Discrete Maths, Homework 15

Problem 1

In the left part of the bipartite graph there are 300 vertices, and in the right one there are 400 vertices. Degress of all vertices in the left part are equal to 4, and degress of all vertices in the right part are equal to 3. Prove that in such a graph there is a matching of size 300.

Per Hall's marriage theorem, there is a matching of size $|L| = 300$ only and only if $\forall S \subseteq L$ the set of neighbours $G(S) \subseteq R$ has at least the same number of vertices as S. In other words, $(|G(S)| \geqslant |S|)$.

Now, let induction go brrr. We need to check whether $|G(S)| \geqslant |S|$ holds for all S.

The base is obviously true:

$$S_0 = \{v_{left_1}\}, |S_0| = 1, G(S_0) = \{v_{right_1}, v_{right_2}, v_{right_3}, v_{right_4}\}, |G(S_0)| = 4,$$

thus $|G(S_0)| > |S|$ is true.

Now, attempt to add some vertex v_{left_i} from the left part of the graph to our S. What can possibly happen?

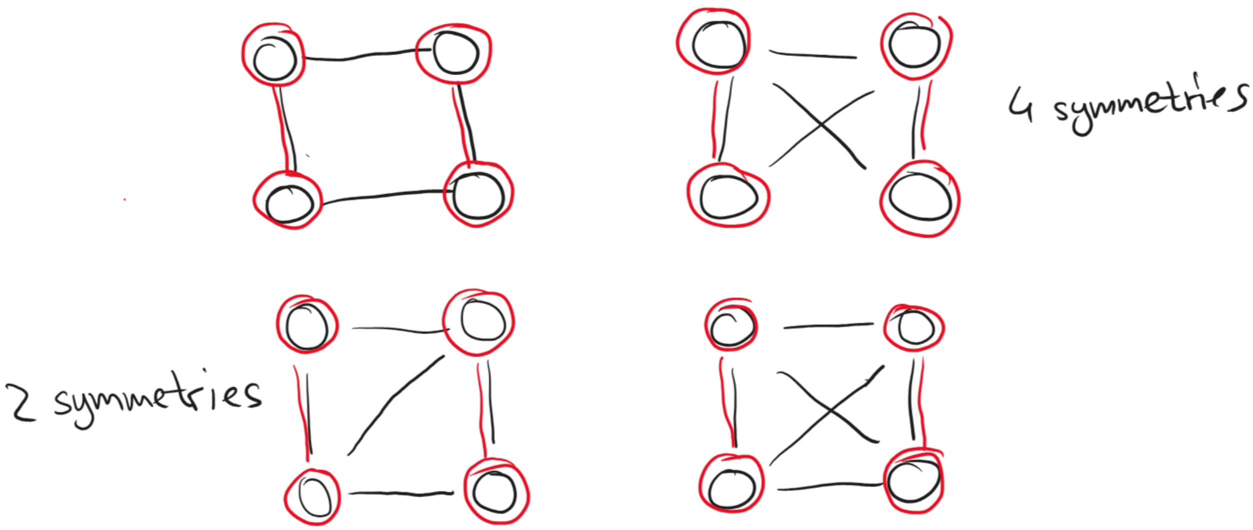
Consider that there are ν vertices in our graph, thus at the very least certainly have 4ν edges and $\left\lceil \frac{4\nu}{3} \right\rceil$ neighbours to our subgraph per the pigeonhole principle. Now, say that we have $\nu + 1$ vertices in our set S. Accordingly, we would have at least $\left\lceil \frac{4\nu}{3} + \frac{1}{3} \right\rceil$ neighbours to our subgraph S.

Therefore, $|G(S)| \geqslant |S|$ for all S and per Hall's marriage theorem, there is always a matching of size $|L| = 300$, q. e. d.

Problem 2

In a graph on 2024 vertices there are at least two edges between each three vertices. Prove that there is a perfect matching (of 1012 edges).

Split the massive group of 2024 vertices into groups of 4 vertices, stripping all edges between those 4 vertices in each group and all other vertices. Within given restrictions, there are only 8 (including symmetries) possible arrangements of edges between 4 vertices such that there are at least two edges between each three vertices. I list all of them in the figure below (reader may easily check that no other arrangements are valid, so the search is exhaustive):



As seen in the figure, it's always possible to choose two matching for each of the subgroups of 4 vertices. Since $2024 \bmod 4$ is 0, there would be no remaining groups after we split all vertices into 506 groups of 4, and all of them would yield 2 matchings regardless of the edge combination, thus yielding $506 \times 2 = 1012$ matchings, which is indeed a perfect one, q. e. d.

Problem 3

In a graph on 101 vertices there is an independent set of size 52. Prove that there is no matching of size 50 in this graph.

Kinda informal proof through logic

Since there is an independent set of size 52, then each of these vertices is a part of a potential independent match.

This leaves $101 - 52 = 49$ vertices to form those matches with, which simply does not allow one to choose a matching of size 50.

Formal-esque proof

We know that $\alpha(G)$ is the maximum size of an independent set in a graph, $\tau(G)$ is the minumum size of a vertex cover, and $\mu(G)$ is the maximum size of a matching, all for graph G.

The following identity from the lectures is true:

$$n - \alpha(G) = \tau(G),$$

where n is the total number of vertices. $\alpha(G)$ is obviously at least 52 per given conditions $\implies \tau(G) = 101 - 52 = 49$.

The following identity is also true:

$$\tau(G) \geq \mu(G)$$

Thus,

$$\mu(G) \leq 49,$$

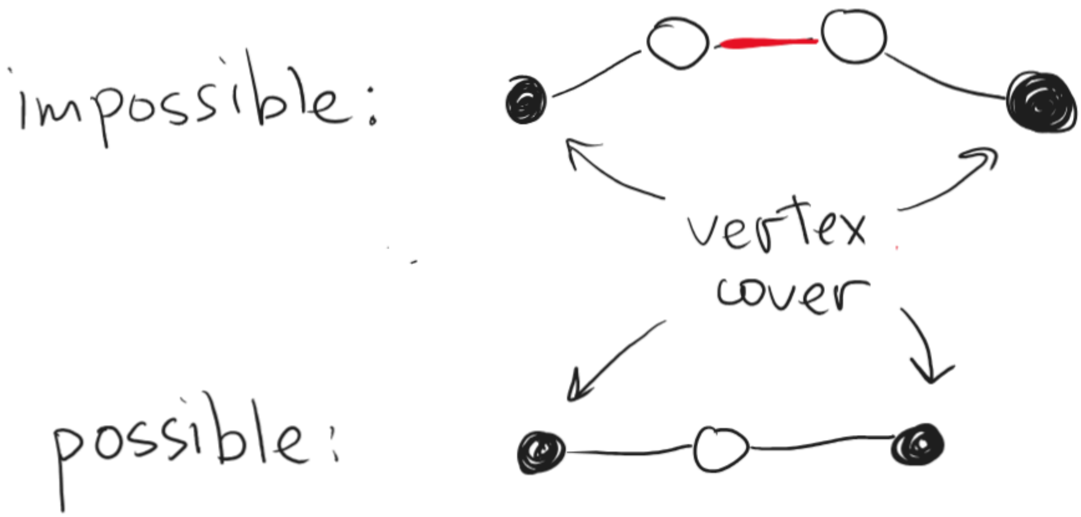
which means that the maximum size of a matching is 49 and physically cannot be 50, q. e. d.

Problem 4

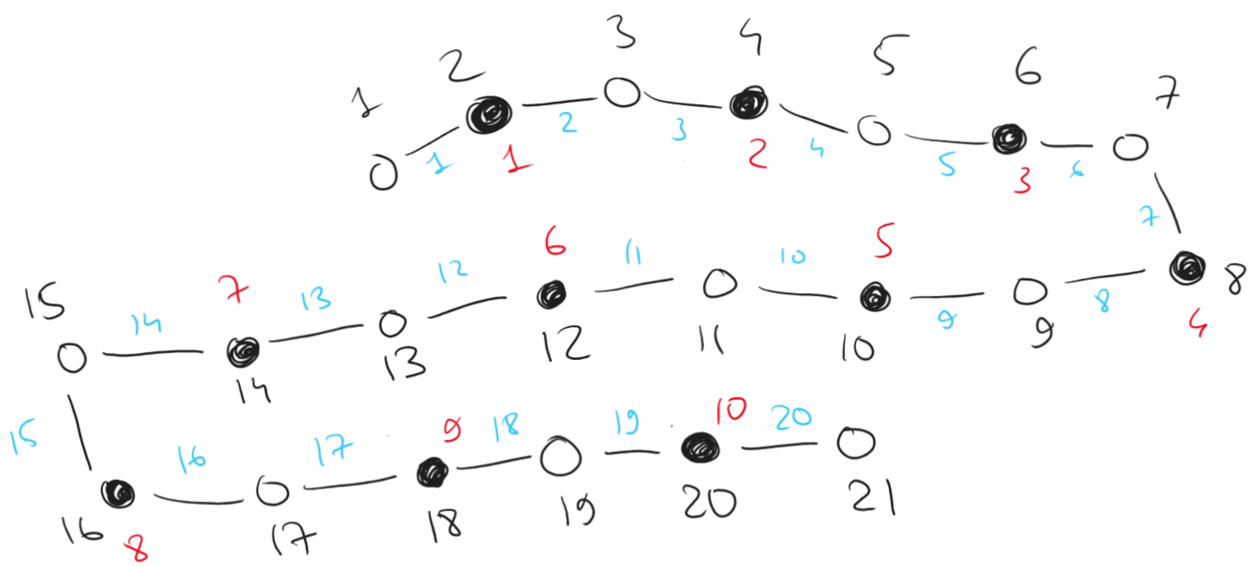
In a graph on n vertices there is a vertex cover of size 10. Prove that in such a graph there is no simple path of length 21. (In a simple path all vertices are different, and the length of the path is the number of edges in it.)

Since there is a vertex cover of size 10, then there is an independent set of at least size $n - 10$. Thus, we require at least $21 - 10 = 11$ vertices in the independent set. Is this possible?

For a simple path to exist, we need an edge between each two consecutive vertices along the path. Two vertices that are not a part of the vertex cover cannot be consecutive in the path, since there can't be an edge between them (see figure below for a visualization).



Therefore, the longest possible simple path through the graph would look similar to the graph in the figure below (there can be any number of edges between any two black nodes [vertex cover] or between any of the black-white pairings but not between white-white nodes, since that goes against the given conditions).



Per the pigeonhole principle, let's build a best-case scenario graph, placing a vertex between each two vertices that are a part of a cover and starting and ending our path with vertices from the independent set. For the simple path of length 21 to exist, we need at least 22 vertices in the graph, which is simply impossible, because there are no two adjacent black vertices we can place a white vertice between, nor can we add any vertices to the beginning or to the end of the graph, thus only resulting in a simple path of length 20 at the very max, q. e. d.