Calculus Homework #6

Problem 8.8

> In general, to prove that the two-variable limit does not exist, it is enough to find two different limits for two different paths.

Subproblem A

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y}{x^3 + y}$$

For $y = mx^3$:

$$\lim_{x \to 0} \frac{x^3 - m^3x^3}{x^3 + m^3x^3} = \lim_{x \to 0} \frac{1 - m^3}{1 + m^3} = \frac{1 - m^3}{1 + m^3}$$

The limit is dependent on $m \Rightarrow$ the limit does not exist.

Subproblem B

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$$

For $y = mx$:

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{mx^2}{(1 + m^2)x^2} = \lim_{x \to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

The limit is dependent on $m \Rightarrow$ the limit does not exist.

Subproblem C

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2 - x^2}{y^2 + x^2}$$

For $y = mx$:

$$\lim_{x\to 0} \frac{m^2x^2 - x^2}{m^2x^2 + x^2} = \lim_{x\to 0} \frac{m^2 - 1}{m^2 + 1} = \frac{m^2 - 1}{m^2 + 1}$$

The limit is dependent on $m \Rightarrow$ the limit does not exist.

Subproblem D

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2y^2 + (x-y)^2}$$

For $y = x$:

$$\lim_{x\to 0} \frac{x^2x^2}{x^2x^2 + (x-x)^2} = \lim_{x\to 0} \frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1$$

For $y = 0$:

$$\lim_{x\to 0} \frac{x^2 0^2}{x^2 0^2 + (x-0)^2} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

There are at least two different limits for two paths $y = x, y = 0 \Rightarrow$ the limit does not exist.

Subproblem E

$$\lim_{\substack{x\to 0\\ y\to 0}} \left(x + y\sin\frac{1}{x}\right)$$

$f_1(x) = \sin\frac{1}{x} \in [-1, 1]$ is a bounded function. $f_2(y) = y$ as $y \to 0$ is an infinitesimal function. Product of a bounded function and an infinitesimal function is infinitesimal. Therefore, $\lim_{\substack{x\to 0\\ y\to 0}} g(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} f_1(x)f_2(y) = 0$

$$\lim_{\substack{x\to 0\\ y\to 0}} \left(x + y\sin\frac{1}{x}\right) = \lim_{\substack{x\to 0\\ y\to 0}} x + \lim_{\substack{x\to 0\\ y\to 0}} y\sin\frac{1}{x} = \lim_{x\to 0} x + \lim_{\substack{x\to 0\\ y\to 0}} g(x,y) = 0 + 0 = 0$$

Answer:

$$\lim_{\substack{x\to 0\\ y\to 0}} \left(x + y\sin\frac{1}{x}\right) = 0$$

Problem 8.9

Find limit of $f(x,y) = \frac{y-2x^2}{y-x^2}$ in point $(0,0)$ along the path $x = \alpha t, y = \beta t, \alpha^2 + \beta^2 \neq 0$. Prove that $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y-2x^2}{y-x^2} \Rightarrow \lim_{t\to 0} \frac{\beta t - 2\alpha^2 t^2}{\beta t - \alpha^2 t^2} = \lim_{t\to 0} \frac{\beta - 2\alpha^2 t}{\beta - \alpha^2 t} = \lim_{t\to 0} \frac{\beta - 0}{\beta - 0} = \lim_{t\to 0} 1 = 1$$

For $y = mx^2$:

$$\lim_{x\to 0} \frac{mx^2 - 2x^2}{mx^2 - x^2} = \lim_{x\to 0} \frac{m-2}{m} = \frac{m-2}{m}$$

Thus, the multivariable limit does not exist.

Answer: $\lim_{\substack{x\to 0\\ y\to 0}} \frac{y-2x^2}{y-x^2} = 1$ along the path of $x = \alpha t, y = \beta t, \alpha^2 + \beta^2 \neq 0$

Problem 8.10

Is the function

$$u(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, \forall x, y\colon x^2 + y^2 \neq 0 \\ 0, \quad \forall x, y\colon x^2 + y^2 = 0 \end{cases}$$

continuous in point $(0,0)$?

$$u'(x,y) = \frac{xy}{x^2+y^2}$$

Per the continuity criterion: for $u(x,y)$ to be continuous in point of closure $(0,0)$ it is required and sufficient that $u(0,0) = \lim_{\substack{x\to 0\\ y\to 0}} u'(x,y)$. This limit of $u'(x,y)$, as proven in **Problem 8.8, Subproblem B** does not exist. Therefore, the function $u(x,y)$ is not continuous.

Answer: $u(x,y)$ is not continuous