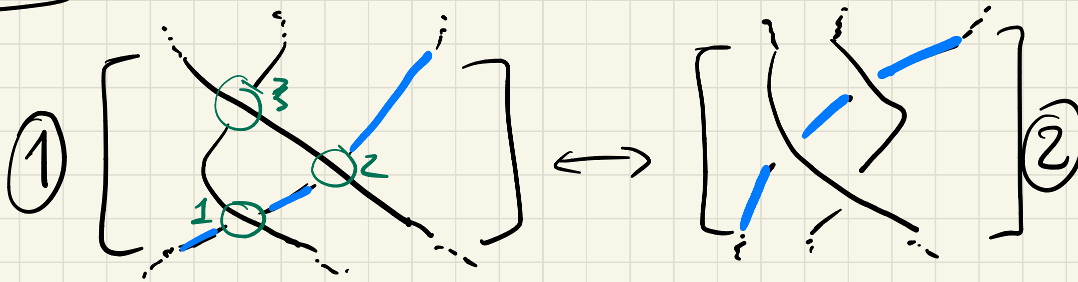
Lecture 6

Coming back to the invariance proof:

$\boxed{R3}$ plan:



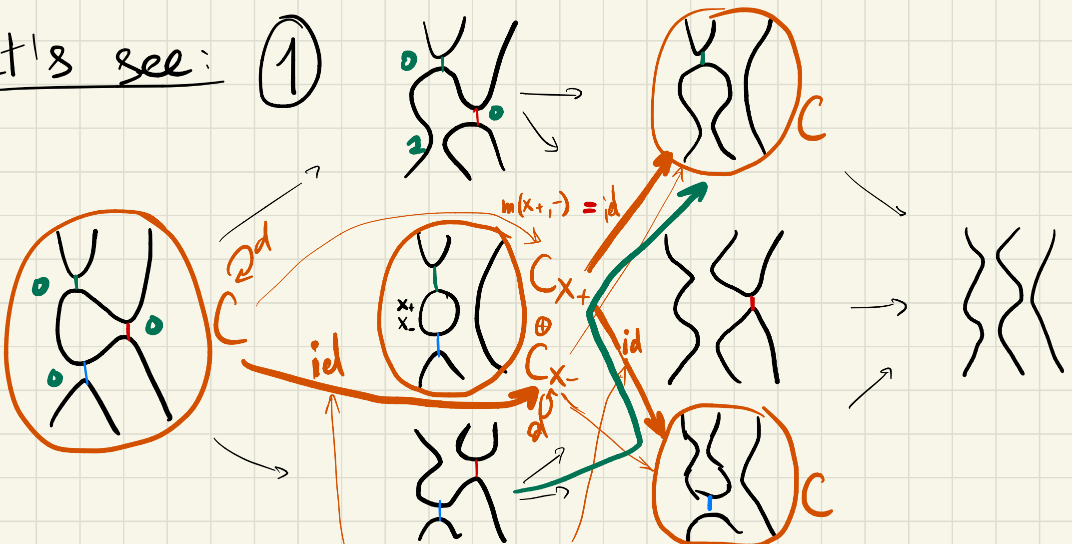
8 terms

8 terms

$\boxed{5 \text{ terms}}$

natural complex corresponding to triple point

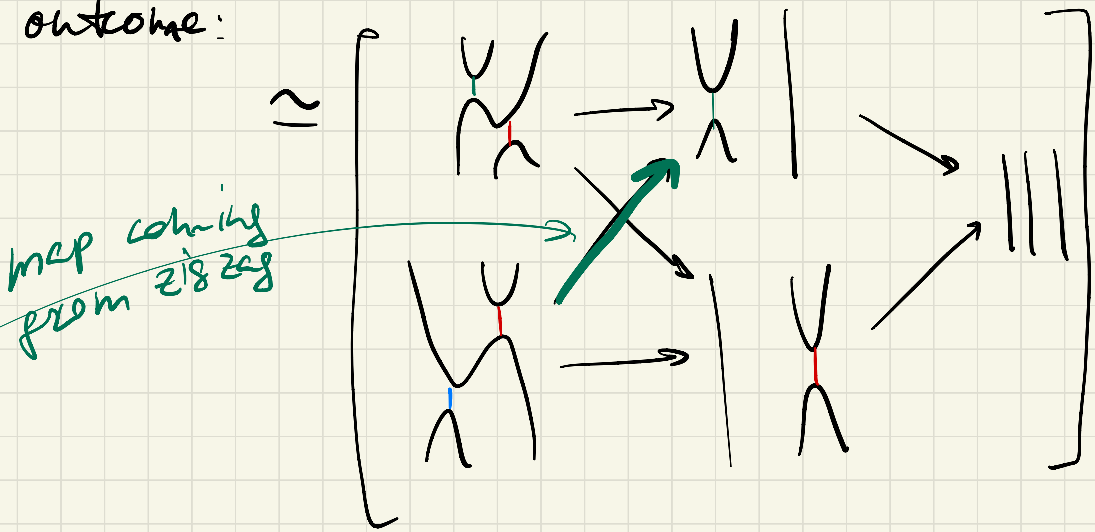
let's see: 1



Cancel first this, (either observe its a quotient contractible complexe, or just use (C1) Lemma)

and then cancel this (using (C1) Lemma)

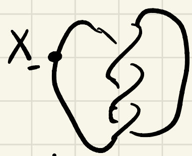
outcome:



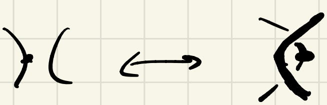
The same works with ②.

∎ (done proving invariance of $Kh(K)$)

Invariance of $\widetilde{Kh}(K)$

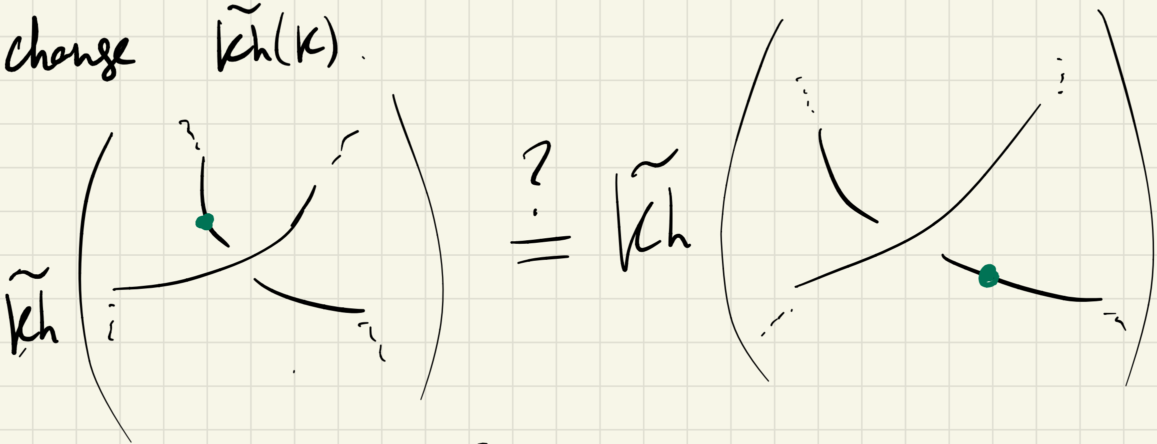


1) Reidemeister moves which don't involve the basepoint: proof of invariance is the same.

2)  What about these moves?

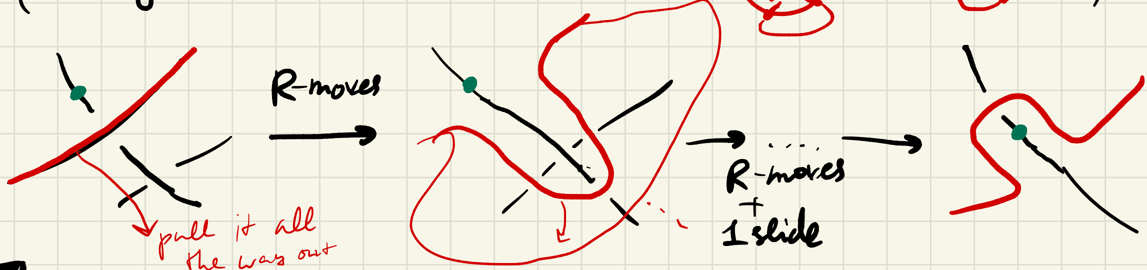
Lemma (which deals with ⁵)

Moving the Bpt through crossings does not change $\widetilde{Kh}(K)$.



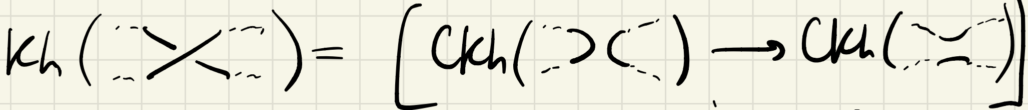
□ idea: work on S^2 instead of the plain.

(to get an extra move )



∎

Rmk



$$\widetilde{Kh}(K) \rightrightarrows HFK(K)$$

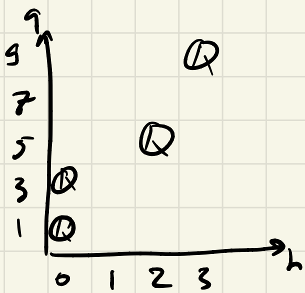
$$\widehat{HF}(\Sigma_2(K)) \qquad I^{\#}(K)$$

spectral sequences

Structure & applications in $Kh(K)$

- $\widetilde{Kh}(K)$ detects the unknot (Kronheimer-Mrowka)
- There are $K_1 \neq K_2$ s.t. $Kh(K_1) = Kh(K_2)$ (Watson)
- $Kh(K)$ is very computable
- Bar-Natan observed certain patterns:

$Kh(\ ;\ \mathbb{Q}) =$



Knight move conjecture (Bar-Natan)

$Kh(K;\mathbb{Q}) = \begin{pmatrix}\mathbb{Q}\\ \mathbb{Q}\end{pmatrix} \oplus$ many knight move pairs

Recently disproved by Manolescu-Marengon.

But still, there is some structure, which can be captured by "deformations" of Kh.

Short-term plan:

- study deformations of $Kh(K)$ [Lee, Bar-Natan]
- apply those to obtain lower bound for slice genus $\left|\frac{s(K)}{2}\right| \leq g_4(K)$

[Rasmussen]

Conjecture Jones poly detects the unknot.

Open question prove that Kh detects the unknot without gauge/Floer theory.

Bar-Natan's deformation

change of notation

$$X_+ \longmapsto 1$$

$$X_- \longmapsto X$$

$$V = \langle X_+, X_- \rangle \longmapsto \mathbb{Z}[X]/(X^2) = \langle 1, X \rangle$$

recall the merge map is multiplication in $\mathbb{Z}[X]/(X^2)$:

$$m: \mathbb{Z}[x]/(x^2) \otimes \mathbb{Z}[x]/(x^2) \longrightarrow \mathbb{Z}[x]/x^2$$

$$1 \otimes x \longmapsto x$$
$$x \otimes 1 \longmapsto x$$
$$1 \otimes 1 \longmapsto 1$$
$$x \otimes x \longmapsto 0$$

So now, the idea is to substitute $\mathbb{Z}[x]/x^2$ by $\boxed{\mathbb{Z}[H,x]/(x^2 - Hx)}$

$$\text{ll as } \mathbb{Z}\text{-module}$$

$$V = \langle 1, x \rangle_{\mathbb{Z}[H]}$$

Then the merge and split maps become:

$$V \otimes_{\mathbb{Z}[H]} V \xrightarrow{m} V$$

$$1 \otimes x \longmapsto x$$
$$x \otimes 1 \longmapsto x$$
$$1 \otimes 1 \longmapsto 1$$
$$x \otimes x \longmapsto Hx$$

deformation of the merge and split maps

$$V \xrightarrow{\Delta} V \otimes_{\mathbb{Z}[H]} V$$

$$1 \longmapsto 1 \otimes x + x \otimes 1 - H \cdot 1 \otimes 1$$
$$x \longmapsto x \otimes x$$

main idea for BN-deformation:

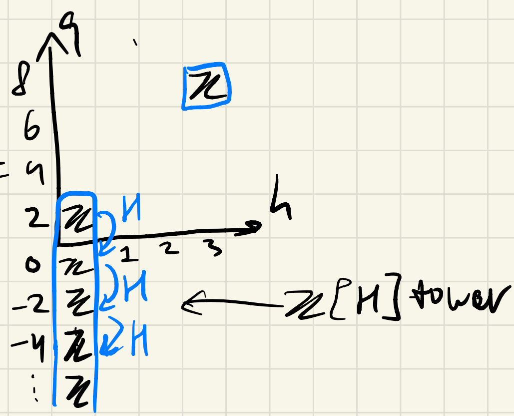
1) work over $\mathbb{Z}[H]$ instead of $\mathbb{Z}$
2) deform the maps as here

The invariance goes through, and we obtain a knot invariant

$$K \rightsquigarrow CBN(K) \overset{H_*}{\rightsquigarrow} BN(K)$$

$CBN(K)$: complex over $\mathbb{Z}[H]$ (so has $\mathbb{Z}[H]$-action on it)

$BN(K)$: coming with $\mathbb{Z}[H]$-action

Ex. $\widetilde{BN}\Big($  $\Big) =$ 

(pick only X on)