Calculus Homework #6

Problem 8.8

In general, to prove that the two-variable limit does not exist, it is enough to find two different limits for two different paths.

Subproblem A

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y}{x^3 + y}$$

x→0y→0lim x3 + yx3 − y

For $y = mx^3$y = mx3:

$$\lim_{x \to 0} \frac{x^3 - mx^3}{x^3 + mx^3} = \lim_{x \to 0} \frac{1 - m}{1 + m} = \frac{1 - m}{1 + m}$$

x→0lim x3 + mx3x3 − mx3 = x→0lim 1 + m1 − m = 1 + m1 − m

The limit is dependent on m m ⇒ the limit does not exist.

Subproblem B

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$$

x→0y→0lim x2 + y2xy

For $y = mx$y = mx:

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{mx^2}{(1 + m^2)x^2} = \lim_{x \to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

x→0lim x2 + m2x2x · mx = x→0lim (1 + m2)x2mx2 = x→0lim 1 + m2m = 1 + m2m

The limit is dependent on m m ⇒ the limit does not exist.

Subproblem C

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2 - x^2}{y^2 + x^2}$$

x→0y→0lim y2 + x2y2 − x2

For $y = mx$y = mx:

$$\lim_{x \to 0} \frac{m^2x^2 - x^2}{m^2x^2 + x^2} = \lim_{x \to 0} \frac{m^2 - 1}{m^2 + 1} = \frac{m^2 - 1}{m^2 + 1}$$

x→0lim m2x2 + x2m2x2 − x2 = x→0lim m2 + 1m2 − 1 = m2 + 1m2 − 1

The limit is dependent on $m \Rightarrow$ m ⇒ the limit does not exist.

Subproblem D

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2y^2}{x^2y^2 + (x - y)^2}$$

x→0y→0lim x2y2 + (x − y)2x2y2

For $y = x$y = x:

$$\lim_{x \to 0} \frac{x^2x^2}{x^2x^2 + (x - x)^2} = \lim_{x \to 0} \frac{x^4}{x^4} = \lim_{x \to 0} 1 = 1$$

x→0lim x2x2 + (x − x)2x2x2 = x→0lim x4x4 = x→0lim 1 = 1

For $y = 0$y = 0:

$$\lim_{x \to 0} \frac{x^2 0^2}{x^2 0^2 + (x - 0)^2} = \lim_{x \to 0} \frac{0}{x^2} = \lim_{x \to 0} 0 = 0$$

x→0lim x202 + (x − 0)2x202 = x→0lim x20 = x→0lim 0 = 0

This should be enough, but in case you can't go along $y = 0$y = 0, a different option for $y = kx,\ k \neq 1$ y = kx, k �= 1:

$$\lim_{x \to 0} \frac{k^2x^4}{k^2x^4 + (1 - k)^2x^2} = \lim_{x \to 0} \frac{x^2}{x^2 + \frac{(1-k)^2}{k^2}} = \lim_{x \to 0} \frac{0}{0 + \frac{(1-k)^2}{k^2}} = 0$$

x→0lim k2x4 + (1 − k)2x2k2x4 = x→0lim x2 + k2(1−k)2x2 = x→0lim 0 + k2(1−k)20 = 0

There are at least two different limits for two paths $y = x,\ y = 0 \Rightarrow$ y = x, y = 0 ⇒ the limit does not exist.

Subproblem E

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(x + y\sin\frac{1}{x}\right)$$

x→0y→0lim (x + y sin x1)

$f_1(x) = \sin\frac{1}{x} \in [-1,\ 1]$f1(x) = sin x1 ∈ [−1, 1] is a bounded function. $f_2(y) = y$f2(y) = y as $y \to 0$y → 0 is an infinitesimal function. Product of a bounded function and an infinitesimal function is infinitesimal. Therefore, $\lim_{\substack{x \to 0 \\ y \to 0}} g(x,\ y) = \lim_{\substack{x \to 0 \\ y \to 0}} f_1(x) f_2(y) = 0$limx→0y→0 g(x, y) = limx→0y→0 f1(x)f2(y) = 0

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(x + y\sin\frac{1}{x}\right) = \lim_{\substack{x \to 0 \\ y \to 0}} x + \lim_{\substack{x \to 0 \\ y \to 0}} y\sin\frac{1}{x} = \lim_{x \to 0} x + \lim_{\substack{x \to 0 \\ y \to 0}} g(x,\ y) = 0 + 0 = 0$$

x→0y→0lim (x + y sin x1) = x→0y→0lim x + x→0y→0lim y sin x1 = x→0lim x + x→0y→0lim

g(x, y) = 0 + 0 = 0

Answer:

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(x + y\sin\frac{1}{x}\right) = 0$$

x→0y→0lim (x + y sin x1) = 0

Problem 8.9

Find limit of $f(x,\ y) = \frac{y - 2x^2}{y - x^2}$f(x, y) = y−x2y−2x2 in point $(0,\ 0)$(0, 0) along the path $x = \alpha t,\ y = \beta t,\ \alpha^2 + \beta^2 \neq 0$x = αt, y = βt, α2 + β2 �= 0. Prove that $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,\ y)$limx→0y→0 f(x, y) does not exist.

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y - 2x^2}{y - x^2} \Rightarrow \lim_{t \to 0} \frac{\beta t - 2\alpha^2 t^2}{\beta t - \alpha^2 t^2} = \lim_{t \to 0} \frac{\beta - 2\alpha^2 t}{\beta - \alpha^2 t} = \lim_{t \to 0} \frac{\beta - 0}{\beta - 0} = \lim_{t \to 0} 1 = 1$$

x→0y→0lim y − x2y − 2x2 ⇒ t→0lim βt − α2t2βt − 2α2t2 = t→0lim β − α2tβ − 2α2t = t→0lim

β − 0β − 0 = t→0lim 1 = 1

For $y = mx^2$y = mx2:

$$\lim_{x \to 0} \frac{mx^2 - 2x^2}{mx^2 - x^2} = \lim_{x \to 0} \frac{m - 2}{m - 1} = \frac{m - 2}{m - 1}$$
x→0lim mx2 − x2mx2 − 2x2 = x→0lim m − 1m − 2 = m − 1m − 2

Thus, the multivariable limit does not exist.

Answer: $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y - 2x^2}{y - x^2} = 1$limx→0y→0 y−x2y−2x2 = 1 along the path of $x = \alpha t,\ y = \beta t,\ \alpha^2 + \beta^2 \neq 0$
x = αt, y = βt, α2 + β2 �= 0

Problem 8.10

Is the function

$$u(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \forall x, y : x^2 + y^2 \neq 0 \\ 0, & \forall x, y : x^2 + y^2 = 0 \end{cases}$$

u(x, y) = ⎩⎨⎧x2 + y2xy,0, ∀x, y : x2 + y2 �= 0∀x, y : x2 + y2 = 0

continuous in point $(0,\ 0)$(0, 0)?

$$u'(x, y) = \frac{xy}{x^2 + y^2}$$
u′(x, y) = x2 + y2xy

Per the continuity criterion: for $u(x, y)$u(x, y) to be continuous in point of closure $(0,\ 0)$(0, 0) it is required and sufficient that $u(0,\ 0) = \lim_{\substack{x \to 0 \\ y \to 0}} u'(x, y)$.u(0, 0) = limx→0y→0 u′(x, y). This limit of $u'(x, y)$u′(x, y),

as proven in **Problem 8.8, Subproblem B** does not exist and **takes different values depending on the path**. Therefore, $(0,\ 0)(0, 0)$ is not the point of closure and the function $u(x,\ y)u(x, y)$ is not continuous.

Answer: $u(x,\ y)u(x, y)$ is not continuous