# Problem 1

Find the number of surjective non-descending functions from $[10]$ to $[7]$. Function $f$ is non-descending if $x \leq y \Rightarrow f(x) \leq f(y)$.

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This function is a surjection if $\forall y \in [7], \exists f(x) = y$. Since the function has to be non-descending, then to maintain the surjective property, we absolutely have to assign each value $y \in \{0,1,2,3,4,5,6\}$ some $x \in \{0,1,2,3,4,5,6,7,8,9\}$ and then have relatively free will to choose what $y$ to assign the remaining $10-7=3$ values, maintaining the non-descending property.

Example: Separating $10$ values collapses to separating $3$ values since it's necessary to have at least one value in each group.

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$$\downarrow$$

$$\mid * * \mid * \mid \mid \mid \mid$$

3 values need to be separated into 7 groups.

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \left(\!\!\binom{7}{3}\!\!\right) = \binom{7+3-1}{3} = \binom{9}{3} = \frac{9!}{3! \cdot 6!} = \frac{7 \cdot 8 \cdot 9}{2 \cdot 3} = 3 \cdot 4 \cdot 7 = 84$$

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**Answer:** $84$

# Problem 2

Prove that

$$\sum_{j=0}^{k} \binom{n+j-1}{j} = \binom{n+k}{k}$$

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$$\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-1}{k} = \binom{n+k}{k}$$

$$\binom{n-1}{0} = 1 = \binom{n}{0} \Rightarrow$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n+1}{2} + \dots + \binom{n+k-1}{k} = \binom{n+k}{k}$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \Rightarrow$$

$$\overbrace{\binom{n}{0} + \binom{n}{1}}^{\binom{n+1}{1}} + \binom{n+1}{2} + \dots + \binom{n+k-1}{k} = \binom{n+k}{k} \Rightarrow$$

$$\overbrace{\binom{n+1}{1} + \binom{n+1}{2}}^{\binom{n+2}{2}} + \dots + \binom{n+k-1}{k} = \binom{n+k}{k} \Rightarrow$$

$$\vdots$$

$$\overbrace{\binom{n+k-1}{k-1}\binom{n+k-1}{k}}^{\binom{n+k}{k}} = \binom{n+k}{k} \Rightarrow$$

$$\binom{n+k}{k} = \binom{n+k}{k}$$

q. e. d.

# Problem 3

How many permutations of ABRACADABRA such that no two A-s are next to each other are there?

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First, count the number of permutations of BBCDRR:

$$\binom{6}{2,1,1,2} = \frac{6!}{2! \cdot 2!} = 2 \cdot 3 \cdot 5 \cdot 6 = 180$$

Afterwards, for X ∈ {B, C, D, R}, there would be $7$ places for $A$: AXAXAXAXAXAXA. Choose 5 of them:

$$\binom{7}{5} = \frac{7!}{2! \cdot 5!} = \frac{7 \cdot 6}{2} = 21$$

Now, the final answer would be:

$$\binom{6}{2,1,1,2}\binom{7}{5} = 180 \cdot 21 = 3780$$

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**Answer:** $3780$

# Problem 4

Compare numbers:

$$\sum_{i=0}^{512} 2^{2i}\binom{1024}{2i} \text{ and } \sum_{i=0}^{511} 2^{2i+1}\binom{1024}{2i+1}$$

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Notice that this is literally a binomial expansion:

$$\sum_{i=0}^{512} 2^{2i}\binom{1024}{2i} - \sum_{i=0}^{511} 2^{2i+1}\binom{1024}{2i+1} =$$

$$= \binom{1024}{1024}2^{1024}(-1)^0 + \binom{1024}{1023}2^{1023}(-1)^1 + \cdots + \binom{1024}{0}2^0(-1)^{1024} = (2-1)^{1024} = 1^{1024} = 1$$

Therefore, the first number is larger.

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**Answer:**

$$\sum_{i=0}^{512} 2^{2i}\binom{1024}{2i}$$