Problem 1

Subproblem A

Show that $\lim_{n\to\infty}\frac{n}{n+1}\neq 2$.

Let's suppose that $\lim_{n\to\infty}\frac{n}{n+1}=2$ and try to find a contradiction:

$$\lim_{n\to\infty}\frac{n}{n+1}=2\Rightarrow\lim_{n\to\infty}(\frac{n}{n+1}-2)=0\Rightarrow\lim_{n\to\infty}(\frac{(n+1)-1}{n+1}-2)=0\Rightarrow$$

$$\lim_{n\to\infty}(1-\frac{1}{n+1}-2)=0\Rightarrow\lim_{n\to\infty}(-\frac{1}{n+1}-1)=0$$

Since $-\frac{1}{n+1}<0$ and $-1<0$, then the sum of these components cannot physically be equal to 0 . Therefore, $\lim_{n\to\infty}\frac{n}{n+1}\neq 2$, q. e. d.

Afterwards, show that $\lim_{n\to\infty}\frac{n}{n+1}=1$. Similarly as above:

$$\lim_{n\to\infty}\left(1-\frac{1}{n+1}-1\right)=\lim_{n\to\infty}\left(-\frac{1}{n+1}\right)$$

Therefore, as per the limit definition, $\forall\varepsilon>0,\exists N\in\mathbb{N}\colon\forall n>N,|x_n-x|<\varepsilon$. For any ε:

$$\left|-\frac{1}{n+1}\right|<\varepsilon\Rightarrow\frac{1}{n+1}<\varepsilon$$

Since $n+1>n$, $\frac{1}{n+1}<\frac{1}{n}$:

$$\frac{1}{n+1}<\frac{1}{n}<\varepsilon$$

Now $\forall n>N$, we can find some $N=\left[\frac{1}{\varepsilon}\right]$, which means that $\forall n>N$, $\left|-\frac{1}{n+1}\right|<\varepsilon$, q. e. d.

Show after which number all elements of the sequence would fall into the $(0.99,1.01)$ interval. In this interval, $\varepsilon=0.01$. As per the equation above,

$$N=\left[\frac{1}{\varepsilon}\right]=\left[\frac{1}{0.01}\right]=100$$

Using Python for calculations, we may check that, truly, starting from the 100th element, all elements of the sequence fall within ε of the limit:

```
98 0.98989898989899
99 0.99
100 0.9900990099009901
101 0.9901960784313726
102 0.9902912621359223
103 0.9903846153846154
104 0.9904761904761905
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Answer: 100

Subproblem B

Assume sequence $\{x_n\}$, where $x_n = \frac{5n^2+10n-3}{8n^2+n+2}$. Show that $\lim_{n\to\infty} x_n \neq 0$.

Similarly as above, assume that $\lim_{n\to\infty} x_n = 0$ and try to find such ε for which there would not be a number after which an infinite number of elements would be less than ε.

Say that $\varepsilon = 0.5$, then $\left|\frac{5n^2+10n-3}{8n^2+n+2}\right| < 0.5 \Rightarrow -0.5 < \frac{5n^2+10n-3}{8n^2+n+2} < 0.5$.

Consider the rightmost part of the equation:

$$\frac{5n^2+10n-3}{8n^2+n+2} < \frac{1}{2}$$

$$10n^2+20n-6 < 8n^2+n+2$$

$$2n^2+19n-8 < 0$$

$$n \in \left(\left[-\frac{19}{4} - \frac{5\sqrt{17}}{4}\right], \left[-\frac{19}{4} + \frac{5\sqrt{17}}{4}\right]\right) \Rightarrow n \in [-9, 0],$$

which means that there are no $n \geq 1$, for which there would be a single element within 0.5 range from $0 \Rightarrow \lim_{n\to\infty} x_n \neq 0$, q. e. d.

Assume that $\lim_{n\to\infty} x_n = \frac{5}{8}$. Then,

$$\lim_{n\to\infty}\left(\frac{5n^2+10n-3}{8n^2+n+2} - \frac{5}{8}\right) = \lim_{n\to\infty}\left(\frac{40n^2+80n-24-40n^2-5n-10}{8(8n^2+n+2)}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{75n-34}{8(8n^2+n+2)}\right)$$

Now, for all ε:

$$\frac{75n-34}{8(8n^2+n+2)} < \frac{75n}{8(8n^2+n+2)} < \frac{75n}{64n^2} = \frac{75}{64n} < \varepsilon$$

Therefore, $\forall n > N$:

$$N = \left[\frac{75}{64\varepsilon}\right]$$

It can be concluded that $\lim_{n\to\infty} x_n = \frac{5}{8}$, q. e. d.

Subproblem C

Prove that $\lim_{n\to\infty} 2^{\frac{n-1}{n^2}} = 1$.

Similarly as above, assume:

$$\lim_{n\to\infty}\left(2^{\frac{n-1}{n^2}} - 1\right) = 0$$

Then, for all ε:

$$2^{\frac{n-1}{n^2}} - 1 < \varepsilon$$

Logarithmize:

$$2^{\frac{n-1}{n^2}} < 2^{\log_2(\varepsilon+1)} \Rightarrow \frac{n-1}{n^2} < \log_2(\varepsilon+1) \Rightarrow$$

$$\frac{n-1}{n^2} < \frac{n}{n^2} = \frac{1}{n} < \log_2(\varepsilon+1) = \frac{\ln(\varepsilon+1)}{\ln 2} \Rightarrow$$

$$\frac{1}{n} < \frac{\ln(\varepsilon+1)}{\ln 2} \Rightarrow n > \frac{\ln 2}{\ln(\varepsilon+1)} \Rightarrow$$

$$n > \log_{\varepsilon+1} 2$$

Logarithm base is always larger than 1 and it would never be undefined since $\varepsilon > 0$. Therefore, $\forall n > N$:

$$N = [\log_{\varepsilon+1} 2]$$

As a result, $\lim_{n\to\infty} 2^{\frac{n-1}{n^2}} = 1$, q. e. d.

Subproblem D

Prove that

$$\lim_{n\to\infty} \frac{6n^4+n^3+3}{2n^4-n+1} = 3$$

Similarly as above,

$$\lim_{n\to\infty} \left(\frac{6n^4+n^3+3}{2n^4-n+1} - 3\right) = \lim_{n\to\infty} \left(\frac{6n^4+n^3+3-6n^4+3n-3}{2n^4-n+1}\right) =$$

$$= \lim_{n\to\infty} \left(\frac{n^3+3n}{2n^4-n+1}\right)$$

Then, $\forall \varepsilon$:

$$\frac{n^3+3n}{2n^4-n+1} < \varepsilon$$

We need to estimate this equation somehow, for all $n \in \mathbb{N}$, assume some $k_1 = 5$:

$$n^3 + 3n < k_1 n^3 = 5n^3 \Rightarrow 4n^3 - 3n > 0 \Rightarrow 4n(n^2 - \frac{3}{4}) > 0,$$

which is always true. The following equation with $k_2 = 1$ would be true for $n = 1$ and onwards:

$$2n^4 - n + 1 \geq n^4 = k_2 n^4$$

Therefore, the following estimation would be true:

$$\frac{n^3+3n}{2n^4-n+1} < \frac{5n^3}{n^4} = \frac{5}{n} < \varepsilon$$

And $\forall n > N$:

$$N = \left[\frac{5}{\varepsilon}\right]$$

In the end,

$$\lim_{n\to\infty} \frac{6n^4+n^3+3}{2n^4-n+1} = 3,$$

q. e. d.

Problem 2

Calculate limits:

Subproblem A

$$\lim_{n\to\infty}\frac{n!}{n^n}=\lim_{n\to\infty}\frac{1\times 2\times 3\times\cdots\times(n-2)\times(n-1)\times n}{n\times n\times n\times\cdots\times n\times n\times n}=$$
$$=\lim_{n\to\infty}\left(\frac{1}{n}\times\frac{2}{n}\times\frac{3}{n}\times\cdots\times\frac{n-2}{n}\times\frac{n-1}{n}\times 1\right)$$

Each (except for the last one) of the terms above is <1. Therefore, the multiplication of all these elements will be less (for $n>1$) than any (!) of the product components, so:

$$0\leq\frac{n!}{n^n}\leq\frac{1}{n}$$

Since we know that $\lim_{n\to\infty}\frac{1}{n}=0$, then according to the squeeze principle:

$$\lim_{n\to\infty}0=\lim_{n\to\infty}\frac{n!}{n^n}=\lim_{n\to\infty}\frac{1}{n}=0$$

Alternatively, just use limit arithmetic: since $\lim_{n\to\infty}\frac{1}{n}=0$, the entire limit is equal to 0.

Answer: $\lim_{n\to\infty}\frac{n!}{n^n}=0$

Subproblem B

$$\lim_{n\to\infty}\sqrt[n^2]{n!}=\lim_{n\to\infty}\sqrt[n^2]{1}\sqrt[n^2]{2}\sqrt[n^2]{3}\ldots\sqrt[n^2]{n}$$

From the binomial formula, we know that $\lim_{n\to\infty}\sqrt[n]{n}=1$. According the squeeze theorem once again,

$$1\leq\sqrt[n^2]{n}\leq\sqrt[n]{n}$$
$$\lim_{n\to\infty}1=\lim_{n\to\infty}\sqrt[n^2]{n}=\lim_{n\to\infty}\sqrt[n]{n}=1$$

and $\forall k\in\mathbb{N}$:

$$1\leq\sqrt[n^2]{k}\leq\sqrt[n]{k}$$
$$\lim_{n\to\infty}1=\lim_{n\to\infty}\sqrt[n^2]{k}=\lim_{n\to\infty}\sqrt[n]{k}=1$$

Limit of a multiplication is equal to the multiplication of limits, which are all defined and are equal to 1. Therefore,

$$\lim_{n\to\infty} \sqrt[n^2]{n!} = \lim_{n\to\infty} \sqrt[n^2]{1} \times \lim_{n\to\infty} \sqrt[n^2]{2} \times \lim_{n\to\infty} \sqrt[n^2]{3} \times \cdots \times \lim_{n\to\infty} \sqrt[n^2]{n} = 1 \times 1 \times 1 \times \cdots \times 1 = 1$$

Answer: $\lim_{n\to\infty} \sqrt[n^2]{n!} = 1$

Subproblem C

$$\lim_{n\to\infty} \sqrt[n]{3^n+1} = \lim_{n\to\infty} 3\sqrt[n]{1+\frac{1}{3^n}}$$

Looking a little bit closer, we can use the squeeze theorem (for $n \geq 5$, as it approaches ∞) to evaluate $\lim_{n\to\infty} \frac{1}{3^n}$:

$$0 \leq \frac{1}{3^n} \leq \frac{1}{n!}$$

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{3^n} = \lim_{n\to\infty} \frac{1}{n!} = 0$$

Substitute the limit back into the original equation:

$$\lim_{n\to\infty} \sqrt[n]{3^n+1} = \lim_{n\to\infty} 3\sqrt[n]{1+0} = 3$$

Answer: $\lim_{n\to\infty} \sqrt[n]{3^n+1} = 3$

Subproblem D

$$\lim_{n\to\infty} \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\ldots\sqrt[2^n]{2} = \lim_{n\to\infty} 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}}$$

Notice that the exponent is almost equal to 1 and for any finite n it would evaluate to:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n} = 1-\frac{1}{2^n}$$

Therefore,

$$\lim_{n\to\infty} \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\ldots\sqrt[2^n]{2} = \lim_{n\to\infty} 2^{1-\frac{1}{2^n}}$$

Same as above, looking a little bit closer, we can use the squeeze theorem (for $n \geq 3$, as it approaches ∞) to evaluate $\lim_{n\to\infty} \frac{1}{2^n}$:

$$0 \leq \frac{1}{2^n} \leq \frac{1}{n!}$$

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{2^n} = \lim_{n\to\infty} \frac{1}{n!} = 0$$

Because of this,

$$\lim_{n\to\infty} \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\ldots\sqrt[2^n]{2} = \lim_{n\to\infty} 2^{1-\frac{1}{2^n}} = \lim_{n\to\infty} 2^{1-0} = 2$$

Answer: $\lim_{n\to\infty} \sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\ldots\sqrt[2^n]{2} = 2$