Calculus, Homework 12

Problem 1

Is it true that for a continuous mapping

Subproblem A

image of a closed set is closed?

> The portmanteau "clopen" looks so cursed honestly, but it's fun to use so don't mind if I do.

We know that $\mathbb{R}$ is clopen per definition. Let's map $\mathbb{R} \mapsto \mathbb{R}$ using a continuous function like e^x. $f(\mathbb{R}) = \mathbb{R}_{>0}$ is not closed since $\exists x = 0, \forall \varepsilon > 0$, the following is true: $0 = x \notin \mathbb{R}_{>0}, x - \varepsilon \notin \mathbb{R}_{>0}$ but $x + \varepsilon \in \mathbb{R}_{>0} \implies$ the image of a closed set is not always closed.

Answer: no.

Subproblem B

image of a bounded set is bounded?

Take two spaces.

The first one E_1 being $\mathbb{N}$ with a discrete metric (distance is equal to 1 if the values are the same, and 0 otherwise). In this space, all numbers are contained within any ball with $r > 0 \implies$ the set is bounded.

The second one E_2 being a classic metric space with a euclidean distance metric on $\mathbb{N}$.

Now consider an obviously continuous mapping $g(x) \mapsto x$ on $E_1 \mapsto E_2$. The function would map values from the first space to the entirety of the second space, which means that $\forall x_i \in E_1, \exists x_j$ such that $f(x_i) < f(x_j) \implies \sup(E_2)$ does not exist $\implies$ the set is unbounded $\implies$ the answer is false.

Answer: no.

Problem 2

Let (E, d) be a metric space, $A, B \subseteq E$ are bounded sets. Prove that $A \cup B$ is bounded.

If a set is bounded, then it is certainly included in some kinda ball with a radius. Thus, for some points x_1, x_2, $A \subseteq \mathcal{U}(a_1, r_1), B \subseteq \mathcal{U}(a_2, r_2)$ with radii r_1, r_2 around points a_1, a_2.

Now, take a point a that would be precisely between points a_1, a_2. There would be some kinda radius defined by the distance $r = d(a_1, a) = d(a_2, a)$. We could construct a new ball centered on the point a with a radius of $r + \max\{r_1, r_2\}$. This ball would guaranteeably contain both balls that surround sets $A, B \implies$ their union would also be contained in some kinda ball and thus would be bounded, q. e. d.

Problem 3

Prove the Jensen inequality. Let $f(x)$ be convex, $\alpha_1, \dots, \alpha_n$ be positive numbers, $\alpha_1 + \dots + \alpha_n = 1$.

Subproblem A

If $\alpha_1, \dots \alpha_n \in \mathbb{Q}$, then

$$f(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \dots + \alpha_n f(x_n)$$

Subproblem B

If the function $f(x)$ is continuous, then

$$f(\alpha_1 x_1 + \dots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \dots + \alpha_n f(x_n)$$

Problem 4

Using knowledge about convex functions from the seminars, prove the inequalities:

Subproblem A

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n} \quad x_1, \ldots, x_n \geq 0$$

Per Jensen's inequality, a function is **concave** $\iff$

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

Let's take all $\alpha_i = \frac{1}{n}$:

$$f\left(\frac{1}{n}x_1 + \cdots + \frac{1}{n}x_n\right) \geq \frac{1}{n}f(x_1) + \cdots + \frac{1}{n}f(x_n)$$

We know from godforbidden *microeconomics* that $\log(x)$ is **concave**; therefore, we may substitute $f = \log$:

$$\log\left(\frac{1}{n}x_1 + \cdots + \frac{1}{n}x_n\right) \geq \frac{1}{n}\log(x_1) + \cdots + \frac{1}{n}\log(x_n)$$

$$\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \log(x_1^{\frac{1}{n}}) + \cdots + \log(x_n^{\frac{1}{n}})$$

$$\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \log(x_1^{\frac{1}{n}} \cdots x_n^{\frac{1}{n}})$$

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

which is the required identity.

Subproblem B

Let $A, B, p, q > 0$ given that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sqrt[p]{A}\sqrt[q]{B} \leq \frac{A}{p} + \frac{B}{q}$$

Problem 5

Prove that a continuous function $f(x)$ is convex if and only if $\forall a, b, c \colon a < b < c$ the inequation

$$\det\begin{pmatrix} a & f(a) & 1 \\ b & f(b) & 1 \\ c & f(c) & 1 \end{pmatrix} \geqslant 0$$

Do some permutations of the determinant:

$$\det\begin{pmatrix} a & f(a) & 1 \\ b & f(b) & 1 \\ c & f(c) & 1 \end{pmatrix} = af(b) + cf(a) + bf(c) - cf(b) - bf(a) - af(c) =$$

$$= (c-b)f(a) + (a-c)f(b) + (b-a)f(c) \geqslant 0$$

$$(b-c)f(a) + (c-a)f(b) + (a-b)f(c) \leqslant 0$$

Drag $f(b)$ out of this forsaken identity:

$$f(b) \leqslant \frac{(c-b)f(a) + (b-a)f(c)}{c-a}$$

Since b is in-between a and c, we could form it using a weighted linear combination of a and c, for some $\lambda \in (0,1)$:

$$b = \lambda a + (1-\lambda)c$$

Now plug in this equation into the equation above:

$$f(b) \leqslant \frac{(c - \lambda a - (1-\lambda)c)f(a) + (\lambda a + (1-\lambda)c - a)f(c)}{c-a}$$

$$f(b) \leqslant \frac{(\lambda c - \lambda a)f(a) + (\lambda a + c - \lambda c - a)f(c)}{c-a}$$

$$f(b) \leqslant \frac{\lambda(c-a)f(a) + (1-\lambda)(c-a)f(c)}{c-a}$$

$$f(b) \leqslant \lambda f(a) + (1-\lambda)f(c)$$

Plug the definition of b back into the equation above:

$$f(\lambda a + (1-\lambda)c) \leqslant \lambda f(a) + (1-\lambda)f(c)$$

$$f(pa + qc) \leqslant pf(a) + qf(c)$$

which is precisely the definition of a convex function. All the transformations were equivalent; therefore, the "if and only if" statement is also proven, q. e. d.

Problem 6

Find intervals on which the function is convex and concave and inflection points and plot the graph of

Subproblem A

$$f(x) = x^2 e^x$$

$$f'(x) = 2xe^x + e^x x^2 = (2x + x^2)e^x$$

Since $e^x > 0$, then the first derivative is only equal to 0 if

$$x(x+2) = 0 \implies x_1 = 0, x_2 = -2$$

$$f''(x) = (2+2x)e^x + (2x + x^2)e^x = (2 + 4x + x^2)e^x$$

Since $e^x > 0$, then the second derivative is only equal to 0 if

$$x^2 + 4x + 2 = 0$$

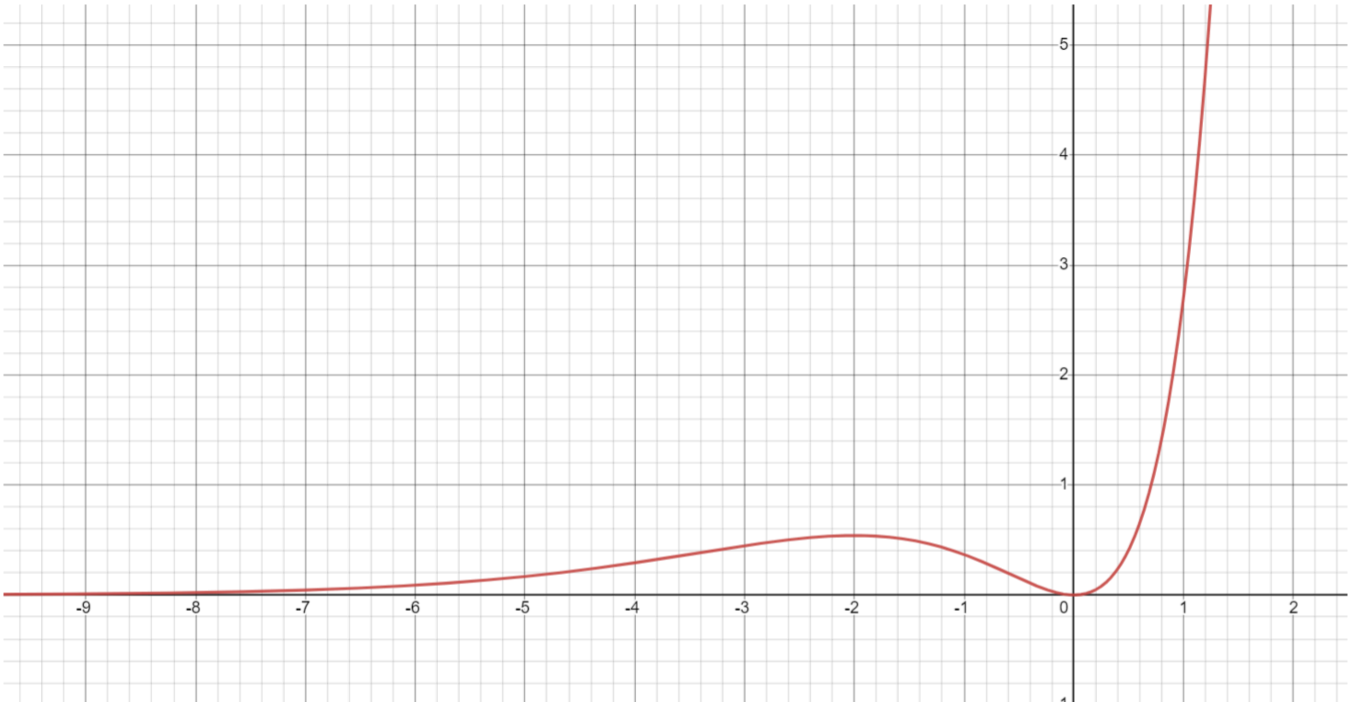
$$x = -2 \pm \sqrt{2}$$

which are the inflection points.

Thus, the function would be:

- ascending on $[0, +\infty)$
- descending on $[-2, 0]$
- ascending on $(-\infty, -2]$
- convex on $[-2+\sqrt{2}, +\infty)$
- concave on $[-2-\sqrt{2}, -2+\sqrt{2}]$
- convex on $(-\infty, -2-\sqrt{2}]$

Corresponding plot



Subproblem B

$$f(x) = 1 + 4x^2 - \frac{2x^4}{3}$$

The function is even $\implies$ I will only consider it on $\mathbb{R}^+$.

$$f'(x) = 8x - \frac{8x^3}{3}$$

Find the zeros in $\mathbb{R}^+$:

$$x\left(1 + \frac{x}{\sqrt{3}}\right)\left(1 - \frac{x}{\sqrt{3}}\right) = 0 \implies x_1 = 0, x_2 = \sqrt{3}$$

$$f''(x) = 8 - 8x^2$$

Find the zeros in $\mathbb{R}^+$:

$$1 - x^2 = 0 \implies (x-1)(x+1) = 0 \implies x = 1$$

Since coefficients next to x^2 and x^3 are negative, we start from a concave and a descending intervals respectively.

The function would be (likewise for $\mathbb{R}^-$ but mirrored):

- descending on $[\sqrt{3}, +\infty)$
- ascending on $[0, \sqrt{3}]$
- concave on $[1, +\infty)$
- convex on $[0, 1]$

Corresponding plot: