Calculus, Homework 10

Problem 1

> Are the following functions differentiable at $(0, 0)$?

Subproblem A

$$f(x,y) = \begin{cases} \frac{2xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

First, let's check whether the function is continuous. For this, limits from all sides (for all paths) have to exist.

Assume $x = p\cos\phi, y = p\sin\phi$, then

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2} = \lim_{p\to 0} \frac{2p^3\cos\phi\sin^2\phi}{p^2(\cos^2\phi + \sin^2\phi)} = \lim_{p\to 0} 2p\cos\phi\sin^2\phi = 0$$

Thus, the function is continuous.

Now, check whether it's differentiable. Per the definition:

$$\lim_{(t,k)\to(0,0)} \frac{f(t,k) - f(0,0) + t - k}{\sqrt{t^2+k^2}} = \lim_{(t,k)\to(0,0)} \frac{\frac{2tk^2}{t^2+k^2} + t - k}{\sqrt{t^2+k^2}} =$$
$$\lim_{(t,k)\to(0,0)} \frac{\frac{2tk^2+t^3+t^2k-tk^2-k^3}{t^2+k^2}}{\sqrt{t^2+k^2}} = \lim_{(t,k)\to(0,0)} \frac{t^3+t^2k+tk^2-k^3}{(t^2+k^2)^{\frac{3}{2}}}$$

Collapse to polar coordinates once again:

$$\lim_{p\to 0} \frac{p^3\cos^3\alpha + p^3\cos^2\alpha\sin^3\alpha + p^3\cos\alpha\sin^2\alpha - p^3\sin\alpha}{(p^2\cos^2\alpha + p^2\sin^2\alpha)^{\frac{3}{2}}} = \lim_{p\to 0} \frac{p^3\mathrm{A}}{p^3} = \mathrm{A},$$

where $\mathrm{A} = \cos^3\alpha + \cos^2\alpha\sin\alpha + \cos\alpha\sin^2\alpha - \sin^3\alpha$. A depends on α, therefore there is no defined limit since it depends on the path one takes. Thus, the function is not differentiable at $(0, 0)$.

Subproblem B

$$f(x,y)=\begin{cases}\dfrac{x|y|}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, \quad (x,y)=(0,0)\end{cases}$$

> The existence of $|x|$ already implies the function wouldn't be differentiable, but let's check it.

Firstly, make sure that the function is continuous:

Assume $x=p\cos\phi, y=p\sin\phi$, then

$$\lim_{(x,y)\to(0,0)}\frac{x|y|}{\sqrt{x^2+y^2}}=\lim_{p\to0}\frac{p|p|\cos\phi|\sin\phi|}{\sqrt{p^2(\cos^2\phi+\sin^2\phi)}}=\lim_{p\to0}\frac{p|p|\cos\phi|\sin\phi|}{|p|}=$$

$$=\lim_{p\to0}p\cos\phi|\sin\phi|=0$$

Now, per the derivative definition, check whether the function is differentiable ($h=\sqrt{t^2+k^2}$):

$$\lim_{(t,k)\to(0,0)}\frac{f(t,k)-f(0,0)+t-k}{\sqrt{t^2+k^2}}=\lim_{(t,k)\to(0,0)}\frac{\frac{t|k|}{\sqrt{t^2+k^2}}+t-k}{\sqrt{t^2+k^2}}=$$

$$=\lim_{(t,k)\to(0,0)}\frac{t|k|+th-kh}{h^2}$$

Transitioning to polar coordinates:

$$\lim_{p\to0}\frac{p\cos\alpha|p\sin\alpha|+p\cos\alpha|p|-p\sin\alpha|p|}{p^2}=\lim_{p\to0}\frac{|p|}{p}(\cos\alpha\sin\alpha+\cos\alpha-\sin\alpha)=$$

$$=\lim_{p\to0}\operatorname{sgn}(p)\mathrm{A}(\alpha)$$

The limit of function $\operatorname{sgn}(p)$ takes different values depending on what side you approach it from. More specifically, $\lim_{p\to0^+}\operatorname{sgn}(p)=1$ and $\lim_{p\to0^-}\operatorname{sgn}(p)=-1$

Subproblem C

$$f(x,y)=\frac{1}{1+x-y}$$

Once again, determine whether the function is continuous: the function is continuous because the limit is easily calculated and constant for all paths: $\lim_{(x,y)\to(0,0)} \frac{1}{1+x-y} = \frac{1}{1+0-0} = 1$.

Now, is the function continuous at $(0,0)$? We need to check whether all partial derivatives exist.

$$\frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h-0} - 1}{h} = \lim_{h\to 0} \frac{1-1-h}{h(1+h)} = -1$$

$$\frac{\partial}{\partial y} f(x,y) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+0-h} - 1}{h} = \lim_{h\to 0} \frac{1-1+h}{h(1+h)} = 1$$

Partial derivatives can be easily calculated $\implies$ they exist and the function is differentiable at $(0,0)$.

Problem 2

Let functions u, v, w be differentiable everywhere, given that $u\colon \mathbb{R}^n \to \mathbb{R}, v\colon \mathbb{R}^m \to \mathbb{R}$ and $\mathbb{R}^k \to \mathbb{R}$ if

Subproblem A

$$f = uvw^2$$

$$(df)_{\mathbf{p}} = \begin{pmatrix} f'_u & f'_v & f'_w \end{pmatrix} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} = \begin{pmatrix} f'_u du & f'_v dv & f'_w dw \end{pmatrix} =$$

$$= \begin{pmatrix} vw^2 du & uw^2 dv & 2uvw dw \end{pmatrix}$$

Subproblem B

$$f = \ln(\sqrt{u^2+v^2})$$

First, let's find f'_u (the derivatives would be symmetric to each other, so we may only find one).

$$f'_u = \frac{(\sqrt{u^2+v^2})'_u}{\sqrt{u^2+v^2}} = \frac{(u^2+v^2)'_u}{2(u^2+v^2)} = \frac{2u}{2(u^2+v^2)} = \frac{u}{u^2+v^2}$$

Similarly,

$$f'_v = \frac{v}{u^2 + v^2}$$

$$(df)_{\mathbf{p}} = \begin{pmatrix} f'_u & f'_v \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} f'_u du & f'_v dv \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{udu}{u^2 + v^2} & \frac{vdv}{u^2 + v^2} \end{pmatrix} = \frac{1}{u^2 + v^2} \begin{pmatrix} udu & vdu \end{pmatrix}$$

Problem 3

> Using l'Hopital's rule, find limits:

Subproblem A

$$\lim_{x \to 0} \frac{x(e^x + 1) - 2(e^x - 1)}{x^3} \xRightarrow{\left[\frac{0}{0}\right]}$$

$$\lim_{x \to 0} \frac{x'(e^x + 1) + x(e^x + 1)' - 2(e^x - 1)'}{(x^3)'} =$$

$$\lim_{x \to 0} \frac{e^x + 1 + xe^x - 2e^x}{3x^2} = \lim_{x \to 0} \frac{(x - 1)e^x + 1}{3x^2} \xRightarrow{\left[\frac{0}{0}\right]}$$

$$\lim_{x \to 0} \frac{(x - 1)'e^x + (x - 1)(e^x)'}{(3x^2)'} =$$

$$\lim_{x \to 0} \frac{e^x + (x - 1)e^x}{6x} = \lim_{x \to 0} \frac{xe^x}{6x} = \frac{e^0}{6} = \frac{1}{6}$$

Subproblem B

$$\lim_{x \to 0^+} x^a \ln(x), \quad a > 0$$

$$\lim_{x \to 0^+} \frac{\ln(x)}{x^{-a}} = \lim_{x \to 0^+} \frac{1}{x} \frac{1}{-ax^{-a-1}} = \lim_{x \to 0^+} \frac{1}{-ax^{-a}} = \lim_{x \to 0^+} \frac{x^a}{-a} = \frac{0}{-a} = 0$$

Subproblem C

$$\lim_{x \to 1} \left(\frac{x}{x - 1} - \frac{1}{\ln(x)} \right) \xRightarrow{x=y+1}$$

$$\lim_{y \to 0} \left(\frac{y + 1}{y} - \frac{1}{\ln(y + 1)} \right) =$$

$$\lim_{y \to 0} \left(\frac{(y + 1)\ln(y + 1) - y}{y \ln(y + 1)} \right) \xRightarrow{\left[\frac{0}{0}\right]}$$

$$\lim_{y\to 0}\frac{(y+1)'\ln(y+1)+(y+1)\ln(y+1)'-y'}{y'\ln(y+1)+y\ln(y+1)'}=$$

$$\lim_{y\to 0}\frac{\ln(y+1)+1-1}{\ln(y+1)+\frac{y}{y+1}}\xRightarrow{\left[\frac{0}{0}\right]}$$

$$\lim_{y\to 0}\frac{\ln(y+1)'}{\ln(y+1)'+\frac{y'(y+1)-y(y+1)'}{(y+1)^2}}=$$

$$\lim_{y\to 0}\frac{\frac{1}{y+1}}{\frac{1}{y+1}+\frac{y+1-y}{(y+1)^2}}=\lim_{y\to 0}\frac{\frac{1}{y+1}}{\frac{1}{y+1}+\frac{1}{(y+1)^2}}=\lim_{y\to 0}\frac{\frac{1}{y+1}}{\frac{y+1+1}{(y+1)^2}}=\lim_{y\to 0}\frac{y+1}{y+2}=\frac{1}{2}$$

Problem 4

Tailorize the following functions up to degree n in point P:

Subproblem A

$$f(x)=e^{\sin(x)}$$

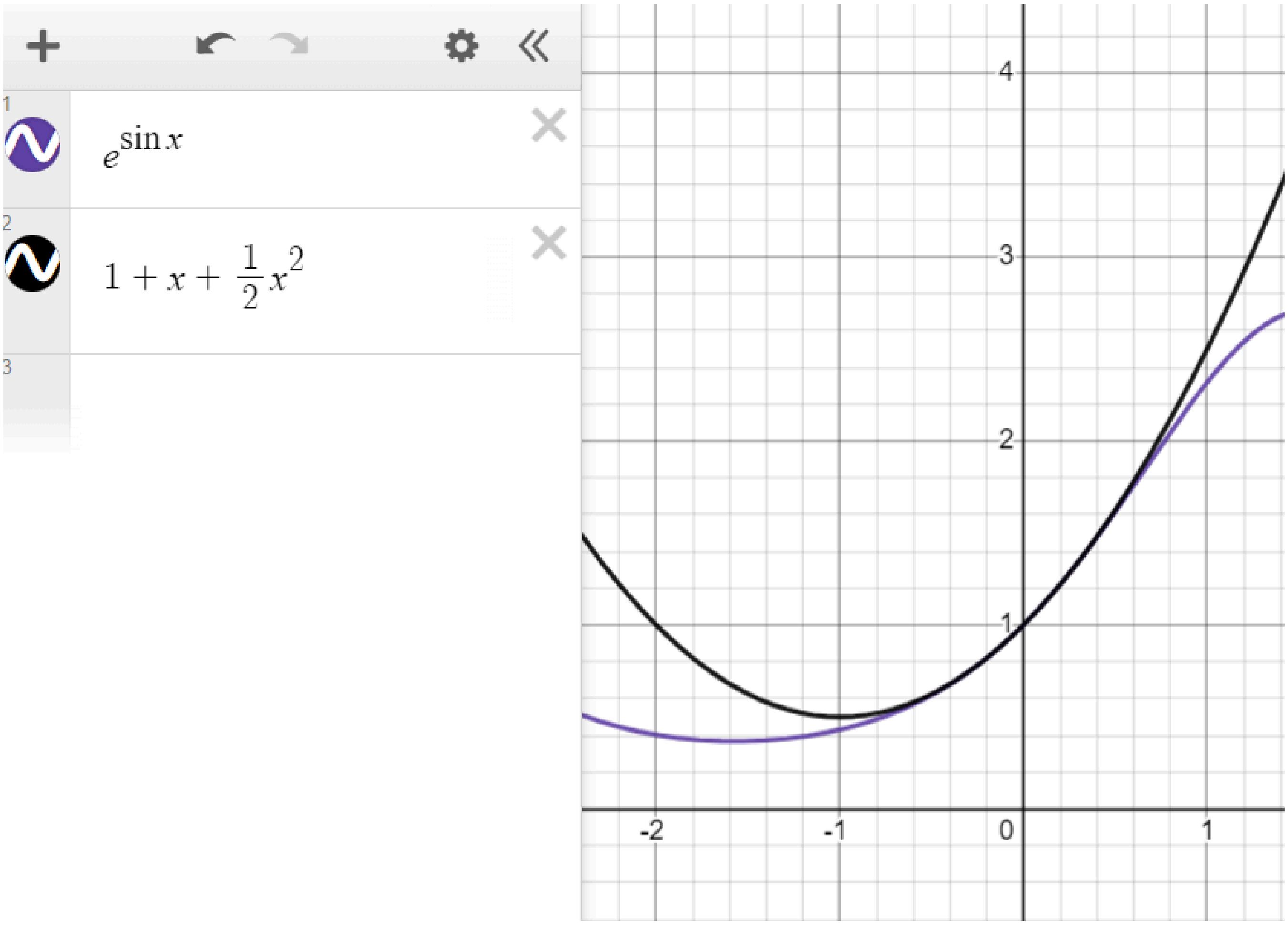
$$T_0^3=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}+O(x^4)$$

$$T_0^3=1+\cos(0)e^{\sin(0)}x+\frac{-\sin(0)e^{\sin(0)}+\cos^2(0)e^{\sin(0)}}{2}x^2+$$

$$+\frac{\overbrace{\overbrace{-3e^{\sin(x)}\sin(x)\cos(x)}^{0}+\overbrace{e^{\sin(x)}\cos^3(x)}^{1}\overbrace{-e^{\sin(x)}\cos(x)}^{-1}}^{0}}{6}+O(x^4)$$

$$T_0^3=1+x+\frac{1}{2}x^2+O(x^4)$$

Visual proof:



Subproblem B

$$f(x, y) = e^{xy}$$

$$T^2_{(0,0)} = f(0,0) + f'_x(0,0)x + f'_y(0,0)y +$$

$$+\frac{1}{2!}(f''_{xx}(0,0)x^2 + 2f'_{xy}(0,0)'xy + f''_{yy}(0,0)y^2)$$

Find the Jacobian matrix (all partial derivatives of first order):

$$\begin{pmatrix} f'_x & f'_y \end{pmatrix} = \begin{pmatrix} e^{xy}y & e^{xy}x \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Find the Hessian matrix (all partial derivatives of second order):

$$\begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} y^2e^{xy} & xye^{xy} + e^{xy} \\ xye^{xy} + e^{xy} & x^2e^{xy} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T^2_{(0,0)} = 1 + 0x + 0y + \frac{1}{2}(0x^2 + 2xy + 0y^2) = 1 + xy + O(x^2)$$

Visual proof: