Problem 1.8

Is $\sqrt{2}+\sqrt{3}$ rational?

Solution

It is known that $\forall x \in \mathbb{Q}\colon x^2 \in \mathbb{Q}$. Therefore, we need to check whether this is true for $\sqrt{2}+\sqrt{3}$. Suppose $\sqrt{2}+\sqrt{3} \in \mathbb{Q}$.

$$(\sqrt{2}+\sqrt{3})^2 = 2 + 2\cdot\sqrt{2}\sqrt{3} + 3 = 5 + 2\sqrt{6}$$

$\sqrt{6} \in \mathbb{R} \Rightarrow 2\sqrt{6} \in \mathbb{R} \Rightarrow 5 + 2\sqrt{6} \in \mathbb{R}$. The sum of a rational number and a product of an irrational and a rational one is irrational. Since $(\sqrt{2}+\sqrt{3})^2 \in \mathbb{R}$, then $\sqrt{2}+\sqrt{3} \notin \mathbb{Q}$, q. e. d.

Answer

$$\sqrt{2}+\sqrt{3} \notin \mathbb{Q}$$

Problem 1.9

Subproblem A

Evaluate $\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \cdots + \frac{1}{(4n-3)(4n+1)}$.

Solution

$$\begin{aligned} a_n = \frac{3n-2}{4n-3} - \frac{3n+1}{4n+1} &= \frac{(3n-2)(4n+1)+(3n+1)(4n-3)}{(4n-3)(4n+1)} \\ &= \frac{12n^2-5n-2-(12n^2-5n-3)}{(4n-3)(4n+1)} = \frac{1}{(4n-3)(4n+1)} \end{aligned}$$

$$\frac{1}{1\cdot5} + \frac{1}{5\cdot9} + \cdots + \frac{1}{(4n-3)(4n+1)} = \underbrace{\frac{1}{1} - \frac{4}{5}}_{a_1} + \underbrace{\frac{4}{5} - \frac{7}{9}}_{a_2} + \underbrace{\frac{7}{9} - \frac{10}{13}}_{a_3} + \cdots + \underbrace{\frac{3n-2}{4n-3} - \frac{3n+1}{4n+1}}_{a_n} =$$

$$= 1 - \frac{3n+1}{4n+1} = \frac{4n+1-3n+1}{4n+1} = \frac{n+2}{4n+1}$$

Answer

$$\frac{n+2}{4n+1}$$

Subproblem B

Evaluate $\frac{1}{2} + \frac{3}{2^2} + \cdots + \frac{2n-1}{2^n}$.

Solution

$$S = \frac{1}{2} + \frac{3}{2^2} + \cdots + \frac{2n-1}{2^n}$$

Multiply the sequence by $\frac{1}{2}$:

$$\frac{1}{2}S = \frac{1}{2^2} + \frac{3}{2^3} + \cdots + \frac{2n-1}{2^{n+1}}$$

Subtract the resulting sequence from the original one:

$$\begin{aligned} S - \frac{1}{2}S &= \frac{1}{2} + \frac{3}{2^2} + \cdots + \frac{2n-1}{2^n} - \frac{1}{2^2} - \frac{3}{2^3} - \cdots - \frac{2n-1}{2^{n+1}} = \\ &= \frac{1}{2} + \underbrace{\frac{3}{4} - \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{5}{8} - \frac{3}{8}}_{\frac{1}{4}} + \cdots + \underbrace{\frac{2n-1}{2^n} - \frac{2n-3}{2^n}}_{\frac{1}{2^{n-1}}} - \frac{2n-1}{2^{n+1}} \end{aligned}$$

Simplify and rearrange the elements:

$$\frac{1}{2}S + \frac{2n-1}{2^{n+1}} - \frac{1}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}$$

The right part of the equation is a geometric progression, therefore we may apply the formula $\frac{q(b_1^k-1)}{q-1}$, where $k=n-1$, $b_1=\frac{1}{2}$, and $q=\frac{1}{2}$.

$$\frac{1}{2}S+\frac{2n-1}{2^{n+1}}-\frac{1}{2}=\frac{\frac{1}{2}(\frac{1}{2}^{n-1}-1)}{\frac{1}{2}-1}=\frac{\frac{1}{2^n}-\frac{1}{2}}{-\frac{1}{2}}=1-\frac{1}{2^{n-1}}$$

Now evaluate S:

$$\frac{1}{2}S+\frac{2n-1}{2^{n+1}}-\frac{1}{2}=1-\frac{1}{2^{n-1}}$$

$$S=-\frac{2n-1}{2^n}-\frac{1}{2^{n-2}}+3=\frac{3\cdot 2^n-2n-3}{2^n}=\frac{3(2^n-1)-2n}{2^n}$$

Answer

$$\frac{3(2^n-1)-2n}{2^n}$$

Problem 1.10

Prove that $\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}$, for $n\geq 2$ by induction.

Solution

Prove the first part of the double inequation:

$$\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$$

Check whether the inequation is true for $n=2$ **(induction base)**:

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}+1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2}=1+\frac{\sqrt{2}}{2}$$

$$\sqrt{2}<1+\frac{\sqrt{2}}{2}\Rightarrow\frac{\sqrt{2}}{2}<1\Rightarrow\sqrt{2}<2,$$

which is true.

Therefore, now we need to prove the **induction hypothesis** that

$$\sqrt{n}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}$$

Check whether the equation would be true if we were to add $\frac{1}{\sqrt{n+1}}$ to it (**induction step**):

$$\sqrt{n}+\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}$$

$$\frac{\sqrt{n}\sqrt{n+1}+1}{\sqrt{n+1}}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}$$

Estimate the equation on the lefthand side above against $\sqrt{n+1}$:

$$\frac{\sqrt{n}(n+1)+\sqrt{n+1}}{n+1}>\sqrt{n+1}$$

$$\sqrt{n}(n+1)+\sqrt{n+1}>n\sqrt{n+1}+\sqrt{n+1}$$

$$\sqrt{n}(n+1)>n\sqrt{n+1}$$

$$\sqrt{n+1}>\sqrt{n}\Rightarrow$$

$$\Rightarrow\sqrt{n+1}<\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n+1}}$$

q. e. d.

Prove the second part of the double inequation:

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}$$

Similarly as above, check the **induction base** for n:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} < 2\sqrt{2}$$

$$\frac{\sqrt{2}+1}{\sqrt{2}} < 2\sqrt{2}$$

$$\frac{2+\sqrt{2}}{2} < 2\sqrt{2}$$

$$2+\sqrt{2} < 4\sqrt{2}$$

$$\frac{2}{3} < \sqrt{2},$$

which is true.

Induction hypothesis:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}} < 2\sqrt{n-1}$$

Induction step: add one last element ($\frac{1}{\sqrt{n}}$) to each half of the hypothesis. This way we would be able to eventually arrive from the **induction base** to any n.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} < 2\sqrt{n-1} + \frac{1}{\sqrt{n}}$$

It would be sufficient to prove the following because due to the **transition property** if $2\sqrt{n-1} + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ would be true, then $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ would also be true.

$$2\sqrt{n-1} + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

$$2\sqrt{n-1}\sqrt{n} < 2n-1$$

$$4n^2 - 4n < 4n^2 - 4n + 1$$

$$0 < 1,$$

q. e. d.

Therefore,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

Lastly,

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n},$$

q. e. d.

Problem 1.11

Given n **skew lines** and an Euclidean plane, find how many regions the lines divide the plane into.

*I will assume that **skew lines** are such lines that no three lines intersect in a single point and which are not parallel to each other because otherwise the task does not make sense. Generally, the term **skew lines** is used for three-dimensional spaces.*

Solution

Evaluate first couple of n to find a pattern.

lines	number of region	comment
0	$a_0 = 1$	1 original plane
1	$a_1 = 2$	we had 1 region, got 2
2	$a_2 = 4$	new line intersects 1 existing line, creating 2 new regions
3	$a_3 = 7$	new line intersects 2 existing lines, creating 3 new regions

4	$a_4 = 11$	new line intersects 3 existing lines, creating 4 new regions
$\vdots$	$\vdots$	$\vdots$
n	$a_n = a_{n-1} + n$	new line intersects $n-1$ existing lines, creating n new regions

Therefore, if there are already n lines and a new $(n+1)$-st line is added, the new line would intersect each of the already existing lines. Those intersections would divide the new line into a certain number $(k+1)$ segments. Each of those segments divides one previous region into two new ones (since the regions are convex polygons). Thus, our hypothesis is correct.

We can look at the formula a little bit closer and see that there is an arithmetic progression inside:

$$a_n = a_{n-1} + n = 1 + 1 + 2 + 3 + 4 + \cdots + n - 1 + n = 1 + \frac{n(n+1)}{2}$$

Answer

$$1 + \frac{n(n+1)}{2}$$