Discrete Maths, Homework 13

Problem 1

Prove that it is possible to enumerate vertices of a connected non-oriented graph on n vertices in with numbers from 1 to n in such a way that for any $1 \leqslant k \leqslant n$ the subgraph induced by the set of vertices from 1 to k is connected.

First, build a spanning tree of a graph. Since the graph is connected, it would always exist.

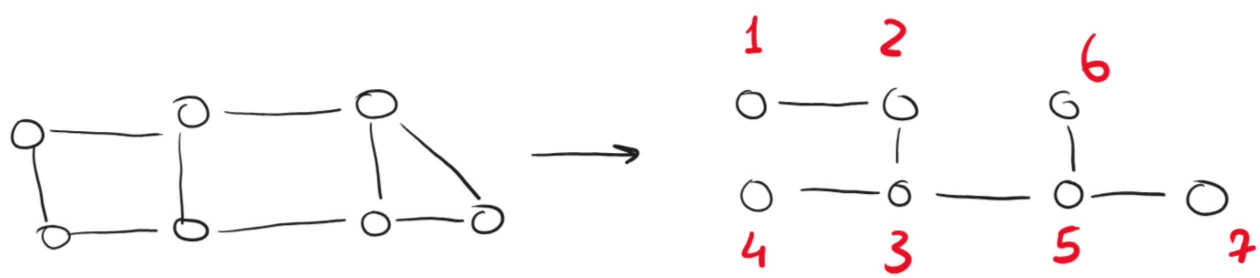
Next, take some leaf (hanging vertex) of the tree and assign the number 1 to it.

Then, move to any adjacent vertex and assign it the number i (which would be connected to the tree formed by $1, 2, \ldots, i-1$ vertices).

> A vertex is **adjacent** if there is a bridge between such vertex and the current tree.

Repeat the process until there are no more vertices.

Illustration of the above process:



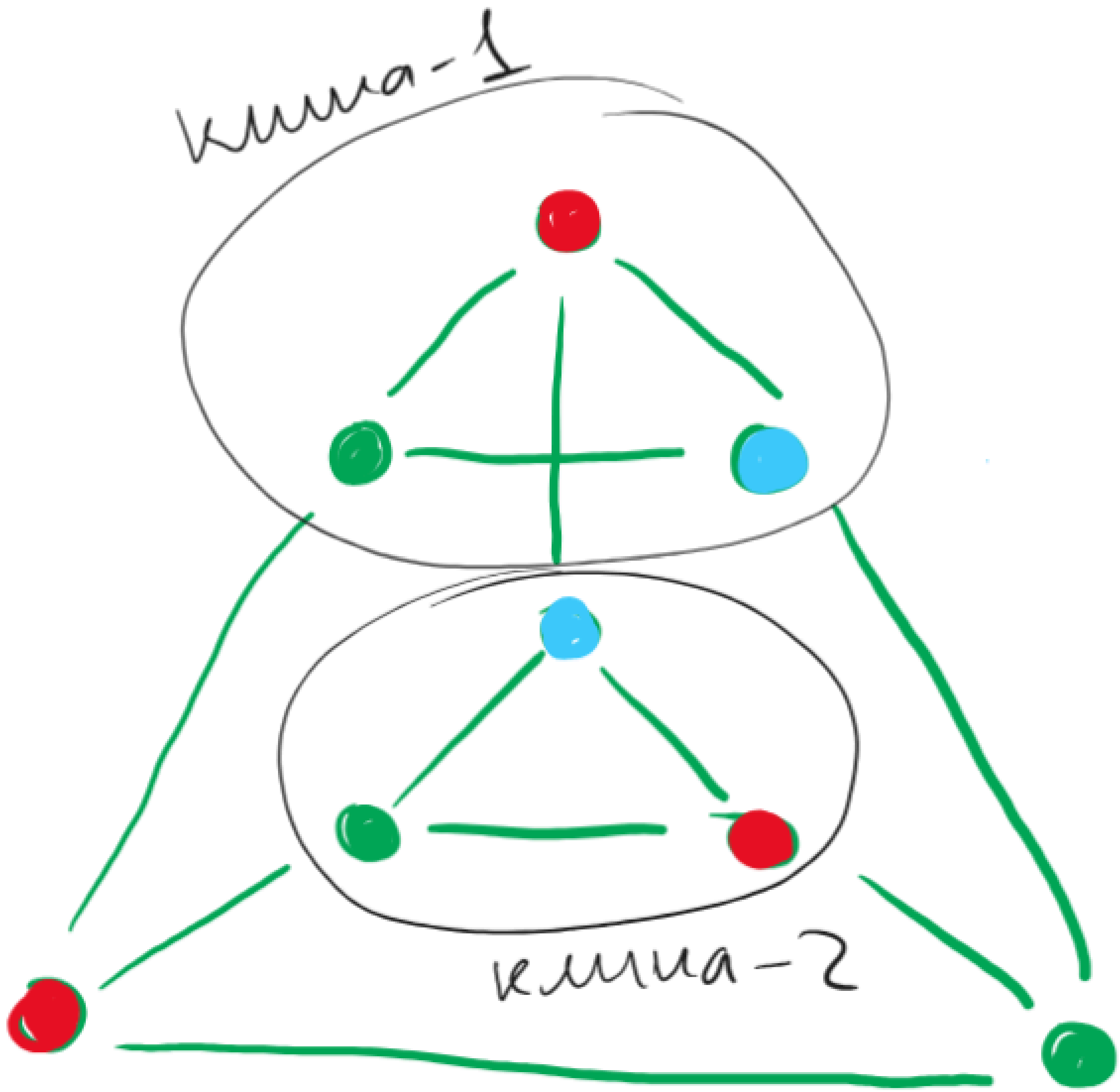
$1 \to (1,2) \to (1,2,3) \to \cdots \to (1,2,3,4,5,6,7)$. All graphs induced by those vertices and edges that exist between them in the spanning tree would be connected since we always add a new vertex that would be connected to the already-existing tree.

Final step: restore all edges in the original graph (which does not affect the connectivity of the graph), thus all subgraphs induced by $(1, \ldots, k)$ vertices would be connected, q. e. d.

Problem 2

Find such a graph on 8 vertices that the degree of each vertex is equal to 3 and there are no independent sets of size 4 in the graph.

Consider the following graph:



There are two cliques of size 3 in the graph, which means that the graph is at least 3-colorable. To try and find an independent set in the graph, we would have to take one vertex from each of the cliques. Other 4 vertices in the cliques couldn't be chosen since they are dependent on other vertices in the clique, thus excluding $4+2=6$ vertices from the graph.

The remaining bottom (see picture) vertices form a clique of size 2, thus making them dependent. Since there are three independent cliques, the maximal independent set consists of 3 vertices. (There are no more than 3 vertices of each color!), q. e. d.

Problem 3

It is known that in a simple non-oriented graph there is an odd number of independent sets. Does it follow that the graph is connected?

Consider connected graph G_1 that has n independent sets. Consider another connected graph G_2 that has k connected sets.

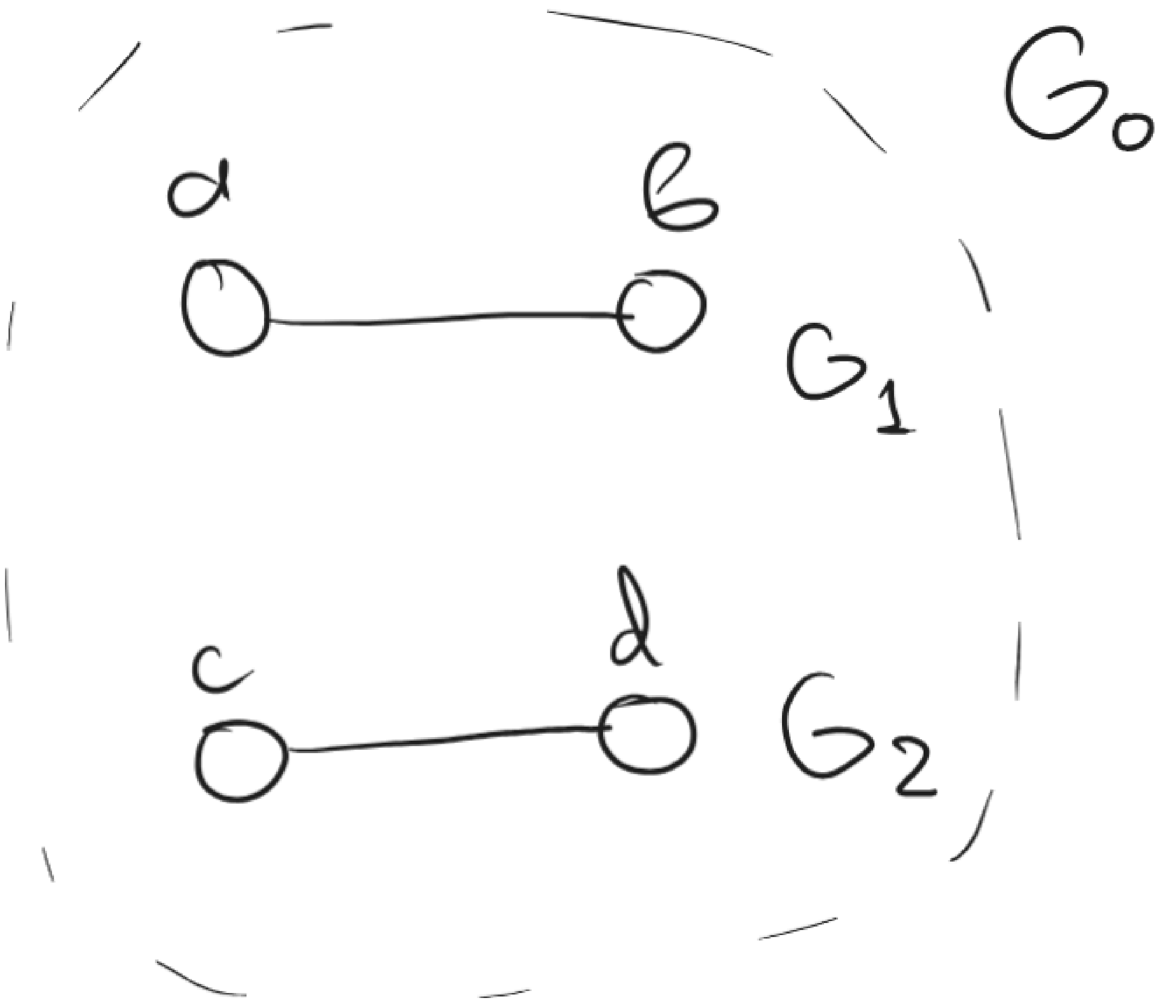
If we merge these graphs into a single one G_0 (not adding any new edges), then to get the number of independent sets in the new graph, we do the following:

```
independent_sets = 0
for every independent_set in G_2: # total k iterations
    # we add each independent set from G_2 to
    # each set from G_2, thus getting n new
    # independent sets each iteration
    independent_sets += n
```

Therefore, the total number of independent sets after merging two disconnected graphs is $n \times k$. Try to find such an example so that $n \times k$ is not even $\implies$ neither n, nor k can be even $\implies$ they both are odd.

Consider two graphs and their independent sets as a counterexample:

- $G_1 \colon \{\varnothing\}, \{a\}, \{b\}$
- $G_2 \colon \{\varnothing\}, \{c\}, \{d\}$



Per our reasoning, the resulting graph G_0 would have $3 \times 3 = 9$ independent sets. We could list them all:

$$\begin{aligned} G_0 \colon & \{\varnothing\}, \{a\}, \{b\} \\ & \{c\}, \{a, c\}, \{b, c\} \\ & \{d\}, \{a, d\}, \{b, d\} \end{aligned}$$

This graph is not connected $\implies$ no, it does not follow, q. e. d.

Problem 4

For which n is there a spanning tree in a boolean cube Q_n, in which all vertices except for 2 have a degree of 2?

For a spanning tree to have all vertices except for 2 have a degree of 2, we need to just find a path that would go through all vertices of the cube (every vertex except for the beginning and the end would have degrees of 2).

Denote each vertex of the cube as follows:

$$\underbrace{010101110001\ldots0101}_{n}$$

Where each n-th 1 in each of bits denotes that the vertex has gone into the n-th dimension and each 0, respectively, denotes that we have not shifted into the according dimension.

Now, go through all words as follows, starting from the 0-th vertex (that has all zeros). Consider an example for 4 dimensions. It could be expanded iteratively to higher dimensions:

$$\begin{gathered}0000 \to 0001 \to 0011 \to 0010 \to \\ 0110 \to 0111 \to 0101 \to 0100 \to \\ 1100 \to 1101 \to 1111 \to 1110 \to \\ 1010 \to 1011 \to 1001 \to 1000\end{gathered}$$

This algorithm can be described as follows:

- start in the 0-th dimension at the only vertex $\underbrace{0\ldots0}_{n}$
- clone this dimension, create a bridge between the last visited vertex in the original dimension and the respective vertex to the last visited in the new dimension
- visit all vertexes in the cloned dimension in the reverse order
- repeat until you run out of dimensions to go into

The first boolean cube with at least 2 vertices is a 1-dimensional cube ($2^1 = 2$). All vertices except for two (so, zero in total) have a degree of 2, thus this statement is valid for such a cube and for all cubes of higher dimensions $(n \geqslant 1)$

Answer: $n \geqslant 1$.