Which are easily proved by Long Exact Sequence for $0 \to C' \to C \to C/C' \to 0$

Lecture 5

Prop Given any complex (C, d), the mapping cone

$$\left[\begin{array}{ccc} C & \xrightarrow{\ id\ } & C \\ \circlearrowleft & & \circlearrowleft \\ -d & & d \end{array}\right]$$

is contractible.

(i.e. $\underset{\text{homot. equivalent}}{\simeq} 0$)

$\square$

$$\begin{tikzcd} C \arrow[loop right, "-d"] \arrow[d, bend right, "id=h"'] \arrow[rr, bend left, "f = 0"] & & 0 \arrow[ll, bend left, "g = 0"] \\ C \arrow[loop right, "d"] \arrow[u, "id"'] & & \end{tikzcd}$$

$= \underline{d}$

- both f & g are chain maps
- $g \circ f = 0 \simeq id$

$\Updownarrow$

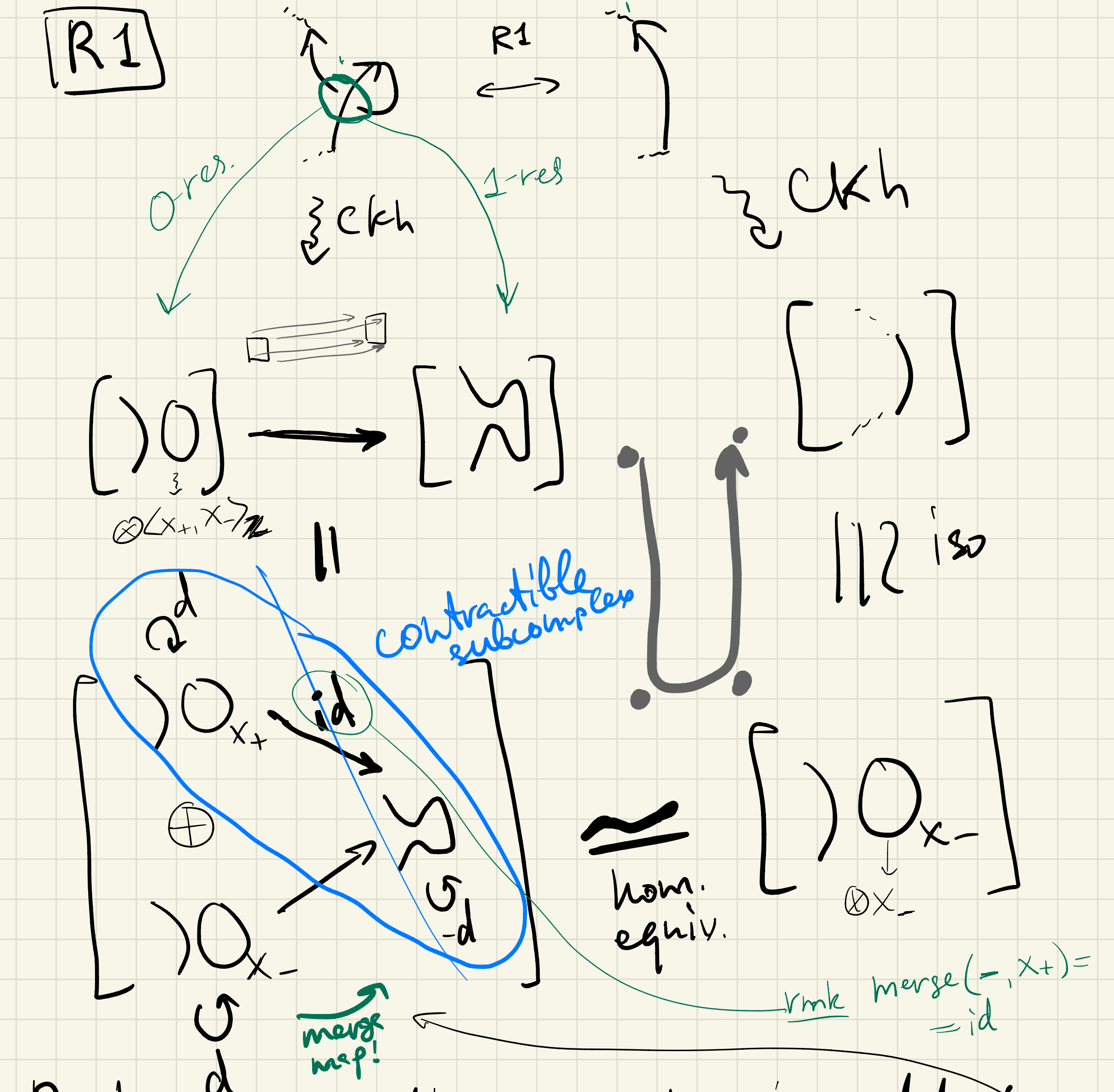
$$\underbrace{id - 0 = h \circ \underline{d} + \underline{d} \circ h}_{\text{indeed true}}$$

∎

Rmk Same is true for any $\left[(C, d) \xrightarrow{\text{iso}} (C', d')\right]$

Coming back to the proof of invariance of $Kh(K)$:

R1



Rmk Enumeration of crossings is needed for signs. For the signs to work out as here, we need to assume that is the first crossing.

P12 Our sign assignment is

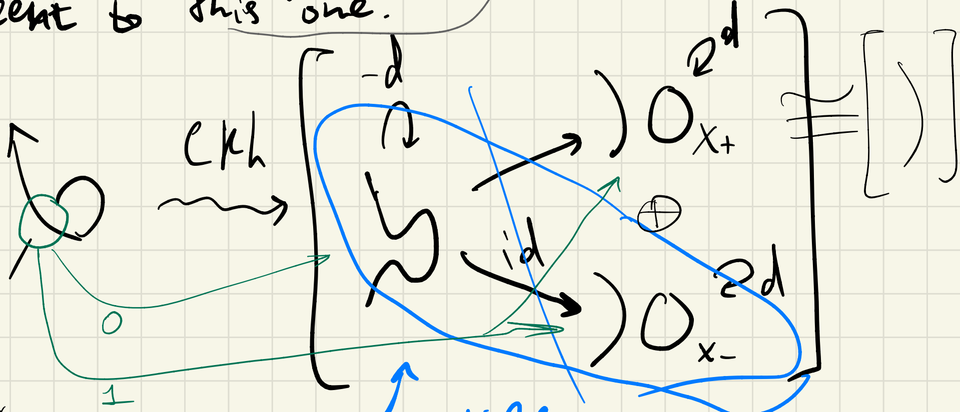
$$(-1)^{\varepsilon} \cdot F_{0110\ldots\boxed{*}\ldots011}$$

1st cross. 2nd cross. 3rd ...

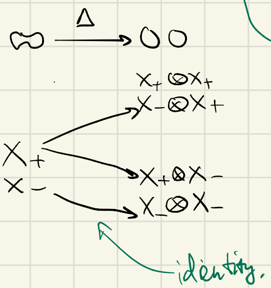
$\varepsilon = \#$ of 1's before $\boxed{*}$

Prove that any other sign assignment, that turns the commutative cube of resolutions to anticommutative, is equivalent to this one.

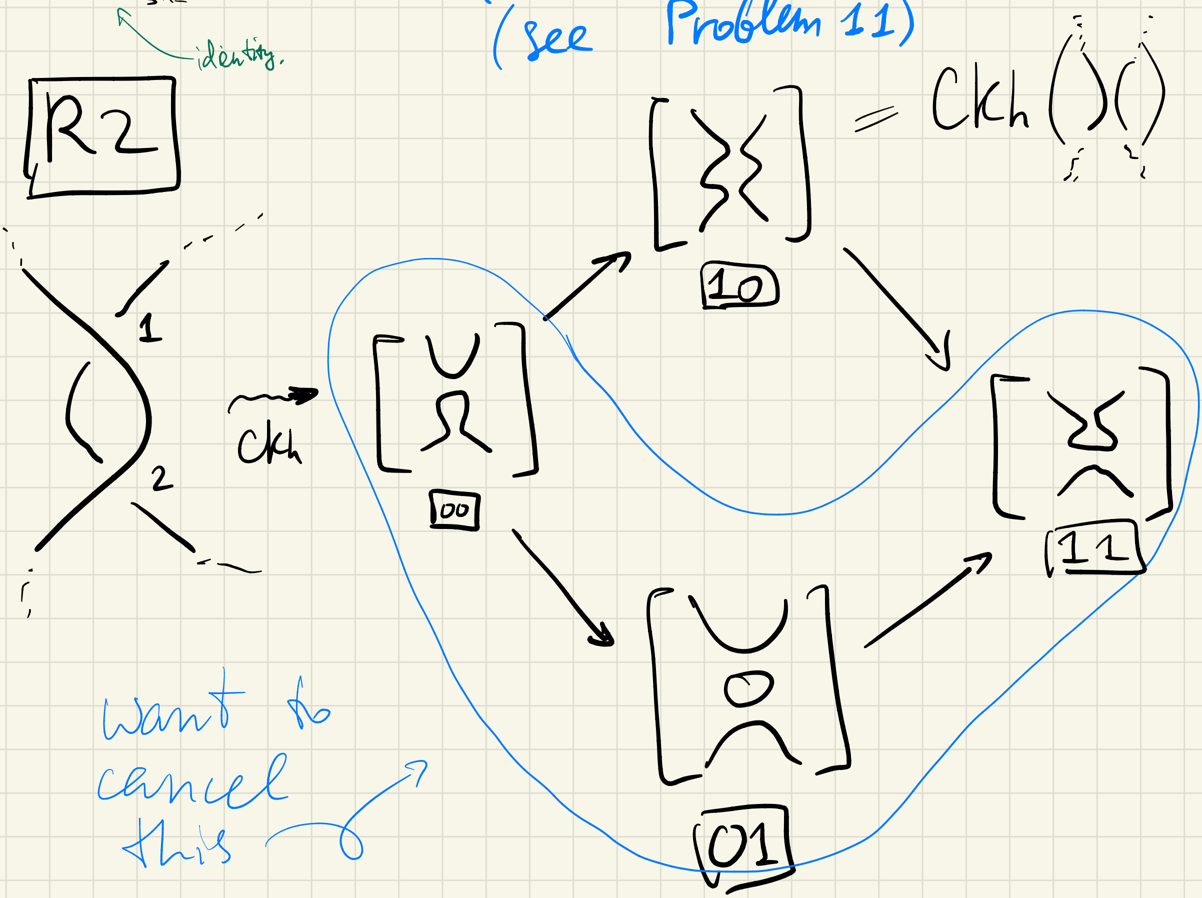
$\boxed{R1'}$



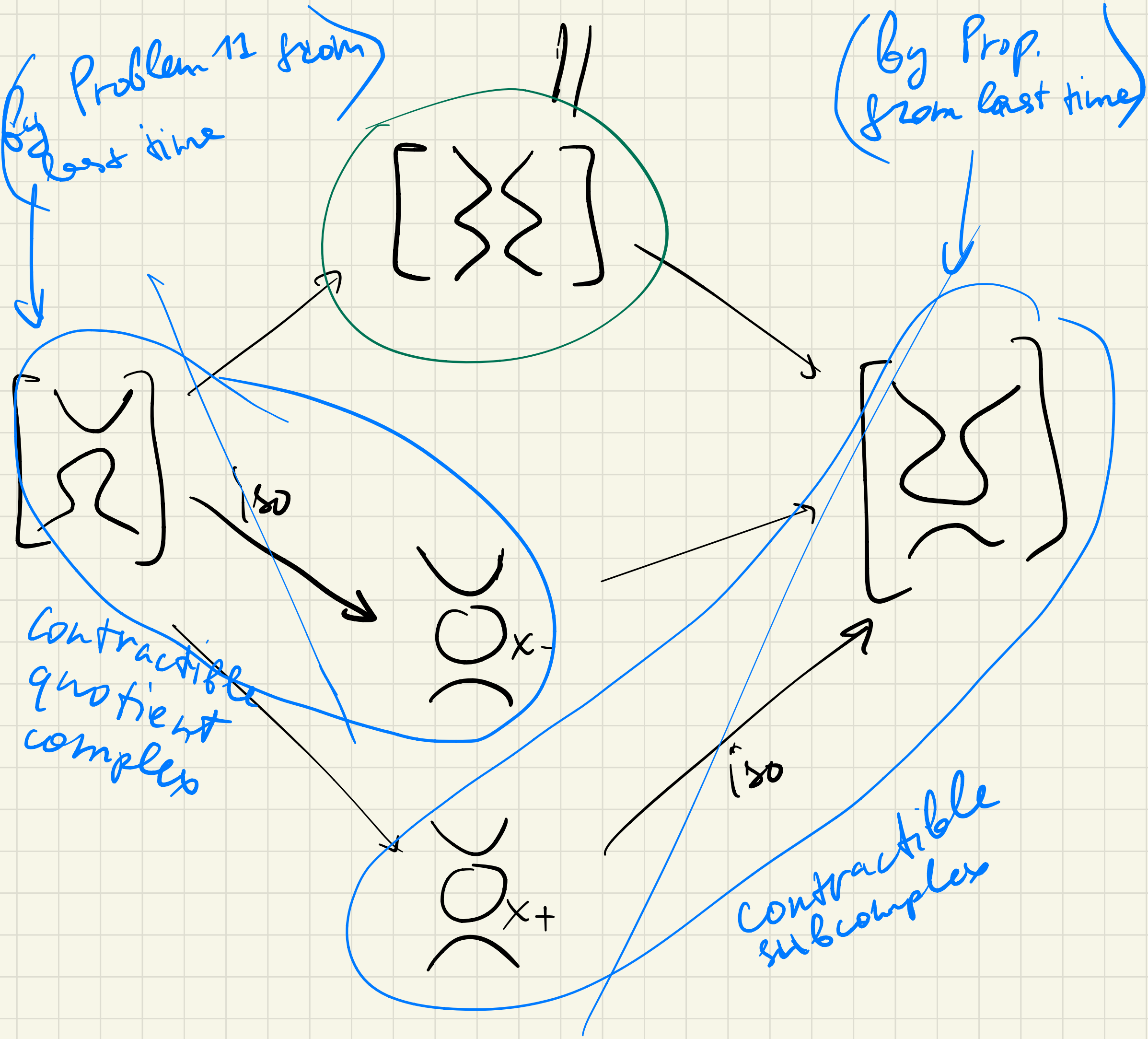
contractible quotient complex (see Problem 11)



$\boxed{R2}$



want to cancel this



To prove R3 we need a different extremely useful tool:

Lemma C1 (cancellation lemma)

Suppose the chain complex (C,d) is of the form

$$
(C,d) = \left[
\begin{tikzcd}
 & (C_1,d_1) \arrow[dl, shift right, "f"'] \arrow[dr, shift left, "h"] & \\
(C_2,d_2) \arrow[ur, shift right, "g"'] \arrow[rr, shift left, "\mathrm{id}"] & & (C_2,-d_2) \arrow[ul, shift left, "e"] \arrow[ll, shift left, "k"]
\end{tikzcd}
\right]
\qquad
\begin{tikzcd}
C_1 \arrow[loop right, "d_1"] \\
C_2 \arrow[u, "F"'] \arrow[loop right, "d_2"]
\end{tikzcd}
$$

Then we can "cancel" if we tweak the differential in (C_1,d_1), namely

$$(C,d) \simeq (C_1, d_1 - g \circ \underleftarrow{\mathrm{id}} \circ h)$$

<u>Rmk 1</u> $(C_2,d_2) \xrightarrow{\mathrm{id}} (C_2,-d_2)$ is neither sub nor quotient complex, so prev. lemmas don't work.

Rmk2 the result holds for iso instead of id:

$$\begin{tikzcd}
& (C_1,d_1) \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "h"] & \\
(C_2,d_2) \arrow[ur, "g"] \arrow[rr, "f", "\cong"'] & & (C_3,d_3) \arrow[ul, bend right, "e"'] \arrow[ll, bend left, "k"]
\end{tikzcd}
\simeq (C_1, d_1 - g\circ f^{-1}\circ h)$$

$\square$

$$\begin{tikzcd}
C_1 \arrow[loop above, "d_1"] \arrow[d, bend right, "f"'] \arrow[dr, "h"] \arrow[rr, bend left, "id"] & & C_1 \arrow[loop above, "d_1 - g\circ id\circ h"] \arrow[ll, "id"'] \arrow[dll, "-id\circ h"] \\
C_2 \arrow[loop below, "d_2"] \arrow[u, bend right, "g"'] \arrow[r, bend left, "id"] & C_2 \arrow[loop below, "-d_2"] \arrow[ul, bend right, "e"'] \arrow[l, bend left, "k"] \arrow[ur, "-g\circ id"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left, "P"] & \bullet \arrow[l, bend left, "I"]
\end{tikzcd}$$

(1) check that $(C_1, d_1 - g\circ id\circ h)$ is indeed a complex: $(d_1 - g\circ id\circ h)\circ(d_1 - g\circ id\circ h) =$

$$= d_1^2 - g\circ id\circ \underbrace{h\circ d_1}_{\substack{= +d_2\circ h \\ - id\circ f}} - \underbrace{d_1\circ g}_{\substack{= -g\circ d_2 \\ - e\circ id}}\circ id\circ h + g\circ id\circ h\circ g\circ id\circ h$$

$$= \underbrace{\left(d_1^2 + g\circ f + e\circ h\right)}_{=0} + g\circ\underbrace{\left(-id\circ d_2 + d_2\circ id + h\circ g\right)}_{=0}\circ h = 0$$

$\boxed{\text{since } d^2 = 0 \text{ in } [\because]}$

P13 check that

(2) Both I & P are chain maps

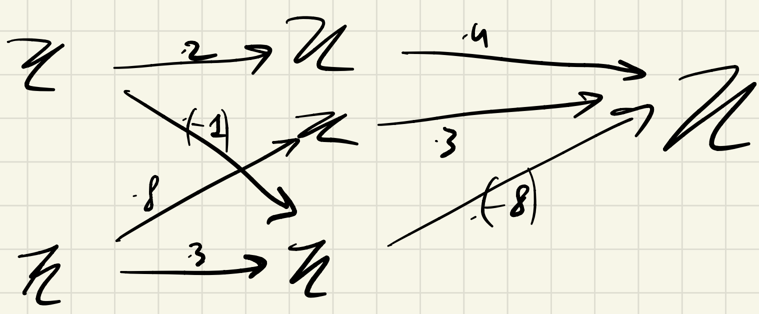
(3) $I\circ P$ and $P\circ I$ are homotopic to identities (can do it over $\mathbb{F}_2$)

∎

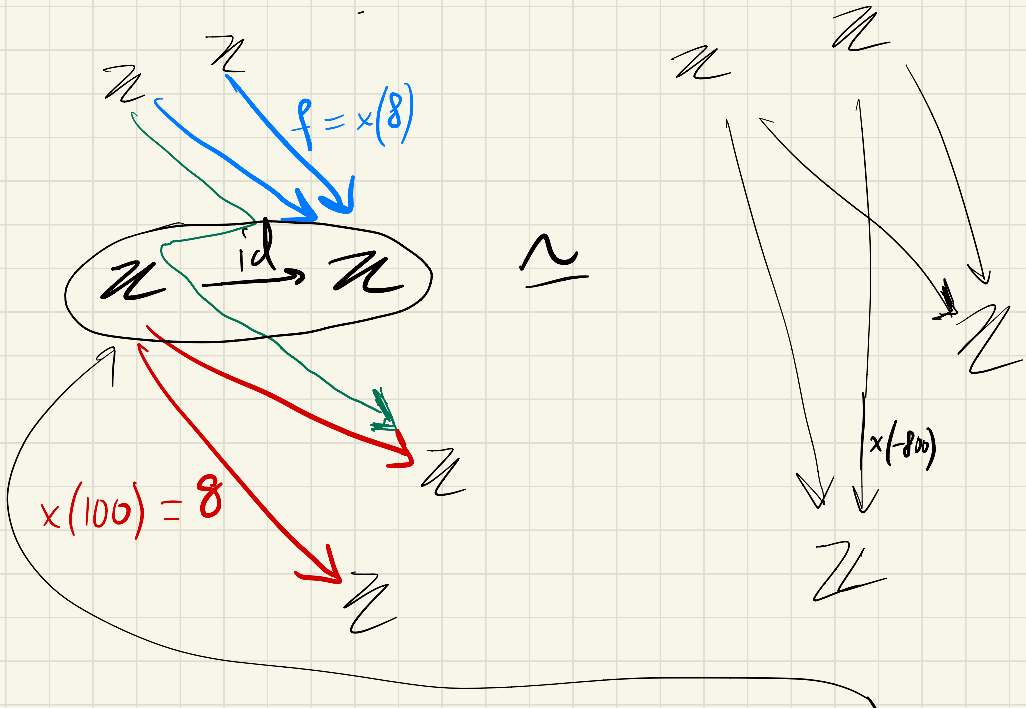
There is a very useful special case:

Lemma C2

(Cancellation of 2 generators) Suppose we represent a chain complex by a graph:



Then if we locate a pair of generators and a map "id" between them, we can erase them from the chain complex at the expense of adding zigzags, with the minus sign.



□ applying cancellation lemma to

This is a super tool to compute homology:

Ex 1

$$\begin{tikzcd} a \arrow[r, "1"] & b \\ c \arrow[ur, "\cancel{1}"] \arrow[r, "1"] & e \\ d \arrow[ur, "1"] & \end{tikzcd} \rightsquigarrow \begin{tikzcd} a \arrow[dr, "-1"] & \\ & e \\ d \arrow[ur, "\cancel{1}"] & \end{tikzcd} \rightsquigarrow a$$

5 dim complex

1 dim homology

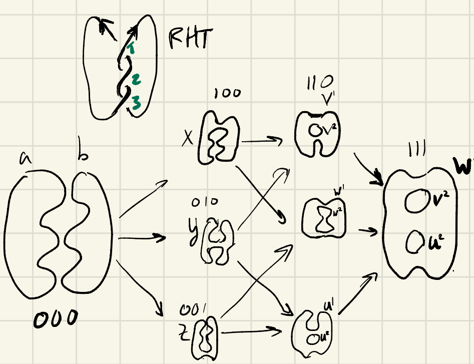
Ex 2

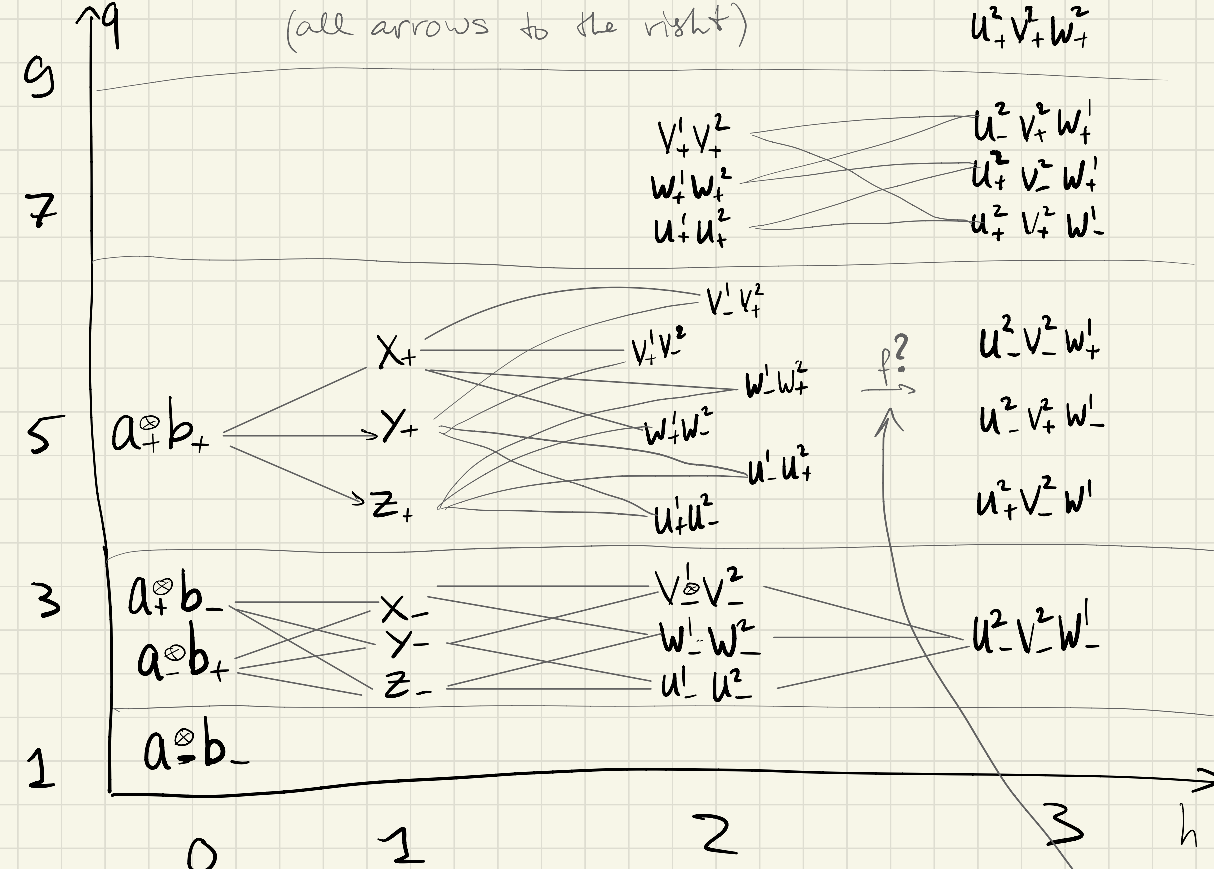
$$\begin{tikzcd} a \arrow[r, "\cancel{1}"] \arrow[dr, "1"] & b \\ c \arrow[ur, "1"] \arrow[dr, "1"] & d \\ e \arrow[ur, "1"] \arrow[r, "1"] & f \end{tikzcd} \simeq \begin{tikzcd} c \arrow[r, "-1"] \arrow[dr, "1"] & d \\ e \arrow[ur, "1"] \arrow[r, "\cancel{1}"] & f \end{tikzcd} \simeq \left[\begin{tikzcd} c \arrow[r, bend left, "-1"] \arrow[r, bend right, "-1"'] \arrow[r, phantom, "+"] & d \end{tikzcd} \right]$$

$= \times(-2)$

$$H_* = \left[0 \quad \mathbb{Z}/2 \right]$$

Hint for P8 (trefoil computation)





What's left to compute Kh( ):

- fill in the arrows that comprise f (according to merge/split maps)
- apply (C2) cancellation a lot of times to compute homology

P14 (easy & useful)

Using the same method, prove → $\widetilde{Kh}$( ) (reduced!) =