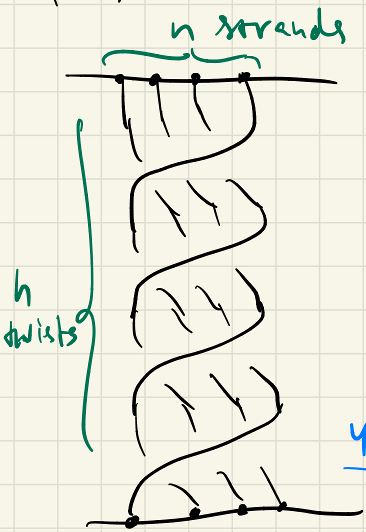
Lecture 4

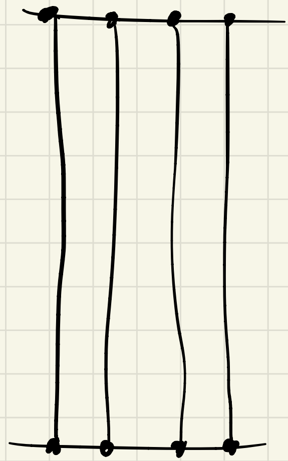
Hint for P3 $\left(u(T(p,q)) \leq \frac{(p-1)(q-1)}{2}\right)$:

First prove that one can unbraid the (n,n)-Braid in $\frac{n(n-1)}{2}$ crossing changes

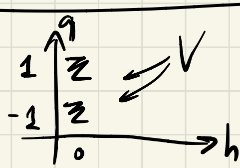
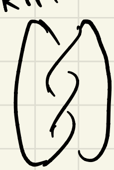
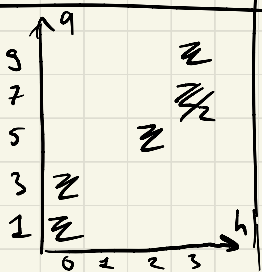


$\frac{n(n-1)}{2}$ crossing changes

4,4 Braid

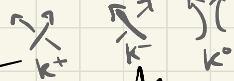


Recap of what we have so far:

	$Kh(K;\mathbb{Z})$	$P_{Kh}(q,h) = \sum_{i,j} (\mathrm{rk}\, Kh^{i,j}) h^i q^j$	$J_K \cdot (q+q^{-1})$
Unknot 		$h^0q^1 + h^0q^{-1} = q+q^{-1}$ $\xrightarrow{h=-1}$	$q+q^{-1}$
RHT 		$q+q^3+h^2q^5+h^3q^9$ $\xrightarrow{h=-1}$	$q+q^3+q^5-q^9$

$$\left(J_U(q) = 1 \, , \quad J_{RHT}(q) = q^2 + q^6 - q^8 \right)$$

Rmk $J_K(q)$ can also be defined by • $J_O = 1$ & oriented skein relation:

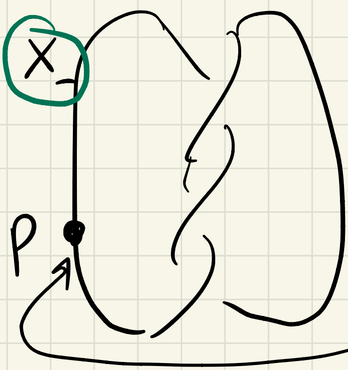


$(-t^{\frac{1}{2}} = q)$ • $t^{-1} J_{K^+} - t J_{K^-} + (t^{-1/2} - t^{1/2}) J_{K_0} = 0$

In other words, $\chi_h(Kh) = \text{Jones}$ is precisely the equality $P_{Kh}(q, -1) = J \cdot (q + q^{-1})$

Q. Is there a way to make this term disappear? Yes!

Reduced version $\widetilde{Kh}(K)$ (for knots)



construction of complex is the same except for each full resolution the circle containing p results not in $\langle X_-, X_+ \rangle_{\mathbb{Z}}$ but rather only $\langle X_- \rangle_{\mathbb{Z}}$, and we get a subcomplex

$$CKh_{X_-} \subset CKh$$

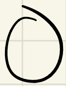
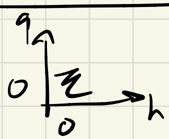
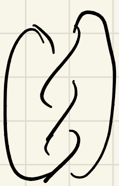
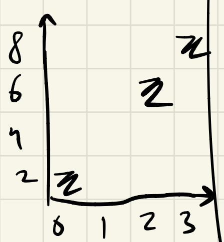
P9 Prove that the subcomplex CKh_{X_-} and the quotient complex CKh/CKh_{X_-} are isomorphic.

(this quotient is denoted by CKh_{X_+})

Rmk $0 \to CKh_- \to CKh \to CKh_+ \to 0 \Rightarrow$ LES $\to \widetilde{Kh} \to Kh \to \widetilde{Kh} \to$

def. $\widetilde{Kh}(K) := q^{1} H_*(CKh_{X_-})$, or

equivalently $\widetilde{Kh}(K) := q^{-1} H_*(CKh_{X_+})$

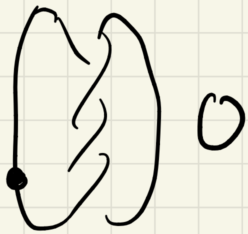
	$\widetilde{Kh}(K)$	$P(q,h) \xrightarrow{h=-1}$	J_K
		1	1
		$q^2+h^2q^6+h^3q^8$	$q^2+q^6-q^8$

- $\chi_h(\widetilde{Kh}(K)) = J_K$ (no $q+q^{-1}$ term!)
- No torsion in $\widetilde{Kh}(\mathrm{RHT})$, so independent

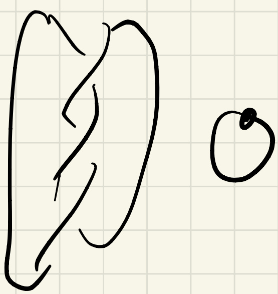
of the coefficients

<u>Q.</u> What happens for links?

It matters which component we reduce:

 $\rightsquigarrow$ $rk(\widetilde{Kh}) = 6$

(over anything)

 $\rightsquigarrow$ $rk(\widetilde{Kh})$ $= 4$ over $\mathbb{Q}$ or $\mathbb{Z}$; $= 6$ over $\mathbb{F}_2$

<u>Fact</u> over $\mathbb{F}_2$ it doesn't matter where to reduce.

Before proving invariance for Kh & $\widetilde{Kh}$ w.r.t. R-moves, we need to cover some algebra motivation working on chain level is better.

$\boxed{Mod^{\mathbb{Z}}}$ DG category of freely generated bigraded chain complexes

Objects free $\mathbb{Z}_q \oplus \mathbb{Z}_h$ finite dim. module over $\mathbb{Z}$, $\boxed{C}$, together with a differential $d: C^{q,h} \longrightarrow C^{q,h+1}$ s.t. $d^2=0$

Morphisms $f \in Mor((C,d),(C',d'))$ are $\mathbb{Z}$-module maps (homomorphisms of abelian groups), endowed with the differential

$$\boxed{D(f) = d' \circ f - (-1)^{h(f)} f \circ d}$$

by how much f raises the h-grading

$C \xrightarrow{f} C'$, $d \circlearrowright C$, $d' \circlearrowright C'$

$D: Mor \circlearrowleft$

Fact

a) Cycles in $Mor((C,d),(C',d'))$ are chain maps (say $h(f) \equiv 0$ mod 2)

b) Cycles mod Boundaries are chain maps up to homotopy
"$d' \circ f - f \circ d = 0$" $\quad Im D = \langle d' \circ h + h \circ d \rangle$

$\rightsquigarrow Ker D / Im D$, $D: Mor \to Mor$

P10 a) $D^2 = 0$

(do it over $\mathbb{F}_2$) b) $D(g \circ f) = D(g) \circ f + (-1)^{h(g)} g \circ D(f)$ (Leibnitz rule)

$C \xrightarrow{f} C' \xrightarrow{g} C''$

These two properties make $Mod^{\mathbb{Z}}$ a dg category

Given a dg category, passing to homology gives an ordinary category: $A \xrightarrow{(Mor, D)} B \rightsquigarrow A \xrightarrow{H_*(Mor,D)} B$

$H_*(Mod^{\mathbb{Z}})$ chain complexes together with chain maps up to homotopy $= H_*(Mor, D)$

Problem: we only want the <u>grading preserving</u> chain maps.

Solution: only pick what we need:

- $\boxed{H_0(\mathcal{M}od^{\mathbb{Z}})}$ chain complexes & bigrading preserving chain maps $\boxed{f: C \to C' \text{ s.t. } d' \circ f - f \circ d = 0}$ up to homotopy $\boxed{f \sim g \text{ is } f - g = d' h + h d}$

- Isomorphism classes of objects in $H_0(\mathcal{M}od^{\mathbb{Z}})$ (objects A & B are iso if $A \underset{g}{\overset{f}{\rightleftarrows}} B$ s.t. $f \circ g = id$, $g \circ f = id$) are presicely chain complexes up to $\underbrace{\text{homotopy equivalence}}_{C \underset{g}{\overset{f}{\rightleftarrows}} C',\ f \circ g \simeq id,\ g \circ f \simeq id}$ (Homology is just a convenient invariant of chain complexes up to homotopy equivalence)

Restatement of the theorem

The isomorphism class of object $CKh(K)$ inside $H_0(\mathcal{M}od^{\mathbb{Z}})$ is a knot invariant. (in other words, if D_0 & D_1 are diagrams of K, we have $CKh(D_0) \simeq CKh(D_1)$)

def *shifted* complex $h^n q^m (C, d)$ is defined to be $(h^n q^m C, (-1)^n d)$

shifted bigraded module

def given a bigrading preserving chain map $f : \begin{tikzcd} C \arrow[r] \arrow[loop below, "d"] & C' \arrow[loop below, "d'"] \end{tikzcd}$,

define the *mapping cone of f* to be the complex $\left(h^{-1}C \oplus C', \begin{pmatrix} d & f \\ 0 & d \end{pmatrix}\right)$

$$\left[\begin{tikzcd} h^{-1}C \arrow[r, "f"] \arrow[loop below, "-d"] & C' \arrow[loop below, "d'"] \end{tikzcd}\right]$$

$h^{-1}(C, d) =$

"$d^2 = 0$" $\Leftrightarrow \begin{cases} (-d)^2 = 0 ✓ \\ (d')^2 = 0 ✓ \\ d' \circ f - f \circ d = 0 \text{ since } f \text{ is a chain map} \end{cases}$

Proposition Given a complex of the form

subcomplex

$$\left[\begin{tikzcd} C \arrow[r, "f"] \arrow[loop below, "d"] & C' \arrow[loop below, "d'"] \end{tikzcd}\right],$$ suppose C' is

contractible, i.e. $C' \underset{\substack{\text{homot.}\\ \text{equiv.}}}{\simeq} 0$. ("cancellation of a contractible subcomplex")

Then
$$\left[\begin{tikzcd} C \arrow[loop below, "d"] \arrow[r, "f"] & C' \arrow[loop below, "d'"] \end{tikzcd}\right] \simeq \left[\begin{tikzcd} C \arrow[loop below, "d"] \end{tikzcd}\right]$$

$\square$ (working up to signs, i.e. let's say we substitute $\mathbb{Z}$ by $\mathbb{Z}/2$. Signs do work out for $\mathbb{Z}$)

- $(C',d') \simeq 0 \Rightarrow$
$$\begin{tikzcd} (C',d') \arrow[loop above, "h"] \arrow[r, bend left, "0"] & 0 \arrow[l, bend left, "0"] \end{tikzcd}$$
$$0 - \boxed{id = d'h + hd'}$$

In other words, $id : (C',d') \to (C',d')$ is null-homotopic

-
$$\begin{tikzcd} C' \arrow[loop right, "d'"] \arrow[rr] & & 0 \\ & & \\ C \arrow[loop right, "d"] \arrow[uu, "f"] \arrow[rr, "id'"] & & C \arrow[loop right, "d"] \arrow[lluu, "h\circ f"] \arrow[ll, bend left, "id"] \end{tikzcd}$$

I map

P map

One can check that I & P are chain maps, and $I \circ P \simeq id$
$P \circ I \simeq id$

$\blacksquare$

P11 prove that

$$\left[\underset{\circlearrowleft d}{C} \xrightarrow{f} \underset{\circlearrowleft d'}{C'} \right] \simeq C'_{\circlearrowleft d'}$$

if $(C, d) \simeq 0$ (i.e. (C,d) contractible)

Remark the above are chain level refinements of statements that

if $C' \subset C$ is a subcomplex, then

- if $C' \simeq 0$ then $H_*(C) \cong H_*(C/C')$
- if $C/C' \simeq 0$ then $H_*(C) \cong H_*(C')$

Which are easily proved by Long Exact Sequence for $0 \to C' \to C \to C/C' \to 0$

<u>Prop</u> $\left[\begin{array}{ccc} C & \xrightarrow{id} & C \\ \circlearrowleft & & \circlearrowleft \\ -d & & d \end{array}\right]$ is contractible. (i.e. $\simeq 0$)