$$P_{NLP} = \begin{cases} \min\limits_{\underset{\sim}{x}} \quad g_0(\underset{\sim}{x}) = x_1^2 + x_2^2 \\ \text{s.t.:} \quad g_1(\underset{\sim}{x}) = -x_1 - x_2 + 0{,}25 \\ \qquad 0 \leq x_1 \leq 1 \\ \qquad 0 \leq x_2 \leq 1 \end{cases}$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

	x_1	x_2
g_0	Linear, Taylor1	Linear, Taylor2
g_1	ConLin, Taylor2	MMA, Taylor1

Mixed Approximation Scheme

$n=2$	: # of variables
$m=2$	: # of responses (incl. objective)
n_sets $=2$	: # of variable sets
m_sets $=2$	: # of response sets
n_ℓ	: # of vars in set $-\ell-$
m_p	: # of responses in set $-p-$

$$P_{ji}^{(k)} = \left.\frac{\partial g_j}{\partial x_i}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left.\frac{\partial x_i}{\partial y_{ij}}\right|_{\underset{\sim}{y}^{(k)}}$$

$$Q_{ji}^{(k)} = \left.\frac{\partial^2 g_j}{\partial x_i^2}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left(\left.\frac{\partial x_i}{\partial y_{ij}}\right|_{\underset{\sim}{y}^{(k)}}\right)^2 + \left.\frac{\partial g_j}{\partial x_i}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left.\frac{\partial^2 x_i}{\partial y_{ij}^2}\right|_{\underset{\sim}{y}^{(k)}}$$

$\boxed{\text{Important sizes}}$

- $P_{ji}^{(k)}, Q_{ji}^{(k)} \longrightarrow [m_p, n_\ell]$
- $y_{ij} \longrightarrow [n_\ell, m_p]$

⊙ Useful formulas

global, passed to solver

- $$\tilde{g}_j^{(k)}(\underset{\sim}{x}) = g_j(\underset{\sim}{x}^{(k)}) + \sum_{\ell=1}^{n\_sets} \tilde{g}_{j\ell}^{(k)}(\underset{\sim}{x}) \qquad (I)$$

local, calculates the contribution of each variable set $-\ell-$ to (I), if Taylor2

- $$\tilde{g}_{j\ell}^{(k)}(\underset{\sim}{x}) = -\sum_{i=1}^{n_\ell} P_{ji}^{(k)} y_{ij}^{(k)} + \frac{1}{2}\sum_{i=1}^{n_\ell} Q_{ji}^{(k)} \left(y_{ij}^{(k)}\right)^2 + \sum_{i=1}^{n_\ell} P_{ji}^{(k)} \cdot y_{ij} - \sum_{i=1}^{n_\ell} Q_{ji}^{(k)} y_{ij}^{(k)} \cdot y_{ij} + \frac{1}{2}\sum_{i=1}^{n_\ell} Q_{ji}^{(k)} \cdot y_{ij}^2 \qquad (II_a)$$

- $$\tilde{g}_{j\ell}^{(k)}(\underset{\sim}{x}) = -\sum_{i=1}^{n_\ell} P_{ji}^{(k)} \cdot y_{ij}^{(k)} + \sum_{i=1}^{n_\ell} P_{ji}^{(k)} \cdot y_{ij} \qquad (II_b)$$

local, calculates the contribution of each variable set $-\ell-$ to (I), if Taylor1 is used

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- $$\frac{\partial \tilde{g}_j^{(k)}}{\partial x_i}(\underset{\sim}{x}) = P_{ji}^{(k)} \cdot \frac{\partial y_{ij}}{\partial x_i} - Q_{ji}^{(k)} y_{ij}^{(k)} \cdot \frac{\partial y_{ij}}{\partial x_i} + Q_{ji}^{(k)} \cdot y_{ij} \cdot \frac{\partial y_{ij}}{\partial x_i} \qquad (III_a)$$

Approximate sensitivities for the cases where Taylor2 is used for $\{x_i, g_j\}$

- $$\frac{\partial \tilde{g}_j^{(k)}}{\partial x_i}(\underset{\sim}{x}) = P_{ji}^{(k)} \cdot \frac{\partial y_{ij}}{\partial x_i} \qquad (III_b)$$

Approximate sensitivities for the cases where Taylor1 is used for $\{x_i, g_j\}$

- $$\frac{\partial^2 \tilde{g}_j^{(k)}}{\partial x_i^2}(\underset{\sim}{x}) = P_{ji}^{(k)} \cdot \frac{\partial^2 y_{ij}}{\partial x_i^2} - Q_{ji}^{(k)} y_{ij}^{(k)} \frac{\partial^2 y_{ij}}{\partial x_i^2} + Q_{ji}^{(k)} \cdot \frac{\partial y_{ij}}{\partial x_i} \cdot \frac{\partial y_{ij}}{\partial x_i} + Q_{ji}^{(k)} \, y_{ij} \, \frac{\partial^2 y_{ij}}{\partial y_i^2} \qquad (IV_a)$$

2^{nd}-order sensitivities of $\{x_i, g_j\}$, if Taylor2 is used

- $$\frac{\partial^2 \tilde{g}_j^{(k)}}{\partial x_i^2}(\underset{\sim}{x}) = P_{ji}^{(k)} \cdot \frac{\partial^2 y_{ij}}{\partial x_i^2} \qquad (IV_b)$$

2^{nd}-order sensitivities of $\{x_i, g_j\}$, if Taylor1 is used

⊙ Building the constituent approximations

$n_\ell = 1$ $\qquad$ $y_{ij} = x_i\,,\quad \frac{\partial x_i}{\partial y_{ij}} = 1$

$$\cdot\ \{g_0, x_1\} \xrightarrow[(\mathrm{II}b)]{\text{Linear, Taylor1}} \tilde{g}_{01}^{(k)}(\underset{\sim}{x}) = -\sum_{i=1}^{n_\ell} P_{ji}^{(k)} y_{ij}^{(k)} + \sum_{i=1}^{n_\ell} P_{ji}^{(k)} y_i = P_{ji}^{(k)}\left(y_{ij} - y_{ij}^{(k)}\right) = \frac{\partial g_0}{\partial x_1}\Big|_{\underset{\sim}{x}^{(k)}} \cdot \left(x_1 - x_1^{(k)}\right)$$

1st variable set

$$\cdot\ \{g_0, x_2\} \xrightarrow[(\mathrm{II}a)]{\text{Linear, Taylor2}} \tilde{g}_{02}^{(k)}(\underset{\sim}{x}) = \ \ldots\ (\mathrm{II}a) \text{ with } \begin{bmatrix} n_\ell = 1 \\ y_{ij} = x_i \\ \frac{\partial x_i}{\partial y_{ij}}\Big|_{y^{(k)}} = 1 \\ \frac{\partial^2 x_i}{\partial y_{ij}^2}\Big|_{y^{(k)}} = \emptyset \end{bmatrix} \ldots = \frac{\partial g_0}{\partial x_2}\Big|_{\underset{\sim}{x}^{(k)}} \left(x_2 - x_2^{(k)}\right) + \frac{1}{2}\frac{\partial^2 g_0}{\partial x_2^2}\Big|_{\underset{\sim}{x}^{(k)}} \cdot \left(x_2 - x_2^{(k)}\right)^2$$

2nd variable set

$$\cdot\ \{g_1, x_1\} \xrightarrow[(\mathrm{II}a)]{\text{ConLin, Taylor2}} \tilde{g}_{11}^{(k)}(\underset{\sim}{x}) = \ldots (\mathrm{II}a) \text{ with } \left[\begin{array}{l} n_\ell = 1 \\ y_{ij} = \begin{cases} x, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} \geq \emptyset \\ \frac{1}{x}, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} < \emptyset \end{cases} \xrightarrow{\frac{\partial g_1}{\partial x_1}\big|_{x^{(k)}} = -1 < \emptyset} y_{11} = \frac{1}{x_1} \\ \frac{\partial x_i}{\partial y_{ij}} = \begin{cases} 1, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} \geq \emptyset \\ -\frac{1}{y_{ij}^2}, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} < \emptyset \end{cases} \Longrightarrow \frac{\partial x_1}{\partial y_{11}}\Big|_{\underset{\sim}{y}^{(k)}} = -\frac{1}{\left(y_{11}^{(k)}\right)^2} = -\left(x_1^{(k)}\right)^2 \end{array}\right] =$$

1st variable set

$$= \underbrace{\frac{\partial g_1}{\partial x_1}\Big|_{\underset{\sim}{x}^{(k)}} \cdot x_1^{(k)}}_{-\sum_{i=1}^{n_\ell} P_{ji}^{(k)} \cdot y_{ij}^{(k)}} + \underbrace{\frac{1}{2}\left[\frac{\partial^2 g_1}{\partial x_1^2}\Big|_{\underset{\sim}{x}^{(k)}} \cdot \left(x_1^{(k)}\right)^4 + \frac{\partial g_1}{\partial x_1}\Big|_{\underset{\sim}{x}^{(k)}} \cdot 2\left(x_1^{(k)}\right)^3\right] \cdot \frac{1}{\left(x_1^{(k)}\right)^2}}_{+\frac{1}{2}\sum_{i=1}^{n_\ell} Q_{ji}^{(k)} \left(y_{ij}^{(k)}\right)^2} - \underbrace{\frac{\partial g_1}{\partial x_1}\Big|_{\underset{\sim}{x}^{(k)}} \left(x_1^{(k)}\right)^2 \cdot \frac{1}{x_1}}_{\sum_{i=1}^{n_\ell} P_{ji}^{(k)} y_{ij}} - \underbrace{\left[\frac{\partial^2 g_1}{\partial x_1^2}\Big|_{\underset{\sim}{x}^{(k)}} \cdot \left(x_1^{(k)}\right)^4 + \frac{\partial g_1}{\partial x_1}\Big|_{\underset{\sim}{x}^{(k)}} \cdot 2\left(x_1^{(k)}\right)^3\right] \cdot \frac{1}{x_1^{(k)}} \cdot \frac{1}{x_1}}_{-\sum_{i=1}^{n_\ell} Q_{ji}^{(k)} y_{ij}^{(k)} y_{ij}} +$$

$$+ \underbrace{\frac{1}{2}\left[\frac{\partial^2 g_1}{\partial x_1^2}\Big|_{\underset{\sim}{x}^{(k)}} \left(x_1^{(k)}\right)^4 + \frac{\partial g_1}{\partial x_1}\Big|_{\underset{\sim}{x}^{(k)}} \cdot 2\left(x_1^{(k)}\right)^3\right]\left(\frac{1}{x_1}\right)^2}_{\frac{1}{2}\sum_{i=1}^{n_\ell} Q_{ji}^{(k)} y_{ij}^2}$$

$$\cdot\ \{g_1, x_2\} \xrightarrow[(\mathrm{II}b)]{\text{MMA, Taylor1}} \tilde{g}_{12}^{(k)}(\underset{\sim}{x}) = \ldots (\mathrm{II}b) \text{ with } \left[\begin{array}{l} n_\ell = 1 \ \left(\begin{array}{l}\text{i.e. only } x_2 \text{ belongs to} \\ \text{variable set 2}\end{array}\right) \\ y_{ij} = \begin{cases} \frac{1}{x_i - L_i^{(k)}}, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} < \emptyset \\ \frac{1}{U_i^{(k)} - x_i}, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} \geq \emptyset \end{cases} \\ \frac{\partial x_i}{\partial y_{ij}} = \begin{cases} -\frac{1}{y_{ij}^2} = -\frac{1}{\left(x_i - L_i^{(k)}\right)^2}, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} < \emptyset \\ \frac{1}{y_{ij}^2} = \frac{1}{\left(U_i^{(k)} - x_i\right)^2}, & \text{if } \frac{\partial g_j}{\partial x_i}\Big|_{\underset{\sim}{x}^{(k)}} \geq \emptyset \end{cases} \end{array}\right]$$

$$= \underbrace{\frac{\partial g_1}{\partial x_2}\Big|_{\underset{\sim}{x}^{(k)}} \cdot \frac{1}{\left(x_2^{(k)} - L_2^{(k)}\right)^3}}_{-\sum_{i=1}^{n_\ell} P_{ji}^{(k)} \cdot y_{ij}^{(k)}} - \underbrace{\frac{\partial g_1}{\partial x_2}\Big|_{\underset{\sim}{x}^{(k)}} \cdot \frac{1}{\left(x_2^{(k)} - L_2^{(k)}\right)^2} \cdot \frac{1}{\left(x_2 - L_2^{(k)}\right)}}_{\sum_{i=1}^{n_\ell} P_{ji}^{(k)} y_{ij}}$$

• $\{g_0, x_1\} \xrightarrow[(\mathrm{III}b)]{\text{Linear, Taylor1}} \dfrac{\partial \tilde{g}_0^{(k)}}{\partial x_1}(\underset{\sim}{x}) = \dots (\mathrm{III}b)$ with $\begin{bmatrix} y_{ij} = x_i \\ \dfrac{\partial y_{ij}}{\partial x_i} = 1 \\ \dfrac{\partial x_i}{\partial y_{ij}} = 1 \end{bmatrix} \dots = \left.\dfrac{\partial g_0}{\partial x_1}\right|_{\underset{\sim}{x}^{(k)}}$.

• $\{g_0, x_2\} \xrightarrow[(\mathrm{III}a)]{\text{Linear, Taylor2}} \dfrac{\partial \tilde{g}_0^{(k)}}{\partial x_2}(\underset{\sim}{x}) = \dots (\mathrm{III}a)$ with $\begin{bmatrix} y_{ij} = x_i \\ \dfrac{\partial y_{ij}}{\partial x_i} = 1 \\ \left.\dfrac{\partial x_i}{\partial y_{ij}}\right|_{y^{(k)}} = 1 \\ \left.\dfrac{\partial^2 x_i}{\partial y_{ij}^2}\right|_{y^{(k)}} = \emptyset \end{bmatrix} \dots = \left.\dfrac{\partial g_0}{\partial x_2}\right|_{\underset{\sim}{x}^{(k)}} + \left.\dfrac{\partial^2 g_0}{\partial x_2^2}\right|_{\underset{\sim}{x}^{(k)}} \cdot (x_2 - x_2^{(k)})$.

• $\{g_1, x_1\} \xrightarrow[(\mathrm{III}a)]{\text{ConLin, Taylor2}} \dfrac{\partial \tilde{g}_1^{(k)}}{\partial x_1}(\underset{\sim}{x}) = \dots (\mathrm{III}a)$ with $\begin{bmatrix} y_{ij} = y_{11} = \dfrac{1}{x_1} \quad \left(\text{cause } \left.\dfrac{\partial g_1}{\partial x_1}\right|_{\underset{\sim}{x}^{(k)}} = -1 < \emptyset\right) \\ \dfrac{\partial y_{11}}{\partial x_1} = -\dfrac{1}{x_1^2} \\ \left.\dfrac{\partial x_1}{\partial y_{11}}\right|_{\underset{\sim}{y}^{(k)}} = -\dfrac{1}{y_{11}^2} = (x_1^{(k)})^2 \\ \left.\dfrac{\partial^2 x_1}{\partial y_{11}^2}\right|_{\underset{\sim}{y}^{(k)}} = \dfrac{2}{(y_{11}^{(k)})^3} = 2(x_1^{(k)})^3 \end{bmatrix} \dots =$

$$= \underbrace{\left.\frac{\partial g_1}{\partial x_1}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left.\frac{\partial x_1}{\partial y_{11}}\right|_{\underset{\sim}{y}^{(k)}} \cdot \frac{\partial y_{11}}{\partial x_1}}_{P_{ji}^{(k)} \cdot \frac{\partial y_{ij}}{\partial x_i}} - \underbrace{\left[\left.\frac{\partial^2 g_1}{\partial x_1^2}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left(\left.\frac{\partial x_1}{\partial y_{11}}\right|_{\underset{\sim}{y}^{(k)}}\right)^2 + \left.\frac{\partial g_1}{\partial x_1}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left.\frac{\partial^2 x_1}{\partial y_{11}^2}\right|_{\underset{\sim}{y}^{(k)}}\right] \cdot y_{11}^{(k)} \cdot \frac{\partial y_{11}}{\partial x_1}}_{-Q_{ji}^{(k)} y_{ij}^{(k)} y_{ij}} +$$

$$+ \underbrace{\left[\left.\frac{\partial^2 g_1}{\partial x_1^2}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left(\left.\frac{\partial x_1}{\partial y_{11}}\right|_{\underset{\sim}{y}^{(k)}}\right)^2 + \left.\frac{\partial g_1}{\partial x_1}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left.\frac{\partial^2 x_1}{\partial y_{11}^2}\right|_{\underset{\sim}{y}^{(k)}}\right] \cdot y_{11} \cdot \frac{\partial y_{11}}{\partial x_1}}_{Q_{ji}^{(k)} y_{ij} \frac{\partial y_{ij}}{\partial x_i}}.$$

• $\{g_1, x_2\} \xrightarrow[(\mathrm{III}b)]{\text{MMA, Taylor1}} \dfrac{\partial \tilde{g}_1^{(k)}}{\partial x_2}(\underset{\sim}{x}) = \dots (\mathrm{III}b)$ with $\begin{bmatrix} y_{ij} = y_{21} = \dfrac{1}{x_2 - L_2^{(k)}} \quad \left(\text{cause } \left.\dfrac{\partial g_1}{\partial x_2}\right|_{\underset{\sim}{x}^{(k)}} = -1 < \emptyset\right) \\ \dfrac{\partial y_{21}}{\partial x_2} = -\dfrac{1}{(x_2 - L_2^{(k)})^2}, \quad \left.\dfrac{\partial x_2}{\partial y_{21}}\right|_{\underset{\sim}{y}^{(k)}} = -\dfrac{1}{(y_{21}^{(k)})^2} = -(x_2^{(k)} - L_2^{(k)})^2 \\ \left.\dfrac{\partial^2 x_2}{\partial y_{21}^2}\right|_{\underset{\sim}{y}^{(k)}} = \dfrac{2}{(y_{21}^{(k)})^3} = 2(x_2^{(k)} - L_2^{(k)})^3 \end{bmatrix} \dots =$

$$= \underbrace{\left.\frac{\partial g_1}{\partial x_2}\right|_{\underset{\sim}{x}^{(k)}} \cdot \left.\frac{\partial x_2}{\partial y_{21}}\right|_{\underset{\sim}{y}^{(k)}} \cdot \left(-\frac{1}{(x_2 - L_2^{(k)})^2}\right)}_{P_{ji}^{(k)} \cdot \frac{\partial y_{ij}}{\partial x_i}}.$$