



Linear Algebra, Homework 15 (in theory)

Problem 1

Let vectors e_1, \dots, e_n and x be defined by their coordinates in some basis:

$$e_1 = (2, 1, -3), e_2 = (3, 2, -5), e_3 = (1, -1, 1), x = (6, 2, -7)$$

Prove that e_1, \dots, e_n is also a basis in this space and find coordinates of vector x in this basis.

Check whether the vectors e_i are linearly independent:

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -3 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -1 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

They are linearly independent; therefore, e_i form a basis, q. e. d.

We need to find what vector gets "sent" to vector x after the basis matrix is applied as follows:

$A\bar{x} = x$, thus:

$$\begin{pmatrix} 2 & 3 & 1 & | & 6 \\ 1 & 2 & -1 & | & 2 \\ -3 & -5 & 1 & | & -7 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -1 & 3 & | & 2 \\ 0 & 1 & -2 & | & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 & | & 2 \\ 0 & -1 & 3 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 & | & 6 \\ 0 & 1 & -3 & | & -2 \\ 0 & 0 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{pmatrix}$$

Answer:

$$\bar{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 2

How would a basis conversion matrix change if one would write the vectors of each of bases in a reversed order?

It is pretty obvious that this matrix would be a symmetry of some sort.

Assume the original basis conversion matrix is something like this (example given for 3 dimensions for simplicity's sake).

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Reversing the order of dimension of one of the bases would have the same effect as reversing (mirroring) lines over their center. Similarly, reversing the order of dimensions of the second basis would have an effect of reversing (mirroring) columns over their center.

Both these transformations could be applied in any order, as shown by the matrix multiplication below:

$$\begin{aligned} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \\ & = \begin{pmatrix} a_{31} & a_{22} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{pmatrix} = \\ & = \begin{pmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix} \end{aligned}$$

Therefore, **the described transformation is identical to a center symmetry of a matrix.**

Problem 3

Let $V = \mathbb{R}[x]_{\leq n}$ be a space of polynomials of degree $\leq n$ from argument x with real coefficients. Systems of polynomials $\{1, x, \dots, x^n\}$ and $\{1, (x-a), \dots, (x-a)^n\}$ for some $a = \text{const}$ are bases in V . Find a transformation matrix from the second basis to the first one and also coordinates of the polynomial $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ in second basis.

Firstly, each polynomial could be represented by a vector like this:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n \mapsto (a_0, a_1, a_2, \dots, a_n)^T$$

Therefore, the matrix for the first basis is

$$A_{n+1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

We need to find some transformation matrix T from basis B to basis A :

$$BT = A$$

We are lucky that $A = E \implies T = B^{-1} \implies$ we just need to find the matrix for the second basis and take its inverse:

$$(x - a)^n = \binom{n}{0} x^0 (-a)^n + \binom{n}{1} x^1 (-a)^{n-1} + \dots + \binom{n}{n} x^n (-a)^0$$

Therefore, every matrix element would be as follows (considering that invalid combinations return 0):

$$t_{ij} = \binom{j-1}{i-1} (-a)^{j-i}$$

$$B_{n+1} = \begin{pmatrix} 1 & -a & a^2 & -a^3 & \dots & \binom{n}{0}(-a)^n \\ 0 & 1 & -2a & 3a^2 & \dots & \binom{n}{1}(-a)^{n-1} \\ 0 & 0 & 1 & -3a & \dots & \binom{n}{2}(-a)^{n-2} \\ 0 & 0 & 0 & 1 & \dots & \binom{n}{3}(-a)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Now, take the inverse, which is **the required transformation matrix**:

$$B_{n+1}^{-1} = \begin{pmatrix} 1 & a & a^2 & a^3 & \dots & \binom{n}{0}a^n \\ 0 & 1 & 2a & 3a^2 & \dots & \binom{n}{1}a^{n-1} \\ 0 & 0 & 1 & 3a & \dots & \binom{n}{2}a^{n-2} \\ 0 & 0 & 0 & 1 & \dots & \binom{n}{3}a^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Now, find the coordinates of the aforementioned polynomial:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n \mapsto (a_0, a_1, a_2, \dots, a_n)^T$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = T \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} \implies \begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} = T^{-1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = B_{n+1} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\begin{pmatrix} a'_0 \\ a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} a_0 + -aa_1 + a^2a_2 + -a^3a_3 + \dots + \binom{n}{0}(-a)^na_n \\ a_1 + -2aa_2 + 3a^2a_3 + \dots + \binom{n}{1}(-a)^{n-1}a_n \\ a_2 + -3aa_3 + \dots + \binom{n}{2}(-a)^{n-2}a_n \\ a_3 + \dots + \binom{n}{3}(-a)^{n-3}a_n \\ \vdots \\ 1 \end{pmatrix}$$

Problem 4

Subspaces U and W in F^4 are defined as set of solutions to HSLUs

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} x_1 + x_3 = 0 \\ x_1 + 3x_2 + x_4 = 0 \end{cases}$$

respectively. Find the basis in $U \cap W$ and basis in $U + W$.

Take the intersection of two HSLUs to find $U \cap W$:

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \implies$$

The solution is

$$\begin{pmatrix} -x_4 \\ 0 \\ x_4 \\ x_4 \end{pmatrix}$$

And the basis in $U \cap W$ would be something like

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Now, find bases in both U and W :

$$U: \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \Rightarrow$$

The solution would be

$$\begin{pmatrix} -2x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

Thus, the basis for U would consist of vectors $\{(-2, 1, 1, 0), (1, -1, 0, 1)\}$.

$$W: \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \Rightarrow$$

The solution would be

$$\begin{pmatrix} -x_3 \\ \frac{1}{3}(x_3 - x_4) \\ x_3 \\ x_4 \end{pmatrix}$$

Thus, the basis for U would consist of vectors $\{(-3, 1, 3, 0), (0, -1, 0, 3)\}$.

Now, $U + W$ would be the linear span of all the basis vectors of U and W . To find the basis, write another matrix:

$$\begin{pmatrix} -2 & 1 & -3 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

Thus, the basis for $U + W$ would consist of vectors $\{(-2, 1, 1, 0), (1, -1, 0, 1), (-3, 1, 3, 0)\}$

Problem 5

Subspaces U and W in F^4 are defined as set of solutions to HSLUs

$$\begin{cases} x_1 + x_2 - x_5 = 0 \\ x_2 + x_3 + x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases} \text{ and } \begin{cases} x_1 + x_3 + x_5 = 0 \\ 2x_2 + 3x_3 + x_4 = 0 \\ x_1 + 2x_2 + x_3 + 2x_4 - x_5 = 0 \end{cases}$$

respectively. Find $\dim(U \cap W)$ and $\dim(U + W)$.

For $W \cap W$, do the same thing as above:

$$\begin{aligned}
 & \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 3 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix} \sim \\
 & \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & \diamond \\ 0 & 1 & 0 & 0 & \diamond \\ 0 & 0 & 1 & 0 & \diamond \\ 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \\
 & \dim(U \cap W) = 5 - 4 = 1
 \end{aligned}$$

We could use the following formula to get the required answer:

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$$

Find dimensions of each of the subspaces:

$$\begin{aligned}
 U: & \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \diamond & \diamond \\ 0 & 1 & 0 & \diamond & \diamond \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \dim U = 5 - 3 = 2 \\
 W: & \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \diamond & \diamond \\ 0 & 1 & 0 & \diamond & \diamond \\ 0 & 0 & 1 & \diamond & \diamond \end{pmatrix} \Rightarrow \dim W = 5 - 3 = 2
 \end{aligned}$$

Therefore, **the answer:**

$$\dim(U + W) = 2 + 2 - 1 = 3$$

Problem 6

Find dimensions of the sum and the intersection of linear spans of vectors in \mathbb{R}^4 :

Subproblem A

$$S = \langle (1, 1, 1, 1), (1, -1, 1, -1), (1, 3, 1, 3) \rangle$$

$$T = \langle (1, 2, 0, 2), (1, 2, 1, 2), (3, 1, 3, 1) \rangle$$

It is obvious that $2 \times (1, 1, 1, 1) - (1, -1, 1, -1) = (1, 3, 1, 3)$, which implies that $\dim S = 2$.

As for T :

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

from which $\dim T = 3$ is obvious.

Now, find the size of $S + T$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 3 \\ 1 & -1 & 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 & 1 & 3 \\ 1 & -1 & 3 & 2 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & \diamond & \diamond \\ 0 & 1 & -1 & 0 & \diamond & \diamond \\ 0 & 0 & 0 & 1 & \diamond & \diamond \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies $\dim(S + T) = 3$

Now per our awesome formula **get the answer**:

$$\dim(S \cap T) = -\dim(S + T) + \dim S + \dim T = -3 + 2 + 3 = 2$$

Subproblem B

$$S = \langle (2, -1, 0, -2), (3, -2, 1, 0), (1, -1, 1, -1) \rangle$$

$$T = \langle (3, -1, -1, 0), (0, -1, 2, 3), (5, -2, -1, 0) \rangle$$

Do the same things all over again:

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & -2 & -1 \\ 0 & 1 & 1 \\ -2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \dim S = 3$$

$$\begin{pmatrix} -3 & 0 & -5 \\ 1 & 1 & 2 \\ 1 & -2 & 1 \\ 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \implies \dim T = 3$$

$$\begin{pmatrix} 2 & 3 & 1 & -3 & 0 & -5 \\ -1 & -2 & -1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 & -2 & 1 \\ -2 & 0 & -1 & 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \diamond & \diamond & \diamond \\ 0 & 1 & 0 & \diamond & \diamond & \diamond \\ 0 & 0 & 1 & \diamond & \diamond & \diamond \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \implies \dim(S + T) = 3$$

Therefore, the **answer** is $\dim(S \cap T) = -\dim(S + T) + \dim S + \dim T = -3 + 3 + 3 = 3$

Problem 7

Find bases of a sum and an intersection of the following linear spans $A = \langle a_1, a_2, a_3 \rangle$ and $B = \langle a_1, a_2, a_3 \rangle$:

Subproblem A

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 2 & -1 & -3 \end{pmatrix}$$

$$(A|B) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 2 & 3 & 1 \\ 1 & -1 & 3 & 2 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & -2 \\ 0 & 1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -2 \end{pmatrix}$$

This implies that $\langle a_1, a_2, b_1 \rangle$ is the basis of $A + B$.

Now, proceed to the intersection and search for the fundamental system of solutions:

$$X = \begin{pmatrix} -2\alpha_3 + 2\beta_2 + 2\beta_3 \\ \alpha_3 + \beta_2 + \beta_3 \\ \alpha_3 \\ \beta_2 + 2\beta_3 \\ \beta_2 \\ \beta_3 \end{pmatrix} \implies$$

$$\begin{pmatrix} \Phi_A \\ \Phi_B \end{pmatrix} = \begin{pmatrix} -2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies \Phi_A = \begin{pmatrix} -2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

To find the basis of $A \cap B$, multiply $A\Phi_A$:

$$A\Phi_A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} -2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ 0 & 5 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

which implies $\{(3, 5, 1)\}$ is the basis of $A \cap B$

Subproblem B

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$

Here we go again...

$$(A| - B) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ -1 & 1 & 1 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This implies that $\langle a_1, a_2, a_3, b_1 \rangle$ is the basis of $A + B$.

Now, proceed to the intersection and search for the fundamental system of solutions:

$$X = \begin{pmatrix} \beta_2 + 2\beta_3 \\ \beta_2 \\ \beta_2 + 2\beta_3 \\ \beta_2 + \beta_3 \\ \beta_2 \\ \beta_3 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \Phi_A \\ \Phi_B \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \Phi_A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \end{pmatrix}$$

To find the basis of $A \cap B$, multiply $A\Phi_A$:

$$A\Phi_A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 2 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$$

which implies $\Phi_{A \cap B} = \{(1, 2, 2, 1, 1), (2, 2, 2, 2, 0)\}$ is the basis of $A \cap B$

Problem 8

I can't take this anymore this is so boring

$$A^T = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad B^T = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & -1 \end{pmatrix}$$

$$A^T \sim \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$X_A = \begin{pmatrix} -\alpha_4 + \alpha_5 \\ \alpha_4 \\ -\alpha_4 - \alpha_5 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} \implies \Phi_A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^T \sim B^T = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 \end{pmatrix}$$

$$X_B = \begin{pmatrix} -\beta_4 + \beta_5 \\ -\beta_4 + \beta_5 \\ \beta_4 - 2\beta_5 \\ \beta_4 \\ \beta_5 \end{pmatrix} \implies \Phi_B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \Phi_A^T \\ \Phi_B^T \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Phi_{A \cap B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which implies that the basis of $A \cap B$ is $\Phi_{A \cap B} = \{(1, 1, 1, 1, 0), (0, 1, 1, 0, 1)\}$

Both bases over field $A \cap B$ generate the same subspace because they are effectively the same (meaning that they both are bases of their linear spans):

$$(1, 2, 2, 1, 1), = (1, 1, 1, 1, 0) + (0, 1, 1, 0, 1)$$

$$(2, 2, 2, 2, 0) = 2 \times (1, 1, 1, 1, 0)$$

Problem 9

Let L_1 and L_2 be subspaces of a finite vector space V . Prove that if $\dim(L_1 + L_2) = 1 + \dim(L_1 \cap L_2)$, then sum $L_1 + L_2$ is equal to one of these subspaces and $L_1 \cap L_2$ to the other.

The formula from the lecture

$$\dim L_1 + \dim L_2 = \dim(L_1 + L_2) + \dim(L_1 \cap L_2)$$

together with

$$\dim(L_1 + L_2) = 1 + \dim(L_1 \cap L_2)$$

implies

$$\dim L_1 + \dim L_2 = 1 + 2 \dim(L_1 \cap L_2)$$

considering that $\dim L_2 \geq \dim L_1$ (per axiom of choice), $\dim L_2 = \dim L_1 + k, k \in \mathbb{Z}^+$

$$2 \dim L_1 - 2 \dim(L_1 \cap L_2) = 1 - k$$

$$\dim L_1 - \dim(L_1 \cap L_2) = \frac{1 - k}{2}$$

Since we have restricted k to be a non-negative number and $\dim L_i \geq 0$ and $\dim L_i \in \mathbb{Z}^+$, the only option for k is 1.

$$\dim L_1 - \dim(L_1 \cap L_2) = 0$$

$$\dim L_1 = \dim(L_1 \cap L_2)$$

$$\dim L_2 = \dim L_1 + 1$$

Now substitute this into the original equation:

$$\dim(L_1 + L_2) = 1 + \dim(L_1 \cap L_2)$$

$$\dim(L_1 + L_2) = 1 + \dim L_1$$

$$\dim L_2 = \dim(L_1 + L_2)$$

We know that $\dim L_1 = \dim(L_1 \cap L_2)$ and $\dim L_2 = \dim(L_1 + L_2)$. Since these are the only two subspaces we're working with, then $L_1 = L_1 \cap L_2$ is true and $L_2 = L_1 + L_2$ is true (or vice versa if $\dim L_1 \geq \dim L_2$), q. e. d.