Lecture 6

Khovanov homology

$\bigcirc \rightsquigarrow V = \langle 1, x \rangle_{\mathbb{Z}} = \mathbb{Z}[x]/(x^2)$

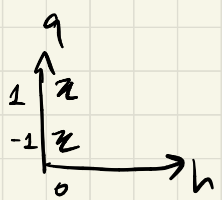
$m: V \otimes_{\mathbb{Z}} V \to V \quad \begin{cases} 1\otimes x \mapsto x & 1\otimes 1 \mapsto 1 \\ x \otimes 1 \mapsto x & x\otimes x \mapsto 0 \end{cases}$

$\Delta: V \to V \otimes_{\mathbb{Z}} V \quad \begin{cases} 1 \mapsto 1\otimes x + x \otimes 1 \\ x \mapsto x \otimes x \end{cases}$

$CKh(K) \rightsquigarrow Kh(K)$

$CKh(K)$: chain complex over $\mathbb{Z}$; $Kh(K)$: abelian group ($\mathbb{Z}$-module)

$Kh(\bigcirc) =$



Bar-Natan homology

$\bigcirc \rightsquigarrow V = \langle 1, x \rangle_{\mathbb{Z}[H]} = \dfrac{\mathbb{Z}[x,H]}{(x^2 - Hx)}$

$m: V \otimes_{\mathbb{Z}[H]} V \to V \quad \begin{cases} 1\otimes x \mapsto x & 1\otimes 1 \mapsto 1 \\ x \otimes 1 \mapsto x & x\otimes x \mapsto Hx \end{cases}$

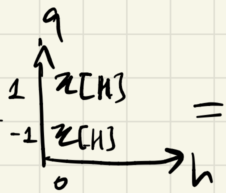
$\Delta: V \to V \otimes_{\mathbb{Z}[H]} V \quad \begin{cases} 1 \mapsto 1\otimes x + x \otimes 1 - H\,1\otimes 1 \\ x \mapsto x \otimes x \end{cases}$

$\boxed{gr(H) = q^{-2}h^{0}}$

$CBN(K) \rightsquigarrow BN(K)$

$CBN(K)$: chain complex over $\mathbb{Z}[H]$; $BN(K)$: $\mathbb{Z}[H]$-module

$BN(\bigcirc) =$



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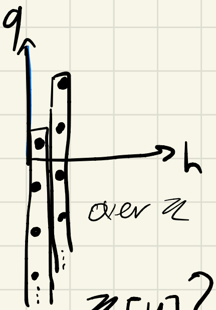


What exactly is a chain complex over $\mathbb{Z}[H]$?

It is a regular chain complex (C, d) over $\mathbb{Z}$ s.t.

1) $\mathbb{Z}[H] \curvearrowright C$ making C a free $\mathbb{Z}[H]$-module
2) the differential $d \circlearrowright C$ is $\mathbb{Z}[H]$-equivariant (means that $H^k(d(x)) = d(H^k x)$)

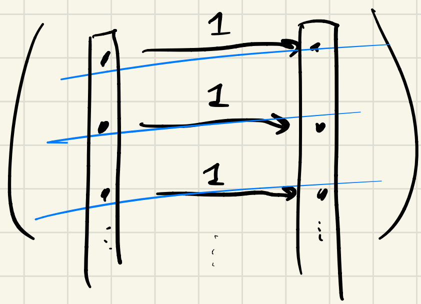
P15 Prove that this structure results in a well-defined $\mathbb{Z}[H]$-action on $H_*(C, d)$.

• The proof of invariance of $BN(k)$ is the same, with the only comment that all chain maps and homotopies are promoted to $\mathbb{Z}[H]$-equivariant maps

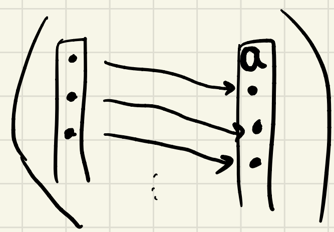
Examples of chain complexes over $\mathbb{Z}[H]$

1) $H_*\left(\mathbb{Z}[H] \xrightarrow{1} \mathbb{Z}[H]\right)$

$\qquad\qquad ||$

H_*  $= 0$

2) $H_*\left(\mathbb{Z}[H] \xrightarrow{H} \mathbb{Z}[H]\right) =$

$= H_*$  $= \langle a \rangle_{\mathbb{Z}} = \mathbb{Z}[H]/(H)$ as a $\mathbb{Z}[H]$-module

$H \cdot a = 0$

3) $H_*\left(\mathbb{Z}[H] \xrightarrow{H^k} \mathbb{Z}[H]\right) = \boxed{\begin{matrix} a \\ Ha \\ \vdots \\ H^{k-1}a \end{matrix}} = \mathbb{Z}[H]/(H^k)$

• One can modify chain complexes over $\mathbb{Z}[H]$ using cancellation lemma (C1), by simply treating it as a chain-complex over $\mathbb{Z}$

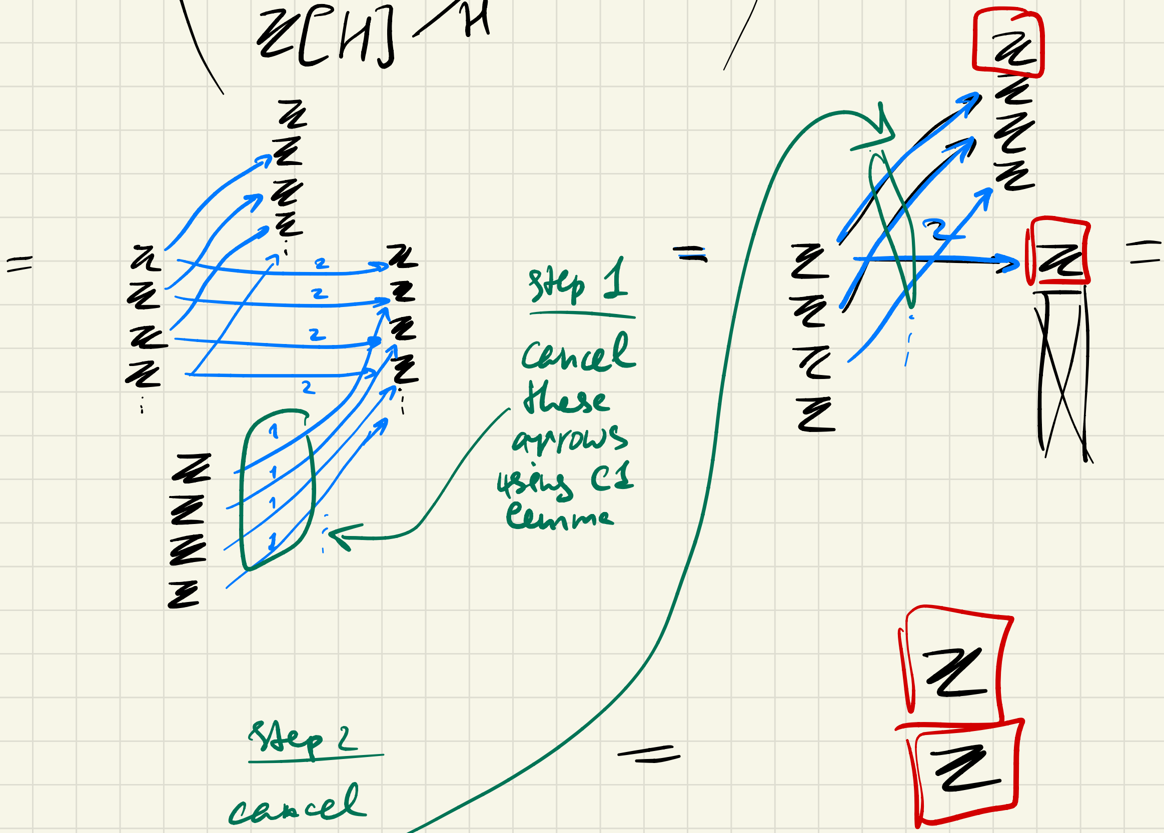
Ex.

$$\left[\begin{tikzcd} \mathbb{Z}[H] \arrow[dr, "H^2"] & & \\ \mathbb{Z}[H] \arrow[r, "1"] \arrow[drr, "2\cdot H^4"'] & \mathbb{Z}[H] & \\ & & \mathbb{Z}[H] \end{tikzcd}\right] \simeq$$

$$\simeq \left[\begin{tikzcd} \mathbb{Z}[H] \arrow[dr, "-2\cdot H^6"] & \\ & \mathbb{Z}[H] \end{tikzcd}\right]$$

homotopy equivalence in the category of chain complexes over $\mathbb{Z}[H]$, i.e. all the chain maps and homotopies are $\mathbb{Z}[H]$-equivariant

$$H_*\left(\begin{tikzcd} & \mathbb{Z}[H] \\ \mathbb{Z}[H] \arrow[ur, "H"] \arrow[r, "2"] & \mathbb{Z}[H] \\ \mathbb{Z}[H] \arrow[ur, "H"'] & \end{tikzcd}\right) =$$



P16 Adapt the cancellations used to compute $Kh\left(\left(\,\right)\right)$, and prove that

- $CBN(\ ) \simeq$

$q \backslash h$	0	1	2	3
9				$\overset{H}{\nearrow}\mathbb{Z}[H]$
7			$\mathbb{Z}[H]$	$\xrightarrow{2} \mathbb{Z}[H]$
5			$\mathbb{Z}[H] \overset{H}{\nearrow}$	
3	$\mathbb{Z}[H]$			
1	$\mathbb{Z}[H]$			

extra differentials in comparison to $Kh(\ )$

- Conclude that $BN(\ ; \mathbb{Z}) =$

$q \backslash h$	0	1	2	3
9				$a\ \mathbb{Z}$
7				$b\ \mathbb{Z}$
5				
3	$\mathbb{Z}[H]$			
1	$\mathbb{Z}[H]$			

$H \cdot a = 2b$

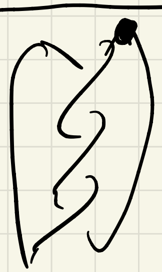
$\uparrow$

$\mathbb{Z}[H]$-action

__Corollary__ $BN(\ ; \mathbb{F}_2) = \mathbb{F}_2[H] \oplus \mathbb{F}_2[H] \oplus \mathbb{F}_2[H]/H \oplus \mathbb{F}_2[H]/H$

But $BN(\ ; \mathbb{Q}) = \mathbb{Q}[H] \oplus \mathbb{Q}[H] \oplus \mathbb{Q}[H]/(H^2)$

Reduced version



$\boxed{Kh}$

$$0 \to CKh^{x} \to CKh \to CKh^{1} \to 0$$

$\langle x \rangle$ on

CKh/CKh^{x}

$\|$

CKh^1

$\wr$

$$\boxed{\begin{tikzcd} Kh \arrow[r] & Kh^1 \arrow[dl, "\delta"] \\ Kh^x \arrow[u] & \end{tikzcd}}$$

- $\mathbb{Z}[x]/x^2$, x^2 has only one root

$\Rightarrow$ $CKh^x \cong CKh^1$

(P? in the past)

__Proposition__ Over $\mathbb{F}_2$ the LES sequence splits, i.e. the map δ is $= 0$.

$\square$

$$CKh_{\mathbb{F}_2} = \Big[\, \overset{\circlearrowleft d_X}{CKh^X_{\mathbb{F}_2}} \xleftarrow{\ \delta\ } \overset{\circlearrowleft d_1}{CKh^1_{\mathbb{F}_2}} \,\Big]$$

$\left(\Rightarrow Kh = \widetilde{Kh} \oplus \widetilde{Kh} \text{ over } \mathbb{F}_2\right)$

(red elements are associated to)

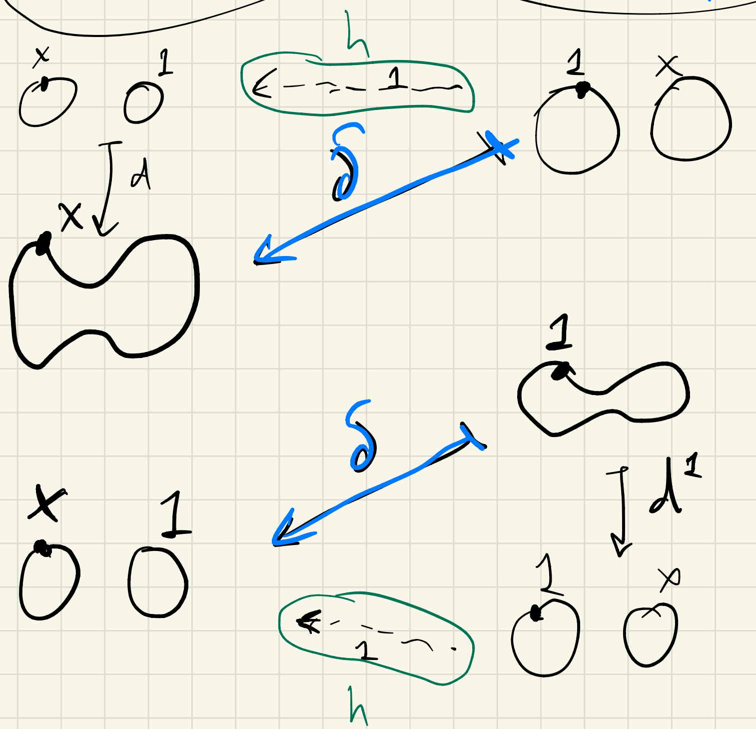
Δ: $\Delta\ O \mapsto OO$

$X \mapsto X \otimes X$ $\qquad$ $1 \mapsto \cancel{1 \otimes X} + X \otimes 1$

m: $1 \otimes X \mapsto X$, $\quad X \otimes X \mapsto 0$

m: $1 \otimes 1 \mapsto 1$, $\quad X \otimes 1 \mapsto \cancel{X}$

Because it's a quotient complex



Pr: Fill in the details, i.e. define rigorously the map $h: CKh^1_{\mathbb{F}_2} \to CKh^X_{\mathbb{F}_2}$ such that $dh + hd = \delta$, i.e. $\delta \simeq 0$

$\boxed{BN}$ $\quad 0 \to CBN^{x} \to CBN \to CBN^{x=0} \to 0$ quotient complex

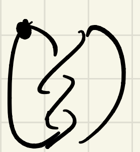
• $\mathbb{Z}[x,H]/(x^2-Hx)$, x^2-Hx has two roots (0 & H) $\Longrightarrow$ CBN^{x} is <u>not</u> isomorphic to $CBN^{x=0}$ on the nose!

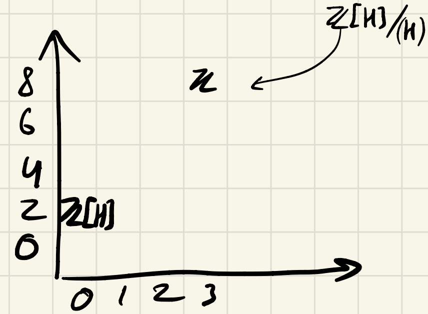
Nevertheless:

<u>Proposition</u> $CBN^{x}(K) \cong CBN^{x=0}(K)$, resulting in <u>one</u> version of reduced Bar-Natan homology $\widetilde{CBN}(K)$.

$\square$ technical, see "Immersed curves in Khovanov homology", Section 3.3.2

$\blacksquare$

<u>P18</u> $\widetilde{BN}\Big($  $\Big) =$



Proposition

(1) $\widetilde{Kh}(K) \underset{\text{over } \mathbb{Z}}{=} \left[\widetilde{BN}(K) \xrightarrow{\times H} \widetilde{BN}(K) \right]$

(2) $Kh(K,\mathbb{Q}) = \left[\widetilde{BN}(K;\mathbb{Q}) \xrightarrow{\times H^2} \widetilde{BN}(K;\mathbb{Q}) \right]$

$\square$ See

"Immersed curves in Khovanov homology"
Section 3.5

$\blacksquare$

Conjecture

$$Kh(K;\mathbb{Z}) = \left[\begin{tikzcd} C\widetilde{BN}(K) \arrow[r, "\times H"] & C\widetilde{BN}(K) \\ C\widetilde{BN}(K) \arrow[ur, "\times 2"] \arrow[r, "\times H"'] & C\widetilde{BN}(K) \end{tikzcd} \right]$$

(Hint: to prove it combine the previous Prop. part (1) together with the proof of $d \simeq 0$ over $\mathbb{Z}/2$)