Calculus, Homework 18

Problem 1

> Integrate (if possible)

Subproblem 1

$$\int (2x^2+1)^3 dx = \int (8x^6+12x^4+6x^2+1)dx =$$

$$8\int x^6 dx + 12\int x^4 dx + 6\int x^2 dx + \int dx =$$

$$\frac{8}{7}x^7 + \frac{12}{5}x^5 + 2x^3 + x + \mathrm{C}$$

Subproblem 2

$$\int (1+\sqrt{x})^4 dx \xRightarrow{u=\sqrt{x},\, du=\frac{dx}{2\sqrt{x}},\, x=u^2,\, dx=2\sqrt{x}du} 2\int u(1+u)^4 dx$$

$$\xRightarrow{u=s-1,\, ds=du} 2\int (s-1)s^4 ds = 2\int s^5 ds + 2\int s^4 ds =$$

$$\frac{s^6}{3} - \frac{2}{5}s^5 + C = \frac{(u+1)^6}{3} - \frac{2}{5}(u+1)^5 + \mathrm{C} =$$

$$\frac{(\sqrt{x}+1)^6}{3} - \frac{2}{5}(\sqrt{x}+1)^5 + \mathrm{C} = \frac{1}{15}(\sqrt{x}+1)^5(5\sqrt{x}-1) + \mathrm{C}$$

Subproblem 3

$$\int \frac{(x+1)(x^2-3)}{3x^2}dx = \frac{1}{3}\int \frac{x^3-3x+x^2-3}{x^2}dx =$$

$$\frac{1}{3}\int\left(x - \frac{3}{x} + 1 - \frac{3}{x^2}\right)dx = \frac{1}{3}\left(\int x dx - \int \frac{3dx}{x} + \int dx - \int \frac{3dx}{x^2}\right) =$$

$$\frac{1}{3}\left(\frac{x^2}{2} - 3\ln x + x + \frac{3}{x}\right) = \frac{x^2}{6} + \frac{x}{3} - \ln(x) + \frac{1}{x} + \mathrm{C}$$

Subproblem 4

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \int \frac{dx}{\sqrt{1 - \frac{x^2}{a^2}}} \xRightarrow{u=\frac{x}{a},\, x=au,\, du=\frac{dx}{a},\, dx=adu}$$

$$\int \frac{du}{\sqrt{1 - u^2}} = \arcsin u = \arcsin\left(\frac{x}{a}\right) + \mathrm{C}$$

Subproblem 5

> Honestly the only idea that came to mind is to somehow factor out the minus using imaginary numbers as well to get the $\arctan$, which appears to have actually worked

$$\int \frac{1}{x^2 - a} dx = -\frac{1}{a} \int \frac{1}{1 + i^2 \frac{x^2}{\sqrt{a}^2}} dx \xRightarrow{u=\frac{ix}{\sqrt{a}},\, x=\frac{\sqrt{a}u}{i},\, du=\frac{i}{\sqrt{a}}dx,\, dx=\frac{\sqrt{a}}{i}du}$$

$$-\frac{i}{\sqrt{a}} \int \frac{1}{1 + u^2} du = -\frac{i \arctan u}{\sqrt{a}} + \mathrm{C}$$

Subproblem 6

> Sorry, no tex-ed long division

$$\int \frac{x^3 + 2x^2 - 3x + 10}{x + 4} dx = \int \left(x^2 - 2x + 5 + \frac{10}{x + 4}\right) dx =$$

$$\int x^2 dx - \int 2x dx + \int 5 dx + \int \frac{10}{x + 4} dx =$$

$$\frac{x^3}{3} - x^2 + 5x + 10 \ln(x + 4) + \mathrm{C}$$

Subproblem 7

$$\int \frac{e^x dx}{\sqrt{1 - e^{2x}}} \xRightarrow{u=e^x,\, du=e^x dx,\, dx=\frac{du}{e^x}}$$

$$\int \frac{du}{\sqrt{1 - u^2}} = \arcsin u = \arcsin e^x + \mathrm{C}$$

Subproblem 8

$$\int x e^{x^2} dx \xRightarrow{u=x^2,\, du=2xdx} \frac{1}{2} \int e^u du = \frac{e^u}{2} + \mathrm{C} = \frac{e^{x^2}}{2} + \mathrm{C}$$

Subproblem 9

> Let's try to rewrite this in a tangent form using the formulas $\sin x = \frac{2\operatorname{tg}\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}, \cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}$

$$\int \frac{dx}{1+\sin(x)} = \int \frac{dx}{1+\frac{2\operatorname{tg}\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1}} = \int \frac{dx}{\frac{\operatorname{tg}^2\frac{x}{2}+2\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}^2\frac{x}{2}+1}}$$

$$\xRightarrow{u=\operatorname{tg}\frac{x}{2},\, du=\frac{1}{2\cos^2\frac{x}{2}}dx,\, dx=\frac{2du}{u^2+1}} 2\int \frac{\frac{du}{u^2+1}}{\frac{u^2+2u+1}{u^2+1}} =$$

$$2\int \frac{du}{(u+1)^2} \xRightarrow{s=u+1, ds=du} 2\int \frac{du}{s^2} = -\frac{2}{s} + \mathrm{C} =$$

$$-\frac{2}{u+1} + \mathrm{C} = -\frac{2}{\operatorname{tg}\frac{x}{2}+1}$$

Subproblem 10

$$\int \frac{x^3dx}{x^8+2} \xRightarrow{u=x^4,\, du=4x^3dx} \frac{1}{4}\int \frac{du}{u^2+2} = \frac{1}{8}\int \frac{du}{\frac{u^2}{2}+1} =$$

$$\xRightarrow{s=\frac{u}{\sqrt{2}},\, ds=\frac{1}{\sqrt{2}}du} \frac{1}{4\sqrt{2}}\int \frac{ds}{s^2+1} =$$

$$\frac{\arctan s}{4\sqrt{2}} + \mathrm{C} = \frac{\arctan\frac{u}{\sqrt{2}}}{4\sqrt{2}} + \mathrm{C} = \frac{\arctan\frac{x^4}{\sqrt{2}}}{4\sqrt{2}} + \mathrm{C}$$

Subproblem 11

> Kinda cheaty way, but it's literally a table integral of a cosecant >:)

$$\int \frac{dx}{\sin(x)} = \int \csc x dx = \ln \operatorname{tg}\frac{x}{2}$$

Subproblem 12

Consider 1-form ω depending on parameters $a, b, c, d \in \mathbb{R}$ and integrate it.

> Strategy: first integrate the form, and then see whether the resulting expression is undefined

Subsubproblem A

$$\omega = \frac{ax+b}{cx+d}dx, \quad c \neq d \neq 0$$

$$\int \frac{ax+b}{cx+d}dx = \int \frac{bc-ad}{c(cx+d)}dx + \int \frac{a}{c}dx =$$

$$\left(b - \frac{ad}{c}\right) \int \frac{dx}{cx+d} + \frac{ax}{c} + \mathrm{C} \xRightarrow{u=cx+d,\, du=cdx}$$

$$\left(b - \frac{ad}{c}\right) \frac{1}{c} \int \frac{du}{u} + \frac{ax}{c} + \mathrm{C} =$$

$$\left(\frac{b}{c} - \frac{ad}{c^2}\right) \ln(cx+d) + \frac{ax}{c} + \mathrm{C}$$

The expression above is true for $c \neq 0$, now when $c = 0$:

$$\int \frac{ax+b}{d}dx = \frac{ax^2+2bx}{2d} + \mathrm{C}$$

Answer:

$$\int \omega = \begin{cases} \left(\frac{bc-ad}{c^2}\right) \ln(cx+d) + \frac{ax}{c} + \mathrm{C} & c \neq 0 \\ \frac{ax^2+2bx}{2d} + \mathrm{C}, & c = 0 \end{cases}$$

Subsubproblem B

$$\omega = \frac{ax^3+bx^2+cx+d}{x^2+1}dx, \quad x \neq i$$

$$\int \frac{ax^3+bx^2+cx+d}{x^2+1}dx = \int \frac{-ax-b+cx+d}{x^2+1}dx + \int axdx + \int bdx =$$

$$= (c-a) \int \frac{x}{x^2+1}dx + (d-b) \int \frac{1}{x^2+1}dx + a \int xdx + b \int dx =$$

$$= \frac{c-a}{2} \ln(x^2+1) + (d-b) \operatorname{arctg} x + \frac{ax^2}{2} + bx + \mathrm{C}$$

> No interesting variations here

Subsubproblem C

$$\omega = \frac{dx}{ax^2+bx+c}, \quad a \neq b \neq c \neq 0$$

$$\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{\frac{4ac-b^2}{4a} + \left(\frac{b}{2\sqrt{a}} + \sqrt{a}x\right)^2}$$

$$\xrightarrow{u=\frac{b}{2\sqrt{a}}+\sqrt{a}x,\, du=\sqrt{a}dx} \frac{1}{\sqrt{a}} \int \frac{du}{\frac{4c-b^2}{4a} + u^2} =$$

$$\frac{4\sqrt{a}}{4ac-b^2} \int \frac{1}{\frac{4a}{4ac-b^2}u^2 + 1} \xrightarrow{s=\frac{2\sqrt{a}u}{\sqrt{4ac-b^2}},\, ds=\frac{2\sqrt{a}}{\sqrt{4ac-b^2}}du}$$

$$\frac{2}{\sqrt{4ac-b^2}} \int \frac{1}{s^2+1} ds = \frac{2 \operatorname{arctg} s}{\sqrt{4ac-b^2}} + \mathrm{C} =$$

$$\frac{2 \operatorname{arctg} \frac{2\sqrt{a}u}{\sqrt{4ac-b^2}}}{\sqrt{4ac-b^2}} + \mathrm{C} = \frac{2 \operatorname{arctg} \frac{2\sqrt{a}(\frac{b}{2\sqrt{a}}+\sqrt{a}x)}{\sqrt{4ac-b^2}}}{\sqrt{4ac-b^2}} + \mathrm{C} =$$

$$\frac{2 \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}}}{\sqrt{4ac-b^2}} + \mathrm{C}$$

This is valid for $4ac \neq b^2$. Now, for $b = \pm 2\sqrt{ac}$:

$$\int \frac{dx}{(\sqrt{a}x \pm \sqrt{c})^2} =$$

$$\frac{1}{\sqrt{a}} \int \frac{1}{(\sqrt{a}x \pm \sqrt{c})^2} d(\sqrt{a}x \pm \sqrt{c}) = -\frac{1}{ax \pm \sqrt{ac}} + \mathrm{C}$$

Now, what if $a = 0$?

$$\int \frac{dx}{bx+c} dx = \frac{1}{b} \int \frac{1}{bx+c} d(bx+c) = \frac{\ln(bx+c)}{b} + \mathrm{C}$$

Finally, what if $a = b = 0$?

$$\int \frac{dx}{c} = \frac{x}{c} + \mathrm{C}$$

Answer:

$$\int \omega = \begin{cases} -\frac{1}{ax \pm \sqrt{ac}} + \mathrm{C}, & b = \pm 2\sqrt{ac} \\ \frac{\ln(bx+c)}{b} + \mathrm{C}, & a = 0 \\ \frac{x}{c} + \mathrm{C}, & a = b = 0 \\ \frac{2 \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}}}{\sqrt{4ac-b^2}} + \mathrm{C}, & \text{otherwise} \end{cases}$$