Calculus Homework #6

Problem 8.8

> In general, to prove that the two-variable limit does not exist, it is enough to find two different limits for two different paths.

Subproblem A

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y}{x^3 + y}$$

For $y = mx^3$:

$$\lim_{x \to 0} \frac{x^3 - m^3x^3}{x^3 + m^3x^3} = \lim_{x \to 0} \frac{1 - m^3}{1 + m^3} = \frac{1 - m^3}{1 + m^3}$$

The limit is dependent on $m \Rightarrow$ the limit does not exist.

Subproblem B

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{x^2 + y^2}$$

For $y = mx$:

$$\lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{mx^2}{(1 + m^2)x^2} = \lim_{x \to 0} \frac{m}{1 + m^2} = \frac{m}{1 + m^2}$$

The limit is dependent on $m \Rightarrow$ the limit does not exist.

Subproblem C

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2 - x^2}{y^2 + x^2}$$

For $y = mx$:

$$\lim_{x \to 0} \frac{m^2x^2 - x^2}{m^2x^2 + x^2} = \lim_{x \to 0} \frac{m^2 - 1}{m^2 + 1} = \frac{m^2 - 1}{m^2 + 1}$$

The limit is dependent on $m \Rightarrow$ the limit does not exist.

Subproblem D

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2y^2}{x^2y^2 + (x - y)^2}$$

For $y = x$:

$$\lim_{x \to 0} \frac{x^2x^2}{x^2x^2 + (x - x)^2} = \lim_{x \to 0} \frac{x^4}{x^4} = \lim_{x \to 0} 1 = 1$$

For $y = 0$:

$$\lim_{x \to 0} \frac{x^2 0^2}{x^2 0^2 + (x - 0)^2} = \lim_{x \to 0} \frac{0}{x^2} = \lim_{x \to 0} 0 = 0$$

This should be enough, but in case you can't go along $y = 0$, a different option for $y = kx, k \neq 1$:

$$\lim_{x \to 0} \frac{k^2x^4}{k^2x^4 + (1 - k)^2x^2} = \lim_{x \to 0} \frac{x^2}{x^2 + \frac{(1-k)^2}{k^2}} = \lim_{x \to 0} \frac{0}{0 + \frac{(1-k)^2}{k^2}} = 0$$

There are at least two different limits for two paths $y = x, y = 0 \Rightarrow$ the limit does not exist.

Subproblem E

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(x + y \sin \frac{1}{x} \right)$$

$f_1(x) = \sin \frac{1}{x} \in [-1, 1]$ is a bounded function. $f_2(y) = y$ as $y \to 0$ is an infinitesimal function. Product of a bounded function and an infinitesimal function is infinitesimal. Therefore, $\lim_{\substack{x \to 0 \\ y \to 0}} g(x, y) = \lim_{\substack{x \to 0 \\ y \to 0}} f_1(x) f_2(y) = 0$

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(x+y\sin\frac{1}{x}\right)=\lim_{\substack{x\to 0\\ y\to 0}}x+\lim_{\substack{x\to 0\\ y\to 0}}y\sin\frac{1}{x}=\lim_{x\to 0}x+\lim_{\substack{x\to 0\\ y\to 0}}g(x,y)=0+0=0$$

Answer:

$$\lim_{\substack{x\to 0\\ y\to 0}}\left(x+y\sin\frac{1}{x}\right)=0$$

Problem 8.9

Find limit of $f(x,y)=\frac{y-2x^2}{y-x^2}$ in point $(0,0)$ along the path $x=\alpha t, y=\beta t, \alpha^2+\beta^2\neq 0$. Prove that $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{y-2x^2}{y-x^2}\Rightarrow\lim_{t\to 0}\frac{\beta t-2\alpha^2t^2}{\beta t-\alpha^2t^2}=\lim_{t\to 0}\frac{\beta-2\alpha^2 t}{\beta-\alpha^2 t}=\lim_{t\to 0}\frac{\beta-0}{\beta-0}=\lim_{t\to 0}1=1$$

For $y=mx^2$:

$$\lim_{x\to 0}\frac{mx^2-2x^2}{mx^2-x^2}=\lim_{x\to 0}\frac{m-2}{m}=\frac{m-2}{m}$$

Thus, the multivariable limit does not exist.

Answer: $\lim_{\substack{x\to 0\\ y\to 0}}\frac{y-2x^2}{y-x^2}=1$ along the path of $x=\alpha t, y=\beta t, \alpha^2+\beta^2\neq 0$

Problem 8.10

Is the function

$$u(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, \forall x,y\colon x^2+y^2\neq 0\\ 0, \quad \forall x,y\colon x^2+y^2=0\end{cases}$$

continuous in point $(0,0)$?

$$u'(x,y)=\frac{xy}{x^2+y^2}$$

Per the continuity criterion: for $u(x,y)$ to be continuous in point of closure $(0,0)$ it is required and sufficient that $u(0,0) = \lim_{\substack{x \to 0 \\ y \to 0}} u'(x,y)$. This limit of $u'(x,y)$, as proven in **Problem 8.8, Subproblem B** does not exist. Therefore, the function $u(x,y)$ is not continuous.

Answer: $u(x,y)$ is not continuous