Linear Algebra, Homework 20

> Find the symmetric bilinear functions associated with quadratic functions:

Problem 1

We know that there is a bijection between all symmetric bilinear forms and matrices of quadratic forms, thus it's really easy to correlate them.

Algorithm

Write out the matrix of a quadratic form, we get:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

These coefficients correspond to coefficients of products $x_i y_j$, where i is the row index and j is the column index, we get:

$$\begin{aligned} \beta(x, y) = & a_{11}x_1y_1 + a_{12}x_1y_2 + a_{13}x_1y_3 + \\ & a_{21}x_2y_1 + a_{22}x_2y_2 + a_{23}x_2y_3 + \\ & a_{31}x_3y_1 + a_{32}x_3y_2 + a_{33}x_3y_3 \end{aligned}$$

Thus, I'll just use this below without much explanation:

Subproblem A

$$Q(x) = x_1^2 + 2x_1x_2 + 2x_2^2 - 6x_1x_3 + 4x_2x_3 - x_3^2$$

$$\begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 2 \\ -3 & 2 & -1 \end{pmatrix}$$

$$\begin{aligned} \beta(x, y) = & x_1y_1 + x_1y_2 - 3x_1y_3 + \\ & x_2y_1 + 2x_2y_2 + 2x_2y_3 + \\ & -3x_3y_1 + 2x_3y_2 - x_3y_3 \end{aligned}$$

Subproblem B

$$x_1x_2 + x_1x_3 + x_2x_3$$

$$\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\beta(x,y) = \frac{1}{2}(x_1y_2 + x_1y_3 + x_2y_1 + x_2y_3 + x_3y_1 + x_3y_2/)$$

Problem 2

Find the symmetric bilinear functions associated with quadratic functions, where $Q(x) = f(x,x)$:

Subproblem A

$$f(x,y) = 2x_1y_1 - 3x_1y_2 - 4x_1y_3 + x_2y_1 - 5x_2y_3 + x_3y_3$$

$$\begin{aligned} Q(x) &= 2x_1^2 - 3x_1x_2 + x_1x_2 - 4x_1x_3 - 5x_2x_3 + x_3^2 = \\ &= 2x_1^2 - 2x_1x_2 - 4x_1x_3 - 5x_2x_3 + x_3^2 \end{aligned}$$

$$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 0 & -\frac{5}{2} \\ -2 & -\frac{5}{2} & 1 \end{pmatrix}$$

$$\begin{aligned} \beta(x,y) = 2x_1y_1 - x_1y_2 - 2x_1y_3 + \\ - x_2y_1 - \frac{5}{2}x_2y_3 + \\ - 2x_3y_1 - \frac{5}{2}x_3y_2 + x_3y_3 \end{aligned}$$

Subproblem B

$$f(x,y) = -x_1y_2 + x_2y_1 - 2x_2y_2 + 3x_2y_3 - x_3y_1 + 2x_3y_3$$

$$\begin{aligned} Q(x) &= -x_1x_2 + x_1x_2 - 2x_2^2 + 3x_2x_3 - x_1x_3 + 2x_3^2 = \\ &= -2x_2^2 + 3x_2x_3 - x_1x_3 + 2x_3^2 \end{aligned}$$

$$\begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & -2 & \frac{3}{2} \\ -\frac{1}{2} & \frac{3}{2} & 2 \end{pmatrix}$$

$$\beta(x,y) = -\frac{1}{2}x_1y_3 - 2x_2y_2 + \frac{3}{2}x_2y_3 - \frac{1}{2}x_3y_1 + \frac{3}{2}x_3y_2 + 2x_3y_3$$

> Normalize the quadratic forms.

Problem 3

$$x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$$

Corresponding matrix of the quadratic form:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Utilizing the Jacobi method, calculate the minors:

$$\delta_1 = 1, \\ \delta_2 = -2 - 1 = -3, \\ \delta_3 = -2 + 2 + 2 + 8 - 1 - 1 = 8$$

Then, the canonical form follows

$$\delta_1 x_1^2 + \frac{\delta_2}{\delta_1}x_2^2 + \frac{\delta_3}{\delta_2}x_3^2 =$$

$$x_1^2 - 3x_2^2 - \frac{8}{3}x_3^3$$

Alternatively, symmetric Gauss works here

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & -3 & -1 \\ 2 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}$$

Problem 4

$$x_1^2 - 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 6x_2x_3$$

Corresponding matrix of the quadratic form:

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -3 \\ 1 & -3 & -3 \end{pmatrix}$$

Utilizing the Jacobi method, calculate the minors:

$$\begin{gathered} \delta_1 = 1, \\ \delta_2 = 0 - 1 = -1, \\ \delta_3 = 0 + 3 + 3 + 0 + 3 - 9 = 0 \end{gathered}$$

Then, the canonical form follows

$$\delta_1 x_1^2 + \frac{\delta_2}{\delta_1} x_2^2 + \frac{\delta_3}{\delta_2} x_3^2 = x_1^2 - x_2^2$$

Alternatively, symmetric Gauss works here

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & -3 \\ 1 & -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$