Problem 2.1

Given sequence $A = (1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 0, 1, 1, 1, 2, \ldots, 2, 0, 2, 3)$, find the number of zeros in the sequence.

Solution

In the patterns below, ? denotes any non-zero digit.

number patterns	number count	zero count	total
$?0$	9	1	9
$?00$	9	2	18
$?0?, ??0$	$2 \times 9 \times 9$	1	162
$?000$	2	3	6
$10??, 1?0?, 1??0$	$3 \times 9 \times 9$	1	243
$1?00, 10?0, 100?$	3×9	2	54
$200?$	9	2	18
$2010, 2020$	2	2	4
$201?$	9	1	9
$2021, 2022, 2023$	3	1	3
Σ			526

Answer

526

Problem 2.2

How many ways are there to choose two squares on a 10×10 board so that they would not have any adjacent corners or edges?

Solution

I will map all the cases on a 10×10 markdown board.

- O - example chosen square;
- X - adjacent square that cannot be chosen;
- V - other possible chosen squares;
- Z - X or V.

Case 1, the first chosen square is a corner:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	O	X	.	.	.	.	.	.	.	V
2	X	X	.	.	.	.	.	.	.	.
3	.	.	.	.	.	.	.	.	.	.
4	.	.	.	.	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	.	.	.	.
7	.	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	.	.
9	.	.	.	.	.	.	.	.	.	.
10	V	.	.	.	.	.	.	.	.	V

There are 4 corners in total and $100 - 4 = 96$ ways to choose the second square $n_{corners} = 4 \times 96 = 384$.

Case 2, the first chosen square is an edge:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	X	O	Z	V	V	V	V	V	V	.
2	Z	X	X	.	.	.	.	.	.	V

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
3	V	.	.	.	.	.	.	.	.	V
4	V	.	.	.	.	.	.	.	.	V
5	V	.	.	.	.	.	.	.	.	V
6	V	.	.	.	.	.	.	.	.	V
7	V	.	.	.	.	.	.	.	.	V
8	V	.	.	.	.	.	.	.	.	V
9	V	.	.	.	.	.	.	.	.	V
10	.	V	V	V	V	V	V	V	V	.

There are 32 edges in total and $100 - 6 = 94$ ways to choose the second square $n_{edges} = 32 \times 94 = 3008$.

Case 3, the first chosen square is one of the centers:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	X	X	X	.	.	.	.	.	.	.
2	X	O	Z	V	V	V	V	V	V	.
3	X	Z	Z	V	V	V	V	V	V	.
4	.	V	V	V	V	V	V	V	V	.
5	.	V	V	V	V	V	V	V	V	.
6	.	V	V	V	V	V	V	V	V	.
7	.	V	V	V	V	V	V	V	V	.
8	.	V	V	V	V	V	V	V	V	.
9	.	V	V	V	V	V	V	V	V	.
10	.	.	.	.	.	.	.	.	.	.

There are 64 centers in total and $100 - 9 = 91$ ways to choose the second square $n_{centers} = 64 \times$

$91 = 5824.$

Here it has to be noted that the order of the chosen squares does not matter, therefore the final result will be the sum of all n-s **divided by 2.**

Total $n = \frac{1}{2}(n_{corners} + n_{edges} + n_{centers}) = \frac{1}{2}(5824 + 3008 + 384) = 4608$

Answer

4608

Problem 2.3

Subproblem A

Prove that $(A_1 \setminus A_2) \times (B_1 \setminus B_2) \subseteq (A_1 \times B_1) \setminus (A_2 \times B_2)$ is true for any sets A_1, A_2, B_1, B_2.

Solution

Assuming that $\forall n_k, m_k \in \mathbb{N}$, define the following:

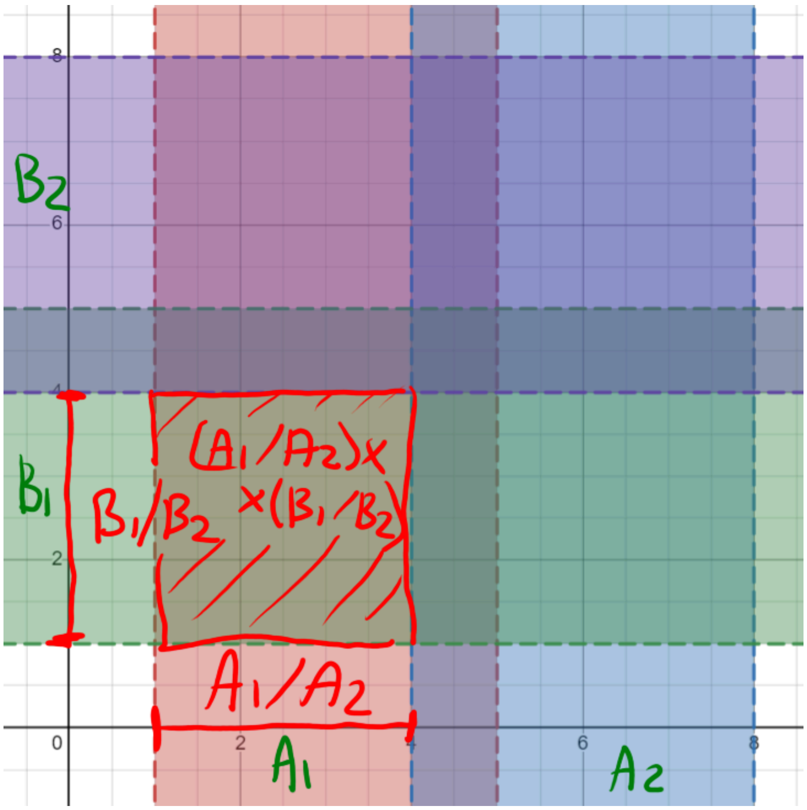
- $a_{0,n_k} \in A_1, A_2$;
- $a_{1,n_k} \in A_1$ and $a_{1,n} \notin A_2$;
- $a_{2,n_k} \in A_2$ and $a_{2,n} \notin A_1$;
- $b_{0,n_k} \in B_1, B_2$;
- $b_{1,n_k} \in B_1$ and $b_{1,n} \notin B_2$;
- $b_{2,n_k} \in B_2$ and $b_{2,n} \notin B_1$;
- $A_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}, a_{0,1}, a_{0,2}, \ldots, a_{0,n_0})$;
- $A_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,n_2}, a_{0,1}, a_{0,2}, \ldots, a_{0,n_0})$;
- $B_1 = (b_{1,1}, b_{1,2}, \ldots, b_{1,m_1}, b_{0,1}, b_{0,2}, \ldots, b_{0,m_0})$;
- $B_2 = (b_{2,1}, b_{2,2}, \ldots, b_{2,m_2}, b_{0,1}, b_{0,2}, \ldots, b_{0,m_0})$.

Then, considering

- $A_1 \setminus A_2 = \{a_{1,1}, a_{1,2}, \ldots, a_{1,n_1}\}$;
- $B_1 \setminus B_2 = \{b_{1,1}, b_{1,2}, \ldots, b_{1,m_1}\}$;

Evaluate the **lefthand side** of the statement above:

$$
\begin{aligned}
(A_1 \setminus A_2) \times (B_1 \setminus B_2) = \{&(a_{1,1}, b_{1,1}), (a_{1,1}, b_{1,2}), \ldots, (a_{1,1}, b_{1,m_1}),\\
&(a_{1,2}, b_{1,1}), (a_{1,2}, b_{1,2}), \ldots, (a_{1,2}, b_{1,m_1}),\\
&\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots,\\
&(a_{1,n_1}, b_{1,1}), (a_{1,n_1}, b_{1,2}), \ldots, (a_{1,n_1}, b_{1,m_1})\}
\end{aligned}
$$



Now, to evaluate the **righthand side** of the statement above, first calculate the following:

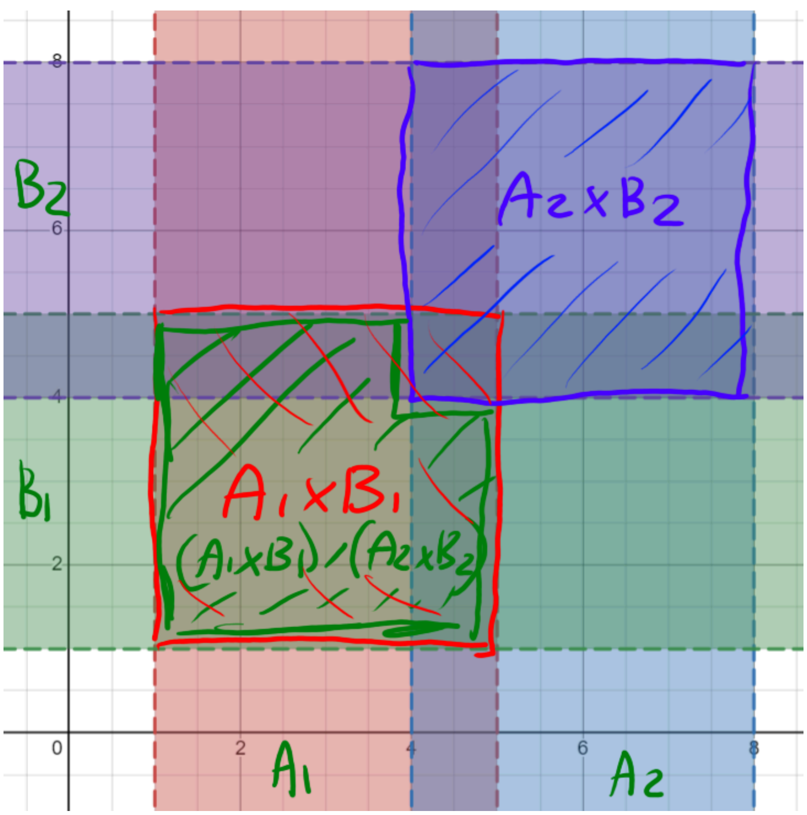
$$
\begin{aligned}
A_1 \times B_1 = \{&(a_{1,1}, b_{1,1}), (a_{1,1}, b_{1,2}), \ldots, (a_{1,1}, b_{1,m_1}), (a_{1,1}, b_{0,1}), (a_{1,1}, b_{0,2}), \ldots, (a_{1,1}, b_{0,m_0}),\\
&(a_{1,2}, b_{1,1}), (a_{1,2}, b_{1,2}), \ldots, (a_{1,2}, b_{1,m_1}), (a_{1,2}, b_{0,1}), (a_{1,2}, b_{0,2}), \ldots, (a_{1,2}, b_{0,m_0}),\\
&\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots,\\
&(a_{1,n_1}, b_{1,1}), (a_{1,n_1}, b_{1,n_1}), \ldots, (a_{1,n_1}, b_{1,m_1}), (a_{1,n_1}, b_{0,1}), (a_{1,n_1}, b_{0,2}), \ldots, (a_{1,n_1}, b_{0,m_0}),\\
&(a_{0,1}, b_{1,1}), (a_{0,1}, b_{1,2}), \ldots, (a_{0,1}, b_{1,m_1}), (a_{0,1}, b_{0,1}), (a_{0,1}, b_{0,2}), \ldots, (a_{0,1}, b_{0,m_0}),\\
&(a_{0,2}, b_{1,1}), (a_{0,2}, b_{1,2}), \ldots, (a_{0,2}, b_{1,m_1}), (a_{0,2}, b_{0,1}), (a_{0,2}, b_{0,2}), \ldots, (a_{0,2}, b_{0,m_0}),\\
&\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots,\\
&(a_{0,n_0}, b_{1,1}), (a_{0,n_0}, b_{1,n_1}), \ldots, (a_{0,n_0}, b_{1,m_1}), (a_{0,n_0}, b_{0,1}), (a_{0,n_0}, b_{0,2}), \ldots, (a_{0,n_0}, b_{0,m_0}), \}
\end{aligned}
$$

$$
\begin{aligned}
A_2 \times B_2 = \{&(a_{2,1}, b_{2,1}), (a_{2,1}, b_{2,2}), \ldots, (a_{2,1}, b_{2,m_2}), (a_{2,1}, b_{0,1}), (a_{2,1}, b_{0,2}), \ldots, (a_{2,1}, b_{0,m_0}),\\
&(a_{2,2}, b_{2,1}), (a_{2,2}, b_{2,2}), \ldots, (a_{2,2}, b_{2,m_2}), (a_{2,2}, b_{0,1}), (a_{2,2}, b_{0,2}), \ldots, (a_{2,2}, b_{0,m_0}),\\
&\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots,\\
&(a_{2,n_2}, b_{2,1}), (a_{2,n_2}, b_{2,n_2}), \ldots, (a_{2,n_2}, b_{2,m_2}), (a_{2,n_2}, b_{0,1}), (a_{2,n_2}, b_{0,2}), \ldots, (a_{2,n_2}, b_{0,m_0}),\\
&(a_{0,1}, b_{2,1}), (a_{0,1}, b_{2,2}), \ldots, (a_{0,1}, b_{2,m_2}), (a_{0,1}, b_{0,1}), (a_{0,1}, b_{0,2}), \ldots, (a_{0,1}, b_{0,m_0}),
\end{aligned}
$$

$$
\begin{gathered}
(a_{0,2}, b_{2,1}), (a_{0,2}, b_{2,2}), \ldots, (a_{0,2}, b_{2,m_2}), (a_{0,2}, b_{0,1}), (a_{0,2}, b_{0,2}), \ldots, (a_{0,2}, b_{0,m_0}), \\
\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \\
(a_{0,n_0}, b_{2,1}), (a_{0,n_0}, b_{2,n_2}), \ldots, (a_{0,n_0}, b_{2,m_2}), (a_{0,n_0}, b_{0,1}), (a_{0,n_0}, b_{0,2}), \ldots, (a_{0,n_0}, b_{0,m_0}), \}
\end{gathered}
$$

And then, considering the above, the following:

$$
\begin{gathered}
(A_1 \times B_1) \setminus (A_2 \times B_2) = \\
\{(a_{1,1}, b_{1,1}), (a_{1,1}, b_{1,2}), \ldots, (a_{1,1}, b_{1,m_1}), (a_{1,1}, b_{0,1}), (a_{1,1}, b_{0,2}), \ldots, (a_{1,1}, b_{0,m_0}), \\
(a_{1,2}, b_{1,1}), (a_{1,2}, b_{1,2}), \ldots, (a_{1,2}, b_{1,m_1}), (a_{1,2}, b_{0,1}), (a_{1,2}, b_{0,2}), \ldots, (a_{1,2}, b_{0,m_0}), \\
\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \\
(a_{1,n_1}, b_{1,1}), (a_{1,n_1}, b_{1,n_1}), \ldots, (a_{1,n_1}, b_{1,m_1}), (a_{1,n_1}, b_{0,1}), (a_{1,n_1}, b_{0,2}), \ldots, (a_{1,n_1}, b_{0,m_0}), \\
(a_{0,1}, b_{1,1}), (a_{0,1}, b_{1,2}), \ldots, (a_{0,1}, b_{1,m_1}), \\
(a_{0,2}, b_{1,1}), (a_{0,2}, b_{1,2}), \ldots, (a_{0,2}, b_{1,m_1}), \\
\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \\
(a_{0,n_0}, b_{1,1}), (a_{0,n_0}, b_{1,n_1}), \ldots, (a_{0,n_0}, b_{1,m_1})\}
\end{gathered}
$$



This makes it obvious that $(A_1 \setminus A_2) \times (B_1 \setminus B_2) \subseteq (A_1 \times B_1) \setminus (A_2 \times B_2)$ is true as every single item from the set on the lefthand side of the statement is included in the righthand side set regardless of the fact whether sets A_1, A_2, B_1, B_2 intersect or not, q. e. d.

Subproblem B

Is $(A_1 \times B_1) \setminus (A_2 \times B_2) \subseteq (A_1 \setminus A_2) \times (B_1 \setminus B_2)$ true for any sets A_1, A_2, B_1, B_2?

Solution

As shown above, $(A_1 \times B_1) \setminus (A_2 \times B_2)$ always contains all elements from $(A_1 \setminus A_2) \times (B_1 \setminus B_2)$.

$(A_1 \times B_1) \setminus (A_2 \times B_2)$ also contains elements from $A_1 \cap B_2$ and $A_2 \cap B_1$, which proves the reversed statement $(A_1 \times B_1) \setminus (A_2 \times B_2) \subseteq (A_1 \setminus A_2) \times (B_1 \setminus B_2)$ false unless $A_1 \cap B_2 = \{\varnothing\}$ or $A_2 \cap B_1 = \{\varnothing\} \Rightarrow$ the statement is irreversible.

Answer

False

Problem 2.4

For any whole positive n, prove the equation:

$$n \cdot 2^0 + (n-1) \cdot 2^1 + (n-2) \cdot 2^2 + \cdots + 1 \cdot 2^{n-1} = 2^{n+1} - 2 - n$$

Solution

Check whether the equation is true for $n = 1$ (**induction base**):

$$1 \cdot 2^0 = 2^{1+1} - 2 - 1$$

$$1 = 1$$

State the **induction hypothesis**:

$$n \cdot 2^0 + (n-1) \cdot 2^1 + (n-2) \cdot 2^2 + \cdots + 1 \cdot 2^{n-1} = 2^{n+1} - 2 - n$$

To check whether the induction hypothesis holds for $n+1$ (**induction step**), add the following expression to both parts of the equation:

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} + 2^n$$

Therefore, we get:

$$\begin{gathered}(n+1) \cdot 2^0 + n \cdot 2^1 + (n-1) \cdot 2^2 + \cdots + 2 \cdot 2^{n-1} + 1 \cdot 2^n = \\ = 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} + 2^n + 2^{n+1} - 2 - n\end{gathered}$$

Rewriting $2^0 + 2^1 + 2^2 + \dots + 2^{n-1} + 2^n$ in binary, we get $1_2 + 10_2 + 100_2 + \dots + 1\underbrace{000\dots0}_{n}{}_2$, which is equal to $\underbrace{111\dots1}_{n+1}{}_2$. Therefore, $2^0 + 2^1 + 2^2 + \dots + 2^{n-1} + 2^n = 2^{n+1} - 1$.

Rewrite the equation further:

$$
\begin{gathered}
(n+1) \cdot 2^0 + n \cdot 2^1 + (n-1) \cdot 2^2 + \dots + 2 \cdot 2^{n-1} + 1 \cdot 2^n = \\
= 2^0 + 2^1 + 2^2 + \dots + 2^{n-1} + 2^n + 2^{n+1} - 2 - n = 2^{n+2} - 1 - 2 - n = \\
= 2^{n+2} - 2 - (n+1) \Rightarrow \\
n \cdot 2^0 + (n-1) \cdot 2^1 + (n-2) \cdot 2^2 + \dots + 1 \cdot 2^{n-1} = 2^{n+1} - 2 - n
\end{gathered}
$$

q. e. d.