Discrete Maths, Homework 21

Problem 1

Consider the following game. A fair 6-sided die is thrown thrice. If 6 never got rolled, then the player loses 1 ruble. If 6 got rolled once, the player wins 1 ruble, if twice, wins 2 rubles, and if thrice, wins 3 rubles. Is this game winning to the player on average?

The chance that the player never rolls a 6 is

$$\frac{5}{6}^3 = \frac{125}{216}$$

The chance that the player rolls a 6 once is

$$\binom{3}{1}\frac{1}{6} \times \frac{5}{6}^2 = \frac{75}{216}$$

The chance that the player rolls a 6 twice is

$$\binom{3}{2}\frac{1}{6}^2 \times \frac{5}{6} = \frac{15}{216}$$

The chance that the player rolls a 6 twice is

$$\frac{1}{6}^3 = \frac{1}{216}$$

Thus, the mathematical expectation of this game is

$$E[\text{game}] = -1 \times \frac{125}{216} + 1 \times \frac{75}{216} + 2 \times \frac{15}{216} + 3 \times \frac{1}{216} = -\frac{17}{216}$$

which means that this game is losing on average.

Problem 2

The probability space is the set of all permutations of $(x_1, \ldots, x_n)$ of elements from 1 to n. All outcomes are equally probable. Find the mathematical expectation of the number of numbers that did not change their position.

Let's find the chance that a number remained on place. This also would give us the proportion of all sequences that have the number with a certain index that remained on place.

This chance is obviously $\frac{1}{n}$ since that's the chance that we have placed the number in its spot.

Thus, the mathematical expectation that a number remained on its spot independent of other values is

$$E[i|x_i = i] = \frac{1}{n}$$

Summarize this all for numbers (there are n of such) to get the required mathematical expectation

$$E[\forall i|x_i = i] = \frac{1}{n} \times n = 1$$

Problem 3

Let X be a random value. It is known that $E[2^X] = 5$. Prove that

$$P[X \geq 6] < \frac{1}{10}$$

Markov's inequality:

$$P[f \geq \alpha] \leq \frac{E[f]}{\alpha}$$

Thus we know that

$$P[X \geq 6] \leq \frac{E[X]}{6}$$

since we are given the value $E[2^X] = 5$, we may transition to this:

$$P[2^X \geq 2^6] \leq \frac{E[2^X]}{2^6}$$

$$P[2^X \geq 64] \leq \frac{5}{2^6} = \frac{5}{64}$$

and since $\frac{5}{64} < \frac{5}{50} < \frac{1}{10}$, we have

$$P[2^X \geq 64] < \frac{1}{10} \implies P[X \geq 6] < \frac{1}{10}$$

Problem 4

Each of the numbers $a_1, \ldots, a_n$ is selected randomly, equally probably, and independently out of numbers $1, 2, \ldots, n$

Subproblem A

Find the mathematical expectation of the number of different numbers among multiset $A = \{a_1, \ldots, a_n\}$.

Let's calculate the number of unique elements. Define a characteristic function as

$$\chi_i = \begin{cases} 1, a_i \notin A[1..i] \setminus a_i, \\ 0, a_i \in A[1..i] \setminus a_i, \end{cases}$$

Thus, the mathematical expectation would be

$$E[\text{unique elements}] = \sum_{i=1}^{n} E[\chi_i]$$

Calculate the number of permutations of numbers $1..n$ of length $i-1$ that would not contain a certain value $k \in [1..i]$. This would yield us $(n-1)^{i-1}$ possible permutations and allow us to get the number of all possible sequences for a possibility to make that some taken k a unique number.

Thus, comparing this to the number of all sequences up to i, we have n^i total possible sequences. Thus, the chance that a certain single number out of a continual sequence is unique would be

$$\frac{(n-1)^{i-1}}{n^i}$$

And if we desire for any number to be unique, we simply multiply this by the count of all numbers:

$$\frac{n(n-1)^{i-1}}{n^i} = \frac{(n-1)^{i-1}}{n^{i-1}}$$

Therefore,

$$E[\chi_i] = \frac{(n-1)^{i-1}}{n^{i-1}}$$

Calculate the geometric progression for starter element $b_1 = 1$ (since there is one unique element in the sequence of length 1) and step $q = \frac{n-1}{n}$:

$$E[\text{unique elements}] = \sum_{i=1}^{n} \frac{(n-1)^{i-1}}{n^{i-1}} = \frac{b_1(1-q^n)}{1-q} = \frac{1(1-\frac{(n-1)}{n}^n)}{1-\frac{(n-1)}{n}} =$$

$$= \frac{1-\frac{(n-1)}{n}^n}{1-\frac{(n-1)}{n}} = \frac{1-\frac{(n-1)}{n}^n}{\frac{n-(n-1)}{n}} = \frac{1-\frac{(n-1)}{n}^n}{\frac{1}{n}} = n\left(1-\left(\frac{n-1}{n}\right)^n\right)$$

Subproblem B

Find the function that takes the form of Cn^{α}, which would be equivalent to the derived mathematical expectation as $n \to \infty$.

Take the limit of the function above

$$\lim_{n\to\infty}\left(n\left(1-\left(\frac{n-1}{n}\right)^n\right)\right) = \lim_{n\to\infty}\left(1-\left(\frac{n-1}{n}\right)^n\right)\lim_{n\to\infty} n =$$

$$= \lim_{n\to\infty}\left(1-\left(1-\frac{1}{n}\right)^n\right)\lim_{n\to\infty} n = \left(1-\frac{1}{e}\right)\lim_{n\to\infty} n \implies$$

$$C = 1-\frac{1}{e}, \quad f(n) = Cn$$