Individual Homework, Variant 17

> $\sim$ denotes a row echelon transformation using Gaussian elimination.

Problem 1

Represent matrix

$$A = \begin{pmatrix} 5 & -18 & 17 & 1 & -6 \\ -17 & -9 & 19 & 38 & -18 \\ -39 & 10 & -18 & 49 & 14 \\ -23 & 17 & -40 & 10 & 38 \\ 15 & 2 & -20 & -37 & 26 \end{pmatrix}$$

as a sum of r matrices of rank 1, where $r = \mathrm{rk} A$.

Reduce the matrix to a row echelon form:

$$A = \begin{pmatrix} 5 & -18 & 17 & 1 & -6 \\ -17 & -9 & 19 & 38 & -18 \\ -39 & 10 & -18 & 49 & 14 \\ -23 & 17 & -40 & 10 & 38 \\ 15 & 2 & -20 & -37 & 26 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -\frac{103}{67} & \frac{2}{67} \\ 0 & 1 & 0 & \frac{13}{67} & -\frac{64}{67} \\ 0 & 0 & 1 & \frac{48}{67} & -\frac{92}{67} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This gives us $r = \mathrm{rk} A = 3$ and we need to represent the last two columns as the sum of the first two. Gladly, the linear combinations of those columns are given in the matrix above:

$$A^{(4)} = \frac{1}{67}\left(-103A^{(1)} + 13A^{(2)} + 48A^{(3)}\right)$$

$$A^{(5)} = \frac{1}{67}\left(2A^{(1)} - 64A^{(2)} - 92A^{(3)}\right)$$

Thus, we may split this matrix into 3 basis matrices of $\mathrm{rk} 1$, expressing vectors from $A^{(1)}$ to $A^{(3)}$ as a linear combination of vectors $A^{(4)}$ and $A^{(5)}$:

$$A_1 = \frac{1}{67}\begin{pmatrix} 335 & 0 & 0 & -515 & 10 \\ -1139 & 0 & 0 & 1751 & -34 \\ -2613 & 0 & 0 & 4017 & -78 \\ -1541 & 0 & 0 & 2369 & -46 \\ 1005 & 0 & 0 & -1545 & 30 \end{pmatrix}$$

$$A_2 = \frac{1}{67}\begin{pmatrix} 0 & -1206 & 0 & -234 & 1152 \\ 0 & -603 & 0 & -117 & 576 \\ 0 & 670 & 0 & 130 & -640 \\ 0 & 1139 & 0 & 221 & -1088 \\ 0 & 134 & 0 & 26 & -128 \end{pmatrix}$$

$$A_3 = \frac{1}{67}\begin{pmatrix} 0 & 0 & 1139 & 816 & -1564 \\ 0 & 0 & 1273 & 912 & -1748 \\ 0 & 0 & -1206 & -864 & 1656 \\ 0 & 0 & -2680 & -1920 & 3680 \\ 0 & 0 & -1340 & -960 & 1840 \end{pmatrix}$$

and

$$A = A_1 + A_2 + A_3$$

Problem 2

In space $\mathbb{R}^3$ there are two bases $\mathbf{e} = (e_1, e_2, e_3)$ and $\mathbf{e}' = (e_1', e_2', e_3')$, where

$$e_1 = (-1, -1, 1), \quad e_2 = (2, 2, 3), \quad , e_3 = (1, -2, -2)$$

$$e_1' = (-1, 7, 6), \quad e_2' = (1, 8, -9), \quad e_3' = (-3, 11, -4)$$

and vector v in basis $\mathbf{e}$ with coordinates $(-1, 2, 5)$.

Translate from the basis (e)

Subproblem A

Prove that sets of vectors $\mathbf{e}$ and $\mathbf{e}'$ are truly bases in $\mathbb{R}^3$

Check for linear dependency:

$$\begin{pmatrix} -1 & 2 & 1 \\ -1 & 2 & 2 \\ 1 & 3 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -3 \\ 7 & 8 & 11 \\ 6 & -9 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -3 \\ 0 & 15 & -10 \\ 0 & 0 & -24 \end{pmatrix}$$

Therefore, the bases $\mathbf{e}, \mathbf{e}'$ are truly bases.

Subproblem B

Find the transformation matrix from basis $\mathbf{e}$ to basis $\mathbf{e}'$

Write all the vectors into columns and reduce rows:

$$\begin{pmatrix} -1 & 2 & 1 & -1 & 1 & -3 \\ -1 & 2 & 2 & 7 & 8 & 11 \\ 1 & 3 & -2 & 6 & -9 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & \frac{71}{5} & \frac{28}{5} & 23 \\ 0 & 1 & 0 & \frac{13}{5} & -\frac{1}{5} & 3 \\ 0 & 0 & 1 & 8 & 7 & 14 \end{pmatrix}$$

which implies that the last 3 columns is the transformation matrix from $\mathbf{e}$ to $\mathbf{e}'$:

$$T^{\mathbf{e}\to\mathbf{e}'} = \frac{1}{5}\begin{pmatrix} 71 & 28 & 115 \\ 13 & -1 & 15 \\ 40 & 35 & 70 \end{pmatrix}$$

Subproblem C

Find coordinates of a vector v in basis $\mathbf{e}'$

To do this, we need to find v' in $v = Tv' \implies v' = T^{-1}v$:

First, find T^{-1} using Gaussian elimination:

$$\begin{pmatrix} \frac{71}{5} & \frac{28}{5} & 23 & 1 & 0 & 0 \\ \frac{13}{5} & -\frac{1}{5} & 3 & 0 & 1 & 0 \\ 8 & 7 & 14 & 0 & 0 & 1 \end{pmatrix} \sim \frac{1}{240}\begin{pmatrix} \frac{1}{240} & 0 & 0 & -119 & 413 & 107 \\ 0 & \frac{1}{240} & 0 & -62 & 74 & 86 \\ 0 & 0 & \frac{1}{240} & 99 & -273 & -87 \end{pmatrix}$$

which implies

$$T^{-1} = \frac{1}{240}\begin{pmatrix} -119 & 413 & 107 \\ -62 & 74 & 86 \\ 99 & -273 & 87 \end{pmatrix}$$

and finally **find the coordinates of the vector**

$$v' = \frac{1}{240}\begin{pmatrix} -119 & 413 & 107 \\ -62 & 74 & 86 \\ 99 & -273 & 87 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \frac{1}{24}\begin{pmatrix} 148 \\ 64 \\ -21 \end{pmatrix}$$

Problem 3

Find the basis and the dimension of each of the subspaces $L_1, L_2, U = L_1 + L_2, W = L_1 \cap L_2$ of space $\mathbb{R}^5$ if L_1 is a linear span of vectors

$$a_1 = (3, 3, 0, -4, 1), \quad a_2 = (-2, 0, 4, -1, 1), \quad a_3 = (-1, 2, 1, 4, 4), \quad a_4 = (-13, -9, 8, 10, -1)$$

and L_2 is a linear span of vectors

$$b_1 = (0, 4, -2, 9, 7), \quad b_2 = (16, 1, -19, -1, -3), \quad b_3 = (-7, -3, 8, 2, 1), \quad b_4 = (2, -5, -3, 3, -1)$$

Basis and dimension for L_1:

$$\begin{pmatrix} 3 & -2 & -1 & -13 \\ 3 & 0 & 2 & -9 \\ 0 & 4 & 1 & 8 \\ -4 & -1 & 4 & 10 \\ 1 & 1 & 4 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies basis of L_1 is $\langle a_1, a_2, a_3 \rangle$ and $\dim L_1 = 3$.

Basis and dimension for L_2:

$$\begin{pmatrix} 0 & 16 & -7 & 2 \\ 4 & 1 & -3 & -5 \\ -2 & -19 & 8 & -3 \\ 9 & -1 & 2 & 3 \\ 7 & -3 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies basis of L_2 is $\langle b_1, b_2, b_3 \rangle$ and $\dim L_2 = 3$.

To find the basis and dimension of $L_1 + L_2$, write all vectors into columns and commense row echelonization:

$$\begin{pmatrix} 3 & -2 & -1 & -13 & 0 & 16 & -7 & 2 \\ 3 & 0 & 2 & -9 & 4 & 1 & -3 & -5 \\ 0 & 4 & 1 & 8 & -2 & -19 & 8 & -3 \\ -4 & -1 & 4 & 10 & 9 & -1 & 2 & 3 \\ 1 & 1 & 4 & -1 & 7 & -3 & 1 & -1 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 2 & -1 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies basis of $L_1 + L_2$ is $\langle a_1, a_2, a_3, b_2 \rangle$ and $\dim L_1 + L_2 = 4$.

> Notice that technically I could have use the basis vectors, but I was too lazy to change the matrices around for the kjgkhbillionth time

Finally, to find the basis and dimension of $L_1 \cap L_2$, write all vectors as columns and find the fundamental system of solutions. We have already almost done that in the previous point, so let's continue:

$$X = \begin{pmatrix} 3\alpha_4 + \beta_3 + 2\beta_4 \\ -2\alpha_4 + \beta_1 - 2\beta_3 - 4\beta_4 \\ -2\beta_1 \\ \alpha_4 \\ \beta_1 \\ -\beta_4 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$$\begin{pmatrix} \Phi_{L_1} \\ \hline \Phi_{L_2} \end{pmatrix} = \left(\begin{array}{cccc} 3 & 0 & 1 & 2 \\ -2 & 1 & -2 & -4 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Finally, find $L_1 \Phi_{L_1} = -L_2 \Phi_{L_2}$:

$$\begin{pmatrix} 0 & 16 & -7 & 2 \\ 4 & 1 & -3 & -5 \\ -2 & -19 & 8 & -3 \\ 9 & -1 & 2 & 3 \\ 7 & -3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 7 & 14 \\ 0 & -4 & 3 & 6 \\ 0 & 2 & -8 & -16 \\ 0 & -9 & -2 & -4 \\ 0 & -7 & -1 & -2 \end{pmatrix}$$

The third and the fourth vectors are proportional to each other, so we may simply drop the fourth one + obviously drop the first one.

Therefore, the basis of $L_1 \cap L_2$ is $\langle (0, -4, 2, -9, -7), (7, 3, -8, -2, -1) \rangle$ and $\dim L_1 \cap L_2 = 2$.

Problem 4

Let U be a subspace in $\mathbb{R}^5$ generated by vectors

$$v_1 = (-12, -1, -5, 13, -10), \quad v_2 = (14, 10, 13, 12, -11), \quad v_3 = (-4, -3, 7, 1, 8), \quad v_4 = (34, 9, 30, -13, 17)$$

Present a basis of some subspace $W \subseteq \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$ and W is not represented by a linear span of just vectors of a standard basis of space $\mathbb{R}^5$.

Check whether the given vectors are linearly dependent and find the basis in U:

$$\begin{pmatrix} -12 & 14 & -4 & 34 \\ -1 & 10 & -3 & 9 \\ -5 & 13 & 7 & 30 \\ 13 & 12 & 1 & -13 \\ -10 & -11 & 8 & 17 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, basis of U is $\langle v_1, v_2, v_3 \rangle$. Let $W = \langle (1,1,1,1,1), (1,0,1,0,1) \rangle$. Check whether this is a complementary subspace to U by checking if $U \oplus W$ form a basis:

$$\begin{pmatrix} -12 & 14 & -4 & 1 & 1 \\ -1 & 10 & -3 & 1 & 0 \\ -5 & 13 & 7 & 1 & 1 \\ 13 & 12 & 1 & 1 & 0 \\ -10 & -11 & 8 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, $U \oplus W$ generate $\mathbb{R}^5$. What remains is to check whether linear spans of W and $\langle e_1, e_2, e_3, e_4, e_5 \rangle$ are the same. They are not because, for instance, it's impossible to find such $n, k, 0 \leqslant i \leqslant 5$ that $e_i = n(1,1,1,1,1) + k(1,0,1,0,1)$, thus $W = \langle (1,1,1,1,1), (1,0,1,0,1) \rangle$ is valid and is the answer to this problem.

Problem 5

In space $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ consider subspaces $U = \langle v_1, v_2 \rangle$ and $W = \langle v_3, v_4 \rangle$, where

$$v_1 = \begin{pmatrix} -13 & -2 \\ -6 & 12 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -11 & 13 \\ 9 & 12 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 11 & -12 \\ -5 & -4 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 6 & 0 \\ 7 & -15 \end{pmatrix}$$

> Note that I will convert all matrices to a vector like follows for convenience's sake:

> $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{pmatrix}$$

Subproblem A

Prove that $\mathrm{Mat}_{2\times 2}(\mathbb{R}) = U \oplus W$.

Transform all matrices to a vector like above and check whether the direct sum forms a basis in $\mathbb{R}^4$ since it was proven on the seminars/lectures that matrices $n \times n$ are isomorphic to vectors in $\mathbb{R}^{n^2}$. It's obvious that the linear spans of U and W have dimensions equal to 2, therefore checking whether $U \oplus W$ forms a basis would be sufficient to prove that $\mathrm{Mat}_{2\times 2}(\mathbb{R}) = U \oplus W$. For this, check linear dependency as many times above (vectors in columns):

$$\begin{pmatrix} -13 & -11 & 11 & 6 \\ -2 & 13 & -12 & 0 \\ -6 & 9 & -5 & 7 \\ 12 & 12 & -4 & -15 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\mathrm{Mat}_{2\times 2}(\mathbb{R}) = U \oplus W$ is indeed true.

Subproblem B

Find the projection of a vector

$$\xi = \begin{pmatrix} -7 & -1 \\ 5 & 5 \end{pmatrix}$$

to subspace W parallel to subspace U.

Write vectors u_1, u_2, w_1, w_2 into columns and solve $Sx = \xi$

$$\begin{pmatrix} -13 & -11 & 11 & 6 \\ -2 & 13 & -12 & 0 \\ -6 & 9 & -5 & 7 \\ 12 & 12 & -4 & -15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 5 \\ 5 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} -13 & -11 & 11 & 6 & -7 \\ -2 & 13 & -12 & 0 & -1 \\ -6 & 9 & -5 & 7 & 5 \\ 12 & 12 & -4 & -15 & 5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Funnily enough,

$$x = v_1 + v_2 + v_3 + v_4$$

Therefore, projection of ξ on W along u is $v_3 + v_4$.

Finally, **the answer**

$$\bar{\xi} = \begin{pmatrix} 11 & -12 \\ -5 & -4 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 7 & -15 \end{pmatrix} = \begin{pmatrix} 17 & -12 \\ 2 & -19 \end{pmatrix}$$