Problem 5.1

Given set U that contains n elements. How many possiblities to choose two subsets A, B in U such that

Subproblem A

Sets $A \cap B = \varnothing$.

For every element in U, we need to distribute this element either to A, to B, or to neither sets. Thus, there are 3 options to distribute each element somewhere and we need to find all words of length n in an alphabet of 3 elements.

Therefore, there are $3 \times 3 \times \cdots \times 3 = 3^n$ possible variants to choose two such subsets.

Answer: 3^n

Subproblem B

Set $A \subseteq B$.

First, choose some set B. For each element in U, either choose this element to be in B or not. In total, similarly as above, find the number of words of length n in the alphabet of 2 elements. There would be 2^n such subsets.

Now, for each B, choose some subset $A \subseteq B$.

Consider subset B of size k. There are $\binom{n}{k}$ ways to choose a subset of size k out of U. Afterwards, there are 2^k ways to choose k elements out of such subset B so that $A \subseteq B$ would hold. In total, it also evaluates to

$$\binom{n}{0}2^0 + \binom{n}{1}2^1 + \cdots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n = (1+2)^n = 3^n$$

> I assume that an easier approach might have been applying the approach in the first task but instead distibuting each of the elements in U to one of three options (either no set, only set B, or both sets). Then, the solution is identical to subproblem A

Answer: 3^n

Problem 5.2

Find the number of such routes from $(0,0)$ to (n,n) that each step is either $\mathcal{S}_1 = (1,0)$ or $\mathcal{S}_2 = (0,1)$ and in each point (x,y) of the route the inequation $|x-y| \le 1$ is true.

Consider a subsequence of steps that sends us from $(k,k) \mapsto (k+1,k+1)$, maintaining the $|x-y| \le 1$ restriction.

Look closer at two options:

- $\mathcal{S}_1$ is first: $(k,k) \xrightarrow{\mathcal{S}_1} (k+1,k)$. The only possible next step is $\mathcal{S}_2$ since otherwise $|x-y| \le 1$ restriction does not hold.
- $\mathcal{S}_2$ is first: $(k,k) \xrightarrow{\mathcal{S}_2} (k,k+1)$. The only possible next step is $\mathcal{S}_2$ since otherwise $|x-y| \le 1$ restriction does not hold.

Therefore, we have only 2 possible options for $(k,k) \mapsto (k+1,k+1)$:

- $(k,k) \xrightarrow{\mathcal{S}_1} (k+1,k) \xrightarrow{\mathcal{S}_2} (k+1,k+1)$
- $(k,k) \xrightarrow{\mathcal{S}_2} (k,k+1) \xrightarrow{\mathcal{S}_1} (k+1,k+1)$

Overall, the number of possible routes would be equal to words of length $n-1$ from the alphabet $\{\mathcal{S}_1, \mathcal{S}_2\}$. In total, the answer would be 2^{n-1}.

Answer: 2^{n-1}

Problem 5.3

Define Z_n as the number of binary words of length n that do not contain two zeros in a row. Prove that $Z_n = F_{n+2}$ for all $n \ge 1$.

Write out how Z_n is calculated to figure out the pattern:

n	**values**	**end with** 0	**end with** 1	**total**
1	$0, 1$	1	1	2
2	$01, 10, 11$	1	2	3
3	$010, 011, 101, 110, 111$	2	3	5
4	$0101, 0110, 0111, 1010, 1011, 1101, 1110, 1111$	3	5	8

n	values	end with 0	end with 1	total
5	$01010, 01011, 0111, 01110, 01111, 10101, 10110,$ $10111, 11010, 11011, 11101, 11110, 11111$	5	8	13
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Awesome, so the way how the next value Z_n is defined is the following, considering that $k_{(n)0}, k_{(n)1}$ are the counts of words that end with $0, 1$ in Z_n:

- Take all $k_{(n-1)0}$ values, then increase the length of the word by 1 by adding one 1 as the last letter.
- Take all $k_{(n-1)1}$ values, then increase the length of the word by 1 by adding either one 1 or one 0 as the last letter.

Now, $Z_n = k_{(n-1)1} + (k_{(n-1)0} + k_{(n-1)1}) = k_{(n-1)1} + Z_{n-1}$. Since per definition $k_{(n-1)1} = k_{(n-2)1} + k_{(n-2)0} = Z_{n-2}$, then $Z_n = Z_{n-2} + Z_{n-1}$. We have calculated the base to be $Z_1 = F_3 = 2$ and $Z_2 = F_4 = 3$. Therefore, since induction base and induction step have been calculated and checked and the definition of the algorithm is deterministic, $Z_n = F_n + F_{n+1} = F_{n+2}, n \geq 1$, q. e. d.

Problem 5.4

Give the combinatorial proof of the equation

$$\sum_{j=0}^{k} \binom{r}{j} \binom{s}{k-j} = \binom{r+s}{k}$$

Suppose there are two sets that contain r and s values each. In how many ways can we create a subset that contains k elements? This is precisely the definition of a combination, therefore the answer to this question would be $\binom{r+s}{k}$.

Alternatively, we could sum the number of subsets of length k. The number of valid options would a subset of j elements from the first set and $k - j$ elements from the second set. Take combinations of $\binom{r}{j}$ and $\binom{s}{k-j}$ respectively and multiply them together as they are dependent. The total would indeed be $\sum_{j=0}^{k} \binom{r}{j} \binom{s}{k-j}$.

Therefore, by double counting proof, we get $\sum_{j=0}^{k} \binom{r}{j} \binom{s}{k-j} = \binom{r+s}{k}$, q. e. d.