Problem 1

Let A be some matrix and $\lambda, \mu \in \mathbb{R}$. Prove that $(\lambda + \mu)A = \lambda A + \mu A$ and $\lambda(\mu A) = (\lambda\mu)A$.

Solution

Suppose:

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

Evaluate $(\lambda + \mu)A$:

$$(\lambda + \mu) \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} = \begin{pmatrix} (\lambda+\mu)a_{11} & (\lambda+\mu)a_{12} & ... & (\lambda+\mu)a_{1n} \\ (\lambda+\mu)a_{21} & (\lambda+\mu)a_{22} & ... & (\lambda+\mu)a_{2n} \\ ... & ... & ... & ... \\ (\lambda+\mu)a_{m1} & (\lambda+\mu)a_{m2} & ... & (\lambda+\mu)a_{mn} \end{pmatrix} = X$$

Evaluate $\lambda A + \mu A$:

$$\lambda \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} + \mu \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & ... & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & ... & \lambda a_{2n} \\ ... & ... & ... & ... \\ \lambda a_{m1} & \lambda a_{m2} & ... & \lambda a_{mn} \end{pmatrix} + \begin{pmatrix} \mu a_{11} & \mu a_{12} & ... & \mu a_{1n} \\ \mu a_{21} & \mu a_{22} & ... & \mu a_{2n} \\ ... & ... & ... & ... \\ \mu a_{m1} & \mu a_{m2} & ... & \mu a_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda a_{11} + \mu a_{11} & \lambda a_{12} + \mu a_{12} & ... & \lambda a_{1n} + \mu a_{1n} \\ \lambda a_{21} + \mu a_{21} & \lambda a_{22} + \mu a_{22} & ... & \lambda a_{2n} + \mu a_{2n} \\ ... & ... & ... & ... \\ \lambda a_{m1} + \mu a_{m1} & \lambda a_{m2} + \mu a_{m2} & ... & \lambda a_{mn} + \mu a_{mn} \end{pmatrix} =$$

$$= \begin{pmatrix} (\lambda+\mu)a_{11} & (\lambda+\mu)a_{12} & ... & (\lambda+\mu)a_{1n} \\ (\lambda+\mu)a_{21} & (\lambda+\mu)a_{22} & ... & (\lambda+\mu)a_{2n} \\ ... & ... & ... & ... \\ (\lambda+\mu)a_{m1} & (\lambda+\mu)a_{m2} & ... & (\lambda+\mu)a_{mn} \end{pmatrix} = X$$

Both parts of the first equation evaluate to the same matrix; therefore, the first equation is true, q. e. d.

Evaluate $\lambda(\mu A)$:

$$\lambda(\mu A) = \lambda \begin{pmatrix} \mu a_{11} & \mu a_{12} & ... & \mu a_{1n} \\ \mu a_{21} & \mu a_{22} & ... & \mu a_{2n} \\ ... & ... & ... & ... \\ \mu a_{m1} & \mu a_{m2} & ... & \mu a_{mn} \end{pmatrix} = \begin{pmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & ... & \lambda\mu a_{1n} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & ... & \lambda\mu a_{2n} \\ ... & ... & ... & ... \\ \lambda\mu a_{m1} & \lambda\mu a_{m2} & ... & \lambda\mu a_{mn} \end{pmatrix} = Y$$

Evaluate $(\lambda\mu)A$:

$$(\lambda\mu)A = \begin{pmatrix} \lambda\mu a_{11} & \lambda\mu a_{12} & ... & \lambda\mu a_{1n} \\ \lambda\mu a_{21} & \lambda\mu a_{22} & ... & \lambda\mu a_{2n} \\ ... & ... & ... & ... \\ \lambda\mu a_{m1} & \lambda\mu a_{m2} & ... & \lambda\mu a_{mn} \end{pmatrix} = Y$$

Both parts of the second equation evaluate to the same matrix as well; therefore, the second equation is also true, q. e. d.

Problem 2

Subproblem A

$$\begin{pmatrix} 1 & 5 & 3 \\ 2 & -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 & 5 \\ -1 & 4 & -2 \\ 3 & -1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 2 + 5 \cdot -1 + 3 \cdot 3 & 1 \cdot -3 + 5 \cdot 4 + 3 \cdot -1 & 1 \cdot 5 + 5 \cdot -2 + 3 \cdot 1. \\ 2 \cdot 2 + -3 \cdot -1 + 1 \cdot 3 & 2 \cdot -3 + -3 \cdot 4 + 1 \cdot -1 & 2 \cdot 5 + -3 \cdot -2 + 1 \cdot 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 6 & 14 & -2 \\ 10 & -19 & 17 \end{pmatrix}$$

Subproblem B

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 3 \\ 2 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 & 2 \\ -2 & -1 & 1 & 2 \\ 2 & 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -4 & 6 & 1 \\ 2 & 2 & -5 & -2 \\ 2 & -2 & 6 & 4 \\ 1 & 3 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 3\cdot1+0\cdot-2+2\cdot2 & 3\cdot2+0\cdot-1+2\cdot1 & 3\cdot-1+0\cdot1+2\cdot1 & 3\cdot2+0\cdot2+2\cdot2 \\ 0\cdot1+1\cdot-2+3\cdot2 & 0\cdot2+1\cdot-1+3\cdot1 & 0\cdot-1+1\cdot1+3\cdot1 & 0\cdot2+1\cdot2+3\cdot2 \\ 2\cdot1+2\cdot-2+0\cdot2 & 2\cdot2+2\cdot-1+0\cdot1 & 2\cdot-1+2\cdot1+0\cdot1 & 2\cdot2+2\cdot2+0\cdot2 \\ 0\cdot1+1\cdot-2+0\cdot2 & 0\cdot2+1\cdot-1+0\cdot1 & 0\cdot-1+1\cdot1+0\cdot1 & 0\cdot2+1\cdot2+0\cdot2 \end{pmatrix} =$$

$$= \begin{pmatrix} 7 & 8 & -1 & 10 \\ 4 & 2 & 4 & 8 \\ -2 & 2 & 0 & 8 \\ -2 & -1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & -4 & 6 & 1 \\ 2 & 2 & -5 & -2 \\ 2 & -2 & 6 & 4 \\ 1 & 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 & 5 & 11 \\ 6 & 4 & -1 & 6 \\ 0 & 0 & 6 & 12 \\ -1 & 2 & 1 & 3 \end{pmatrix}$$

Problem 3

Find all (2×2) matrices B that commute with matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, i. e. for which $AB = BA$.

Solution

Let $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Evaluate AB and BA:

$$AB = \begin{pmatrix} 2b_{11} - b_{21} & 2b_{12} - b_{22} \\ b_{11} & b_{12} \end{pmatrix}$$

$$BA = \begin{pmatrix} 2b_{11} + b_{12} & -b_{11} \\ 2b_{21} + b_{22} & -b_{21} \end{pmatrix}$$

Therefore, for $AB = BA$, the following system of equations has to be true:

$$\begin{cases} 2b_{11} - b_{21} = 2b_{11} + b_{12} \\ 2b_{12} - b_{22} = -b_{11} \\ b_{11} = 2b_{21} + b_{22} \\ b_{12} = -b_{21} \end{cases} \Rightarrow \begin{cases} -b_{21} = b_{12} \\ b_{11} = b_{22} - 2b_{12} \\ b_{11} = 2b_{21} + b_{22} \\ b_{12} = -b_{21} \end{cases} \Rightarrow \begin{cases} b_{12} = -b_{21} \\ b_{11} = b_{22} + 2b_{21} \end{cases}$$

Thus, considering $b_{21} = x$, $b_{22} = y$, matrix $B = \begin{pmatrix} y + 2x & -x \\ x & y \end{pmatrix}$.

Problem 4

Evaluate the following expression:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n$$

Solution

Evaluate matrices for $n = 1, 2$ to try and figure out the pattern:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^2 = \begin{pmatrix} \cos^2\alpha - \sin^2 a & -2\sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

It appears that $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$. This shall be the induction hypothesis. Its base for $n = 1$ has been already proven in the beginning of the solution, so only the induction step has to be checked.

Prove that $\begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos((n+1)\alpha) & -\sin((n+1)\alpha) \\ \sin((n+1)\alpha) & \cos((n+1)\alpha) \end{pmatrix}$.

$$\begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos n\alpha\cos\alpha - \sin n\alpha\sin\alpha & -\sin n\alpha\cos\alpha - \sin\alpha\cos n\alpha \\ \sin n\alpha\cos\alpha + \sin\alpha\cos n\alpha & \cos n\alpha\cos\alpha - \sin n\alpha\sin\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} \cos((n+1)\alpha) & -\sin((n+1)\alpha) \\ \sin((n+1)\alpha) & \cos((n+1)\alpha) \end{pmatrix}$$

Therefore, $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n = \begin{pmatrix} \cos n\alpha & -\sin n\alpha \\ \sin n\alpha & \cos n\alpha \end{pmatrix}$, q. e. d.

Problem 5

Evaluate the following expression:

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}^k$$

Solution

Evaluate matrices for $n = 1, 2$ to try and figure out the pattern:

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}^1 = \begin{pmatrix} \lambda_1^1 & & & 0 \\ & \lambda_2^1 & & \\ & & \ddots & \\ 0 & & & \lambda_n^1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}^2 = \begin{pmatrix} \lambda_1 \cdot \lambda_1 + 0 + ... + 0 & \lambda_1 \cdot 0 + 0 \cdot \lambda_2 + 0 + ... + 0 & \dots & \lambda_1 \cdot 0 + ... + 0 + 0 \cdot \lambda_n \\ 0 \cdot \lambda_1 + \lambda_2 \cdot 0 + ... + 0 & 0 + \lambda_2 \cdot \lambda_2 + 0 + ... + 0 & \dots & 0 + \lambda_2 \cdot 0 + ... + 0 \cdot \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 \cdot \lambda_1 + 0 + ... + \lambda_n \cdot 0 & 0 + 0 \cdot \lambda_2 + ... + \lambda_n \cdot 0 & \dots & 0 + ... + 0 + \lambda_n \cdot \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$$

Similarly as in Problem 4, prove the hypothesis that

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$$

via induction. The induction base is true as previously described. We need to check whether the induction step is true:

$$\begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda_1^k \cdot \lambda_1 + 0 + ... + 0 & \lambda_1^k \cdot 0 + 0 \cdot \lambda_2 + 0 + ... + 0 & \dots & \lambda_1^k \cdot 0 + ... + 0 + 0 \cdot \lambda_n \\ 0 \cdot \lambda_1 + \lambda_2 \cdot 0 + ... + 0 & 0 + \lambda_2^k \cdot \lambda_2 + 0 + ... + 0 & \dots & 0 + \lambda_2^k \cdot 0 + ... + 0 \cdot \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 \cdot \lambda_1 + 0 + ... + \lambda_n^k \cdot 0 & 0 + 0 \cdot \lambda_2 + ... + \lambda_n^k \cdot 0 & \dots & 0 + ... + 0 + \lambda_n^k \cdot \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^{k+1} & & & 0 \\ & \lambda_2^{k+1} & & \\ & & \ddots & \\ 0 & & & \lambda_n^{k+1} \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}^k = \begin{pmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$$

q. e. d.

Problem 6

Evaluate the following expression:

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n$$

Solution

Similarly as in Problems 5, 6, evaluate the expression for $n = 1, 2$:

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^1 = \begin{pmatrix} \lambda^1 & 1 \cdot \lambda^{1-1} \\ 0 & \lambda^1 \end{pmatrix}^1$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^2 = \begin{pmatrix} \lambda \cdot \lambda + 1 \cdot 0 & \lambda \cdot 1 + 1 \cdot \lambda \\ 0 \cdot \lambda + \lambda \cdot 0 & 0 \cdot 1 + \lambda \cdot \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda^{n-1} \\ 0 & \lambda^2 \end{pmatrix}$$

Induction base is already proven, now we need to prove that $\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$:

$$\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^n \cdot \lambda + n\lambda^{n-1} \cdot 0 & n\lambda^{n-1} \cdot \lambda + 1 \cdot \lambda \\ 0 \cdot \lambda^n + \lambda^n \cdot 0 & 0 \cdot n\lambda^{n-1} + \lambda^n \cdot \lambda \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda^n \cdot \lambda + n\lambda^{n-1} \cdot 0 & n\lambda^{n-1} \cdot \lambda + 1 \cdot \lambda \\ 0 \cdot \lambda^n + \lambda^n \cdot 0 & 0 \cdot n\lambda^{n-1} + \lambda^n \cdot \lambda \end{pmatrix} = \begin{pmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

q. e. d.

Problem 7

Calculate H^n of the following matrix:

$$H = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Solution

Attempt to square the matrix to see what happens:

$$H^2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

As seen from the result matrix, the non-zero elements "shift" diagonally by one index on multiplication. We may write an equation that we think would determine the second matrix (consider h_{ij} an element of the result matrix) and then prove it via induction:

$$h_{ij} = \begin{cases} 1, \text{if } j - i = n \\ 0, \text{otherwise} \end{cases}$$

The induction base is already proven ($H^1 = H$). Therefore, we need to check whether the induction step is true.

For some $n < k$, where k is the matrix's dimension, we multiply H^n by H (positions are represented accurately in each of the multiplication steps):

$$H^n \cdot H = \begin{pmatrix} 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} = H^{n+1}$$

Therefore, the induction step is true, q. e. d.

NB: The equation that determines the result matrix is identical to a zero matrix if $n \geq k$.

Answer: $H^n = \left(h_{ij}\right)$, where: $h_{ij} = \begin{cases} 1, \text{if } j - i = n \\ 0, \text{otherwise} \end{cases}$