Calculus, Homework 19

Problem 1

Let $\omega_{k,m} = x^k \ln^m(x)$. Find the integral of this form for all k, m. (Present a recurrent formula).

$$\int x^k \ln^m(x)dx = \int_m$$

$$u = \ln^m(x), dv = x^k dx, du = m\ln^{m-1}(x)\frac{1}{x}dx, v = \frac{x^{k+1}}{k+1}$$

$$\frac{\ln^m(x)x^{k+1}}{k+1} - \int \frac{m\ln^{m-1}(x)x^{k+1}}{(k+1)x}dx =$$

$$\frac{\ln^m(x)x^{k+1}}{k+1} - \frac{m}{k+1}\int \ln^{m-1}(x)x^k dx$$

$$\int_m = \frac{\ln^m(x)x^{k+1}}{k+1} - \frac{m}{k+1}\int_{m-1}$$

$$\int_1 = \int x^k \ln(x)dx$$

$$u = \ln(x), dv = x^k dx, du = \frac{1}{x}dx, v = \frac{x^{k+1}}{k+1}$$

$$\int_1 = \frac{\ln(x)x^{k+1}}{k+1} + \frac{1}{k+1}\int x^k dx = \frac{\ln(x)x^{k+1}}{k+1} + \frac{x^{k+1}}{(k+1)^2}$$

Problem 2

> Find the following integrals:

> Integration by parts formula for reference: $\int fdg = fg - \int gdf$

Subproblem A

$$\int e^{ax}\sin(bx)dx$$

$$f = \sin(bx), df = b\cos(bx)dx, dg = e^{ax}dx, g = \frac{e^{ax}}{a} \implies$$

$$\frac{e^{ax}}{a}\sin(bx) - \int \frac{e^{ax}}{a} b\cos(bx)dx$$

$$f = \cos(bx), df = -b\sin(bx)dx, dg = e^{ax}dx, g = \frac{e^{ax}}{a} \implies$$

$$\frac{e^{ax}}{a}\sin(bx) - \frac{b}{a}\left(\frac{e^{a}x}{a}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)dx\right)$$

> We have arrived at the integral that we've started with, so we may express it out and arrive at the solution:

$$\frac{e^{ax}}{a}\sin(bx) - \frac{b}{a}\left(\frac{e^{ax}}{a}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)dx\right) = \int e^{ax}\sin(bx)dx$$

$$\left(\frac{b^2}{a^2}+1\right)\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a}\sin(bx) - \frac{be^{ax}}{a^2}\cos(bx) + C$$

$$\int e^{ax}\sin(bx)dx = \frac{\frac{e^{ax}}{a}\sin(bx) - \frac{be^{ax}}{a^2}\cos(bx)}{\left(\frac{b^2}{a^2}+1\right)} + C$$

$$= \frac{(a\sin(bx) - b\cos(bx))e^{ax}}{a^2+b^2} + C$$

> Now we need to consider two cases. First, for $a = 0$, and second, for $a = b = 0$:

First, $a = 0$:

$$\int \sin(bx)dx = -\frac{\cos(bx)}{b}$$

Second, $a = b = 0$:

$$\int 0dx = 0$$

Subproblem B

$$\int x\cos(3x)dx$$

$$f = x, dg = \cos(3x)dx, df = 1dx, g = \frac{\sin 3x}{3}dx$$

$$\frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3xdx = \frac{1}{3}x\sin 3x - \frac{1}{9}\int \sin 3xd(3x) =$$

$$\frac{1}{3}x\sin 3x - \frac{1}{9}\cos(3x) + C$$

Subproblem C

$$\int \frac{x\cos(x)}{\sin^2(x)}dx$$

$$f = x, dg = \frac{\cos(x)}{\sin^2(x)}, g = -\frac{1}{\sin(x)}, df = dx$$

$$-\frac{x}{\sin(x)} - \int \frac{dx}{\sin(x)} = -\frac{x}{\sin(x)} - \ln \operatorname{tg} \frac{x}{2}$$

Subproblem D

$$\int x^3 e^{-x^2} dx$$

$$u = x^2, du = 2xdx$$

$$\frac{1}{2}\int e^{-u} udu$$

$$f = u, df = 1du, dg = e^{-u}du, g = -e^{-u}$$

$$-\frac{1}{2}e^{-u}u + \frac{1}{2}\int e^{-u}du = -\frac{1}{2}e^{-u}u - \frac{1}{2}\int e^{-u}d(-u)$$

$$= -\frac{1}{2}e^{-u}u - \frac{1}{2}e^{-u} + C = -\frac{1}{2}e^{-x^2}(x^2+1) + C$$

Subproblem E

$$\int e^{\sqrt{x}}dx$$

$$u = \sqrt{x}, du = \frac{1}{2u}dx$$

$$2\int e^u udu$$

$$f = u, dg = e^u du, df = 1du, g = e^u$$

$$2e^u u - 2\int e^u du = 2e^u u - 2e^u + C =$$

$$2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$

Subproblem F

$$\int \sin(\ln x)dx$$

$$u = \ln x, du = \frac{1}{e^u}dx, dx = e^u du, x = e^u$$

$$\int \sin(u)e^u du$$

> We have solved this integral above, so we may simply use

$$\frac{1}{2}e^u(\sin(u) - \cos(u)) = \frac{1}{2}x(\sin\ln x - \cos\ln x)$$

Problem 3

Subproblem A

$$\int \frac{x^3+1}{x^3-5x^2+6x}dx$$

$$\int \frac{x^3+1}{x^3-5x^2+6x}dx = \int \left(1 + \frac{5x^2-6x+1}{x^3-5x^2+6x}\right) dx =$$

$$\int \left(1 + \frac{5x^2-6x+1}{x(x-3)(x-2)}\right) dx = \int \left(1 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}\right) dx$$

$$5x^2 - 6x + 1 = A(x-2)(x-3) + Bx(x-3) + Cx(x-2) = \\ Ax^2 - 5Ax + 6A + Bx^2 - 3Bx + Cx^2 - 2Cx$$

$$\begin{cases} A + B + C = 5 \\ -5A - 3B - 2C = -6 \\ 6A = 1 \end{cases} \implies \begin{cases} B + C = \frac{29}{6} \\ 3B + 2C = \frac{31}{6} \\ A = \frac{1}{6} \end{cases} \implies \begin{cases} A = \frac{1}{6} \\ B = -\frac{9}{2} \\ C = \frac{28}{3} \end{cases}$$

$$\int \left(1 + \frac{1}{6x} - \frac{9}{2(x-2)} + \frac{28}{3(x-3)}\right) dx =$$

$$\int dx + \frac{1}{6}\int \frac{dx}{x} - \frac{9}{2}\int \frac{dx}{x-2} + \frac{28}{3}\int \frac{dx}{x-3} =$$

$$x + \frac{1}{6}\ln x - \frac{9}{2}\ln(x-2) + \frac{28}{3}\ln(x-3) + C$$

Subproblem B

$$\int \frac{x}{x^3-3x+2}dx$$

$$\int \frac{x}{(x-1)^2(x+2)}dx = \int \left(\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+2}\right) dx$$

$$x = A(x+2) + B(x-1)(x+2) + C(x-1)^2 =$$
$$Ax + 2A + Bx^2 + Bx - 2B + Cx^2 - 2Cx + C$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 1 \\ 2 & -2 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & \frac{2}{9} \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{9} \end{array}\right)$$

$$\frac{1}{3}\int \frac{dx}{(x-1)^2} + \frac{2}{9}\int \frac{dx}{x-1} - \frac{2}{9}\int \frac{dx}{x+2}$$

$$-\frac{1}{3}\frac{1}{x-1} + \frac{2}{9}\ln(x-1) - \frac{2}{9}\ln(x+2) + C$$

Subproblem C

$$\int \frac{dx}{(x+1)(x+2)^2(x+3)^3}$$

$$\int \left(\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{x+3} + \frac{E}{(x+3)^2} + \frac{F}{(x+3)^3}\right) dx$$

$$1 = A(x+2)^2(x+3)^3 + B(x+1)(x+2)(x+3)^3 + C(x+1)(x+3)^3 +$$
$$+D(x+1)(x+2)^2(x+3)^2 + E(x+1)(x+2)^2(x+3) + F(x+1)(x+2)^2 =$$

> No torturing myself here unfortunately, the matrix instantly

$$\begin{pmatrix} 108 & 54 & 27 & 36 & 12 & 4 \\ 216 & 135 & 54 & 96 & 28 & 8 \\ 171 & 126 & 36 & 97 & 23 & 5 \\ 67 & 56 & 10 & 47 & 8 & 1 \\ 13 & 12 & 1 & 11 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies$$

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ 2 \\ -1 \\ -\frac{17}{8} \\ -\frac{5}{4} \\ -\frac{1}{2} \end{pmatrix}$$

$$\frac{1}{8}\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2} - \int \frac{dx}{(x+2)^2} -$$

$$-\frac{17}{8}\int \frac{dx}{x+3} - \frac{5}{4}\int \frac{dx}{(x+3)^2} - \frac{1}{2}\int \frac{dx}{(x+3)^3}dx =$$

$$\frac{1}{8}\ln(x+1)+2\ln(x+2)-\frac{17}{8}\ln(x+3)+\\ \frac{1}{x+2}+\frac{5}{4}\frac{1}{x+3}+\frac{1}{4}\frac{1}{(x+3)^2}+C$$

Subproblem D

$$\int\frac{dx}{x(x+1)(x^2+1)}$$

$$\int\left(\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}\right)dx$$

$$1=A(x^3+x^2+x+1)+Bx(x^2+1)+(x^2+x)(Cx+D)=\\ (A+B+C)x^3+(A+C+D)x^2+(A+B+D)x+A$$

$$\begin{pmatrix}1&0&0&0\\1&1&0&1\\1&0&1&1\\1&1&1&0\end{pmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\implies\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}1\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}$$

$$\int\frac{dx}{x}-\frac{1}{2}\int\frac{dx}{x+1}-\frac{1}{2}\int\frac{x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$\ln x-\frac{1}{2}\ln(x+1)-\frac{1}{4}\int\frac{d(x^2+1)}{x^2+1}-\frac{1}{2}\operatorname{arctg}x$$

$$\ln x-\frac{1}{2}\ln(x+1)-\frac{1}{4}\ln(x^2+1)-\frac{1}{2}\operatorname{arctg}x+C$$

Subproblem E

$$\int\frac{dx}{x^4+x^2+1}$$

$$\int\left(\frac{dx}{(x^2-x+1)(x^2+x+1)}\right)=\int\left(\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2+x+1}\right)dx$$

$$1=Ax(x^2+x+1)+B(x^2+x+1)+Cx(x^2-x+1)+D(x^2-x+1)=\\ B+D+(A+C)x^3+(A+B-C+D)x^2+(A+B+C-D)x$$

$$\begin{pmatrix}0&1&0&1\\1&1&1&-1\\1&1&-1&1\\1&0&1&0\end{pmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\implies\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix}$$

$$\frac{1}{2}\int\frac{-x+1}{x^2-x+1}dx+\frac{1}{2}\int\frac{x+1}{x^2+x+1}dx$$

$$\frac{1}{2}\left(\int \frac{dx}{2(x^2-x+1)} - \int \frac{2x-1}{2(x^2-x-1)}dx + \right.$$

$$\left. + \int \frac{2x+1}{2(x^2+x+1)}dx + \int \frac{dx}{2(x^2+x+1)}\right)$$

$$\frac{1}{2}\left(\int \frac{dx}{2(x^2-x+1)} - \int \frac{d(x^2-x-1)}{2(x^2-x-1)} + \right.$$

$$\left. + \int \frac{d(x^2+x+1)}{2(x^2+x+1)} + \int \frac{dx}{2(x^2+x+1)}\right)$$

$$\frac{1}{4}(\ln(x^2+x+1) - \ln(x^2-x-1)) +$$

$$+\frac{1}{4}\left(\int \frac{dx}{(x^2-x+1)} + \int \frac{dx}{(x^2+x+1)}\right)$$

> Evaluate the following integral:

$$\int \frac{dx}{(x^2 \pm x + 1)} = \int \frac{dx}{(x \pm \frac{1}{2})^2 + \frac{3}{4}} = \frac{4}{3}\int \frac{dx}{\frac{4}{3}(x \pm \frac{1}{2})^2 + 1}$$

$$= \frac{4}{3}\int \frac{dx}{(\frac{2}{\sqrt{3}}(x \pm \frac{1}{2}))^2 + 1} = \frac{4}{3}\frac{\sqrt{3}}{2}\int \frac{d(\frac{2}{\sqrt{3}}(x \pm \frac{1}{2}))}{(\frac{2}{\sqrt{3}}(x \pm \frac{1}{2}))^2 + 1} + C =$$

$$\frac{2}{\sqrt{3}}\int \frac{d(\frac{2}{\sqrt{3}}(x \pm \frac{1}{2}))}{(\frac{2}{\sqrt{3}}(x \pm \frac{1}{2}))^2 + 1} = \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x \pm \frac{1}{2})\right)$$

> And now the final integral:

$$\frac{1}{4}(\ln(x^2+x+1) - \ln(x^2-x-1)) +$$

$$+\frac{1}{4}\left(\frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x - \frac{1}{2})\right) + \frac{2}{\sqrt{3}} \operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x + \frac{1}{2})\right)\right) + C =$$

$$\frac{1}{4}(\ln(x^2+x+1) - \ln(x^2-x-1)) +$$

$$+\frac{1}{2\sqrt{3}}\left(\operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x - \frac{1}{2})\right) + \operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x + \frac{1}{2})\right)\right) + C$$

Subproblem F

$$\int \frac{dx}{x^6+1}$$

$$\int \frac{dx}{(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)(x^2+1)} =$$

$$\int \left(\frac{Cx+D}{x^2-\sqrt{3}x+1} + \frac{Ex+F}{x^2+\sqrt{3}x+1} + \frac{Ax+B}{x^2+1} \right) dx$$

$$\begin{gathered} 1 = -B + D - F + (-A + C - E)x^5 + (-B + \sqrt{3}C + D + \sqrt{3}E - F)x^4 + \\ +(A + 2C + \sqrt{3}D - 2E + \sqrt{3}F)x^3 + (B + \sqrt{3}C + 2D + \sqrt{3}E - 2F)x^2 + \\ +(-A + C + \sqrt{3}D - E + \sqrt{3}F)x \end{gathered}$$

$$\begin{pmatrix} 0 & -1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & \sqrt{3} & -1 & \sqrt{3} \\ 0 & 1 & \sqrt{3} & 2 & \sqrt{3} & -2 \\ 1 & 0 & 2 & \sqrt{3} & -2 & \sqrt{3} \\ 0 & -1 & \sqrt{3} & 1 & \sqrt{3} & -1 \\ -1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies$$

$$\begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{3} \\ -\frac{1}{2\sqrt{3}} \\ \frac{1}{3} \\ -\frac{1}{2\sqrt{3}} \\ -\frac{1}{3} \end{pmatrix}$$

$$\int \left(\frac{-\frac{x}{2\sqrt{3}} + \frac{1}{3}}{x^2-\sqrt{3}x+1} + \frac{\frac{x}{2\sqrt{3}} + \frac{1}{3}}{x^2+\sqrt{3}x+1} \right) dx + \frac{1}{3} \int \frac{dx}{x^2+1}$$

Solve for

$$\int \frac{\pm\frac{x}{2\sqrt{3}} + \frac{1}{3}}{x^2 \pm \sqrt{3}x+1} dx = \frac{1}{12} \int \frac{1}{x^2 \pm \sqrt{3}x+1} dx \pm \frac{1}{4\sqrt{3}} \int \frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1} dx$$

First part:

$$\begin{gathered} \int \frac{dx}{x^2 \pm \sqrt{3}x+1} = 4 \int \frac{dx}{(2x \pm \sqrt{3})^2+1} = 2 \int \frac{d(2x \pm \sqrt{3})}{(2x \pm \sqrt{3})^2+1} = \\ 2 \operatorname{arctg}(2x \pm \sqrt{3}) + C \end{gathered}$$

Second part:

$$\int \frac{2x \pm \sqrt{3}}{x^2 \pm \sqrt{3}x+1} dx = \int \frac{d(2x \pm \sqrt{3})}{x^2 \pm \sqrt{3}x+1} = \ln(x^2 \pm \sqrt{3}x+1) + C$$

Third part:

$$\int \frac{dx}{x^2+1} = \operatorname{arctg}(x) + C$$

Finally:

$$\int \left(\frac{-\frac{x}{2\sqrt{3}} + \frac{1}{3}}{x^2 - \sqrt{3}x + 1} + \frac{\frac{x}{2\sqrt{3}} + \frac{1}{3}}{x^2 + \sqrt{3}x + 1} \right) dx + \frac{1}{3} \int \frac{dx}{x^2+1} =$$

$$\frac{1}{6}(\operatorname{arctg}(2x + \sqrt{3}) + \operatorname{arctg}(2x - \sqrt{3})) +$$
$$\frac{1}{4\sqrt{3}} \left(\ln(x^2 + \sqrt{3}x + 1) - \ln(x^2 - \sqrt{3}x + 1) \right) + \frac{1}{3} \operatorname{arctg} x + C$$

Subproblem G

$$\int \frac{dx}{x^5 - x^4 + x^3 - x^2 + x - 1}$$

I'm insanely lazy, so notice that

$$-\frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx - \frac{1}{2} \int \frac{dx}{x^2 - x + 1} + \frac{1}{3} \int \frac{dx}{x-1}$$

First two integrals we have already calculated and the last one is terribly easy, so the answer is:

$$-\frac{1}{6} \ln(x^2 + x + 1) + \frac{1}{3} \ln(1 - x) - \frac{\sqrt{3}}{3} \operatorname{arctg} \left(\frac{2\sqrt{3}}{3} x - \frac{\sqrt{3}}{3} \right)$$