Calculus, Homework 16

Problem 1

Determine whether series $(x_n), (y_n), (z_n)$ converge:

$$x_n = \frac{n^2}{2^{n-1}}, \quad y_n = \left(\frac{n+1}{8n-1}\right)^n, \quad z_n = nz^{n-1},\ z > 0$$

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d'Alembert criteria:

$$x_n = \frac{n^2}{2^{n-1}}, \quad x_{n+1} = \frac{(n+1)^2}{2^n}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^n} \frac{2^{n-1}}{n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{2} = \frac{1}{2}$$

which implies that the series converges.

$$y_n = \left(\frac{n+1}{8n-1}\right)^n, \quad y_{n+1} = \left(\frac{n+2}{8n+7}\right)^{n+1}$$

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lim_{n\to\infty} \left(\frac{n+2}{8n+7}\right)^{n+1} \left(\frac{8n-1}{n+1}\right)^n =$$

$$\lim_{n\to\infty} \left(\frac{1+\frac{2}{n}}{8+\frac{7}{n}}\right)^{n+1} \left(\frac{8-\frac{1}{n}}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty} \left(\frac{1+\frac{2}{n}}{8+\frac{7}{n}}\right) \times 1 = \frac{1}{8}$$

which also implies the series converges.

$$z_n = nz^{n-1}, \quad z_{n+1} = (n+1)z^n$$

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \lim_{n\to\infty} \frac{(n+1)z^n}{nz^{n-1}} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})z^n}{z^{n-1}} = \lim_{n\to\infty} \frac{z^n}{z^{n-1}} = z$$

which implies that the series converges if $z < 1$ and diverges if $z \geq 1$. The series diverges if $z = 1$ since we get a series of we get a series (n) which is a sequence of all natural numbers, which diverges.

Problem 2

Using the radical Cauchy criteria, determine whether series (x_n) converges:

$$x_n = \frac{x^n}{a_n}, \quad n \geq 1$$

where $x > 0$ and (a_n) is a sequence of positive numbers with a limit $\lim_{n\to\infty} a_n = a$.

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{x^n}{a_n}} = \lim_{n\to\infty} \frac{x}{\sqrt[n]{a_n}} = \lim_{n\to\infty} \frac{x}{\sqrt[n]{a}} = x$$

which implies the series converges if $x < 1$ and diverges if $x > 1$. As for $x = 1$, the series also diverges since we would approach a sum larger than $\frac{1}{a}$, which itself diverges.

Problem 3

Using the d'Alembert criteria, determine whether series (x_n) converges, where

$$x_n = \frac{(nx)^n}{n!}, \quad x > 0, n \geq 0$$

$$x_n = \frac{(nx)^n}{n!}, \quad x_{n+1} = \frac{((n+1)x)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{((n+1)x)^{n+1}}{(n+1)!} \frac{n!}{(nx)^n} = \lim_{n\to\infty} \frac{(n+1)^n x}{n^n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n x = ex$$

which implies the series converges if $x < \frac{1}{e}$, diverges if $x > \frac{1}{e}$ and is indeterminable through d'Alembert if $x = \frac{1}{e}$.

Problem 4

Consider series (x_n) where

$$x_n = (ab)^n, \quad n \geq 0$$

and a, b are two different positive numbers. Using the radical Cauchy criteria and d'Alembert's criteria determine whether the series converges.

d'Alembert:

$$x_n = (ab)^n, \quad x_{n+1} = (ab)^{n+1}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(ab)^{n+1}}{(ab)^n} = ab$$

which implies that the series converges when $ab < 1$ and diverges when $ab \geq 1$ (diverges when it is equal to one since then we get a sequence of ones, which diverges).

Cauchy:

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{(ab)^n} = ab$$

implies the same.

Problem 5

Let $0 < r < 1, \alpha \in \mathbb{R}$. Prove using the radical Cauchy criteria that series (x_n) where

$$x_n = r^n |\sin(n\alpha)|, n \geq 1$$

converges.

Is it possible to prove the convergence of this series using the d'Alembert's criteria?

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{r^n |\sin(n\alpha)|} = r \lim_{n\to\infty} \sqrt[n]{|\sin(n\alpha)|} < 0$$

which implies that the series converges since $r < 1, |\sin(n\alpha)| \leq 1$.

$$x_n = r^n |\sin(n\alpha)|, \quad x_{n+1} = r^{n+1} |\sin((n+1)\alpha)|$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{r^{n+1} |\sin((n+1)\alpha)|}{r^n |\sin(n\alpha)|} = r \lim_{n\to\infty} \left| \frac{\sin(\alpha(n+1))}{\sin(\alpha n)} \right|$$

the sine division is indeterminable so no, I don't think it's possible to prove this using d'Alembert's unless we use some ridiculous substitution.

Problem 6

Consider series (x_n) where

$$x_n = \frac{1}{n!}\left(\frac{n}{e}\right)^n, \quad n \geq 1$$

using Raabe's criterion, determine the convergence of this series.

Raabe's criterion tells us that the series (x_n) converges if for large enough n the following is true:

$$n\left(\frac{x_n}{x_{n+1}} - 1\right) > 1$$

$$x_n = \frac{1}{n!}\left(\frac{n}{e}\right)^n, \quad x_{n+1} = \frac{1}{(n+1)!}\left(\frac{n+1}{e}\right)^{n+1}$$

$$\lim_{n\to\infty}\left(\frac{n}{n!}\left(\frac{n}{e}\right)^n \frac{(n+1)!}{1}\left(\frac{e}{n+1}\right)^{n+1} - n\right) = \lim_{n\to\infty}\left(\frac{nn^n(n+1)!e^{n+1}}{n!e^n(n+1)^{n+1}} - n\right) =$$

$$\lim_{n\to\infty}\left(\frac{n^{n+1}(n+1)e}{(n+1)^{n+1}} - n\right) = \lim_{n\to\infty}\left(\frac{n^{n+1}e}{(n+1)^n} - n\right) = \lim_{n\to\infty}\left(\frac{ne}{(1+\frac{1}{n})^n} - n\right) =$$

$$\lim_{n\to\infty}\left(\frac{ne - n(1+\frac{1}{n})^n}{(1+\frac{1}{n})^n}\right) = \frac{1}{2}$$

> originally I made an assumption that this limit is zero although it isn't somehow

which implies that the series diverges.

Problem 7

Consider series (x_n), where

$$x_n = \frac{n!x^n}{(x+a_1)(2x+a_2)\cdots(nx+a_n)}, \quad n \geq 1, x > 0$$

and (a_n) is a sequence of positive numbers with a limit of $\lim_{n\to\infty} a_n = a$. Using Raabe's criteria, determine the convergence of this series.

$$n\left(\frac{x_n}{x_{n+1}}-1\right)>1$$

$$x_n=\frac{n!x^n}{(x+a_1)(2x+a_2)\cdots(nx+a_n)}$$

$$x_{n+1}=\frac{(n+1)!x^{n+1}}{(x+a_1)(2x+a_2)\cdots(nx+a_n)((n+1)x+a_{n+1})}$$

$$\lim_{n\to\infty} n\left(\frac{n!x^n}{(x+a_1)(2x+a_2)\cdots(nx+a_n)}\frac{(x+a_1)(2x+a_2)\cdots(nx+a_n)((n+1)x+a_{n+1})}{(n+1)!x^{n+1}}-1\right)=$$

$$=\lim_{n\to\infty} n\left(\frac{n!x^n((n+1)x+a_{n+1})}{(n+1)!x^{n+1}}-1\right)=\lim_{n\to\infty}\left(n\frac{((n+1)x+a)}{x(n+1)}-1\right)=$$

$$=\lim_{n\to\infty} n\left(\left(1+\frac{a}{x(n-1)}\right)-1\right)=\frac{a}{x}$$

which implies that the series diverges when $\frac{a}{x}<1$ and converges when $\frac{a}{x}>1$.