

also up to mirroring (changing all crossings $X \leftrightarrow X$ in the diagram)
and up to orientation reversing

$\pi_1(k)$ is a strong invariant, but
very hard to compare $\pi_1(k)$ to $\pi_1(k')$

(there is a sense in which it is
a complete invariant, see
"peripheral subgroup" page on wiki.)

Much better invariant is representations:

hom $(\pi_1(k), G)$ / conjugation

Before Alexander & Jones poly came,
taking G to be some finite group
produced the most computable/strong
knot invariants

Modern strong knot invariants (hard to discover easier to compute)

- Alex./Conway polynomial $\Delta_K(t)$ (1923)
- Jones polynomial $J_K(q)$ (1984)
 - easily distinguishes LHT from RHT
 - gives lower bounds for $c(K)$
- HOMFLY-pt poly

Since 2000's:

- Knot Floer homology $\widehat{\text{HFK}}(K)$ (Ozsváth-Szabó)
 - categorifies $\Delta_K(t)$
 - functorial!
 - Detects $g_3(K)$, fiberedness
- Khovanov homology $\text{Kh}(K)$
 - categorifies $J_K(q)$
 - functorial!
 - combinatorial proof of Milnor conjecture
 - detects the unknot
 - in some sense unifies knot homologies

Plan: 1) $J_K(q)$

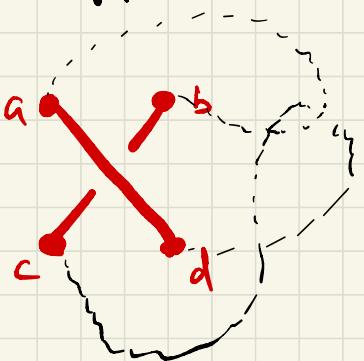
2) $\text{Kh}(K)$, proof of Milnor conjecture

3) $\text{Kh}(T)$, T is a tangle e.g. 

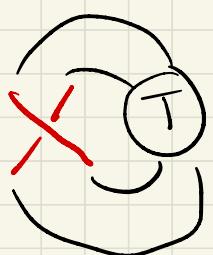
4) Floer-theoretic techniques to study 4-ended tangles and their Kh

Some conjectures towards which this curriculum is aimed:

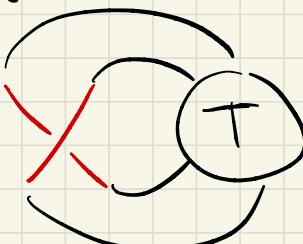
Cosmetic Crossing Conjecture

Suppose we locate a crossing in a knot such that a is connected to b, c is connected to d, and knot is not of the form 

Then the crossing change changes the knot: (i.e. the crossing is not cosmetic)

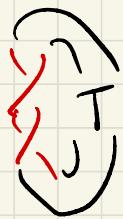


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Generalized Cosmetic Crossing Conjecture

Under the same conditions all the knots below are different



2 twists



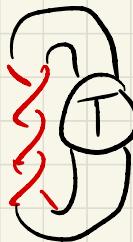
1 twist



-1 twist



-2 twists



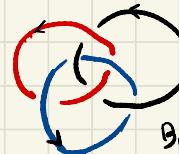
-3 twists

Jones polynomial

Oriented link is a bunch of oriented knots possibly linked with other, e.g.



Hopf link



Borromean rings

Jones polynomial

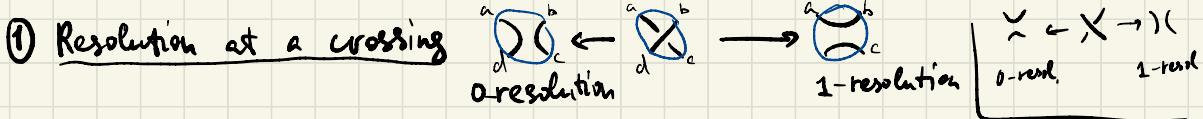
$J: \{ \text{oriented links} \} \rightarrow \mathbb{Z}[q, q^{-1}]$

$$\text{Diagram } \xrightarrow{\text{1}} \text{cube of resolutions} \xrightarrow{\text{2}} q^2 + q^6 - q^8$$

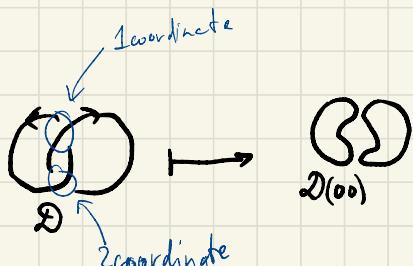
Construction:

Diagram $\xrightarrow{\text{1}}$ orientation doesn't matter

cube of resolutions $\xrightarrow{\text{2}}$ $J_k(q) = \sum_{\text{all resolutions}} (\dots)$



Link diagram with n crossings results in 2^n full resolutions:



cube of resolutions

To each resolution $\bar{D}(\bar{v})$, $\bar{v} \in \{0,1\}^n$, we associate a

polynomial $J_{\bar{D}(\bar{v})} = (q+q^{-1})^{|\bar{D}(\bar{v})|-1} \cdot (-q)^{|\bar{v}|}$, $\bigcirc \bigcirc \mapsto (q+q^{-1})^2 \cdot (-q)^{|\bar{v}|}$

$|\bar{D}(\bar{v})|$ is the # of circles in $\bar{D}(\bar{v})$

$|\bar{v}| = \sum v_i$ is the # of 1's in \bar{v} ("height" of the vertex)

Then we sum over all resolutions:

$$\sum_{\bar{v} \in \{0,1\}^n} (q+q^{-1})^{|\bar{D}(\bar{v})|-1} \cdot (-q)^{|\bar{v}|}$$

($q+q^{-1}$)

Examples

1) \rightarrow $\begin{matrix} \bar{D}(0) & \bigcirc \bigcirc \\ \bar{D}(1) & \text{trefoil knot} \end{matrix} = q^{-1}$

$(q+q^{-1}) \cdot (-q)$

2) $\sim (q+q^{-1})^0 \cdot (-q)^0 = 1$

discrepancy!

hyper-cubes

$\mathbb{R}^2 \rightarrow [0,1]^2$

$\mathbb{R}^3 \rightarrow [0,1]^3$

$\mathbb{R}^4 \rightarrow [0,1]^4$

$\mathbb{R}^5 \rightarrow [0,1]^5$

$\mathbb{R}^6 \rightarrow [0,1]^6$

$\mathbb{R}^7 \rightarrow [0,1]^7$

$\mathbb{R}^8 \rightarrow [0,1]^8$

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\mathbb{R}^{181}

Renormalization saves the day:

$$J_0(q) = \left[\sum_{\sigma \in \{0,1\}^n} (q+q^{-1})^{|\{i| \sigma_i=1\}| - 1} \cdot (-q)^{|\{i| \sigma_i=0\}|} \right] \cdot (-1)^{n_-} q^{(n_+ - 2n_-)}$$

Where $n_- = \#$ of negative crossings 

$n_+ = \#$ of positive crossings 

Now

$$\text{Diagram with } n_- = 0, n_+ = 2 \mapsto \begin{matrix} & \circ & \circ \\ & \downarrow & \downarrow \\ \circ & \text{---} & \circ \end{matrix} \left[\begin{matrix} (q+q^{-1}) \\ + \\ -q \end{matrix} \right] \cdot (-1)^0 \cdot q^2 = 1 \quad \checkmark$$

Hopf link (positive)

$$\text{Diagram with } n_+ = 2, n_- = 0 \mapsto \begin{matrix} & \circ & \circ \\ & \downarrow & \downarrow \\ \circ & \text{---} & \circ \end{matrix} \left[\begin{matrix} (q+q^{-1}) \\ 0(00) \\ 0(10) \\ -q \end{matrix} \right] \mapsto \begin{matrix} & \circ & \circ \\ & \downarrow & \downarrow \\ \circ & \text{---} & \circ \end{matrix} \left[\begin{matrix} (q+q^{-1})(-q) \\ 0(11) \end{matrix} \right] \mapsto (q+q^{-1}-q-q+q^3+q)q^2 = q^2 + q^5$$

P5 $J_{\text{RHT}}(q) = q^2 + q^6 - q^8$ and $J_{\text{LHT}}(q) = q^{-2} + q^{-6} - q^{-8}$

This distinguishes RHT from LHT, in the view of the following theorem:

Theorem Given an oriented link \overrightarrow{L} and its two diagrams D_1, D_2 , the polynomials $J_{D_1}(q)$ and $J_{D_2}(q)$ coincide. Thus $J_{\overrightarrow{L}}(q)$ is a link invariant.

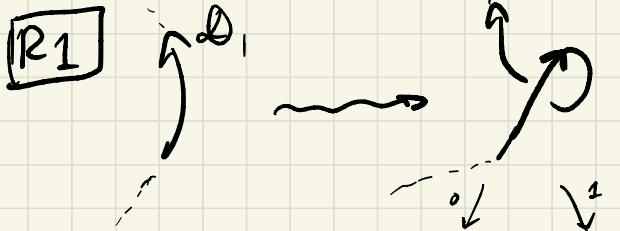
Rmk 1 $q \rightarrow -t^{\frac{1}{2}}$ gives classical repr. of Jones poly
For knots all degrees of q are even, so J is a polynomial in $q^2 = t$

Rmk 2 for knots $J_k(q)$ doesn't depend on orientation, but for links it does.

Proof $D_2 \xrightarrow{\text{R-moves}} \dots \xrightarrow{\text{R-moves}} D_1$
Reidemeister moves

So we need to check invariance of $J_{\overrightarrow{L}}(q)$

w.r.t. R-moves:



$$J_{D_2} = \left(\cancel{J_{D_1}(q+q^{-1})} + \cancel{J_{D_1}(q)^2} \right) \cdot q^{\frac{1}{2}} = J_{D_1}$$

Since we have positive extra crossings

renormalization
 $n_+(D_2) - n_+(D_1) = 1$

R^2

$$J_{D_2} = \underbrace{\left(J_{D_2(00)} + J_{D(01)} + J_{D(11)} + \left(J_{D(10)} \right) \right)}_{\text{Should cancel}} \cdot \text{norm.}$$

$D(01)$

P6 prove invariance w.r.t. R^3 move