Problem 1

Determine whether the sequences are fundamental (Cauchy seequences).

Preamble

A sequence is fundamental (Cauchy) if $\forall \varepsilon > 0, \exists N$ such that $\forall n, m \geq N, |a_n - a_m| < \varepsilon$.

Subproblem A

$$x_n = \frac{1+4n^2}{2+2n^2}$$

Permute stuff around:

$$\frac{1+4n^2}{2+2n^2} = \frac{4+4n^2-3}{2+2n^2} = 2 - \frac{3}{2+2n^2}$$

Compare some elements x_n and x_m (per the triangle inequation), assuming $m > n$:

$$|x_n - x_m| = \left|2 - \frac{3}{2+2n^2} - 2 + \frac{3}{2+2m^2}\right| = \frac{3}{2+2n^2} + \frac{3}{2+2m^2} < \frac{6}{2+2n^2} = \frac{3}{1+n^2} < \frac{3}{n^2}$$

Now, $\forall n, m \geq N$ and $\forall \varepsilon > 0$:

$$\frac{3}{N^2} < \varepsilon \Rightarrow \frac{3}{\varepsilon} < N^2 \Rightarrow N^2 > \frac{3}{\varepsilon} \Rightarrow N > \sqrt{\frac{3}{\varepsilon}}$$

Therefore, starting $N = \left[\sqrt{\frac{3}{\varepsilon}}\right]$, the distance between any two values will be less than some arbitrary value ε, which means that the sequence is fundamental (Cauchy).

Answer: a Cauchy sequence.

Subproblem B

$$a_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

Compare some elements a_n and a_m (considering $m < n$) and use some funny tricks like a telescopic sum to properly compare everything:

$$|a_n - a_m| = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{m^2}\right) = \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{n^2} <$$

$$< \frac{1}{m} \cdot \frac{1}{m+1} + \frac{1}{m+1} \cdot \frac{1}{m+2} + \cdots + \frac{1}{n-1} \cdot \frac{1}{n} =$$

$$= \left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{m} - \frac{1}{n} < \frac{1}{m}$$

Now, $\forall n, m \geq N$ and $\forall \varepsilon > 0$:

$$\frac{1}{N} < \varepsilon$$

Therefore, starting $N = \left[\frac{1}{\varepsilon}\right]$, the distance between any two values will be less than some arbitrary value ε, which means that the sequence is fundamental (Cauchy).

Answer: a Cauchy sequence.

Subproblem C

$$a_n = 1 + \frac{1}{4} + \frac{1}{7} + \cdots + \frac{1}{3n-2}$$

Compare some elements a_n and a_m ($n = m + k, k \in \mathbb{N}$):

$$|a_n - a_m| = 1 + \frac{1}{4} + \frac{1}{7} + \cdots + \frac{1}{3n-2} - \left(1 + \frac{1}{4} + \frac{1}{7} + \cdots + \frac{1}{3m-2}\right) =$$

$$= \underbrace{\frac{1}{3(m+1)-2} + \frac{1}{3(m+2)-2} + \cdots + \frac{1}{3(m+k)-2}}_{k} >$$

$$> \underbrace{\frac{1}{3n-2} + \frac{1}{3n-2} + \cdots + \frac{1}{3n-2}}_{k} = \frac{k}{3n-2} > \frac{k}{3n}$$

Try to find a counterexample, when $\nexists N \; \forall \varepsilon$. Say that $k = m$, then $|a_n - a_m| > \frac{k}{3n} = \frac{m}{3 \cdot 2m} = \frac{1}{6}$ and then $\exists \varepsilon = 0.1$, for which $|a_n - a_m| < 0.1$ contradicts $|a_n - a_m| > \frac{1}{6}$. Therefore, the sequence is not fundamental (not Cauchy).

Answer: **not** a Cauchy sequence.

Problem 2

Find the limit

$$\lim_{n \to \infty} \left(1 + \frac{3}{2n}\right)^{13n+11}$$

On the lecture it was proven that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Let's try using this special limit and permute stuff around:

$$\lim_{n\to\infty}\left(1+\frac{3}{2n}\right)^{13n+11} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{2}{3}n}\right)^{\frac{39}{2}(\frac{2}{3}n)+11} = \lim_{n\to\infty}\left(\left(1+\frac{1}{\frac{2}{3}n}\right)^{\frac{39}{2}(\frac{2}{3}n)}\left(1+\frac{1}{\frac{2}{3}n}\right)^{11}\right) =$$

$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{\frac{2}{3}n}\right)^{\frac{2}{3}n}\right)^{\frac{39}{2}}\lim_{n\to\infty}\left(1+\frac{1}{\frac{2}{3}n}\right)^{11} = e^{\frac{39}{2}}(1+\frac{3}{2}\lim_{n\to\infty}\frac{1}{n}) = e^{\frac{39}{2}}(1+\frac{3}{2}\cdot 0) = e^{\frac{39}{2}}$$

Answer:

$$e^{\frac{39}{2}}$$

Problem 3

Find all partial limits of a sequence

$$\{x_n\} = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \ldots, \frac{1}{2^n}, \frac{2^n-1}{2^n}, \ldots$$

Split the original sequence into two:

$$\{a_n\} = \frac{1}{2^n},\ \ \{b_n\} = \frac{2^n-1}{2^n}$$

It was proven in previous homeworks that $\lim_{n\to\infty}\frac{1}{2^n} = 0$. Therefore, $\liminf_{n\to\infty}\{x_n\} = 0$.

$$\{b_n\} = \frac{2^n-1}{2^n} = 1 - \frac{1}{2^n}$$

$$\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right) = 1 - \lim_{n\to\infty}\frac{1}{2^n} = 1$$

Therefore, $\lim_{n\to\infty}\sup\{x_n\} = 1$.

There are no other partial limits since there are no other unaccounted elements in the sequence.

Answer:

$$\lim_{n\to\infty}\inf\{x_n\} = 0, \lim_{n\to\infty}\sup\{x_n\} = 1$$

Problem 4

Formulate a number sequence that has $a_1, a_2, \ldots, a_p$ as its partial limits.

Consider the following sequence:

$$\begin{gathered}
\{x_1\} = a_1, \\
\{x_2\} = a_2, \\
\{x_3\} = a_3, \\
\vdots \\
\{x_p\} = a_p, \\
\{x_{p+1}\} = a_1, \\
\{x_{p+2}\} = a_2, \\
\{x_{p+3}\} = a_3, \\
\vdots \\
\{x_{2p}\} = a_p, \\
\{x_{2p+1}\} = a_1, \\
\{x_{2p+2}\} = a_2, \\
\{x_{2p+3}\} = a_3, \\
\vdots \\
\{x_{3p}\} = a_p \\
\vdots \quad \vdots \quad \vdots \\
\{x_{(n-1)p}\} = a_p, \\
\{x_{(n-1)p+1}\} = a_1, \\
\{x_{(n-1)p+2}\} = a_2, \\
\{x_{(n-1)p+3}\} = a_3, \\
\vdots \\
\{x_{np}\} = a_p
\end{gathered}$$

To calculate partial limits of this sequence, we may split the sequence into p subsequences, grouping together elements that are equal to each other. Limits of each of these monotonous, stagnant subsequences would be some value from $\{a_1, a_2, a_3, \ldots a_p\}$, which is precisely what we wanted.

Problem 5

Formulate such a number sequence $\{a_n\}$ so that each element is also its partial limit. What other partial limits does this sequence certainly have?

Consider the following sequence:

$$\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \dots\}$$

The sequence is defined as follows: we count all numbers from 1 to n and add them to the sequence, then increase n by 1 and repeat the process indefinitely.

In this case, there would be an infinite number of each of the natural numbers, so it is always possible to find such a subsequence $\{a_{(k)n}\}$ that would be stagnant and only consist of the same $k \in \mathbb{N}$, defining every single partial limit. Therefore, for $\forall k \in \mathbb{N}, \lim_{n\to\infty} a_{(k)n} = k$.

Since by our sequence definition, it only contains natural numbers, any its subsequences would have a limit that is a natural number (albeit it's pretty boring, yeah). Since every single $k \in \mathbb{N}$ and $k \in a_n$, then there cannot be any other partial limits since every single possibility is already accounted for.

Answer: no other partial limits.

Problem 6

Find $\limsup x_n$ and $\liminf x_n$, where

Subproblem A

$$x_n = \frac{n^2}{n^2+1} \cos\frac{2\pi n}{3}$$

First of all, state the following for $n \in \mathbb{N}$:

- $\cos\left(\frac{2\pi}{3} + 2\pi n\right) = \cos\left(\frac{4\pi}{3} + 2\pi n\right)$
  - Therefore, if $n \bmod 3 \neq 0$, then we get the first subsequence $x_{(1)n} = \frac{n^2}{n^2+1}\cos\frac{2\pi}{3} = \frac{1}{2}\frac{n^2}{n^2+1}$
- $\cos(2\pi n) = 1$
  - In this case, if $n \bmod 3 = 0$, then we get the second subsequence $x_{(2)n} = \frac{n^2}{n^2+1}\cos(0) = \frac{n^2}{n^2+1}$

Calculate the limit of the second subsequence:

$$\{x_{(2)n}\} \lim_{x\to\infty} \frac{n^2}{n^2+1} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+\lim_{x\to\infty}\frac{1}{n^2}} = \frac{1}{1+0} = 1$$

Using limit arithmetic, calculate the limit of the first subsequence:

$$\lim_{x\to\infty}\{x_{(1)n}\} = \lim_{x\to\infty}\frac{1}{2}\frac{n^2}{n^2+1} = \frac{1}{2}\lim_{x\to\infty}\{x_{(2)n}\} = \frac{1}{2}$$

Therefore, choose corresponding values from the partial limits and get the answer:

Answer:

$$\limsup x_n = 1, \;\; \liminf x_n = \frac{1}{2}$$

Subproblem B

$$x_n = (-1)^{n+1}\left(1+\frac{2}{n}\right)^{3n} + \sin\frac{\pi n}{6}$$

Calculate the following for later:

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{1}{2}n}\right)^{\left(\frac{1}{2}n\right)6} = e^6$$

In this sequence, key values depend on whether n is odd and whether it is divisible by 12.

- $n \bmod 2 = 0 \Rightarrow (-1)^{n+1} = -1$
- $n \bmod 2 \neq 0 \Rightarrow (-1)^{n+1} = 1$

Next:

- $n \bmod 12 = 0, 6 \Rightarrow \sin\frac{\pi n}{6} = 0$
- $n \bmod 12 = 1, 5 \Rightarrow \sin\frac{\pi n}{6} = \frac{1}{2}$
- $n \bmod 12 = 2, 4 \Rightarrow \sin\frac{\pi n}{6} = \frac{\sqrt{3}}{2}$
- $n \bmod 12 = 3 \Rightarrow \sin\frac{\pi n}{6} = 1$
- $n \bmod 12 = 7, 11 \Rightarrow \sin\frac{\pi n}{6} = -\frac{1}{2}$
- $n \bmod 12 = 8, 10 \Rightarrow \sin\frac{\pi n}{6} = -\frac{\sqrt{3}}{2}$
- $n \bmod 12 = 9 \Rightarrow \sin\frac{\pi n}{6} = -1$

Therefore, let's evaluate $\lim_{x\to\infty}\{x_{(\alpha)n}\} = x$ for each of the $\alpha = n \bmod 12$, where $\{x_{(\alpha)n}\}$ is a subsequence of the original sequence with corresponding α-s:

- $\alpha = 0, 6 \Rightarrow x = -e^6$
- $\alpha = 1, 5 \Rightarrow x = e^6 + \frac{1}{2}$
- $\alpha = 2, 4 \Rightarrow x = -e^6 + \frac{\sqrt{3}}{2}$

- $\alpha = 3 \Rightarrow x = e^6 + 1$
- $\alpha = 7, 11 \Rightarrow x = e^6 - \frac{1}{2}$
- $\alpha = 8, 10 \Rightarrow x = -e^6 - \frac{\sqrt{3}}{2}$
- $\alpha = 9 \Rightarrow x = e^6 - 1$

Awesome, now choose corresponding values from the partial limits and get the answer:

Answer:

$$\limsup x_n = e^6 + 1, \;\; \liminf x_n = -e^6 - \frac{\sqrt{3}}{2}$$

Subproblem C

$$x_n = \frac{n}{n+1} \sin^2 \frac{\pi n}{4}$$

Similarly as above, check how $\sin^2 \frac{\pi n}{4}$ behaves for $n \bmod 8$:

- $n \bmod 8 = 0, 4 \Rightarrow \sin^2 \frac{\pi n}{4} = 0$
- $n \bmod 8 = 1, 3, 5, 7 \Rightarrow \sin^2 \frac{\pi n}{4} = \frac{1}{2}$
- $n \bmod 8 = 2, 6 \Rightarrow \sin^2 \frac{\pi n}{4} = 1$

Therefore, we may split our sequence into some number of subsequences (take 3, for example), and then the partial limits would be one of the coefficients above multiplied by $\lim_{x \to \infty} \frac{n}{n+1} = \lim_{x \to \infty} \frac{1}{1+\frac{1}{n}} = 1$.

Therefore, the partial limits would be $0, \frac{1}{2}, 1$. Write the answer:

Answer:

$$\limsup x_n = 1, \;\; \liminf x_n = 0$$