Problem 4.1

Function f from set X to set Y is such that for $A \subseteq X, B \subseteq X$, the following is true:

$$f^{-1}[f[A]] = f^{-1}[f[B]]$$

Does the equation $A = B$ follow out of this?

Solution

Since $f^{-1}[f[A]] = f^{-1}[f[B]]$, then images of A, B should be equal: $f[A] = f[B]$. Find a counterexample: f is a function that returns modulo 2. Let $A = \{2\}$, $B = \{4\}$. $f[A] = f[B] = \{0\} \Rightarrow f^{-1}[f[A]] = f^{-1}[f[B]]$, but $A \neq B$. Therefore, no, the equation does not follow.

Answer: no :(

Problem 4.2

Function f is defined on the set $A \cup B$ and takes arguments from set Y. If one were to replace the symbol ? in the following equation:

$$f[A \triangle B] \ ? \ f[A] \triangle f[B]$$

with one of the symbols $\subseteq$, $\supseteq$, they would get a statement. What of these two statements are true for any f?

Solution

Let A_0, B_0 be sets of all values which are included in A but not in B and vice versa. Let $X = A \cap B$.

Rewrite the equation:

$$f[A \triangle B] \ ? \ f[A] \triangle f[B] = f[A_0 \cup X \triangle B_0 \cup X] \ ? \ f[A_0 \cup X] \triangle f[B_0 \cup X] =$$

$$= f[A_0 \cup B_0] \ ? \ f[A_0 \cup X] \triangle f[B_0 \cup X] = f[A_0] \cup f[B_0] \ ? \ f[A_0] \cup f[X] \triangle f[B_0] \cup f[X]$$

Now formalize. The first set is a union of images of A_0, B_0. The second set is a symmetric difference of two unions: one of images of A_0, X and other of images of B_0, X.

Counterexample: It is possible for values from the image of X to be present in the image of

A_0 or B_0. In this case, $f[A] \triangle f[B]$ may exclude some values from $f[A_0]$ or $f[B_0]$, which fall in both $f[A_0]$ and $f[X]$ as well as in both $f[B_0]$ and $f[X]$. Since all other remaining values would be a part of either $f[A_0]$ or $f[B_0]$, the righthand side of the equation would be a subset of the lefthand side of the equation, and not vice versa.

Answer: no ($\subseteq$) :(/ yes ($\supseteq$) :)

Problem 4.3

Does a surjection f from the set of words of length 9 in the alphabet $\{0, 1\}$ to the set of words of length 3 in the alphabet $\{0, 1, 2, 3, 4\}$, for which the preimage of the set

$$\{(0, 0, 0), (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4)\}$$

has a cardinarity of 400, exist?

Solution

Cardinality of the first set is $2^9 = 512$. Cardinality of the second set is $5^3 = 125$. Per the original condition, 400 values from the first set are mapped to 5 values in the second set. It is required to map the remaining $512 - 400 = 112$ values from the first set to $125 - 5 = 120$ values from the second set.

Is it possible to map the values in such a way the function would remain a surjection? Per the surjection definition, for each value in the second set there should be at least one value in the first one, which is not true $\Rightarrow$ no.

Answer: no :(

Problem 4.4

How many 6-digit numbers in which there is an equal number of odd and even digits are there?

There should be 3 odd and even numbers each.

There are $\binom{6}{3}$ position combinations of the even digits.

Then, there are 5 ways to fill each of the 3 even slots and 5 ways to fill each of the 3 odd slots. Taking into account that a ninth of the numbers would have a leading zero, the final answer would be:

$$\frac{9}{10}\binom{6}{3} \times 5^3 \times 5^3 = \frac{9 \times 6! \times 5^6}{10 \times 3! \times 3!} = \frac{2^4 \times 3^4 \times 5^7}{2^3 \times 3^2 \times 5} = 2 \times 3^2 \times 5^6 = 281250$$

Answer: 281250