Discrete Mathematics, Homework 10

Problem 1

Subproblem A

Is it true that the composition of reflexive relations over set A is reflexive?

Per the definition of composition, $(a, a) \in R \circ R$ only if there is $x \in A$ so that $(a, x) \in R$ and $(x, a) \in R$. Since R is reflexive, then take $x = a$ and then since $(a, a) \in R$ and (duh) $(a, a) \in R$ then $(a, a) \in R \circ R$, q. e. d.

Answer: yes, it is true.

Subproblem B

Is it true that the composition of antireflexive relations over set A is antireflexive?

Per the definition of antireflexiveness, $(a, a) \notin R$. Prove by counter-example:

$$R_1 = \{(a, x)\} \;\; R_2 = \{(x, a)\} \;\; R_1 \circ R_2 = \{(a, a)\}$$

$R_1 \circ R_2$ is reflexive, therefore no, it is not true.

Answer: no, it is not true.

Problem 2

Give an example of a relation that is not reflexive, nor antireflextive, nor symmetric, nor antisymmetric, nor transitive, but is total.

Consider the following relation over $\{0, 1, 2\}$, we will define it everywhere, **so it would be total**:

	0	1	2
0	1	1	0
1	1	0	1
2	1	0	0

$$R = \{(0,0),(0,1),(1,0),(2,0),(1,2)\}$$

Check for each property:

- not reflexive since $\exists(1,1) \notin R$, whereas all $(a,a) \in R$ is required;
- not anti-reflexive since $\exists(0,0) \in R$, whereas all $(a,a) \notin R$ is required;
- not symmetric since $\exists(2,0) \in R$ and $\exists(0,2) \notin R$, whereas $\forall a \neq b, (a,b) \in R\colon (b,a) \in R$ is required;
- not anti-symmetric since $\exists(1,0),(0,1) \in R$, whereas $\forall a \neq b, (a,b) \in R\colon (b,a) \notin R$ is required;
- not transitive since $\exists(0,1),(1,2) \in R$ and $(0,2) \notin R$, whereas $\forall(a,b),(b,c) \in R\colon (a,c) \in R$ is required.

Problem 3

How many antisymmetric non-transitive binary relations are there on the set of $\{1,2,3\}$?

Можно сделать перебор ручками, но я сделаю перебор питоном (комментарии прилагаются):

```
def is_antisymmetric(relation: list) -> bool:
    relation1 = relation.copy()

    # убираем пары (a, a), потому что они нерелевантны
    for t in range(1, 4):
        if (t, t) in relation1:
            relation1.remove((t, t))

    # проверяем, что условие антисимметричности выполнено для всех
    # для каждой пары (a, b) нет (b, a)
    if all(t[::-1] not in relation1 for t in relation1):
        return True
```

```
    return False

def is_transitive(relation: list) -> bool:
    # условие транзитивности: если (a, b) есть и при b = c (c, d) (b, d)
    # тоже есть, тогда (a, d) должно быть в отношении, иначе бан
    for a, b in relation:
        for c, d in relation:
            if b == c and ((a, d) not in relation):
                return False

    return True

def mask_array(array: list, mask: list) -> list:
    # просто функция бинарной маски
    result = []
    for i in range(len(array)):
        if mask[i]:
            result.append(array[i])

    return result

# генерируем все пары бинарного отношения {1, 2, 3}^2
relation_select = [(i, j) for i in range(1, 4) for j in range(1, 4)]
mask_length = len(relation_select)

counter = 0
for i in range(0, 2 ** mask_length):
    # перебор всех вариантов через бинарную маску
    mask = [char == "1" for char in bin(i)[2:].zfill(mask_length)]
    masked = mask_array(relation_select, mask)
    if not is_transitive(masked) and is_antisymmetric(masked):
        counter += 1
        print(masked)

print(counter)
```

```
[(2, 3), (3, 1)]
[(2, 3), (3, 1), (3, 3)]
[(2, 2), (2, 3), (3, 1)]
[(2, 2), (2, 3), (3, 1), (3, 3)]
[(2, 1), (3, 2)]
... [54 lines omitted]
[(1, 1), (1, 2), (2, 2), (3, 1), (3, 3)]
[(1, 1), (1, 2), (2, 2), (2, 3)]
[(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)]
[(1, 1), (1, 2), (2, 2), (2, 3), (3, 1)]
[(1, 1), (1, 2), (2, 2), (2, 3), (3, 1), (3, 3)]
```

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64
```

Okay, I actually came up with a better solution ($\overline{X}$ denotes an inverted bit in the table):

	1	2	3
1	X	A	B
2	$\overline{A}$	X	C
3	$\overline{B}$	$\overline{C}$	X

X are currently irrelevant, let's consider all relations that are transitive by assigning required values to A, B, C and then we could add all combinations out of the set $\{(1,1),(2,2),(3,3)\}$ to the relation without affecting its transitivity or anti-symmetry. Using a binary mask, there would be 2^3 such combinations, which we would need to later multiply by all valid relations dependent on A, B, C.

Find all anti-symmetric relations by filling the table above: for each pair $(A, \overline{A})$ there are three options that maintain antisymmetry: $(0,0),(1,0),(0,1)$. Therefore, by applying a ternary mask, there would be $3^3 = 27$ such relations.

We know that an empty relation is transitive and relations consisting of a single element are transitive, thus removing $1+6$ possible relations. Now we need to check 20 remaining ones, which is totally doable by hand:

```
[(3, 1), (3, 2)] - trans (no connections)
[(2, 3), (3, 1)] - non-trans (connection and only 2 elements)
[(2, 1), (3, 2)] - non-trans (connection and only 2 elements)
[(2, 1), (3, 1)] - trans (no connections)
[(2, 1), (3, 1), (3, 2)] - trans
[(2, 1), (2, 3)] - trans (no connections)
[(2, 1), (2, 3), (3, 1)] - trans
[(1, 3), (3, 2)] - non-trans (connection and only 2 elements)
[(1, 3), (2, 3)] - trans (no connections)
[(1, 3), (2, 1)] - non-trans (connection and only 2 elements)
[(1, 3), (2, 1), (3, 2)] - non-trans (lacking (1, 2), for instance)
[(1, 3), (2, 1), (2, 3)] - trans
[(1, 2), (3, 2)] - trans (no connections)
[(1, 2), (3, 1)] - non-trans (connection and only 2 elements)
[(1, 2), (3, 1), (3, 2)] - trans
```

```
[(1, 2), (2, 3)] - non-trans (connection and only 2 elements)
[(1, 2), (2, 3), (3, 1)] - non-trans (lacking (2, 1), for instance)
[(1, 2), (1, 3)] - trans (no connections)
[(1, 2), (1, 3), (3, 2)] - trans
[(1, 2), (1, 3), (2, 3)] - trans
```

There are 8 non-transitive antisymmetric relations, and now we may add any of the 8 subsets of $\{(1,1),(2,2),(3,3)\}$ to it, maintaining its non-transitivity and anti-symmetry, thus we get the final answer of $8 \times 8 = 64$.

Answer: 64

Problem 4

There are 2 elements in a relation. How many elements could exist in its transitive closure?

Transitive closure R^* always contains R, so at the least it may contain 2 elements.

For each ordered pair of different tuples in relation R, we need to add at max a single tuple (in case this tuple doesn't already exist). Therefore, the maximum number elements added to the transitive closure on the top of the already-existing $|R|$ elements will be equal to $|R|(|R|-1)$ (that's the number of pairs of tuples).

Therefore, $|R^*| \leq |R| + |R|(|R|-1) = |R| + |R|^2 - |R| = |R|^2 \Rightarrow |R^*| \leq |R|^2$

Thus, the cardinality of the transitive closure set may not be more than $|R|^2 = 2^2 = 4$.

Now, prove by example that each of the possibilities for $|R*| = 2, 3, 4$ is possible:

- For $A = \{0,1\}, R = \{(0,0),(0,1)\}$, $R^* = R = \{(0,0),(0,1)\}, |R^*| = 2$
- For $A = \{0,1,2\}, R = \{(0,1),(1,2)\}$, $R^* = \{(0,1),(1,2),(0,2)\}, |R^*| = 3$
- For $A = \{0,1\}, R = \{(0,1),(1,0)\}$, $R^* = \{(0,0),(0,1),(1,0),(1,1)\}, |R^*| = 4$

Thus, we get $|R^*| = 2, 3, 4$.

Answer: 2, 3, or 4.