Calculus, Homework 13

Problem 1

Is it possible to study the point $(0,0)$ for extremities for function $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ using the studied methods?

In order to use our method for point $(0,0)$, we would need to calculate the Hessian matrix $\mathbb{H}_{(0,0)}$ for this point.

Hessian matrix consists of second-order derivatives and since our function $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ contains (x^2+y^2) in the denominator, all its derivatives would have a denominator that is a power of (x^2+y^2).

Since the function is undefined if the denominator is equal to 0, we can't use our method since plugging in $(x,y)=(0,0)$ yields a division by zero: $(0^2+0^2)=0 \implies$ the Hesse matrix would be undefined, rendering us unable to use this method of extremum-searching.

Problem 2

Study functions for local extremums:

Subproblem A

$$f(x,y) = x^3 + 3xy^2 - 15x - 12y$$

Find stationary points:

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15 = 0 \\ \frac{\partial f}{\partial y} = 6xy - 12 = 0 \end{cases} \implies \begin{cases} (x+y)^2 = 9 \\ xy = 2 \end{cases} \implies \begin{cases} \left[\begin{array}{l} x+y=3 \\ x+y=-3 \end{array} \right. \\ xy = 2 \end{cases}$$

We get two points, $a=(1,2), b=(-1,-2)$.

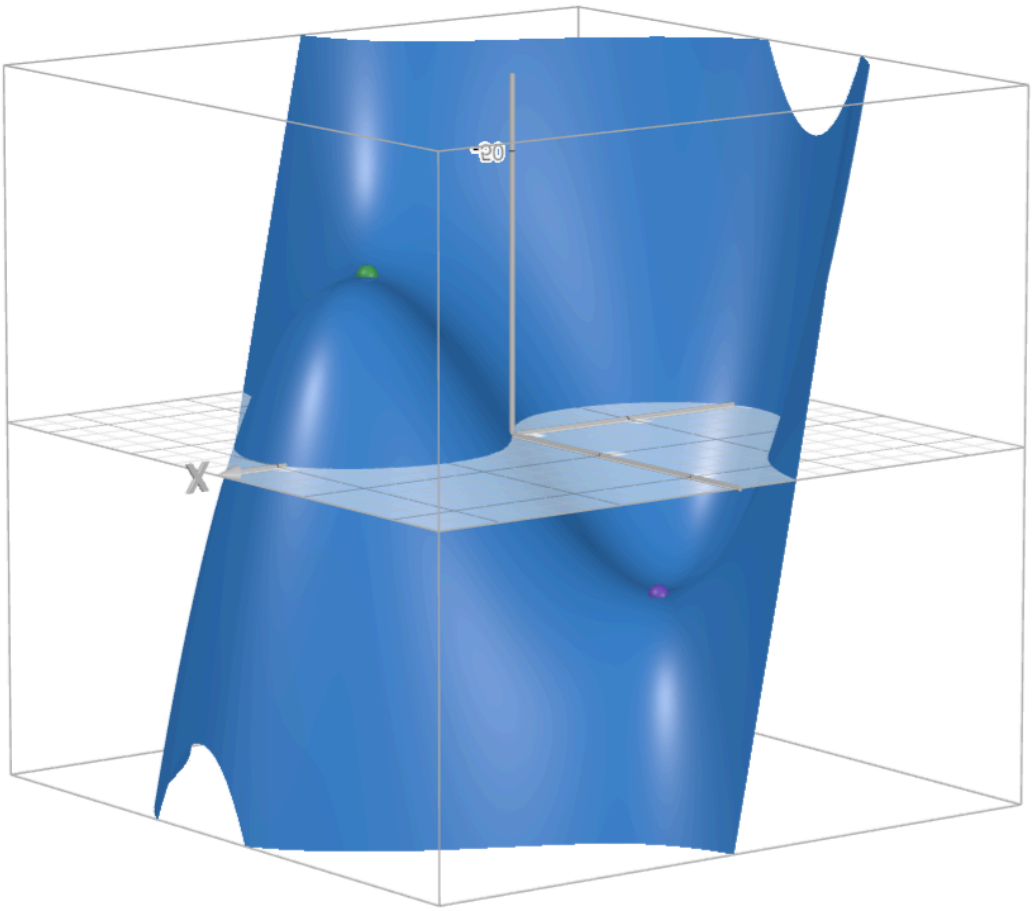
Find second-order derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 6y, \quad \frac{\partial^2 f}{\partial y^2} = 6x$$

Hessian matrices:

$$\mathbb{H}_a = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix} \sim \begin{pmatrix} 6 & 0 \\ 0 & -18 \end{pmatrix}, \quad \mathbb{H}_b = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix} \sim \begin{pmatrix} -6 & 0 \\ 0 & 18 \end{pmatrix}$$

The elements on the diagonals of both matrices have different signs, so there are no extremums, as it could also be seen below.



Subproblem B

$$f(x, y) = x^2 + xy + y^2 - 2x - y$$

Find stationary points:

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + y - 2 = 0 \\ \frac{\partial f}{\partial y} = 2y + x - 1 = 0 \end{cases} \Longrightarrow \begin{cases} x + y = 1 \\ x + 2y = 1 \end{cases} \Longrightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$$

We get one point, $a = (1, 0)$.

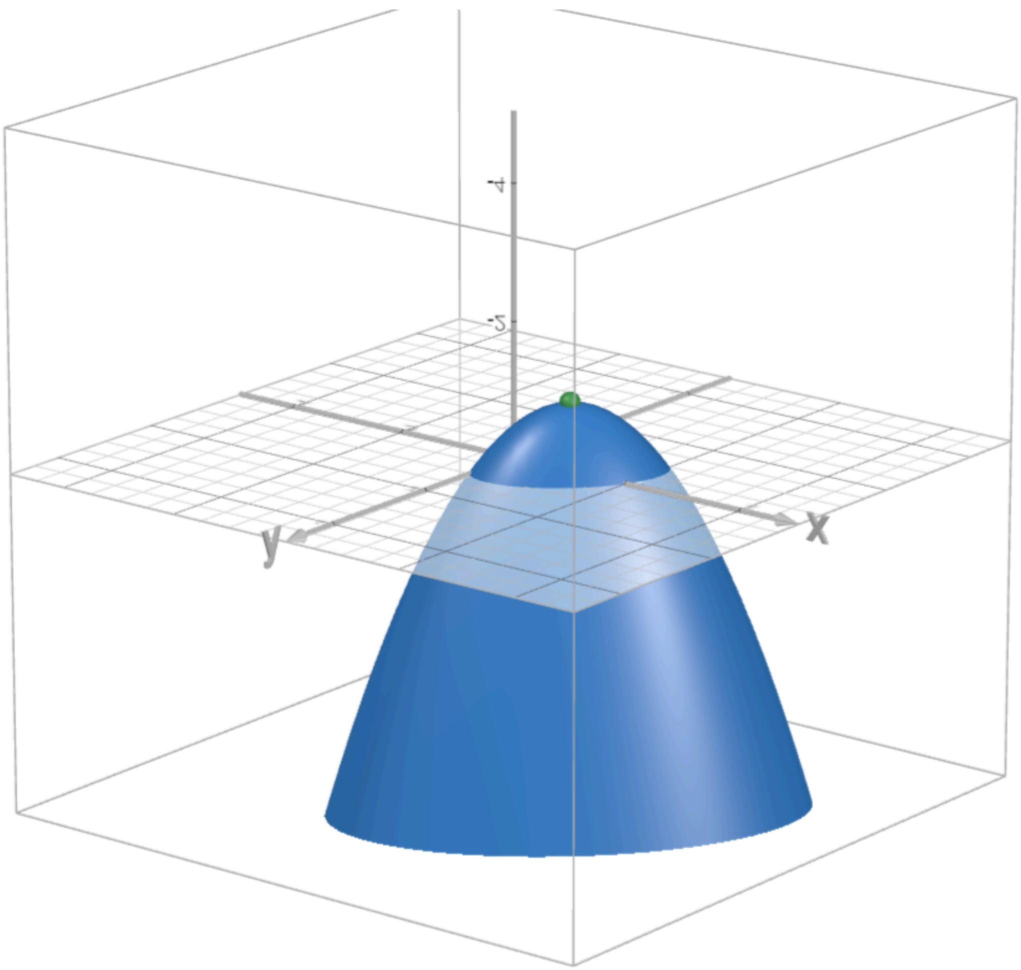
Find second-order derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 1, \quad \frac{\partial^2 f}{\partial y^2} = 2$$

Hessian matrix:

$$\mathbb{H}_a = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$$

The elements on the diagonals of the matrix are positive, so there is a single minimum at $a = (1, 0)$, as it could be seen below:



Subproblem C

$$f(x,y) = 3xy - x^2 - y^2 - 10x + 5y$$

Find stationary points:

$$\begin{cases} \frac{\partial f}{\partial x} = 3y - 2x - 10 = 0 \\ \frac{\partial f}{\partial y} = 3x - 2y + 5 = 0 \end{cases} \implies \begin{cases} x + y = 5 \\ 3y - 2x = 10 \end{cases} \implies \begin{cases} x = 1 \\ y = 4 \end{cases}$$

We get one point, $a = (1,4)$.

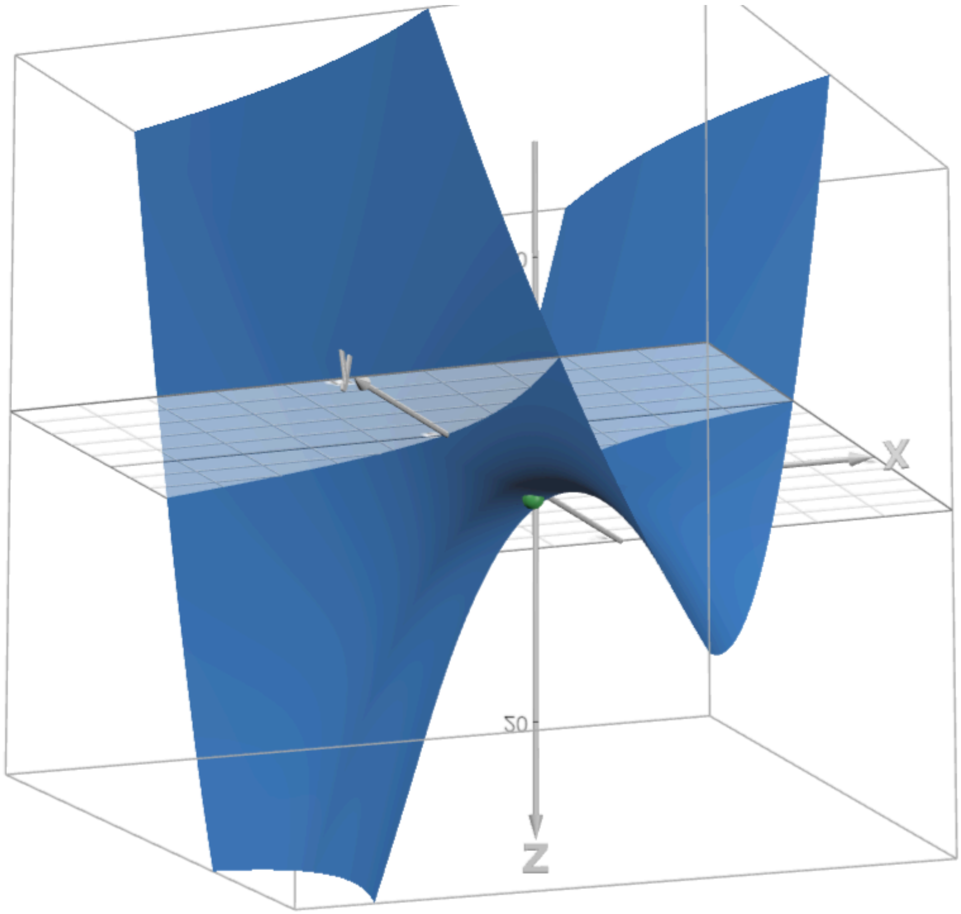
Find second-order derivatives:

$$\frac{\partial^2 f}{\partial x^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 3, \quad \frac{\partial^2 f}{\partial y^2} = -2$$

Hessian matrix:

$$\mathbb{H}_a = \begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 \\ 0 & \frac{5}{2} \end{pmatrix}$$

The elements on the diagonals of the matrix are different signs, so there are no extremums, as it could be seen below:



Problem 3

Depending on $\lambda \in \mathbb{R}$, study the point $(0, 0, 0)$ for extremums for function

$$f(x, y, z) = 5x^2 + y^2 + \lambda z^2 + 4xy - 2xz - 2yz$$

Find all first-order derivatives:

$$\frac{\partial f}{\partial x} = 10x + 4y - 2z, \quad \frac{\partial f}{\partial y} = 2y + 4x - 2z, \quad \frac{\partial f}{\partial z} = 2\lambda z - 2x - 2y$$

Find all second-order derivatives to build a Hessian matrix:

$$\frac{\partial^2 f}{\partial x^2} = 10, \quad \frac{\partial^2 f}{\partial x \partial y} = 4, \quad \frac{\partial^2 f}{\partial x \partial z} = -2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 4, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial y \partial z} = -2$$

$$\frac{\partial^2 f}{\partial z \partial x} = -2, \quad \frac{\partial^2 f}{\partial z \partial y} = -2, \quad \frac{\partial^2 f}{\partial z^2} = 2\lambda$$

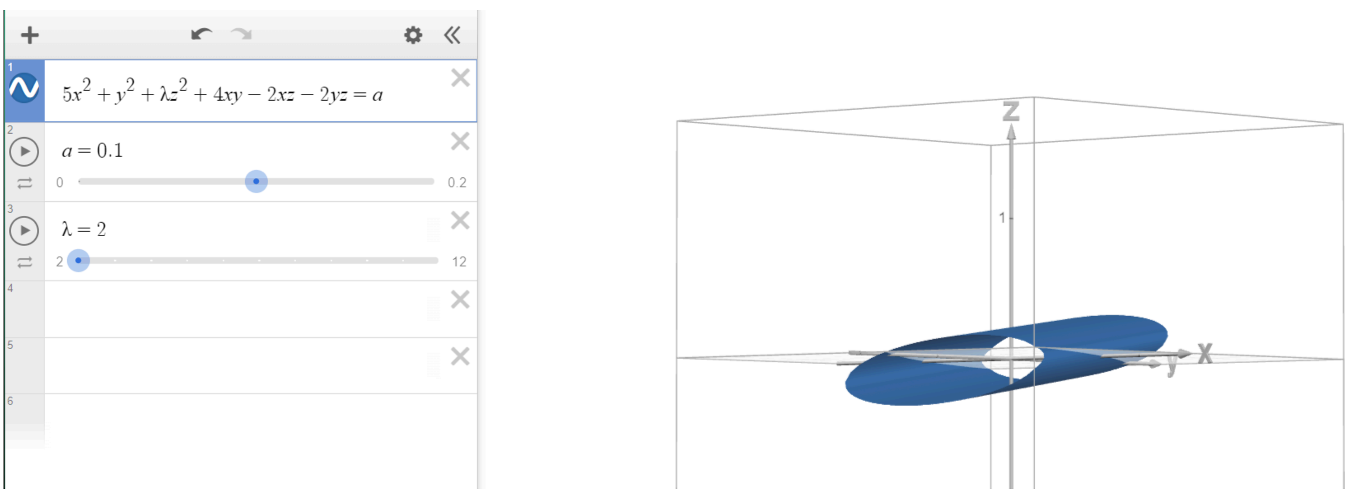
$$\mathbb{H}_{(0,0,0)} = \begin{pmatrix} 10 & 4 & -2 \\ 4 & 2 & -2 \\ -2 & -2 & 2\lambda \end{pmatrix} \sim \begin{pmatrix} 10 & 0 & -2 \\ 0 & 0.4 & -1.2 \\ -2 & -1.2 & 2\lambda \end{pmatrix}$$

$$\sim \begin{pmatrix} 10 & 0 & 0 \\ 0 & 0.4 & -1.2 \\ 0 & -1.2 & 2\lambda - 0.4 \end{pmatrix} \sim \begin{pmatrix} 10 & 0 & 0 \\ 0 & \frac{2}{5} & 0 \\ 0 & 0 & 2\lambda - 4 \end{pmatrix}$$

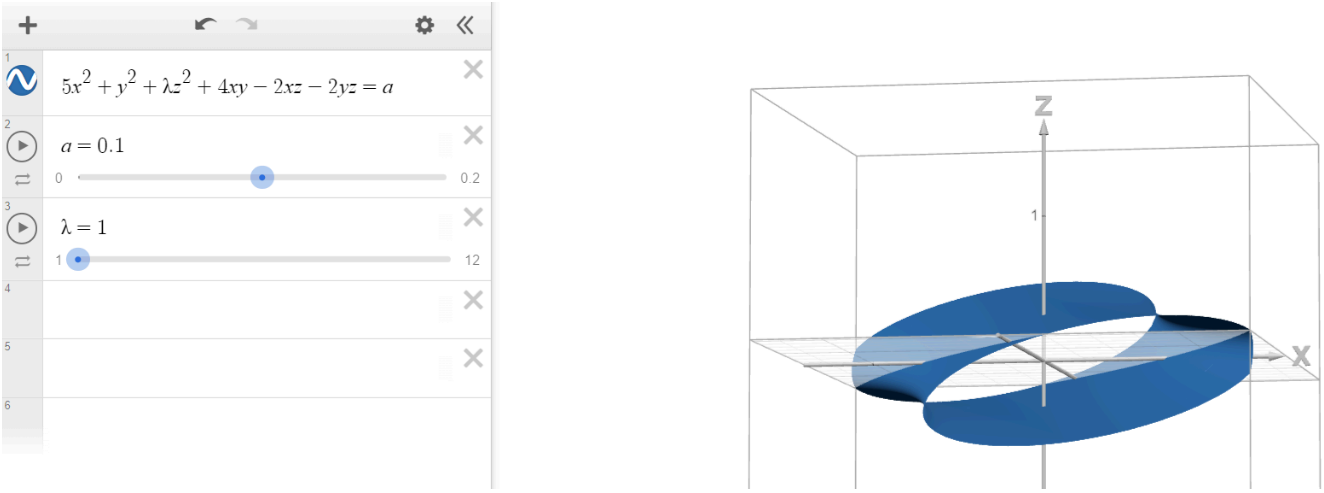
For an extremum (minimum, specifically) to exist, we need $\operatorname{sgn} 10 = \operatorname{sgn} 0.4 = \operatorname{sgn}(2\lambda - 4) \implies 2\lambda > 4 \implies \lambda > 2$

If you think this is unvisualizeable, it isn't:

If we ever so slightly deviate from the point $(0,0,0)$, which is achieved when $a = 0$ (which is the fourth dimension), at $\lambda = 2$, we get a tube, implying point $(0,0,0)$ isn't an extremum.



If we lower λ, this tube becomes more and more flared out at towards its openings:



If we raise λ above 2, this tube becomes a closed ellipsoid, implying that as $a \to 0$, our shape approaches a single point, which is the extremum in of this three-dimensional function.