The earring correspondence on the pillowcase

(instantons + Bounding cochains)

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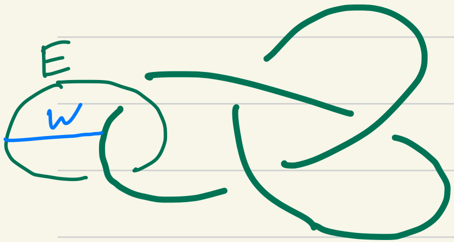
Joint with Guillem Cazassus, Chris Herald and Paul Kirk

Reduced singular instanton homology



$K \xrightarrow[\text{via singular connections}]{\text{(Kronheimer and Mrowka)}} I^\natural(K)$

on $P \xrightarrow{SO(3)} S^3 \setminus K \cup E$

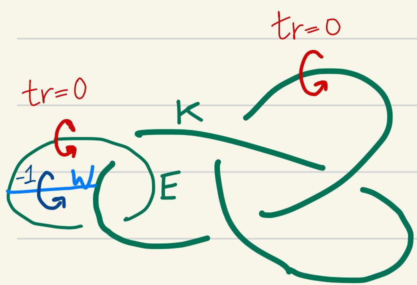
$(w_2(P) = PD(w)$ + $O(2)$-reduction & $\mathbb{Z}_2$-orbifold str. near $K \cup E)$

- Key facts $\left.\begin{array}{l} \widetilde{Kh}(mK) \rightrightarrows I^\natural(K) \\ I^\natural(K) \cong KHI(K) \end{array}\right\} \Rightarrow$ $\widetilde{Kh}$ detects the unknot

- From the viewpoint of representations

Traceless $SU(2)$-character variety satisfying w_2-condition

$$R(S^3, K\cup E, W) = \left\{ \begin{array}{ll} \text{representations} & \rho : \pi_1(S^3 \setminus K\cup E\cup W) \to SU(2) \\ \underline{\text{traceless}} & \mathrm{Tr}\,\rho(\mu_K) = \mathrm{Tr}\,\rho(\mu_E) = 0 \\ \underline{w_2\text{-condition}} & \rho(\mu_W) = -1 \end{array} \right\} \Big/ \text{conj.}$$



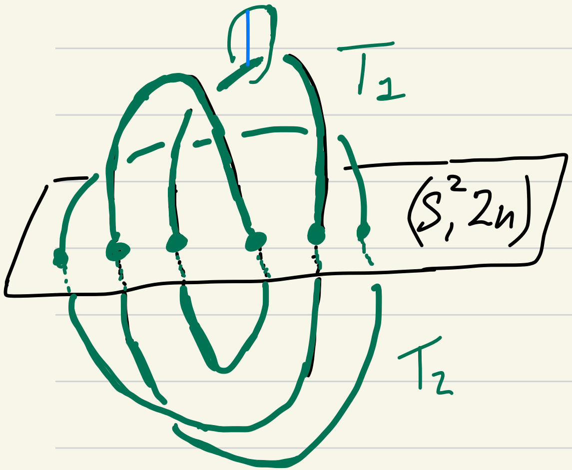
. notation $R^\natural(K) := R(S^3, K\cup E, W)$

. perturbed $R^\natural_\pi(K) := R_\pi(S^3, K\cup E, W)$

<u>key point</u> $CI^\natural(K)$ is generated by points $R^\natural_\pi(K)$

Atiyah-Floer conjecture

Given n-Bridge decomposition



$$
\begin{array}{ccc}
(D^3, T_1) & & R^{\natural}_{\pi}(T_1) \quad \sim \text{Lagrangian} \\
\uparrow & & \downarrow \\
(S^2, 2n) & \rightsquigarrow & R(S^2, 2n) \quad \sim \text{symplectic (ABG)} \\
\downarrow & & \uparrow \\
(D^3, T_2) & & R_{\pi}(T_2) \quad \sim \text{Lagrangian}
\end{array}
$$

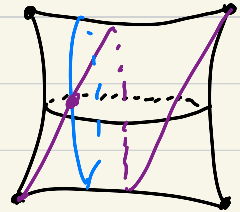
Conjecture $\quad HF\left(R^{\natural}_{\pi}(T_1), R_{\pi}(T_2)\right) \cong I^{\natural}(K)$

- $R(S^2, 2n)$ is stratified of top dim $= 4n-6$, Lagrangians as well

$n = 2 \Rightarrow 4n - 6 = 2$

Pillowcase homology (Hedden, Herald, Kirk)

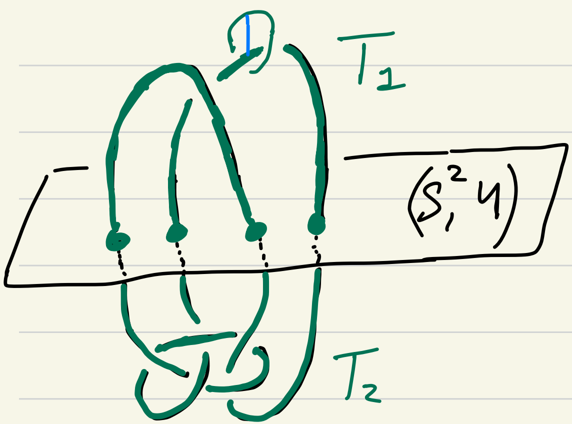
- $R(S^2, 4) = T/\mathbb{Z}/2 =$



$\leftarrow$ The pillowcase P

2-sphere with four $\mathbb{Z}/2$-orbifold singularities

- tangles need not be trivial

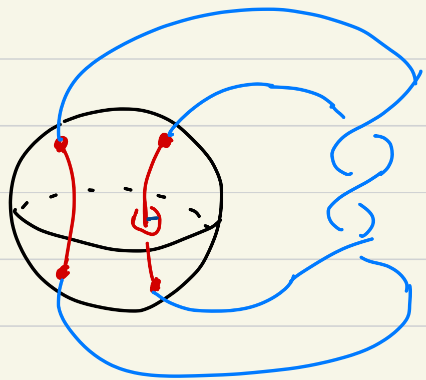


- $R_\pi^\natural(T_1)$ compact immersed curve in P (HK)
- $R_\pi(T_2)$ non-compact immersed curve in P

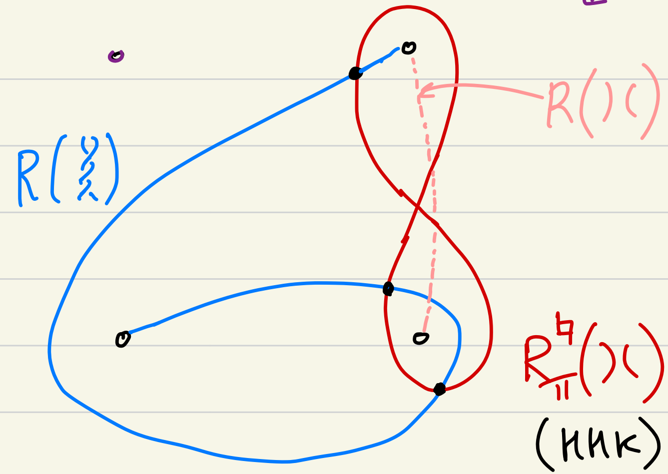
$\Rightarrow$ $HF(R_\pi^\natural(T_1), R_\pi(T_2))$ makes sense!

inside the smooth part $P^* \subset P$

Trefoil example



$\rightsquigarrow$



- $HF(R^\natural()(,\, R(\wr)) = \mathbb{F}^3 \cong I^\natural(\text{trefoil})$

- Generalizes to 2-bridge knots $I^\natural(K(p,q)) = \mathbb{F}^p$

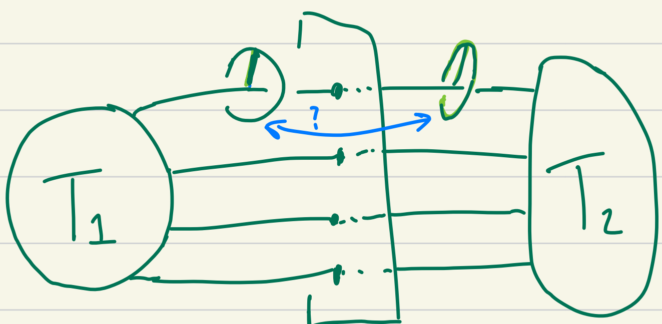
- Many other supporting computations, including $(4,5)$-torus knot

- Proving Pillowcase homology is well-defined
- $HF(R^{\natural}_{\pi}(T_1), R_{\pi}(T_2)) \overset{?}{\cong} I^{\natural}(K)$

} difficult for many reasons

- Each difficulty is an open-ended research direction
- We focus on <u>dependence on the earring location</u>

$$HF(R^{\natural}_{\pi}(T_1), R_{\pi}(T_2)) \overset{?}{\cong} HF(R_{\pi}(T_1), R^{\natural}_{\pi}(T_2))$$



Lagrangian correspondence (Weinstein)

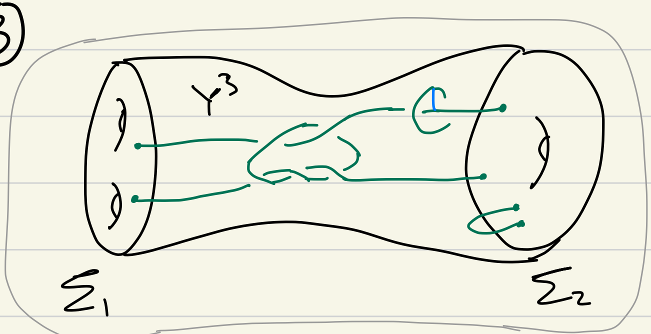
from (M,ω) to (N,ω) is an immersed Lagrangian

$$L \looparrowright M^- \times N \iff \begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ M & & N \end{tikzcd}$$

E.g. ① Lagrangian in N: $\quad pt \leftarrow L \rightarrow N$

② Diagonal $\quad M \leftarrow \Delta \rightarrow M$

③



tangle cobordism

$\overset{R}{\rightsquigarrow}$

$$\begin{tikzcd} & R^{\natural}_{\pi}(Y,T) \arrow[dl] \arrow[dr] & \\ R(\Sigma_1, 2) & & R(\Sigma_2, 4) \end{tikzcd}$$

(singular) Lagrangian correspondence

- Lag. corr. "transfers" Lagrangians by geometric composition

$$
\begin{tikzcd}
 & L \arrow[dl] \arrow[dr, "i"] & & L_{12} \arrow[dl, "j"'] \arrow[dr, "k"] & \\
pt & & M_1 & & M_2
\end{tikzcd}
\quad \overset{\text{intuitively}}{\rightsquigarrow} \quad
\begin{tikzcd}
L \arrow[dr, "k \circ j^{-1} \circ i"] & \\
 & M_2
\end{tikzcd}
$$

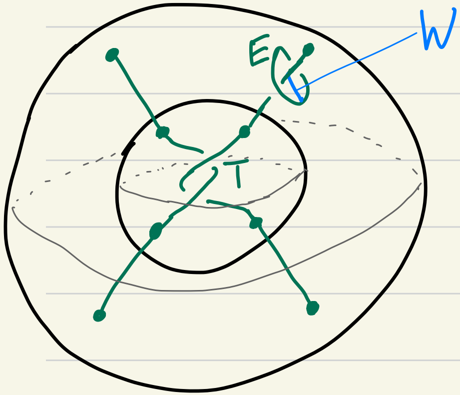
- Rigorously, in general, correspondences compose via fiber product

$$
L_{01} \circ L_{12} = \{ (\ell, \ell') \mid i(\ell) = j(\ell') \}
$$

$$
\begin{tikzcd}
 & & L_{01} \circ L_{12} \arrow[dl] \arrow[dr] & & \\
 & L_{01} \arrow[dl] \arrow[dr, "i"] & & L_{12} \arrow[dl, "j"'] \arrow[dr] & \\
M_0 & & M_1 & & M_2
\end{tikzcd}
$$

- Lagrangian
- Immersed if certain transversality assumption is met

Adding an earring $\Longleftrightarrow$ composing with Lagrangian corresp.



$$\begin{tikzcd}[column sep=small]
(P^3, T) & & (S^2\times I,\ 4pt\times I \cup E,\ W) & \\
& (S^2,4)^{in} \arrow[ul, hook] \arrow[ur, hook] & & (S^2,4)^{out} \arrow[ul, hook']
\end{tikzcd}$$

$$\Big\downarrow R$$

Denote $V = R^\natural(S^2\times I,\ 4pt\times I)$

$$R^\natural(T) \overset{*}{=} R(T)\circ V$$

$$\begin{tikzcd}
& R^\natural(T) \arrow[dl] \arrow[dr] & & \\
R(T) \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\
& P^{in} & & P^{out}
\end{tikzcd}$$

focus

Perturbations ($s \in \mathbb{R}$)

$$V_s = \{ \rho : \pi_1(S^2 \times I \setminus A \cup E \cup W \cup p \cup q) \longrightarrow SU(2) \} / \text{conj}$$

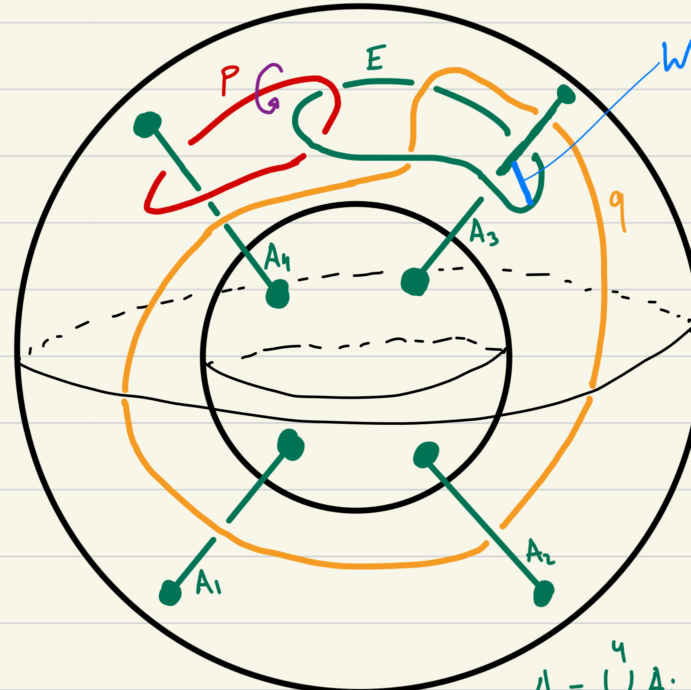
satisfying:

- traceless around green
- -1 around blue
- holonomy perturbed around p and q

$$\begin{cases} \rho(M_p) = e^{s \cdot \mathrm{Im}(\rho(\lambda_p))} \\ \rho(M_q) = e^{s \cdot \mathrm{Im}(\rho(\lambda_q))} \end{cases}$$

(* $\mathrm{Im}(a+bi+cj+dk) = bi+cj+dk$)

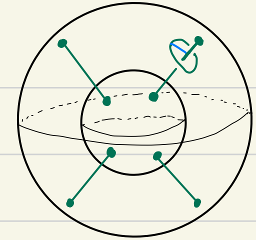
$s=0 \Rightarrow$ unperturbed



$A = \bigcup_{i=1}^{4} A_i$

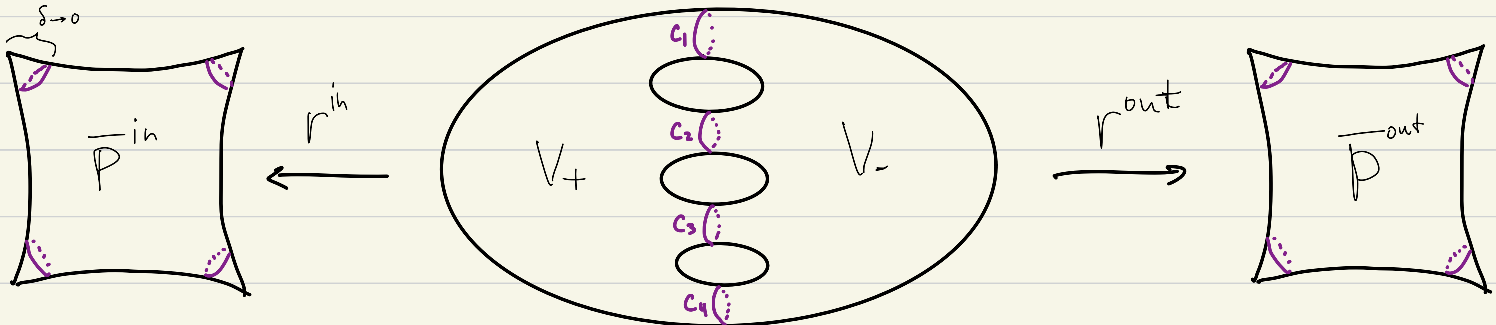
Theorem (Cazassus, Herald, Kirk, K)

$$P^{in} \xleftarrow{\ r^{in}\ } V_s \xrightarrow{\ r^{out}\ } P^{out}$$

$(S^{in}, 4) \hookrightarrow$  $\hookleftarrow (S^{out}, 4)$

1) V_s smooth genus 3 surface

2) (r^{in}, r^{out}) misses the corners, and is arbitrarily close to

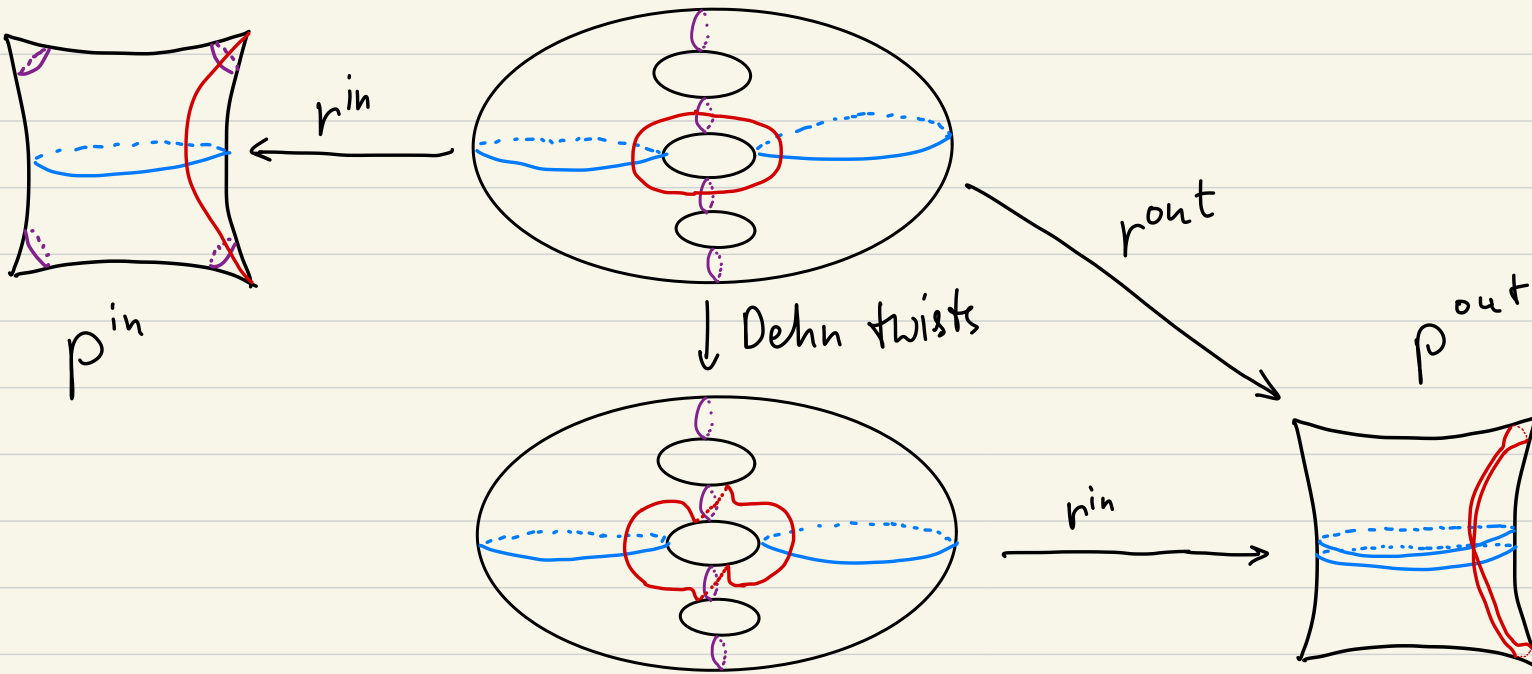


r^{in} bijectively sends $V_+ \to \bar{P}$, $V_- \to \bar{P}$

$r^{out} = r^{in} \circ$ (Dehn twists along all c_i)

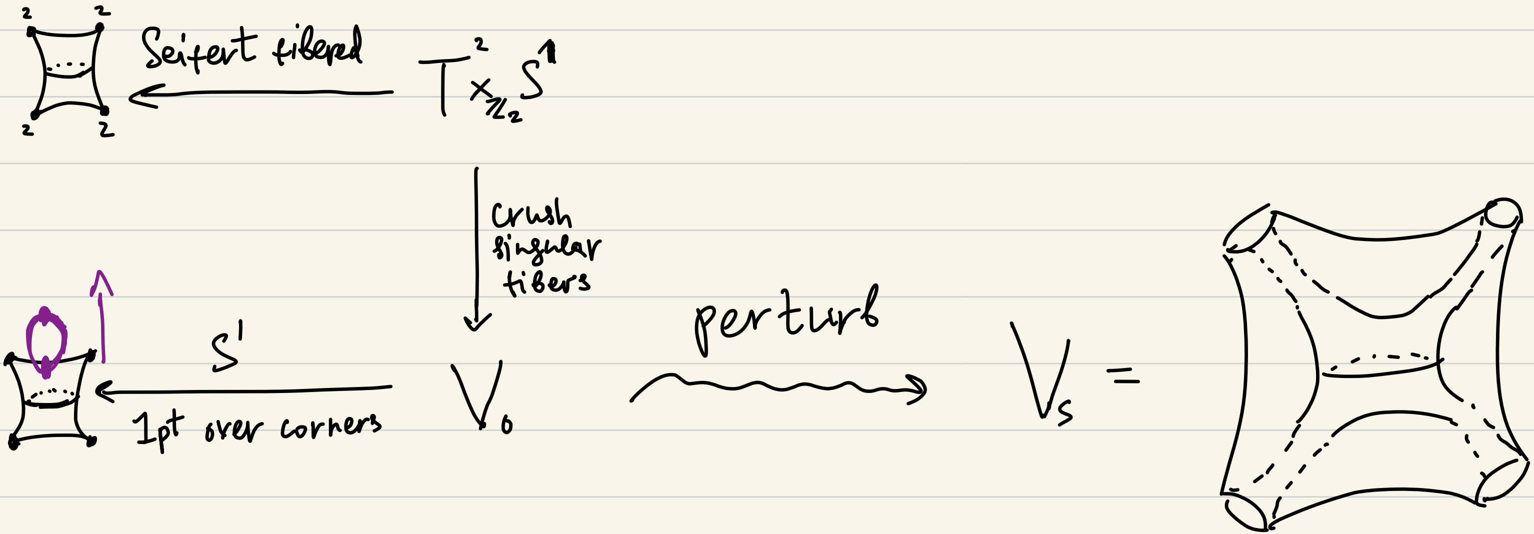
Action on curves

- doubles compact curves
- turns non-compact ones into figure eights



Remarks on the proof

- How perturbation works:



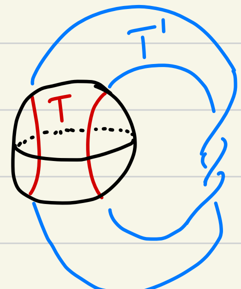
- [missing the corners] is the key step

(Thats why corners turn into circles)

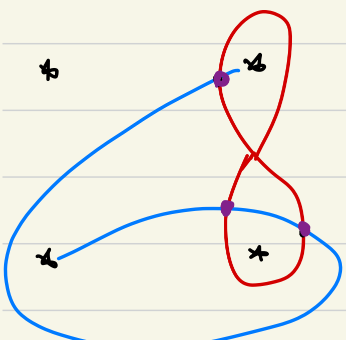
Insight into dependence on the earring location

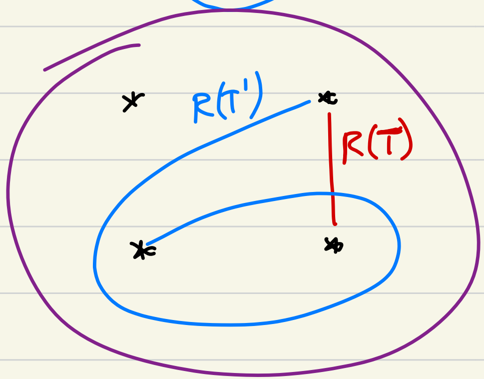
- On simple examples it works

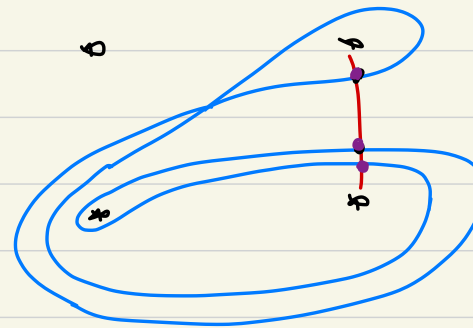
$$HF\left(R^\natural_\pi(T), R(T')\right) = \mathbb{F}^3$$



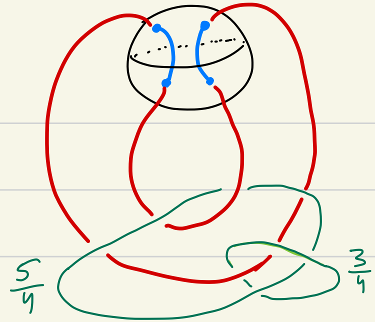
$$HF\left(R(T), R^\natural_\pi(T')\right) = \mathbb{F}^3$$







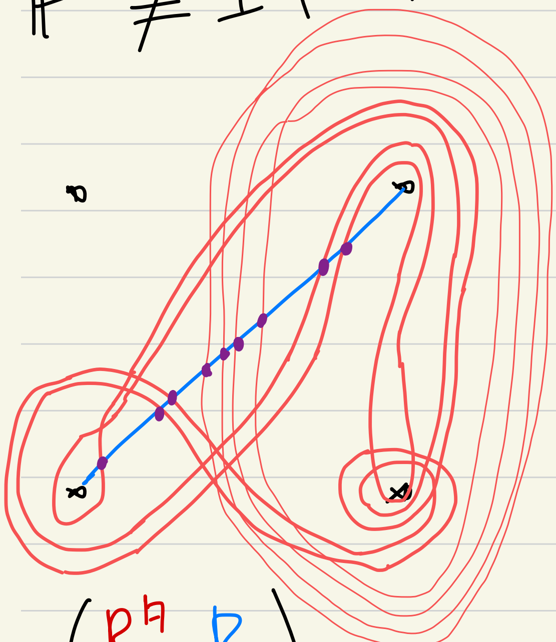
(4,5)-torus knot



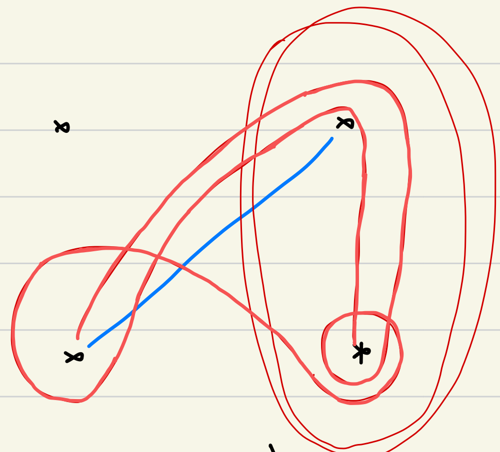
$\mathbb{F}^9 \not\cong I^4(T_{4,5})$

extra differential!

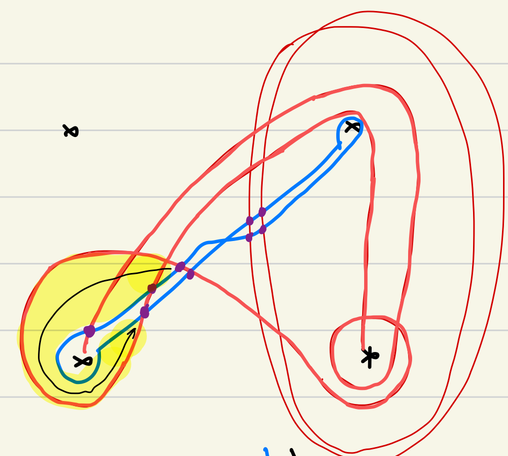
$\mathbb{F}^7 \cong I^4(T_{4,5})$



(R^4, R)



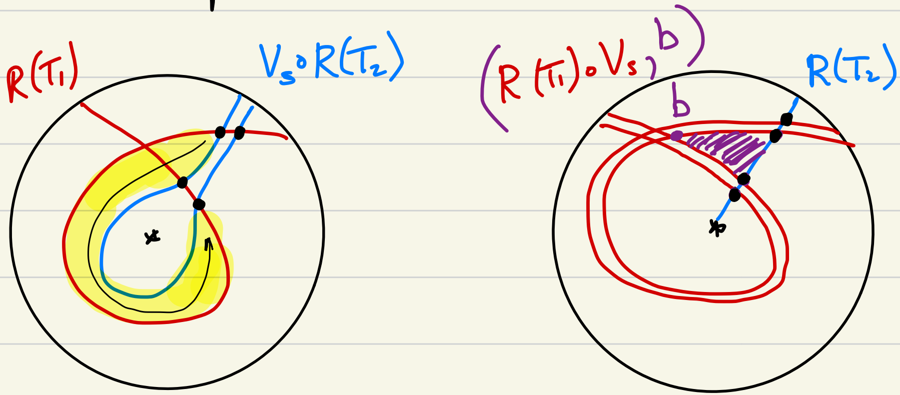
(R, R)



(R, R^4)

Q. What is the reason for discrepancy?

$$\begin{tikzcd} R(T_1) \arrow[dr] & & V_s \arrow[dl] \arrow[dr] & & R(T_2) \arrow[dl] \\ & P & & P & \end{tikzcd}$$

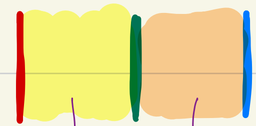


$b \in CF(R(T_1) \circ V_s)$

A. The described action of V_s on curves does not induce a well-defined functor $\mathcal{F}(V_s) : \mathcal{W}(P^*) \to \mathcal{W}(P^*)$.

Bounding cochains must be added.*

Q. Where does b come from?



Quilted Floer homology (Wehrheim-Woodward) $\quad HF_*\left(R(T_1) \rightarrow P \leftarrow V_s \rightarrow P \leftarrow R(T_2) \right)$

- Recovers both $HF(R(T_1), V_s \circ R(T_2))$ $\quad HF(R(T_1) \circ V_s, R(T_2))$ if everything embedded!

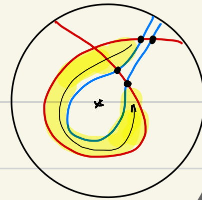
- Our case: everything immersed

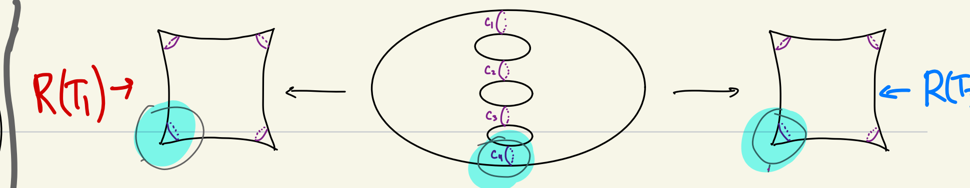
$\Longrightarrow$ **A.** figure eight bubbles produce b (Bottman-Wehrheim)

Rmk Fukaya has an alternative approach.

The bigon from

$CF\left(R(T_1),\ V_s \circ R(T_2)\right)$



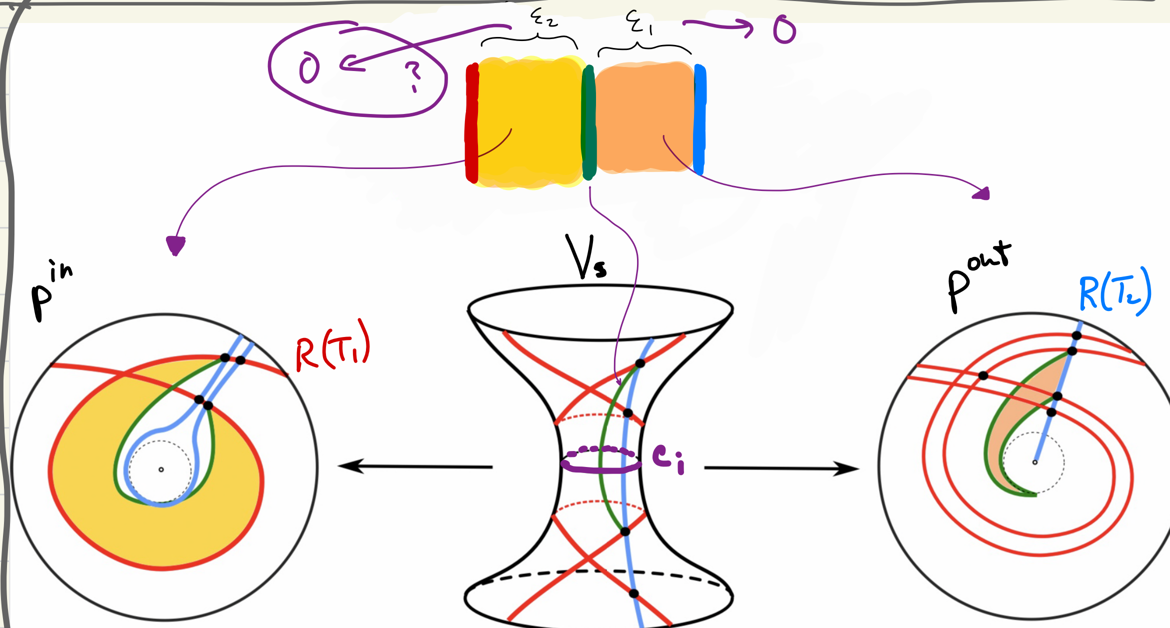


↓magnify↓

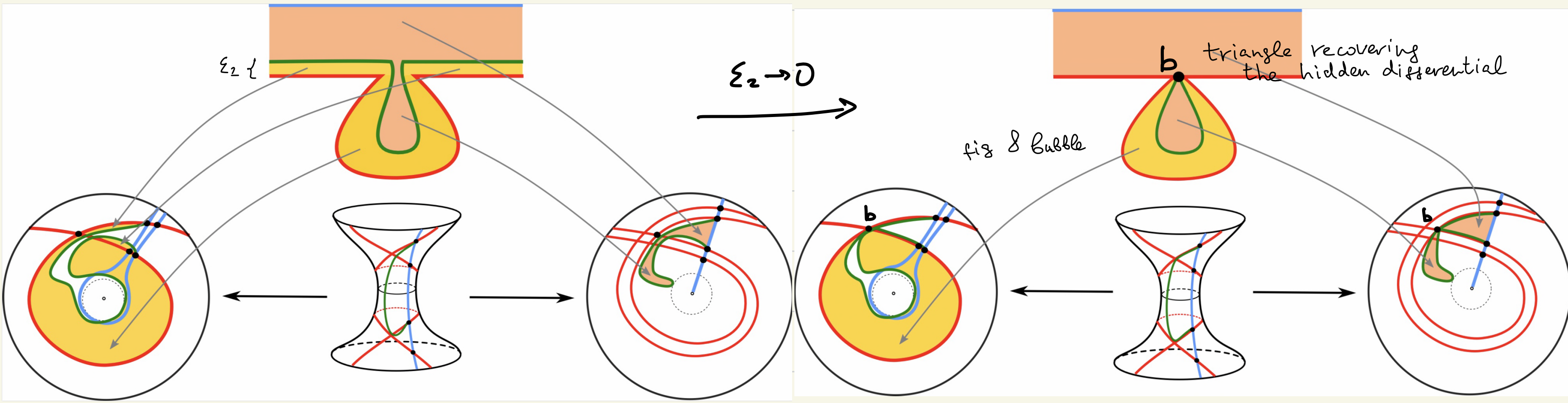
↑ $\varepsilon_1 \to 0$

$CF\left(R(T_1),\ V_s,\ R(T_2)\right)$

The corresponding quilt



$\varepsilon_2 \to 0$ limit has a figure eight bubble



- We identified the homotopy class of the bubble
- Pillowcase homology has to be upgraded
- Other bounding cochains must be added, in line with Floer field theory (Wehrheim-Woodward)