Calculus, Homework 17

Problem 1

Using Largange's Theorem and the first comparison criteria, prove that the following series (x_n) diverge:

> Honestly, I don't quite get the point of the task

Subproblem A

$$x_n = \frac{1}{n^{1+\sigma}}, \quad \sigma > 0$$

We need to find some convergent series, the derivative for which is equal to the series below:

$$f(n+1) - f(n) = f'(n+\theta), \quad 0 < \theta < 1$$

Let:

$$f'(n+\theta) = \frac{1}{(n+\theta)^{\sigma+1}} > \frac{1}{n^{\sigma+1}}$$

This is eerily similar to integration, so:

$$\int \frac{dn}{n^{\sigma+1}} = -\frac{1}{\sigma n^{\sigma}} + C$$

Thus we have

$$f(n) = -\frac{1}{\sigma n^{\sigma}}$$

and since for this series exists some number N, for which we can define

$$y_n := f(n+1) - f(n)$$

such that $\forall n > N$:

$$x_n < y_n$$

take

$$y_n = f(n+1) - f(n)$$

$$y_n = -\frac{1}{\sigma n^\sigma} + \frac{1}{\sigma n^\sigma \times n} = -\frac{1}{\sigma n^\sigma}\left(1 - \frac{1}{n}\right)$$

This is a telescopic sum, so sum of (y_n) would be

$$\frac{1}{\sigma} - \frac{1}{\sigma(n+1)^\sigma}$$

the limit of this is

$$\lim_{n\to\infty}\left(\frac{1}{\sigma} - \frac{1}{\sigma(n+1)^\sigma}\right) = \frac{1}{\sigma}$$

thus this series converges since it has a non-infinite sum.

Now given that

$$\left(\frac{1}{(n+\theta)^{\sigma+1}}\right)$$

can be treated similarly since we'd get another telescopic sum, then we know that the following series within our given conditions are almost similar:

$$\left(\frac{1}{(n+1)^{\sigma+1}}\right), \quad \left(\frac{1}{n^{\sigma+1}}\right)$$

Thus we may conclude that

$$\left(\frac{1}{n^{\sigma+1}}\right)$$

also converges per the first comparison criteria.

Subproblem B

$$x_n = \frac{1}{n \ln n}$$

Consider

$$f(x) := \ln(\ln(x))$$

on $[n, n+1]$ and since $f(x)$ is differentiable, we get that per Lagrange theorem

$$\exists n+\theta \in (n, n+1), \quad 0<\theta<1$$

$$f(n+1)-f(n)=f'(n+\theta) \implies$$

$$\ln(\ln(n+1))-\ln(\ln(n))=\frac{1}{(n+\theta)\ln(n+\theta)}$$

Since we know that $n+\theta<n+1$, then

$$y_n=\ln(\ln(n+1))-\ln(\ln(n))>\frac{1}{(n+1)\ln(n+1)}$$

which means that per the first comparison criteria we get that since (y_n) diverges since telescopically we get a single expression that approaches infinity as n approaches infinity, therefore the series

$$\left(\frac{1}{n\ln n}\right)$$

also diverges

Problem 2

Determine absolute convergence of the following series (x_n):

Subproblem A

$$x_n=\frac{x^n}{n!}$$

Find d'Alembert's variant:

$$\mathfrak{D}_n=\frac{|x_{n+1}|}{|x_n|}=\frac{|x^{n+1}n!|}{|(n+1)!x^n|}=\left|\frac{x}{n+1}\right|$$

$$\lim_{n\to\infty}\mathfrak{D}_n=0 \implies$$

the series converges absolutely $\forall x$.

Subproblem B

$$x_n=xn^{n-1}$$

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d'Alembert's variant:

$$\mathfrak{D}_n = \frac{|x_{n+1}|}{|x_n|} = \frac{|x(n+1)^n|}{|xn^{n-1}|} = \left(\frac{n+1}{n}\right)^n \times n$$

$$\lim_{n\to\infty} n\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} ne = \infty \implies$$

the series diverges absolutely $\forall x$.

Subproblem C

$$x_n = \frac{x^n}{n^s}, \quad s > 0, x \neq -1$$

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d'Alembert's variant:

$$\mathfrak{D}_n = \frac{|x^{n+1}|n^s}{(n+1)^s|x^n|} = |x|\left(\frac{n}{n+1}\right)^s = |x|\left(1+\frac{1}{n}\right)^s$$

$$\lim_{n\to\infty} |x|\left(1+\frac{1}{n}\right)^s = |x|$$

which implies that the series converges absolutely for $-1 < x < 1$ and diverges for $x > 1 \vee x < -1$. Additionally, when $x = \pm 1$, the series converges absolutely if $s > 1$ and diverges absolutely if $s \leq 1$ since then it would be a Dirichet's series.

Subproblem D

$$x_n = n!\frac{x^n}{n^n}$$

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d'Alembert's variant:

$$\mathfrak{D}_n = \frac{|x_{n+1}|}{|x_n|} = \frac{|(n+1)!x^{n+1}n^n|}{|(n+1)^{n+1}n!x^n|} = \frac{|n!(n+1)x^{n+1}n^n|}{|(n+1)^n(n+1)n!x^n|} = |x|\left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} |x|\left(\frac{n}{n+1}\right)^n = \frac{|x|}{e}$$

which implies that the series converges absolutely for $-e < x < e$ and diverges absolutely for $x > e \vee x < -e$. In case when $x = \pm e$, we may use the limit test to determine whether the series diverges:

$$\lim_{n\to\infty} n!\frac{|e|^n}{n^n} = \lim_{n\to\infty}\left(\frac{|e|}{n}\times\frac{2|e|}{n}\times\cdots\times\frac{(n-1)|e|}{n}\times\frac{n|e|}{n}\right)$$

We understand that from some k, $\frac{k|e|}{n}$ is greater than 1. Therefore, this limit is equal to ∞ and the series diverges absolutely for $x = \pm e$.

Subproblem E

$$x_n = \frac{(nx)^n}{n!}, \quad x \neq -\frac{1}{e}$$

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d'Alembert's variant:

$$\mathfrak{D}_n = \frac{|x_{n+1}|}{|x_n|} = \frac{(n+1)^n(n+1)|x|^{n+1}n!}{n!(n+1)n^n|x|^n} = |x|\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}|x|\left(1+\frac{1}{n}\right)^n = |x|e$$

which implies that the series converges for $-\frac{1}{e} < x < \frac{1}{e}$ and diverges absolutely for $x > \frac{1}{e} \vee x < -\frac{1}{e}$. We don't need to determine absolutely convergence for $x = -\frac{1}{e}$, so consider the case for $x = \frac{1}{e}$. Similarly to above, use the limit test:

$$\lim_{n\to\infty}\frac{n^n}{e^n n!} = \lim_{n\to\infty}\left(\frac{n}{e}\times\frac{n}{2e}\times\cdots\times\frac{n}{(n-1)e}\times\frac{n}{ne}\right)$$

We understand that from some k, $\frac{n}{ke}$ is greater than 1. Therefore, this limit is equal to ∞ and the series diverges absolutely for $x = \frac{1}{e}$.

Subproblem F

$$x_n = \frac{x^n}{1-x}, \quad x \neq \pm 1$$

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d'Alembert's variant:

$$\mathfrak{D}_n = \frac{|x_{n+1}|}{|x_n|} = \frac{|x^{n+1}||1-x|}{|x||x^n|} = |1-x|$$

From here, it's obvious that the series converges for $|x| < 1$ and diverges for $|x| > 1$.