk</u>

! $(a-b)(a+b) = a^2 - b^2$

$$f(0) = \frac{\sqrt{1+0} - \sqrt{1-0}}{0} = \frac{0}{0}$$ so we need to simplify:

switch

called "rationalization" or "multiplying by a conjugate"

$$\lim_{x \to 0} \left(\frac{\overset{a}{\sqrt{1+x}} - \overset{b}{\sqrt{1-x}}}{x} \right) \cdot \left(\frac{\overset{a}{\sqrt{1+x}} + \overset{b}{\sqrt{1-x}}}{\sqrt{1+x} + \sqrt{1-x}} \right)$$

<u>numerator:</u> $(\underset{a}{\sqrt{1+x}} - \underset{b}{\sqrt{1-x}})(\underset{a}{\sqrt{1+x}} + \underset{b}{\sqrt{1-x}}) = \underset{a^2}{\sqrt{1+x}^2} - \underset{b^2}{\sqrt{1-x}^2} =$

$= (1+x) - (1-x) = 2x$

<u>denominator:</u> $x(\sqrt{1+x} + \sqrt{1-x})$

$$\lim_{x \to 0} \frac{2\cancel{x}}{\cancel{x}(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{\sqrt{1+0} + \sqrt{1-0}} = \frac{2}{2} = 1$$

ex. $$\lim_{x\to 16} \frac{\overset{a}{4}-\overset{b}{\sqrt{x}}}{16x-x^2} \times \frac{(\overset{a}{4}+\overset{b}{\sqrt{x}})}{(4+\sqrt{x})}$$

$f(16) = \frac{0}{0}$

$$\lim_{x\to 16} \frac{\overset{(a^2)}{4^2}-\overset{(b^2)}{\sqrt{x}^2}}{(16x-x^2)(4+\sqrt{x})} =$$

$$= \lim_{x\to 16} \frac{\cancel{16-x}}{x\cancel{(16-x)}(4+\sqrt{x})} =$$

$$= \lim_{x\to 16} \frac{1}{x(4+\sqrt{x})} = \frac{1}{16(4+\sqrt{16})}$$

plug in 16

$$= \frac{1}{128}$$

ex. $\lim\limits_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{1}{t}\right)$

$f(0)=\frac{1}{0}-\frac{1}{0}=\infty-\infty$ so

we need to simplify.

First find common denominator, and only then multiply by a conjugate:

$$\lim_{t\to 0}\left(\frac{1}{t\sqrt{1+t}}-\frac{\sqrt{1+t}}{t\sqrt{1+t}}\right)=$$

$$=\lim_{t\to 0}\left(\frac{\overset{a}{1}-\overset{b}{\sqrt{1+t}}}{t\sqrt{1+t}}\right)\cdot\frac{\overset{a}{1}+\overset{b}{\sqrt{1+t}}}{1+\sqrt{1+t}}$$

how conjugate

$$=\lim_{t\to 0}\frac{\overbrace{1}^{a^2}-\overbrace{(1+t)}^{b^2}}{t\sqrt{1+t}\,(1+\sqrt{1+t})}=\lim_{t\to 0}\frac{-\cancel{t}}{\cancel{t}\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$=\lim_{t\to 0}\frac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})}\underset{\text{plug in}}{=}\frac{-1}{1\cdot 2}=-\frac{1}{2}$$

· Read 2.3 for help.

$\lim_{x \to a} f(x)$

· Change in plans: we don't cover 2.4 (rigorous definition of limits)

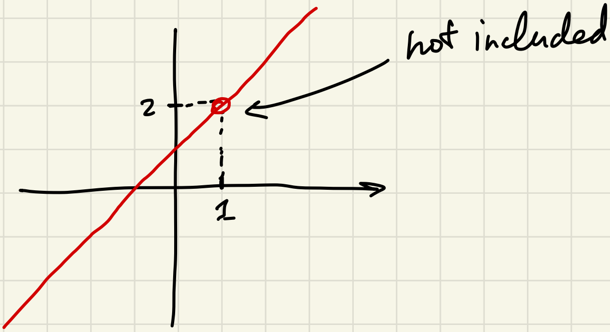
· Exam 1 info

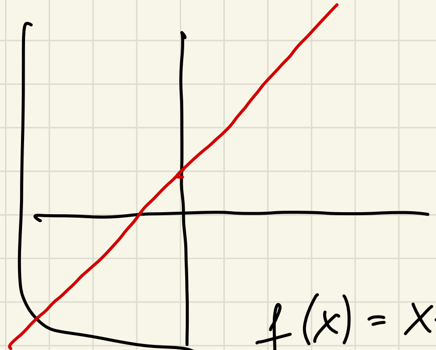
- online, the usual zoom meeting
- 50 min.
- harder than practice exams
- ! exam 1 will include work (only answers are not enough!)

show steps!

- 25th September
- (includes 2.5, 2.6 ... To be determined)

Continuity (2.5)





$f(x) = x+1$

$$f(x) = \frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{(x-1)} = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{DNE} & \text{if } x = 1 \end{cases}$$

Q. How to describe the difference between?

definition function $f(x)$ is continuous at point (number) $x=a$ if $\lim_{x \to a} f(x) = f(a)$

How to find out if $f(x)$ is continuous at a?

3 rules for $f(x)$ to be continuous at $x=a$

we need three things:

$$f(a) = \lim_{x \to a} f(x)$$

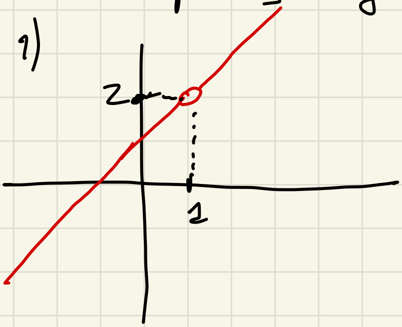
(1) existence of

(3)

2) existence of

Examples of discontinuous functions:

1)



$f(x) = \frac{x^2-1}{x-1}$ at $x=1$

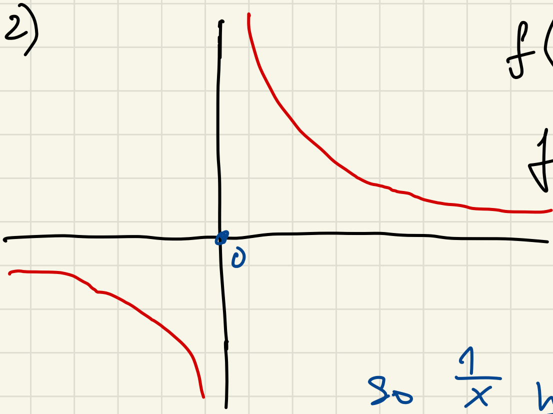
not continuous at $x=1$

because rule (1) fails

($f(1)$ DNE!)

remark: $\lim_{x\to 1} f(x) = 2$, so (2) rule holds.

2)



$f(x) = \frac{1}{x}$ at $x=0$

fails rules (1) & (2)

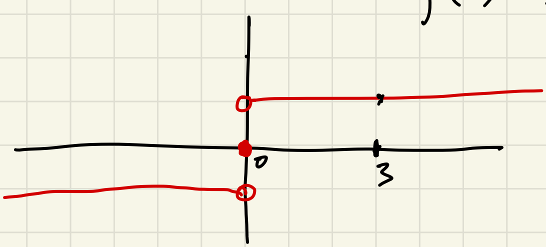
- $f(0)$ DNE $\to$ fails (1)
- $\lim_{x\to 0} f(x)$ DNE $\to$ fails (2)

so $\frac{1}{x}$ not continuous at $x=0$

---

3)

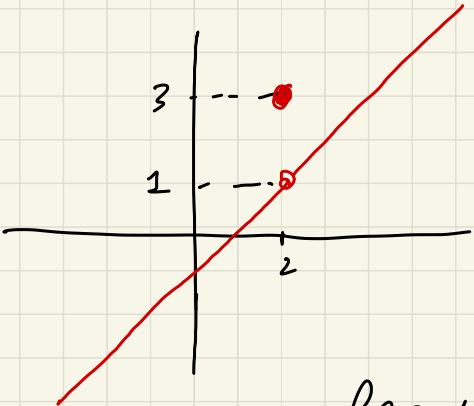
$f(x) = \text{sgn}(x)$



- at $x=0$ discontinuous, bc fails (2)

- at $x=3$ $\lim_{x\to 3} f(x) = 1 = f(3)$

so $f(x)$ is continuous at $x=3$

4) 

$$f(x) = \begin{cases} x-1 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$$

$f(x)$ at $x=2$

__not__ continuous,

because $\lim_{x\to 2} f(x) = 1$

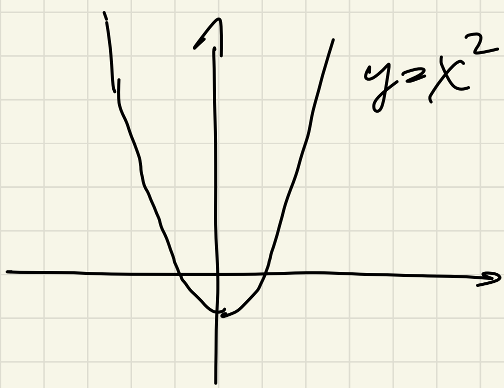
$f(2) = 3$

so the rule (3) fails

__Visual intuition__ continuity of $f(x)$ at $x=a$ is being able to draw the graph without lifting your pencil

5) $y = x^2 - 1$

is continuous at all points



- continuity is a property of a function $f(x)$ <u>and</u> a point $x=a$

(input: $f(x)$, a output: $f(x)$ continuous or not at $x=a$)

6) given $f(x) = \frac{1}{x-3}$ is f continuous at a) $x=0$? b) $x=3$?

a) at $x=0$

(1) $f(0) = \frac{1}{-3}$ ✓

(2) $\lim\limits_{x \to 0} \frac{1}{x-3} = \frac{1}{-3}$ ✓

(3) (1) = (2) ✓

therefore continuous

b) at $x=3$

(1) $f(3) = \frac{1}{0}$ ✗

DNE!

So rule (1) fails, so not continuous.

remark · value $f(a)$ cannot be ∞ or $-\infty$.

· but a limit can:

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = +\infty, \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0} \frac{1}{x} \text{ DNE}$$

7) $f(x) = \begin{cases} x^3, & x < 0 \\ \sqrt{x}, & 0 \le x < 4 \\ x+4, & 4 \le x \end{cases}$

Is $f(x)$ continuous at a) $x=0$? b) $x=4$?

a) (1) $f(0) = \sqrt{0} = 0$ ✓

(2) $\lim_{x \to 0} f(x) = 0$ ✓

$\lim_{x \to 0^+} f(x) = \sqrt{0} = 0$

$\lim_{x \to 0^-} f(x) = 0^3 = 0$

one-sided limits agree

b) ~~(1)~~ $f(4) = 4+4 = 8$

(2) $\lim_{x \to 4} f(x)$ DNE ✗

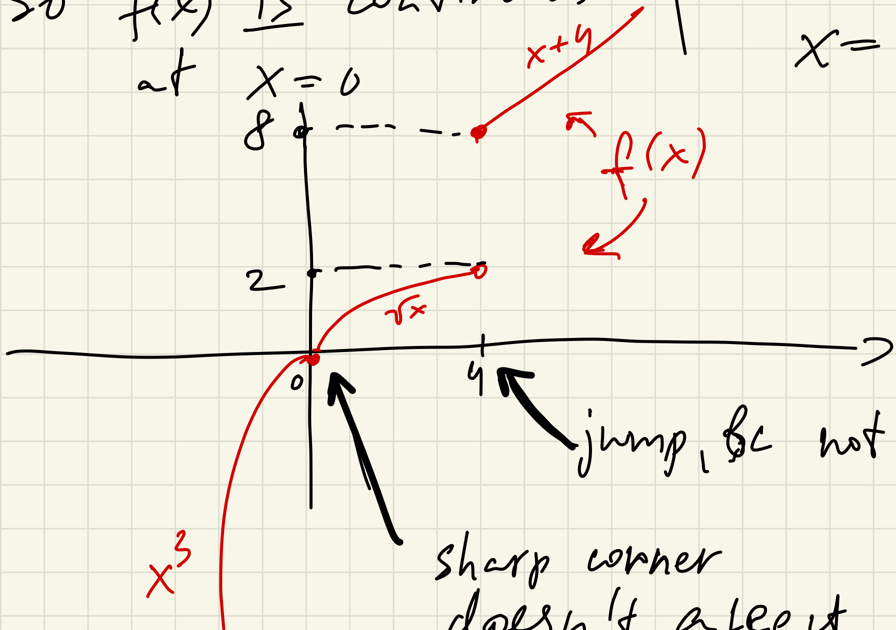
$\lim_{x \to 4^+} f(x) = 4+4 = 8$

$\lim_{x \to 4^-} f(x) = \sqrt{4} = 2$

(3) (1) = (2) ✓

So $f(x)$ is continuous at $x=0$

So $f(x)$ is not continuous at $x=4$



jump, bc not continuous

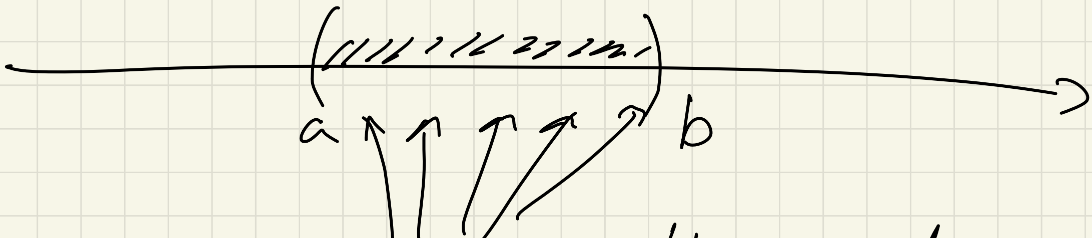
sharp corner doesn't affect continuity, $f(x)$ is continuous at $x=0$.

So far, we talked about continuity of $f(x)$ at $x=a$,

Now, let's define

Continuity of f(x) on an interval

definition $f(x)$ is continuous on an open interval (a,b) if it's continuous at every point in (a,b) (that is continuous at all x such that $a < x < b$)



if $f(x)$ is continuous at all those points, then we say $f(x)$ is continuous on (a,b)

Problem!

Find the maximal intervals such that $f(x) = \frac{1}{x^2-4}$ is continuous (that is find all points where $f(x)$ is continuous)

$$f(x) = \frac{1}{x^2-4}$$

can't check every single point, so instead we find those points where $f(x)$ is not continuous. For this we check where rules (1), (2), (3) may fail.

rule (1) ($f(a)$ exists) fails at $x^2-4=0$ so $x=2$, $x=-2$

rule (2) ($\lim\limits_{x\to a} f(x)$ exists) can fail when we divide by 0 or at the boundary of piecewise

$\lim\limits_{x\to a} \left(\frac{1}{x^2-4}\right)$ DNE at $a=2$, $a=-2$ possibly

rule (3) ($f(a) = \lim\limits_{x\to a} f(x)$) for all a except 2 and -2 we have $\frac{1}{a^2-4} = \lim\limits_{x\to a} \frac{1}{x^2-4}$

Answer $f(x)$ is ~~not~~ not continuous at $x=-2$ and $x=2$, (bc rule (1) fails)

and so

$$\boxed{f(x) \text{ is continuous on } (-\infty,-2)\cup(-2,2)\cup(2,\infty)}$$

Problem 2 Find all points where $f(x) = \begin{cases} x^3, & x<0 \\ \sqrt{x}, & 0 \le x < 4 \\ \frac{1}{x+4}, & 4 \le x \end{cases}$

is continuous.

$f(x)$ is piecewise. So we find discontinuities for each of

(1) "$f(a)$ exists" is true for all three

✓ – x^3 is defined for all $x<0$

✓ – $\sqrt{x}$ is defined for all $\underbrace{0\le x<4}_{!}$

✓ – $\frac{1}{x+4}$ is defined for all $4\le x$

(2) "$\lim_{x\to a} f(x)$ exists"

may fail at endpoints: (check division by 0 as well)

$\underline{x=0}$

check one-sided limits, and set

$\lim_{x\to 0} f(x)=0$

$\underline{x=4}$

check one-sided limits, and set

✗ $\lim_{x\to 4} f(x)$ DNE

So $f(x)$ is **not** continuous at $x=4$

(3) $f(a)=\lim_{x\to a} f(x)$

. true for $x \neq 0$ and 4

. for $x=0 \longrightarrow f(0) = \lim_{x \to 0} f(x)$ ✓

$x=4 \longrightarrow$ lim DNE ✗

<u>answer</u> $f(x)$ is not contin. at $x=4$,

and so $\boxed{\text{it } \underline{\text{is continuous}} \text{ on } (-\infty, +4) \cup (4, +\infty)}$

def $f(x)$ is continuous at $x=a$

if
$$f(a) = \lim_{x \to a} f(x)$$

(1) value exists

(3) they are equal

(2) limit exists

Problem For what $x=a$ the function

$$f(x) = \begin{cases} \frac{1}{x-2}, & 1 \le x \quad (A) \\ 2x-3, & x < 1 \quad (B) \end{cases}$$

is continuous?

Cannot analyze each $x=a$, so we find those numbers $x=a$ where $f(x)$ is not continuous. For that we analyze each function (A) & (B) on their domains, and then analyze the endpoint $x=1$ as well.

(A) $\underbrace{\frac{1}{x-2} \text{ on } (1,+\infty)}$.

boundary not included since we will analyze it later

Where can continuity fail?

(1) "$f(a)$ exists" can fail due to division by 0, square roots of negatives, logarithms of negative

$x-2=0$

$x=2$ rmk $x=2$ is on $(1,+\infty)$, so we do exclude it

(2) "$\lim\limits_{x\to a} f(x)$ exists" can possibly fail due to division by zero

$\to$ $x=2$ possibly bad point

(3) "$f(a)=\lim\limits_{x\to a} f(x)$" holds everywhere on $(1,+\infty)$ except at $x=2$

$\boxed{\text{So } \frac{1}{x-2} \text{ on } (1,+\infty) \text{ is not continuous at } x=2}$

(B) $2x-3$ on $(-\infty, 1)$

continuous everywhere, since rules (1),(2),(3) don't fail } missed steps here

$x=1$

✓ (1) "$f(a)$ exists"

$$f(1) = \frac{1}{1-2} = -1$$

✓ (2) "$\lim_{x\to a} f(x)$ exists"

$$\lim_{x\to 1} f(x) = -1$$

1 is the boundary, so we analyze one-sided limits:

$$\lim_{x\to 1^+} f(x) = \frac{1}{1-2} = -1$$

$$\lim_{x\to 1^-} f(x) = 2\cdot 1 - 3 = -1$$

✓ (3) "$f(a) = \lim_{x\to a} f(x)$"

$$-1 = -1$$

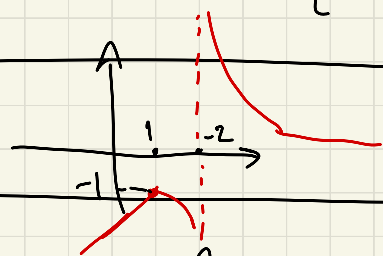
$$f(x) = \begin{cases} \frac{1}{x-2}, & 1 \le x \quad (A) \\ 2x-3, & x < 1 \quad (B) \end{cases}$$

So $f(x)$ is continuous at $x=1$

In conclusion, $f(x)$ is not continuous only at $x=2$. So

$f(x)$ is continuous on $(-\infty,2)\cup(2,\infty)$

answer



Faster ways to tell if $f(x)$ is continuous or not at $x=a$

Facts:

- linear functions are continuous everywhere on $(-\infty,\infty)$

ex.: $f(x)=2x-7$, $f(x)=-3$, ...

- polynomials are continuous everywhere on $(-\infty,\infty)$

ex.: $f(x)=-7x^4+2x-17x^8+231$

ex. $\log_2(X)$ is cont. on $(0,\infty)$

- power functions, roots, exponential functions, trig functions, logarithms are always continuous on their domain (not necessarily $(-\infty,\infty)$)

! • If $f(x)$ and $g(x)$ are continuous at $X=a$, then the following functions are also continuous at $X=a$

1) $f(x)+g(x)$
2) $f(x)-g(x)$
3) $f(x)\cdot g(x)$
4) $\dfrac{f(x)}{g(x)}$ provided $g(a)\neq 0$
5) $C\cdot f(x)$ (where C is constant)
6) $f(g(x))$ (composition of functions)

Problem Show that $e^x \cdot \sin(x) + \frac{x}{e^x}$ is continuous on $(-\infty, \infty)$.

- e^x and $\sin(x)$ are cont. on $(-\infty, \infty)$, and so by 3) $e^x \cdot \sin(x)$ is cont. as well
- x and e^x are cont. on $(-\infty, \infty)$, so by 4) $\frac{x}{e^x}$ is cont. on $(-\infty, \infty)$, since e^x is never zero.
- $e^x \cdot \sin(x)$ cont. on $(-\infty, \infty)$
  $\frac{x}{e^x}$ cont. on $(-\infty, \infty)$

So by 1) $e^x \cdot \sin(x) + \frac{x}{e^x}$ is cont. on $(-\infty, \infty)$

rmk

$\left.\begin{array}{l} e^x \text{ cont.} \\ 1 \text{ cont.} \end{array}\right\} \xrightarrow{1)} e^x + 1$ is continuous as well ex. $f(x) = \dfrac{x}{e^x - 1}$

$e^x - 1 = 0$

$e^x = 1$

$x = 0$

continuous everywhere except at $x=0$, since $f(0)$ DNE

So continuous on $(-\infty, 0) \cup (0, \infty)$

→ Intermidiet value thm 2.5

→ limits at ∞ 2.6

(HB asymptotes)

ex. $e^x + \sqrt{x}$

e^x cont. on $(-\infty, \infty)$

$\sqrt{x}$ cont. on $[0, +\infty)$

→ $e^x + \sqrt{x}$ is cont. on $[0, +\infty)$ since for negative numbers

$e^x + \sqrt{x}$ DNE

(still 2.5)

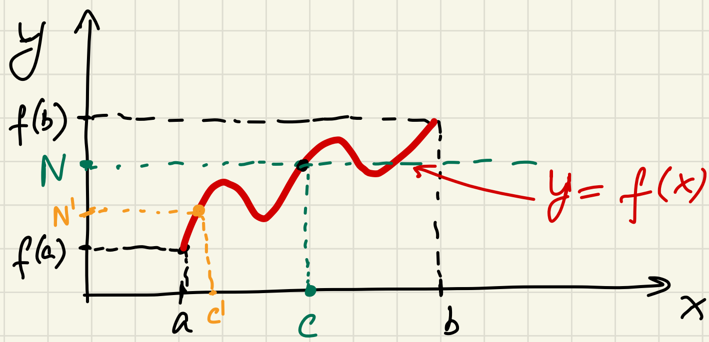
Intermediate Value Theorem (IVT)

If 1) $f(x)$ is continuous on some interval $[a,b]$,

and 2) N is a number between $f(a)$ and $f(b)$,

then there exists some c on $[a,b]$ such that $f(c) = N$.

Illustration:



In other words, continuous function $f(x)$ attains all possible y-values between $f(a)$ and $f(b)$

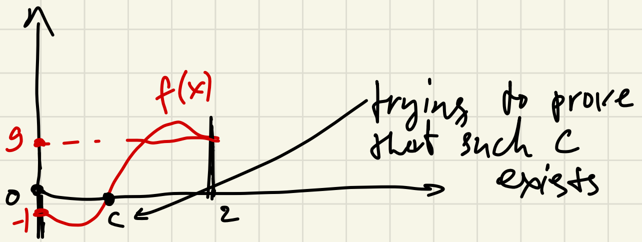
Problem 1 / Show (Prove) that $f(x) = x^3 + x - 1$ has $f(x) = 0$ for some x in $[0,2]$.

same question

(Show that $x^3 + x - 1 = 0$ has a solution on $[0,2]$

$f(x) = x^3 + x - 1$ $\quad N = 0 \quad [a,b] = [0,2]$

Solution: apply IVT for
- $f(x) = x^3 + x - 1$
- $N = 0$
- $a = 0$
- $b = 2$



✓1) check if $f(x)$ is continuous on $[0,2]$.

$f(x)$ is polynomial, so continuous everywhere

✓2) check if 0 is between $f(0)$ and $f(2)$

$f(0) = 0^3 + 0 - 1 = -1$

$f(2) = 2^3 + 2 - 1 = 9$

0 is between these!

So, applying IVT now gives us some $x = c$ on $[0,2]$ such that $f(c) = 0$

Remark this was an example of a **proof** = a sequence of arguments that justify a claim.

In order to **apply** a theorem, you need to check its conditions.

Remark IVT doesn't tell us what c is, it only proves its existence.

Problem 2

Prove that $e^x = 3 - 2x$ has a solution.

solution: cannot solve for x, so we use IVT.

- $f(x) = ?$
- $N = ?$
- $[a, b] = ?$

First, set everything on one side, and let $f(x)$ be equal to that side.

$$\underbrace{e^x - 3 + 2x}_{\substack{\parallel \\ f(x)}} = \underbrace{0}_{\substack{\parallel \\ N}}$$

. $[a,b] = ?$ will worry about it later.

Conditions of IVT:

1) $f(x)$ is continuous everywhere, since it is a sum of e^x and a linear function.

2) want: show that $N=0$ is between $f(a)$ and $f(b)$. But we are not given a & b!

When not given an interval, we must come up with a suitable one:

. $N=0$, so we want $f(a)$ to be $(-)$ and $f(b)$ to be $(+)$ (or vice versa)

For this try <u>simple</u> numbers...

guess: $a=0$, $b=1$

$$f(0) = e^0 - 3 + 2\cdot 0 = -2 < 0$$

$$f(1) = e^1 - 3 + 2\cdot 1 = e - 1 > 0$$

$(e = 2.71\ldots)$

So applying IVT to $f(x) = e^x - 3 + 2x$, $[a,b] = [0,1]$, $N=0$ gives us existence of some c on $[0,1]$

such that $e^{c} - 3 + 2c = 0$

so $e^{c} = 3 - 2c$

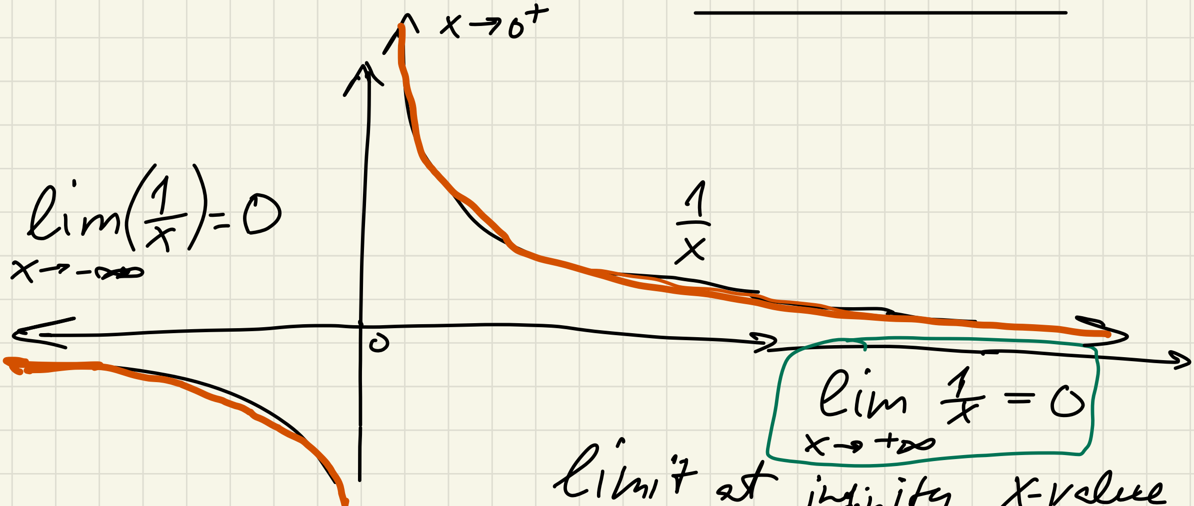
this is what we were asked to find in the beginning

---

infinite limits, y-value goes to ∞

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right) = +\infty$$

$$\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0$$



$$\boxed{\lim_{x \to +\infty} \frac{1}{x} = 0}$$

limit at infinity, x-value goes to ∞

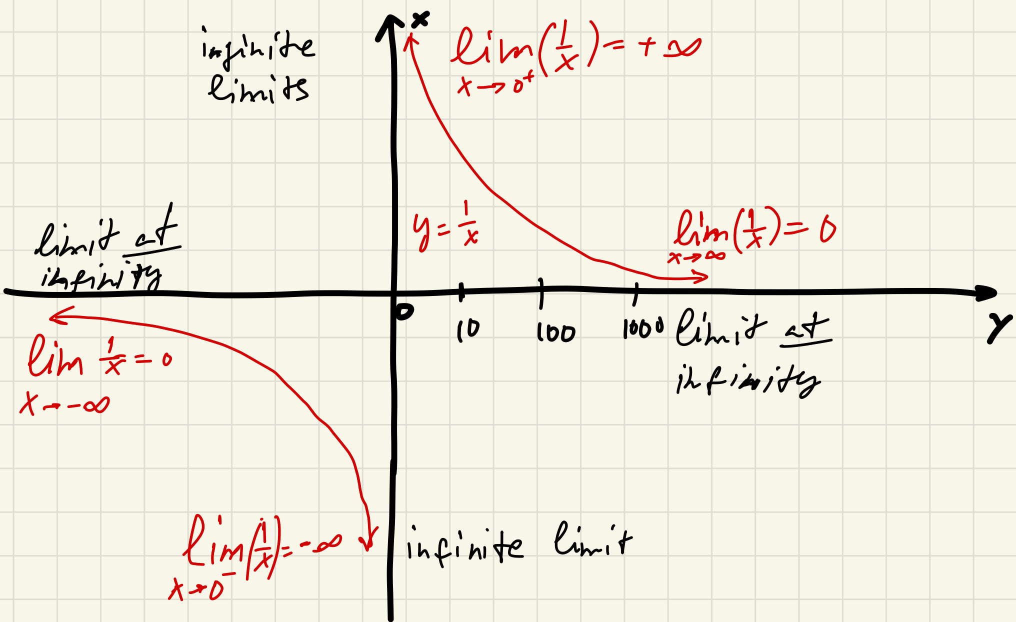
2.6 Limits at infinity

def. The limit as x goes to $+$ infinity of $f(x)$ is the value C that $f(x)$ approaches as x gets arbitrarily big.

Notation: $\lim_{x \to \infty} f(x) = C$

Similarly, when x gets arbitrarily negative large, we have $\lim_{x \to -\infty} f(x)$

Rmk do not confuse them with infinite limits



algebraic explanation of limits at ∞:
of $f(x) = \frac{1}{x}$

x	10	100	1000	$\to \infty$
$f(x)$	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\to 0$

explains $\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0$

x	-10	-100	-1000	$\to -\infty$
$f(x)$	$\frac{1}{-10}$	$\frac{1}{-100}$	$\frac{1}{-1000}$	$\to 0$

explains $\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0$

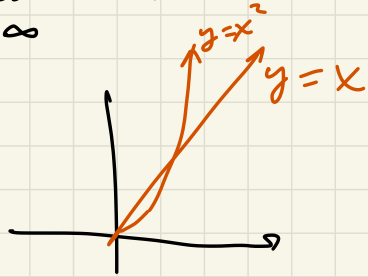
<u>Important fact</u> if A is any number,
then $\lim_{\substack{x \to \infty \\ \text{or} \\ x \to -\infty}} \frac{A}{x^N} = 0$ if $N > 0$
(ex: $N = \frac{1}{2}$, $N = 7$ ...)

going to use over & over

<u>ex.</u> $\lim_{x \to \infty} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{\infty}} = \frac{1}{\infty} = 0$

<u>another fact</u>

$\lim_{x \to \infty} x^N = \infty$ if $N > 0$



<u>ex.</u>
$\lim_{x \to \infty} x^3 + x = \infty + \infty =$
$= \infty$

ex.2 (typical more complicated example)

$$\lim_{x \to \infty} \frac{x^2+x+1}{3x^3+2x+5}$$

1st step: "substitute" $\frac{\infty^2+\infty+1}{3\infty^3+2\infty+5} = \frac{\infty}{\infty}$

(for $\frac{1}{x}$ we would get $\frac{1}{\infty} = 0$)

simplify

2nd step: $\boxed{\text{divide by the highest power of } x \text{ in the denominator}}$

key step

$$\lim_{x \to \infty} \frac{x^2+x+1 \quad \div x^3}{3x^3+2x+5 \quad \div x^3} = \lim_{x \to \infty} \frac{\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{3+2\frac{1}{x^2}+\frac{5}{x^3}}$$

now substitute:

$$\frac{\cancel{\frac{1}{\infty}}^{0}+\cancel{\frac{1}{\infty^2}}^{0}+\cancel{\frac{1}{\infty^3}}^{0}}{3+2\cdot\underbrace{\cancel{\frac{1}{\infty^2}}}_{0}+\underbrace{\cancel{\frac{5}{\infty^3}}}_{0}} = \frac{0+0+0}{3+0+0} =$$

$$= \frac{0}{3} = \boxed{0} \longleftarrow \text{answer}$$

ex.3 $\lim\limits_{x\to\infty} \dfrac{4x^4-x^2-1}{-x^4+x}$

step 1 sub ∞ : $\dfrac{4\infty^4-\infty^2-1}{-(\infty)^4+\infty} = \dfrac{\infty}{-\infty}$,

so we need to simplify

step 2

$$\lim_{x\to\infty} \frac{4x^4-x^2-1 \quad \div x^4}{-x^4+x \quad \div x^4} = \lim_{x\to\infty} \frac{4-\frac{1}{x^2}-\frac{1}{x^4}}{-1+\frac{1}{x^3}}$$

$$= \frac{4 - \cancel{\frac{1}{\infty^2}}^{\to 0} - \cancel{\frac{1}{\infty^4}}^{\to 0}}{-1+\cancel{\frac{1}{\infty^3}}_{\to 0}} = \frac{4}{-1} = \boxed{-4}$$

answer

ex. 4 $\lim\limits_{x\to\infty} \dfrac{x^2-1 \quad \div x}{x+2 \quad \div x} =$

$$\lim_{x\to\infty} \frac{x-\cancel{\frac{1}{x}}^{\to 0}}{1+\cancel{\frac{2}{x}}_{\to 0}} = \frac{\infty-\cancel{\frac{1}{\infty}}^{\to 0}}{1+\cancel{\frac{2}{\infty}}_{\to 0}} = \frac{\infty}{1} = \infty$$

Shortcut:

two polynomials

$$\lim_{x\to\infty} \frac{ax^k + \overset{\text{lower degrees}}{\ldots}}{ax^m + \underset{\text{lower degrees}}{\ldots}} = \begin{cases} 0 \\ \text{number} \\ +\infty \\ -\infty \end{cases}$$

1) 0 if $m>k$

2) equal to $\frac{a_1}{a_2}$ if $k=m$

3) $+\infty$ or $-\infty$ if $m<k$

In order to decide between $+\infty$ and $-\infty$, substitute $x=1000$, and check the sign

Don't recommend using this, since sometimes the shortcut doesn't work, if fraction includes roots, e^x, $\sin x$ ...

ex. 5 $\lim\limits_{t \to \infty} \dfrac{\sqrt{t} + (\sqrt{t} \cdot t^2)}{2t - (\sqrt{t} \cdot t^2)} =$

sub ∞ gives $\dfrac{\infty}{\infty - \infty}$, so need to simplify.

$$= \lim_{t \to \infty} \frac{t^{\frac{1}{2}} + t^{\frac{5}{2}}}{2t^1 - t^{\frac{5}{2}}} \begin{matrix} \div t^{\frac{5}{2}} \\ \div t^{5/2} \end{matrix}$$

$$= \lim_{t \to \infty} \frac{\frac{t^{1/2}}{t^{5/2}} + 1}{2\frac{t}{t^{5/2}} - 1} = \lim_{t \to \infty} \frac{\cancelto{0}{\frac{1}{t^2}} + 1}{2\cancelto{0}{\frac{1}{t^{3/2}}} - 1} =$$

$$= \frac{1}{-1} = -1$$

$$\boxed{\frac{t^{1/2}}{t^{5/2}} = \frac{1}{t^{5/2 - \frac{1}{2}}} = \frac{1}{t^2}}$$

how I got $t^{5/2}$:

$$\boxed{\sqrt{t} \cdot t^2 = t^{1/2} \cdot t^2 = t^{\frac{1}{2} + 2} = t^{\frac{1}{2} + \frac{4}{2}} = t^{\frac{5}{2}}}$$

ex. 6

$$\lim_{x \to \infty} \frac{8 - x^3}{\sqrt{1 + 4x^6}} \quad \begin{matrix} \div x^3 \\ \div x^3 \end{matrix}$$

(Because $\sqrt{x^6} = x^3$)

$$\lim_{x \to \infty} \frac{\frac{8}{x^3} - 1}{\frac{1}{x^3}\sqrt{1 + 4x^6}} =$$

$$= \lim_{x \to \infty} \frac{\frac{8}{x^3} - 1}{\frac{\sqrt{1+4x^6}}{\sqrt{x^6}}} =$$

($\sqrt{x^6} = |x^3|$, but in our case $\sqrt{x^6} = x^3$ because $x \to \infty$, and so we know $x > 0$)

$$= \lim_{x \to \infty} \frac{\frac{8}{x^3} - 1}{\sqrt{\frac{1+4x^6}{x^6}}} = \lim_{x \to \infty} \frac{\overbrace{\frac{8}{x^3}}^{0} - 1}{\sqrt{\underbrace{\frac{1}{x^6}}_{0} + 4}} =$$

$$= \frac{-1}{\sqrt{4}} = \boxed{\frac{-1}{2}}$$

answer!

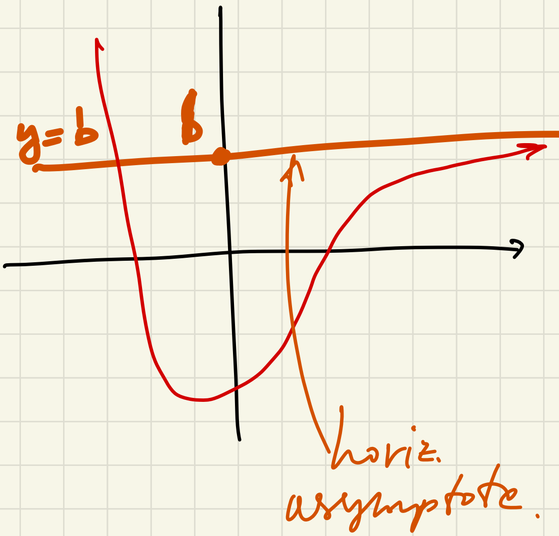
Q. $$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{\sqrt{3+4x^4}} \qquad \begin{matrix} \div x^2 \\ \div x^2 \end{matrix}$$

By which degree to divide?

x^2! Because $\sqrt{x^4} = x^2$

def $f(x)$ has a horizontal asymptote $y=b$ if

$$\lim_{\substack{x\to +\infty \\ \text{or} \\ -\infty}} = b$$

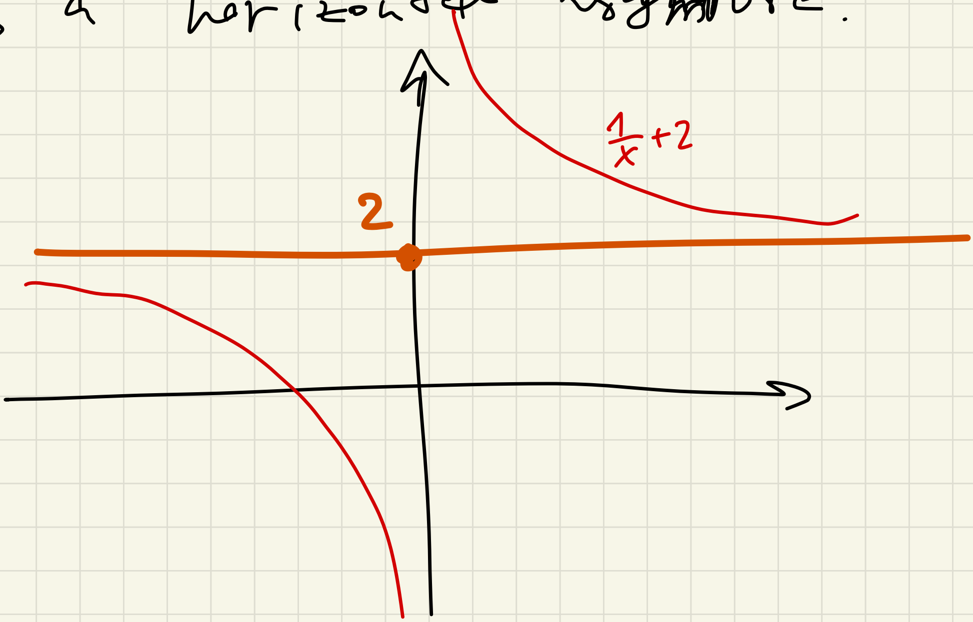


ex. find hor. asymptote of $\frac{1}{x}+2$.

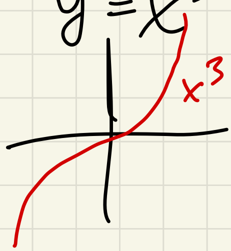
Solution $$\lim_{x\to\infty} \frac{1}{x}+2 = \cancel{\frac{1}{\infty}}^{\to 0}+2 = 2$$

$$\lim_{x\to-\infty} \frac{1}{x}+2 = \underbrace{\frac{1}{-\infty}}_{0} + 2 = 2$$

so we conclude that $y=2$ is a horizontal asymptote.



ex: horiz. asymptotes of $y=x^3$?



$$\lim_{x\to\infty} x^3 = \infty^3 = \infty$$

$$\lim_{x\to-\infty} = (-\infty)^3 = -\infty$$

not numbers, so no horizontal asymptotes

<u>def</u> $f(x)$ has <u>vertical asymptote</u>

$x=a$ if $\lim\limits_{\substack{x\to a^+ \\ \text{or} \\ a^-}} = \begin{matrix} +\infty \\ \text{or} \\ -\infty \end{matrix}$

<u>upshot</u>

horizonta asymptotes $\longleftrightarrow$ limit <u>at</u> ∞

$y=b \longleftrightarrow \lim\limits_{x\to\infty} f(x) = b$

vertical asymptotes $\longleftrightarrow$ infinite limits

$x=a \longleftrightarrow \lim\limits_{x\to a^+ \text{ or } a^-} f(x) = \begin{matrix} +\infty \\ \text{or} \\ -\infty \end{matrix}$

<u>ex.</u> $f(x) = \dfrac{x-2}{x-3}$

- horiz. asymptotes? $\lim\limits_{x\to\infty} \dfrac{x-2 \;\div x}{x-3 \;\div x} =$

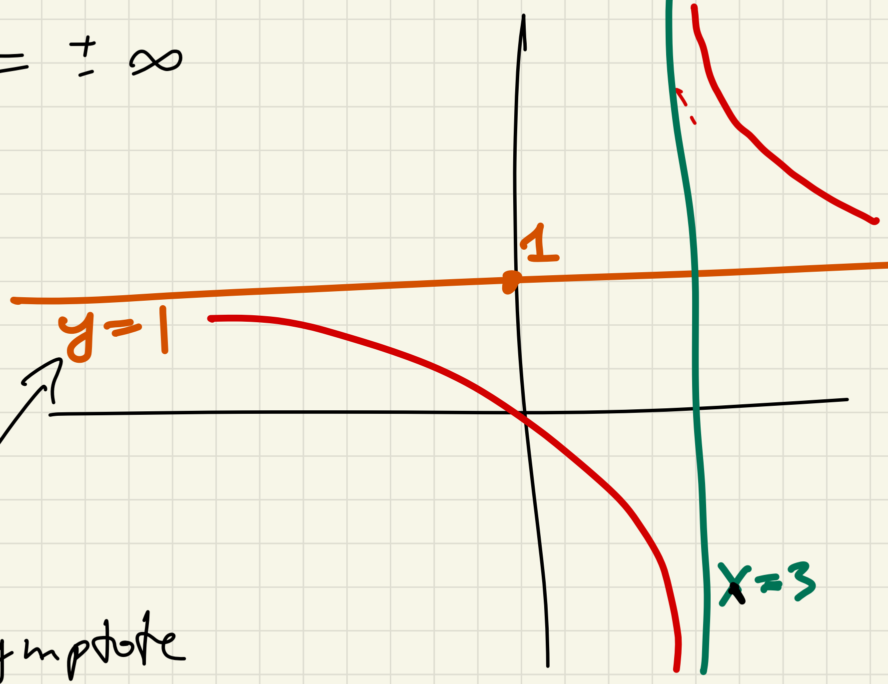
(same for $\lim\limits_{x\to -\infty} \dots$) $= \lim\limits_{x\to\infty} \dfrac{1-\cancel{2/x}\to 0}{1-\cancel{3/x}\to 0} = \dfrac{1}{1} = 1$

so we have $y=1$ hor. asymptote

- vertical asymptotes?

$$\lim_{x \to a^{\pm}} \frac{x-2}{x-3} = \pm\infty$$

$\boxed{a=3}$!

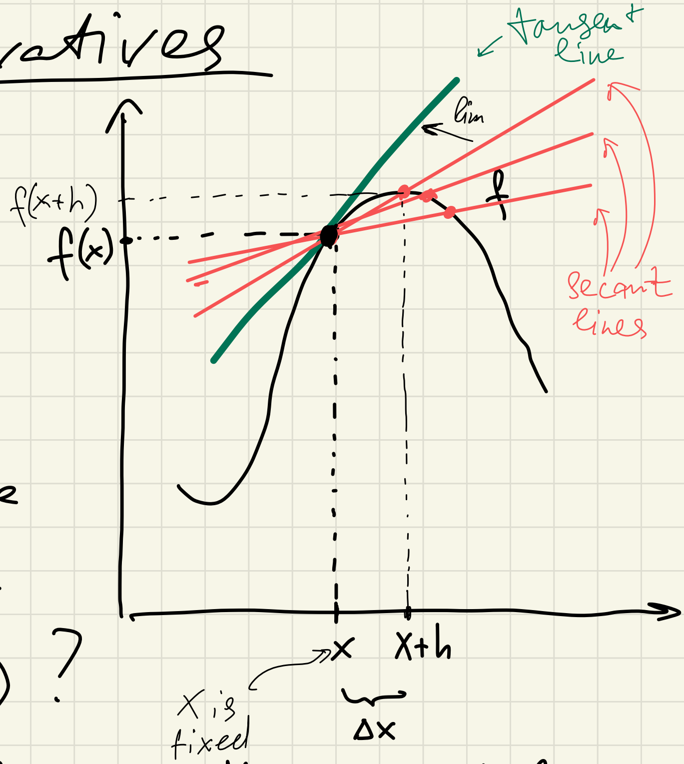


hor. asymptote

vertical asymptote

2.7. Derivatives

Q. Given a function f and a number x, what is the slope (derivative) of f at point $(x, f(x))$?



Idea: it is a slope of the tangent line.

But how to find the tangent line?

A. Tangent is the limit of secant lines

$$\boxed{\text{slope of the secant line}} = \frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{\cancel{x}+h-\cancel{x}} = \frac{f(x+h)-f(x)}{h}$$

Thus, to find the slope of tangent line, we take the limit

$$\boxed{\text{slope of the tangent line at } (x, f(x))} \underset{\text{definition}}{=\!=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

" derivative!

Def. Given a function f, the derivative at x is given by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Important: $f'(x)$ gives slope of $f(x)$ at the point $(x, f(x))$

Def. a line $y = mx + b$ is tangent to $f(x)$ at $(a, f(a))$ if

1) $y = mx + b$ passes through $(a, f(a))$
2) $m = f'(a)$

Remark: • sometimes you see $f'(x)$, and sometimes $f'(a)$. This just means a instead of x:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

this is a derivative at $x = a$

• $f'(x)$ is a derivative at a point x, and therefore it is a function.

ex 1 Compute $f'(x)$ for $f(x) = 2x+1$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

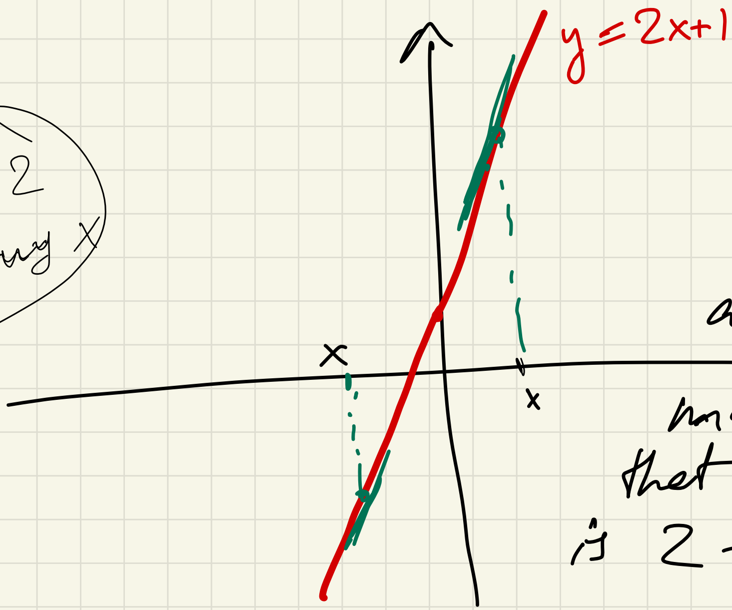
by definition

plug $(x+h)$ for x

$$f(x+h) = 2(x+h)+1 =$$
$$= 2x+2h+1$$

$$= \lim_{h \to 0} \frac{\overbrace{\cancel{2x}+2h\cancel{+1}}^{f(x+h)} - \overbrace{(\cancel{2x}+\cancel{1})}^{f(x)}}{h} =$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}} = \boxed{2} \leftarrow \text{answer!}$$



$f'(x) = 2$ for any x

at any point x, the slope of $2x+1$ is 2, and so it makes sense that its derivative is 2 everywhere

Ex. 2 $f(x) = x^2 + 1$, $f'(x) = ?$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$(x+h)^2 = x^2 + 2xh + h^2$

$$= \lim_{h \to 0} \frac{\overbrace{(x+h)^2 + 1}^{f(x+h)} - \overbrace{(x^2+1)}^{f(x)}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 \cancel{+1} - \cancel{x^2} \cancel{-1}}{h} =$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} =$$

$$= \lim_{h \to 0} (2x + h) = 2x + 0 = \boxed{2x}$$

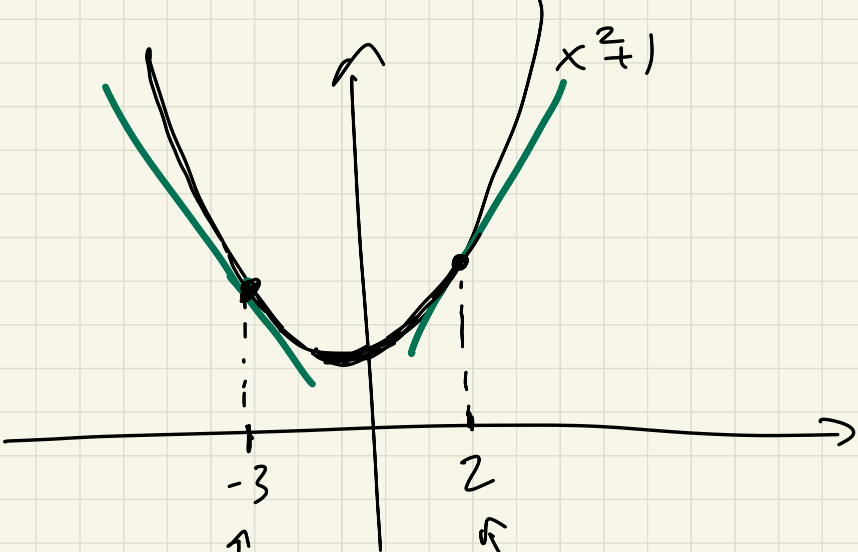
treat as a constant

plug in $h = 0$

answer

$f(x) = x^2 + 1$

$f'(x) = 2x$ ← slope of $f(x)$ at point x

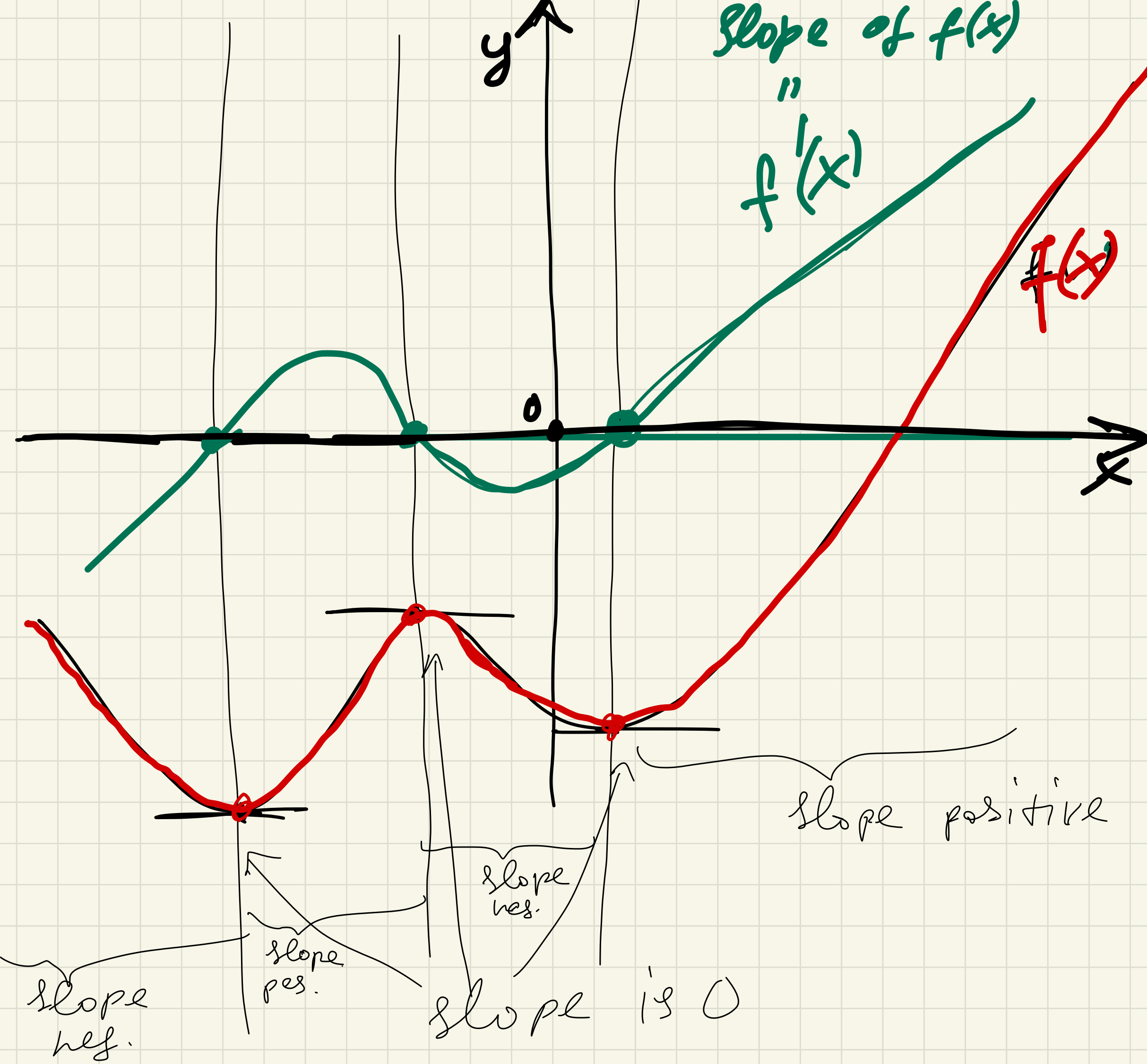


slope here $f'(-3) = 2\cdot(-3) =$
$= -6$

slope here $f'(2) = 2\cdot 2 = 4$

$f'(x)$ is a function that gives <u>slope</u> of $f(x)$ at each point.

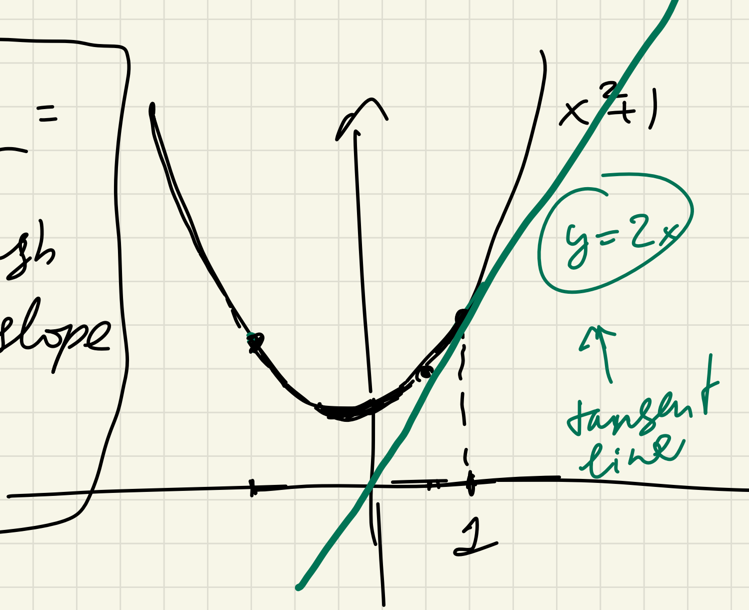
<u>Graphical example 3</u>



Ex.4 find tangent line to

$$f(x) = x^2+1 \quad \text{at} \quad x=1$$

Tangent line = the line through $(a, f(a))$ with slope $f'(a)$



in our case: $f(x)=x^2+1$, $f'(x)=2x$

$a=1$

$y=mx+b$ solve for m & b.

1) $f'(x)=2x$ ← done in ex.

2) slope = derivative

$$m = f'(1) = 2 \cdot 1 = 2$$

3) $y=2x+b$ has to go through $(a, f(a))$ "

$(1, f(1))$ "

$(1, 1^2+1)$

goes through $x=1$, $y=2$

$$2 = 2 \cdot 1 + b$$

$\downarrow$ $(1, 2)$

$2 - 2 = b$

$0 = b$

tangent line is $\boxed{y = 2 \cdot x + 0}$

So if we are asked to find tangent line to $f(x)$ at $x = a$, we do

1) compute $f'(x)$

sub a into $f'(x)$

2) slope: $m = f'(a)$

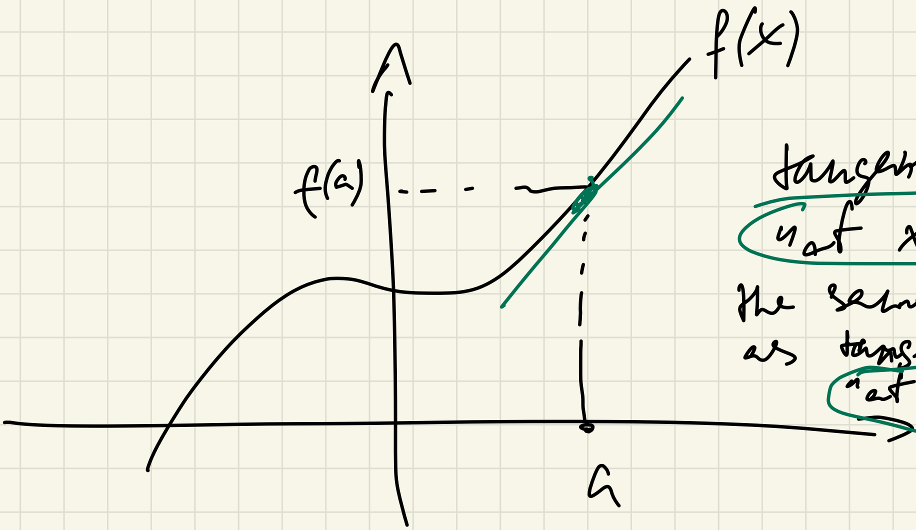
3) solve for b the fact that $y = f'(a) \cdot x + b$ goes through $(a, f(a))$

1) Understand where and why you lost your points

2) If you got <70 curved grade, you need change something

catch up with material

change the way you study



tangent line "at $x=a$" is the same thing as tangent line "at $(a, f(a))$"

Continueing derivatives (2.7 + 2.8)

def. Tangent line to a function $f(x)$ at $x=a$ is the line of slope $f'(a)$ that goes through the point $(a, f(a))$

ex. Find the tangent line to $y=\sqrt{x}$ at $x=1$.

1) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$

$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})\cdot(\sqrt{x+h}+\sqrt{x})}{h\cdot(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0} \frac{(\cancel{x}+h)-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})} =$

$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} =$

steps: 1) $f'(x) = \ldots$

2) finding the tangent line $y = mx+b$ by
- slope $= f'(1)$
- goes through $(1, f(1))$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

sub $h=0$

$f(x) = \sqrt{x}$ ← function

$f'(x) = \frac{1}{2\sqrt{x}}$ ← derivative, gives the slope of $f(x)$ at each point

2) tangent line to $\sqrt{x}$ at $x=1$

$y = mx + b$ → has slope $f'(1)$

→ goes through $(1, f(1))$

- slope $= m = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

- $y = \frac{1}{2}x + b$ goes throug $(1, \underset{f(1)}{\sqrt{1}}) = (\underset{x}{1}, \underset{y}{1})$

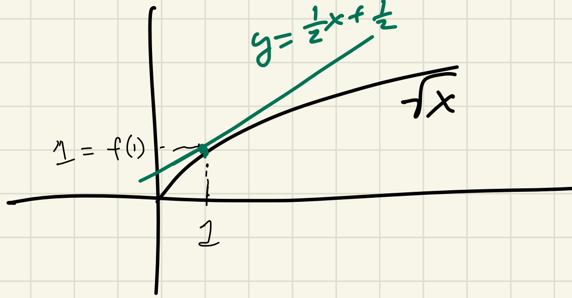
plug in & solve for b

$1 = \frac{1}{2} \cdot 1 + b$

$\frac{1}{2} = b$

So tangent line to $\sqrt{x}$ at $x = 1$

is $\quad y = \frac{1}{2}x + \frac{1}{2}$

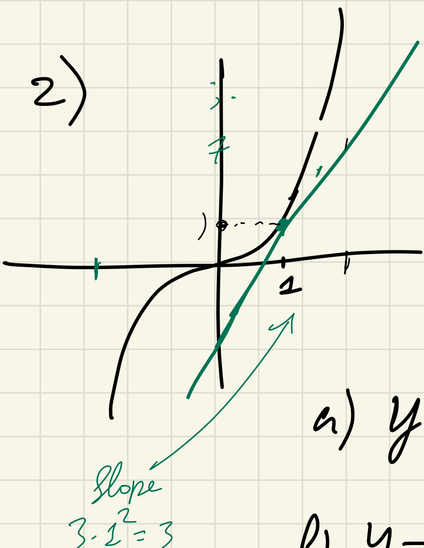


<u>ex.</u> $f(x) = x^3$. Find tangent line at $x = 1$

1) $f'(x) = \underbrace{\lim_{h \to 0} \frac{\dots}{\dots} = \dots}_{\text{missed steps}} = 3x^2$

you need a formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

2)



tangent line at $x = 1$

a) slope $= f'(1)$

b) goes through $(1, f(1))$

a) $y = mx + b \qquad m = f'(1) = 3 \cdot 1^2 = 3$

b) $y = 3x + b$ goes through $(1, f(1)) = (1, 1^3) =$
$= (1, 1)$

$1 = 3 \cdot 1 + b$

$-2 = b$

$\rightarrow y = 3x - 2$

formula for the tangent line

Tangent line to $f(x)$ at $x=a$ has the formula

$$\boxed{y = f'(a)(x-a) + f(a)}$$

so, to find the tangent line you only need $f'(a)$ and $f(a)$.

ex. $f(x) = \dfrac{1}{x-3}$. Find the tangent line at $x=4$?

$$y = f'(4)(x-4) + f(4)$$

we only need these

· $f(4) = \dfrac{1}{4-3} = \dfrac{1}{1} = 1$

· $f'(4) = ?$ $\quad f'(x) = \lim\limits_{h \to 0} \dfrac{\overbrace{\dfrac{1}{x+h-3}}^{f(x+h)} - \overbrace{\dfrac{1}{x-3}}^{f(x)}}{h} =$

$$= \lim_{h \to 0} \frac{\dfrac{(\cancel{x-3}) - (\cancel{x}+h\cancel{-3})}{(x+h-3)(x-3)}}{h} =$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{(x+h-3)(x-3)\cancel{h}} =$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-3)(x-3)} \underset{\substack{\text{sub}\\ h=0}}{=} -\frac{1}{(x-3)^2} = f'(x)$$

$$\boxed{f'(4) = -\frac{1}{(4-3)^2} = -1}$$

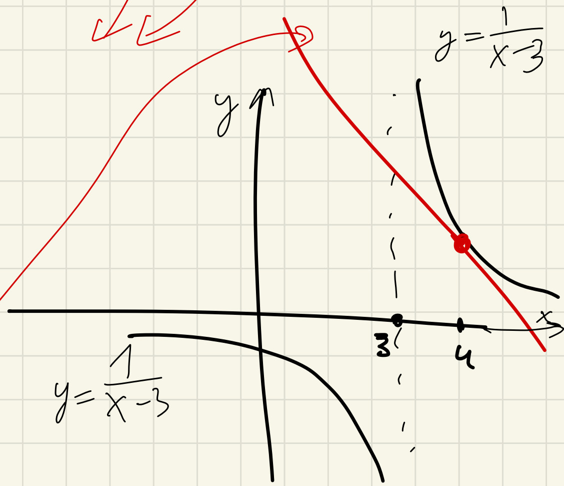
tangent line to $f(x)$ at $x=4$

$$y = f'(4)(x-4) + f(4)$$

$$y = (-1)(x-4) + 1$$

$$\boxed{y = 5 - x}$$

tangent line



def. $f(x)$ is differentiable at $x=a$ if $f'(a)$ exists ex. Where is $f(x) = \dfrac{1}{x-3}$ differentiable?

Strategy: compute $f'(x)$, and find its domain.

$$f'(x) = -\frac{1}{(x-3)^2}$$

see the previous example

Where does this $f'(x)$ exist?
(or whats the domain of $f'(x)$?)

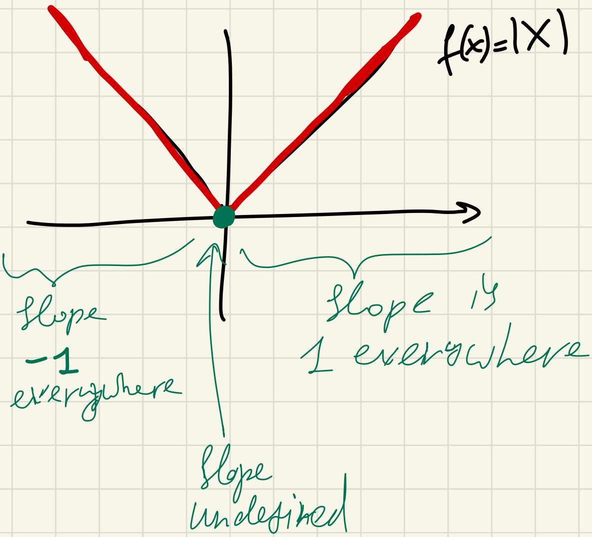
Everywhere except $x=3$, since we divide by $(x-3)^2$.

So domain of $f'(x) = -\dfrac{1}{(x-3)^2}$ is

$$(-\infty, 3) \cup (3, \infty)$$

and so $f(x) = \dfrac{1}{x-3}$ is differentiable everywhere except $x=3$.

ex. (continuous but not differentiable at $x=0$)

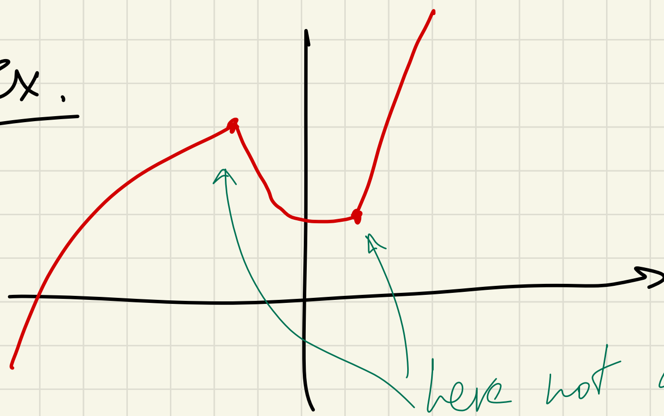


it turns out that

$$(|x|)' = \begin{cases} 1 & \text{if } x>0 \\ \text{DNE} & \\ -1 & \text{if } x<0 \end{cases}$$

at $x=0$ $f(x)=|x|$ is not differentiable, because there is a "change in direction"

ex.



Remark

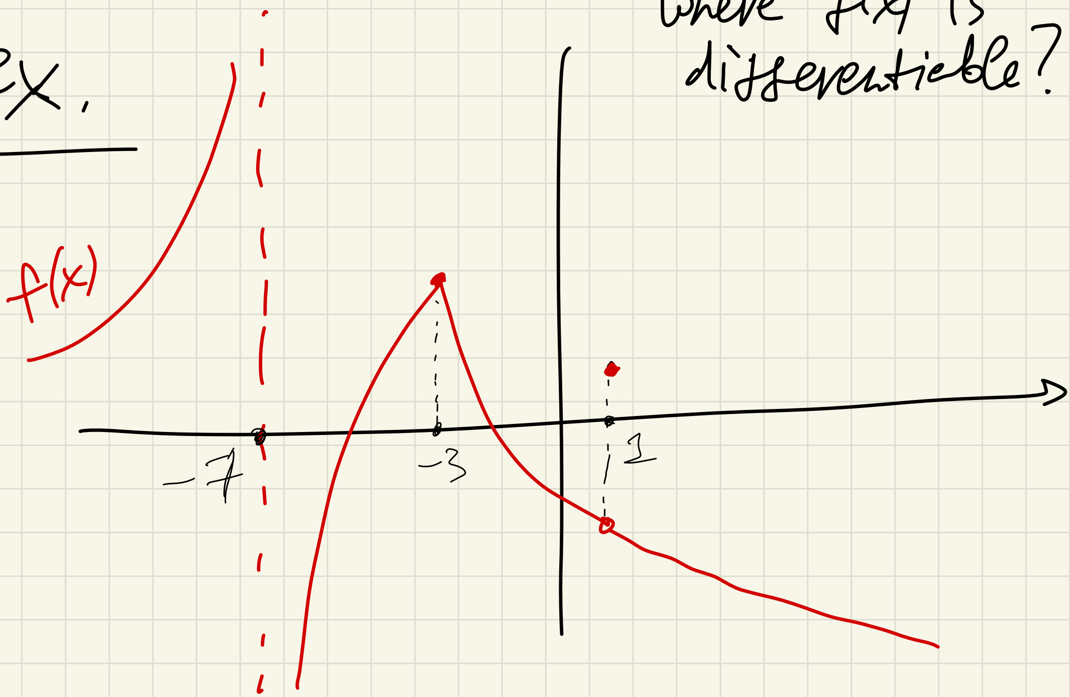
1) If $f(x)$ is differentiable at $x=a$, then it is continuous at $x=a$!

2) Vice versa not true!

$|x|$ continuous at $x=0$
not differentiable at $x=0$

ex.

Where $f(x)$ is differentiable?



Intuition: everywhere where $f(x)$ is "smooth".

answer:

$$(-\infty,-7)\cup(-7,-3)\cup(-3,1)\cup(1,+\infty)$$

(1 is excluded, since $f(x)$ is not even continuous there)

Computing derivatives in a faster/more algebraic way

Chapter 3

3.1 Polynomials and exponentials

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

New notation instead of $f'(x)$ we will frequently write $\frac{d}{dx}[f(x)]$

Both are derivatives!

Rules (shortcuts) for taking derivatives:

① $\boxed{\frac{d}{dx}[c] = 0}$, where c is any constant

② $\boxed{\frac{d}{dx}[x^n] = n \cdot x^{n-1}}$, where n is any non-zero number

"power rule"

- $\frac{d}{dx}[x^4] = 4 \cdot x^3$, $\frac{d}{dx}[x^{3/61}] = \frac{3}{61} \cdot x^{-\frac{58}{61}}$
- When x is in denominator, move it to numerator.

Example: $\frac{d}{dx}\left[\frac{1}{x^2}\right] = \frac{d}{dx}[x^{-2}] = (-2) \cdot x^{-3} = \frac{-2}{x^3}$

- if x is under radical, turn into exponent

Example: $\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{\frac{1}{3}}] = \frac{1}{3}x^{-\frac{2}{3}}$

$\left(\frac{1}{3} - 1 = -\frac{2}{3}\right)$

"ignore the constant"

③ $\boxed{\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]}$, where c is any constant

"derivative of a sum = sum of derivatives"

④. $\boxed{\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]}$

- the same for subtraction

Warning: cannot do this for multiplication or division.

⑤. $\boxed{\frac{d}{dx}[e^x] = e^x}$

Examples

- $\frac{d}{dx}[x^2] \underset{②}{=} 2 \cdot x^1 = \underline{2x}$ ② power rule

- $\frac{d}{dx}[5x^2] \overset{③}{=} 5\frac{d}{dx}[x^2] \overset{②}{=} 5 \cdot 2x =$ ignoring constant

  $= \underline{10 \cdot x}$

- $\frac{d}{dx}\left[\frac{1}{7\sqrt[3]{x}}\right] = \frac{d}{dx}\left[\frac{1}{7} \cdot \frac{1}{x^{1/3}}\right] =$

  $= \frac{d}{dx}\left[\frac{1}{7}x^{-\frac{1}{3}}\right] \underset{③}{=} \frac{1}{7}\frac{d}{dx}[x^{-\frac{1}{3}}] \underset{②}{=}$

$$= \frac{1}{7} \cdot \left(-\frac{1}{3}\right) x^{-\frac{1}{3}-1} = \underline{-\frac{1}{21} \cdot x^{-\frac{4}{3}}}$$

- $\frac{d}{dx}[13] \overset{①}{=} 0$, $\frac{d}{dx}[0] \overset{①}{=} 0$

$$\frac{d}{dx}[z] = 0$$ (if z does not depend on x)

- $\frac{d}{dx}[m \cdot x] \overset{③}{=} m \cdot \frac{d}{dx}[x^1] = m \cdot 1 \cdot x^0 = \underline{m}$

parameter, treat as a constant

"derivate term by term"

- $\frac{d}{dx}[6x^2 + x + 1] \overset{④}{=} \frac{d}{dx}[6x^2] + \frac{d}{dx}[x] + \frac{d}{dx}[1]$

$$= 6 \cdot 2x + 1 + 0 = \underline{12x+1}$$

$$\left(\frac{d}{dx}[x] = \frac{d}{dx}[x^1] = 1 \cdot x^0 = 1 \right)$$

<u>Claim</u> We can find the derivative of any polynomial!

$$\frac{d}{dx}[17x^5 - 60x^4 + 3x^3 + 8] =$$

$= \left(\text{do it term by term, according to rule ④}\right) =$

$$= \frac{d}{dx}[17x^5] - \frac{d}{dx}[60x^4] + \frac{d}{dx}[3x^3] + \frac{d}{dx}[8] = 17\cdot 5\cdot x^{5-1} - 60\cdot 4x^{4-1} + 3\cdot 3\cdot x^{3-1} + 0 = \underline{85x^4 - 240x^3 + 9\cdot x^2}$$

For general coefficients (parameters) you set

term by term　　power rule

$$\frac{d}{dx}[ax^n + bx^{n-1} + \dots + gx + h] = (\text{rule ④ \& ②})$$

$$= \underbrace{a\cdot n\cdot x^{n-1} + b\cdot(n-1)x^{n-2} + \dots + g + 0}$$

- $f(x) = 6x^3 - \frac{1}{\sqrt{x}} + 3e^x$

$$f'(x) = \frac{d}{dx}\left[6x^3 - \frac{1}{\sqrt{x}} + 3e^x\right] \overset{④}{=} \left(\text{find } \frac{d}{dx} \text{ term by term}\right)$$

$$= \frac{d}{dx}[6x^3] - \frac{d}{dx}\left[\frac{1}{\sqrt{x}}\right] + \frac{d}{dx}[3e^x] \overset{③}{=}$$

("ignore constants")

$$= 6\frac{d}{dx}[x^3] - \frac{d}{dx}[x^{-\frac{1}{2}}] + 3\frac{d}{dx}[e^x] =$$

"power rule, degree goes down"

(2)

$$= 6\cdot 3x^2 - \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} + 3\cdot e^x -$$

$$= \boxed{18x^2 + \frac{1}{2}x^{-\frac{3}{2}} + 3e^x}$$

Remark you need to use the limit definition of $f'(x)$ in 2.7, 2.8, and whenever you are explicitly asked to. But starting with 3.1 you can use shortcuts def. Second derivative is the derivative of derivative:

$$f''(x) = \frac{d}{dx}\left[f'(x)\right]$$

ex. $f(x) = 7x^6$. $f''(x) = ?$

$$f'(x) = 7 \cdot 6 \cdot x^5 = 42 \cdot x^5$$

$$f''(x) = \frac{d}{dx}\left[f'(x)\right] = 42 \cdot 5 \cdot x^4 = 210x^4$$

def. Third derivative $f'''(x)$ is derivative of $f''(x)$.

ex. $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

fourth derivative

$f''''(x) = 0$

3.2 Product & quotient rules

for sums we derivate "term by term"

Recall: $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

This is not true for products:

$$\frac{d}{dx}[f(x)\cdot g(x)] \neq \frac{d}{dx}[f(x)]\cdot\frac{d}{dx}[g(x)]$$

Instead:

$(f\cdot g)' = f'\cdot g + f\cdot g'$

Product rule: $\boxed{\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)\cdot g'(x)}$

Quotient rule: $\boxed{\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}$

$\left(\frac{f}{g}\right)' = \frac{f'g - f\cdot g'}{g^2}$

ex. $\frac{d}{dx}[\underbrace{x^2}_{f}\cdot\underbrace{e^x}_{g}] = \underbrace{(x^2)'\cdot e^x}_{f'\cdot g} + \underbrace{x^2\cdot(e^x)'}_{f\cdot g'} =$

$= 2x\cdot e^x + x^2\cdot e^x$

ex. $\frac{d}{dx}\left[\frac{1}{x+3}\right] = \frac{(1)'\cdot(x+3) - 1\cdot(x+3)'}{(x+3)^2} =$

$1 = f$, $x+3 = g$

$(x)' = 1$ $(3)' = 0$

$\frac{f'\cdot g - f\cdot g'}{g^2}$

$= \frac{0\cdot(x+3) - 1\cdot(1+0)}{(x+3)^2} =$

$= \boxed{-\frac{1}{(x+3)^2}}$

ex. $\frac{d}{dx}\left[(7x+5x^2)e^x\right] = \frac{d}{dx}\left[7x+5x^2\right]\cdot e^x + (7x+5x^2)\cdot\frac{d}{dx}\left[e^x\right]$

$7x+5x^2 = f$, $e^x = g$

$f'\cdot g + f\cdot g'$

$= (7+10x)\cdot e^x + (7x+5x^2)\cdot e^x =$

$= \boxed{e^x(7+17x+5x^2)}$

ex. $\frac{d}{dx}\left[(x^3+x^2+x)\cdot(e^x+3x)\right] = f'\cdot g + f\cdot g' =$

$x^3+x^2+x = f$, $e^x+3x = g$

$= \frac{d}{dx}\left[x^3+x^2+x\right]\cdot(e^x+3x) + (x^3+x^2+x)\cdot\frac{d}{dx}\left[e^x+3x\right] =$

do not forget do put parenthesis

$$= (3x^2+2x+1)(e^x+3x) + (x^3+x^2+x)(e^x+3) =$$

$$= 3x^2e^x + 2x\cdot e^x + e^x + 9x^3 + 6x^2 + 3x$$

$$+ x^3e^x + x^2e^x + xe^x + 3x^3 + 3x^2 + 3x$$

$$= \boxed{e^x(x^3+4x^2+3x+1) + 12x^3 + 9x^2 + 6x}$$

ex. $\dfrac{d}{dx}\left[\dfrac{\overbrace{1-e^x}^{f}}{\underbrace{1+e^x}_{g}}\right] =$ quotient rule $\dfrac{f'\cdot g - f\cdot g'}{g^2}$

$$= \frac{\overbrace{\frac{d}{dx}[1-e^x]}^{f'}\cdot\overbrace{(1+e^x)}^{g} - \overbrace{(1-e^x)}^{f}\overbrace{\frac{d}{dx}[1+e^x]}^{g'}}{\underbrace{(1+e^x)^2}_{g^2}} =$$

$$= \frac{(0-e^x)(1+e^x) - (1-e^x)(0+e^x)}{(1+e^x)^2} =$$

$$= \frac{-e^x - \cancel{e^{2x}} - e^x + \cancel{e^{2x}}}{(1+e^x)^2} = \frac{-2e^x}{(1+e^x)^2}$$

Sometimes we have to use two rules in a sequence:

ex. $$\frac{d}{dt}\left[\frac{\overset{f}{7-t}}{\underset{g}{t\cdot e^t}}\right] =$$

quotient rule
$$\frac{f'\cdot g - f\cdot g'}{g^2}$$

$$= \frac{\frac{d}{dt}[7-t]\cdot t\cdot e^t - (7-t)\frac{d}{dt}[t\cdot e^t]}{(t\cdot e^t)^2} =$$

When you take derivative, don't forget parenthesis

product rule
$f'\cdot g + f\cdot g'$

$$= \frac{(0-1)\cdot t\cdot e^t - (7-t)\left(\frac{d}{dt}[t]\cdot e^t + t\cdot\frac{d}{dt}[e^t]\right)}{t^2 e^{2t}}$$

Because $e^t\cdot e^t = e^{t+t} = e^{2t}$

$$= \frac{-t e^t - (7-t)(e^t + t\cdot e^t)}{t^2\cdot e^{2t}} =$$

watch out for signs!

$$= \frac{\cancel{-te^t} - 7e^t - 7t\cdot e^t \cancel{+te^t} + t^2 e^t}{t^2\cdot e^{2t}} =$$

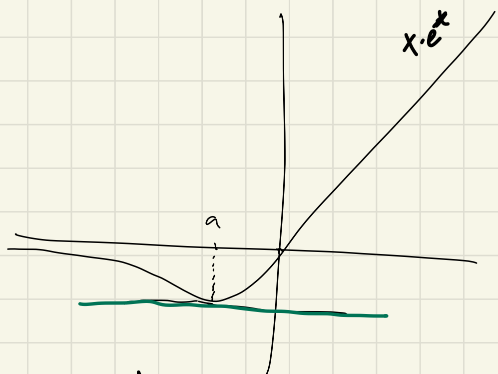
$$= \frac{-7e^t - 7te^t + t^2e^t}{t^2 e^{2t}} = \frac{\cancel{e^t}(-7-7t+t^2)}{\cancel{e^t}\cdot e^t \cdot t^2} =$$

$$= \frac{t^2 - 7t - 7}{e^t \cdot t^2}$$

Sometimes you are asked to find tangent lines of particular slopes:

<u>ex.</u> Find the tangent line to $f(x) = x \cdot e^x$ that is horizontal.

(usually: tangent line is
$y = f'(a)(x-a) + f(a)$
and we just find)



Here we are not given a! We have to find a first!

Instead we are given that the tangent line is horizontal, that is $f'(a) = 0$

- $f'(x) = \frac{d}{dx}\left[x \cdot e^x\right] \overset{\text{product rule}}{=} (x)' \cdot e^x + x \cdot (e^x)' =$

$$= e^x + x \cdot e^x = \boxed{e^x(1+x)}$$

a·b=0 means that either a=0 or b=0

- $f'(x) = 0$, so $e^x \cdot (1+x) = 0$

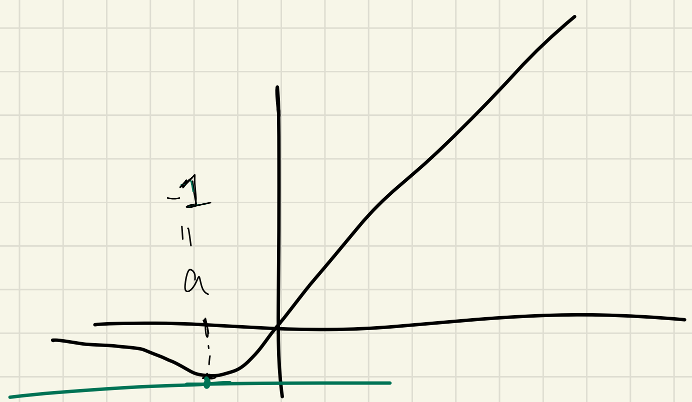
tangent line horizontal

never 0! So 1+x has to be 0.

$$1 + x = 0$$

$$x = -1$$

← this is point a



So, we found that $f'(x)=0$ is satisfied (tangent line horizontal)

at $X=-1$

(formula for tangent line is $\boxed{y=f'(a)(x-a)+f(a)}$)

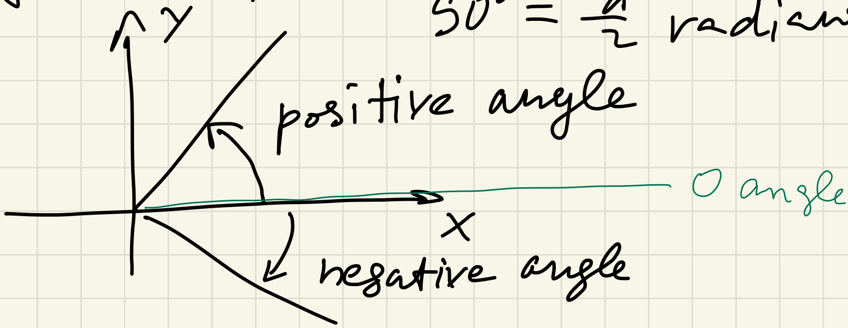
So $y=\underbrace{f'(-1)}_{=0}(x-(-1))+f(-1)$

$y=0\cdot\cancel{(x+1)}+(-1)\cdot e^{-1}$

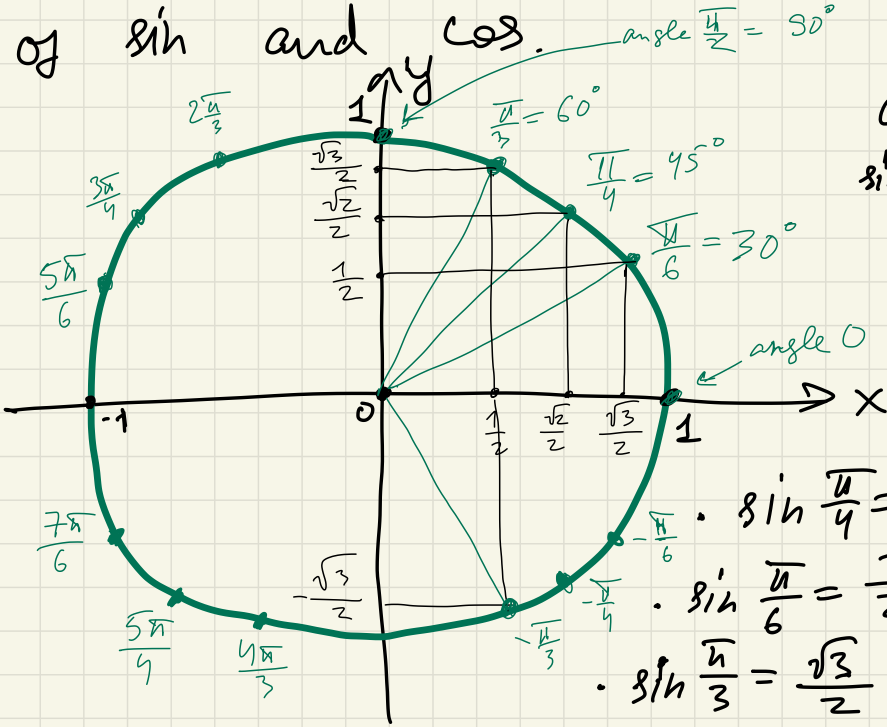
$\boxed{y=-\frac{1}{e}}$ ← the horizontal tangent line to $f(x)=x\cdot e^{x}$

Appendix D: Trig functions.

- We measure angles in radians, for example $180° = \pi$ radians
  $90° = \frac{\pi}{2}$ radians



- Unit circle gives specific values of sin and cos.



cos = x-values
sin = y-values

- $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$
- $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
- $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$

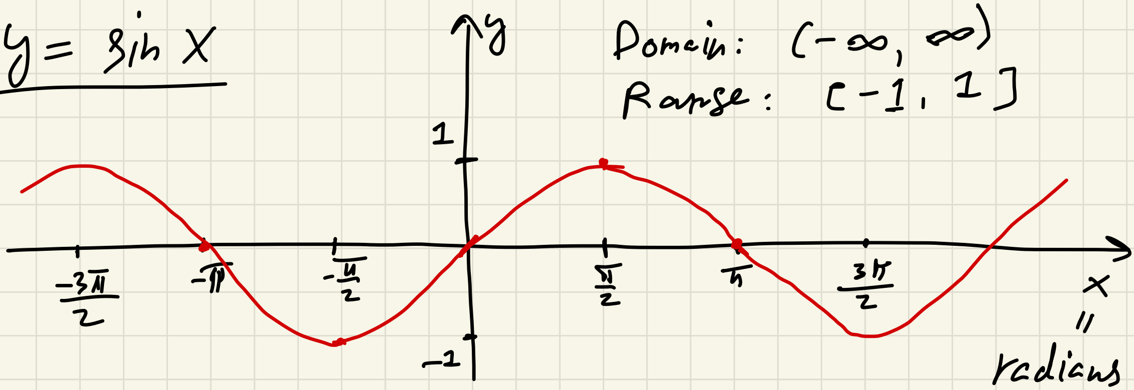
- $\sin 0 = 0$, $\cos 0 = 1$ · $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$
- $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$ · $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

Graphs of trig functions

$y = \sin x$

Domain: $(-\infty, \infty)$

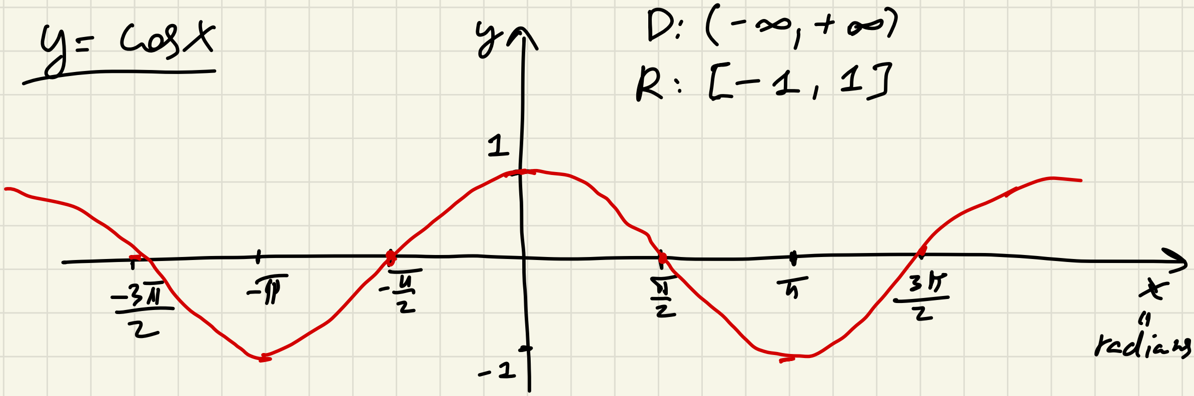
Range: $[-1, 1]$



$y = \cos x$

D: $(-\infty, +\infty)$

R: $[-1, 1]$



Further trig functions:

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{(cosecant) } \csc(x) = \frac{1}{\sin x}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad \text{(secant) } \sec(x) = \frac{1}{\cos x}$$

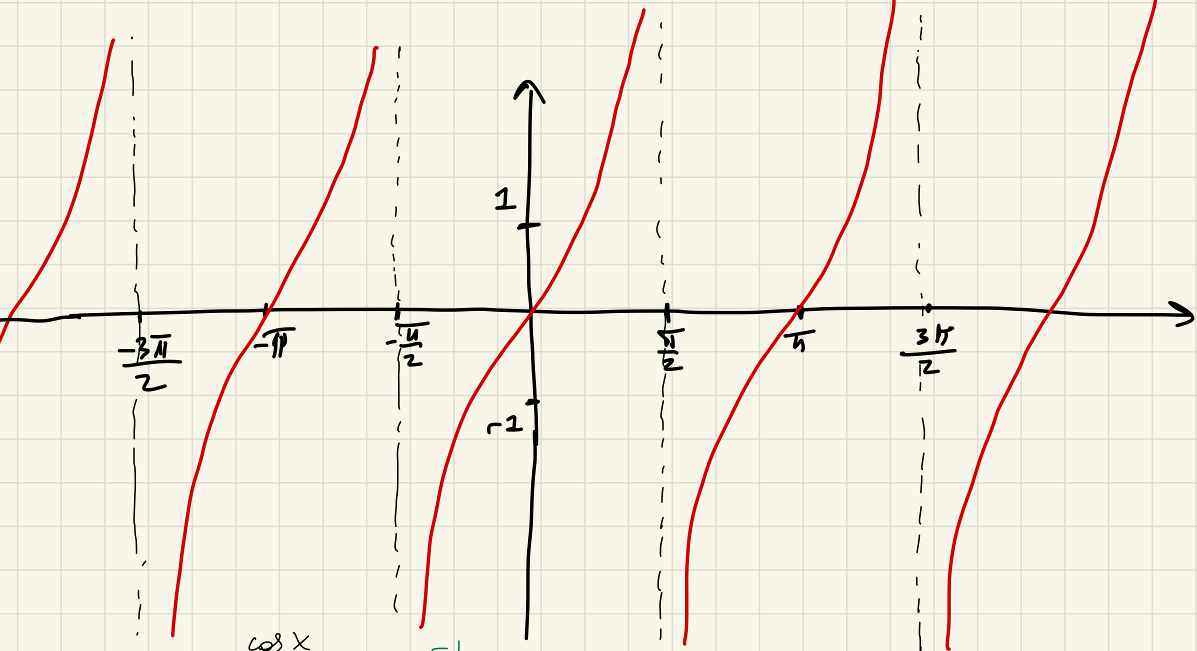
$y = \tan x$

$= \frac{\sin x}{\cos x}$, $\cos x = 0$ if $x = \frac{\pi}{2} + k\cdot\pi$

D: all numbers except $x = \frac{\pi}{2} + k\cdot\pi$ where k is an integer

In short, $\boxed{D: x \neq \frac{\pi}{2} + \pi\cdot k}$

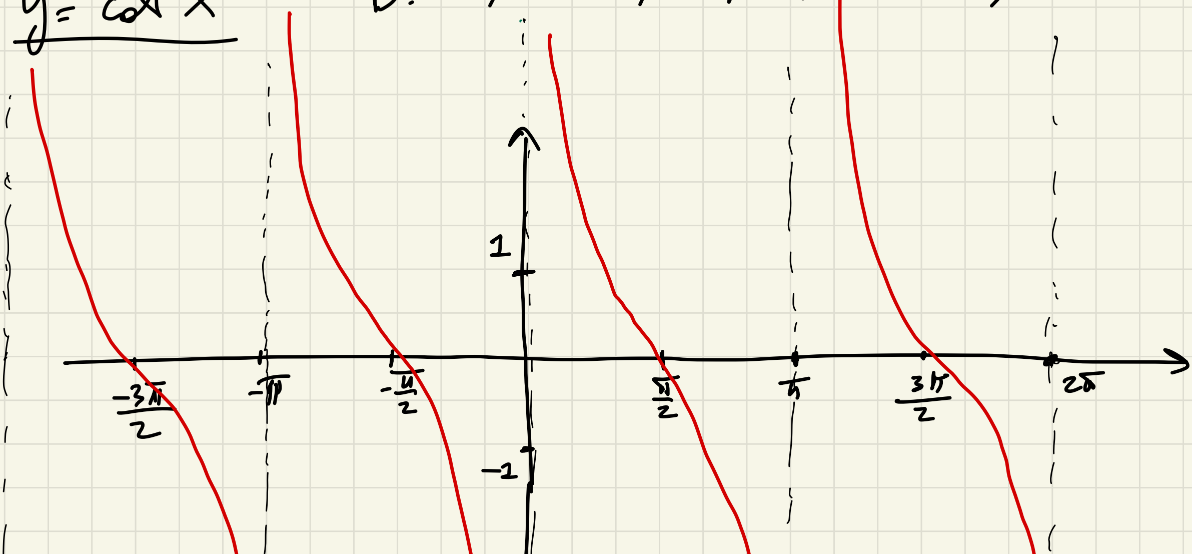
R: $(-\infty, \infty)$



$= \frac{\cos x}{\sin x}$, $\sin x = 0$ if $x = \pi\cdot k$

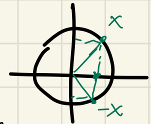
$y = \cot x$

D: $x \neq \pi\cdot k$, R: $(-\infty, \infty)$



Important trig formules

① $\boxed{\sin^2(x) + \cos^2(x) = 1}$ ← most important



② $\sin(-x) = -\sin(x)$

③ $\cos(-x) = \cos(x)$

clear from the unit circle perspective

④ $\begin{cases} \sin(x \pm 2\pi \cdot k) = \sin x \\ \cos(x \pm 2\pi \cdot k) = \cos x \end{cases}$ ← sin & cos are 2π - periodic

(k is an integer)

Ex. If $\sin x = \frac{1}{3}$ and $0 \le x \le \frac{\pi}{2}$, find $\tan x$.

$\tan x = \dfrac{\sin x}{\cos x}$

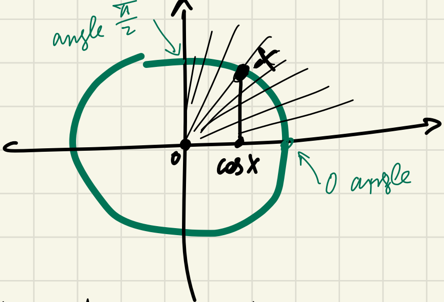
- $\sin x = \frac{1}{3}$ ← given
- $\cos x = ?$ Let's find it using formula ①

$\sin^2 x + \cos^2 b = 1$

$\cos^2 x = 1 - \sin^2 x = 1 - \frac{1}{9} = \frac{8}{9}$

$\cos^2 x = \frac{8}{9}$, and so $\cos x = \sqrt{\frac{8}{9}}$ or $\cos x = -\sqrt{\frac{8}{9}}$

But we know $0 \le x \le \frac{\pi}{2}$, so $\cos x$



has to be positive, and so we obtain

$$\boxed{\cos x = \sqrt{\frac{8}{5}}}$$

(read Appendix D for more examples)

$$\tan x = \frac{\sin x}{\cos x} = \frac{\frac{1}{3}}{\sqrt{\frac{8}{5}}} =$$

$$= \frac{1}{3 \cdot \frac{\sqrt{8}}{\sqrt{3}}} = \frac{1}{\cancel{3} \cdot \frac{\sqrt{8}}{\cancel{3}}} = \boxed{\frac{1}{\sqrt{8}}}$$

$$\left(\text{also } \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{4 \cdot 2}} = \frac{1}{2\sqrt{2}}\right)$$

Ex. $\sec(\varphi) = -1.5$, $\frac{\pi}{2} < \varphi < \pi$,

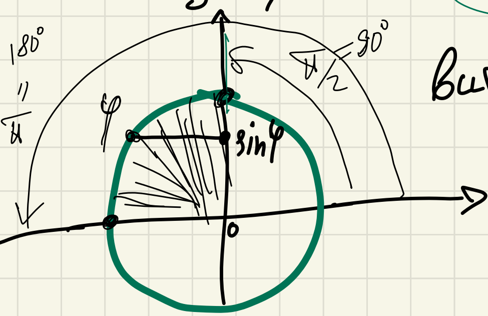
$\tan \varphi = ?$

- $\tan \varphi = \frac{\sin \varphi}{\cos \varphi}$ so we need to find $\sin$ & $\cos$.

- $\sec \varphi = \frac{1}{\cos \varphi} = -1.5 = -\frac{3}{2}$, so $\cos\varphi = \dfrac{1}{-\frac{3}{2}} = -\dfrac{2}{3}$

• $\sin^2\varphi = 1 - \cos^2\varphi = 1 - \left(-\frac{2}{3}\right)^2 = 1 - \frac{4}{9} =$

$= \frac{5}{9}$, so $\sin\varphi = \frac{\sqrt{5}}{3}$ or $\sin\varphi = -\frac{\sqrt{5}}{3}$



But we know $\frac{\pi}{2} < \varphi < \pi$,

and so $\sin\varphi > 0$, and so

$$\boxed{\sin\varphi = \frac{\sqrt{5}}{3}}$$

• $\tan\varphi = \dfrac{\sin\varphi}{\cos\varphi} = \dfrac{\frac{\sqrt{5}}{3}}{-\frac{2}{3}} =$

$= -\dfrac{\sqrt{5}}{\cancel{3}}\dfrac{\cancel{3}}{2} = -\dfrac{\sqrt{5}}{2}$

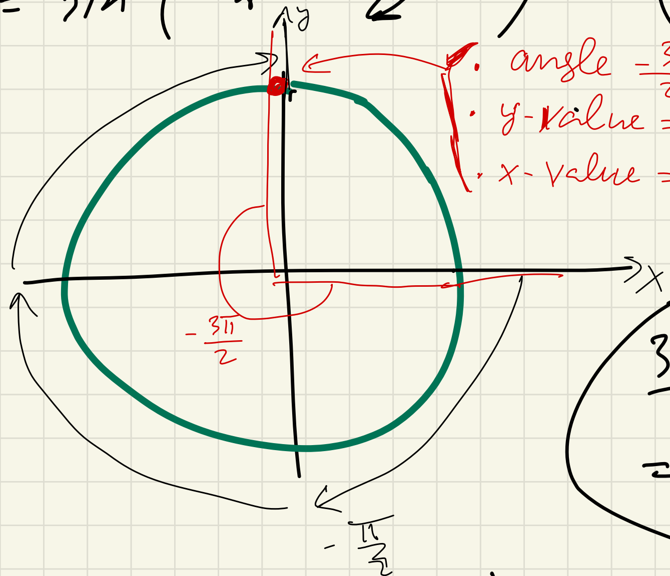
ex. Find $\sin\left(-\frac{7\pi}{2}\right)$

1 way

$\sin\left(-\frac{7}{2}\pi\right) = \sin\left(-\left(3+\frac{1}{2}\right)\cdot\pi\right) = \left(4 \text{ formula}\right)$

$= \sin\left(-3\pi - \frac{1}{2}\pi + 2\pi\right) =$

$= \sin\left(-\pi - \frac{1}{2}\pi\right) = \boxed{\sin\left(-\frac{3}{2}\pi\right) = 1}$



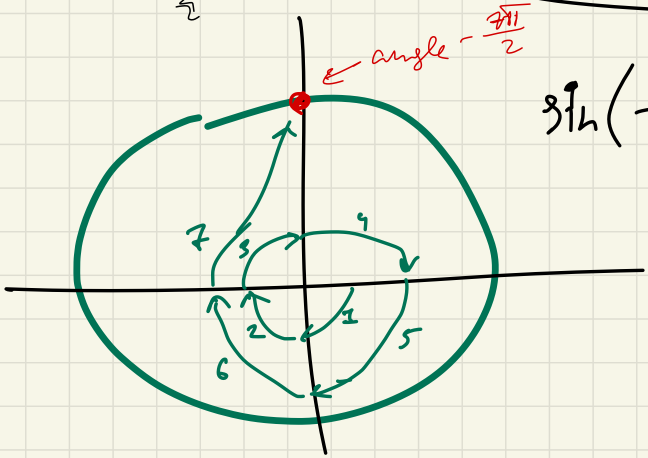
- angle $-\frac{3\pi}{2}$
- y-value $= \sin\left(-\frac{3\pi}{2}\right) = 1$
- x-value $= \cos\left(-\frac{3\pi}{2}\right) = 0$

3 way $\sin\left(-\frac{7}{2}\pi\right) = 4 =$

$= \sin\left(-\frac{7}{2}\pi + 4\pi\right) =$

$= \sin\left(\frac{\pi}{2}\right) = 1$

2 way



$\sin\left(-\frac{7\pi}{2}\right) = 1$

$\shortparallel$

7 times $-\frac{\pi}{2}$

Ex. $\cos\left(\frac{8\pi}{3}\right) = ?$

using $\frac{8}{3} = 2 + \frac{2}{3}$

$$\cos\left(\frac{8\pi}{3}\right) = \cos\left(\left(2 + \frac{2}{3}\right)\pi\right) = \cos\left(2\pi + \frac{2}{3}\pi\right) =$$

using (4)

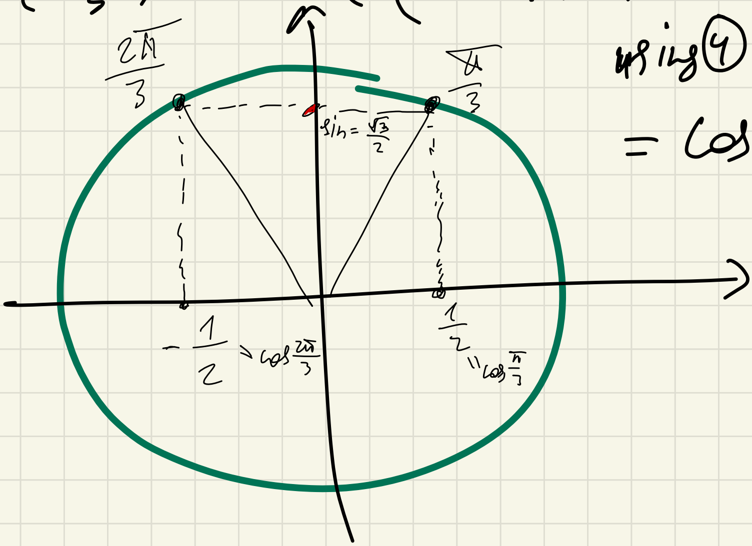
$$= \cos\left(\cancel{2\pi} + \frac{2}{3}\pi - \cancel{2\pi}\right)$$

$$\parallel$$

$$\cos\left(\frac{2}{3}\pi\right)$$

$$\parallel$$

$$-\frac{1}{2}$$



- memorize
- practice (HW will have examples)
- read Appendix D
- google videos

3.3 Derivatives of trig functions

1. $\boxed{\frac{d}{dx}[\sin(x)] = \cos(x)}$

need to remember these

2. $\boxed{\frac{d}{dx}[\cos(x)] = -\sin(x)}$

Ex. 1 What is derivative of $\tan x$?

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{d}{dx}\left[\frac{\overset{f}{\sin x}}{\underset{g}{\cos x}}\right] \underset{\text{quotient rule}}{=} \frac{\overbrace{(\sin x)'\cos x}^{f'\cdot g} - \overbrace{\sin x\cdot(\cos x)'}^{f\cdot g'}}{\underbrace{(\cos x)^2}_{g^2}} =$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} =$$

$$= \boxed{\frac{1}{\cos^2 x}} = \boxed{\sec^2 x}$$

quotient rule

$$\boxed{\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'\cdot g - f\cdot g'}{g^2}}$$

functions

3. $\boxed{\frac{d}{dx}[\tan x] = \sec^2 x = \frac{1}{\cos^2 x}}$

4. $\boxed{\frac{d}{dx}[\cot x] = -\csc^2(x) = -\frac{1}{\sin^2 x}}$

5. $\boxed{\frac{d}{dx}[\sec x] = \sec(x)\cdot\tan x = \frac{\sin x}{\cos^2 x}}$

6. $\boxed{\frac{d}{dx}[\csc x] = -\csc(x)\cdot\cot(x) = -\frac{\cos x}{\sin^2 x}}$

The last 4 formulas one can deduce by using quotient rule, just as we did in Ex. 1

Ex. 2 $f(x) = \dfrac{x\cdot e^x}{\cot(x)}$, $f'(x) = ?$ (don't need to simplify)

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{\overset{f}{x\cdot e^x}}{\underset{g}{\cot x}}\right] \underset{\text{quotient rule}}{=} \qquad \frac{f'\cdot g - f\cdot g'}{g^2}$$

$$= \frac{\frac{d}{dx}[\underset{f}{x}\cdot\underset{g}{e^x}]\cdot\cot x - x\cdot e^x\cdot\frac{d}{dx}[\cot x]}{\cot^2 x} =$$

product rule

$$= \frac{(\overset{f'}{(x)'}\cdot \overset{g}{e^x} + \overset{f}{x}\cdot\overset{g'}{(e^x)'})\cdot\cot x - x\cdot e^x\left(-\frac{1}{\sin^2 x}\right)}{\cot^2 x} =$$

<u>Remark</u> By convention we write $\sin^2 x$ instead of $(\sin x)^2$

$$= \frac{(e^x + x \cdot e^x) \cdot \cot x + x \cdot e^x \cdot \left(\frac{1}{\sin^2 x}\right)}{\cot^2 x}$$

Ex. 3 Find tangent line to $f(x) = \frac{1}{\cos x}$ at $x = \frac{\pi}{3}$.

tangent line to $y = f(x)$ at $x = a$ has equation

$y = f'(a)(x-a) + f(a)$

tangent line equation: $y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)$

now we only need to find these

- $f\left(\frac{\pi}{3}\right) = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

- $f'\left(\frac{\pi}{3}\right) = ?$ For this we first find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left[\frac{1^{f}}{\cos x_{g}}\right] = \frac{(1)'\cos x - 1 \cdot (\cos x)'}{\cos^2 x} =$$

quotient rule

(or look up $\frac{d}{dx}[\sec(x)]$)

$$= \frac{0 \cdot \cos x - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$$

confirms formula 5

$$f'\left(\frac{\pi}{3}\right) = \frac{\sin\frac{\pi}{3}}{\cos^2\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{\left(\frac{1}{2}\right)^2} =$$

$$= \frac{\sqrt{3}\cdot 4}{2} = \boxed{2\sqrt{3}}$$

- tangent line to $f(x) = \frac{1}{\cos x}$ at $x = \frac{\pi}{3}$

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right)$$

$$y = 2\sqrt{3}\left(x - \frac{\pi}{3}\right) + 2$$

$$\boxed{y = 2\sqrt{3}\cdot x + 2 - \frac{2\sqrt{3}\pi}{3}}$$

Before we go into chain rule (3.4), we cover compositions of functions

def. Given two functions $f(x)$ and $g(x)$, composition $f \circ g = \boxed{f(g(x))}$ is a function that is obtained by plugging in $g(x)$ for x into $f(x)$.

Examples:

- $f(x) = 3x + 1$, $g(x) = 5 - x$

$$f(g(x)) = 3 \cdot \underbrace{x}_{g(x)} + 1 = 3(5-x)+1$$

$$= -3x + 16$$

$$g(f(x)) = 5 - \underbrace{x}_{f(x)} = 5 - (3x+1) = 4 - 3x$$

$$f(f(x)) = 3(3x+1)+1 = 9x+4$$

$$g(g(x)) = 5-(5-x) = x$$

$$f(\underbrace{g(f(x))}_{\text{start with this}}) = f(\underbrace{5-(3x+1)}_{g(f(x))}) =$$

$$= 3(5-(3x+1))+1 = 15-9x-3+1 =$$

$$= 13 - 9x$$

- $f(x) = e^x$ $g(x) = \sin x - 3x$

$$f(g(x)) = e^{(\sin x - 3x)}$$

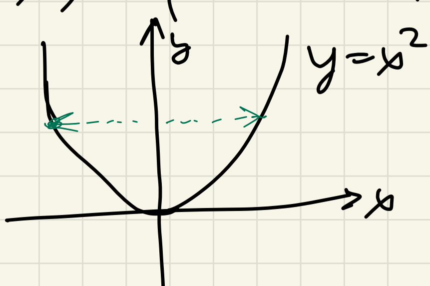
$$g(f(x)) = \sin(e^x) - 3 \cdot e^x$$

<u>def.</u> · $f(x)$ is called <u>even</u> if
$f(-x) = f(x)$.

· $f(x)$ is called <u>odd</u> if
$f(-x) = -f(x)$

<u>Ex.</u> x^2 is it odd/even/neither?

$f(-x) = (-x)^2 = x^2 = f(x)$, so even!

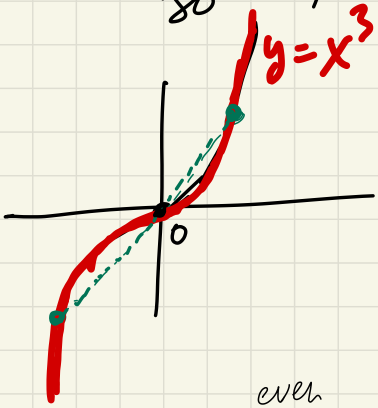


__fact__ "$f(x)$ is even" is equivalent to the graph of $f(x)$ being symmetric with respect to y-axis.

__Ex.__ x^3 is it odd/even/neither?

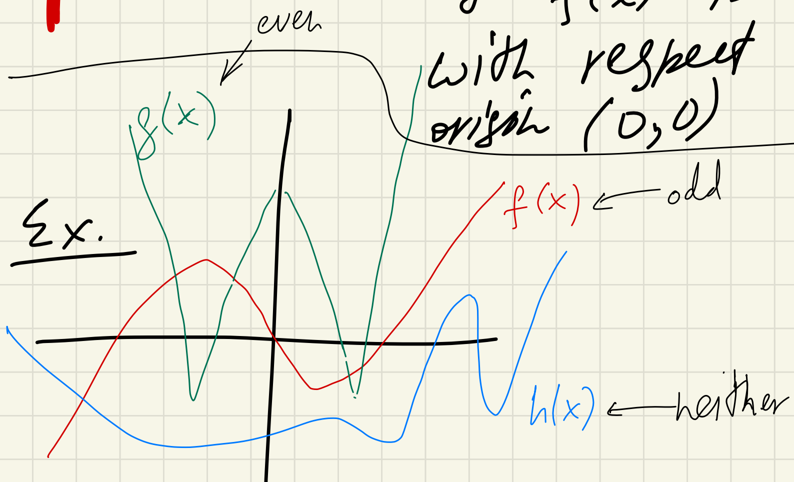
$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

so it is odd



__fact__ "$f(x)$ is odd" is equivalent to saying that graph of $f(x)$ is symmetric with respect to the origin $(0,0)$

__Ex.__



- $f(x)$ is odd, since its graph is symmetric about $(0,0)$

- $g(x)$ is even

- $h(x)$ is neither odd nor even

Ex. • x^2+x^3 is neither

$f(-x) = (-x)^2 + (-x)^3 = x^2 - x^3 \neq f(x)$, so not even
$\neq -f(x)$, so not odd

- $\cos x$ is even $(\cos(x) = \cos(-x))$

- $\sin x$ is odd $(\sin(-x) = -\sin(x))$

$\boxed{\text{Next week, on Oct 21st, there will be Exam 2}}$

more info will be in announcement today...

3.4 Chain rule

The most complicated topic in the course, so please read the textbook.

Chain rule is a systematic way to compute derivatives of compositions

$$\boxed{\text{Chain rule: } \frac{d}{dx}\Big[\overbrace{f(g(x))}^{\text{composition of } f \,\&\, g}\Big] = \overbrace{f'(g(x))}^{\text{composition of } f' \text{ and } g} \cdot g'(x)}$$

Ex. 1 $y(x) = (x+3)^{100}$ ← cannot derivate using previous rules

$y'(x) = ?$

So we apply the chain rule:

Step 1 express function as a composition of simpler functions

$$(x+3)^{100} = f(g(x))$$

$$f(x) = x^{100}$$

$$g(x) = x+3$$

Step 2 compute $f'(x)$ and $g'(x)$.

(preparation for chain rule)

$$f(x) = x^{100} \qquad g(x) = x+3$$

$$f'(x) = 100x^{99} \qquad g'(x) = 1$$

Step 3 (executing chain rule)

$$\frac{d}{dx}\left[(x+3)^{100}\right] = \frac{d}{dx}\left[f(g(x))\right] = \left(\text{use chain rule}\right)$$

$$= \underbrace{f'(g(x))}_{\text{composition of } f'(x) \text{ and } g(x)} \cdot g'(x) = \underbrace{100(x+3)^{99}}_{f'(g(x))} \cdot \underbrace{1}_{g'(x)} =$$

$$= \boxed{100(x+3)^{99}}$$

Ex2 $y(x) = \sin(2x+1)$

step 1 $\sin(2x+1) = f(g(x))$

outside function: $f(x) = \sin x$ inside function: $g(x) = 2x+1$

step2 $f'(x) = \cos x$ $g'(x) = 2$

step3 $\frac{d}{dx}[\sin(2x+1)] = \frac{d}{dx}[f(g(x))] =$

chain rule $=$ $f'(g(x)) \cdot g'(x) = \boxed{\cos(2x+1) \cdot 2}$

Remark Use chain rule only if product/quotient rules don't work.

Ex. 3 $\quad f(x) = 2^x, \quad f'(x) = ?$

Remark $\quad (x^n)' = n \cdot x^{n-1}$

$$(e^x)' = e^x$$

But we do not know how to derivate 2^x.

In particular, neither of these rules work:

$$(2^x)' \cancel{=} x \cdot 2^{x-1}$$

$$(2^x)' \cancel{=} 2^x$$

So, to compute $(2^x)'$ we will chain rule in a clever way:

$\boxed{\text{first}}$ let's transform 2^x:

$$f(x) = 2^x = (\underbrace{e^{\ln 2}}_{\substack{\| \text{ by def. of } \ln \\ 2}})^x = e^{\ln(2) \cdot x}$$

$\boxed{\text{second}}$ we use the chain rule:

use h since f is already taken

$$e^{\ln(2) \cdot x} = h(g(x))$$

$$h(x) = e^x \qquad g(x) = \ln(2)\cdot x$$

$$h'(x) = e^x \qquad g'(x) = \ln(2)$$

$$\left(\begin{array}{l} (7x)' = 7\cdot(x)' = 7 \\ (\ln(2)\cdot x)' = \ln(2)\cdot(x)' = \ln(2) \end{array} \right)$$

now we are ready to use chain rule:

$$\frac{d}{dx}[2^x] = \frac{d}{dx}[e^{\ln(2)\cdot x}] = \frac{d}{dx}[h(g(x))] =$$

$$\overset{\text{chain rule}}{=} h'(g(x))\cdot g'(x) = e^{\ln(2)\cdot x}\cdot \ln(2) =$$

$$= \boxed{2^x \cdot \ln(2)}$$

~~memorize~~ ~~add to cheat sheet~~

In general, we have

$$\boxed{\frac{d}{dx}[a^x] = \ln(a)\cdot a^x \quad \text{for any constant } a > 0}$$

Remark here a is constant, and so it cannot depend on x!

A more complicated example would be

$(x+3)^x$ ← this what we will cover in 2–3 classes!

To memorize / have on a cheat-sheet:

⓪ $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (limit def. of derivative)

power rule

① $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$, for any $n \neq 0$

"ignoring" constant

② $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$, for any constant c

linearity

③ $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

product rule

④ $\frac{d}{dx}[f(x) \cdot g(x)] = f' \cdot g + f \cdot g'$

quotient rule

⑤ $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f' \cdot g - f \cdot g'}{g^2}$

⑥ $\frac{d}{dx}[c] = 0$, for any constant c

⑦ $\frac{d}{dx}[e^x] = e^x$

chain rule

⑧ $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

⑨ $\frac{d}{dx}[a^x] = \ln(a) \cdot a^x$

⑩ $\frac{d}{dx}[\ln x] = \frac{1}{x}$

More examples on chain rule:

Ex. 1 $f(x) = \sqrt{\cos(x^2)}$, $f'(x) = ?$

$$\frac{d}{dx}\left[\sqrt{\underbrace{\cos(x^2)}_{g(x)}}\right] = \frac{d}{dx}\left[h(g(x))\right]$$

$h(x) = \sqrt{x} = x^{\frac{1}{2}}$ $\qquad$ $g(x) = \cos(x^2)$

$h'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\qquad$ $g'(x) = -2x \cdot \sin(x^2)$

(by power rule)

subproblem $\quad g'(x) = \frac{d}{dx}\left[\cos(\underbrace{x^2}_{v(x)})\right] = u'(v(x)) \cdot v'(x) =$

$= -\sin(x^2) \cdot 2x$

$u(x) = \cos x$ $\qquad$ $v(x) = x^2$

$u'(x) = -\sin x$ $\qquad$ $v'(x) = 2x$

$$\frac{d}{dx}\left[\sqrt{\cos(x^2)}\right] = \frac{d}{dx}\left[h(g(x))\right] =$$

$$= h'(g(x)) \cdot g'(x) = \cancel{\frac{1}{2}} (\cos(x^2))^{-\frac{1}{2}} \cdot \sin(x^2) \cdot (-\cancel{2}x) =$$

$$= \boxed{\frac{-x \cdot \sin(x^2)}{\sqrt{\cos(x^2)}}}$$ ← answer

<u>ex. 2</u> $f(x) = e^{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x}$

find values of x where $f(x)$ has a horizontal tangent line.

• "horizontal tangent line" is equivalent to $f'(x) = 0$. So we need to find $f'(x)$.

$$\cdot\ f'(x) = \frac{d}{dx}\left[e^{\overbrace{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x}^{g(x)}}\right] = \text{(chain rule)}$$

$$\left(\begin{array}{ll} h(x) = e^x & g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x \\ h'(x) = e^x & g'(x) = x^2 - 3x + 2 \end{array} \right)$$

$$= h'(g(x)) \cdot g'(x) = e^{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x} \cdot (x^2 - 3x + 2)$$

- $f'(x) = 0$

$$(x^2 - 3x + 2) \cdot \underbrace{e^{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x}}_{\text{never } 0 \text{ (always } > 0)} = 0$$

$$x^2 - 3x + 2 = 0$$

$$x = 2$$

or

$$x = 1$$

← answer

Ex. 3 $f(x) = \left(\frac{x-1}{2x+1}\right)^8$

$f'(x) = ?$

$$f'(x) = \frac{d}{dx}\left[\left(\underbrace{\frac{x-1}{2x+1}}_{g(x)}\right)^8\right] = \text{(chain rule)}$$

$$\left(h(x) = x^8 \qquad g(x) = \frac{x-1}{2x+1} \right)$$

$$h'(x) = 8x^7 \qquad g'(x) = \frac{d}{dx}\left[\frac{x-1}{2x+1}\right] = \text{(quotient rule)}$$

$$= \frac{1(2x+1) - 2(x-1)}{(2x+1)^2} =$$

$$= \frac{3}{(2x+1)^2}$$

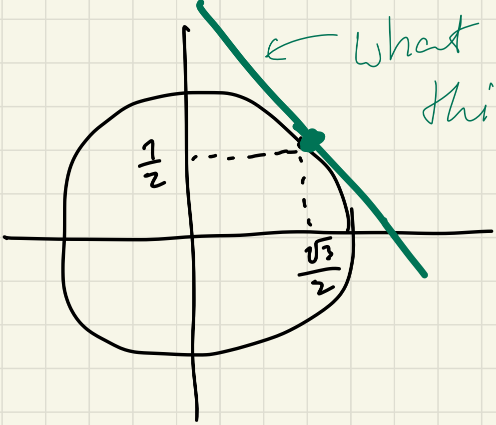
$$= h'(g(x)) \cdot g'(x) = 8\left(\frac{x-1}{2x+1}\right)^7 \cdot \frac{3}{(2x+1)^2} =$$

$$= \boxed{\frac{24(x-1)^7}{(2x+1)^9}} \longleftarrow \text{answer}$$

3.5 Implicit differentiation

This is a method used to find slopes of curves that are not functions, but equations.

Ex. 4 Find equation of tangent line to $x^2 + y^2 = 1$ at $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



← what is the equation for this tangent line.

<u>remark</u> if the curve is given by $y=f(x)$, then the slope of tangent line is $f'(x)$.

~~$y=f(x)$~~

<u>Q.</u> But what do we do in this case?

$$x^2+y^2=1$$

A. Implicit differentiation.

differentiate both sides with respect to x

$$\frac{d}{dx}\left[x^2+y^2\right]=\frac{d}{dx}[1]$$

$$\frac{d}{dx}\left[x^2\right]+\frac{d}{dx}\left[y^2\right]=\frac{d}{dx}[1]$$

$$\frac{d}{dy}\left[y^2\right]=2y$$

$$\frac{d}{dx}\left[y^2\right]\neq 2y$$

$$2x+\boxed{?}=0$$

How we find $\frac{d}{dx}\left[y^2\right]$? It is <u>not</u> $2y$, because we derivate with respect to x, not y. Instead, we treat y as y(x)!

chain rule

$$\frac{d}{dx}\left[(y(x))^2\right] = f'(g(x)) \cdot g'(x) = \underbrace{2 \cdot y(x)}_{f'(g(x))} \cdot \underbrace{y'(x)}_{g'(x)}$$

outside function: $f(x) = x^2 \downarrow f'(x) = 2x$

inside function: $g(x) = y(x) \downarrow g'(x) = y'(x)$

$$= 2y \cdot y'$$

$$x^2 + y^2 = 1$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[1\right]$$

chain rule

$$2x + 2y \cdot y' = 0$$

slope that we want

(solve for y')

$$2yy' = -2x$$

$$\boxed{y' = -\frac{x}{y}}$$

now we plug in our point

$$\boxed{\underset{\text{at}\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)}{\text{slope}} = \underset{\text{at}\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)}{y'} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}}$$

tangent line $y = mx + b$

- slope $= m = -\sqrt{3}$

$$y = -\sqrt{3}\,x + b$$

- goes through $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$$\frac{1}{2} = -\sqrt{3} \cdot \frac{\sqrt{3}}{2} + b$$

$$\frac{1}{2} = -\frac{3}{2} + b$$

$$2 = b$$

$$\boxed{y = -\sqrt{3}\,x + 2}$$ $\leftarrow$ answer

3.5 Continueing implicit differentiation

<u>Ex. 1</u> Find the equation of tangent line to $x^3+y^3 = 6xy$ at $(\overset{x}{3}, \overset{y}{3})$.

<u>Remark</u> $\frac{dy}{dx}$ is the same thing as $y' = y'(x)$.

<u>Strategy</u> ① to find slope (y') we use implicit differentiation

② find equation based on a slope & point

①

Implicit differentiation: $x^3 + y^3 = 6xy$

$\frac{d}{dx}\left[(y(x))^3\right] =$

chain rule for $f(y(x))$, $f(x) = x^3$

$= f'(y(x))\,y'(x) =$

$= 3y(x)^2 \cdot y'(x) =$

$= 3y^2 \cdot y'$

$$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] = \frac{d}{dx}[\underbrace{6x}_{f}\,\underbrace{y(x)}_{g}]$$

chain rule $y = y(x)$

product rule

$$3x^2 + 3y^2 \cdot y' = \frac{d}{dx}[6x] \cdot y + 6x \cdot \frac{d}{dx}[y]$$

$$3x^2 + 3y^2 \cdot y' = 6 \cdot y + 6x \cdot y'$$

Now we solve for y'

$$3y^2y' - 6xy' = 6y - 3x^2$$

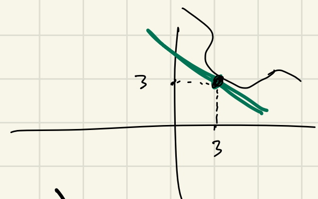
$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$\boxed{y' = \frac{6y - 3x^2}{3y^2 - 6x}}$$

(2) tangent line $\boxed{y = mx + b}$ • has slope y' at $(3,3)$, so

we compute using

$$m = y'_{\text{at } (3,3)} = \frac{6 \cdot 3 - 3 \cdot 3^2}{3 \cdot 3^2 - 6 \cdot 3} = \frac{-9}{9} = -1$$



$$\boxed{y = -x + b}$$

• goes through $(3,3)$

$$3 = -3 + b$$

$$6 = b$$

$\boxed{y = -x + 6}$ ← answer!

Detailed $\frac{d}{dx}[y^3] \overset{y = y(x)}{=\!=} \frac{d}{dx}\left[(y(x))^3\right] =$

$= \frac{d}{dx}\left[f(y(x))\right] \overset{\text{chain rule}}{=\!=} f'(y(x)) \cdot y'(x) =$

where $f(x) = x^3$ $\qquad f'(x) = 3x^2$

$$= 3(y(x))^2 \cdot y'(x) = 3y^2 \cdot y'$$

Ex. 2 Find y' at $(0, \frac{\pi}{2})$ (x, y)

if $\underline{\sin(xy) = \cos(x+y)}$

This is an equation, not a function, $(y = f(x))$ so we do implicit differentiation

$$\frac{d}{dx}\Big[\underbrace{\sin}_{\text{outside function } f}(\underbrace{x \cdot y}_{\text{inside function } g})\Big] = \frac{d}{dx}\Big[\cos(x+y)\Big]$$

$f(x) = \sin(x)$
$f'(x) = \cos(x)$

chain rule   chain rule

$$\overset{f'(g(x))}{\cos(xy)} \cdot \frac{d}{dx}[\overset{g'(x)}{xy(x)}] = -\sin(x+y)\frac{d}{dx}[x+y]$$

product rule   term by term

$$\cos(xy) \cdot \big((x)' \cdot y + x \cdot (y)'\big) = -\sin(x+y)(1 + y')$$

$$\cos(xy) \cdot (y + x \cdot y') = -\sin(x+y)(1+y')$$

Now we solve for y'

$$\cos(xy) \cdot y + \cos(xy) x \cdot y' = -\sin(x+y) - \sin(x+y) \cdot y'$$

$$\sin(x+y) y' + \cos(xy) xy' = -\sin(x+y) - \cos(xy) \cdot y$$

$$y'(\sin(x+y) + \cos(xy)x) = -\sin(x+y) - \cos(xy)y$$

$$y' = \frac{-\sin(x+y) - \cos(xy)\cdot y}{\sin(x+y) + \cos(xy)\cdot x}$$

$$y'\Big|_{\text{at } (0,\frac{\pi}{2})} = \frac{-\sin\left(0+\frac{\pi}{2}\right) - \cos\left(0\cdot\frac{\pi}{2}\right)\cdot\frac{\pi}{2}}{\sin\left(0+\frac{\pi}{2}\right) + \cos\left(0\cdot\frac{\pi}{2}\right)\cdot 0}$$

$$= \frac{-1 - \left(1\cdot\frac{\pi}{2}\right)}{1 + \cancel{1\cdot 0}} = \boxed{-1-\frac{\pi}{2}}$$

answer

Ex. 3 Find y'' in terms of x & y, if $x^4 + y^4 = 16$ ← equation, so we do implicit differentiation

$$\frac{d}{dx}[x^4] + \frac{d}{dx}[y^4_{(x)}] = \frac{d}{dx}[16]$$

chain rule

$$4x^3 + 4y^3\cdot y' = 0$$

$$4y^3y' = -4x^3$$ solve for y'

$$\boxed{y' = -\frac{x^3}{y^3}}$$

because we need y'', we derivate one more time

$$\frac{d}{dx}[y'] = \frac{d}{dx}\left[-\frac{x^3}{y^3}\right]$$ quotient rule

$$y'' = -\left(\frac{\frac{d}{dx}[x^3]\cdot y^3 - x^3\cdot\frac{d}{dx}[y^3]}{y^6}\right)$$

chain rule

sub!

$$y'' = -\frac{3x^2y^3 - x^3(3y^2\cdot y')}{y^6}$$

$$y'' = -\frac{3x^2y^3 - x^3\left(3y^2\cdot\left(-\frac{x^3}{y^3}\right)\right)}{y^6}$$

$$a\cdot b\cdot\frac{c}{d} = \frac{a\cdot b\cdot c}{d}$$

we have to simplify further:

$$y'' = -\frac{3x^2y^3 + \frac{3x^3\cdot y^2\cdot x^3}{y^3}}{y^6}$$

$$y'' = -\frac{3x^2y^3 + 3x^6y^{-1}}{y^6}$$

to get rid of y^{-1}, I multiply both num. & denom. by y

$$y'' = -\frac{(3x^2y^3 + 3x^6y^{-1})y}{y^6 \cdot y}$$

$$y'' = -\frac{3x^2y^4 + 3x^6}{y^7}$$

$$y'' = -3x^2\left(\frac{y^4 + x^4}{y^7}\right)$$

remember initial equation $x^4 + y^4 = 16$

$$y'' = -3x^2\frac{16}{y7}$$

$$\boxed{y'' = -48\frac{x^2}{y^7}}$$

3.6 Derivatives of log functions & logarithmic differentiation.

① $$\boxed{\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x}}$$

remember & add to the cheat-sheet

proof: $y(x) = \ln x$ is equivalent to (by definition of ln)

$$e^y = x$$

differentiate

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

chain rule

$$e^y \cdot y' = 1$$

$$y' = \frac{1}{e^y}$$

remember $y = \ln x$

$$y' = \frac{1}{e^{\ln x}}$$

$$\boxed{y'(x) = \frac{1}{x}}$$

② $$\boxed{\frac{d}{dx}[\log_b x] = \frac{1}{x \cdot \ln(b)}}$$ $(\ln x = \log_e x)$

Ex 1 $f(x) = \ln(2x+1)$, $f'(x) = ?$

$$\frac{d}{dx}[\underbrace{\ln}_{\text{outside function } h}(\underbrace{2x+1}_{\text{inside function } (g(x))})] \underset{\text{chain rule!}}{=} \underbrace{\frac{1}{2x+1}}_{h'(g(x))} \cdot \underbrace{2}_{g'(x)} = \frac{2}{2x+1}$$

$h(x) = \ln x$ $\quad g(x) = 2x+1$

$h'(x) = \frac{1}{x}$ $\quad g'(x) = 2$

Ex 2 $$\frac{d}{dx}[x \cdot \ln(x)] \underset{\text{product rule}}{=} \frac{d}{dx}[x] \cdot \ln x +$$

$$+ x \cdot \frac{d}{dx}[\ln x] = 1 \cdot \ln x + x \cdot \frac{1}{x} =$$

$$= \boxed{\ln x + 1}$$

Ex 3 $$\frac{d}{dx}[\underbrace{\sqrt{\ln x}}_{h(g(x))}] \underset{\text{chain rule}}{=} h'(g(x)) \cdot g'(x)$$

$||$

$h(x) = x^{\frac{1}{2}} = \sqrt{x}$ $\quad g(x) = \ln x$ $\quad //$

$h'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ $\quad g'(x) = \frac{1}{x}$ $\quad \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x}$

$\quad \parallel \frac{1}{2\sqrt{x}}$

$// \left(a^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}\right)$

$$\boxed{\frac{1}{2\sqrt{\ln x} \cdot x}}$$

Logarithmic differentiation

It helps to take derivatives of functions with x in the exponent.

Best explained on examples:

Ex. 4 $\quad f(x) = x^x, \quad f'(x) = ?$

Nothing what we learned works here!

(it is not x^a, not a^x, chain rule also doesn't work
$x^x \neq f(g(x))$)

So we do logarithmic differentiations:

step1 write as $\boxed{y(x) = x^{x}}$

3 main formulas for $\ln(x)$

① $\ln(ab) = \ln(a) + \ln(b)$

② $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

③ $\boxed{\ln(a^{p}) = p \cdot \ln a}$

step2 take $\ln(\cdot)$ of Both sides to drop the exponent

$$\boxed{\ln(y) = \ln(x^{x})}$$

used ③

here there is (x) which we surpress

$$\boxed{\ln(y) = x \cdot \ln(x)}$$

step3 do implicit differentiation $\frac{d}{dx}[\cdot]$ of Both sides

$\ln(y) = h(y(x))$

$h(x) = \ln(x)$ $\quad y(x)$

$h'(x) = \frac{1}{x}$ $\quad y'(x)$

$$\boxed{\frac{d}{dx}\left[\ln(y(x))\right] = \frac{d}{dx}\left[x \cdot \ln(x)\right]}$$

outside inside

chain rule

product rule (see Ex.2)

$h'(y(x)) \cdot y'(x) =$

$$\boxed{\frac{1}{y} \cdot y' = \ln x + x \cdot \frac{1}{x}}$$

step4 solve for $y'(x)$

$$\boxed{y' = y(\ln x + 1)}$$

↓ remember $y = x^x$

$$\boxed{y' = x^x \cdot (\ln x + 1)}$$

$$\boxed{y' = x^x \cdot \ln x + x^x}$$

Ex. 5 $f(x) = (2x+3)^{\sin x}$, $f'(x) = ?$

$$y = (2x+3)^{\sin x}$$

↓ $\ln(-)$ of both sides

$$\ln(y) = \ln\left((2x+3)^{\sin x}\right)$$

with $a = 2x+3$ and $p = \sin x$

↓ $\ln(a^p) = p\ln(a)$

$$\ln(y) = \sin(x) \cdot \ln(2x+3)$$

↓ $\frac{d}{dx}[-]$ of both sides

$$\frac{d}{dx}\left[\ln(y(x))\right] = \frac{d}{dx}\left[\sin(x) \cdot \ln(2x+3)\right]$$

↓ chain rule ↓ product rule

$$\frac{1}{y} \cdot y' = \frac{d}{dx}[\sin x] \cdot \ln(2x+3) + \sin x \cdot \frac{d}{dx}[\ln(2x+3)]$$

↓ see Ex. 1, chain rule

$$\frac{y'}{y} = \cos(x) \cdot \ln(2x+3) + \sin(x)\left(\frac{2}{2x+3}\right)$$

$$y' = y\left(\cos(x)\cdot \ln(2x+3) + \sin(x)\cdot \frac{2}{2x+3}\right)$$

$$\boxed{y' = (2x+3)^{\sin x}\left(\cos(x)\cdot \ln(2x+3) + \frac{2\sin(x)}{2x+3}\right)}$$

Logarithmic differentiation is also used to find derivatives of large products/quotients:

<u>Ex.6</u> (ex. 7 from the textbook)

Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+5)^5}$

<u>Idea:</u> We <u>can</u> use quotient and then product rules, but it will be lengthy. Instead, let's first take $\ln(-)$ of both sides:

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+5)^5}$$

$$\ln(y) = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+5)^5}\right) =$$

$$= \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln\left((3x+5)^5\right) =$$

$$= \ln\left(x^{3/4}\right) + \ln\left((x^2+1)^{\frac{1}{2}}\right) - 5\ln(3x+5) =$$

$$\boxed{\ln(y) = \frac{3}{4}\cdot\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+5)}$$

$\downarrow$ $\frac{d}{dx}(-)$ of both sides

$\downarrow$ chain rule

$$\frac{y'}{y} = \frac{3}{4}\frac{d}{dx}\left[\ln x\right] + \frac{1}{2}\frac{d}{dx}\left[\ln(x^2+1)\right] - 5\frac{d}{dx}\left[\ln(3x+5)\right]$$

use chain rule (see Ex. 1)

$$\frac{y'}{y} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - 5\frac{3}{3x+5}$$

$$y' = y\left(\cdot \cdot - - —— \right)$$

remember $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+5)^5}$

$$\boxed{y' = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+5)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+5}\right)}$$

(logarithmic differentiation)

Upshot: take $\ln(-)$ of both sides if

① there is x in the exponent (ex 4 & 5)

② there is large quotient/product (ex 6)

read 3.6

+

do practice problems specified in the last HW

- Exams are graded
- by 4pm I will calculate and post
  $X = \boxed{\text{the total grade (HW+quizzes+exams)}}$
- Sunday, Oct 25 — last day to withdraw
- If $X \leq 70$, think carefully about dropping/not dropping the class, and make an informed decision.

3.8. Exponential growth & decay

Today we will use $y(t)$ to denote function.
We will also frequently use $\frac{dy}{dt}$ istead of $y'(t)$

<u>Differential equation</u> is an equation
like $(y'(t))^2 + y(t) = 0$
which involve derivatives.

Fact *

differential equation

$$\boxed{y'(t) = k \cdot y(t) \quad (k \text{ is constant})}$$

(derivative $y'(t)$ is proportional to $y(t)$)

has unique solution, exponential function

$$\boxed{y(t) = C \cdot e^{kt}} \quad (C \text{ is any constant})$$

Moreover, $C = y(0)$

proof we want $C \cdot e^{kt}$ to satisfy

$$\underbrace{\frac{d}{dt}\left[C \cdot e^{kt}\right]}_{y'(t)} \underset{\substack{\uparrow \\ \text{chain rule}}}{=} C \cdot k \cdot e^{kt} = k \cdot \underbrace{C \cdot e^{kt}}_{y(t)}$$

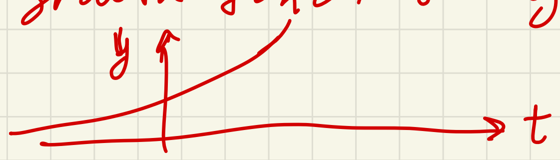
inside

outside

so we obtain $y'(t) = k \cdot y(t)$, and this is what we wanted.

---

* exp. growth functions $y(t) = C \cdot e^{kt}$, $k > 0$



Ex 1 (exponential growth)

- A population of Bacteria initially contains 100 cells ← $y(0)$
- This population grows with rate proportional to size
  (rate = derivative)
- Contains 400 cells after 1 hour

Q1 Find the function representing # of Bacteria after t hours.

Setting up the mathematically:

We know:

$y(t)$ = population at time t

- $y(0) = 100$
- $y'(t) = k \cdot y(t)$ ← size of sample at t hours

  ↖ growth rate = derivative!
- $y(1) = 400$

$y(t) = ?$

(by Fact *)

$\frac{dy}{dt} = k \cdot y$ implies that $y(t)$ is exponential

$$\boxed{y(t) = C \cdot e^{k \cdot t}}$$

Now we only need to find C & k, and we do this by plugging in 0 & 1

$y(0) = 100$, so $C \cdot \underbrace{e^{k \cdot 0}}_{=1} = 100$, so $\boxed{C = 100}$

$y(1) = 400$ so $100 \cdot e^{k \cdot 1} = 400$ so $e^k = 4$

so $\boxed{k = \ln(4)}$

$$y(t) = 100 \cdot e^{\ln(4) \cdot t} = 100 \cdot \left(e^{\ln(4)}\right)^t = 100 \cdot 4^t$$

$$\boxed{y(t) = 100 \cdot 4^t}$$

used $\left(e^a\right)^b = e^{a \cdot b}$

Q. What if we are given $y(2)$ instead of $y(0)$

$\begin{cases} y(1) = 400 \\ y(2) = 1600 \end{cases} \to \begin{cases} C \cdot (e^k)^1 = 400 & ① \\ C \cdot (e^k)^2 = 1600 & ② \end{cases}$ divide ② by ① $\longrightarrow$ $\begin{cases} \frac{②}{①} & \frac{\cancel{C}(e^k)^2}{\cancel{C}\,\cancel{e^k}} = 4 \\ ① & C \cdot e^k = 400 \end{cases} \to \begin{cases} e^k = 4 \\ C \cdot e^k = 400 \end{cases} \to \boxed{\begin{cases} k = \ln 4 \\ C = 100 \end{cases}}$

Q2 Find # of bacteria after 3h.

$y(3) = 100 \cdot 4^3 = \boxed{6400}$

Q3 Find rate of growth of population after 3h.

derivative!

$$y'(t) = \frac{d}{dt}\left[100 \cdot 4^t\right] = 100 \cdot 4^t \cdot \ln(4)$$

$$y'(3) = 100 \cdot 4^3 \cdot \ln(4) = \boxed{6400 \cdot \ln(4)}$$

Q4 when does the population reach 10000 cells?

$$y(t) = 10000$$

$$100 \cdot 4^t = 10000$$

$$4^t = 100$$

$$\boxed{t = \log_4 100}$$

Ex2 (exponential decay)

. The half-life of a radiactive element is 30y.
. A sample is initially 100 mg.

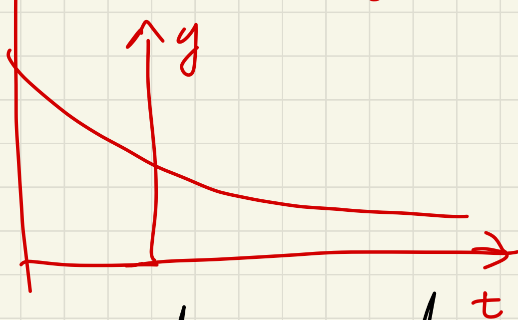
Q1 Find the formula for the mass after t years.

Q2 How much is left after 100 years?

Q3 After how long will 1 mg be left?

☆ the half-life is the time it takes for a sample to become half the initial size

☆ $\boxed{\text{decay}}$, Half-life means means that the function is $y = C \cdot e^{kt}$ where k is negative



① half life is given, that means we have exp decay, and so $\underbrace{y(t) = C \cdot e^{kt}}$

. $y(0) = 100$ so $C \cdot e^{k \cdot 0} = 100$ so $C = 100$

. (half-life) $y(30) = 100 \cdot \frac{1}{2} = 50$

$$100 \cdot e^{k \cdot 30} = 50$$

$$e^{k \cdot 30} = \frac{1}{2}$$

$$k \cdot 30 = \ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) = -\ln(2)$$

$$\boxed{k = -\frac{\ln(2)}{30}}$$

$$y(t) = 100 \cdot e^{-\frac{\ln(2)}{30} \cdot t} =$$

$$= 100 \cdot \left(e^{\ln(2)}\right)^{-\frac{t}{30}} = 100 \cdot 2^{-\frac{t}{30}}$$

$$\boxed{y(t) = 100 \cdot 2^{-\frac{t}{30}}}$$

2 $\boxed{y(100) = 100 \cdot 2^{-\frac{100}{30}}}$

3 we solve

$$y(t) = 1$$

$$100 \cdot 2^{-\frac{t}{30}} = 1$$

$$2^{-\frac{t}{30}} = \frac{1}{100}$$

$$-\frac{t}{30} = \log_2 \frac{1}{100}$$

$$\boxed{t = -30 \cdot \log_2\left(\frac{1}{100}\right)}$$

Read 3.8

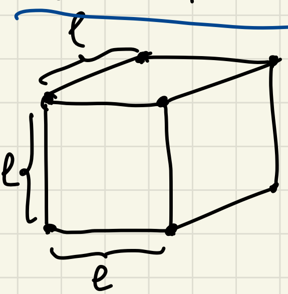
Do HW (due on Tuesday)

3.9 Related rates

Word problems that relate rate of change of one quantity to another. Typically, you are given a rate, you must find another rate.

☆ Rate of change of quantity y = derivative y' or $\frac{dy}{dx}$

Ex. 1 A cube has sides increasing at a rate 6 cm/sec. When the side is 10 cm long, what is the rate that volume is changing at?



- $\ell(t)$ = side length

t is time

- $V(t)$ = volume

given: $\ell'(t) = 6$

Q: When $\ell(t) = 10$, what is $V'(t) = ?$

How to solve related rate problem

1. Draw picture, define variables, formulate the Q. in terms of variables

$$V = \ell^3$$

2. Find a formula relating the variables

$$\frac{d}{dt}[V(t)] = \frac{d}{dt}[\ell^3(t)]$$

$$V' = 3\ell^2 \cdot \ell'$$

3. Differentiate with respect to time t both sides.

★ remember: $\frac{d}{dt}[y] = y'$

(chain rule) → $\frac{d}{dt}[y^3] = 3y^2 \cdot y'$

$$V' = 3 \cdot (10)^2 \cdot 6 =$$

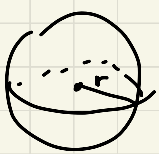
$$= 1800 \text{ cm}^3/\text{sec}$$

4. Plug in numbers and solve for rate you need

Formules to remember / write down:

- 

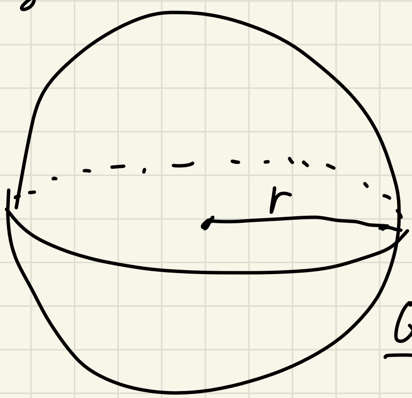
area $= \pi r^2$
circumference $= 2\pi r$
} fun fact: related by taking derivative

- 

volume $= \frac{4}{3}\pi r^3$
surface area $= 4\pi r^2$

Ex. 2 The volume of a sphere is increasing at rate $5\,cm^3/sec$. When the radius is $4\,cm$, what is the rate of change of the radius?

1.



$r(t) = $ radius

$V(t) = $ volume

$V'(t) = 5$

Q. When $r = 4$, what is r'?

2. $$V = \frac{4}{3}\pi r^3$$

$$\downarrow \frac{d}{dt}[-]$$

3. $$V' = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$V' = \frac{4}{3}\pi \frac{d}{dt}[r^3]$$

chain rule!

$$V' = \frac{4}{3}\pi \cdot 3 \cdot r^2 \cdot r'$$

$$\frac{d}{dt}\left[r(t)^3\right] =$$

$f(x) = x^3$, $r(t)$

$f'(x) = 3r^2$, $r'(t)$

$$= \frac{d}{dt}\left[f(r(t))\right] \overset{\text{ch. rule}}{=}$$

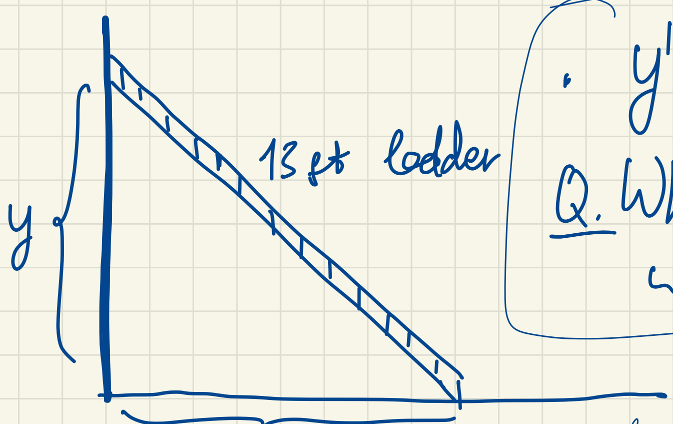
$$= f'(r(t)) \cdot r'(t) =$$

$$= \boxed{3 \cdot r^2 \cdot r'}$$

4. $5 = \frac{4}{\cancel{3}} \cdot \cancel{3} \cdot \cancel{\pi} \cdot 4^2 \cdot r'$ ← want

$$\boxed{\frac{5}{64\pi} \text{ cm/sec} = r'}$$

Ex. 3 A 13 ft ladder is sliding down a wall and distance between the top of the ladder and the ground is decreasing at a rate 1 ft/s. When base of the ladder is 12 ft from the wall, what is the rate of change of the distance from the base of the ladder to the wall?



x distance from the base to the wall

$y' = -1$ (y is decreasing, so y' has to be neg.)

Q. When $x = 12$, what is $x' = ?$

$$x(t)^2 + y(t)^2 = 13^2$$

$\downarrow \frac{d}{dt}[-]$, using chain rule

want

$$2x \cdot x' + 2 \cdot y \cdot y' = 0$$

$$12^2 + y^2 = 13^2$$

$$y^2 = 25$$

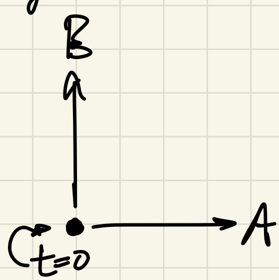
$$y = 5$$

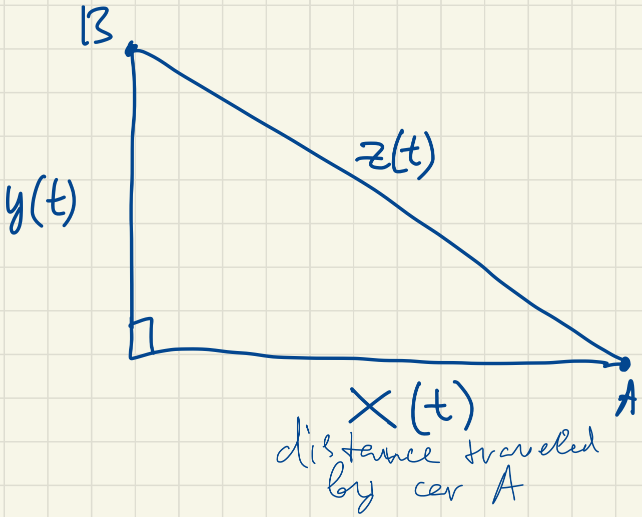
$$2 \cdot 12 \cdot x' + 2 \cdot y \cdot (-1) = 0$$

$$24 \cdot x' + 2 \cdot 5(-1) = 0$$

$$x' = \frac{10}{24} = \frac{5}{12} \text{ ft/s}$$

<u>Ex.4</u> 2 cars leave the same place at the same time. Car A driving east at 60 mph, car B driving north at 80 mph. 1 hr later, what's the rate of change of distance between the cars?





$x'(t) = 60$ mph

$y'(t) = 80$ mph

What is $z'(1)$?

$$x^2 + y^2 = z^2$$

$$\Big\downarrow \frac{d}{dt}[-]$$

$$2x \cdot x' + 2yy' = 2z \cdot z' \leftarrow \text{want}$$

$$z' = \frac{(x \cdot x' + y \cdot y')}{z}$$

$$z'(1) = \frac{x(1) \cdot x'(1) + y(1) \cdot y'(1)}{z(1)} =$$

$$= \frac{60 \cdot 60 + 80 \cdot 80}{100} = \boxed{100 \text{ mph}}$$

$x'(1) = 60$

$y'(1) = 80$

$x(1) = \underset{\substack{\uparrow \\ \text{speed}}}{60} \cdot \underset{\substack{\uparrow \\ \text{time}}}{1} = 60$

$y(1) = 80 \cdot 1 = 80$

$z(1) = \sqrt{x(1)^2 + y(1)^2} =$

$= \sqrt{60^2 + 80^2} = 100$

3.10 Linear approximation

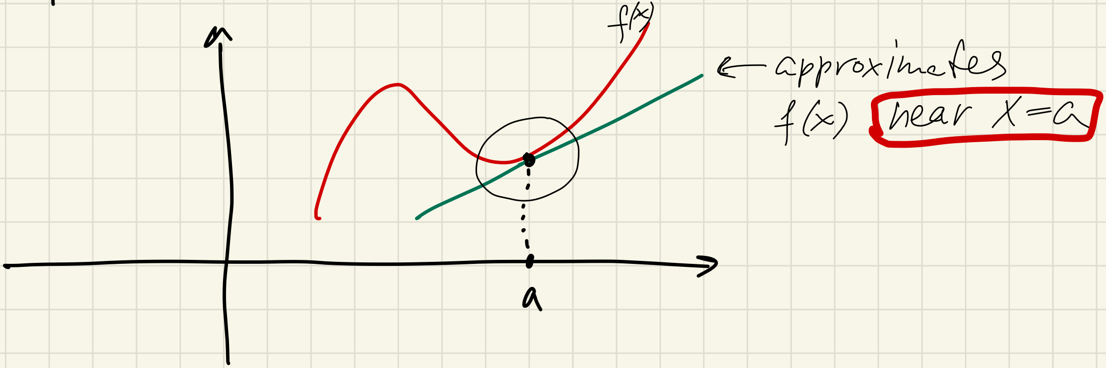
(1) We approximate functions $f(x)$ by <u>linear functions</u>

Linearization (linear approximation)

input: $f(x)$ function
$x=a$ point

output: $L(x) = \underbrace{f(a)+f'(a)(x-a)}_{\text{eq. of tangent line}}$

Linearization of $f(x)$ at $x=a$ is the equation of the tangent line.



Ex. 1 Find linearization (linear approximation) of $f(x) = x^2 - 4$ at $x=1$.

Use it to approximate $f(\frac{3}{2})$. (meaning compute $L(\frac{3}{2})$)

$$L(x) = f(1) + f'(1)(x-1)$$

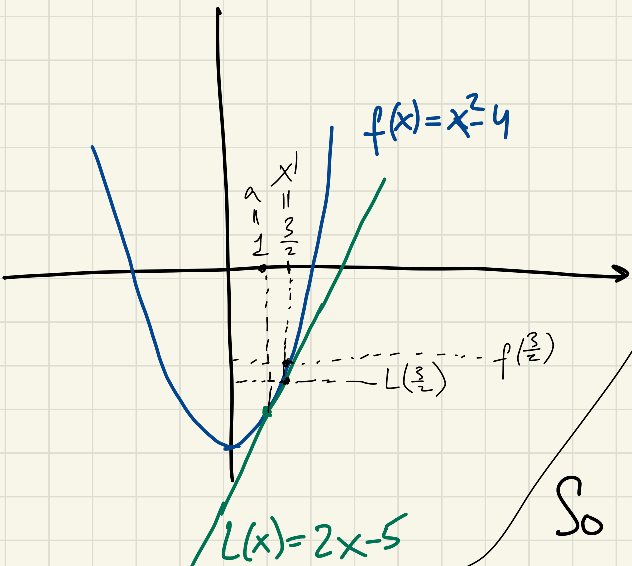
$$f(1) = 1^2 - 4 = -3$$

$$f'(1) = 2 \cdot 1 = 2 \qquad f'(x) = \frac{d}{dx}\left[x^2 - 4\right] = 2x$$

$$L(x) = -3 + 2(x-1)$$

$$\boxed{L(x) = 2x - 5}$$

Since $\frac{3}{2}$ is close to 1, we can use $L(\frac{3}{2})$ to approximate $f(\frac{3}{2})$



$$L\left(\tfrac{3}{2}\right) = 2 \cdot \tfrac{3}{2} - 5 =$$

$$= \boxed{-2} \leftarrow \text{approx.}$$

$$f\left(\tfrac{3}{2}\right) = \left(\tfrac{3}{2}\right)^2 - 4 = \tfrac{9}{4} - 4 =$$

$$= -\tfrac{7}{4} \leftarrow \text{actual value}$$

So $f\left(\tfrac{3}{2}\right) - L\left(\tfrac{3}{2}\right) =$

$$= -\tfrac{7}{4} - (-2) = \tfrac{1}{4} \leftarrow \text{error}$$

$\boxed{\text{Linearization } L(x) \text{ of } f(x) \text{ at } x=a \text{ is used to approximate by } L(b) \text{ y-values } f(b) \text{ for } b \text{ near } a}$

$a=1$
$b=\frac{3}{2}$
$f(x)=x^2-4$

Linearization can be used to approximate values of functions which otherwise would require calculator.

Ex.2 Find approximate of $\sqrt{10}$.

Step 1 shift the perspective from number to function

function $\boxed{f(x)=\sqrt{x}}$ so that $\sqrt{10}=f(10)$

$f'(x)=\frac{1}{2\sqrt{x}}$ ← will need later

Step 2 Find a number a close to 10, such that we know $f(a)$ & $f'(a)$

point $\boxed{a=9}$ ⟶ $f(9)=\sqrt{9}=3$
$f'(9)=\frac{1}{2\sqrt{9}}=\frac{1}{6}$

Step 3 Find lin. approx. of $\underbrace{f(x)}_{\sqrt{x}}$ at $\underbrace{x=a}_{x=9}$

$L(x)=f(9)+f'(9)(x-9)$

$$L(x) = 3 + \frac{1}{6}(x-9)$$

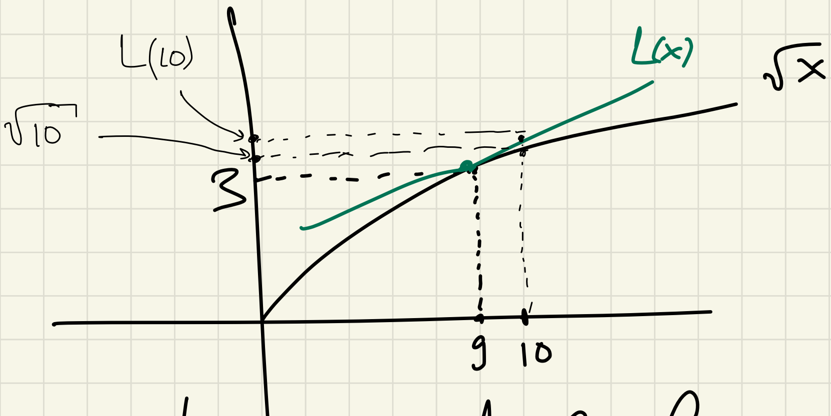
$$L(x) = \frac{1}{6}x + 3 - \frac{9}{6}$$

$$\boxed{L(x) = \frac{1}{6}x + \frac{3}{2}}$$

<u>Step 4</u> use $L(10)$ to approximate $\sqrt{10}$

$$L(10) = \frac{1}{6}\cdot 10 + \frac{3}{2} = \frac{10}{6} + \frac{9}{6} =$$

$$= \boxed{\frac{19}{6}}$$ is a good approximation for $\sqrt{10}$



<u>remark</u> we used 9 because it

is close to 10.

$$\left(\frac{19}{6}\right)^2 = \frac{361}{36}$$ ← pretty close to 10

Ex. 3 approximate $\ln\left(\frac{3}{2}\right)$

Step 1 $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$

Step 2 $a = 1$, $\ln(1) = 0$ (since $e^0 = 1$)

↑ close to $\frac{3}{2}$, and we know its ln

Step 3 $L(x) = f(1) + f'(1)(x-1)$

$$L(x) = \underbrace{\ln(1)}_{0} + \frac{1}{1}(x-1)$$

$$\boxed{L(x) = x - 1}$$

Step 4 $L\left(\frac{3}{2}\right) = \frac{3}{2} - 1 = \frac{1}{2}$

② We also sometimes like to estimate errors. We do it using <u>differentials</u>

<u>Differential</u> of $f(x)$ is a linear function

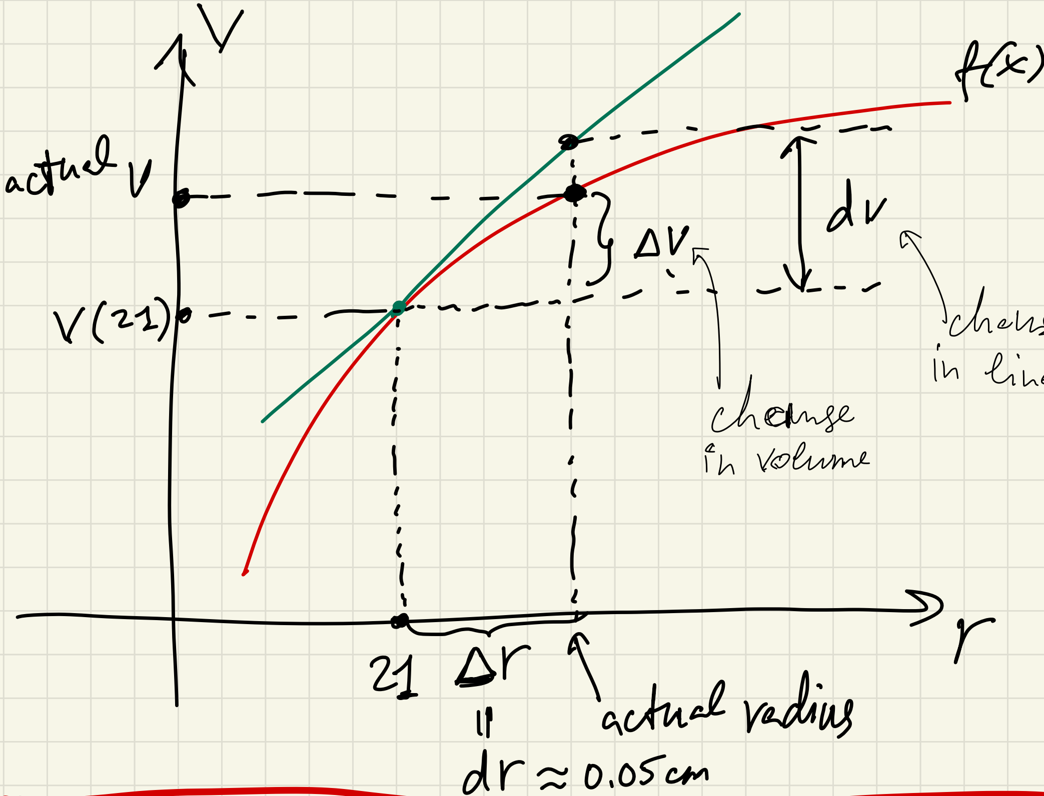
$$\underbrace{dy}_{\text{output}} = f'(x) \underbrace{dx}_{\text{input}} \qquad \left(\text{Because } f'(x) = \frac{dy}{dx}\right)$$

<u>example</u>
suppose $f(x) = x^2$ and we are interested in differential at $x=3$ $\left(f'(x) = 2x,\ f'(3) = 6 \right)$

$$dy = f'(3)\,dx$$

$$\boxed{dy = \underline{6\,dx}}$$ ← linear function

dy	dx
6	1
12	2
-6	-1



$\boxed{dy \text{ is used to approximate } \Delta y}$

Ex. The radius of a sphere was measured to be 21 cm with a $\underbrace{\text{possible error}}_{\Delta x = dx}$ at most 0.05 cm

What is the error in using this radius to compute the

Volume of the sphere?

solution

r - radius

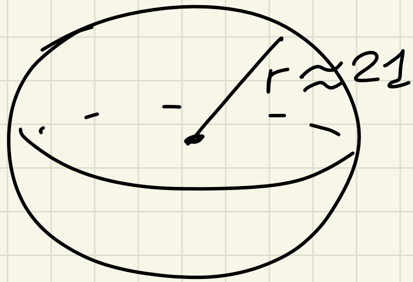
V - volume

$$\frac{d}{dr}\left[\frac{4}{3}\pi r^3\right] = \frac{4}{3}\pi \frac{d}{dr}\left[r^3\right] = \frac{4}{3}\pi\, 3r^2$$

formula

$$\boxed{V = \frac{4}{3}\pi r^3}$$

differentiate $\frac{d}{dr}[-]$



$$\frac{dV}{dr} = 4\pi r^2$$

differential

$$\boxed{\underbrace{dV}_{\text{approx. for error in volume } \Delta V} = 4\pi \underbrace{r}_{\text{radius}}{}^2 \underbrace{dr}_{\text{possible error in radius}}}$$

plus in numbers

$$\boxed{dV = 4\pi \cdot 21^2 \cdot 0.05 = \pi \cdot \frac{441}{5}}$$

So the error in the calculated volume is about $\boxed{\pi \cdot \frac{441}{5}\ \text{cm}^3}$

because dy only approximates Δy

Note: Relative error $= \dfrac{\text{error}}{\text{total volume}}$

is approximated by $\boxed{\dfrac{dy}{y}}$

$$\frac{\Delta V}{V} \approx \boxed{\frac{dV}{V}} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = 3\,\frac{dr}{r} =$$

$$= 3 \cdot \frac{0.05}{21} = \frac{0.05}{7} = \boxed{\frac{5}{100 \cdot 7}} \text{ relative error}$$

or $\boxed{\dfrac{5}{7}\,\%}$

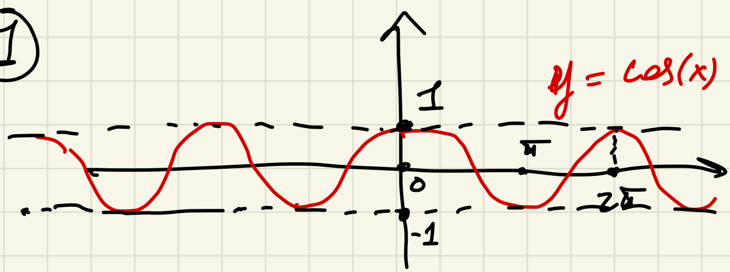
4.1 Maximum & minimum values

def. Let c be a number in the domain D of a function $f(x)$. Then we call $f(c)$

- (global) **absolute maximum** value of f on D if $f(c) \geq f(x)$ for all x in D
- (global) **absolute minimum** value of f on D if $f(c) \leq f(x)$ for all x in D

Examples

①



$D = (-\infty, \infty)$

abs. max. $= 1$

abs. min. $= -1$

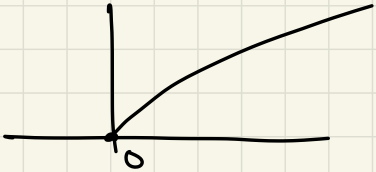
② $y = x^2$

$D = (-\infty, \infty)$

abs max. DNE

abs. min. $= 0$

③ $y=\sqrt{x}$



$D=[0,\infty)$

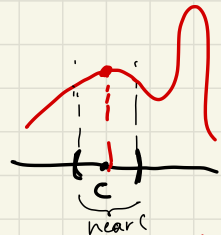
abs. max. DNE

abs. min. $=0$

def. The number $f(c)$ is called

- local maximum value of f

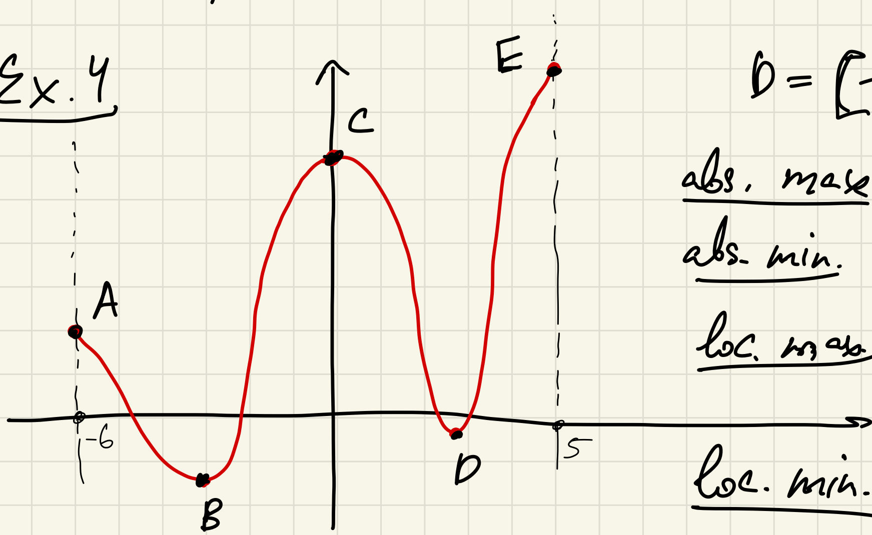
  if $f(c) \geq f(x)$ for x near c



- local minimum value of f

  if $f(c) \leq f(x)$ for x near c

Ex. 4



$D=[-6,5]$

abs. max. E

abs. min. B

loc. max. C, A, E

not abs. max.

loc. min. B, D

Remark: loc./abs. max./min. is always $f(c)$, (y-value)

attained at c (x-value)

Q. Where are loc. min. of $f(x)$?
asks for x-values

Q. What are loc. min. of $f(x)$?
asks for y-values

remark abs. max. is always loc. max. as well

def. A critical number (point) of a function f is a number c in the domain of f such that

either (a) $\underbrace{f'(c)=0}$, or (b) $\underbrace{f'(c) \text{ DNE}}$,

Ex 5 $f(x) = 4 + \frac{1}{3}x - \frac{1}{2}x^2$

find crit numbers?

(a) $f'(x) = \frac{1}{3} - x = 0$

$$\boxed{x = \frac{1}{3}}$$

(b) $\underbrace{f'(x) \text{ DNE}}$

no such x

Ex 6 $f(x) = e^{x^2-x}$, crit. numbers?

$$f'(x) = \frac{d}{dx}\left[e^{x^2-x}\right] \underset{\text{chain rule}}{=} e^{x^2-x} \cdot (2x-1)$$

(a) $\underbrace{e^{x^2-x}}_{\text{never } 0} \cdot (2x-1) = 0$

$2x-1 = 0$

$\boxed{x = \frac{1}{2}}$

(b) $\underbrace{e^{x^2-x}(2x-1) \text{ DNE}}_{\text{no such } x}$

Ex. 7 $f(x) = \dfrac{1}{7-x}$, crit numbers?

$f'(x) = \dfrac{d}{dx}\left[\dfrac{1}{7-x}\right] \underset{\substack{\text{(chain rule)} \\ \text{or} \\ \text{(quoitent rule)}}}{=} -\dfrac{1}{(7-x)^2}\cdot(-1) =$

$= \dfrac{1}{(7-x)^2}$ DNE at $x=7$

So $\boxed{x=7}$ is the crit. number

(nothing else since $\dfrac{1}{(7-x)^2} = 0$ doesn't have solutions)
$\downarrow$
$1 = 0$

Ex. 8 $f(x) = \dfrac{x^2-1}{x+3}$ crit. numbers?

$$f'(x) \underset{\text{quotient rule}}{=} \frac{2x(x+3) - (x^2-1)\cdot 1}{(x+3)^2} =$$

$$= \frac{x^2+6x+1}{(x+3)^2}$$

crit. numbers:

ⓑ $f'(x)$ DNE at $x=-3$

ⓐ

$$\frac{x^2+6x+1}{(x+3)^2} = 0 \to \text{roots of } x^2+6x+1=0$$

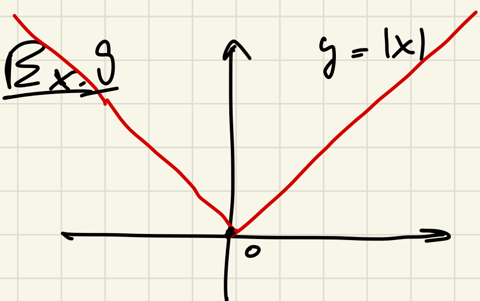
$$x = \frac{-6+\sqrt{36-4}}{2}$$

$$x = \frac{-6-\sqrt{36-4}}{2}$$

Fermat's theorem

$x=c$ loc. max./min for $f(x)$ $\overset{\text{"implies"}}{\Longrightarrow}$ $x=c$ is a critical number or an endpoint

Ex. 9

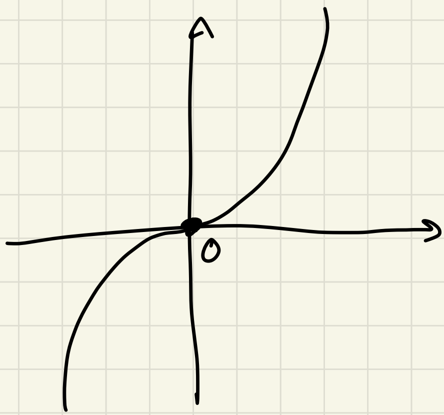


at $x=0$ $|x|$ has a loc. min., so by [Fermat's theorem] 0 is a crit. number for $|x|$

Warning

loc. max./min. $\cancel{\Longleftarrow}$ crit. number

Ex. 10 $f(x)=x^3$ $\quad f'(x)=3x^2=0$



$\boxed{x=0 \text{ not loc. max./min}}$ $\quad \boxed{x=0 \text{ is a crit. number}}$

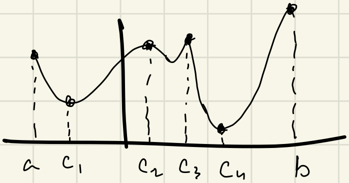
The closed interval method

To find absolute max. & min. values of a continuous function on a closed interval $[a,b]$:

Step 1 Find crit. numbers of f on $[a,b]$: $c_1, c_2, \dots, c_n$

ⓐ $f'(x)=0$
ⓑ $f'(x)$ DNE

Step 2 Compute $f(c_1), f(c_2), \dots, f(c_n), f(a), f(b)$



values at crit. points

values at endpoints

Step 3 The largest value in step 2 — <u>abs. max.</u>

The smallest value in step 2 — <u>abs. min.</u>

Ex. 11 $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$

find abs. max. & abs. min. on $[-2, 3]$

check: $f(x)$ is continuous, so we use the closed int. method.

Step 1 $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$\underbrace{f'(x) = 0}$ or $\underbrace{f'(x) \text{ DNE}}$

$12x(x^2 - x - 2) = 0$ no such x

$x = 0$ or $x^2 - x - 2 = 0$ → $\boxed{\begin{matrix} x = 0 \\ x = -1 \\ x = 2 \end{matrix}}$ crit. numbers

all on $[-2, 3]$

Step 2 values at crit. pts

$f(0) = 3 \cdot 0^4 - 4 \cdot 0^3 - 12 \cdot 0^2 + 1 = 1$

$f(-1) = 3 \cdot (-1)^4 - 4 \cdot (-1)^3 - 12 \cdot (-1)^2 + 1 = -4$

$f(2) = \ldots = -31$

values at endpoints $\begin{cases} f(-2) = \dots = 33 \\ f(3) = \dots = 28 \end{cases}$

Step 3

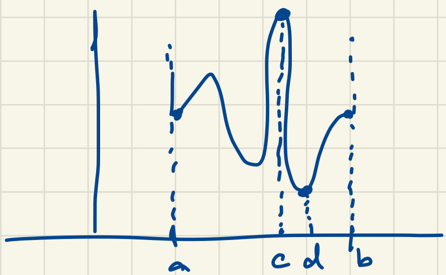
largest value 33 — abs. max. attained at -2

smallest value -31 — abs. min. attained at 2

Why the closed interval method works?

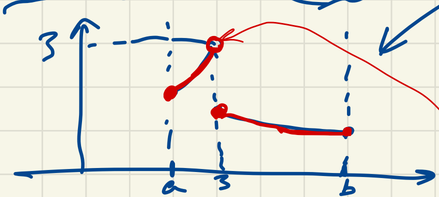
Extreme value thm

If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains the abs. max. value $f(c)$ and the absolute min. value $f(d)$ at some numbers c & d on $[a,b]$



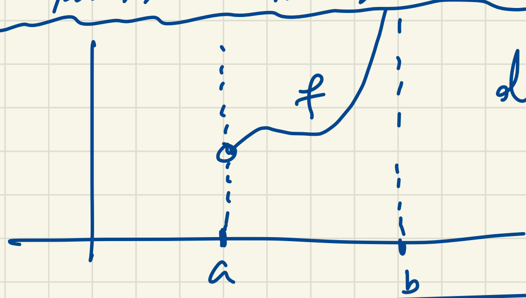
Examples of failure of this thm if some conditions are not met:

1) if $f(x)$ is not contin. doesn't attain abs. max.,



Because this point is excluded

2) if interval is not closed: (a,b)



doesn't attain
nor abs. max.
neither abs. min.

So, whenever using closed int. method, need to check that $f(x)$ is cont. & interval is closed

Ex. 3 $f(x) = x^{-2} \cdot \ln(x)$
find abs. max/min on $[\frac{1}{2}, 4]$

- $f(x)$ is contin. on $[\frac{1}{2}, 4]$
- $[\frac{1}{2}, 4]$ is closed

So we use the closed int. method:

Step 1 Find criti. numbers:

$$f'(x) = \frac{d}{dx}\left[x^{-2}\ln(x)\right] =$$

$$= -2(x^{-3})\ln(x) + x^{-2}\cdot\frac{1}{x} =$$

$$= -2x^{-3}\ln(x) + x^{-3} = x^{-3}(-2\ln(x)+1)$$

$$f'(x) = \frac{-2\ln(x)+1}{x^3}$$

$\underline{f'(x)=0}$

$$-2\ln(x)+1=0$$

$$\ln(x) = \frac{1}{2}$$

$$\boxed{x = e^{\frac{1}{2}}} = \sqrt{e}$$

$\underbrace{f'(x) \text{ DNE}}$

$x<0$

$x=0$

not on $[\frac{1}{2},4]$

$$\frac{1}{2} \leq 1 \leq \sqrt{e} \leq e \leq 4$$

<u>Step 2</u> Compute values of $f(x)$

- at critical numbers on the interval $[\frac{1}{2},4]$

$$f(e^{\frac{1}{2}}) = (e^{\frac{1}{2}})^{-2}\ln(e^{\frac{1}{2}}) = \frac{1}{e}\cdot\frac{1}{2} = \frac{1}{2e}$$

. at endpoints

$$f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^{-2} \ln\left(\tfrac{1}{2}\right) = 4 \cdot \ln\left(\tfrac{1}{2}\right)$$

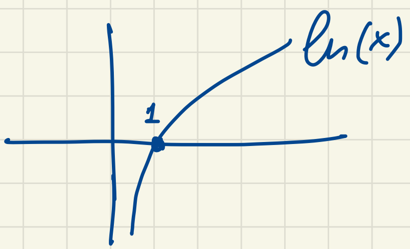
$$f(4) = (4)^{-2} \ln(4) = \frac{1}{16} \ln(4)$$

Step 3

$$\underline{\frac{1}{2e}} \qquad \underline{4\ln\left(\tfrac{1}{2}\right)} \qquad \underline{\frac{1}{16}\ln(4)}$$

Choose largest & smallest.

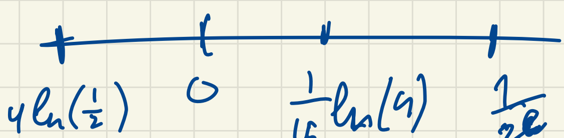
$$\cdot\ 4\ln\left(\tfrac{1}{2}\right) = -4\ln\left(2^{-1}\right) = -4\ln(2) < 0$$



$\cdot\ \frac{1}{2e} > 0$

$\cdot\ \frac{1}{16}\ln(4) > 0$

So we know $\boxed{4\ln\left(\tfrac{1}{2}\right) \text{ is } \underline{\text{abs. min.}}}$



Remaining Q.: $\frac{1}{2e} > \frac{1}{16}\ln(4)$

$$\frac{8}{e} > \ln(4)$$

So

$$\boxed{\frac{1}{2e} \text{ is the abs. max.}}$$

$$8 > \underset{\substack{\| \\ 3\cdot 2}}{6} > \underbrace{e}_{\substack{\wedge \\ 3}} \cdot \underbrace{\ln(4)}_{\substack{\wedge \\ 2}} \quad (*)$$

$(*)$ $\ln(4) < 2$

$2e^{(-)}$

$$4 < e^2$$

$$4 = 2^2 < e^2$$

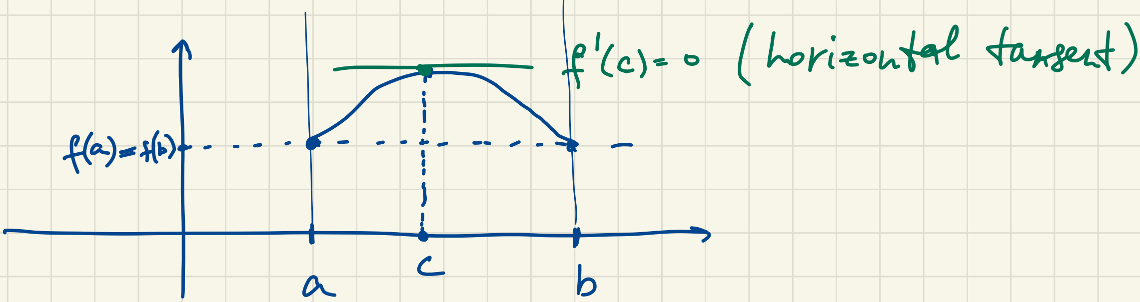
$$2 < \underset{\substack{\| \\ 2.718\ldots}}{e}$$

4.2 Mean Value theorem

Rolle's Theorem

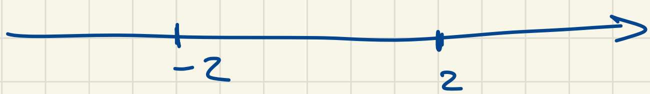
If
- $f(x)$ is differentiable on $[a,b]$
- and $f(a)=f(b)$

then there exists c on $[a,b]$ such that $f'(c)=0$



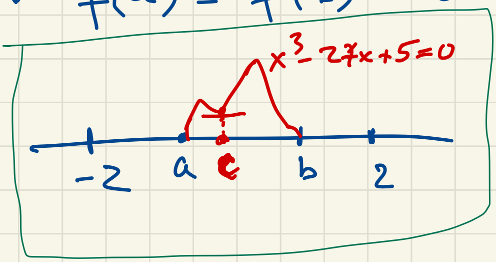
Problem (how Rolle's thm is used)

Prove that there is at most one x on $[-2,2]$ such that $x^3-27x+5=0$



We show it by assuming the contrary, and arriving at contradiction (proof by contradiction)

- assume there is more than one x on $[-2,2]$ such that $\underbrace{x^3-27x+5}_{f(x)}=0$
- then take two of them: a and b on $[-2,2]$
- $f(a)=f(b)=0$



Use Rolle's:

- $f(a)=f(b)$
- $x^3-27x+5$ differentiable

- So $f'(c)=0$ for some c on $[a,b]$
- $f'(x)=3x^2-27$ has root on $[a,b]$, which itself is between $\underline{[-2,2]}$

$f'(x)=0$

- $3x^2-27=0$

$3(x^2-9)=0$

$3(x-3)(x+3)=0$

$x=3$ → neither of these

$x=-3$ → are on $[-2,2]$!

that's contradiction

- our assumption is false, and so there is no more than one x on $[-2,2]$ s.t. $x^3 - 27x + 5 = 0$

Problem

Prove that "A is true".

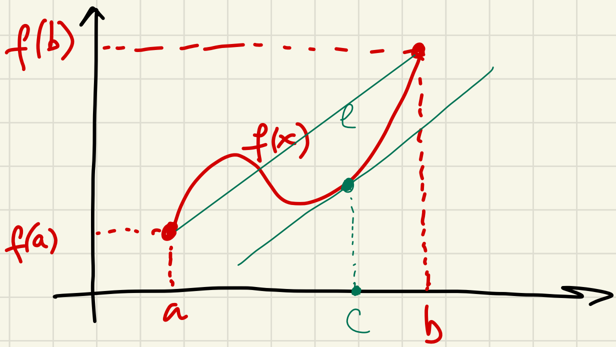
Proof by contradiction

- assume "A is not true"
- arrive at contradiction
- conclude that assumption was incorrect, and therefore A is true

Continuing with 4.2

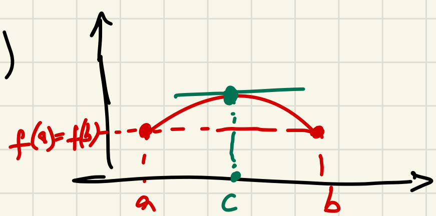
Mean Value Theorem (MVT)

If $f(x)$ is differentiable on $[a,b]$ then there exists some c on $[a,b]$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$ (slope of line ℓ)



remark: Rolle's thm is a special case of MVT, when $f(a)=f(b)$

$f'(c) = \dfrac{f(b)-f(a)}{b-a} = 0$



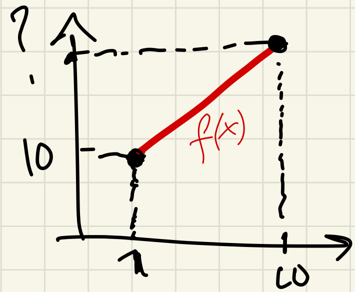
What MVT is used for?

① Problems where we relate $f'(x)$ to values of $f(x)$

Ex. 1 If $f(x)$ is differentiable on $[1, 10]$, $f'(x) = 3$ everywhere, and $f(1) = 10$, then what is the value $f(10)$?

solution: since $f(x)$ is diff. on $[1, 10]$, let's use MVT:

$$\underbrace{f'(c)}_{3} = \frac{f(10) - \overset{10}{f(1)}}{10 - 1}$$



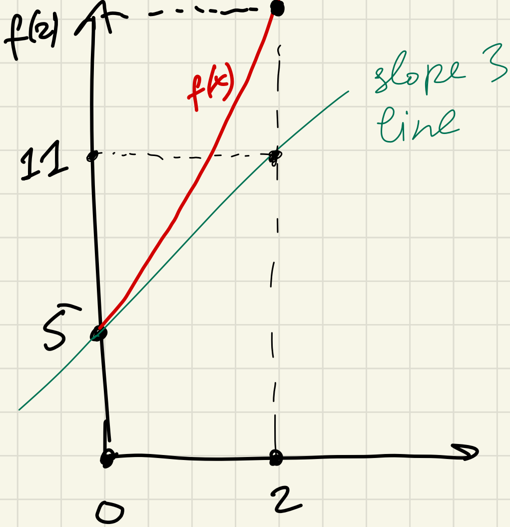
$$3 = \frac{f(10) - 10}{9}$$

$$27 = f(10) - 10$$

$$\boxed{37 = f(10)}$$

Ex. 2 Let $f(x)$ be differentiable on $[0, 2]$ and let $f(0) = 5$. If $f'(x) \geq 3$ on $[0, 2]$, What is the smallest that $f(2)$ can be?

solution:



$f(x)$ differentiable on $[0,2]$ $\longrightarrow$

$\longrightarrow$ we use MVT

$$3 \le \boxed{f'(c)} \overset{?}{=} \frac{f(2) - \overbrace{f(0)}^{5}}{2-0}$$

$$3 \le \frac{f(2)-5}{2}$$

$$6 \le f(2) - 5$$

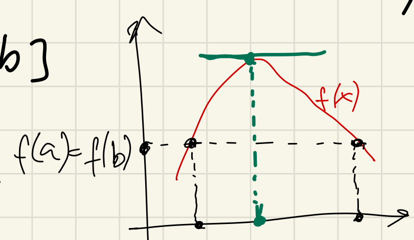
$$\boxed{11} \le f(2)$$

answer

Recall Rolle's thm (MVT when $f(a)=f(b)$)

- $f(x)$ is differentiable on $[a,b]$
- $f(a)=f(b)$

Then there is c on $[a,b]$ such that



a c b

$$\boxed{f'(c) = 0}$$

Ex. 3 Show that there's exactly one x on $[-2, 2]$ such that $x^3 + e^x = 0$.

Solution Solving $x^3+e^x=0$ is impossible! Instead, we don't find the root, but rather prove that it exists!

Step 1 prove that there cannot be two (or more) roots of $x^3+e^x=0$ on $[-2,2]$

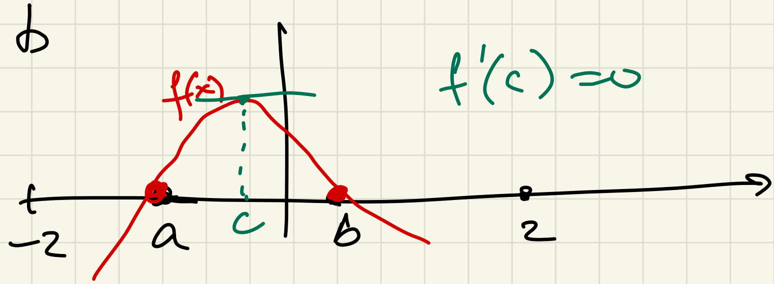
Step 2 prove that there is at least one root.

$$\#\text{ roots}: \quad \underbrace{\cancel{0} \quad 1}_{\substack{\text{Step 2} \\ \text{(IVT)}}} \quad \overbrace{\cancel{2} \quad \cancel{3} \quad \cancel{\cdot} \quad \cancel{\cdot} \quad \cancel{\cdot}}^{\substack{\text{(Rolle's)} \\ \text{Step 1}}}$$

step 1 prove that there cannot be two roots of $\underbrace{x^3+e^x}_{f(x)}=0$ on $[-2,2]$.

Proof by contradiction:

- Assume there are two roots of $f(x)=0$ on $[-2,2]$. Denote two roots by a & b



- $\cdot\ f(x)=x^3+e^x$ is differentiable on $[a,b]$
  $\cdot\ f(a)=f(b)$ (because both are 0)

Rolle's

there is c on $[a,b]$ s.t.

$$f'(c)=0$$

• Let's see if this is possible

$f'(x) = 3x^2 + e^x$ so

$$f'(x) = 0$$

$$\boxed{0 < 3x^2 + e^x} = 0$$

$$\geq 0 \qquad > 0$$

doesn't have any solutions!

So $f'(c) = 0$ cannot exist, so we arrive at $\boxed{\text{contradiction}}$

• So the assumption that the two roots a & b exist is false

• so there cannot be two or more roots for $x^3 + e^x = 0$ on $[-2, 2]$

Step 2 prove that there

is at least one root
of $x^3+e^x=0$ on $[-2,2]$

<u>solution</u>: use IVT (intermidiate value thm)

<u>IVT</u>

- $f(x)$ is cont. on $[a,b]$

then $f(x)$ attains all values between $f(a)$ & $f(b)$.

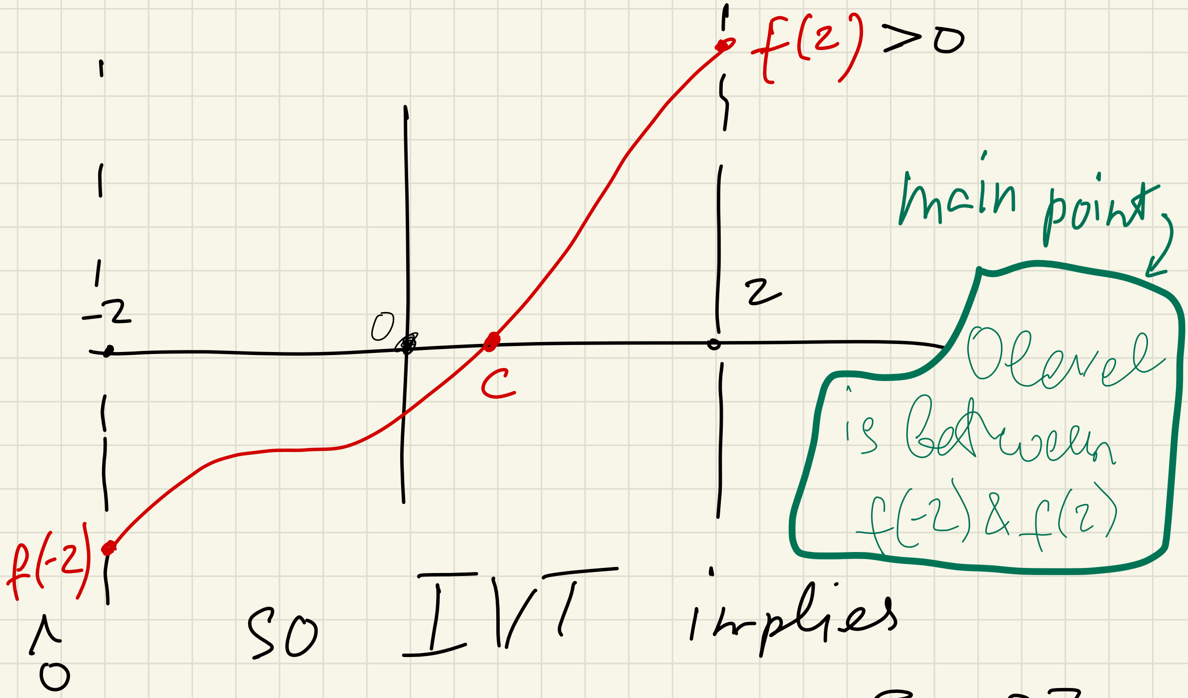
<u>IVT in our case</u> $a=-2$, $b=2$

- $f(x)=x^3+e^x$ <u>is contin.</u>

So $f(x)$ attains all value between

$$f(-2)=(-2)^3+e^{-2}=-8+\underbrace{\frac{1}{e^2}}_{<1}<0$$

$$f(2)=2^3+e^2=8+e^2>0$$

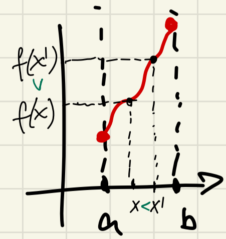


so IVT implies existence of c on $[-2,2]$ such that $f(c)=0$, and that's what we wanted.

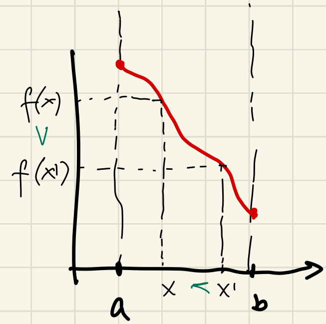
4.3 How derivatives affect the shape of a graph

__def.__

- $f(x)$ is increasing on (a,b)
  if for every $x<x'$ we have $f(x)<f(x')$



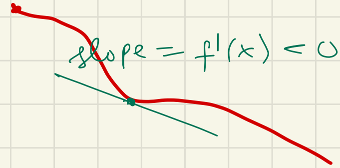
- $f(x)$ is decreasing on (a,b)
  if for every $x<x'$ we have $f(x)>f(x')$



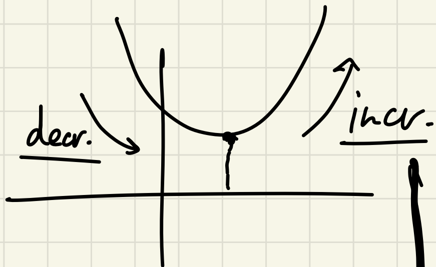
__Key facts to remember__ if $f(x)$ is differentiable:

1. $f(x)$ is increasing on (a,b) follows from $f'(x)>0$ everywhere on (a,b)

2. $f(x)$ is decreasing on (a,b) follows from $f'(x)<0$ everywhere on (a,b)



Ex. 1 Given $f(x) = x^2 - 4x + 5$, find the intervals where $f(x)$ is increasing / decreasing.



Strategy: $\boxed{\text{increasing/decreasing test}}$

Step 1 find where $f'(x) = 0$ or $f'(x)$ DNE (crit. numbers of $f(x)$)

Step 2 Intervals complimentary to crit. numbers are either decr. / incr. intervals.

To decide, plug in x-values from each resulting interval into $f'(x)$

Step 1 $f'(x) = 2x - 4$

$f'(x) = 0$

$2x - 4 = 0$

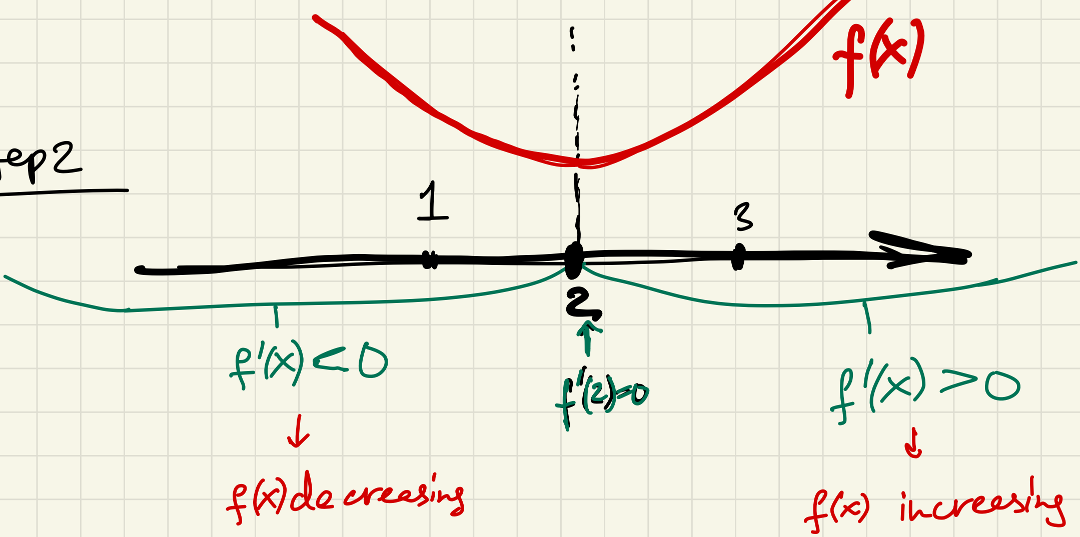
$2x = 4$

$x = 2$

$f'(x)$ DNE

no such x

step2



$f'(1) = 2 \cdot 1 - 4 = -2 < 0$, so $f'(x) < 0$ on $(-\infty, 2)$

$f'(3) = 2 \cdot 3 - 4 = 2 > 0$, so $f'(x) > 0$ on $(2, \infty)$

Answer $x^2 - 4x + 5$ is

decreasing on $(-\infty, 2)$

increasing on $(2, +\infty)$

Ex.2 $f(x) = \dfrac{1}{x^2}$. Find where decreasing/increasing?

step 1 $f'(x) = \dfrac{d}{dx}\left[x^{-2}\right] = -2x^{-3} =$

$$= \frac{-2}{x^3}$$

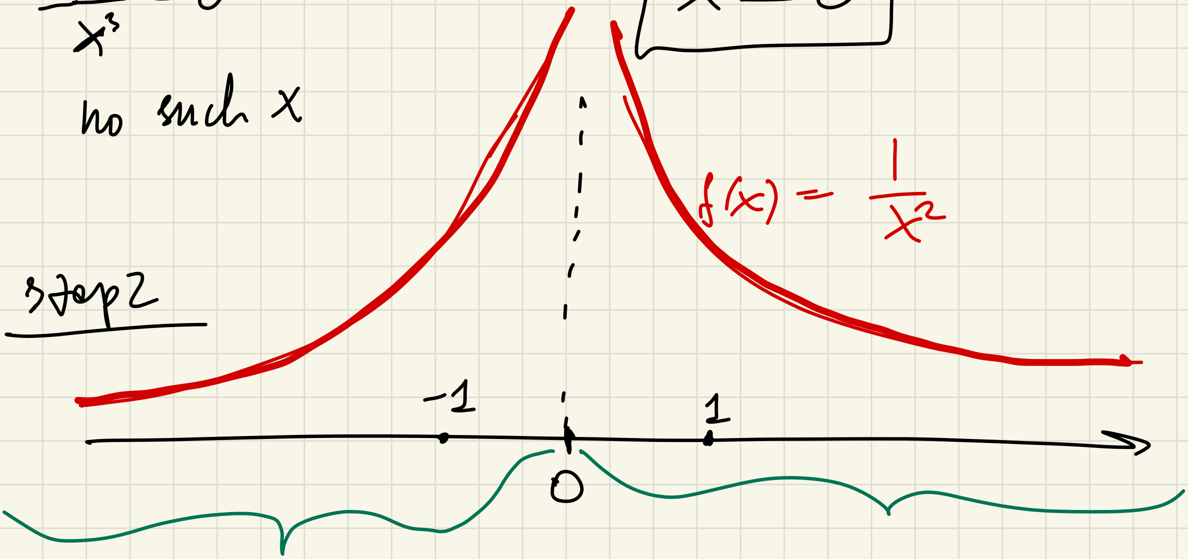
$\underbrace{f'(x)=0}$

$\frac{-2}{x^3} = 0$

no such x

$\underbrace{f'(x) \text{ DNE}}$

$\boxed{x=0}$



$f'(x) > 0$

$\downarrow$

$f(x)$ is increasing

$f'(x) < 0$

$\downarrow$

$f(x)$ is decreasing

- $f'(1) = -\frac{2}{1^3} = -2 < 0$, so

$f(x)$ is <u>decreasing</u> on $(0, +\infty)$

- $f'(-1) = -\frac{2}{(-1)^3} = -\frac{2}{-1} = 2 > 0$, so

$f(x)$ increasing on $(-\infty, 0)$

<u>Ex. 3</u> $f(x) = e^x$. Where incr./decr.?

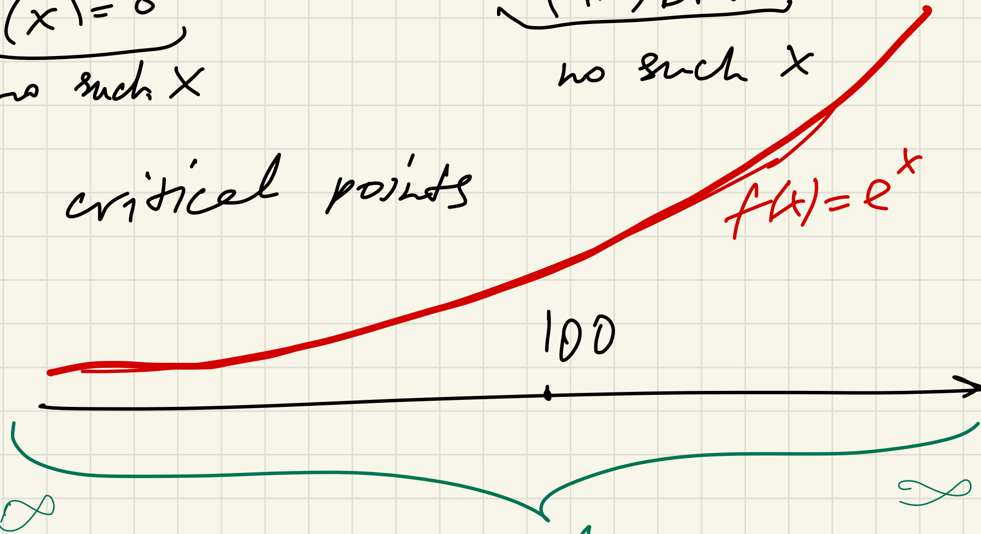
<u>Step 1</u>

$f'(x) = e^x$

$\underbrace{f'(x) = 0}_{\text{no such } x}$ $\underbrace{f'(x) \text{ DNE}}_{\text{no such } x}$

So <u>no</u> critical points

<u>Step 2</u>



$f'(x) > 0$

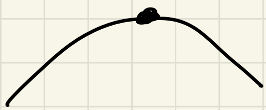
$\downarrow$

$f(x)$ is increasing on $(-\infty, \infty)$

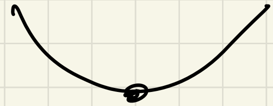
$f'(100) = e^{100} > 0$

Recall:

local max



local min.



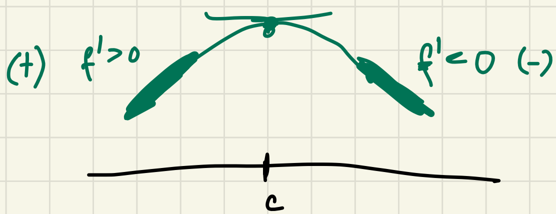
Q. How do we find loc. max/min?

First derivative test

If $x=c$ is a crit. number $\left(\begin{array}{c} f'(c)=0 \\ \text{or} \\ f'(c) \text{ DNE} \end{array} \right)$

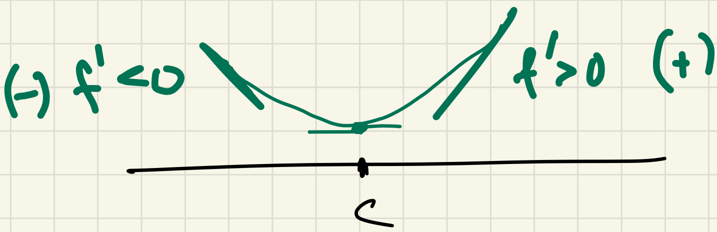
then

1) if $f'(x)$ switches from $+$ to $-$ at c,
   then c is a loc. max. for $f(x)$



2) if $f'(x)$ switches from $-$ to $+$ at c,

then c is a <u>loc. min.</u> for $f(x)$



(if neither of the above, then there is no conclusion)

<u>Ex. 4</u> $f(x) = (x+1)^5 (x-1)^3$

Find where it is incr./decr., and also find loc. max./min.

<u>step 1</u> find crit. numbers

$$f'(x) = \frac{d}{dx}\left[(x+1)^5 (x-1)^3\right] = \begin{pmatrix}\text{product}\\ \text{rule}\end{pmatrix}$$

$$= \frac{d}{dx}\left[(x+1)^5\right](x-1)^3 + (x+1)^5 \frac{d}{dx}\left[(x-1)^3\right] = \begin{pmatrix}\text{chain}\\ \text{rule}\end{pmatrix}$$

$$= 5(x+1)^4 \cdot 1 \cdot (x-1)^3 + (x+1)^5 \cdot 3(x-1)^2 \cdot 1 =$$

(factor out $(x+1)^4 \cdot (x-1)^2$

$$= (x+1)^4(x-1)^2\big(5(x-1) + 3(x+1)\big) =$$

$$= \boxed{(x+1)^4(x-1)^2(8x-2) = f'(x)}$$

$\underline{f'(x) = 0}$

$x+1=0 \rightarrow x=-1$

$x-1=0 \rightarrow x=1$

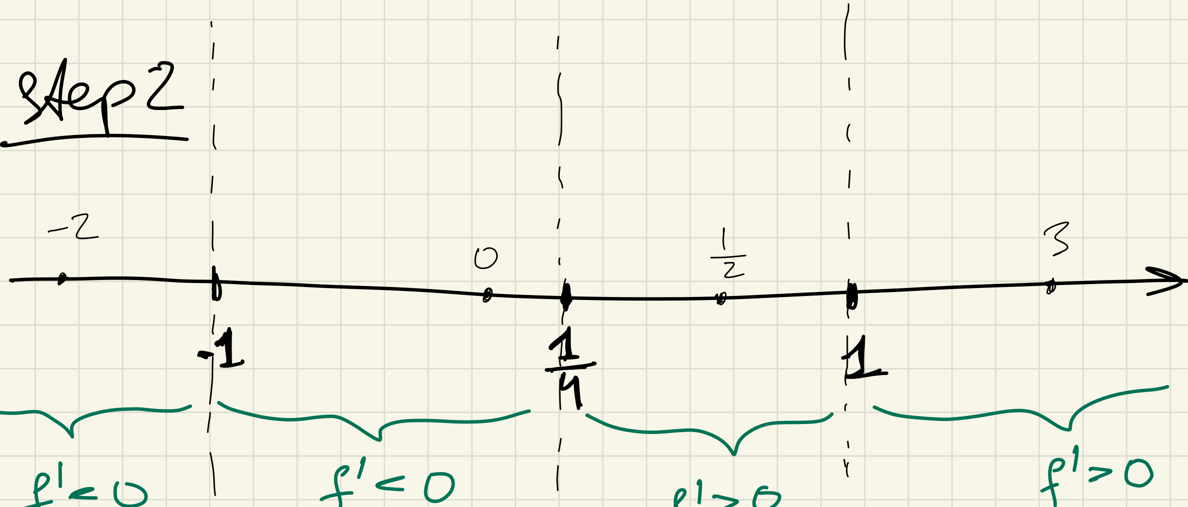
$8x-2=0 \rightarrow x=\frac{1}{4}$

crit. numbers

$\underline{f'(x) \text{ DNE}}$

no such x

Step 2



$f'<0$	$f'<0$	$f'>0$	$f'>0$
(−)	(−)	(+)	(+)
f(x) decr	f(x) decr.	f(x) incr.	f(x) incr.

To find where f(x) incr./decr.:

• $f'(3) = (3+1)^4(3-1)^2(8\cdot 3-2) > 0$, and

so by incr./decr. test we know

$f'(x) > 0$ on $(1, +\infty)$

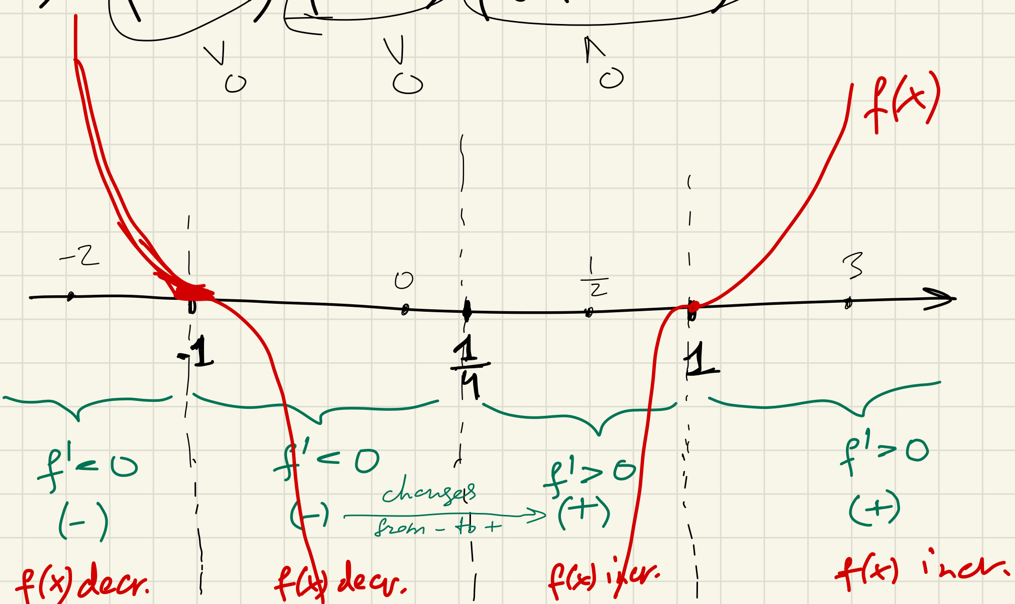
• $f'(\frac{1}{2}) = \underset{>0}{(\frac{1}{2}+1)^4}\,\underset{>0}{(\frac{1}{2}-1)^2}\,\underset{>0}{(8\cdot\frac{1}{2}-2)} > 0$

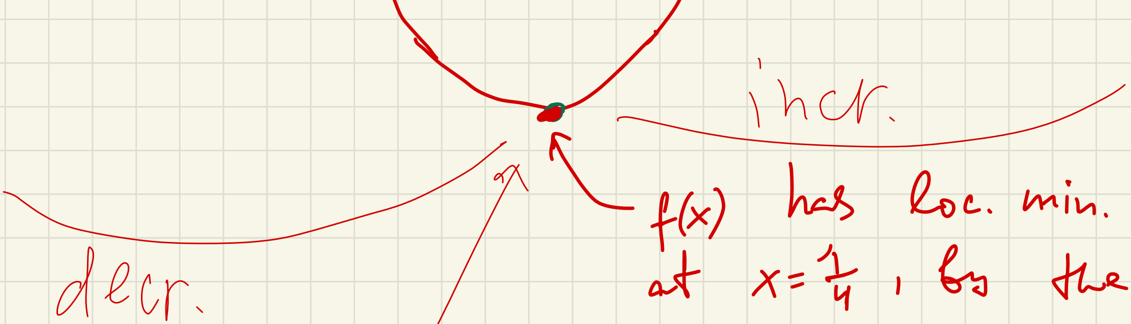
squared!

so $f'(x) > 0$ on $(\frac{1}{4}, 1)$

• $f'(0) = \underset{>0}{(0+1)^4}\,\underset{>0}{(0-1)^2}\,\underset{<0}{(8\cdot 0-2)} < 0$

• $f'(-2) = \underset{>0}{(-2+1)^4}\,\underset{>0}{(-2-1)^2}\,\underset{<0}{(8\cdot(-2)-2)} < 0$



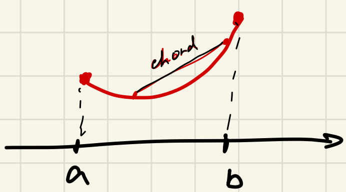


f(x) has loc. min. at $x=\frac{1}{4}$, by the first deriv. test

($f'(x)$ changes from (−) to (+))

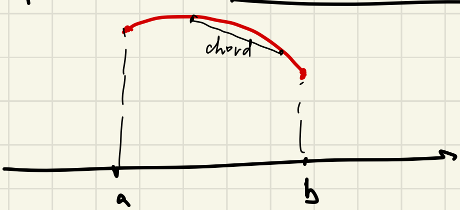
Concavity

$f(x)$ is concave up on (a,b) if $f''(x)>0$ for all x on (a,b)



(it is equiv. to every "chord" being above) ← intuition

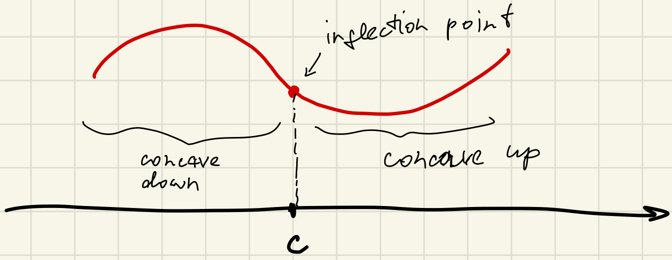
$f(x)$ is concave down on (a,b) if $f''<0$ for all x on (a,b)



(it is equiv. to every "chord" being below)

intuition

- $x = c$ is a point of <u>inflection</u> if $f''(x)$ switches from + to − or from − to + at $x=c$



<u>Ex.</u> Find where $f(x) = 3x^5 - 5x^3 + 15$ is concave up/down & find points of inflection.

Procedure here is similar as before, but with $f''(x)$ instead of $f'(x)$

- $f'(x) = 15x^4 - 15x^2$

$$f''(x) = 60x^3 - 30x = 60x\left(x^2 - \tfrac{1}{2}\right) = 60x\left(x - \sqrt{\tfrac{1}{2}}\right)\left(x + \sqrt{\tfrac{1}{2}}\right)$$

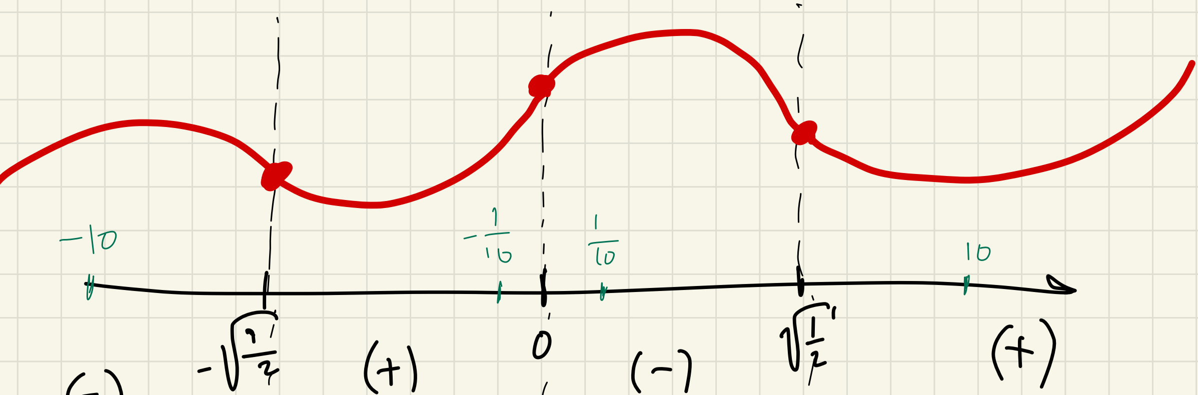
$f''(x) = 0 \longrightarrow$

$x = 0$

$x = \sqrt{\frac{1}{2}}$

$x = -\sqrt{\frac{1}{2}}$

points where $f''(x) = 0$



(−) (+) (−) (+)

$f''<0$ | $f''>0$ | $f''<0$ | $f''>0$

check by plugging $10, \frac{1}{10}, -\frac{1}{10}, -10$ into f''

$$f''(x) = 60x\left(x-\sqrt{\tfrac{1}{2}}\right)\left(x+\sqrt{\tfrac{1}{2}}\right)$$

conc. down | conc. up | conc down | conc. up

all three are inflection points, because at each point $f''(x)$ switches the sign

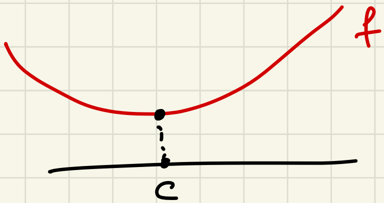
(finishing 4.3)

Second derivative test

If $x = c$ is a crit. number $(f'(c)=0 \text{ or } f'(c) \text{ DNE})$ for $f(x)$,

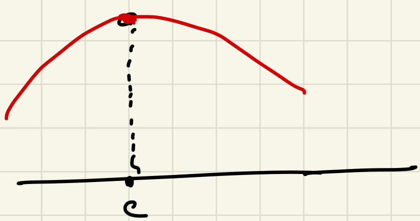
then

1) if $f''(c) > 0$, then $f(c)$ is loc. min.



(• crit. number
• concave up
imply loc. min)

2) if $f''(c) < 0$, then $f(c)$ is loc. max.



(• crit. number
• concave down
imply loc. max.)

Remark if $f''(c) = 0$ there is no conclusion

Ex. 1 $f(x) = \cos x + \sin x$ has

crit. numbers at $x = \frac{\pi}{4}$ & $x = \frac{5\pi}{4}$

(there is more, but we focus two)

Use 2^{nd} deriv. test to find whether these are loc. min. or max.

<u>solution</u> need the second deriv.:

$$f(x) = \cos x + \sin x$$

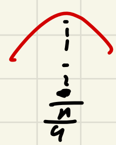
$$f'(x) = -\sin x + \cos x$$

$$f''(x) = -\cos x - \sin x$$

$\boxed{x = \frac{\pi}{4}}$ (by the way, it is a crit. number because $f'(\frac{\pi}{4}) = 0$)

$$f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} - \sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} =$$

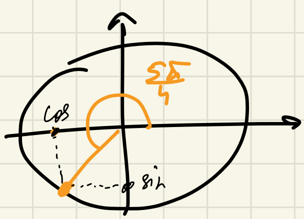
$$= -\sqrt{2} < 0,$$

so <u>concave down</u> 

,

so <u>loc. max.</u> at $x = \frac{\pi}{4}$

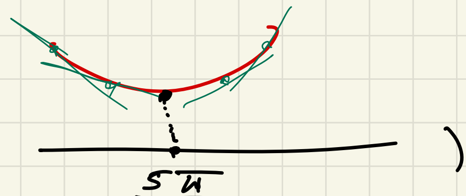
$\boxed{x = \frac{5\pi}{4}}$



$$f''\left(\frac{5\pi}{4}\right) = -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} =$$

$$= -\left(-\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2} > 0$$

so concave up



so loc. min. at $x = \frac{5\pi}{4}$

4.4 L'Hospital's Rule (LH rule)

Given two functions $f(x)$ & $g(x)$ such that

$\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ ("indeterminate form")

then $\boxed{\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}}$

Note:

- a can be ∞ or $-\infty$
- If $\lim \neq \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then DO NOT use LH!

(you will get a wrong answer)

- on HW/quizzes/exams, you have to say that you are using "LH rule"

Ex. 2

Compute $\lim_{x \to 0} \frac{\sin x}{x} \xrightarrow[x=0]{\text{sub}} \frac{\sin 0}{0} = \frac{0}{0}$, So we can use LH rule

$\frac{d}{dx}[-]$ $\frac{d}{dx}[-]$

$\parallel$ LH rule

$\lim_{x \to 0} \frac{\cos x}{1}$

$\parallel$

$\frac{\cos 0}{1} = \frac{1}{1} = \boxed{1}$

Ex. 3 $\lim_{x \to 1} \frac{\ln x}{x-1} \xrightarrow[x=1]{\text{sub}} \frac{\ln 1}{1-1} = \frac{0}{0}$

$\parallel$ LH rule

$$\lim_{x \to 1} \frac{\frac{1}{x}}{1}$$

sub $\cancel{x}=1$

$$\frac{\frac{1}{1}}{1} = \boxed{1}$$

"sub" (∞ = very large number)

$x = \infty$

<u>Ex. 4</u> $\lim_{x \to \infty} \frac{x^2}{e^{2x}} \longrightarrow \frac{\infty^2}{e^{2\cdot\infty}} = \frac{\infty}{\infty}$

|| LH rule

$\frac{d}{dx}[\cdot]$ using chain rule

$\lim_{x \to \infty} \frac{2x}{2 \cdot e^{2x}} \longrightarrow \frac{2\cdot\infty}{2\cdot e^{2\cdot\infty}} = \frac{\infty}{\infty}$

|| LH rule

$$\lim_{x \to \infty} \frac{2}{2 \cdot 2 \cdot e^{2x}}$$

"substitute" ∞ (really large number)

$$\frac{2}{2 \cdot 2 \cdot e^{2\cdot\infty}} = \frac{1}{2 \cdot e^{2\cdot\infty}} = \boxed{0}$$

super-large

__Extra trick 1__ If you see $\infty - \infty$, find common denominator to get a fraction and only then use LH rule.

__Ex. 5__ $$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{(x-1)}\right) \quad \xrightarrow[x=1]{\text{sub}} \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

so we find common denominator to obtain a fraction

$$\lim_{x\to 1}\left(\frac{(x-1)}{\ln(x)\cdot(x-1)} - \frac{\ln(x)}{\ln(x)\cdot(x-1)}\right)$$

$$\Downarrow$$

$$\lim_{x\to 1}\left(\frac{x-1-\ln(x)}{\ln(x)(x-1)}\right) \xrightarrow[x=1]{\text{sub}} \frac{0}{0}$$

now we $\underset{\text{LH rule}}{\underline{\text{can}}}$ use LH rule!

$$\left(\frac{d}{dx}\left[\ln(x)(x-1)\right] \overset{\text{prod. rule}}{=\!=} \frac{1}{x}(x-1) + \ln x = 1 - \frac{1}{x} + \ln x\right)$$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x}\right) \xrightarrow[x=1]{\text{sub}} \frac{0}{0}$$

$\Vert$ LH rule

$$\lim_{x\to 1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}+\frac{1}{x}}\right) \xrightarrow[]{\substack{\text{sub}\\ x=1}} \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

answer!

<u>Extra trick 2</u> if you see $(\pm\infty)\times 0$, create a fraction and apply the LH rule

<u>Ex. 6</u> $\lim_{x\to\infty} x^3 \cdot e^{-x^2} \xrightarrow[x=\infty]{\text{sub}} (\infty)^3 \cdot e^{-(\infty^2)} = (\infty)\cdot(0)$

because e^{-10000} is close to 0

Now we throw e^{-x^2} to $\Vert$

denominator:

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} \xrightarrow[x=\infty]{\text{sub}} \frac{(\infty)}{(\infty)}$$

and so now we can apply LH rule!

$$\overset{LH}{=}$$

chain rule

$$\lim_{x\to\infty} \frac{3x^2}{2x\cdot e^{x^2}} \longrightarrow \frac{\infty}{\infty}$$

product rule

$$\overset{LH}{=}$$

$$\lim_{x\to\infty} \frac{6x}{4x^2 e^{x^2} + 2e^{x^2}} \longrightarrow \frac{\infty}{\infty}$$

$$\overset{LH}{=}$$

check this at home

$$\lim_{x\to\infty} \frac{6}{(8x^3 e^{x^2} + 8\cdot x\cdot e^{x^2} + 4xe^{x^2})} \longrightarrow \frac{6}{\infty} = \boxed{0}$$

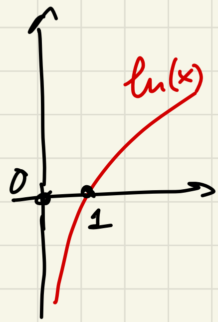
answer

Ex. 1

$$\lim_{x\to 0^+} x^2\ln(x) \xrightarrow[x=0]{\text{"sub"}} \underbrace{\overset{x^2 \to}{0}\cdot\overset{\ln(x) \to}{(-\infty)}}$$

indeterminate form

$$\Big\| \quad \text{create a fraction}$$



$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x^2} \qquad \frac{-\infty}{\infty}, \text{ so we use LH}$$

$$\Big\|\,\text{LH}$$

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-2/x^3} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{x^3}{-2} = \lim_{x\to 0^+} \frac{x^2}{-2}$$

$$\downarrow \text{ sub } x=0$$

$$\frac{0}{-2}$$

$$\|$$

$$\boxed{0}$$

4.5 Curve sketching

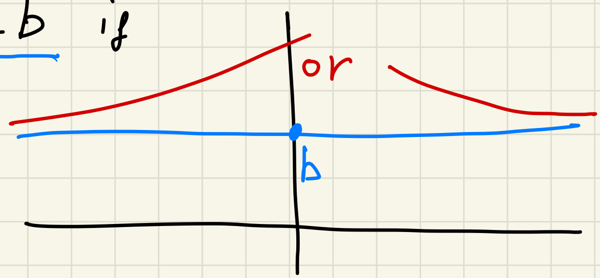
You are given a function $f(x) =$ formula, and asked to sketch the graph.
To do this, follow these steps:

1. Find domain

2. Horizontal & vertical asymptotes

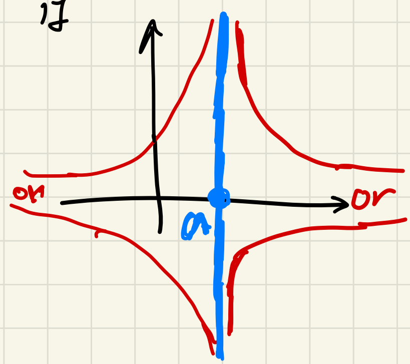
- horiz. asymptote $y=b$ if

$$\lim_{x \to \pm \infty} f(x) = b$$



- vertical asymptote at $x=a$ if

$$\lim_{x \to a^{+ \text{or} -}} f(x) = \pm \infty$$



3. Decreasing/increasing & loc max./min.
(By finding $f'<0$, $f'>0$ & 1st/2nd deriv. test)

After ①, ②, ③ you can sketch the graph,

key points: loc. max./min.

④ x & y intercepts

⑤ concavity & inflection points

$\Big\}$ sometimes

Ex. 2 Sketch $f(x) = x^4 - 8x^2 + 8$

① Domain: $(-\infty, \infty)$ because polynomial

② horiz. asymptotes

$$\lim_{x\to\infty} (x^4 - 8x^2 + 8) = \infty$$ not a number

dominating terms

$$\lim_{x\to-\infty} (x^4 - 8x^2 + 8) = \infty$$ not a number

and so there are no horiz. asymptotes

vertical asymptotes

to have one, we need

$$\lim_{x\to a^{+\text{or}-}} (x^4 - 8x^2 + 8) = \pm\infty$$

but

$$\lim_{x\to a^{+\text{or}-}} (x^4 - 8x^2 + 8) =$$

$$= a^4 - 8a^2 + 8$$

it is never $\pm\infty$!

So no vertical asymptotes.

In other words: to have vert. asymptote we need to be dividing by 0 or have $\ln(x)$, & this is not the case for

$x^4 - 8x^2 + 8 = f(x)$

③ find crit. points:

$$\underbrace{f'(x) = 4x^3 - 16x = 0}$$

$$\underbrace{f'(x) \text{ DNE}}$$

no such x

$$4x(x^2-4)=0$$

$$4x(x-2)(x+2)=0$$

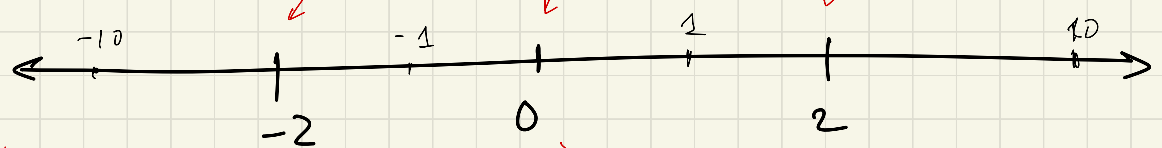
$x=0$

$x=2$

$x=-2$

by 1st deriv. test

loc. min. — loc. max — loc. min



$f'<0$	$f'>0$	$f'<0$	$f'>0$
(−)	(+)	(−)	(+)
f(x) decr.	f(x) incr.	f(x) decr.	f(x) incr.

· because at $x=10$ $\quad f'(10) = 4\cdot 10(10-2)(10+2) > 0$

· at each $x=2$, $x=-2$, $x=0$ the sign changes, because the degrees are odd

$$4x^{1}(x-2)^{1}(x+2)^{1} = f'(x)$$

(alternatively substitute 10, 1, -1, -10 into $f'(x)$)

④ x & y intercepts:

y-intercept: $0^4 - 8\cdot 0^2 + 8 = 8$

x-intercept: $(x^2)^2 - 8x^2 + 8 = 0$ (solve this for x^2 as a quadr. equation)

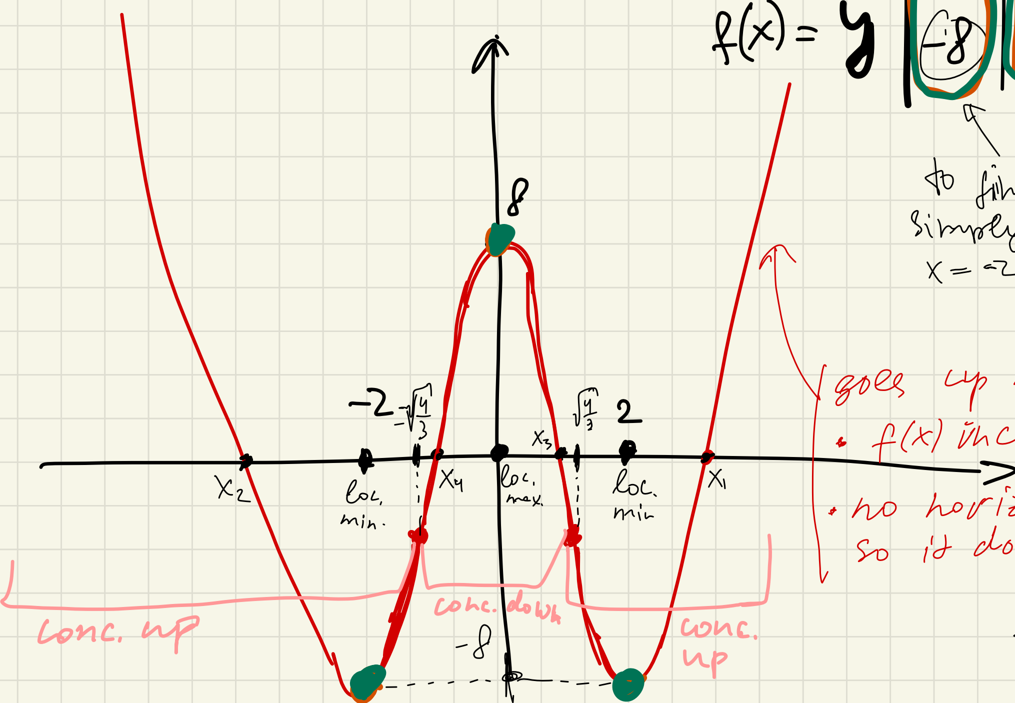
$$x^2 = 4 \pm \sqrt{16-8} = 4 \pm \sqrt{8}$$

$$x_1 = \sqrt{4+\sqrt{8}}$$
$$x_2 = -\sqrt{4+\sqrt{8}}$$
$$x_3 = \sqrt{4-\sqrt{8}}$$
$$x_4 = -\sqrt{4-\sqrt{8}}$$

loc. min./max.:

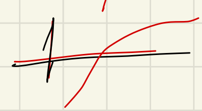
x	-2	0	2
$f(x) = y$	-8	8	-8

to find these simply plug in $x = -2, 0, 2$



goes up since $f(x)$ increasing $\nearrow \infty$

- no horiz. asympt., so it doesn't flatten



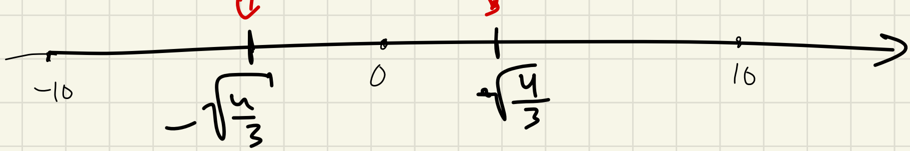
the key is to first depict loc. max./min points (both x & y values)

⑤ inflection points & concavity

$f''(x) = ?$

$$f(x) = x^4 - 8x^2 + 8$$
$$f'(x) = 4x^3 - 16x$$
$$f''(x) = 12x^2 - 16 = 0$$
$$12\left(x^2 - \frac{4}{3}\right) = 0$$
$$x = \sqrt{\frac{4}{3}}$$
$$x = -\sqrt{\frac{4}{3}}$$

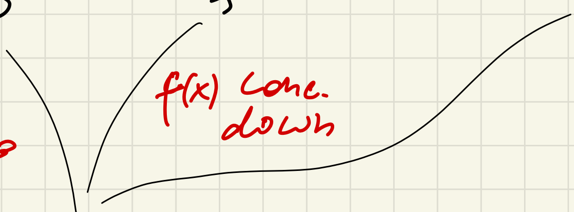
inflection points, since f'' changes sign there



$f'' > 0$ — $f(x)$ conc. up

$f'' < 0$ — $f(x)$ conc. down

$f'' > 0$ — $f(x)$ conc. up



plug in $-10, 0, 10$ into $f''(x)$

Warning $f''(x)=0$ are not inflection points!

$\boxed{\text{Inflection points are those where } f'' \text{ changes sign}}$

Ex. 3 Sketch $f(x)=\dfrac{x^2}{x^2-1}$

1 Domain: $x^2-1 \neq 0$

$x \neq 1$ so $\boxed{(-\infty,-1)\cup(-1,1)\cup(1,\infty)}$

$x \neq -1$

2 horiz. asymptotes

$$\lim_{x\to\infty} \frac{x^2}{x^2-1} \begin{matrix}\div x^2\\ \div x^2\end{matrix} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{x^2}} = \frac{1}{1-0} = 1$$

$\left(\frac{\infty}{\infty}\right)$ "divide by the highest degree in denominator"

$$\lim_{x\to-\infty} \frac{x^2}{x^2-1} \begin{matrix}\div x^2\\ \div x^2\end{matrix} = \lim_{x\to-\infty} \frac{1}{1-\frac{1}{x^2}} = \frac{1}{1-0} = 1$$

$\boxed{y=1 \text{ is a hor. assymptote}}$

vertical asymptotes

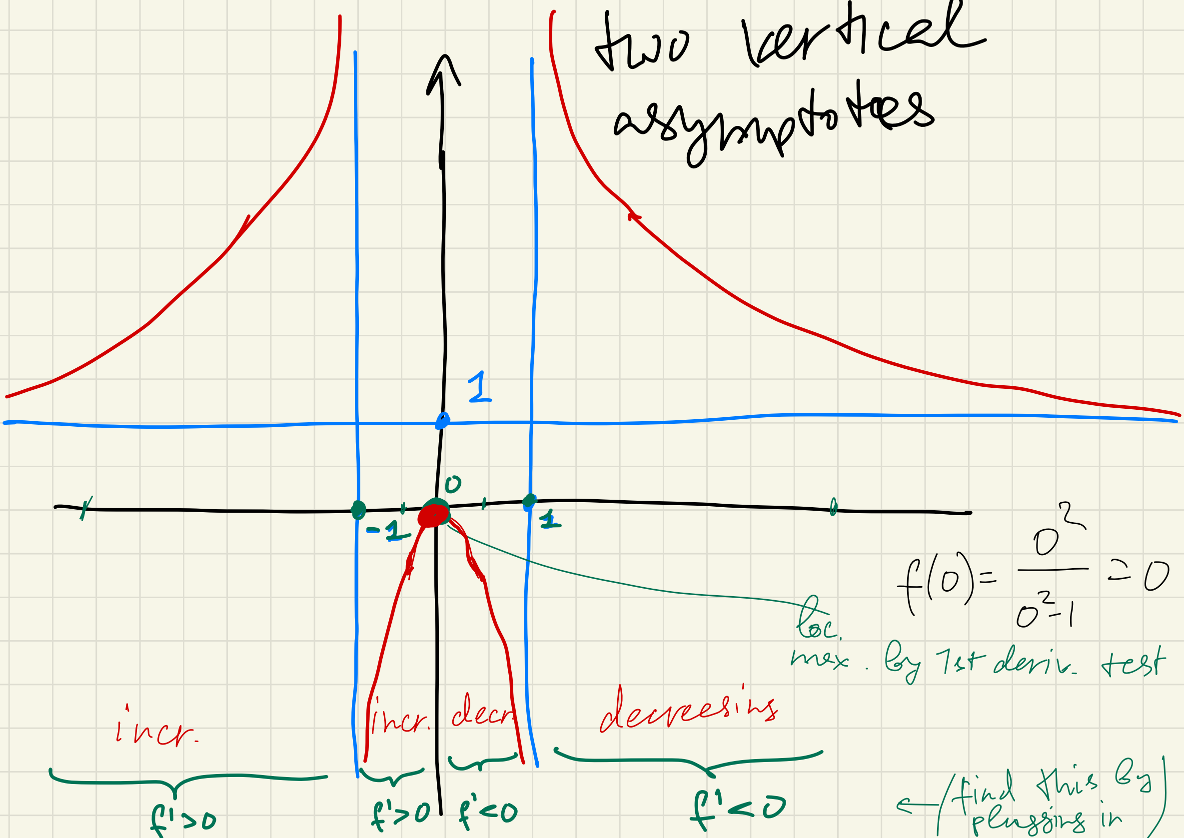
$$\lim_{x\to a^{+\text{or}-}} \frac{x^2}{x^2-1} = \pm\infty$$ happens when

denominator $= 0$ $\longrightarrow$ $x=1$, $x=-1$

check: $\lim_{x\to 1^{+\text{or}-}} \frac{x^2}{x^2-1} = \frac{1}{1-1} = \frac{1}{0} = \pm\infty$ ✓

$\lim_{x\to -1^{+\text{or}-}} \frac{x^2}{x^2-1} = \frac{1}{1-1} = \frac{1}{0} = \pm\infty$ ✓

two vertical asymptotes



$f(0) = \frac{0^2}{0^2-1} = 0$

← (find this by plugging in)

$(10, \frac{1}{2}, -\frac{1}{2}, -10)$

(3) to find loc. max./min. & incr./decr. intervals we study $f'(x) = \frac{d}{dx}\left[\frac{x^2}{x^2-1}\right] \underset{\text{quotient rule}}{=} \frac{2x(x^2-1) - x^2 \cdot 2x}{(x^2-1)^2} =$

$$= \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

crit. points $\quad f'(x) = 0 \longrightarrow -2x = 0 \rightarrow x = 0$

$f'(x)$ DNE $\longrightarrow (x^2-1)^2 = 0 \rightarrow x = 1$, $x = -1$

Ex. 4 Sketch $f(x) = x \cdot e^x$

(1) Domain: $(-\infty, +\infty)$

(2) HORIZ. asympt.

$$\lim_{x \to \infty} x e^x = \underbrace{\infty \cdot e^{\infty}}_{\text{both are very large numbers}} = \infty$$

$$\lim_{x \to -\infty} x e^x \longrightarrow (-\infty) \cdot (e^{-\infty}) = (-\infty) \cdot 0$$

$\|$

$$\lim_{x\to-\infty} \frac{x}{e^{-x}} \longrightarrow \frac{-\infty}{e^{-(-\infty)}} = \frac{-\infty}{\infty}$$

chain rule

so we can use LH

$$\Big\| LH$$

$$\lim_{x\to-\infty} \frac{1}{-e^{-x}} \longrightarrow \frac{1}{-e^{-(-\infty)}} = \frac{1}{-e^{\infty}} = \frac{1}{\infty} = \boxed{0}$$

upshot $\lim_{x\to-\infty} xe^{x} = 0$, so

$\boxed{y=0}$ is a hor. asymptote

(xe^{x} approaches it on the left $x\to-\infty$)

vertical asymptotes

$\lim_{x\to a^{+\text{or}-}} xe^{x} = \pm\infty$ never happens! Since

$\quad = a\cdot e^{a}$ ← never $\pm\infty$

so no vertical asymptotes

(3) $f'(x) = \frac{d}{dx}[xe^{x}] \overset{\text{prod. rule}}{=} e^{x} + xe^{x}$

crit. points $\underbrace{e^{x}+xe^{x}=0}$ $\qquad$ $\underbrace{e^{x}+xe^{x} \text{ DNE}}_{\text{no such } x}$

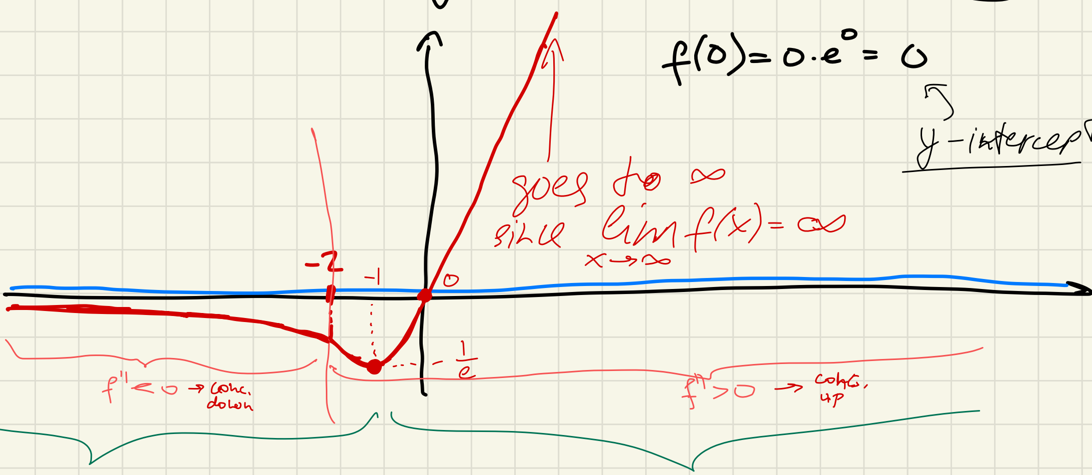
$e^{x}(1+x)=0$

$\boxed{x=-1}$

$$y = f(-1) = -1 \cdot e^{-1} = -\frac{1}{e}$$

$$f(0) = 0 \cdot e^0 = 0$$

y-intercept



$f' < 0$

$\left(f'(-10) = e^{-10} - 10 \cdot e^{-10} < 0 \right)$

$= -9 \cdot e^{-10}$

f(x) decr.

$f' > 0$

$\left(f'(10) = e^{10} + 10 \cdot e^{10} > 0 \right)$

f(x) incr.

$$f'' = \frac{d}{dx}\left[e^x(1+x)\right] = e^x + (1+x)e^x = (2+x)e^x$$

$$f'' = 0 \longrightarrow x = -2$$

4.7 Optimization

Word problems that want you to maximize or minimize some quantity give some constraint.

Strategy

1. Quantity to be maximized/minimized $\longrightarrow$ function of 2 variables
2. Constraint $\to$ make the above 1 variable function
3. Find local max./min. of 1 var. function

and pick from those.

Ex. Suppose two positive numbers (x, y) sum to 10, find values of numbers that minimize the sum squares.

1. Want to minimize $S = x^2 + y^2$

2. Constraint $x+y=10 \to y=10-x$

$$\to \quad S(x) = x^2 + (10-x)^2$$

chain rule!

3. $S'(x) = 2x + 2(10-x)\cdot(-1) = 4x-20$

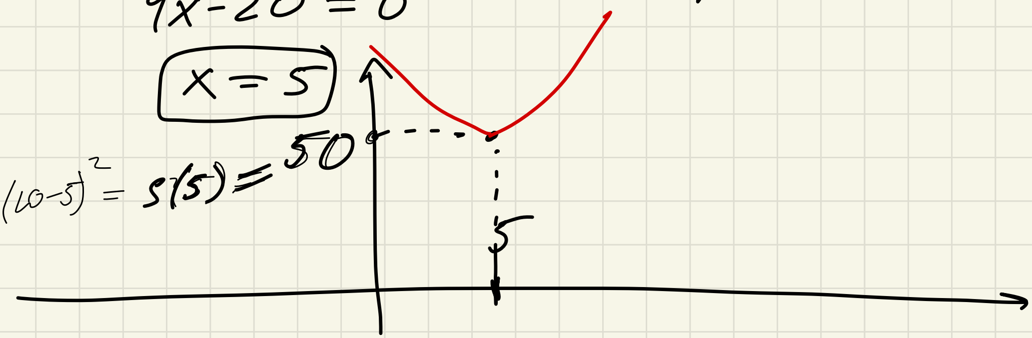
$\underbrace{S'(x)=0}$

$4x-20=0$

$\boxed{x=5}$

$\underbrace{S'(x) \text{ DNE}}_{\text{no such } x}$

$5^2+(10-5)^2 = S(5) = 50$



$\underbrace{\qquad\qquad}_{S'<0}$

$(4\cdot 0-20<0)$

S decr.

$\underbrace{\qquad\qquad}_{S'>0}$

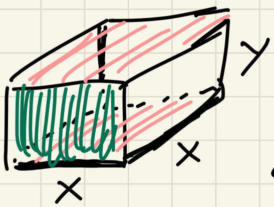
$(4\cdot 10-20>0)$

S incr.

the minimum of $S = x^2+y^2$ given $x+y=10$ is $\boxed{S=50}$

Continuing 4.7 (Optimization)

Ex. 1 Given rectangular box with square base whose volume is $1000\,cm^3$, find dimensions of the box that minimize the amount of material needed to make it.



Quantity we want to minimize:

$2\cdot \text{Area(base)} + 4\cdot \text{area(side)}$ (surface area)

to minimize: $S = 2\cdot x^2 + 4\cdot x\cdot y$

constraint: $1000 = x^2\cdot y \longrightarrow y = \dfrac{1000}{x^2}$

and so

$$S = 2x^2 + 4x\cdot \frac{1000}{x^2}$$

$$S = 2x^2 + \frac{4000}{x}$$

We want to minimize

$$S'(x) = 4x - \frac{4000}{x^2}$$

Crit points

$$\underbrace{S'(x) = 0}$$

$$4x - \frac{4000}{x^2} = 0$$

$$\frac{4x^3 - 4000}{x^2} = 0$$

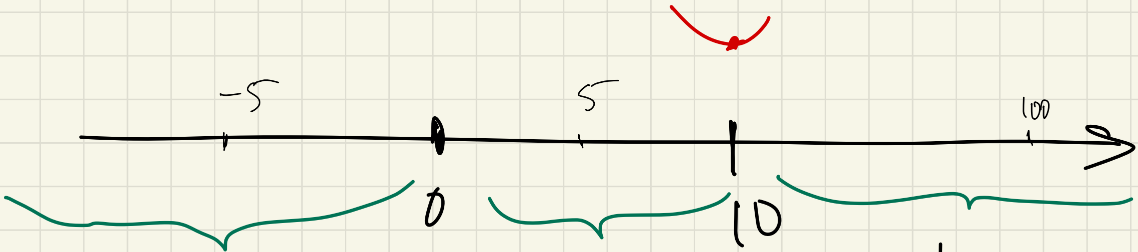
$$4x^3 - 4000 = 0$$

$$x^3 = 1000$$

$$x = 10$$

$$\underbrace{S'(x) \text{ DNE}}$$

$$x = 0$$



$S' < 0$ $S' < 0$ $S' > 0$

find out by plugging $x = -5, 5, 100$ into $S'(x)$

loc. min. for S (by 1st deriv. test)

Therefore

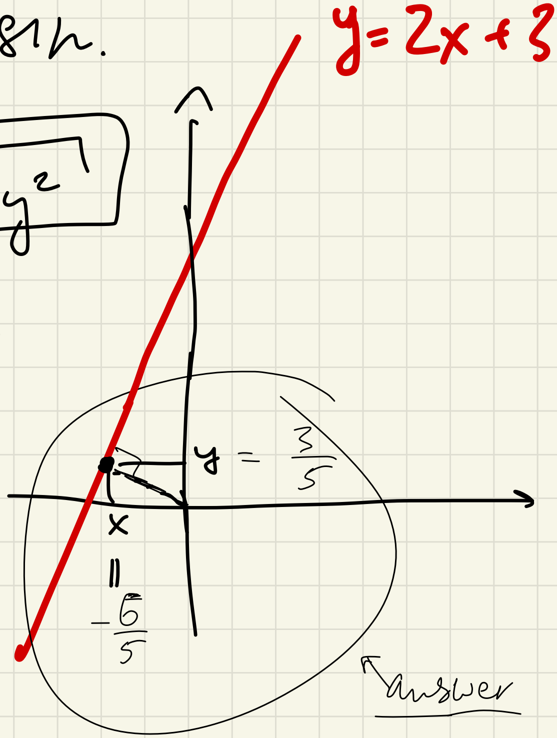
$$S = 2x^2 + 4xy$$

is minimized when

$$x = 10$$

$$y = \frac{1000}{10^2} = 10$$

Ex.2 Find the point on line $y = 2x+3$ with smallest distance to origin.



to minize: $\boxed{d = \sqrt{x^2+y^2}}$

(distance from (x,y) to $(0,0)$)

constraint: $y = 2x+3$

$$d = \sqrt{x^2 + (2x+3)^2}$$

Distance is minimized whenever its square is minimized. So, we can work with the square of distance

$$D = x^2 + (2x+3)^2$$

chain rule

$$D'(x) = 2x + 2\cdot(2x+3)\cdot 2 = 2x + 8x + 12$$

$$= 10x + 12$$

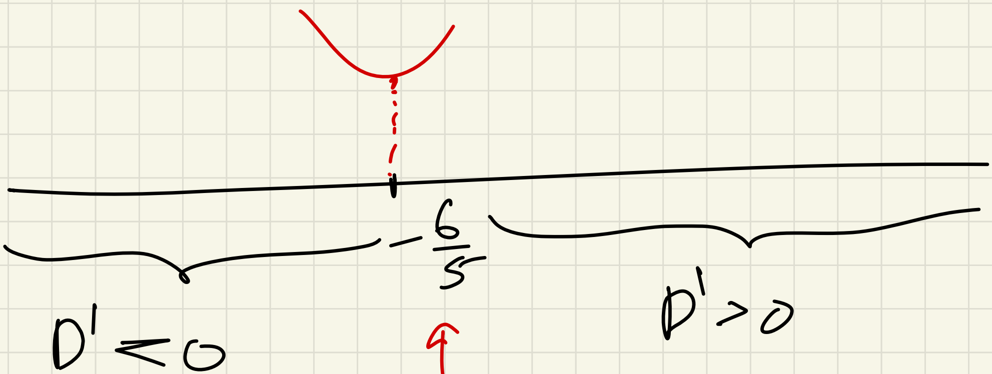
crit. points

- $10x + 12 = 0$

$$x = -\frac{12}{10} = -\frac{6}{5}$$

- $10x + 12$ DNE

no such x



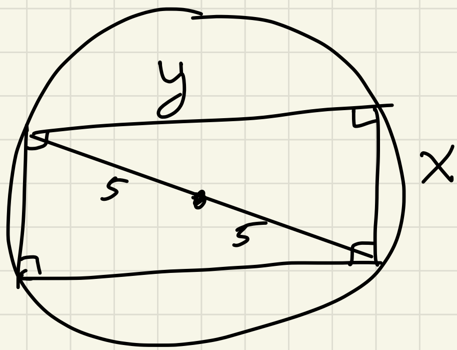
loc. min.
by 1st deriv. test

and so the distance is smallest at

$$x = -\frac{6}{5}$$

$$y = 2\left(-\frac{6}{5}\right) + 3 = +\frac{3}{5}$$

Ex. 3 Find dimensions of the largest area rectangle that is inscribed into circle of radius 5.



to maximize: $A = xy$

constraint $x^2 + y^2 = 100$

$\hookrightarrow \quad y = \sqrt{100 - x^2}$

$$A = x \cdot \sqrt{100 - x^2}$$

$$A'(x) = \frac{d}{dx}\left[x\left(100 - x^2\right)^{\frac{1}{2}}\right] = \text{(product rule)}$$

chain rule

$$= \sqrt{100 - x^2} + x \frac{-2x}{\sqrt{(100 - x^2)}} \frac{1}{2} =$$

(common denominator)

$$= \frac{100-x^2}{\sqrt{100-x^2}} - \frac{x^2}{\sqrt{(100-x^2)}} = \frac{100-2x^2}{\sqrt{100-x^2}}$$

crit. points

$A' = 0$

$100 - 2x^2 = 0$

$50 = x^2$

$\sqrt{50} = x$

$-\sqrt{50} = x$

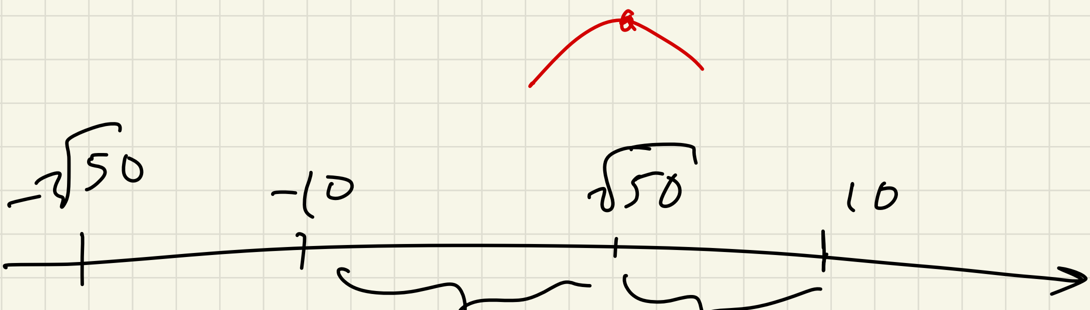
A' DNE

$100 - x^2 = 0$

$x = 10$

$x = -10$

cannot be negative



$A' > 0$ $\quad$ $A' < 0$

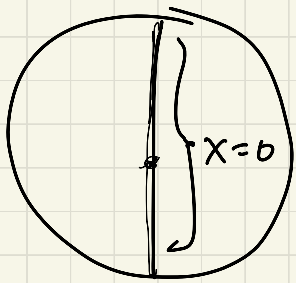
side of a rectangle cannot be negative

loc. max., and so the answer

x cannot be 10, because the diameter is 10 ($x=10$ implies $y=0$)

$$x = \sqrt{50}$$

$$y = \sqrt{100 - \sqrt{50}^2} = \sqrt{50}$$



answer.

4.9 Antiderivatives

Antiderivate for $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$

Ex. 1 Find antiderivative $F(x)$ for $f(x) = \cos x$.

$$\frac{d}{dx}\left[F(x) \right] = \cos x$$

$$F(x) = \sin x$$

Ex. 2 Find antiderivative $F(x)$ for $f(x) = \frac{1}{x}$.

$\ln x$ (since $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$)

Important note: antiderivatives are not unique, but they are unique up to a constant, meaning all antiderivatives are of the form $F(x) + C$

(Because derivative of constant is 0)

Ex. 3 $f(x) = 2x$. $F(x) = ?$

$$F(x) = x^2, \quad F(x) = x^2 + 1, \quad F(x) = x^2 - \frac{1}{e},$$

are all antiderivatives. So we write all of them as $\boxed{F(x) = x^2 + C}$

family of all possible antiderivatives

Fact if $\boxed{f(x) = x^n}$ where $n \neq -1$, then

antiderivatives $\rightarrow$ $\boxed{F(x) = \frac{x^{n+1}}{n+1} + C}$

$\left(\text{since } \frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + C\right] = = (n+1)\frac{x^n}{n+1} + 0 = = x^n \right)$

Even though antiderivatives are not unique, they become unique if some conditions are added.

Simple example $f'(x) = 2x$, $f(0) = 10$, then $f(x) = ?$ $f(x) = x^2 + C$, $f(0) = 0^2 + C = 10 \rightarrow \boxed{C = 10}$ $\rightarrow \boxed{f(x) = x^2 + 10}$

Ex. 4 $f'(x) = \frac{1}{2\sqrt{x}}$, $f(4) = 3$. $f(x) = ?$ $\uparrow$ antiderivative of $f'(x)$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} \longrightarrow \left(\begin{array}{c}\text{By the formula}\\ \text{above}\end{array}\right)$$

$$\rightarrow f(x) = \frac{1}{2}\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \right) =$$

$$= \frac{1}{\cancel{2}} \frac{x^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + \underbrace{\frac{1}{2}C}_{\text{spans all numbers}} = \sqrt{x} + \underbrace{C}_{\text{spans all numbers as well}}$$

are not equal as functions, but equal as families of functions (thats why I was able to remove $\frac{1}{2}$)

$$f(4) = 3 \rightarrow \sqrt{4} + C = 3$$

$$2 + C = 3$$

$C = 1$

$$f(x) = \sqrt{x} + 1$$

Ex. 5 Find $f(x)$ if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$, $f(1) = 1$

$$f''(x) = 12x^2 + 6x - 4$$

$\downarrow$ antiderivative

$$f'(x) = 12\frac{x^3}{3} + 6\frac{x^2}{2} - 4x + C =$$

$$= 4x^3 + 3x^2 - 4x + C$$

$\downarrow$ antiderivative

$$f(x) = 4\frac{x^4}{4} + 3\frac{x^3}{3} - 4\frac{x^2}{2} + C \cdot x + D$$

different constant from C

$$= x^4 + x^3 - 2x^2 + Cx + D$$

$f(0) = 4 \rightarrow 0^4 + 0^3 - 2 0^2 + C \cdot 0 + D = 4$

$$D = 4$$

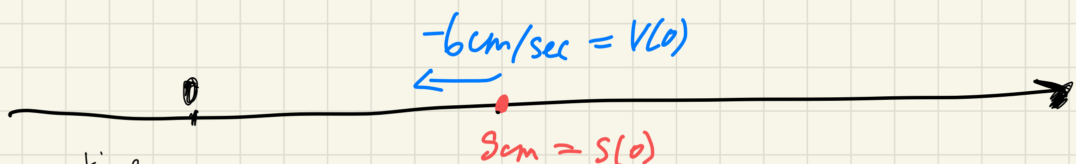
$$f(1) = 1 \rightarrow \quad 1^4 + 1^3 - 2\,1^2 + C\cdot 1 + 4 = 1$$

$$2 - 2 + C + 4 = 1$$

$$C = -3$$

$$\boxed{f(x) = x^4 + x^3 - 2x^2 - 3x + 4}$$

Ex. 6 A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/sec, and its initial position on the line is $s(0) = 9$ cm. Find its position function $s(t)$.



time

$a(t) = 6t + 4$

$s(t) = ?$

$$\underbrace{s'(t) = v(t)}$$

derivative of position function is the velocity function

$$\underbrace{s''(t) = v'(t) = a(t)}$$

derivative of velocity function is the acceleration function

to memorize!

$$a(t) = 6t + 4$$

antiderivative

$$v(t) = 6\frac{t^2}{2} + 4t + C$$

$$= 3t^2 + 4t + C$$

$$v(0) = 3 \cdot 0^2 + 4 \cdot 0 + C = -6$$

$$C = -6$$

$$v(t) = 3t^2 + 4t - 6$$

antiderivative

$$S(t) = 3\frac{t^3}{3} + 4\frac{t^2}{2} - 6\cdot t + D$$

$$= t^3 + 2t^2 - 6t + D$$

$$S(0) = 0^3 + 2 0^2 - 6\cdot 0 + D = 9$$

$$D = 9$$

$$\rightarrow \boxed{S(t) = t^3 + 2t^2 - 6t + 9}$$

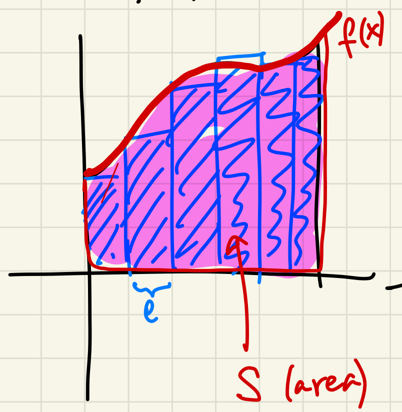
Upshot • if you are given $f'(x)$, you need <u>one</u> extra condition to find $f(x)$

• if you are given $f''(x)$, you

need two extra conditions to find $f(x)$

5.1 Areas

- Consider area S between x-axis & $f(x)$
- We will approximate it by rectangles



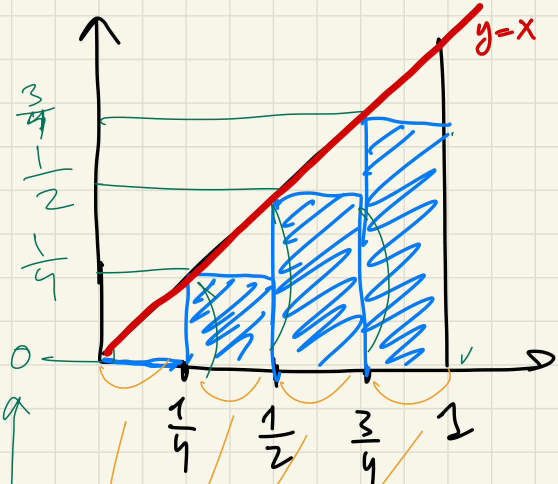
$\to$ take limit as $l \to 0$, and get the desired area S

- left endpoints
- right endpoints
- midpoints

3 methods

Ex. 1 Approximate area under $y = x$ on $[0, 1]$ using 4 rectangles, using all the three ways here

case 1 left-endpoints



rectangles have

- base $= \frac{1}{4}$ (since we have 4 rect.)
- height is y-value of left endpoint of the subinterval

subintervals: $[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1]$

heights: $f(0) = 0$ $\quad f(\frac{1}{4}) = \frac{1}{4}$ $\quad f(\frac{1}{2}) = \frac{1}{2}$ $\quad f(\frac{3}{4}) = \frac{3}{4}$

bases

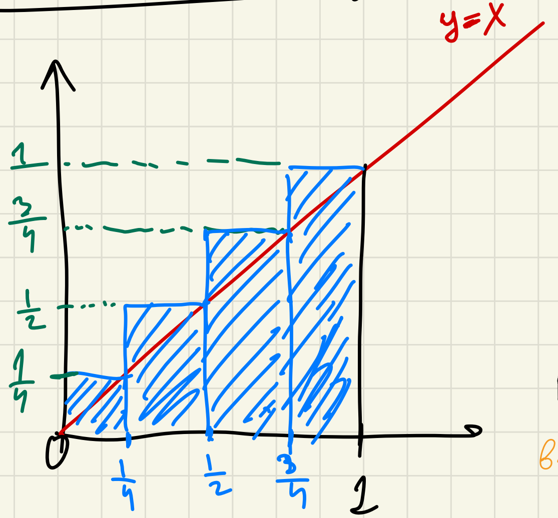
$$L_4 = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} =$$

heights

$$= \frac{4}{16} + \frac{1}{8} = \boxed{\frac{3}{8} = L_4}.$$

approximation of area under $y=x$ on $[0,1]$ using 4 rectangles and left endpoints

Case 2: right endpoints



approximation is

$\boxed{R_4 = \text{sum of areas of rectangles with heights} = f(\text{right endpoint})}$

$$R_4 = \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{4}\cdot 1 =$$

Bases

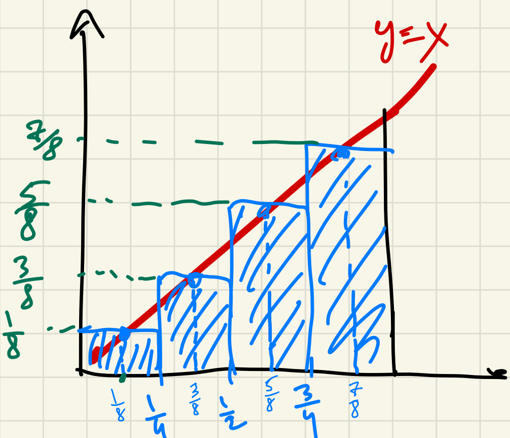
heights

$$= \frac{4}{16} + \frac{3}{8} = \frac{2}{8} + \frac{3}{8} = \boxed{\frac{5}{8}}$$

# of intervals

R_4, approximation of area using right endpoints

Case 3: midpoints



approximation is

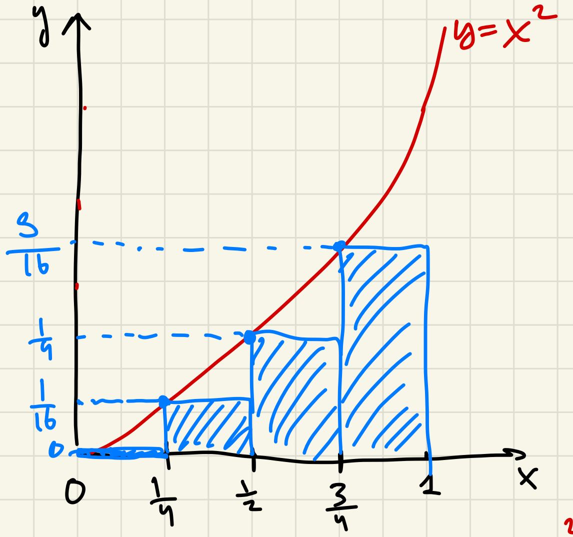
M_4 = sum of areas of rectangles with height = f(midpoint)

$$M_4 = \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{3}{8} + \frac{1}{4}\cdot\frac{5}{8} + \frac{1}{4}\cdot\frac{7}{8} =$$

$$= \frac{16}{4\cdot 8} = \boxed{\frac{1}{2}}$$

Recall from last time:

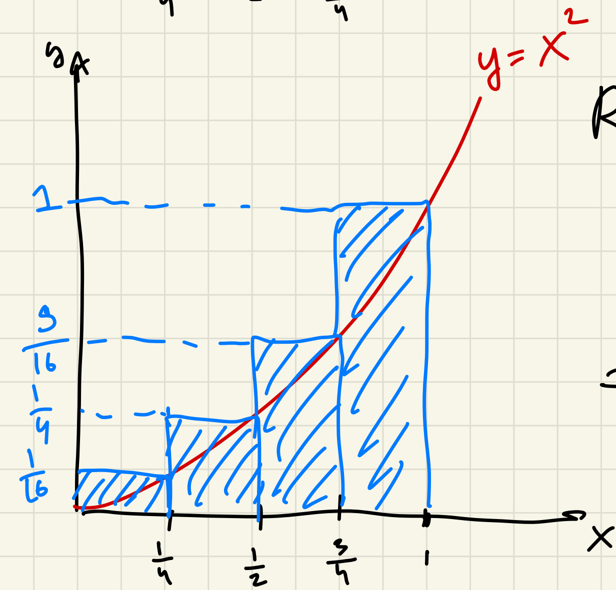
Ex. 1 Approximate the area under the graph of $y=x^2$ on $[0,1]$ using 4 rectangles, and left & right endpoints.



left-endpoints approximation

$$L_4 = \frac{1}{4}\cdot \overset{f(0)}{0} + \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{16} = \frac{14}{4\cdot 16} = \frac{7}{32}$$

$$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^2$$



$$R_4 = \frac{1}{4}\cdot\frac{1}{16} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{16} + \frac{1}{4}\cdot 1 = \frac{10}{4\cdot 16} + \frac{5}{16} = \frac{15}{32}$$

(still 5.1)

<u>remark</u> areas are always positive!

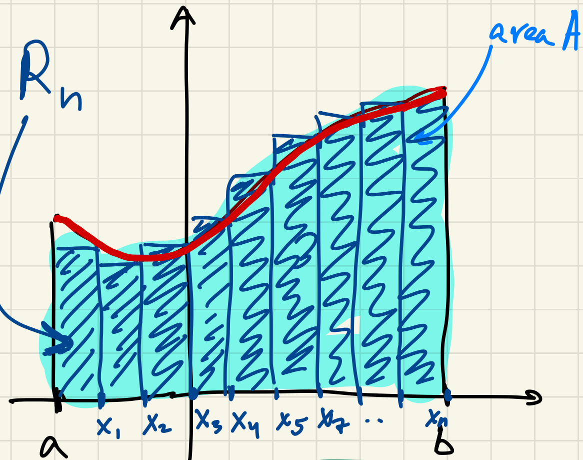
<u>def.</u> The <u>area</u> of the region that lies under the graph of continuous function $f(x)$ is the limit of the left-endpoint (or right-endpoint) sums:

$$\boxed{A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n}$$

$$\lim_{n\to\infty}\left(f(x_1)\cdot\Delta x + \dots + f(x_n)\cdot\Delta x\right)$$

$\Delta x = \frac{(b-a)}{n}$ base of subintervals

$n = \#$ of subinterval



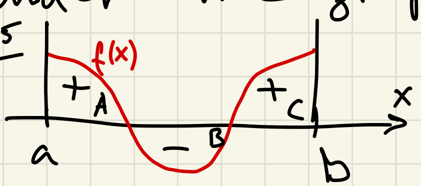
key idea: the larger is the # of subintervals, the better is the approximation

5.2

We learned to approximate areas, but how do we compute areas precisely? For this we need definite integrals:

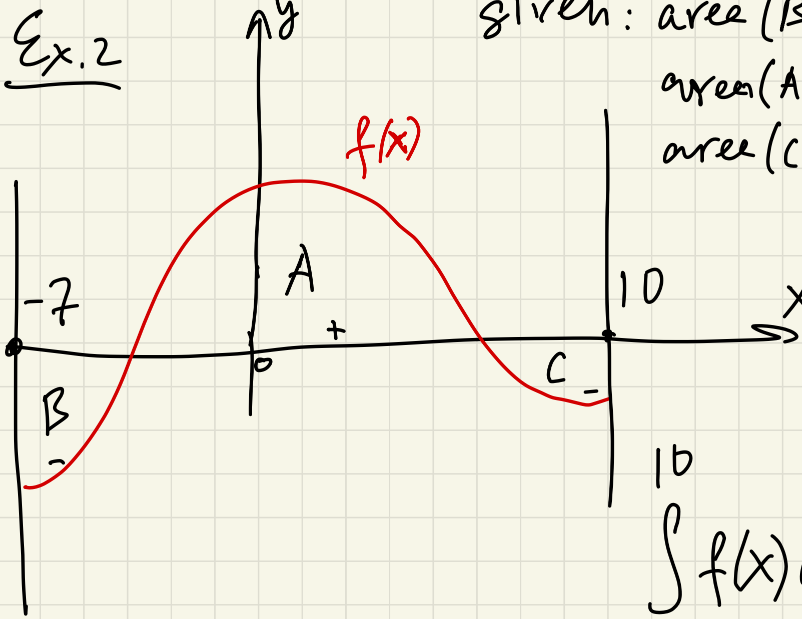
<u>def.</u> The <u>definite integral</u> of $f(x)$ on $[a,b]$ is the <u>signed area</u> under the graph. Notation: $\int_a^b f(x)\,dx$

<u>Ex. 1.5</u>



$$\int_a^b f(x)\,dx = \text{area}(A) - \text{area}(B) + \text{area}(C)$$

Ex. 2



given: area(B) = 6
area(A) = 20
area(C) = 5

$$\int_{-7}^{10} f(x)\,dx =$$

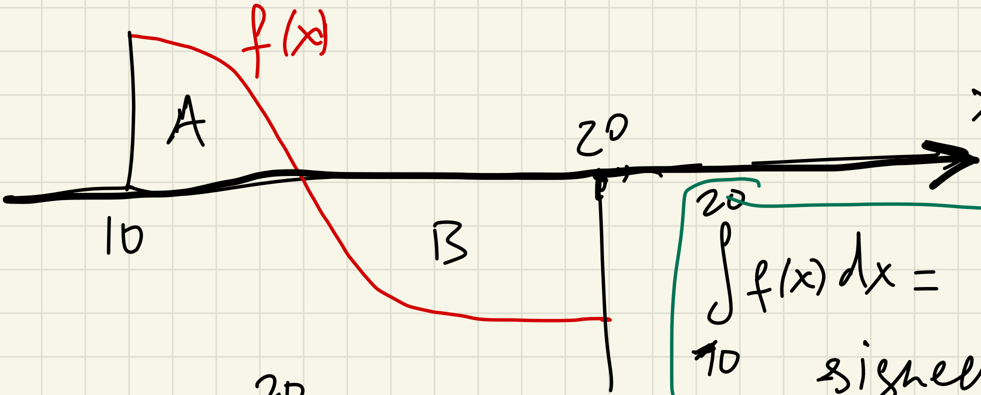
$$= -\text{area}(B) + \text{area}(A) - \text{area}(C) =$$

$$= -6 + 20 - 5 = 9$$

• If $\int_a^b f(x)\,dx$ exists, then we say that $f(x)$ is <u>integrable</u> on $[a,b]$

<u>Important fact</u> If $f(x)$ is continuous, then $f(x)$ is integrable.

Ex. 3



$$\int_{10}^{20} f(x)\,dx = \text{sum of } \underline{\text{signed}} \text{ areas}$$

given: $$\int_{10}^{20} f(x)\,dx = -3$$

$$\text{area}(A) = 5$$

Q. $\text{Area}(B) = ?$

A. $\boxed{8}$ because

$$\int_{10}^{20} f(x)\,dx = \text{area}(A) - \text{area}(B)$$

$$-3 = 5 - \text{area}(B)$$

$$\boxed{8 = \text{area}(B)}$$

Important: areas are always positive! That's why $\boxed{\text{area}(B) = 8}$

Now, signed area can be negative, and in this case we have signed-area$(B) = -8$

Fundamental Theorem of Calculus (FTC)

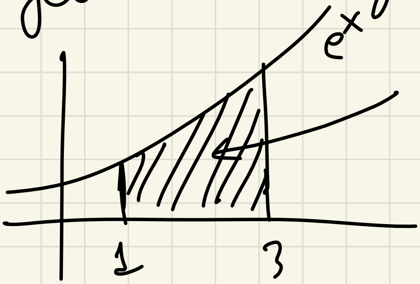
If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

where F is any antiderivative of $f(x)$

Ex. 4 compute $\int_1^3 e^x dx$?

geometrically



~~area~~ = $\int_1^3 e^x dx = ?$

$f(x) = e^x$ has antiderivative $F(x) = e^x + C$

we choose $C = 0$ (can choose any)

$$\int_1^3 e^x dx \overset{FTC}{=} F(3) - F(1) =$$

$$= \boxed{e^3 - e^1}$$ ← answer

notation $F(b) - F(a) = F(x) \Big|_a^b$

$$\int_1^3 e^x\,dx = e^x\Big|_1^3 = e^3 - e$$

5.3 Fundamental thm of calculus (FTC)

Fundamental Theorem of Calculus, Part II

If $f(x)$ is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

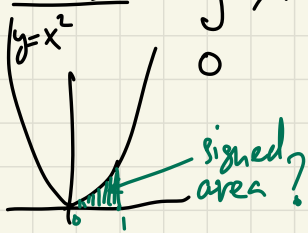
(sometimes written as $F(x)\Big|_a^b$)

where $F(x)$ is an antiderivative of $f(x)$

It allow us to compute definite integrals in those cases when we can find antiderivative.

Ex 1

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \boxed{\frac{1}{3}}$$



$F(x) = \frac{x^3}{3} + C$, we choose $C=0$

(can choose any)

what if we choose $C = 17.5$?

Then $\displaystyle\int_0^1 x^2 dx = \left(\frac{x^3}{3} + 17.5\right)\Bigg|_0^1 = \overbrace{\left(\frac{1^3}{3} + 17.5\right)}^{F(1)} -$

$$- \underbrace{\left(\frac{0^3}{3} + 17.5\right)}_{F(0)} =$$

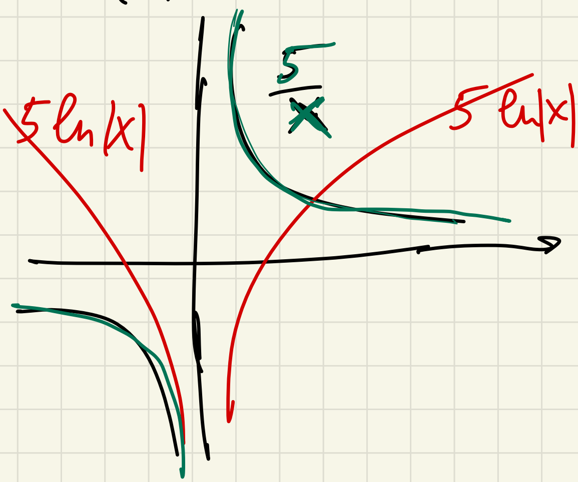
$$= \frac{1^3}{3} - \frac{0^3}{3} + \cancel{17.5} - \cancel{17.5} = \frac{1}{3}$$

Ex. 2 $\displaystyle\int_1^e \frac{5}{x} dx = 5 \cdot \ln|x| \Bigg|_1^e = 5\ln e - 5\ln 1 =$

$f(x) = \frac{5}{x}$

$$= 5 \cdot 1 - 5 \cdot 0 = \boxed{5}$$

$F(x) = 5 \cdot \ln|x|$



remember: antiderivative of $\frac{1}{x}$ is $\ln|x| + C$

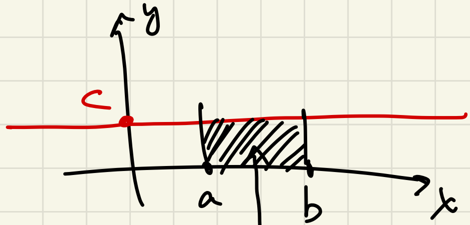
HW: Read 5.3

Properties of integrals (from 5.2)

(1) $$\int_a^b c\,dx = c\cdot(b-a)$$

constant

signed-area of



(2) $$\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

(3) $$\int_a^b 5\cdot f(x)\,dx = 5\int_a^b f(x)\,dx$$

any constant, not a function!

Ex. 3 $$\int_0^4 (x^3-6x)\,dx \underset{by (2)}{=} \int_0^4 x^3\,dx - \int 6x\,dx \overset{FTC}{=}$$

$F_1 = \dfrac{x^4}{4}$ $\quad F_2 = 6\dfrac{x^2}{2} = 3\cdot x^2$

$$= \left(\frac{x^4}{4}\Big|_0^4\right) - \left(3x^2\Big|_0^4\right) =$$

$$= \left(\frac{4^4}{4} - \cancel{\frac{0^4}{4}}\right) - \left(3\cdot 4^2 - \cancel{3\cdot 0^2}\right) =$$

$$= 4^3 - 3\cdot 4^2 = 64 - 48 = \boxed{16}$$

<u>Ex. 4</u> $\displaystyle\int_0^{\pi/4} \sec x(\sec x - \tan x)\,dx =$

$$= \int_0^{\pi/4} (\sec^2 x - \sec x \tan x)\,dx \overset{\text{by (2)}}{=\!=}$$

$$= \int_0^{\pi/4} \sec^2 x\,dx - \int_0^{\pi/4} \sec x \tan x\,dx \underset{(FTC)}{=\!=}$$

$\downarrow$ $F(x) = \tan x$ $\left(\text{since } \frac{d}{dx}(\tan x) = \sec^2 x\right)$

$\downarrow$ $F(x) = \sec x$ $\left(\text{since } \frac{d}{dx}[\sec x] = \sec x \cdot \tan x\right)$

$$= \tan x \Big|_0^{\pi/4} - \sec x \Big|_0^{\pi/4} =$$

$$= \left(\tan\frac{\pi}{4} - \tan 0\right) - \left(\sec\frac{\pi}{4} - \sec 0\right) =$$

$$= (1 - 0) - (\sqrt{2} - 1) =$$

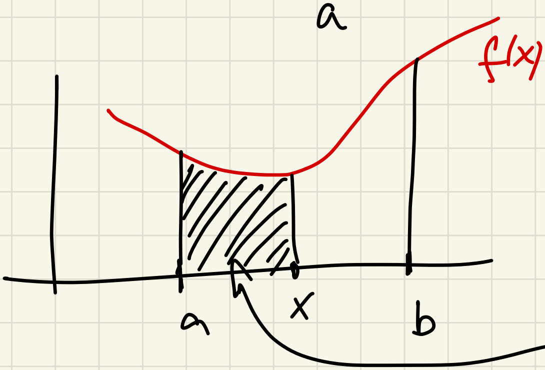
$$\left(\sec x = \frac{1}{\cos x}, \tan x = \frac{\sin x}{\cos x}\right)$$

$$= \boxed{2 - \sqrt{2}}$$

Fundamental Theorem of Calculus, Part I

Suppose $f(x)$ is continuous on $[a,b]$.

Then $g(x) = \int_a^x f(t)\,dt$ is antiderivative of $f(x)$.



$g(x)$ is the signed-area under $f(x)$ from a to x

Rmk in other words, $g'(x) = f(x)$.

Ex. 5 Find $g'(x)$ if $g(x) = \int_0^x \sec(t)\,dt$.

By FTC (I) we know that $g(x)$ is antideriv. of $\sec(x)$, so $\boxed{g'(x) = \sec x}$

Ex. 6 Find $g'(e^4)$ if $g(x) = \int_1^x \sqrt{\ln t}\, dt$.

By FTC (I) we know that $g(x)$ is antideriv. of $\sqrt{\ln(x)}$, so

$$g'(x) = \sqrt{\ln(x)}$$

so

$$g'(e^4) = \sqrt{\ln e^4} = \sqrt{4} = \boxed{2}$$

What do we do if we have

$$g(x) = \int_a^{h(x)} f(t)\,dt \; ? \quad \left(\text{instead of } \int_a^x f(t)\,dt\right)$$

Turns out that by chain rule we have

$$g'(x) = \frac{d}{dx}\left(\int_a^{h(x)} f(t)\,dt\right) = f(h(x)) \cdot h'(x)$$

remember

Ex. 7 Find $g'(x)$ if $g(x) = \int_0^{x^3} t^4\,dt$.

$h(x)$, $f(t)$

$$g'(x) = \frac{d}{dx}\left(\int_0^{x^3} t^4\,dt\right) = f(h(x))\,h'(x) =$$

$h(x)$

$$= f(x^3)\cdot 3x^2 = (x^3)^4\cdot 3x^2 = x^{12}\cdot 3x^2$$

$$= \boxed{3\cdot x^{14}}$$

Ex. 8 Find $g'\left(\frac{\pi}{4}\right)$ for $g(x) = \int_0^{\sin x} t^2\,dt$.

$$g'(x) = \frac{d}{dx}\left(\int_0^{\sin x} t^2\,dt\right) = f(h(x))\cdot h'(x) =$$

$h(x)$, $f(t)$

$$= (\sin x)^2\cdot \cos x$$

$$g'\left(\frac{\pi}{4}\right) = \left(\sin\frac{\pi}{4}\right)^2\cdot\cos\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2\cdot\frac{1}{\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$$

Ex. 3 If $g(x)=\int_0^x (1-t^2)e^{t^2}\,dt$, find where $g(x)$ is increasing.

$g(x)$ increasing $\Longleftrightarrow$ $g'(x)>0$, so we need to find $g'(x)$.

$$g'(x) = \frac{d}{dx}\left(\int_0^x (1-t^2)e^{t^2}\,dt\right) \overset{FTC(I)}{=\!=} (1-x^2)e^{x^2}$$

critical numbers:

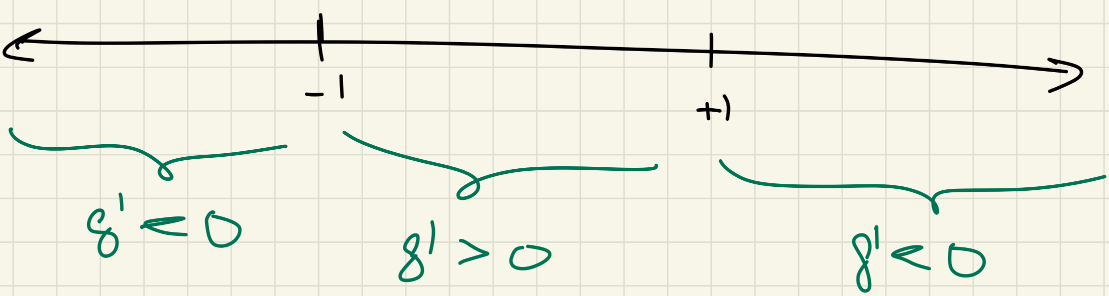
$\underbrace{g'(x)=0}$

$$(1-x^2)e^{x^2}=0$$

$$1-x^2=0$$

$$x=1$$
$$x=-1$$

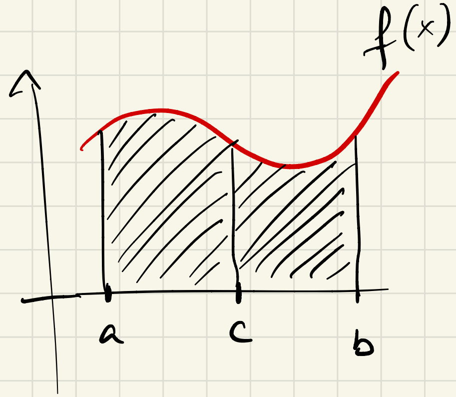
$\underbrace{g'(x)\ \text{DNE}}_{\text{no such } x}$



g(x) decr. g(x) incr. g(x) decr.

$\boxed{g(x) \text{ incr. on } (-1,1)}$

More useful properties of definite integral:



Property(1)

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

It is useful when dealing with piecewise functions:

Ex.1 $\int_{-1}^{1} |x| dx$ we don't know antiderivative... So we do this

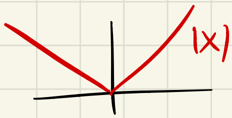
$$\| $$

$$\int_{-1}^{0} |x| dx + \int_{0}^{1} |x| dx$$

chose 0 since this is the cut point for $|x|$

$$=$$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



$$\int_{-1}^{0} (-x)\,dx + \int_{0}^{1} x\,dx$$

$$=$$

$$\frac{-x^2}{2}\Bigg|_{-1}^{0} + \frac{x^2}{2}\Bigg|_{0}^{1}$$

$$=$$

$$\left(-\frac{0^2}{2} - \frac{-(-1)^2}{2}\right) + \left(\frac{1^2}{2} - \frac{0^2}{2}\right)$$

$$=$$

$$\left(0 + \frac{1}{2}\right) + \left(\frac{1}{2} - 0\right)$$

$$=$$

$$1$$

$\underline{Ex.2}$

$$\int_0^2 |2x-1|\,dx =$$

$$\left(|2x-1| = \begin{cases} 2x-1, & 2x-1 \geq 0 \xrightarrow{2x\geq 1} x \geq \frac{1}{2} \\ -(2x-1), & 2x-1 < 0 \rightarrow x < \frac{1}{2} \end{cases} \right)$$

$$= \int_0^{\frac{1}{2}} |2x-1|\,dx + \int_{\frac{1}{2}}^2 |2x-1|\,dx =$$

$$= \int_0^{\frac{1}{2}} (1-2x)\,dx + \int_{\frac{1}{2}}^2 (2x-1)\,dx =$$

$$= \left(-x^2+x\right)\Big|_0^{1/2} + \left(x^2-x\right)\Big|_{\frac{1}{2}}^2 =$$

$$= \left(-\frac{1}{4}+\frac{1}{2}\right) + \left(4-2-\left(\frac{1}{4}-\frac{1}{2}\right)\right) =$$

$$= \boxed{\frac{5}{2}}$$

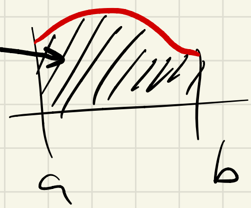
<u>Note</u> Since abs. value is always positive, integrals of abs. value are also positive.

This generalizes to

<u>Property (2)</u> Comparison properties.

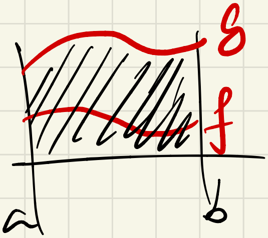
a) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

signed area under positive functions is positive



b) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then

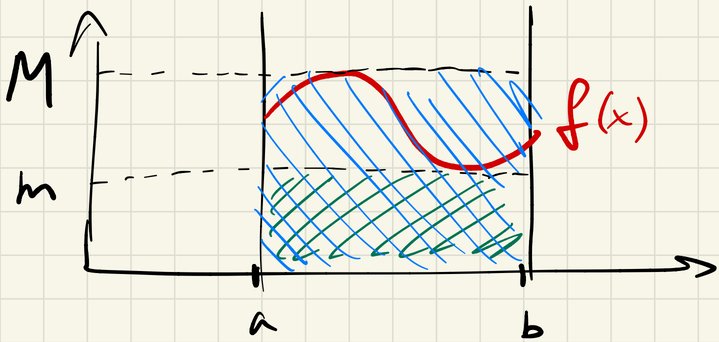
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$



c) If $m \le f(x) \le M$ for $a \le x \le b$,

constants

then
$$\underline{m(b-a)} \le \int_a^b f(x)\,dx \le \underline{M(b-a)}$$



<u>Ex.3</u> (Sec. 5.2 #55)

Verify inequality without computing the integral

$$\int_0^4 (x^2-4x+4)\,dx \ge 0$$

(Idea: if we prove that $x^2-4x+4 \ge 0$, then we are done by property 2a))

$x^2-4x+4 = (x-2)^2 \geq 0$, and so

the inside function is positive, and

so the integral is positive by 2a)

__Ex. 4__ (5.2 #57)

Prove that

$$2 \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$$

(Idea: find abs. min. & abs. max. of $\sqrt{1+x^2}$ on $[-1,1]$ and then use prop. 2c))

__crit. points:__ $f'(x) = \frac{d}{dx}\left(\sqrt{1+x^2}\right) \underset{\text{rule}}{\overset{\text{chain}}{=\!=}} \frac{1}{2\sqrt{1+x^2}} \cdot 2x =$

$$= \frac{x}{\sqrt{1+x^2}}$$

$\underbrace{f'(x) = 0}$ ← sources of crit. points → $\underbrace{f'(x) \text{ DNE}}$

$\frac{x}{\sqrt{1+x^2}} = 0$

$1+x^2 = 0$
no such x

$1+x^2 < 0$
no such x

$\boxed{x=0}$ only one crit. pt

$f(0) = \sqrt{1+0^2} = 1$

$f(1) = \sqrt{1+1^2} = \sqrt{2}$

$f(-1) = \sqrt{1+(-1)^2} = \sqrt{2}$

endpoints

closed interval method to find abs min. & abs max.

1 is abs. min. and $\sqrt{2}$ is abs. max.

$1 = m$, $\sqrt{2} = M$

$$\underset{m\cdot(b-a)}{1\cdot(1-(-1))} \overset{2c)}{\leq} \int_{-1}^{1} \sqrt{1+x^2}\,dx \overset{2c)}{\leq} \underset{M(b-a)}{\sqrt{2}(1-(-1))}$$

$$2 \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$$

and that's what we wanted

5.4 Indefinite integrals

Definite integral

$$\int_0^2 f(x)\,dx$$

2 ← upper limit

0 ← lower limit

spits out a number = signed-area

Indefinite integral

("all antiderivatives")

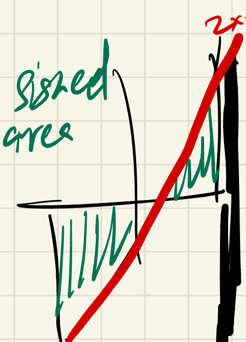
no upper/lower limits!

$$\int f(x)\,dx = F(x) + C$$

spits out a family of functions

Ex. 5

$$\int_{-2}^{3} (2x-1)\,dx =$$



(By FTC)

$$= \left(x^2 - x\right)\Big|_{-2}^{3} =$$

$$= \left(3^2 - 3\right) - \left((-2)^2 - (-2)\right) =$$

$$= 9 - 3 - 4 - 2 = 0$$

$$\int (2x-1)\,dx = \underbrace{x^2 - x + C}$$

family of all antiderivatives

$$\left\{ \begin{array}{c} x^2 - x + 0 \\ x^2 - x + \frac{1}{7} \\ x^2 - x - 1000 \\ \vdots \end{array} \right\}$$

To find indefinite integrals you

need to memorize the table of antiderivatives (p. 403 in the textbook)

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$

- $\int \frac{1}{x} dx = \ln|x| + C$

- $\int e^x dx = e^x + C$

- $\int a^x dx = \frac{a^x}{\ln a} + C$

  some number

- $\int \sin x \, dx = -\cos x + C$, $\int \cos x \, dx = \sin x + C$

$\int \sec^2 x \, dx = \tan x + C$, $\int \csc^2 dx = -\cot x + C$

$\int \sec x \cdot \tan x \, dx = \sec x + C$, $\int \csc x \cdot \cot x \, dx = -\csc x + C$

<u>HW</u>: memorize / write on a cheat-sheet all formules (p. 403)

---

a couple of examples of indefinite integrals:

<u>Ex. 1</u> $\int \left(x^2 - \sec(x) \cdot \tan(x) \right) dx =$

$$x^2 \downarrow \tfrac{1}{3}x^3 \qquad \sec(x)\tan(x) \downarrow \sec(x)$$

$= \frac{1}{3}x^3 - \sec(x) + C$

<u>Ex 2</u> $\int \frac{1 + \sqrt{x} + x}{x} dx =$

$= \int \frac{1}{x} dx + \int \frac{\sqrt{x}}{x} dx + \int \frac{x}{x} dx =$

$$= \int \frac{1}{x}\,dx + \int \frac{1}{\sqrt{x}}\,dx + \int 1\,dx =$$

$\ln|x|$

$x^{-\frac{1}{2}}$

$\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2x^{\frac{1}{2}} = 2\sqrt{x}$

x

$$= \ln|x| + 2\sqrt{x} + x + C$$

$$\left(\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1\right)$$

rmk $\frac{\sqrt{x}}{x} = \frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\cdot\sqrt{x}} = \frac{1}{\sqrt{x}}$

Net change theorem

(FTC applied to word problems)

If $f(x)$ represents rate of change (derivative) of some physical quantity, then

$$\boxed{\int_a^b \underset{\text{rate of change}}{f(x)}\,dx = \underbrace{\overset{\text{quantity}}{F(b) - F(a)}}_{\text{net change of}}}$$

the quetity on [a,b]

displacement – net change in position (distance)

- position (or distance) $\xrightarrow{\text{deriv.}}$
- velocity – rate of change in distance $\downarrow$ deriv.
- acceleration – rate of change in velocity

As a result, we have (By Net Change Theorem)

$$\int_a^b \underset{\uparrow \text{velocity}}{v(t)}\, dt = \text{displacement on } [a,b]$$

Ex. 3 A particle is moving on a line with velocity $\underset{\uparrow \text{time}}{v(t)} = 2t - 1$ m/s.

Find its displacement after 5 seconds.

Displacement on $[0,5] =$

$$= \int_0^5 v(t)\, dt =$$

(since $v(t)$ is the derivative of distance function)

$$= \int_0^5 (2t-1)\,dt \overset{\text{FTC}}{=} t^2 - t \Big|_0^5 =$$

$$= (5^2-5) - (0^2-0) = \boxed{20\text{ m}}$$

Ex 4 A particle moves on a line with acceleration $a(t) = 2t-1$ m/s². If its initial velocity $v(0) = -6$ m/s, find the displacement of particle between $t=0$ & $t=5$.

(net change thm)

$$\text{Displacement on } [0,5] =$$

$$= \int_0^5 v(t)\,dt = \int_0^5 t^2 - t - 6\,dt \quad \underline{\underline{\text{FTC}}}$$

Subproblem: find $v(t)$ based on $\underset{2t-1}{a(t)}$ & $\underset{-6}{v(0)}$

$v(t) = ?$ Well, $v'(t) = a(t) = 2t-1$

So $v(t) = t^2 - t + C$

$v(0) = -6$, so $C = -6$, so

$v(t) = t^2 - t - 6$

$$= \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Bigg|_0^5 =$$

$$= \left(\frac{5^3}{3} - \frac{5^2}{2} - 6\cdot 5\right) - \cancel{\left(\frac{0^3}{3} - \frac{0^2}{2} - 6\cdot 0\right)} =$$

$$= \frac{2\cdot 5^3 - 3\cdot 5^2 - 6\cdot 5\cdot 6}{6} =$$

$$= \frac{250 \overbrace{- 75 - 180}^{-255}}{6} = \boxed{-\frac{5}{6}}$$

5.5 Substitution rule

chain rule backwards, or in other words trick to integrate complicated functions

Substitution method

$$\int f(x)\,dx$$

key step, finding $v(x)$ $\quad || \quad$ 1

$$\boxed{dv = v'(x)\,dx}$$

is the reason why we want this

$$\int g(v(x)) \cdot k \cdot v'(x)\,dx$$

function $\quad$ constant

|| 2 $\quad$ substitution $\boxed{v'dx = dv}$

$$k \cdot \int g(v) \cdot dv$$

integrate with respect to v $\quad$ 3

sub $v(x)$ $\quad$ 4

Ex. 5

guess $v(x)$

$$\int x e^{x^2} dx$$

$v(x) = x^2$

$dv = v'dx =$

$= 2x\,dx$

$$\int \underbrace{e^{x^2}}_{g(x)} \cdot \frac{1}{2} \cdot \underbrace{2x}_{\propto v'}\,dx$$

|| $\quad g(x) = e^x$

$$\frac{1}{2}\int e^v dv$$

||

$$\frac{1}{2}(e^v + C)$$

||

$$\boxed{\frac{1}{2}e^{x^2} + C}$$

Ex. 1

$$\int x\sqrt{1-3x^2}\,dx =$$

$V(x)$

up to a constant -6

$V'(x) = -6x$

$$dv = v'(x)\,dx = -6x\,dx$$

$$= \int \sqrt{V(x)} \cdot \left(-\frac{1}{6}\right) \cdot \underbrace{(-6x)\,dx}_{} = \text{(substitution)}$$

$$= -\frac{1}{6}\int \underbrace{\sqrt{v}}_{v^{\frac{1}{2}}}\,dv = \text{(integration)}$$

$$= -\frac{1}{6}\, v^{\frac{3}{2}} \cdot \frac{2}{3} + C = \text{(sub } V(x) \text{ back)}$$

$$= \boxed{-\frac{1}{9}\left(1-3x^2\right)^{3/2} + C}$$

Ex 2

$$\int 6\cdot(2x+1)^5\,dx =$$

$V(x)$

$V'(x) = 2$ up to a constant ✓

$$dv = v'\,dx = 2\,dx$$

$$= \int (v(x))^5 \cdot 3 \cdot \underbrace{2\,dx} =$$

$$= 3\int v^5\, dv = 3\frac{v^6}{6} + C =$$

$$= \frac{(2x+1)^6}{2} + C$$

The same works for definite integrals, but the limits of integration change:

$$\boxed{\int_a^b f(x)\,dx = k\int_{v(a)}^{v(b)} g(v)\,dv}$$

<u>Ex 3</u> $\displaystyle\int_1^e \frac{(\ln x)^3}{x}\,dx =$ $\left(\begin{aligned} v(x) &= \ln(x) \\ dv &= v'dx = \\ &= \frac{1}{x}dx \end{aligned} \right)$

$v(x)$

✓ $\frac{1}{x} = v'(x)$

$$= \int_1^e (v(x))^3 \underbrace{\frac{1}{x}dx} = \text{(substitution, limits change!)}$$

$$= \int_{v(1)=\ln 1}^{v(e)=\ln e} v^3\, dv = \int_0^1 v^3\, dv = \text{(FTC)}$$

$$= \frac{v^4}{4}\Big|_0^1 = \frac{1^4}{4} - \frac{0^4}{4} = \boxed{\frac{1}{4}}$$

other way

$$\frac{v^4}{4}\Big|_0^1 \overset{\substack{\text{sub } v(x) \text{ back} \\ \text{change limits back}}}{=} \frac{(\ln x)^4}{4}\Big|_1^e = \frac{(\ln e)^4}{4} - \frac{(\ln 1)^4}{4}$$

$$= \frac{1^4}{4} - \frac{0^4}{4} = \boxed{\frac{1}{4}}$$

Ex 4

$$\int_0^{\pi/2} \frac{\cos x}{\underbrace{1+\sin x}_{v(x)}} dx =$$

$$dv = v'(x)dx = \cos x\, dx$$

$$= \int_0^{\pi/2} \frac{1}{v(x)} \cdot \underbrace{\cos x \, dx} = \begin{pmatrix} v\left(\frac{\pi}{2}\right) = 1 + \sin\frac{\pi}{2} = 2 \\ v(0) = 1 + \sin 0 = 1 \end{pmatrix}$$

$$= \int_1^2 \frac{1}{v} \, dv = \ln|v| \Big|_1^2 = \ln 2 - \ln 1 =$$

$$= \boxed{\ln 2}$$

Ex.5 $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2} \, dx = \int_1^2 e^{v(x)} \cdot (-1)\left(-\frac{1}{x^2}\right) dx =$

$v(x)$

$v'(x) = -\frac{1}{x^2}$ ✓ up to a constant

$dv = v' \, dx = -\frac{1}{x^2} \, dx$

$$= (-1)\int_{v(1)}^{v(2)} e^{v}\,dv = -\int_{1}^{\frac{1}{2}} e^{v}\,dv \underset{(FTC)}{=}$$

$$= -\left(e^{v}\Big|_{1}^{\frac{1}{2}}\right) =$$

$$= -\left(e^{\frac{1}{2}} - e^{1}\right) =$$

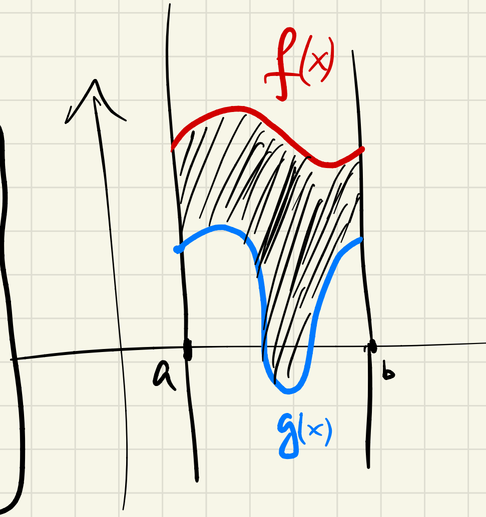
$$= \boxed{e - \sqrt{e}}$$

6.1 Area between curves

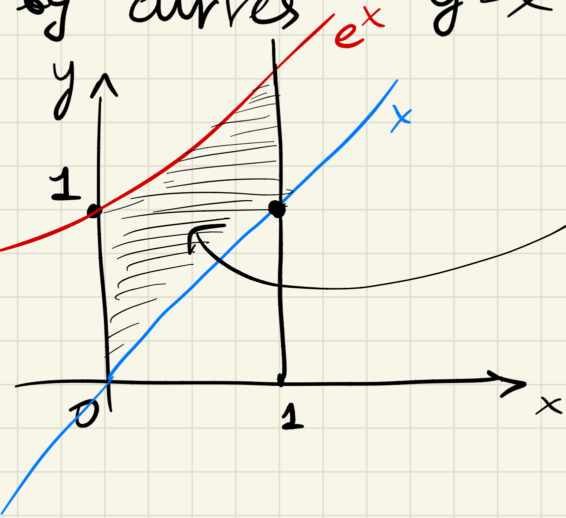
main formule:

$$\text{area} = \int_a^b (\underset{\text{top curve}}{f(x)} - \underset{\text{bottom curve}}{g(x)})\,dx$$

where $f(x)$ = top curve
$g(x)$ = bottom curve



Ex. 6 Find the area of region bounded by curves $y=x$, $y=e^x$, $x=0$, $x=1$



$$\text{area} = \int_0^1 \underset{\text{top}}{e^x} - \underset{\text{bottom}}{x}\,dx \overset{(FTC)}{=\!=} $$

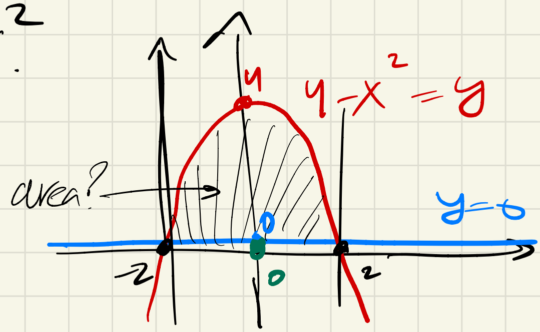
$$= \left(e^x - \frac{x^2}{2}\right)\Bigg|_0^1 =$$

$$= \left(e^{1} - \frac{1^2}{2}\right) - \left(e^{0} - \frac{0^2}{2}\right) =$$

$$= e - \frac{1}{2} - 1 = \boxed{e - \frac{3}{2}}$$

Ex. 1 Find area bounded by $y=0$ and $y=4-x^2$.

$$\text{area} = \int_{?}^{?} 4-x^2-0\,dx$$



in order to find limits of integration, we find x-values of intersection points:

- $f(x)=g(x)$

$$4-x^2=0$$

$$x^2=4$$

$$x=2$$
$$x=-2$$

- also, we can plug in $x=0$ in order to determine the top curve

$$f(0) > g(0)$$
$$4-0^2 > 0$$

so $4-x^2$ is the top curve

any point on $[-2,2]$

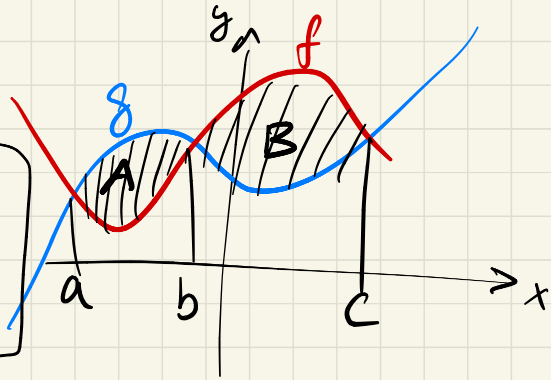
area $= \int_{-2}^{2} (\underbrace{4-x^2}_{f(x)}) - (\underbrace{0}_{g(x)})\,dx =$ (FTC)

$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \left(4\cdot 2 - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} =$$

$$= \boxed{\frac{32}{3}}$$

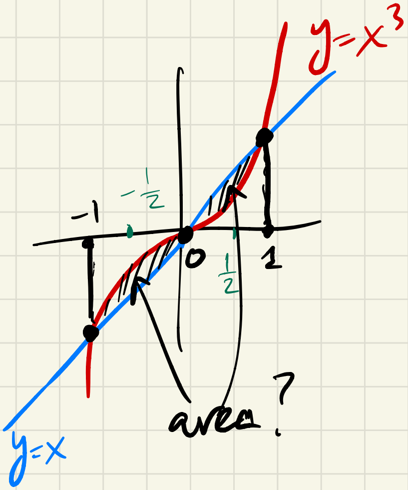
<u>Remark:</u> if top & bottom curves switch, break up the integral:



$$\text{area}(A) + \text{area}(B) = \int_a^b g(x) - f(x)\,dx + \int_b^c f(x) - g(x)\,dx$$

Ex. 2 Find area bounded by curves $y = \underbrace{x}_{g}$ and $y = \underbrace{x^3}_{f}$

• intersection points:



$$x = x^3$$
$$0 = x^3 - x$$
$$0 = x(x^2 - 1)$$
$$0 = x(x-1)(x+1)$$

$$\boxed{\begin{array}{l} x = -1 \\ x = 0 \\ x = 1 \end{array}}$$

• which is top & where? We pick $-\frac{1}{2}$ on $[-1, 0]$ and $\frac{1}{2}$ on $[0, 1]$

$f(\frac{1}{2}) < g(\frac{1}{2})$ so $\underset{\substack{\downarrow \\ x}}{g}$ is the top on $[0, 1]$

$$\left(\frac{1}{2}\right)^3 < \frac{1}{2}$$

$$\frac{1}{8} < \frac{1}{2}$$

$f(-\frac{1}{2}) > g(-\frac{1}{2})$ so $\overset{\substack{x^3 \\ \uparrow}}{f}$ is the top curve on $[-1, 0]$

$$\left(-\frac{1}{2}\right)^3 > -\frac{1}{2}$$

$$-\frac{1}{8} > -\frac{1}{2}$$

$$\text{area} = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx \underset{\text{(FTC)}}{=}$$

$$= \left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Bigg|_{-1}^{0} + \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{0}^{1} =$$

$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{2}}$$

6.5 Average value of a function

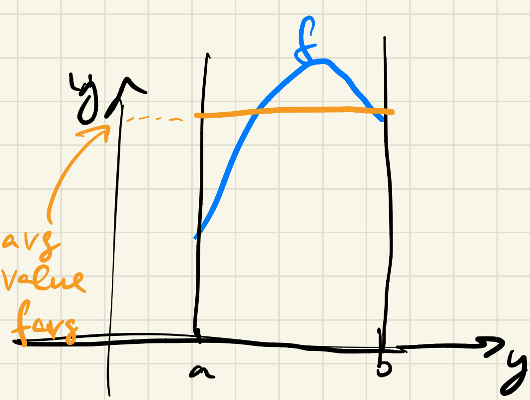
<u>input</u>: • function $f(x)$

• interval $[a,b]$

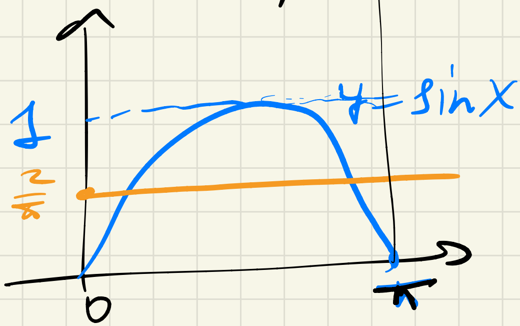
<u>output</u>: average value of $f(x)$ on $[a,b]$

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

main formula to remember



Ex. 3 Compute avg value of $f(x) = \sin x$ on $[0, \pi]$?



$$f_{avg} = \frac{1}{\pi - 0} \int_0^{\pi} \sin x \, dx = (FTC)$$

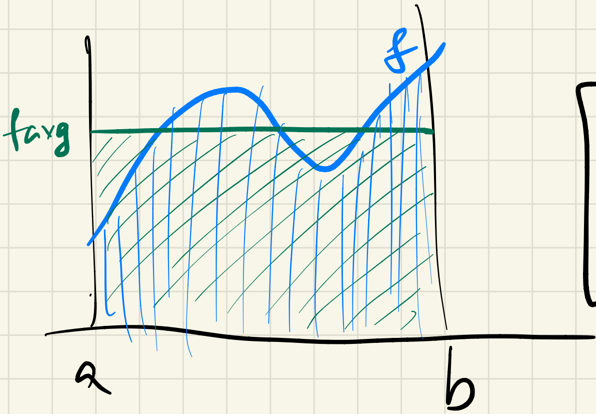
$$= \frac{1}{\pi} (-\cos x) \Big|_0^{\pi} = \frac{1}{\pi}\left(-(-1) - (-1)\right)$$

$$= \boxed{\frac{2}{\pi}}$$

What is avg value?

$$f_{avg} = \frac{1}{b-a} \underbrace{\int_a^b f(x) \, dx}_{\text{signed area}}$$

This means that for positive $f(x)$ we have that



$\boxed{\text{area under } f(x) \text{ is the same as area und } f_{avg}}$

why?

$$\text{area under } f(x) = \int_a^b f(x)\,dx$$

$$\text{area under } y = f_{avg} \text{ is } f_{avg} \cdot (b-a) =$$

$$= \cancel{\frac{1}{b-a}} \int_a^b f(x)\,dx \cancel{(b-a)} = \int_a^b f(x)\,dx$$

$\boxed{\text{As such, if we have water with level } f(x), \text{ the gravity will make the water be on the level } f_{avg}}$

geometric/physical meaning

Ex. 4 In a city the temperature (in F°) t hours after 8am is modeled by the function

$$T(t) = 50 + 14 \cdot \sin\frac{\pi t}{12}$$

temp. time

Find the average temp. during the period from 8am to 8pm

$$T_{avg} = \frac{1}{12-0}\int_0^{12}\left(50 + 14\sin\left(\frac{\pi t}{12}\right)\right)dt$$

hours after 8am

$v(t) = \frac{\pi t}{12}$

$$= \left(\frac{1}{12}\right)\left(\int_0^{12} 50\,dt + 14\int_0^{12}\sin\left(\frac{\pi t}{12}\right)dt\right) =$$

$dv = v'dt = \frac{\pi}{12}dt$

$$\frac{1}{12}\left(12\cdot 50 + 14\int_0^{12} \underbrace{\sin(v(t))}_{g(v)}\underbrace{\left(\frac{12}{\pi}\right)}_{k}\underbrace{\left(\frac{\pi}{12}\right)dt}_{\boxed{v'(t)\,dt}}\right) =$$

$$= 50 + \frac{14}{12}\,\frac{12}{\pi}\int_{v(0)=0}^{v(12)=\pi} \sin v \,\boxed{dv} =$$

$$= 50 + \frac{14}{\pi}\left(-\cos v\right)\Big|_0^{\pi} =$$

$$= 50 + \frac{14}{\pi}\left(-(-1) - (-1)\right) =$$

$$= \boxed{\left(50 + \frac{28}{\pi}\right)F^\circ}$$