Individual Homework 1 (variant 20)

Problem 1

> *Today, my son asked "Can I have a book mark?" and I burst into tears. 11 years old and he still doesn't know my name is Brian.*

Given:

$$A = \begin{pmatrix} -5 & 4 & 3 \\ 0 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 3 & -4 \\ -5 & 1 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}, \quad D = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$$

Calculate:

$$\operatorname{tr}(A^T A)DBB^T + \operatorname{tr}((7BA^T + 7AB^T)D + D(-7AB^T + 4BA^T))(A+B)(A^T - B^T) - 9C^2 - 18CD - 9D^2$$

I shall split this monstrosity into three parts and calculate each one separately (I will omit some partial calculations as otherwise it would take an eternity to tex everything):

Part 1

$$\operatorname{tr}(A^T A)DBB^T$$

$$A^T = \begin{pmatrix} -5 & 0 \\ 4 & 0 \\ 3 & 6 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} -5 & 0 \\ 4 & 0 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} -5 & 4 & 3 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 25 & -20 & -15 \\ -20 & 16 & 12 \\ -15 & 12 & 45 \end{pmatrix}$$

$$\operatorname{tr} A^T A = 25 + 16 + 45 = 86$$

$$DB = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -5 & 3 & -4 \\ -5 & 1 & 7 \end{pmatrix} = \begin{pmatrix} -20 & 12 & -16 \\ -45 & 9 & 63 \end{pmatrix}$$

$$B^T = \begin{pmatrix} -5 & -5 \\ 3 & 1 \\ -4 & 7 \end{pmatrix}$$

$$DBB^T = \begin{pmatrix} -20 & 12 & -16 \\ -45 & 9 & 63 \end{pmatrix} \begin{pmatrix} -5 & -5 \\ 3 & 1 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 100+36+64 & 100+12-112 \\ 225+27-252 & 225+9+441 \end{pmatrix} = \begin{pmatrix} 200 & 0 \\ 0 & 675 \end{pmatrix}$$

$$\operatorname{tr}(A^T A)DBB^T = 86 \begin{pmatrix} 200 & 0 \\ 0 & 675 \end{pmatrix} = \begin{pmatrix} 17200 & 0 \\ 0 & 58050 \end{pmatrix}$$

Part 2

$$\operatorname{tr}((7BA^T+7AB^T)D+D(-7AB^T+4BA^T))(A+B)(A^T-B^T)$$

$$BA^T=\begin{pmatrix}-5 & 3 & -4\\ -5 & 1 & 7\end{pmatrix}\begin{pmatrix}-5 & 0\\ 4 & 0\\ 3 & 6\end{pmatrix}=\begin{pmatrix}25 & -24\\ 50 & 42\end{pmatrix}$$

$$AB^T=(BA^T)^T=\begin{pmatrix}25 & 50\\ -24 & 42\end{pmatrix}$$

$$7BA^T+7AB^T=7\left(\begin{pmatrix}25 & -24\\ 50 & 42\end{pmatrix}+\begin{pmatrix}25 & 50\\ -24 & 42\end{pmatrix}\right)=\begin{pmatrix}350 & 182\\ 182 & 588\end{pmatrix}$$

$$(7BA^T+7AB^T)D=\begin{pmatrix}350 & 182\\ 182 & 588\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}=\begin{pmatrix}1400 & 1638\\ 728 & 5292\end{pmatrix}$$

$$-7AB^T=-7\begin{pmatrix}25 & 50\\ -24 & 42\end{pmatrix}=\begin{pmatrix}-175 & -350\\ 168 & -294\end{pmatrix}$$

$$4BA^T=4\begin{pmatrix}25 & -24\\ 50 & 42\end{pmatrix}=\begin{pmatrix}100 & -96\\ 200 & 168\end{pmatrix}$$

$$-7AB^T+4BA^T=\begin{pmatrix}-75 & -446\\ 368 & -126\end{pmatrix}$$

$$D(-7AB^T+4BA^T)=\begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}\begin{pmatrix}-75 & -446\\ 368 & -126\end{pmatrix}=\begin{pmatrix}-300 & -1784\\ 3312 & -1134\end{pmatrix}$$

$$\operatorname{tr}((7BA^T+7AB^T)D+D(-7AB^T+4BA^T))=(1400+5292)+(-300-1134)=5258$$

$$A+B=\begin{pmatrix}-5 & 4 & 3\\ 0 & 0 & 6\end{pmatrix}+\begin{pmatrix}-5 & 3 & -4\\ -5 & 1 & 7\end{pmatrix}=\begin{pmatrix}-10 & 7 & -1\\ -5 & 1 & 13\end{pmatrix}$$

$$A^T-B^T=\begin{pmatrix}-5 & 0\\ 4 & 0\\ 3 & 6\end{pmatrix}-\begin{pmatrix}-5 & -5\\ 3 & 1\\ -4 & 7\end{pmatrix}=\begin{pmatrix}0 & -5\\ 1 & -1\\ 7 & -1\end{pmatrix}$$

$$(A+B)(A^T-B^T)=\begin{pmatrix}-10 & 7 & -1\\ -5 & 1 & 13\end{pmatrix}\begin{pmatrix}0 & -5\\ 1 & -1\\ 7 & -1\end{pmatrix}=\begin{pmatrix}0 & -56\\ 92 & -39\end{pmatrix}$$

$$\operatorname{tr}((7BA^T+7AB^T)D+D(-7AB^T+4BA^T))(A+B)(A^T-B^T)=5258\begin{pmatrix}0 & -56\\ 92 & -39\end{pmatrix}=$$
$$\begin{pmatrix}0 & -294448\\ 483726 & -205062\end{pmatrix}$$

Part 3

$$-9C^2 - 18CD - 9D^2$$

$$C^2 = \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 32 & 36 \\ 36 & 41 \end{pmatrix}$$

$$CD = \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 16 & 36 \\ 16 & 45 \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}$$

$$-9C^2 - 18CD - 9D^2 = -9C^2 = -9\left(\begin{pmatrix} 32 & 36 \\ 36 & 41 \end{pmatrix} + \begin{pmatrix} 32 & 72 \\ 32 & 90 \end{pmatrix} + \begin{pmatrix} 16 & 0 \\ 0 & 81 \end{pmatrix}\right) =$$
$$= \begin{pmatrix} -720 & -972 \\ -612 & -1908 \end{pmatrix}$$

Finale

$$\begin{pmatrix} 17200 & 0 \\ 0 & 58050 \end{pmatrix} + \begin{pmatrix} 0 & -294448 \\ 483726 & -205062 \end{pmatrix} + \begin{pmatrix} -720 & -972 \\ -612 & -1908 \end{pmatrix} = \begin{pmatrix} 16480 & -295420 \\ 483124 & -148920 \end{pmatrix}$$

Problem 2

> *Two fish in a tank. One says: 'How do you drive this thing?'*

Find all possible values of AB if

$$A+B=\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}+\begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} a_{11} & a_{12}+b_{12} & a_{13}+b_{13} & a_{14}+b_{14} \\ a_{12}-b_{12} & a_{22} & a_{23}+b_{23} & a_{24}+b_{24} \\ a_{13}-b_{13} & a_{23}-b_{23} & a_{33} & a_{34}+b_{34} \\ a_{14}-b_{14} & a_{24}-b_{24} & a_{34}-b_{34} & a_{44} \end{pmatrix}=\begin{pmatrix} -60 & 10 & -36 & 8 \\ -28 & -58 & 42 & -10 \\ -14 & -18 & -8 & 30 \\ 28 & 52 & -54 & 10 \end{pmatrix}$$

Notice that:

$$\begin{cases} a_{12}+b_{12}=10 \\ a_{12}-b_{12}=-28 \end{cases} \Rightarrow \begin{cases} a_{12}=-9 \\ b_{12}=19 \end{cases}$$

$$\begin{cases} a_{13}+b_{13}=-36 \\ a_{13}-b_{13}=-14 \end{cases} \Rightarrow \begin{cases} a_{13}=-25 \\ b_{13}=-11 \end{cases}$$

$$\begin{cases} a_{14}+b_{14}=8 \\ a_{14}-b_{14}=28 \end{cases} \Rightarrow \begin{cases} a_{14}=18 \\ b_{14}=-10 \end{cases}$$

$$\begin{cases} a_{23}+b_{23}=42 \\ a_{23}-b_{23}=-18 \end{cases} \Rightarrow \begin{cases} a_{23}=12 \\ b_{23}=30 \end{cases}$$

$$\begin{cases} a_{24}+b_{24}=-10 \\ a_{24}-b_{24}=52 \end{cases} \Rightarrow \begin{cases} a_{24}=21 \\ b_{24}=-31 \end{cases}$$

$$\begin{cases} a_{34}+b_{34}=30 \\ a_{34}-b_{34}=-54 \end{cases} \Rightarrow \begin{cases} a_{34}=-12 \\ b_{34}=42 \end{cases}$$

$$\begin{cases} a_{11}=-60 \\ a_{22}=-58 \\ a_{33}=-8 \\ a_{44}=10 \end{cases}$$

$$A = \begin{pmatrix} -60 & -9 & -25 & 18 \\ -9 & -58 & 12 & 21 \\ -25 & 12 & -8 & -12 \\ 18 & 21 & -12 & 10 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 19 & -11 & -10 \\ -19 & 0 & 30 & -31 \\ 11 & -30 & 0 & 42 \\ -10 & 31 & -42 & 0 \end{pmatrix}$$

Now multiply (sorry it physically doesn't fit in the pdf properly haha...):

$$AB = \begin{pmatrix} -60 \times 0 + -9 \times -19 + -25 \times 11 + 18 \times -10 & -60 \times 19 + -9 \times 0 + -25 \times -30 + 18 \times 31 & -60 \times \\ -9 \times 0 + -58 \times -19 + 12 \times 11 + 21 \times -10 & -9 \times 19 + -58 \times 0 + 12 \times -30 + 21 \times 31 & -9 \times \\ -25 \times 0 + 12 \times -19 + -8 \times 11 + -12 \times -10 & -25 \times 19 + 12 \times 0 + -8 \times -30 + -12 \times 31 & -25 \times \\ 18 \times 0 + 21 \times -19 + -12 \times 11 + 10 \times -10 & 18 \times 19 + 21 \times 0 + -12 \times -30 + 10 \times 31 & 18 \times \end{pmatrix}$$

$$= \begin{pmatrix} -284 & 168 & -366 & -171 \\ 1024 & 120 & -2523 & 2392 \\ 196 & -607 & 1139 & -458 \\ -631 & 1012 & 12 & -1335 \end{pmatrix},$$

which is the answer.

Problem 3

> *Crime in multi-storey car parks. That is wrong on so many different levels.*

Matrix A is shown in a form of $A = CJD$, where:

$$C = \begin{pmatrix} 1 & 8 & -8 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{pmatrix}$$

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -8 & -56 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculate DC and find matrix $S = E + A + \cdots + A^{2021}$.

$$DC = \begin{pmatrix} 1\times 1 + -8\times 0 + -56\times 0 & 1\times 8 + -8\times 1 + -56\times 0 & 1\times -8 + -8\times -8 + -56\times 1 \\ 0\times 1 + 1\times 0 + 8\times 0 & 0\times 8 + 1\times 1 + 8\times 0 & 0\times -8 + 1\times -8 + 8\times 1 \\ 0\times 1 + 0\times 0 + 1\times 0 & 0\times 8 + 0\times 1 + 1\times 0 & 0\times -8 + 0\times -8 + 1\times 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

Now, mess around with matrices like A^n:

$$A^n = (CDJ)^n = \underbrace{(CJD)(CJD)\dots(CJD)}_{n} = CJ\underbrace{(EJ)(EJ)\dots(EJ)}_{n-1}D = C\underbrace{JJ\dots J}_{n}D = CJ^nD$$

By induction, we can prove that

$$J^n = \begin{pmatrix} (-1)^n & (-1)^{n+1}n & (-1)^n\sum_{i=1}^{n-2} n \\ 0 & (-1)^n & (-1)^{n+1}n \\ 0 & 0 & (-1)^n \end{pmatrix} = \begin{pmatrix} (-1)^n & (-1)^{n+1}n & (-1)^n\frac{n(n-1)}{2} \\ 0 & (-1)^n & (-1)^{n+1}n \\ 0 & 0 & (-1)^n \end{pmatrix}$$

Then,

$$\sum_{i=0}^{n} J^i = \begin{pmatrix} \sum_{i=0}^{n}(-1)^n & \sum_{i=0}^{n}(-1)^{n+1}n & \sum_{i=0}^{n}(-1)^n\frac{n(n-1)}{2} \\ 0 & \sum_{i=0}^{n}(-1)^n & \sum_{i=0}^{n}(-1)^{n+1}n \\ 0 & 0 & \sum_{i=0}^{n}(-1)^n \end{pmatrix} =$$

$$= \begin{pmatrix} n \mod 2 & (-1)^n \lfloor \frac{n}{2} \rfloor & (-1)^{n+1} \left(\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} i + \sum_{i=1}^{\lfloor \frac{n-2}{2} \rfloor} i \right) \\ 0 & n \mod 2 & (-1)^n \lfloor \frac{n}{2} \rfloor \\ 0 & 0 & n \mod 2 \end{pmatrix}$$

And,

$$\sum_{i=0}^{2023} J^i = \begin{pmatrix} 1 & -1011 & 1022121 \\ 0 & 1 & -1011 \\ 0 & 0 & 1 \end{pmatrix}$$

Finally, per the distribution property:

$$\sum_{n=0}^{2023} A^n = C \left(\sum_{i=0}^{2023} J^i \right) D = \begin{pmatrix} 1 & -1011 & 1005945 \\ 0 & 1 & -1011 \\ 0 & 0 & 1 \end{pmatrix},$$

which is the answer.

Problem 4

> *Police arrested two kids yesterday. One was drinking battery acid, the other was eating fireworks. They charged one – and let the other one off.*

Check that matrix

$$S = \begin{pmatrix} 36 & 42 & -12 \\ -30 & -35 & 10 \\ -6 & -7 & 2 \end{pmatrix}$$

takes the form of $S = uv^T$ for some $u, v \in \mathbb{R}^3$ and find $\mathrm{tr} S^{13}$.

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$S = uv^T = \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}$$

By finding greatest common divisors, notice that

$$u = \begin{pmatrix} 6 \\ -5 \\ -1 \end{pmatrix}, \quad v = \begin{pmatrix} 6 \\ 7 \\ -2 \end{pmatrix},$$

which means that this property holds.

Therefore, considering that $v^T u = \mathrm{tr} S = 36 - 35 + 2 = 3$,

$$S^{13} = u\underbrace{v^T uv^T \dots u}_{12}v^T = (\mathrm{tr} S)^{12} S$$

At last,

$$\mathrm{tr} S^{13} = \mathrm{tr}((\mathrm{tr} S)^{12} S) = (\mathrm{tr} S)^{13} = 3^{13} = 1594323,$$

which is the answer.

Problem 5

> *What do you call a fish with no eyes? A fsh.*

Subproblem A

$$\begin{cases} -3x_1 + 2x_2 + 3x_3 - 5x_4 = 0 \\ -6x_1 + 4x_2 + 6x_3 - 10x_4 = 4 \\ -8x_1 + 5x_2 + 7x_3 - 12x_4 = -5 \\ 7x_1 - 4x_2 - 5x_3 + 9x_4 = -5 \end{cases}$$

Divide the second line by 2 and see that the first and the second lines are the same, which means that the matrix is inconsistent.

$$\left(\begin{array}{cccc|c} -3 & 2 & 3 & -5 & 0 \\ -6 & 4 & 6 & -10 & 4 \\ -8 & 5 & 7 & -12 & -5 \\ 7 & -4 & -5 & 9 & -5 \end{array}\right) \sim \left(\begin{array}{cccc|c} -3 & 2 & 3 & -5 & 0 \\ -3 & 2 & 3 & -5 & 4 \\ -8 & 5 & 7 & -12 & -5 \\ 7 & -4 & -5 & 9 & -5 \end{array}\right)$$

Answer: the matrix is inconsistent.

Subproblem B

$$\begin{cases} -3x_1 + 2x_2 + 3x_3 - 5x_4 = 14 \\ -6x_1 + 4x_2 + 6x_3 - 10x_4 = 28 \\ -8x_1 + 5x_2 + 7x_3 - 12x_4 = 36 \\ 7x_1 - 4x_2 - 5x_3 + 9x_4 = -30 \end{cases}$$

$$\left(\begin{array}{cccc|c} -3 & 2 & 3 & -5 & 14 \\ -6 & 4 & 6 & -10 & 28 \\ -8 & 5 & 7 & -12 & 36 \\ 7 & -4 & -5 & 9 & -30 \end{array}\right) \sim \left(\begin{array}{cccc|c} -3 & 2 & 3 & -5 & 14 \\ -8 & 5 & 7 & -12 & 36 \\ 7 & -4 & -5 & 9 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -\frac{2}{3} & -1 & \frac{5}{3} & -\frac{14}{3} \\ -8 & 5 & 7 & -12 & 36 \\ 7 & -4 & -5 & 9 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -\frac{2}{3} & -1 & \frac{5}{3} & -\frac{14}{3} \\ 0 & -\frac{1}{3} & -1 & \frac{4}{3} & -\frac{4}{3} \\ 0 & \frac{2}{3} & 2 & -\frac{8}{3} & \frac{8}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & -\frac{2}{3} & -1 & \frac{5}{3} & -\frac{14}{3} \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 2 & 6 & -8 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -\frac{2}{3} & -1 & \frac{5}{3} & -\frac{14}{3} \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & -2 \\ 0 & 1 & 3 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Awesome, now write the general solution:

$$\begin{cases} x_1 = x_4 - x_3 - 2 \\ x_2 = 4 - 3x_3 + 4x_4 \end{cases}$$

Example of a partial solution:

$$\begin{cases} x_1 = x_4 - x_3 - 2 \\ x_2 = 4 - 3x_3 + 4x_4 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$$