Calculus, Homework 3

Problem 2.3

$$\int_0^4 dz \int_{-z}^{z} dx \int_0^{\sqrt{z^2-x^2}} z^2xy^2dy = \int_0^4 dz \int_{-z}^{z} dx \left(z^2x\frac{y^3}{3}\Big|_0^{\sqrt{z^2-x^2}}\right)$$

$$= \int_0^4 dz \int_{-z}^{z} z^2x(z^2-x^2)\sqrt{z^2-x^2}dx$$

$$= \int_0^4 dz \int_{-z}^{z} (z^4x - z^2x^3)\sqrt{z^2-x^2}dx$$

$$\boxed{x = z\sin u, \quad dx = z\cos u du, \quad u = \arcsin(\tfrac{x}{z})}$$

$$= \int_0^4 dz \int_{-z}^{z} (z^5\sin u - z^5\sin^3 u)\sqrt{z^2 - z^2\sin^2 u}z\cos u du$$

$$= \int_0^4 dz \int_{-z}^{z} (z^5\sin u - z^5\sin^3 u)z^2\cos^2 u du$$

$$= \int_0^4 dz \int_{-z}^{z} (z^7\sin u - z^7\sin^3 u)(1-\sin^2 u)^2 du$$

$$= \int_0^4 dz \int_{-z}^{z} (z^7\sin u - 2z^7\sin^3 u + z^7\sin^5 u)du$$

$$\boxed{w = \cos u, \quad dw = -\sin w du}$$

$$= \int_0^4 dz \left((-z^7\cos\arcsin(\tfrac{x}{z}))\Big|_{-z}^{z} - 2z^7\int_{-z}^{z}\sin^3 u du + z^7\int_{-z}^{z}\sin^5 u du \right)$$

$$\boxed{\begin{array}{l}\text{at this point I realized that all the following integrals will eval to zero}\\ \text{since cosarcsin(x/z) evals to zero with boundaries -z and z}\end{array}}$$

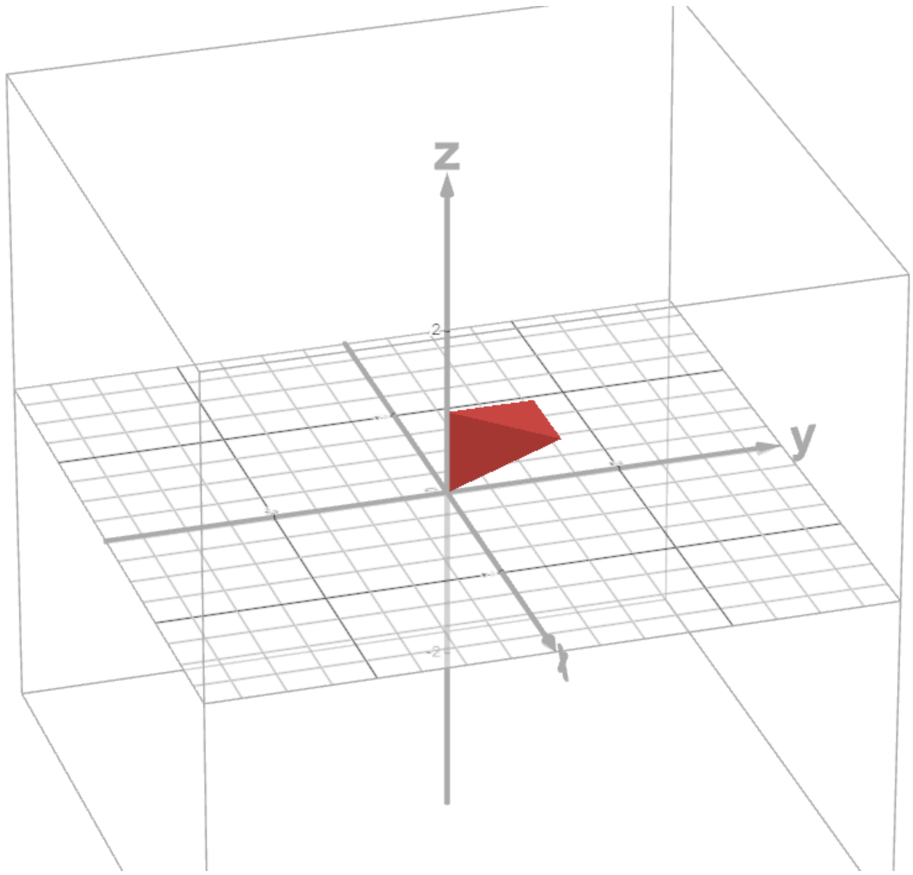
$$= \int_0^4 dz \left(\cancel{(-z^7\sqrt{1-\tfrac{x^2}{z^2}})\Big|_{-z}^{z}} \cancel{- 2z^7\int_{-z}^{z}(1-\cos^2 u)\sin u du} + \cancel{z^7\int_{-z}^{z}(1-\cos^4 u)\sin} \right.$$

$$= \int_0^4 0dx = 0$$

Problem 2.4

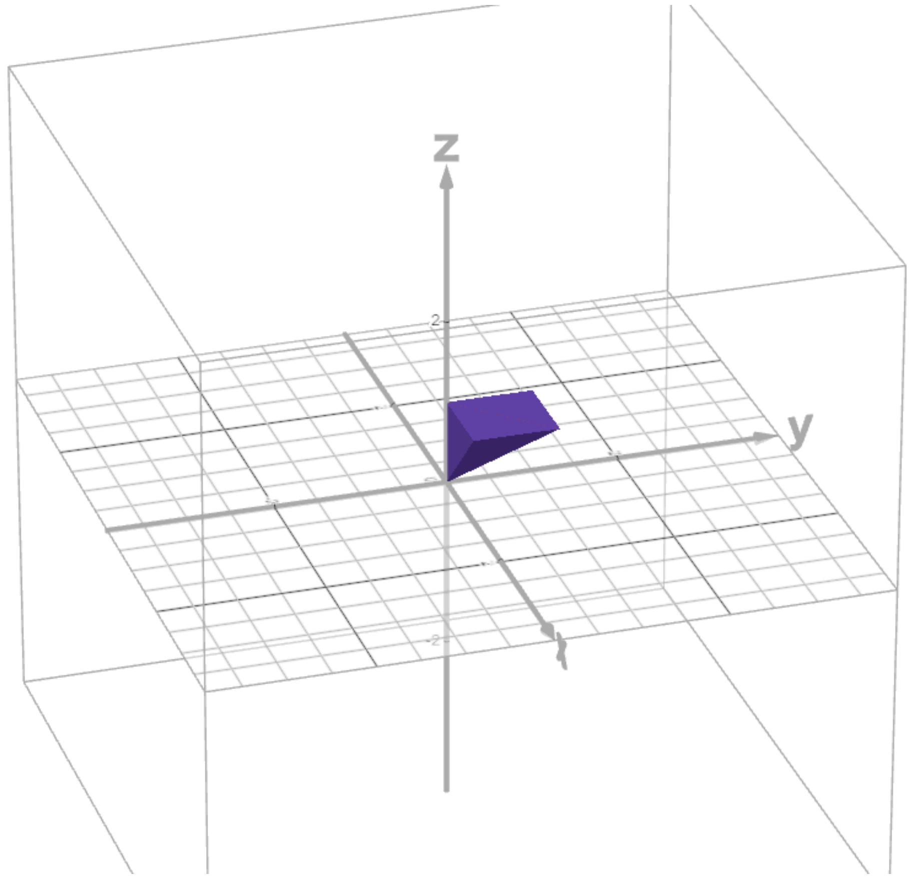
These are the given boundaries:

$$\begin{cases} 0 \leq x \leq 1 \\ x \leq y \leq 1 \\ y \leq z \leq 1 \end{cases}$$



These are the boundaries that I have changed the integral to be

$$\begin{cases} 0 \leq z \leq 1 \\ 0 \leq y \leq z \\ 0 \leq x \leq z \end{cases}$$



The shape below has twice the volume of the shape above.

$$\int_0^1 dx \int_x^1 dy \int_y^1 e^{z^3} dz = \frac{1}{2}\int_0^1 dz \int_0^z dy \int_0^z e^{z^3} dx$$

$$= \frac{1}{2}\int_0^1 dz \int_0^z dy (xe^{z^3}) \Big|_0^z$$

$$= \frac{1}{2}\int_0^1 dz \int_0^z ze^{z^3} dy$$

$$= \frac{1}{2}\int_0^1 dz (yze^{z^3}) \Big|_0^z$$

$$= \int_0^1 \frac{1}{2} z^2 e^{z^3} dz$$

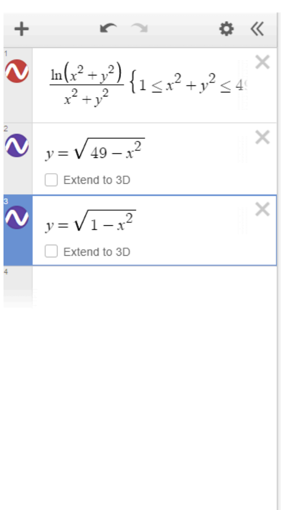
$$\boxed{z^3 = u, \quad du = 3z^2 dz \implies dz = \frac{du}{3z^2}}$$

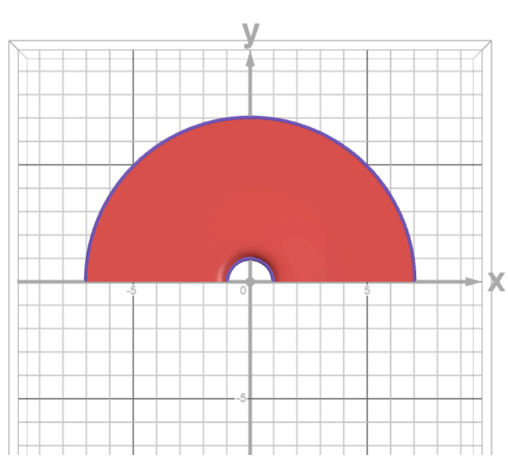
$$= \frac{1}{6}\int_0^1 e^u du = \frac{e^u}{6}\Big|_0^1 = \frac{e^{z^3}}{6}\Big|_0^1 = \frac{1}{6}(e-1)$$

Problem 3.3a

$$\iint\limits_{\substack{1 \le x^2+y^2 \le 49 \\ y \ge 0}} \frac{\ln(x^2+y^2)}{x^2+y^2} dxdy$$

Firstly, visualize the graph since this is actually really easy to do in this case.





The most optimal way to calculate it would be to take a polar coordinate replacement. We need to differentiate through radii $r \in [1, 7]$ and through angles $\theta \in [0, \pi]$.

Thus, we replace

$$x = r\cos\theta, y = r\sin\theta$$

$$\iint\limits_{\substack{1 \le x^2+y^2 \le 49 \\ y \ge 0}} \frac{\ln(x^2+y^2)}{x^2+y^2} dxdy = \int\limits_0^\pi \int\limits_1^7 \frac{\ln(r^2\cos^2\theta + r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} rdrd\theta$$

$$= \int\limits_0^\pi \int\limits_1^7 \frac{\ln(r^2)}{r} drd\theta$$

$$\boxed{u = \ln(r^2), \quad du = \frac{2r}{r^2}dr = \frac{2}{r}dr \implies dr = \frac{1}{2}rdu}$$

$$= \frac{1}{2}\int\limits_0^\pi \int\limits_1^7 udud\theta = \frac{1}{4}\int\limits_0^\pi (\ln(r^2)^2)\Bigg|_1^7 d\theta$$

$$= \frac{1}{4}\int\limits_0^\pi (2\ln(7))^2 - \cancel{2\ln(1)^2} d\theta$$

$$= \ln^2(7)\theta \Bigg|_0^\pi = \ln^2(7)\pi$$