also up to mirroring (changing all crossings $\times \leftrightarrow \times$ in the diagram)

and up to orientation reversing

$\pi_1(K)$ is a strong invariant, but

very hard to compare $\pi_1(K)$ to $\pi_1(K')$

(There is a sense in which it is a complete invariant, see "peripheral subgroup" page on wiki.)

Much better invariant is representations:

$$\boxed{\hom(\pi_1(K), G) / \text{conjugation}}$$

Before Alexander & Jones poly came, taking G to be some finite group produced the most computable/strong knot invariants

Modern strong knot invariants (hard to discover / easier to compute)

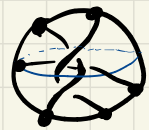
- Alex./Conway polynomial $\Delta_K(t)$ (1923)
- Jones polynomial $J_K(q)$ (1984)
  - <u>easily</u> distinguishes LHT from RHT
  - gives lower bounds for $c(K)$
- HOMFLY-pt poly

Since 2000's:

- Knot Floer homology $\widehat{HFK}(K)$ (Ozsváth-Szabó)
  - categorifies $\Delta_K(t)$
  - functorial!
  - Detects $g_3(K)$, fiberedness
- Khovanov homology $Kh(K)$
  - categorifies $J_K(q)$
  - functorial!
  - combinatorial proof of Milnor conjecture
  - detects the unknot
  - in some sense unifies knot homologies

Plan: 1) $J_K(q)$

2) $Kh(K)$, proof of Milnor conjecture

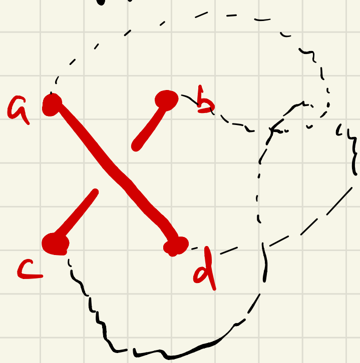
3) $Kh(T)$, T is a tangle e.g. 

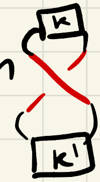
4) Floer-theoretic techniques to study 4-ended tangles and their Kh

Some conjectures towards which this curriculum is aimed:

Cosmetic Crossing Conjecture

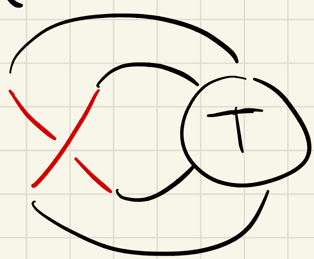
Suppose we locate a crossing in a knot



such that a is connected to b, c is connected to d, and knot is not of the form 

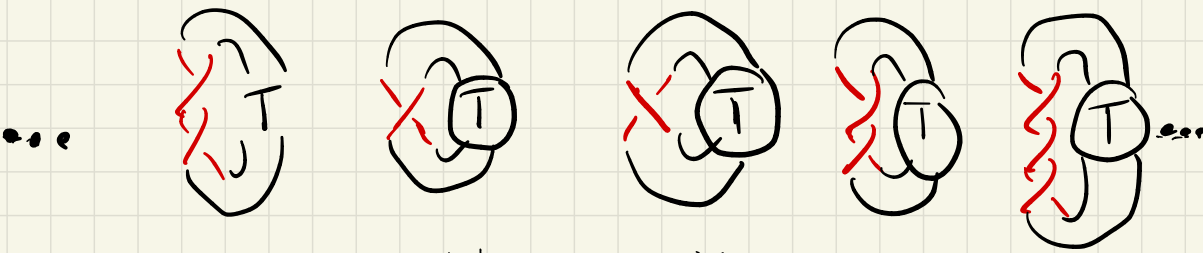
Then the crossing change changes the knot: (i.e. the crossing is not cosmetic)

$\neq$



Generalized Cosmetic Crossing Conjecture

Under the same conditions all the knots below are different

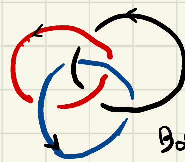


2 twists   1 twist   -1 twist   -2 twists   -3 twists

Jones polynomial

<u>Oriented link</u> is a bunch of oriented knots possibly linked with other, e.g.

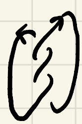
Hopf link



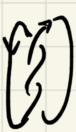
Borromean rings

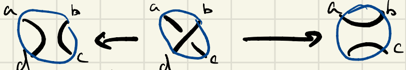
<u>Jones polynomial</u>

$J: \{\text{oriented links}\} \longrightarrow \mathbb{Z}[q, q^{-1}]$

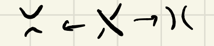
 $\longmapsto q^2 + q^6 - q^8$

Construction:

Diagram  $\xmapsto[\text{orientation doesn't matter}]{1}$ cube of resolutions $\xmapsto{2} J_k(q) = \sum_{\text{all resolutions}} (\dots)$

① Resolution at a crossing 

0-resolution 1-resolution



0-resol. 1-resol.

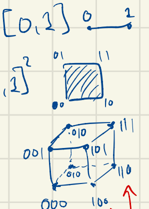
Link diagram with n crossings results in 2^n full resolutions: $= |\{0,1\}^n|$

hyper-cubes

$\mathbb{R} \supset [0,1]$

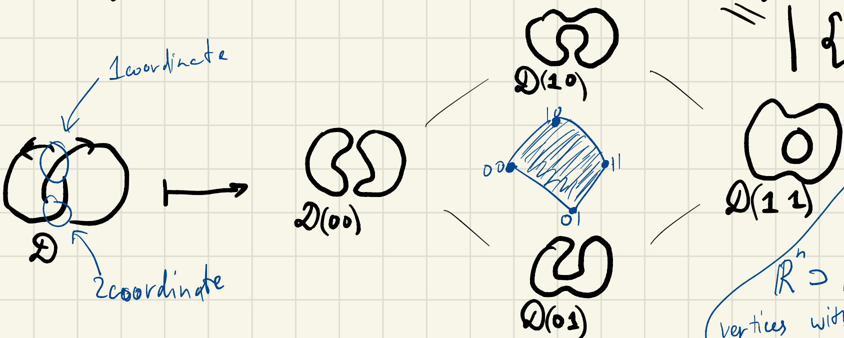
$\mathbb{R}^2 \supset [0,1]^2$

$\mathbb{R}^3 \supset [0,1]^3$



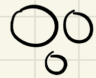
coordinates of vertices

$\mathbb{R}^n \supset [0,1]^n$ has vertices with coordinates $\underbrace{(0 \text{ or } 1, 0 \text{ or } 1, 0 \text{ or } 1, \dots, 0 \text{ or } 1)}_{n \text{ coordinates}}$



cube of resolutions

② To each resolution $\mathcal{D}(\bar v)$, $\bar v \in \{0,1\}^n$, we associate a polynomial

$$J_{\mathcal{D}(\bar v)} = (q+q^{-1})^{|\mathcal{D}(v)|-1} \cdot (-q)^{|v|},$$  $\mapsto (q+q^{-1})^2 \cdot (-q)^{|v|}$

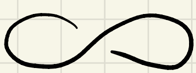
$|\mathcal{D}(\bar v)|$ is the # of circles in $\mathcal{D}(v)$

$|\bar v| = \sum_i v_i$ is the # of 1's in $\bar v$ ("height" of the vertex)

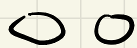
Then we sum over all resolutions:

$$\sum_{\bar v \in \{0,1\}^n} (q+q^{-1})^{|\mathcal{D}(\bar v)|-1} \cdot (-q)^{|v|}$$

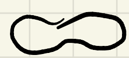
Examples

1)  $\longrightarrow$

$(\cancel{q+}q^{-1})$

$\mathcal{D}(0)$ 

$= q^{-1}$

$\mathcal{D}(1)$ 

$\cancel{(q+q^{-1})^0} \cdot (\cancel{-q})$

2)  $\rightsquigarrow (q+q^{-1})^0 \cdot (-q)^0 = 1$

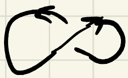
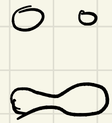
discrepancy!

Renormalization saves the day:

$$\boxed{J_D(q) = \left[\sum_{v\in\{0,1\}^n} (q+q^{-1})^{|D(v)|-1} \cdot (-q)^{|v|}\right] \cdot (-1)^{n_-}\, q^{(n_+-2n_-)}}$$

where $n_- = \#$ of negative crossings 

$n_+ = \#$ of positive crossings 

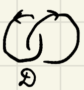
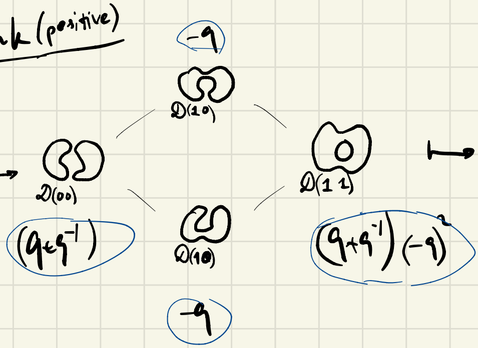
Now  $\mapsto$ 

$$\left[\begin{matrix} (\cancel{q}+q^{-1}) \\ + \\ \cancel{-q} \end{matrix}\right] \cdot (-1)^0 \cdot q^1 = 1 \quad ✓$$

$n_-=0$

$n_+=1$

Hopf link (positive)

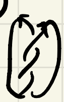
 $\mapsto$ 

$$\mapsto \left(q+q^{-1}-\cancel{q}-\cancel{q}+q^3+\cancel{q}\right) q^2 = $$
$$= q^1 + q^5$$

$n_+=2$

$n_-=0$

P5

J$(q) = q^2+q^6-q^8$ and J$(q) = q^{-2}+q^{-6}-q^{-8}$

This distinguishes RHT from LHT, in the view of the following theorem:

Theorem Given an oriented link $\vec{L}$ and its two diagrams $\mathcal{D}_1, \mathcal{D}_2$, the polynomials $J_{\mathcal{D}_1}(q)$ and $J_{\mathcal{D}_2}(q)$ coincide. Thus $J_{\vec{L}}(q)$ is a link invariant

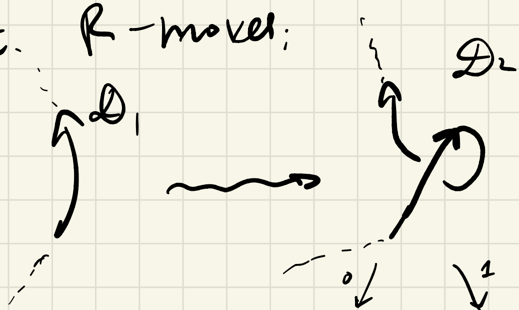
Rmk 1 $q \to -t^{\frac{1}{2}}$ gives classical repr. of Jones poly.
For knots all degrees of q are even, so J is a polynomial in $q^2 = t$

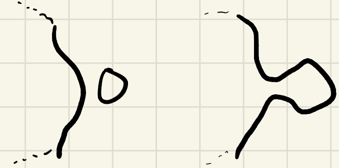
Rmk2 for knots $J_K(q)$ doesn't depend on orientation, but for links it does.

proof $\mathcal{D}_1 \longrightarrow \longrightarrow \longrightarrow \longrightarrow \mathcal{D}_2$
Reidemeister moves

So we need to check invariance of $J_{\vec{L}}(q)$ w.r.t. R-moves:

$\boxed{R1}$

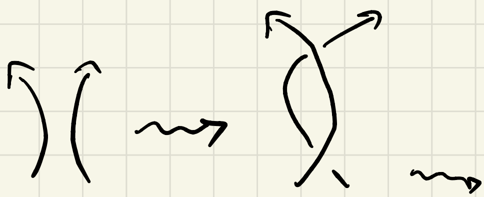


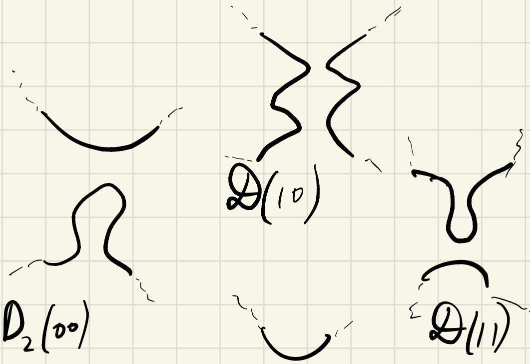


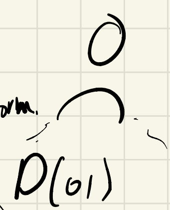
since we have 1 extra positive crossing

$$J_{D_2} = \left(\cancel{J_{D_1}\cdot(q+q^{-1})} + \cancel{J_{D_1}\cdot(-q)^2} \right)\cdot \underbrace{q^1}_{\substack{\text{renormalization} \\ n_+(D_2) - n_+(D_1) = 1}} = J_{D_1}$$

$\boxed{R2}$







$$J_{D_2} = \Big(\underbrace{J_{D_2(00)} + J_{D(01)} + J_{D(11)}}_{\text{should cancel}} + J_{D(10)} \Big) \cdot \text{renorm.}$$

P6 prove invariance w.r.t. R3 move