Homework 3

Problem 3.1

Per definition, functions from A to B are subsets of $A \times B$, and therefore, theoretical set operations are applicable to them. Let f, g be two functions from A to B.

Is it true that:

Subproblem A

Their union is also a function?

In other words, we need to prove that their union ($f \cup g$) is a subset of $A \times B$.

Let $\forall n, m \in \mathbb{N} \colon a_n \in A, b_m \in B$; therefore,

$$A \times B = \begin{Bmatrix} (a_1, b_1) & \dots & (a_1, b_m) \\ \vdots & \ddots & \vdots \\ (a_n, b_1) & \dots & (a_n, b_m) \end{Bmatrix}$$

Take some $A', A'' \subseteq A$ and some $B', B'' \subseteq B$ such that $\forall x, f \colon A' \mapsto B', g \colon A'' \mapsto B''$.

Now, for some $K_p = \{k \mid k \in \mathbb{N}\}$ (different p-s are selected for each set), the following is true:

- for $K_{p_1} \colon \{a_k \mid k \in \mathbb{N}, 1 \leq k \leq n\} = A'$
- for $K_{p_2} \colon \{a_k \mid k \in \mathbb{N}, 1 \leq k \leq n\} = A''$
- for $K_{p_3} \colon \{b_k \mid k \in \mathbb{N}, 1 \leq k \leq n\} = B''$
- for $K_{p_4} \colon \{b_k \mid k \in \mathbb{N}, 1 \leq k \leq n\} = B''$

If we take the union of these sets, then the domans and ranges "merge" (merging together pairs from either f, g, or both):

- $\text{Dom}\ (f \cup g) = \text{Dom}\ f \cup \text{Dom}\ g = A' \cup A''$;
- $\text{Range}\ (f \cup g) = \text{Range}\ f \cup \text{Range}\ g = B' \cup B''$.

Therefore, $(f \cup g)(x) \colon A' \cup A'' \mapsto B' \cup B''$, which means that any $(x, y) \in A' \cup A'' \times B' \cup B''$ since $(x, y) \in f \vee (x, y) \in g$ So, the union of two functions is a function since the range and domain of the resulting function are subsets of $A \times B$, q. e. d.

Subproblem B

Their intersection is also a function?

In other words, we need to prove that their union ($f \cap g$) is a subset of $A \times B$.

`-- || -- This part is identical to above, so it was omitted -- || --`

If we take the intersection of these sets, then the domans and ranges intersectand only pairs that previously were in both sets remain:

- $\text{Dom}\,(f \cap g) = \text{Dom}\,f \cap \text{Dom}\,g = A' \cap A''$;
- $\text{Range}\,(f \cap g) = \text{Range}\,f \cap \text{Range}\,g = B' \cap B''$.

Therefore, $(f \cap g)(x) \colon A' \cap A'' \mapsto B' \cap B''$, which means that any $(x, y) \in A' \cap A'' \times B' \cap B''$ since $(x, y) \in f \lor (x, y) \in g$. So, the intersection of two functions is a function since the range and domain of the resulting function are subsets of $A \times B$, q. e. d.

Problem 2

Prove that function $f \colon (x_0, x_1, \dots, x_{n-1}) \mapsto (x_0 + 0, x_1 + 1, \dots, x_{n-1} + n - 1)$ is a bijection from the set of non-descending integer sequences to the set of strictly ascending integer sequences.

Since the function has a clearly range and domain, it's total and not partial.

According to bijection definition, the function should be both a surjection and an injection:

- injection definition: $f(x) \colon A \mapsto B, \forall x, y \colon x \neq y \Rightarrow f(x) \neq f(y)$
- surjection definition: $f(x) \colon A \mapsto B, \text{Range}\,f = B, \forall y \in B, \exists x \in A, (x, y) \in f$

Consider the sequence $(x_0, x_1, \dots, x_{n-1})$, which is non-descending, $\Rightarrow x_0 \leq x_1 \leq \dots \leq x_{n-1}$.

Add 1 to every element of the equation and compare to x_0:

$$x_0 < x_0 + 1 \leq x_1 + 1 \leq \dots \leq x_{n-1} + 1$$

Omit $x_0 + 1$ from the equation to get the first part of the result inequation:

$$x_0 + 0 < x_1 + 1 \leq \cdots \leq x_{n-1} + 1$$

By induction, repeat the same steps over and over again to arrive at the following inequation:

$$x_0 + 0 < x_1 + 1 < \cdots < x_{n-1} + n - 1$$

Next, since the same constant is added to each element x_n depending on its position, then the action that happens is a kind of "a parallel shift" that "shifts" all the values to a set of different ones linearly, so it's impossible for multiple values to map to the same value $\Rightarrow$ $f\colon (x_0, x_1, \ldots, x_{n-1}) \mapsto (x_0 + 0, x_1 + 1, \ldots, x_{n-1} + n - 1), \forall x_k, x_p \colon x_k \neq x_p \Rightarrow f(x_k) \neq f(x_p) \Rightarrow$ the function is an injection.

Lastly, say that $(b_1, b_2, \ldots b_{n+1})$ is some sequence. Then we can restore the original sequence using an inverse function f^{-1} that subtracts n from every n-th number. Since the inverse of a linear function is also linear, then the entire $\operatorname{Range} f$ can be mapped to $\operatorname{Range} f^{-1}$ and vice versa via linear transformations. The field of any sequence of numbers of any length is reachable through linear transformations, so $\operatorname{Range} f = (x_0 + 0, x_1 + 1, \ldots, x_{n-1} + n - 1)$ includes all possible sequences and $\forall y \in B, \exists x \in A, (x, y) \in f$, which was just described above.

The function f is both an injection and a surjection $\Rightarrow$ the function is a bijection, q. e. d.

Problem 3 (it has been annihilated so there is nothing)

Problem 4

Set A consists of triangles, the sides of which are all natural numbers (and not equal to 0), and their perimeter is equal to 2020.

Set B consists of triangles, the sides of which are all natural numbers (and not equal to 0), and their perimeter is equal to 2023. What is larger, $|A|$ or $|B|$?

$$A = \{(a_A, b_A, c_A) \mid a_A + b_A + c_A = 2020, a_A + b_A > c_A, a_A + c_A > b_A, b_A + c_A > a_A\}$$

$$B = \{(a_B, b_B, c_B) \mid a_B + b_B + c_B = 2023, a_B + b_B > c_B, a_B + c_B > b_B, b_B + c_B > a_B\}$$

Add 1 to each of the sides in set A, then $a_A + 1 + b_A + 1 + c_A + 1 = 2023$ (all the equations could be conserved because we would add 2 to its lefthand side, which more than we add to the righthand side). Then, we could map every single element from $A \mapsto$ some element from B:

$$(a_A, b_A, c_A) \mapsto (a_A + 1, b_A + 1, c_A + 1)$$

Attempt to find some value in B that is not in A. It is obvious that for some $a_B = a_A + 1 \geq 2$ since $a_A \geq 1$. Therefore, find such a triple $\in B$ that for $a_A = 1$ and a triangle with a perimeter of 2023 exists:

$$(a_B, b_B, c_B) = (1, 1011, 1011)$$

Since $1 + 1011 > 1011$, $1011 + 1011 > 1$, $1 + 1011 > 1011$, then this is an unmapped value in set $B \Rightarrow$ there are more elements in B than in $A \Rightarrow |B| > |A|$

Answer: $|B| > |A|$