Calculus, Homework 7

> Calculate the following limits:

Problem 1

Subproblem A

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2}=\lim_{x\to 2}\frac{(x+3)(x-2)}{(x-1)(x-2)}=\lim_{x\to 2}\frac{x+3}{x-1}=\frac{2+3}{2-1}=5$$

Answer:

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2}=5$$

Subproblem C

$$\lim_{x\to 1}\frac{x^5-3x^4+3x^3-x^2}{x^4-6x^2+8x-3}=\lim_{x\to 1}\frac{x^2(x^3-3x^2+3x-1)}{x^4-6x^2+8x-3}=\lim_{x\to 1}\frac{x^2(x-1)^3}{x^4-6x^2+8x-3}=$$
$$\lim_{x\to 1}\frac{x^2(x-1)^3}{(x+3)(x-1)^3}=\lim_{x\to 1}\frac{x^2}{(x+3)}=\frac{1^2}{1+4}=\frac{1}{4}$$

Answer:

$$\lim_{x\to 1}\frac{x^5-3x^4+3x^3-x^2}{x^4-6x^2+8x-3}=\frac{1}{4}$$

Subproblem F

$$\lim_{x\to 3}\frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}=\lim_{x\to 3}\frac{x+13-4x-4}{(x^2-9)(\sqrt{x+13}+2\sqrt{x+1})}=$$
$$\lim_{x\to 3}\frac{-3(x-3)}{(x-3)(x+3)(\sqrt{x+13}+2\sqrt{x+1})}=\lim_{x\to 3}\frac{-3}{(x+3)(\sqrt{x+13}+2\sqrt{x+1})}=$$
$$\lim_{x\to 3}\frac{-3}{(x+3)(\sqrt{x+13}+2\sqrt{x+1})}=\frac{-3}{6\times(\sqrt{3+13}+2\sqrt{3+1})}=-\frac{1}{2\times 8}=-\frac{1}{16}$$

Answer:

$$\lim_{x\to 3}\frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}=-\frac{1}{16}$$

Problem 2

Subproblem A

$$\lim_{x\to 1}\frac{\sin\frac{\pi x}{2}}{x}=\frac{\sin\frac{\pi}{2}}{1}=1$$

Answer:

$$\lim_{x\to 1}\frac{\sin\frac{\pi x}{2}}{x}=1$$

Subproblem C

$$\lim_{x\to 0}\frac{\operatorname{tg}x+\operatorname{tg}2x+\cdots+\operatorname{tg}nx}{\operatorname{arctg}x}=\lim_{x\to 0}\sum_{k=1}^{n}\frac{\operatorname{tg}kx}{\operatorname{arctg}x}$$

> Since $\operatorname{tg}x=x+O(x)$ and $\operatorname{arctg}x=x+O(x)$, then

$$\lim_{x\to 0}\frac{\operatorname{tg}kx}{\operatorname{arctg}x}=\lim_{x\to 0}\frac{kx+O(x)}{x+O(x)}=\lim_{x\to 0}(k+O(1))=k$$

> Alternatively, since $\lim_{x\to 0}\frac{\operatorname{tg}x}{x}=\lim_{x\to 0}\frac{\operatorname{arctg}x}{x}=1$
>
> $$\lim_{x\to 0}\frac{\operatorname{tg}kx}{\operatorname{arctg}x}=\lim_{x\to 0}\frac{kx\operatorname{tg}kx}{kx\operatorname{arctg}x}=k\lim_{x\to 0}\frac{\operatorname{tg}kx}{kx}\lim_{x\to 0}\frac{x}{\operatorname{arctg}x}=k\times 1\times 1^{-1}=k$$

Therefore,

$$\lim_{x\to 0}\sum_{k=1}^{n}\frac{\operatorname{tg}kx}{\operatorname{arctg}x}=\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

Answer:

$$\lim_{x\to 0}\frac{\operatorname{tg}x+\operatorname{tg}2x+\cdots+\operatorname{tg}nx}{\operatorname{arctg}x}=\frac{n(n+1)}{2}$$

Subproblem D

$$\lim_{x\to 0}\frac{\operatorname{tg}x-\sin x}{\sin^3 x}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{\sin^3 x\cos x}=\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x\cos x}=$$

> Since $1-\cos x=2\sin^2\frac{x}{2}, \sin x\cos x=\frac{1}{2}\sin 2x, \sin x=x+O(x)$, then:

$$\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}\sin 2x\sin x}=\lim_{x\to 0}\frac{2(\frac{x}{2}+O(x))^2}{\frac{1}{2}(2x+O(x))(x+O(x))}=4\lim_{x\to 0}\frac{\frac{x^2}{4}+\frac{x}{2}O(x)+O(x^2)}{2x^2+2xO(x)+xO(x)+O(x^2)}=$$

$$4\lim_{x\to 0}\frac{\frac{x^2}{4}+O(x^2)}{2x^2+O(x^2)}=\frac{4}{8}\lim_{x\to 0}\frac{x^2+O(x^2)}{x^2+O(x^2)}=\frac{1}{2}$$

> Alternatively, since $\lim_{x\to 0}\frac{\sin x}{x}=1$
>
> $$\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}\sin 2x\sin x}=\frac{1}{2}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\frac{x^2}{2^2}}\frac{x}{\sin x}\frac{2x}{\sin 2x}=\frac{1}{2}\times 1^2\times 1^{-1}\times 1^{-1}=\frac{1}{2}$$

Answer:

$$\lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{\sin^3 x}=\frac{1}{2}$$

Subproblem E

$$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a}\xRightarrow{z=x-a}\lim_{z\to 0}\frac{2\sin\frac{z}{2}\cos(\frac{z}{2}+a)}{z}=$$

$$\lim_{z\to 0}\frac{2(\frac{z}{2}+O(z))\cos(\frac{z}{2}+a)}{z}=\lim_{z\to 0}\frac{(z+O(z))\cos(\frac{z}{2}+a)}{z}=$$

$$\lim_{z\to 0}\frac{z\cos(\frac{z}{2}+a)+O(z)\cos(\frac{z}{2}+a)}{z}=\lim_{z\to 0}\left(\cos\left(\frac{z}{2}+a\right)+O(1)\cos\left(\frac{z}{2}+a\right)\right)=$$

$$\cos\left(\frac{0}{2}+a\right)+0=\cos(a)$$

> Alternatively,
>
> $$\lim_{z\to 0}\frac{\sin\frac{z}{2}}{\frac{z}{2}}\cos\left(\frac{z}{2}+a\right)=1\times\cos\left(\frac{0}{2}+a\right)=\cos a$$

Answer:

$$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\cos(a)$$

Subproblem F

$$\lim_{x\to 1}\frac{\ln(x^2+\cos\frac{\pi x}{2})}{\sqrt{x}-1}=\lim_{y\to 0}\frac{\ln((y+1)^2+\cos(\frac{\pi y}{2}+\frac{\pi}{2}))}{\sqrt{y+1}-1}=$$

> Since $\cos x=-\sin\left(x+\frac{\pi}{2}\right)$, $\sin x=x-\frac{x^3}{6}+O(x^3)$

$$\lim_{y\to 0}\frac{\ln((y+1)^2-\sin\frac{\pi y}{2})}{\sqrt{y+1}-1}=\lim_{y\to 0}\frac{\ln((y+1)^2-\frac{\pi y}{2}+\frac{\pi^3y^3}{8}+O(y^3))}{\sqrt{y+1}-1}=\lim_{y\to 0}\frac{\ln(1+y^2+(2-\frac{\pi}{2})y+\frac{\pi^3y^3}{8}+O(y^3))}{\sqrt{y+1}-1}$$

> Since $\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+O(x^3),\ln(1+x)=x+O(x)$

$$\lim_{y\to 0}\frac{y^2+(2-\frac{\pi}{2})y+\frac{\pi^3y^3}{8}+O(y^3)}{1+\frac{y}{2}-\frac{y^2}{8}+\frac{y^3}{16}-1+O(y^3)}=\lim_{y\to 0}\frac{y+(2-\frac{\pi}{2})+\frac{\pi^3y^2}{8}+O(y^2)}{\frac{1}{2}-\frac{y}{8}+\frac{y^2}{16}+O(y^2)}=\frac{0+(2-\frac{\pi}{2})+0+0}{\frac{1}{2}-0+0+0}=4-\pi$$

> Alternatively, since $\lim_{x\to 0}\frac{(y+1)^\alpha-1}{y}=\alpha,\lim_{x\to 0}\frac{\sin x}{x}=1$
>
> $$\lim_{y\to 0}\frac{\ln((y+1)^2-\sin\frac{\pi y}{2})}{\sqrt{y+1}-1}=\lim_{y\to 0}\frac{y}{(y+1)^{\frac{1}{2}}-1}\frac{(y+2-\frac{\pi}{2}\frac{\sin\frac{\pi y}{2}}{\frac{\pi y}{2}})\ln(1+y(y+2-\frac{\pi}{2}\frac{\sin\frac{\pi y}{2}}{\frac{\pi y}{2}}))}{(y+2-\frac{\pi}{2}\frac{\sin\frac{\pi y}{2}}{\frac{\pi y}{2}})y}=$$
>
> $$\left(\frac{1}{2}\right)^{-1}\times\lim_{y\to 0}\left(y+2-\frac{\pi}{2}\frac{\sin\frac{\pi y}{2}}{\frac{\pi y}{2}}\right)\times 1=2\lim_{y\to 0}\left(y+2-\frac{\pi}{2}\times 1\right)=2\left(0+2-\frac{\pi}{2}\right)=4-\pi$$

Answer:

$$\lim_{x\to 1}\frac{\ln(x^2+\cos\frac{\pi x}{2})}{\sqrt{x}-1}=4-\pi$$

Subproblem G

$$\lim_{x\to+0}\frac{\sqrt{1-e^{-x}}-\sqrt{1-\cos x}}{\sqrt{\sin x}}=\lim_{x\to+0}\frac{\sqrt{1-e^{-x}}}{\sqrt{\sin x}}-\lim_{x\to+0}\frac{\sqrt{1-\cos x}}{\sqrt{\sin x}}$$

> Since $e^x=1+x+O(x),\sin x=x+O(x)$

$$\lim_{x\to+0}\frac{\sqrt{1-e^{-x}}}{\sqrt{\sin x}}=\lim_{x\to+0}\sqrt{\frac{1-1+x+O(x)}{x+O(x)}}=\lim_{x\to+0}\sqrt{\frac{x+O(x)}{x+O(x)}}=1$$

> Alternatively, since $\lim_{x\to 0}\frac{e^x-1}{x}=1$
>
> $$\lim_{x\to+0}\sqrt{\frac{1-e^{-x}}{\sin x}}=\lim_{x\to+0}\sqrt{\frac{x(e^{-x}-1)}{x\sin x}}=\lim_{x\to+0}\sqrt{\frac{(e^{-x}-1)}{x}\frac{x}{\sin x}}=\sqrt{1\times 1^{-1}}=1$$

$$\lim_{x\to+0}\frac{\sqrt{1-\cos x}}{\sqrt{\sin x}}=\lim_{x\to+0}\sqrt{\frac{1-\cos x}{\sin x}}=\lim_{x\to+0}\sqrt{\frac{1-\cos x}{\sin^2 x}\sin x}=$$

> Since $\lim_{x\to+0}\frac{1-\cos x}{\sin^2 x}=1-\frac{x^2}{2}$

$$\sqrt{\lim_{x\to+0}\frac{1-\cos x}{\sin^2 x}\lim_{x\to+0}\sin x}=\sqrt{\left(1-\frac{x^2}{2}\right)\lim_{x\to+0}\sin x}=\sqrt{\left(1-\frac{x^2}{2}\right)\times 0}=0$$

Therefore,

$$\lim_{x\to+0}\frac{\sqrt{1-e^{-x}}}{\sqrt{\sin x}}-\lim_{x\to+0}\frac{\sqrt{1-\cos x}}{\sqrt{\sin x}}=1-0=1$$

Answer:

$$\lim_{x\to+0}\frac{\sqrt{1-e^{-x}}-\sqrt{1-\cos x}}{\sqrt{\sin x}}=1$$

Subproblem H

$$\lim_{x\to1}\frac{\ln(2x^2-x)}{\ln(x^4+x^2-x)}=\lim_{x\to1}\frac{\ln x+\ln(2x-1)}{\ln x+\ln(x^3+x-1)}\xrightarrow{x=y+1}\lim_{y\to0}\frac{\ln(y+1)+\ln(2y+1)}{\ln(y+1)+\ln((y+1)^3+y+1-1)}=$$

> Since $\ln(x+1)=x-\frac{x^2}{2}+\frac{x^3}{3}+O(x^3)$ and $\ln(x+1)=x+O(x)$

$$\lim_{y\to0}\frac{\ln(y+1)+\ln(2y+1)}{\ln(y+1)+\ln(y^3+3y^2+4y+1)}=\lim_{y\to0}\frac{y-\frac{y^2}{2}+\frac{y^3}{3}+O(y^3)+2y-2y^2+\frac{8y^3}{3}+O(y^3)}{y-\frac{y^2}{2}+\frac{y^3}{3}+O(y^3)+y^3+3y^2+4y+O(y^3)}=$$

$$\lim_{y\to0}\frac{3y-\frac{5y^2}{2}+3y^3+O(y^3)}{5y+\frac{5y^2}{2}+\frac{4y^3}{3}+O(y^3)}=\lim_{y\to0}\frac{3-\frac{5y}{2}+3y^2+O(y^2)}{5+\frac{5y}{2}+\frac{4y^2}{3}+O(y^2)}=\frac{3-0+0+0}{5+0+0+0}=\frac{3}{5}$$

> Alternatively, since $\lim_{x\to0}\frac{\ln(1+x)}{x}=1$
>
> $$\lim_{y\to0}\frac{\frac{y\ln(y+1)}{y}+\frac{2y\ln(2y+1)}{2y}}{\frac{y\ln(y+1)}{y}+\frac{(y^3+3y^2+4y)\ln(y^3+3y^2+4y+1)}{y^3+3y^2+4y}}=\lim_{y\to0}\frac{y+2y}{y+y^3+3y^2+4y}=\lim_{y\to0}\frac{3}{5+y^2+3y}=\frac{3}{5}$$

Answer:

$$\lim_{x\to1}\frac{\ln(2x^2-x)}{\ln(x^4+x^2-x)}=\frac{3}{5}$$