Individual Homework №4, Linear Algebra

Problem 1

Given matrix

$$A = \begin{pmatrix} 14+7i & -12-8i \\ 18+12i & -16-13i \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$$

find all values $x \in \mathbb{C}$, for which matrix $A - xE$ is irreversible.

For a matrix to be irreversible, its determinant has to be equal to 0:

$$|A - xE| = \det \begin{pmatrix} 14+7i-x & -12-8i \\ 18+12i & -16-13i-x \end{pmatrix} = 0$$

$$\det \begin{pmatrix} 14+7i-x & -12-8i \\ 18+12i & -16-13i-x \end{pmatrix} = 0$$

Thus,

$$(14+7i-x)(-16-13i-x) - (18-12i)(-12-8i) = 0$$

$$-91i^2 + 6ix - 294i + x^2 + 2x - 224 + 216 - 96i^2 = 0$$

$$91 + 6ix - 294i + x^2 + 2x - 224 + 216 + 96 = 0$$

$$x^2 + 2x + 6ix + 179 - 294i = 0$$

$$x^2 + x(2+6i) + 179 - 294i$$

$$x^2 + (2+6i)x - (8-6i) = -187 + 300i$$

$$(x + (1+3i))^2 = -187 + 300i$$

$$x = \pm\sqrt{-187+300i} - (1+3i)$$

Answer:

$$x = \pm\sqrt{-187+300i} - (1+3i)$$

Problem 2

Calculate

$$\sqrt[4]{-18-18\sqrt{3}i}$$

$$z_0 = 18\sqrt[4]{-1-\sqrt{3}i}$$

$$z_1 = -1-\sqrt{3}i$$

Get the trigonometric form:

$$z_1 = |z_1|(\cos\phi + i\sin\phi)$$

$$|z_1| = \sqrt{(-1)^2+(-\sqrt{3})^2} = 2$$

$$\cos\phi = -\frac{1}{2}$$

$$\sin\phi = -\frac{\sqrt{3}}{2}$$

$$\phi = -\frac{2\pi}{3}$$

$$z_1 = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\left(-\frac{2\pi}{3}\right)\right)$$

Find such w_i that $w_i^4 = z_1$.

$$w_i = \sqrt[4]{|z_1|}\left(\cos\frac{\phi+2\pi k}{4} + i\sin\frac{\phi+2\pi k}{4}\right)$$

$$w_i = \sqrt[4]{2}\left(\cos\frac{-\frac{2\pi}{3}+2\pi k}{4} + i\sin\frac{-\frac{2\pi}{3}+2\pi k}{4}\right)$$

$$w_i = \sqrt[4]{2}\left(\cos\left(-\frac{\pi}{6}+\frac{\pi k}{2}\right) + i\sin\left(-\frac{\pi}{6}+\frac{\pi k}{2}\right)\right)$$

Now find all 4 roots and thus get **the answer**:

$$w_1 = \sqrt[4]{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$w_2 = \sqrt[4]{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$w_3 = \sqrt[4]{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$w_4 = \sqrt[4]{2}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

Problem 3

Given vectors

$$v_1 = \begin{pmatrix} 8 \\ -7 \\ 1 \\ -1 \\ -4 \end{pmatrix} \quad v_2 = \begin{pmatrix} -88 \\ 82 \\ -12 \\ 5 \\ 45 \end{pmatrix} \quad v_3 = \begin{pmatrix} 200 \\ -200 \\ 30 \\ a \\ -104 \end{pmatrix}$$

prove that they are linearly independent for all values of parameter a, and for each a, complement these vectors to a basis of the entire $\mathbb{R}^5$ space.

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \vec{0}$$

Calculate the matrix's rank, first reducing the matrix to a row echelon form:

$$\begin{pmatrix} 8 & -88 & 200 \\ -7 & 82 & -200 \\ 1 & -12 & 30 \\ -1 & 5 & a \\ -4 & 45 & 104 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -12 & 30 \\ 0 & -2 & 10 \\ 0 & 8 & -40 \\ 0 & -7 & a+30 \\ 0 & -3 & 224 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -12 & 30 \\ 0 & -2 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & a-5 \\ 0 & 0 & 209 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -12 & 30 \\ 0 & -2 & 10 \\ 0 & 0 & 209 \\ 0 & 0 & a-5 \\ 0 & 0 & 0 \end{pmatrix}$$

At this point, we may add $\frac{-a+5}{209}$ times row 3 to row 4 regardless of value a and get a matrix of rank 3 (there are 3 non-zero rows in this form of the matrix), which proves that these 3 vectors are linearly independent.

Now build a basis from this. Try to add the following vectors to the basis and check whether the determinant of the resulting matrix would be 0 or not:

$$v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Resulting matrix:

$$\begin{pmatrix} 1 & 8 & -88 & 200 & 0 \\ 0 & -7 & 82 & -200 & 0 \\ 0 & 1 & -12 & 30 & 0 \\ 0 & -1 & 5 & a & 0 \\ 0 & -4 & 45 & 104 & 1 \end{pmatrix}$$

Row echelon form:

$$\begin{pmatrix} 1 & 8 & -88 & 200 & 0 \\ 0 & 1 & -12 & 30 & 0 \\ 0 & 0 & -2 & 10 & 0 \\ 0 & 0 & -7 & a+30 & 0 \\ 0 & 0 & -3 & 224 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 8 & -88 & 200 & 0 \\ 0 & 1 & -12 & 30 & 0 \\ 0 & 0 & -7 & a+30 & 0 \\ 0 & 0 & 0 & -\frac{2}{7}(a-5) & 0 \\ 0 & 0 & 0 & \diamond(a) & 1 \end{pmatrix}$$

> Here we have some kinda value $\diamond(a)$ that depends on a that I was too lazy to calculate since the next form is

$$\begin{pmatrix} 1 & 8 & -88 & 200 & 0 \\ 0 & 1 & -12 & 30 & 0 \\ 0 & 0 & -7 & a+30 & 0 \\ 0 & 0 & 0 & -\frac{2}{7}(a-5) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Determinant of a row echelon form is equal to the product of the values on the diagonal, and since there are no zeros, $\det(v^{(i)}) \neq 0$ and the **set of vectors** $v_i \ \forall i = 1 \dots 5$ **is the basis.**

Problem 4

Subspace $U \subseteq \mathbb{R}^5$ is defined as a linear combination of vectors

$$v_1 = \begin{pmatrix} 19 \\ 14 \\ 27 \\ 10 \\ 28 \end{pmatrix} \quad v_2 = \begin{pmatrix} -14 \\ -7 \\ -15 \\ 3 \\ -13 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ -6 \\ -2 \\ -5 \\ -4 \end{pmatrix} \quad v_4 = \begin{pmatrix} -13 \\ -9 \\ -20 \\ -9 \\ -21 \end{pmatrix}$$

Subproblem A

Choose a basis in U out of these vectors.

As per usual, the row echelon form (v_3, v_1, v_2, v_4):

$$\begin{pmatrix} -1 & 19 & -14 & -13 \\ -6 & 14 & -7 & -9 \\ -2 & 27 & -15 & -20 \\ -5 & 10 & 3 & -9 \\ -4 & 28 & -13 & -21 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 19 & -14 & -13 \\ 0 & -100 & 77 & 69 \\ 0 & -11 & 13 & 6 \\ 0 & -85 & 73 & 56 \\ 0 & -48 & 43 & 31 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 19 & -14 & -13 \\ 0 & -100 & 77 & 69 \\ 0 & 0 & 7.55 & -2.65 \\ 0 & 0 & 4.53 & -1.59 \\ 0 & 0 & 6.04 & -2.12 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 19 & -14 & -13 \\ 0 & -100 & 77 & 69 \\ 0 & 0 & 7.55 & -2.65 \\ 0 & 0 & 4.53 & -1.59 \\ 0 & 0 & 6.04 & -2.12 \end{pmatrix}$$

> All three last rows are proportional to each other!

$$\begin{pmatrix} -1 & 19 & -14 & -13 \\ 0 & -100 & 77 & 69 \\ 0 & 0 & 7.55 & -2.65 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, we get that the first three (v_3, v_1, v_2) vectors form a basis.

Subproblem B

Out of vectors

$$u_1 = \begin{pmatrix} 3 \\ -13 \\ 1 \\ -14 \\ -5 \end{pmatrix} \quad u_2 = \begin{pmatrix} -13 \\ 2 \\ -16 \\ 6 \\ -12 \end{pmatrix}$$

choose those which lie within the span U and find their composition in the found basis.

We need to just solve $Ax = u_i$ for each of the vectors and a matrix $(A|u_i) = (v^{(3)}, v^{(1)}, v^{(2)}|u_i)$:

First vector

$$\left(\begin{array}{ccc|c} -1 & 19 & -14 & 3 \\ -6 & 14 & -7 & -13 \\ -2 & 27 & -15 & 1 \\ -5 & 10 & 3 & -14 \\ -4 & 28 & -13 & -5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 19 & -14 & 3 \\ 0 & -100 & 77 & -31 \\ 0 & -11 & 13 & 5 \\ 0 & -85 & 73 & -29 \\ 0 & -48 & 43 & -17 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 19 & -14 & 3 \\ 0 & -100 & 77 & -31 \\ 0 & 0 & 151 & -53 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now actually get the solution out of the system:

$$\begin{cases} -x_3 + 19x_1 - 14x_2 = 3 \\ -100x_1 + 77x_2 = -31 \\ 151x_2 = -53 \end{cases}$$

$$\begin{cases} x_3 = \frac{403}{151} \\ x_1 = \frac{6}{151} \\ x_2 = -\frac{53}{151} \end{cases}$$

Finally,

$$u_1 = \frac{1}{151}(6v_1 - 53v_2 + 403v_3)$$

Second vector

$$\left(\begin{array}{ccc|c} -1 & 19 & -14 & -13 \\ -6 & 14 & -7 & 2 \\ -2 & 27 & -15 & -16 \\ -5 & 10 & 3 & 6 \\ -4 & 28 & -13 & -12 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 19 & -14 & -13 \\ 0 & -100 & 77 & 80 \\ 0 & -11 & 13 & 10 \\ 0 & -85 & 73 & 71 \\ 0 & -48 & 43 & 40 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 19 & -14 & -13 \\ 0 & -100 & 77 & 80 \\ 0 & 0 & 453 & 120 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 151 & 40 \end{array}\right)$$

This system of equations has no solutions.

Problem 5

Find the basis and dimension of the subspace $U \subseteq \mathbb{R}^5$, which is the set of all solutions to the following system:

$$\begin{cases} 6x_1 - 3x_2 + 2x_3 - 5x_4 - x_5 = 0 \\ 9x_1 + 2x_2 + 9x_3 - 19x_4 - 4x_5 = 0 \\ 10x_1 - 4x_2 + 6x_3 - 18x_4 - 4x_5 = 0 \\ 14x_1 + x_2 + 9x_3 - 12x_4 - 2x_5 = 0 \end{cases}$$

As per usual, row echelon form and gaussian elimination $(x^{(5)}, x^{(3)}, x^{(2)}, x^{(4)}, x^{(1)})$:

$$\begin{pmatrix} -1 & 2 & 3 & -5 & 6 \\ -4 & 9 & 2 & -19 & 9 \\ -4 & 6 & -4 & -18 & 10 \\ -2 & 9 & 1 & -12 & 14 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 3 & -5 & 6 \\ 0 & 1 & 14 & 1 & -15 \\ 0 & -2 & 8 & 2 & -14 \\ 0 & 5 & 7 & -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 3 & -5 & 6 \\ 0 & 1 & 14 & 1 & -15 \\ 0 & 0 & 36 & 4 & -4 \\ 0 & 0 & -63 & -7 & 77 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 3 & -5 & 6 \\ 0 & 1 & 14 & 1 & -15 \\ 0 & 0 & 9 & 1 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} -x_5 + 2x_3 - 3x_2 - 5x_4 + 6x_1 = 0 \\ x_3 + 14x_2 + x_4 - 15x_1 = 0 \\ 9x_2 + x_4 - 11x_1 = 0 \end{cases}$$

$$\begin{cases} x_5 = -\frac{32}{9}x_4 - \frac{17}{9}x_1 \\ x_3 = \frac{5}{9}x_4 - \frac{19}{9}x_1 \\ x_2 = -\frac{1}{9}x_4 + \frac{11}{9}x_1 \end{cases}$$

Establish a fundamental system of solutions:

x_1	x_2	x_3	x_4	x_5
1	$\frac{11}{9}$	$-\frac{19}{9}$	0	$-\frac{17}{9}$
0	$-\frac{1}{9}$	$\frac{5}{9}$	1	$-\frac{32}{9}$

Thus, **the basis** would consist of two vectors:

$$e_1 = \begin{pmatrix} 1 \\ \frac{11}{9} \\ -\frac{19}{9} \\ 0 \\ -\frac{17}{9} \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 \\ -\frac{1}{9} \\ \frac{5}{9} \\ 1 \\ -\frac{32}{9} \end{pmatrix}$$

The **dimension** of the resulting set would be equal to the number of basis vectors. Thus, $U \in \mathbb{R}^2$ and $\dim U = 2$.