Calculus, Homework 2

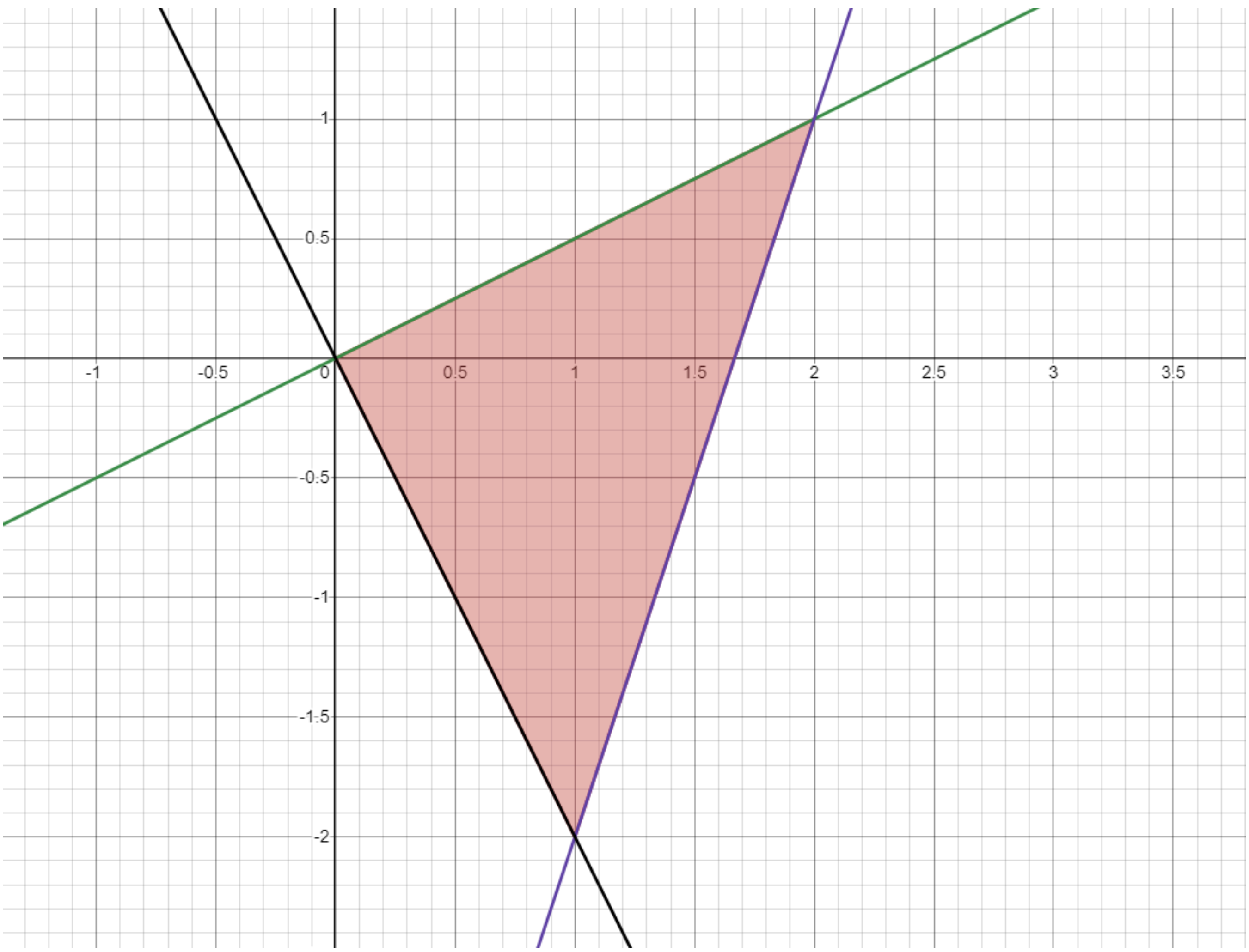
Problem 1.4

Calculate integral

$$\iint\limits_D x^2 y dx dy$$

where D is a closed triangle with vertices $(0, 0), (2, 1), (1, -2)$.

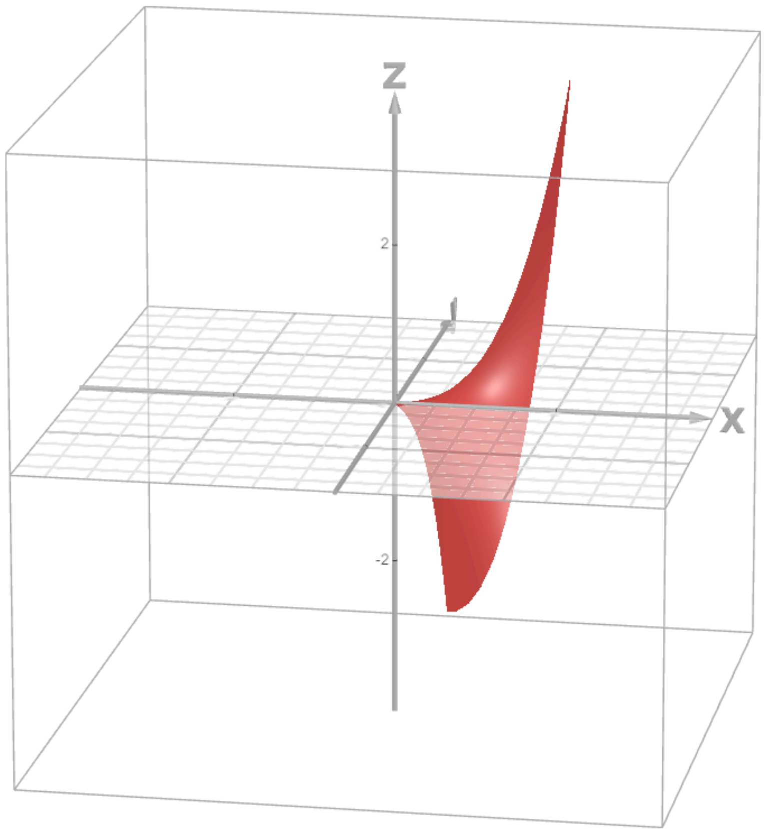
The bounding box of the triangle would be denoted by the following three lines:



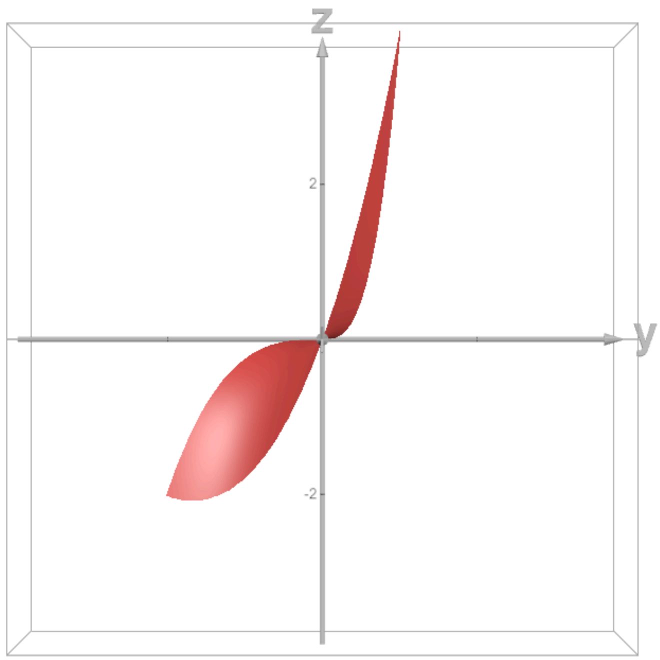
which are

$$\begin{cases} 2y = x & \text{green} \\ \frac{y}{3} + \frac{5}{3} = x & \text{purple} \\ -\frac{y}{2} = x & \text{black} \end{cases}$$

The figure we need to calculate is



which I will split into two parts horizontally over $y = 0$.



Thus we'd have

$$\underbrace{\int_{-2}^{0} dy \int_{2y}^{\frac{y}{3}+\frac{5}{3}} x^2 y dx}_{\mathcal{I}_\mathcal{A}} + \underbrace{\int_{0}^{1} dy \int_{-\frac{y}{2}}^{\frac{y}{3}+\frac{5}{3}} x^2 y dx}_{\mathcal{I}_\mathcal{B}}$$

$$\begin{aligned}
\mathcal{I}_\mathcal{A} &= \int_0^1 dy \int_{2y}^{\frac{y}{3}+\frac{5}{3}} x^2 y dx \\
&= \int_0^1 \left(\frac{x^3 y}{3}\right)\Bigg|_{2y}^{\frac{y}{3}+\frac{5}{3}} dy \\
&= \int_0^1 \left(\frac{(\frac{y}{3}+\frac{5}{3})^3 y}{3} - \frac{(2y)^3 y}{3}\right) dy \\
&= \int_0^1 \left(-\frac{215y^4}{81} + \frac{5y^3}{27} + \frac{25y^2}{27} + \frac{125y}{81}\right) dy \\
&= -\frac{215y^5}{405} + \frac{5y^4}{108} + \frac{25y^3}{81} + \frac{125y^2}{162}\Bigg|_0^1 \\
&= \frac{193}{324}
\end{aligned}$$

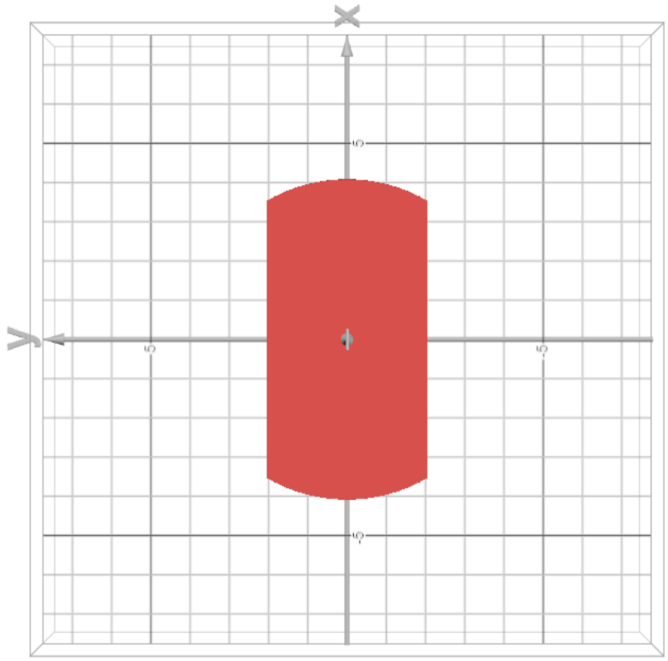
$$
\begin{aligned}
\mathcal{I}_{\mathcal{B}} &= \int_{-2}^{0} dy \int_{-\frac{y}{2}}^{\frac{y}{3}+\frac{5}{3}} x^2 y dx \\
&= \int_{-2}^{0} \left(\frac{x^3 y}{3}\right) \Bigg|_{-\frac{y}{2}}^{\frac{y}{3}+\frac{5}{3}} dy \\
&= \int_{-2}^{0} \left(\frac{(\frac{y}{3}+\frac{5}{3})^3 y}{3} - \frac{(-\frac{y}{2})^3 y}{3}\right) dy \\
&= \int_{-2}^{0} \left(\frac{35y^4}{648} + \frac{5y^3}{27} + \frac{25y^2}{27} + \frac{125y}{81}\right) dy \\
&= \frac{35y^5}{3240} + \frac{5y^4}{108} + \frac{25y^3}{81} + \frac{125y^2}{162} \Bigg|_{-2}^{0} \\
&= -\frac{82}{81}
\end{aligned}
$$

$$
\mathcal{I}_{\mathcal{A}} + \mathcal{I}_{\mathcal{B}} = \frac{193}{324} - \frac{82}{81} = -\frac{5}{12}
$$

Problem 1

Bring triple integral $\iiint\limits_D f(x, y, z) dxdydz$ to one of iterated ones, where $D = \{(x, y, z) | y^2 \leq z \leq 4, x^2 + y^2 \leq 16\}$.

First way I suggest we bound the integral between yz surfaces, differentiating by x.



We would have to define the bounding arcs of the circle, which would be

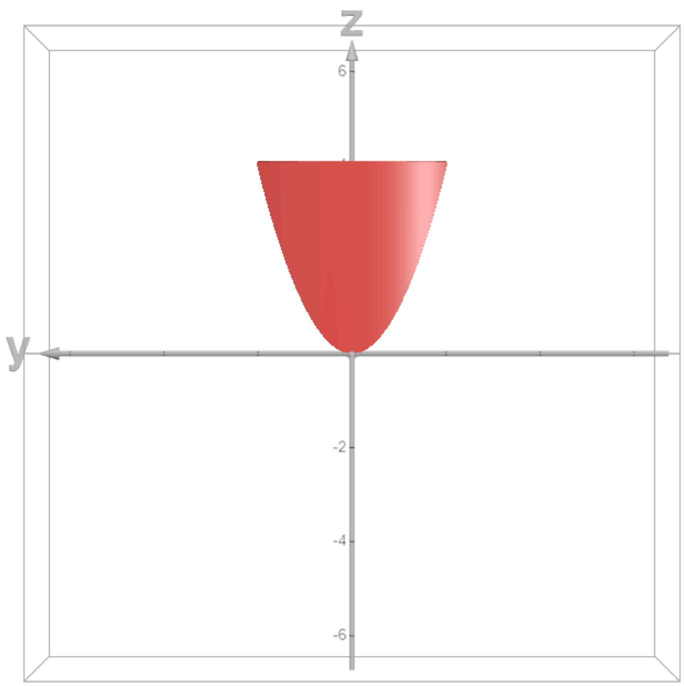
$$x^2 \leq 16 - y^2$$

$$\begin{cases} x \leq -\sqrt{16 - y^2} \\ x \geq \sqrt{16 - y^2} \end{cases}$$

Thus our integral would be

$$\iiint_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} f(x, y, z) dx \dots$$

Now to bound the area between xy-surfaces, differentiating by z.



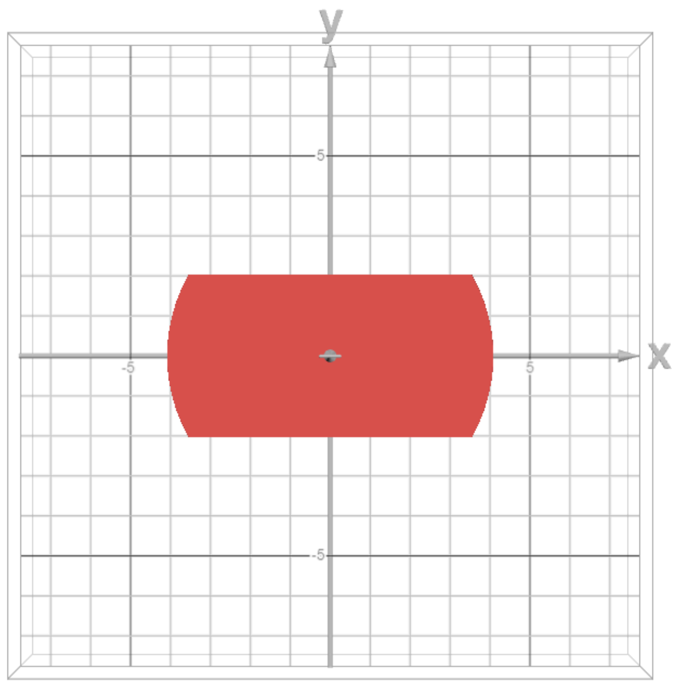
This would be simply

$$y^2 \leq z \leq 4$$

We would get

$$\iint_{y^2}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} f(x,y,z)dxdz \ldots$$

Then bound the area parallel between xz-plane, differentiating by y. the integral between the following lines:



If we were to analytically compute it, we would have to take

$$y^2 \leq 4 \implies -2 \leq y \leq 2$$

Thus we may add the last set of boundaries to our integral.

$$\int_{-2}^{2} \int_{y^2}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} f(x,y,z) dxdzdy$$

Problem 2

Change the order of integration all possible ways:

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{x^2+y} f(x,y,z)dz$$

Firstly, let's list all 6 $(3! = 6)$ possible ways determining the order in which we bound the area between surfaces (here any variables from xyz denote the arguments of a function that defines the plane that we use, whereas the missing variable out of three denotes the dimension we differentiate by).

$$\underset{dz}{xy} \mapsto \underset{dx}{yz} \mapsto \underset{dy}{xz}$$

$$\underset{dz}{xy} \mapsto \underset{dy}{xz} \mapsto \underset{dx}{yz}$$

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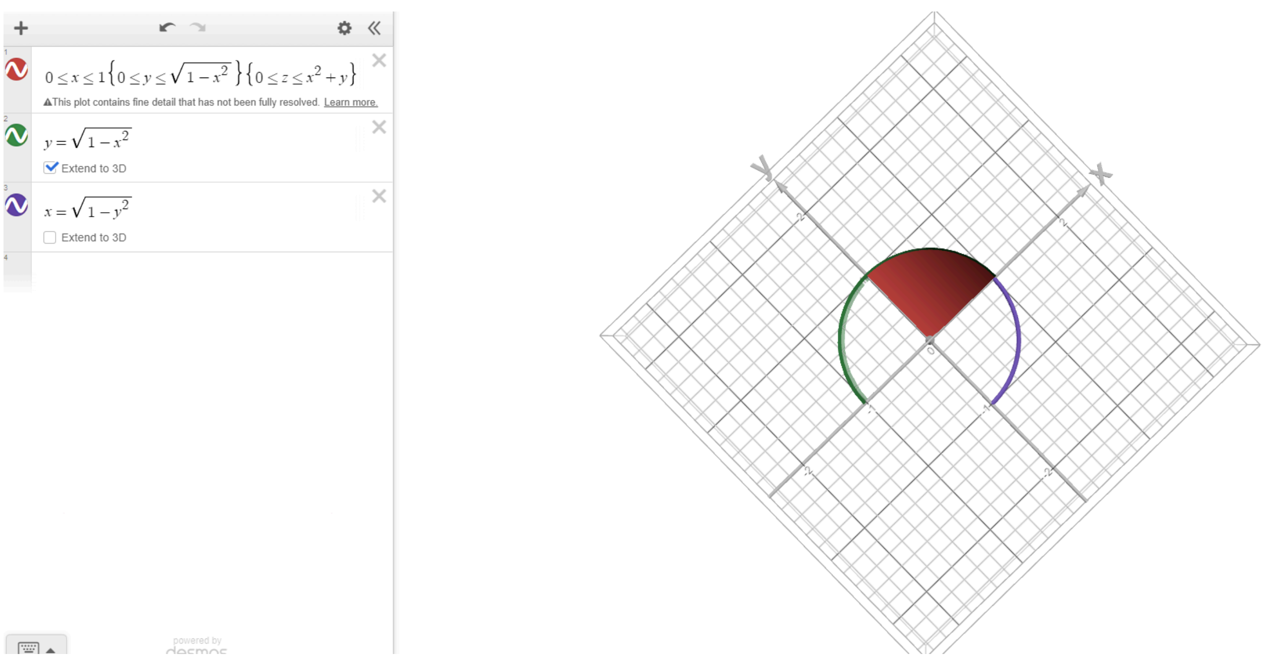
$$\underset{dy}{xz} \mapsto \underset{dx}{yz} \mapsto \underset{dz}{xy}$$

and consider them sequentially.

$$\underset{dz}{xy} \mapsto \underset{dx}{yz} \mapsto \underset{dy}{xz}$$

First operation remains, second operation is symmetrical to the original integral bounds, third operation is also symmetrical, thus we get:

$$\int_0^1 dy \int_0^{\sqrt{1-y^2}} dx \int_0^{x^2+y} f(x,y,z)dz$$



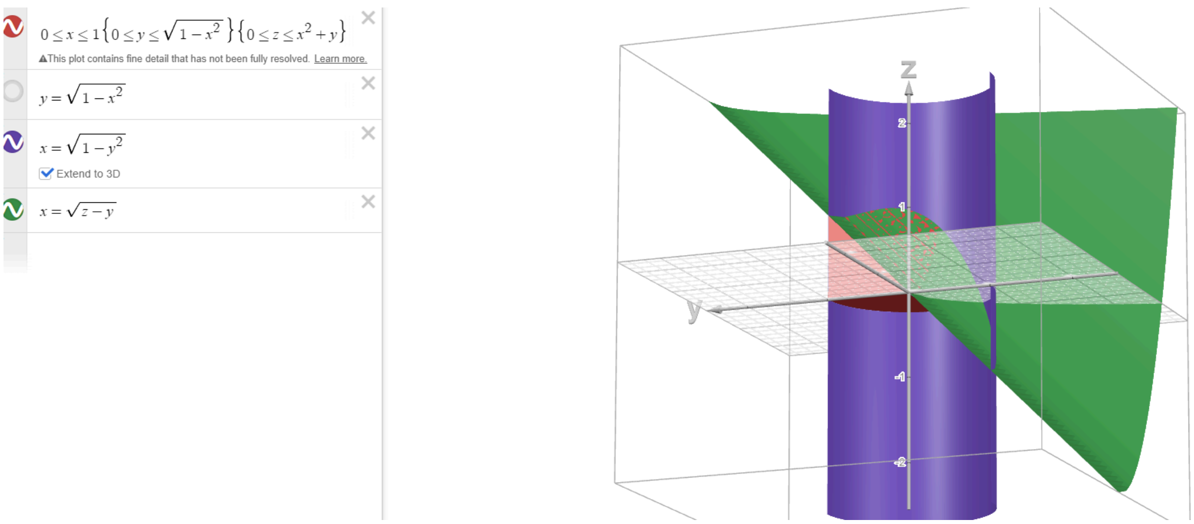
$$\underset{dz}{xy} \mapsto \underset{dy}{xz} \mapsto \underset{dx}{yz}$$

This is the original integral.

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{x^2+y} f(x,y,z)dz$$

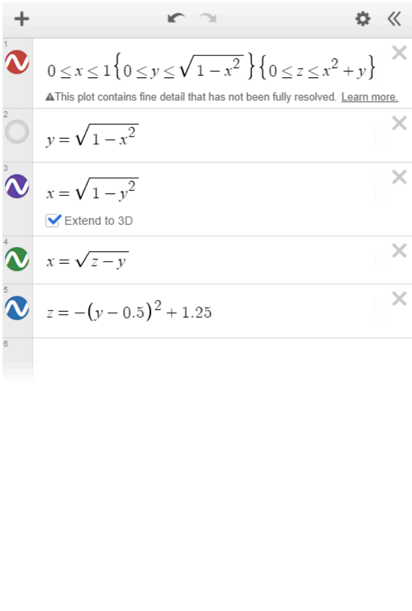
$$\underset{dx}{yz} \mapsto \underset{dz}{xy} \mapsto \underset{dy}{xz}$$

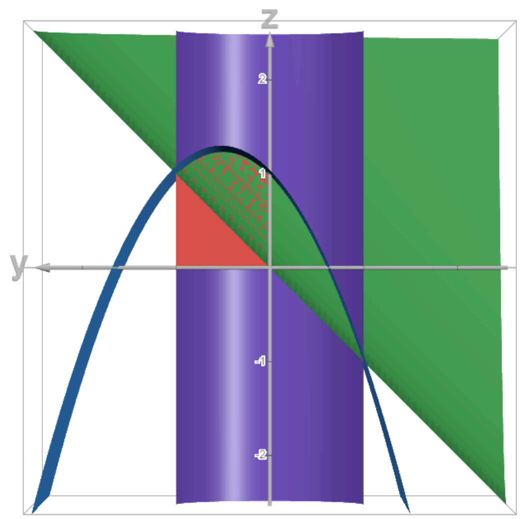
Bounding the integral between yz-argument planes, we get one above-mentioned plane $x = \sqrt{1-y^2}$ and one plane we derive from $z = x^2 + y \implies x = \sqrt{z-y}$, for which we take the positive root since we operate in a non-negative eighth part of the cartesian space.



For the bounds from xy arguments, we should take $z = 0$ as the lower bound and find the upper bound as follows, which could be seen below:

$$\begin{cases} z = x^2 + y \\ x = \sqrt{1-y^2} \end{cases} \implies z = 1 - y^2 + y$$





which leaves us with the necessity to bound y with $x = 0$ and $x = 1$

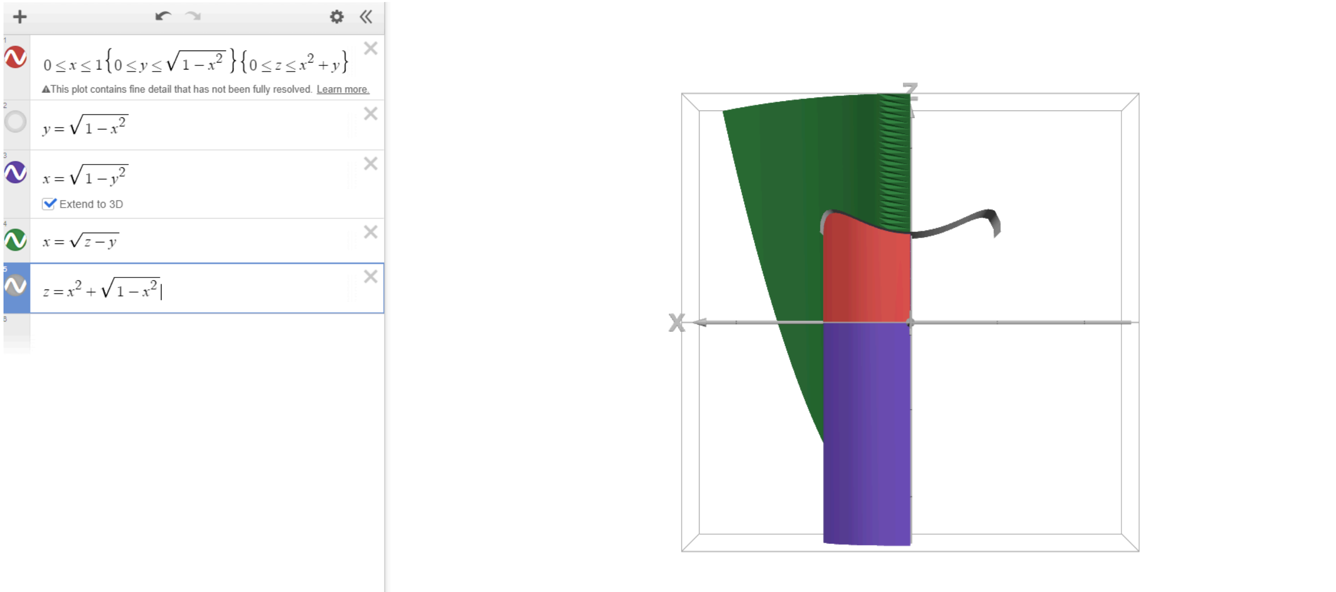
$$\int_0^1 dy \int_0^{1-y^2+y} dz \int_{\sqrt{z-y}}^{\sqrt{1-y^2}} f(x,y,z)dx$$

However, since $\sqrt{z-y}$ doesn't evaluate in real numbers when $0 \le z \le y$, then we also need to add another integral for this case when z is bounded in $[0, y]$ and x goes down to zero

$$\int_0^1 dy \int_0^{1-y^2+y} dz \int_{\sqrt{z-y}}^{\sqrt{1-y^2}} f(x,y,z)dx$$
$$+ \int_0^1 dy \int_0^{y} dz \int_0^{\sqrt{1-y^2}} f(x,y,z)dx$$

$$\underset{dx}{yz} \mapsto \underset{dy}{xz} \mapsto \underset{dz}{xy}$$

The first step would be the same, the next boundary for the function of z with argument x we could get as follows:



$$\begin{cases} z = x^2 + y \\ y = \sqrt{1-x^2} \end{cases} \implies z = x^2 + \sqrt{1-x^2}$$

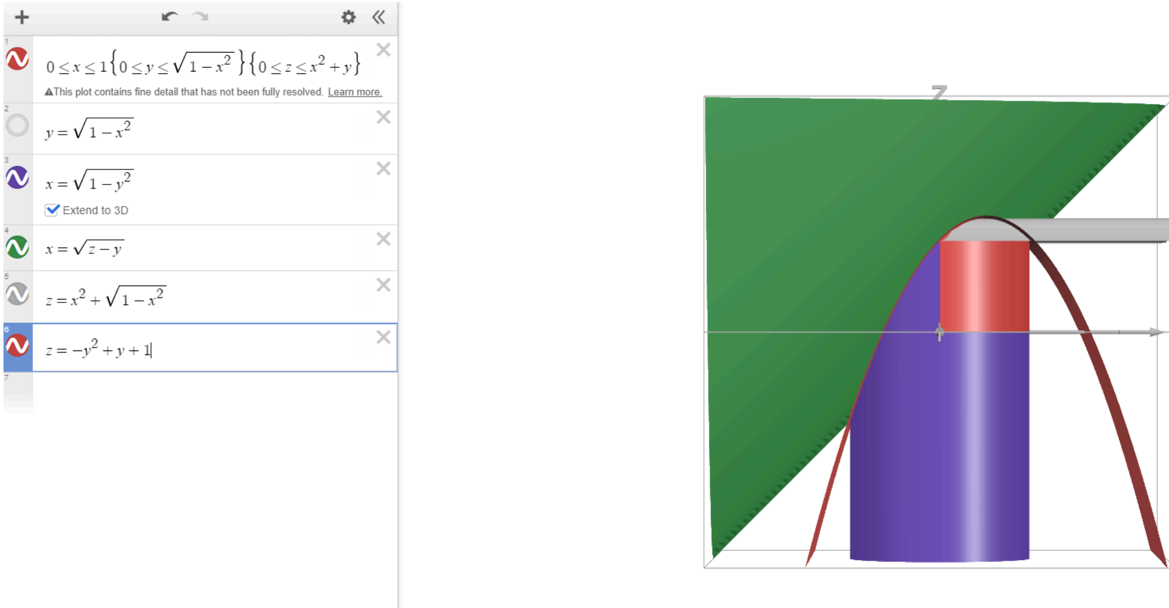
with 0 being the lower bound.

Finally, with the upper boundary being the vertex of the aforementioned parabola $-y^2+y+1$, we may get that its y coordinate is

$$(-y^2+y+1)'_y = -2y+1 = 0 \implies y = \frac{1}{2}$$

implying that

$$z = -\left(\frac{1}{2}\right)^2 + \frac{1}{2} + 1 = \frac{5}{4}$$



Thus we get

$$\int_0^{\frac{5}{4}} dz \int_0^{x^2+\sqrt{1-x^2}} dy \int_{\sqrt{z-y}}^{\sqrt{1-y^2}} f(x,y,z)dx$$

Similarly as above, we're missing a piece for when $z \leq y \leq 1$, which we should include, bounding x below with 0.

$$\int_0^{\frac{5}{4}} dz \int_0^{x^2+\sqrt{1-x^2}} dy \int_{\sqrt{z-y}}^{\sqrt{1-y^2}} f(x,y,z)dx$$
$$+ \int_0^1 dz \int_1^z dy \int_0^{\sqrt{1-y^2}} f(x,y,z)dx$$

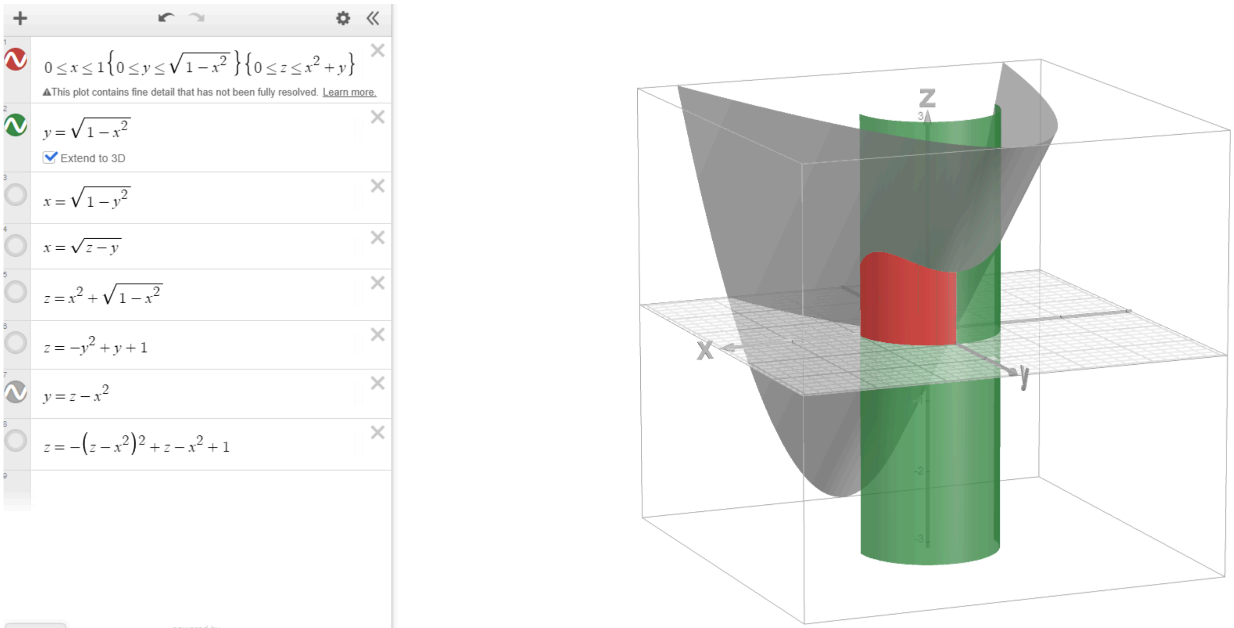
$$\underset{dy}{xz} \mapsto \underset{dz}{xy} \mapsto \underset{dx}{yz}$$

Firstly, bound using upper bound that we already have calculated above, which is

$$y = \sqrt{1-x^2}$$

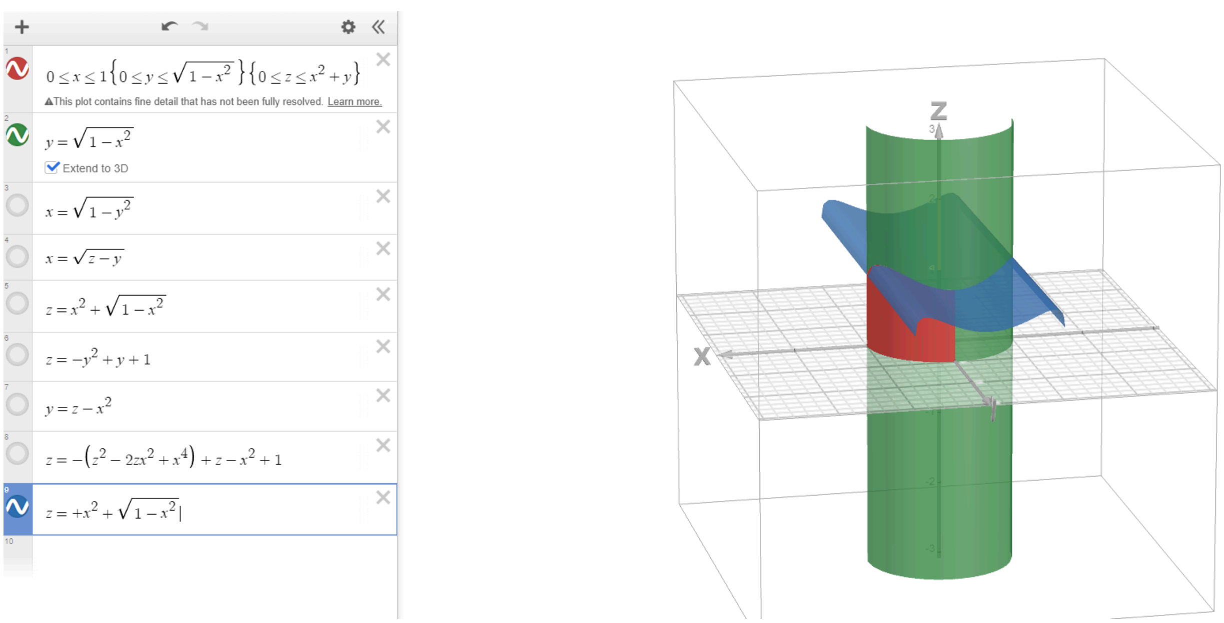
and lower bound

$$z = x^2 + y \implies y = z - x^2$$

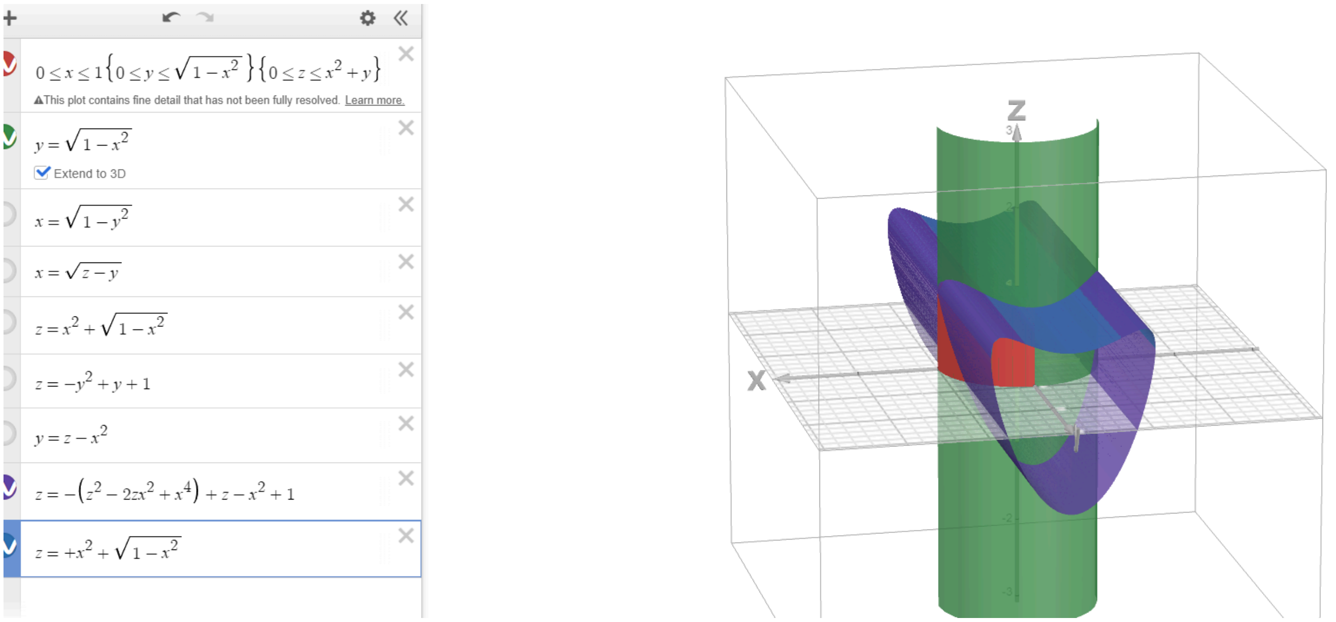


Secondly, we bound with 0 below and a re-rendering of the well-known parabola from above.

$$z = -\sqrt{1-x^2}^2 + \sqrt{1-x^2} + 1 = -1 + x^2 + \sqrt{1-x^2} + 1 = x^2 + \sqrt{1-x^2}$$



Also, a heart easter egg:



The last boundary would simply be $[0,1]$.

$$\int_0^1 dx \int_0^{x^2+\sqrt{1-x^2}} dz \int_{z-x^2}^{\sqrt{1-x^2}} f(x,y,z)dy$$

However in this case we will have included an extra piece because $z-x^2$ may take negative values of y, which could be mediated simply by subtracting this piece bounded above by 0 for y off of the integral:

$$\int_0^1 dx \int_0^{x^2+\sqrt{1-x^2}} dz \int_{z-x^2}^{\sqrt{1-x^2}} f(x,y,z)dy \\ -\int_0^1 dx \int_0^{x^2+\sqrt{1-x^2}} dz \int_{z-x^2}^{0} f(x,y,z)dy$$

$$\underset{dy}{xz} \mapsto \underset{dx}{yz} \mapsto \underset{dz}{xy}$$

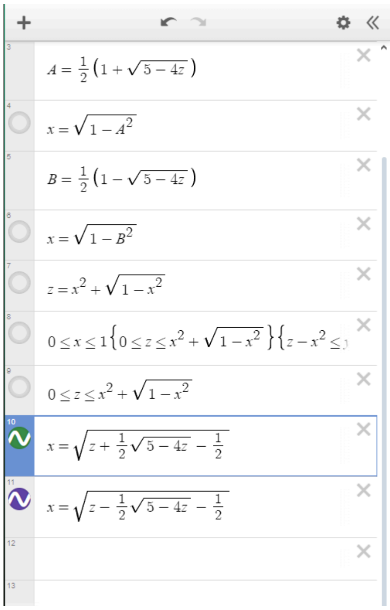
The first boundary would be the same as the previous point, the second one would be bounded below by 0, whereas the upper bound would be a rendition of the following:

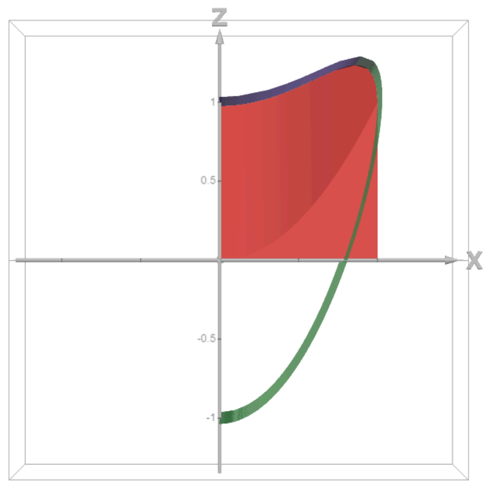
$$\begin{cases} y=\sqrt{1-x^2} \\ z=1-y^2+y \end{cases} \implies z=x^2+\sqrt{1-x^2}$$

Solving the above for z would yield us positive

$$x=\sqrt{z+\frac{1}{2}\sqrt{5-4z}-\frac{1}{2}}$$

$$x=\sqrt{z-\frac{1}{2}\sqrt{5-4z}-\frac{1}{2}}$$



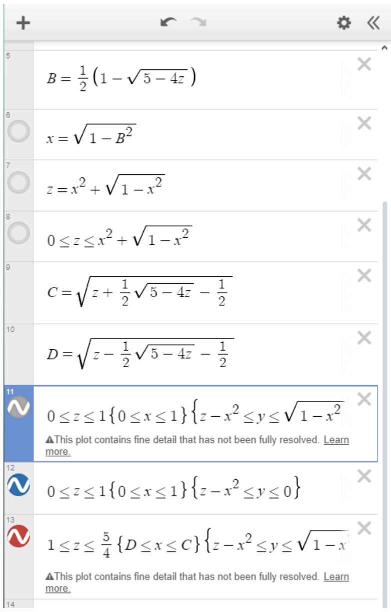


which are upper and lower bounds respectively for $1\le z\le\frac{5}{4}$, whereas for $0\le z\le 1$ we would simply be bound in $[0,1]$. Thus, we get

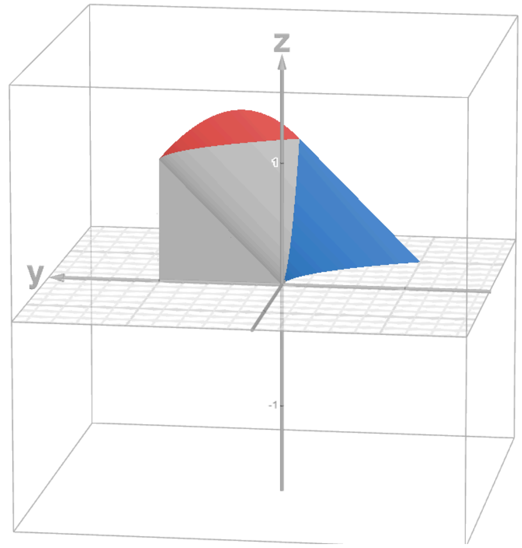
> Remembering to subtract the last extra for when $y\le 0$

$$\int_1^{\frac{5}{4}}dz\int_{\sqrt{z-\frac{1}{2}\sqrt{5-4z}-\frac{1}{2}}}^{\sqrt{z+\frac{1}{2}\sqrt{5-4z}-\frac{1}{2}}}dx\int_{z-x^2}^{\sqrt{1-x^2}}f(x,y,z)dy$$
$$+\int_0^1 dz\int_0^1 dx\int_{z-x^2}^{\sqrt{1-x^2}}f(x,y,z)dy$$
$$+\int_0^1 dz\int_0^1 dx\int_{z-x^2}^{0}f(x,y,z)dy$$

Visualization:



5
$B = \frac{1}{2}\left(1 - \sqrt{5 - 4z}\right)$
6
$x = \sqrt{1 - B^2}$
7
$z = x^2 + \sqrt{1 - x^2}$
8
$0 \le z \le x^2 + \sqrt{1 - x^2}$
9
$C = \sqrt{z + \frac{1}{2}\sqrt{5 - 4z} - \frac{1}{2}}$
10
$D = \sqrt{z - \frac{1}{2}\sqrt{5 - 4z} - \frac{1}{2}}$
11
$0 \le z \le 1\{0 \le x \le 1\}\{z - x^2 \le y \le \sqrt{1 - x^2}$
This plot contains fine detail that has not been fully resolved. Learn more.
12
$0 \le z \le 1\{0 \le x \le 1\}\{z - x^2 \le y \le 0\}$
13
$1 \le z \le \frac{5}{4}\{D \le x \le C\}\{z - x^2 \le y \le \sqrt{1 - x}$
This plot contains fine detail that has not been fully resolved. Learn more.
14



z
y
1
-1