Homework 20, Discrete Maths

Problem 1

Choose an arbitary function $f\colon \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$. All outcomes are equally probable. Are events "$f(1) = f(2)$" and "$f(2) = f(3)$" independent?

> Technically this might be sufficient since it's kinda obvious, but obviousness here frequently doesn't work, so I'll also present a stricter approach. Values $f(n)$ are independent from one another, and every value of $f(n)$ for any n is equally probable with a probability of $\frac{1}{n}$. Let's fix the value of $f(2) = \text{const}$. Then, the first event is dependent only on the value of $f(1)$, and the second event is dependent only on the value of $f(3)$, which means they're independent.

For the events to be independent, the following has to be true:

$$P[A \cap B] = P[B] \cdot P[A]$$

What is the probability of $P[B]$ and $P[A]$ independently? Kinda follow the approach above. Say, that we have already chosen $f(2)$, and need to choose $f(1)$ for event A and $f(3)$ for event B. For the events to happen, our chosen values for $f(1), f(3)$ have to be the same as $f(2)$. $f(2)$ has already been chosen out of n equally probable options, so the chance of choosing the same option as $f(2)$ for each of these cases (independently, for now) is $\frac{1}{n}$.

> Note that the above explanations apply symmetrically if we were to first choose $f(1)$ or $f(3)$, respectively, and only then choose $f(2)$ for each of the cases.

Thus,

$$P[A] = \frac{1}{n}, \quad P[B] = \frac{1}{n}$$

Now, what are the chances of the event $A \cap B$ occurring? This event would be called "$f(1) = f(2) = f(3)$". Let's calculate its chance of occurring: fix $f(1)$ to some value. We need to calculate the chance that $f(2), f(3)$ would be the same as $f(1)$, so we need to choose a specific number out of n options, twice, thus giving us a probability of $\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$.

Thus,

$$P[A \cap B] = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} = P[A] \cdot P[B]$$

which means that the events are independent from each other.

Problem 2

In a lotto, four numbers out of $\{1, 2, \ldots, 16\}$ are chosen randomly and with equally probable chances. Find the probability of an event A "there is no 13 among the chosen numbers" under a condition B that "there is no 1 among the chosen numbers".

The probability space is all sequences with non-repeating numbers of length 4 from the set above. This gives us

$$A_{16}^{4} = \frac{16!}{(16-4)!} = 16 \times 15 \times 14 \times 13$$

total possible sequences.

The number of sequences when event A or event B occur are (we just remove a single number from the pool of possible options):

$$A_{15}^{4} = \frac{15!}{(15-4)!} = 15 \times 14 \times 13 \times 12$$

Now, we know that

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Event $A \cap B$ could be defined as "there is neither 13 or 1 among the chosen numbers", which gives us

$$A_{14}^{4} = \frac{14!}{(14-4)!} = 14 \times 13 \times 12 \times 11$$

possible options.

Now,

$$P[B] = \frac{15 \times 14 \times 13 \times 12}{16 \times 15 \times 14 \times 13} = \frac{3}{4}$$

$$P[A \cap B] = \frac{14 \times 13 \times 12 \times 11}{16 \times 15 \times 14 \times 13} = \frac{3}{4} \cdot \frac{11}{15} = \frac{33}{60}$$

Finally,

$$P[A|B] = \frac{\frac{33}{60}}{\frac{3}{4}} = \frac{11}{15}$$

which is the final answer.

Problem 3

Four people A, B, C, and D form a queue in a random order (all options are equally probable). Find the conditional probability that A precedes B (event X) if it is known that A precedes C (event Y).

We once again need to use the formula from above:

$$P[X|Y] = \frac{P[X \cap Y]}{P[Y]}$$

To find the number of possible options, I will just list them all. There are obviously $4!$ permutations, resulting in 24 options:

```
      Y | X & Y
ABCD  V     V
ABDC  V     V
ACBD  V     V
ACDB  V     V
ADBC  V     V
ADCB  V     V
BACD  V     X
BADC  V     X
BCAD  X     X
BCDA  X     X
BDAC  V     X
BDCA  X     X
CABD  X     X
CADB  X     X
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CBAD  X       X
CBDA  X       X
CDAB  X       X
CDBA  X       X
DABC  V       V
DACB  V       V
DBAC  V       X
DBCA  X       X
DCAB  X       X
DCBA  X       X
```

We have 8 options for $A \cap B$ and 12 options for B.

Thus,

$$P[Y] = \frac{12}{24} = \frac{1}{2}, \quad P[X \cap Y] = \frac{8}{24} = \frac{1}{3}$$

and

$$P[X|Y] = \frac{P[X \cap Y]}{P[Y]} = \frac{\frac{8}{24}}{\frac{1}{2}} = \frac{16}{24} = \frac{2}{3}$$

Problem 4

In the first box there are 9 chips enumerated from 1 to 9. In the second box there are 10 chips, enumerated from 2 to 11. Chips in each box differ from each other only by numbers. A random box is chosen with equal probabilities, and then a random chip is chosen with equal probabilities from the chosen box. What is the probability that a box with 10 chips is chosen (event A) if a chip with number 7 was pulled out (event B)?

Let's denote chosen chips with two coordinates: its box number and its own number. Formula from above strikes again:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

The chance that we choose either boxes is $\frac{1}{2}$. Thus, we may split the field of probabilities into two parts. The first part would have each of the events "chips with certain coordinates is pulled"

$$(1,1),(1,2),\ldots(1,9)$$

occur with a probability of $\frac{1}{2}\times\frac{1}{9}=\frac{1}{18}$ since we divide this half into 9 equally probable events.

Whereas similar events for the second box

$$(2,2),(2,3),\ldots(2,11)$$

each occur with a probability of $\frac{1}{2}\times\frac{1}{10}=\frac{1}{20}$ since here we have 10 equally probable options.

Thus, the chance of pulling a chip with number 7 is equal to the sum of respective probabilities from each of the boxes:

$$P[B]=\frac{1}{18}+\frac{1}{20}=\frac{19}{180}$$

The chance of a certain chip with number 7 from the second box is simply

$$P[A\cap B]=\frac{1}{20}$$

Finally,

$$P[A|B]=\frac{\frac{1}{20}}{\frac{19}{180}}=\frac{9}{19}$$

which is the final answer.