## Problem 1

Let $A$ be some matrix and $\lambda, \mu \in$ \R. Prove that $(\lambda+\mu)A = \lambda A + \mu A$ and $\lambda(\mu A) = (\lambda\mu)A$.

### Solution

Suppose:

$$A = ( a_{11} \quad a_{12} \quad \dots \quad a_{1n} \, a_{21} \quad a_{22} \quad \dots \quad a_{2n} \dots \quad \dots \quad \dots \quad \dots \, a_{m1} \quad a_{m2} \quad \dots \quad a_{mn} )$$

Evaluate $(\lambda+\mu)A$:

$$(\lambda+\mu) ( a_{11} \quad a_{12} \quad \dots \quad a_{1n} \, a_{21} \quad a_{22} \quad \dots \quad a_{2n} \dots \quad \dots \quad \dots \quad \dots \, a_{m1} \quad a_{m2} \quad \dots \quad a_{mn} ) = ( (\lambda+\mu)a_{11} \quad (\lambda+\mu)a_{12} \quad \dots \quad (\lambda+\mu)a_{1n} \, (\lambda+\mu)a_{21} \quad (\lambda+$$

Evaluate $\lambda A + \mu A$:

$$\lambda ( a_{11} \quad a_{12} \quad \dots \quad a_{1n} \, a_{21} \quad a_{22} \quad \dots \quad a_{2n} \dots \quad \dots \quad \dots \quad \dots \, a_{m1} \quad a_{m2} \quad \dots \quad a_{mn} ) + \mu ( a_{11} \quad a_{12} \quad \dots \quad a_{1n} \, a_{21} \quad a_{22} \quad \dots \quad a_{2n} \dots \quad \dots \quad \dots \quad \dots \, a_{m1}$$

$$= ( \lambda a_{11} \quad \lambda a_{12} \quad \dots \quad \lambda a_{1n} \, \lambda a_{21} \quad \lambda a_{22} \quad \dots \quad \lambda a_{2n} \dots \quad \dots \quad \dots \quad \dots \, \lambda a_{m1} \quad \lambda a_{m2} \quad \dots \quad \lambda a_{mn} ) + ( \mu a_{11} \quad \mu a_{12} \quad \dots \quad \mu a_{1n} \, \mu a_{21} \quad \mu a_{22} \quad \dots \quad \mu a_{2n} \dots$$

$$= ( \lambda a_{11} + \mu a_{11} \quad \lambda a_{12} + \mu a_{12} \quad \dots \quad \lambda a_{1n} + \mu a_{1n} \, \lambda a_{21} + \mu a_{21} \quad \lambda a_{22} + \mu a_{22} \quad \dots \quad \lambda a_{2n} + \mu a_{2n} \dots \quad \dots \quad \dots \quad \dots \, \lambda a_{m1} + \mu a_{m1} \quad \lambda a_{m2} + \mu a_{m2} \quad \dots \quad \lambda a$$

$$= ( (\lambda+\mu)a_{11} \quad (\lambda+\mu)a_{12} \quad \dots \quad (\lambda+\mu)a_{1n} \, (\lambda+\mu)a_{21} \quad (\lambda+\mu)a_{22} \quad \dots \quad (\lambda+\mu)a_{2n} \dots \quad \dots \quad \dots \quad \dots \, (\lambda+\mu)a_{m1} \quad (\lambda+\mu)a_{m2} \quad \dots \quad (\lambda+\mu)a_{mn} ) =$$

Both parts of the first equation evaluate to the same matrix; therefore, the first equation is true, q. e. d.

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Evaluate $\lambda(\mu A)$:

$$\lambda(\mu A) = \lambda ( \mu a_{11} \quad \mu a_{12} \quad \dots \quad \mu a_{1n} \, \mu a_{21} \quad \mu a_{22} \quad \dots \quad \mu a_{2n} \dots \quad \dots \quad \dots \quad \dots \, \mu a_{m1} \quad \mu a_{m2} \quad \dots \quad \mu a_{mn} ) = ( \lambda\mu a_{11} \quad \lambda\mu a_{12} \quad \dots \quad \lambda\mu a_{1n} \, \lambda\mu a_{21} \quad \lambda\mu a_{22}$$

Evaluate $(\lambda\mu)A$:

$$(\lambda\mu)A = ( \lambda\mu a_{11} \quad \lambda\mu a_{12} \quad \dots \quad \lambda\mu a_{1n} \, \lambda\mu a_{21} \quad \lambda\mu a_{22} \quad \dots \quad \lambda\mu a_{2n} \dots \quad \dots \quad \dots \quad \dots \, \lambda\mu a_{m1} \quad \lambda\mu a_{m2} \quad \dots \quad \lambda\mu a_{mn} ) = Y$$

Both parts of the second equation evaluate to the same matrix as well; therefore, the second equation is also true, q. e. d.

## Problem 2

### Subproblem A

$$( 1 \quad 5 \quad 3 \, 2 \quad -3 \quad 1 ) \cdot ( 2 \quad -3 \quad 5 \, -1 \quad 4 \quad -2 \, 3 \quad -1 \quad 1 ) =$$

$$= ( 1 \cdot 2 + 5 \cdot -1 + 3 \cdot 3 \quad 1 \cdot -3 + 5 \cdot 4 + 3 \cdot -1 \quad 1 \cdot 5 + 5 \cdot -2 + 3 \cdot 1 . \, 2 \cdot 2 + -3 \cdot -1 + 1 \cdot 3 \quad 2 \cdot -3 + -3 \cdot 4 + 1 \cdot -1 \quad 2 \cdot 5 + -3 \cdot -2 + 1 \cdot 1 ) =$$

$$= ( 6 \quad 14 \quad -2 \, 10 \quad -19 \quad 17 )$$

### Subproblem B

$$( 3 \quad 0 \quad 2 \, 0 \quad 1 \quad 3 \, 2 \quad 2 \quad 0 \, 0 \quad 1 \quad 0 ) \cdot ( 1 \quad 2 \quad -1 \quad 2 \, -2 \quad -1 \quad 1 \quad 2 \, 2 \quad 1 \quad 1 \quad 2 ) + ( 0 \quad -4 \quad 6 \quad 1 \, 2 \quad 2 \quad -5 \quad -2 \, 2 \quad -2 \quad 6 \quad 4 \, 1 \quad 3 \quad 0 \quad 1 ) =$$

$$= ( 3 \cdot 1 + 0 \cdot -2 + 2 \cdot 2 \quad 3 \cdot 2 + 0 \cdot -1 + 2 \cdot 1 \quad 3 \cdot -1 + 0 \cdot 1 + 2 \cdot 1 \quad 3 \cdot 2 + 0 \cdot 2 + 2 \cdot 2 \, 0 \cdot 1 + 1 \cdot -2 + 3 \cdot 2 \quad 0 \cdot 2 + 1 \cdot -1 + 3 \cdot 1 \quad 0 \cdot -1 + 1 \cdot 1 + 3 \cdot 1$$

$$= ( 7 \quad 8 \quad -1 \quad 10 \, 4 \quad 2 \quad 4 \quad 8 \, -2 \quad 2 \quad 0 \quad 8 \, -2 \quad -1 \quad 1 \quad 2 ) + ( 0 \quad -4 \quad 6 \quad 1 \, 2 \quad 2 \quad -5 \quad -2 \, 2 \quad -2 \quad 6 \quad 4 \, 1 \quad 3 \quad 0 \quad 1 ) = ( 7 \quad 4 \quad 5 \quad 11 \, 6 \quad 4 \quad -1$$

## Problem 3

Find all $(2 \times 2)$ matrices $B$ that commute with matrix $A = ( 2 \quad -1 \, 1 \quad 0 )$, i. e. for which $AB = BA$.

### Solution

Let $B = ( b_{11} \quad b_{12} \, b_{21} \quad b_{22} )$.

Evaluate $AB$ and $BA$:

$$AB = ( 2b_{11} - b_{21} \quad 2b_{12} - b_{22} \, b_{11} \quad b_{12} )$$

$$BA = ( 2b_{11} + b_{12} \quad -b_{11} \, 2b_{21} + b_{22} \quad -b_{21} )$$

Therefore, for $AB = BA$, the following system of equations has to be true:

$$\{ \, 2b_{11} - b_{21} = 2b_{11} + b_{12} \, 2b_{12} - b_{22} = -b_{11} \, b_{11} = 2b_{21} + b_{22} \, b_{12} = -b_{21} \quad \Rightarrow \{ \, -b_{21} = b_{12} \, b_{11} = b_{22} - 2b_{12} \, b_{11} = 2b_{21} + b_{22} \, b_{12} = -b_{21} \quad \Rightarrow \{ \, b_{12} = -b_{21} \, b_{11} = b$$

Thus, considering $b_{21} = x$, $b_{22} = y$, matrix $B = ( y + 2x \quad -x \, x \quad y )$

## Problem 4

Evaluate the following expression:

$$( \cos\alpha \quad -\sin\alpha \, \sin\alpha \quad \cos\alpha )^n$$

### Solution

Evaluate matrices for $n = 1, 2$ to try and figure out the pattern:

$$(\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha)^1 = (\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha)$$

$$(\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha)^2 = (\cos^2\alpha - \sin^2 a \quad -2\sin\alpha\cos\alpha \ 2\sin\alpha\cos\alpha \quad \cos^2\alpha - \sin^2\alpha) = (\cos 2\alpha \quad -\sin 2\alpha \ \sin 2\alpha \quad \cos 2\alpha)$$

It appears that $(\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha)^n = (\cos n\alpha \quad -\sin n\alpha \ \sin n\alpha \quad \cos n\alpha)$. This shall be the induction hypothesis. Its base for $n = 1$ has been already proven in the beginning of the solution, so only the induction step has to be checked.

Prove that $(\cos n\alpha \quad -\sin n\alpha \ \sin n\alpha \quad \cos n\alpha)(\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha) = (\cos((n+1)\alpha) \quad -\sin((n+1)\alpha) \ \sin((n+1)\alpha) \quad \cos((n+1)\alpha))$.

$$(\cos n\alpha \quad -\sin n\alpha \ \sin n\alpha \quad \cos n\alpha)(\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha) = (\cos n\alpha\cos\alpha - \sin n\alpha\sin\alpha \quad -\sin n\alpha\cos\alpha - \sin\alpha\cos n\alpha \ \sin n\alpha\cos\alpha + \sin\alpha\cos n\alpha$$

$$= (\cos((n+1)\alpha) \quad -\sin((n+1)\alpha) \ \sin((n+1)\alpha) \quad \cos((n+1)\alpha))$$

Therefore, $(\cos\alpha \quad -\sin\alpha \ \sin\alpha \quad \cos\alpha)^n = (\cos n\alpha \quad -\sin n\alpha \ \sin n\alpha \quad \cos n\alpha)$, q. e. d.

## Problem 5

Evaluate the following expression:

$$\left(\lambda_1 \quad 0 \quad \lambda_2 \quad \ddots \quad 0 \quad \lambda_n\right)^k$$

### Solution

Evaluate matrices for $n = 1, 2$ to try and figure out the pattern:

$$\left(\lambda_1 \quad 0 \quad \lambda_2 \quad \ddots \quad 0 \quad \lambda_n\right)^1 = \left(\lambda_1^1 \quad 0 \quad \lambda_2^1 \quad \ddots \quad 0 \quad \lambda_n^1\right)$$

$$\left(\lambda_1 \quad 0 \quad \lambda_2 \quad \ddots \quad 0 \quad \lambda_n\right)^2 = \left(\lambda_1 \cdot \lambda_1 + 0 + \ldots + 0 \quad \lambda_1 \cdot 0 + 0 \cdot \lambda_2 + 0 + \ldots + 0 \quad \ldots \quad \lambda_1 \cdot 0 + \ldots + 0 + 0 \cdot \lambda_n \ 0 \cdot \lambda_1 + \lambda_2 \cdot 0 + \ldots + 0 \quad 0\right.$$

Similarly as in Problem 4, prove the hypothesis that $\left(\lambda_1 \quad 0 \quad \lambda_2 \quad \ddots \quad 0 \quad \lambda_n\right)^k = \left(\lambda_1^k \quad 0 \quad \lambda_2^k \quad \ddots \quad 0 \quad \lambda_n^k\right)$ via induction. The induction base is true as previously described. We need to check whether the induction step is true:

$$\left(\lambda_1^k \quad 0 \quad \lambda_2^k \quad \ddots \quad 0 \quad \lambda_n^k\right)\left(\lambda_1 \quad 0 \quad \lambda_2 \quad \ddots \quad 0 \quad \lambda_n\right) =$$

$$= \left(\lambda_1^k \cdot \lambda_1 + 0 + \ldots + 0 \quad \lambda_1^k \cdot 0 + 0 \cdot \lambda_2 + 0 + \ldots + 0 \quad \ldots \quad \lambda_1^k \cdot 0 + \ldots + 0 + 0 \cdot \lambda_n \ 0 \cdot \lambda_1 + \lambda_2 \cdot 0 + \ldots + 0 \quad 0 + \lambda_2^k \cdot \lambda_2 + 0 + \ldots + 0 \quad \ldots \quad 0 + \lambda_2^k \cdot 0 + \ldots + 0 \cdot \lambda\right.$$

Therefore, $\left(\lambda_1 \quad 0 \quad \lambda_2 \quad \ddots \quad 0 \quad \lambda_n\right)^k = \left(\lambda_1^k \quad 0 \quad \lambda_2^k \quad \ddots \quad 0 \quad \lambda_n^k\right)$, q. e. d.

## Problem 6

Evaluate the following expression:

$$(\lambda \quad 1 \ 0 \quad \lambda)^n$$

### Solution

Similarly as in Problems 5, 6, evaluate the expression for $n = 1, 2$:

$$(\lambda \quad 1 \ 0 \quad \lambda)^1 = (\lambda^1 \quad 1 \cdot \lambda^{1-1} \ 0 \quad \lambda^1)^1$$

$$(\lambda \quad 1 \ 0 \quad \lambda)^2 = (\lambda \cdot \lambda + 1 \cdot 0 \quad \lambda \cdot 1 + 1 \cdot \lambda \ 0 \cdot \lambda + \lambda \cdot 0 \quad 0 \cdot 1 + \lambda \cdot \lambda) = (\lambda^2 \quad 2\lambda^{n-1} \ 0 \quad \lambda^2)$$

Induction base is already proven, now we need to prove that $(\lambda^n \quad n\lambda^{n-1} \ 0 \quad \lambda^n)^n (\lambda \quad 1 \ 0 \quad \lambda) = (\lambda^{n+1} \quad (n+1)\lambda^n \ 0 \quad \lambda^{n+1})$:

$$(\lambda^n \quad n\lambda^{n-1} \ 0 \quad \lambda^n)^n (\lambda \quad 1 \ 0 \quad \lambda) = (\lambda^n \cdot \lambda + n\lambda^{n-1} \cdot 0 \quad n\lambda^{n-1} \cdot \lambda + 1 \cdot \lambda \ 0 \cdot \lambda^n + \lambda^n \cdot 0 \quad 0 \cdot n\lambda^{n-1} + \lambda^n \cdot \lambda) =$$

$$= (\lambda^n \cdot \lambda + n\lambda^{n-1} \cdot 0 \quad n\lambda^{n-1} \cdot \lambda + 1 \cdot \lambda \ 0 \cdot \lambda^n + \lambda^n \cdot 0 \quad 0 \cdot n\lambda^{n-1} + \lambda^n \cdot \lambda) = (\lambda^{n+1} \quad (n+1)\lambda^n \ 0 \quad \lambda^{n+1})$$

Therefore, $(\lambda \quad 1 \ 0 \quad \lambda)^n = (\lambda^{n+1} \quad (n+1)\lambda^n \ 0 \quad \lambda^{n+1})$, q. e. d.

## Problem 7

Calculate $H^n$ of the following matrix:

$$H = \left(0 \quad 1 \quad 0 \quad \ldots \quad 0 \ 0 \quad 0 \quad 1 \quad \ldots \quad 0 \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots 0 \quad 0 \quad 0 \quad \ldots \quad 1 \ 0 \quad 0 \quad 0 \quad \ldots \quad 0\right)$$

### Solution

Attempt to square the matrix to see what happens:

$$H^2 = \left(0 \quad 1 \quad 0 \quad \ldots \quad 0 \ 0 \quad 0 \quad 1 \quad \ldots \quad 0 \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots 0 \quad 0 \quad 0 \quad \ldots \quad 1 \ 0 \quad 0 \quad 0 \quad \ldots \quad 0\right)^2 = \left(0 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 0 \ 0 \quad 0 \quad 0 \quad 1 \quad \ldots \quad 0 \vdots \quad \vdots \quad \vdots \quad \vdots\right.$$

As seen from the result matrix, the non-zero elements "shift" diagonally by one index on multiplication. We may write an equation that we think would determine the second matrix (consider $h_{ij}$ an element of the result matrix) and then prove it via induction:

$$h_{ij} = \{\, 1, \mathrm{if}\ j-i=n\ 0, \mathrm{otherwise}$$

The induction base is already proven ($H^1 = H$). Therefore, we need to check whether the induction step is true.

For some $n < k$, where $k$ is the matrix's dimension, we multiply $H^n$ by $H$ (positions are represented accurately in each of the multiplication steps):

$$H^n=\begin{pmatrix} 0 & \dots & 1 & 0 & 0 & \dots & 0 \ 0 & \dots & 0 & 1 & 0 & \dots & 0 \ 0 & \dots & 0 & 0 & 1 & \dots & 0 \ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \ 0 & \dots & 0 & 0 & 0 & \dots & 1 \ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \ 0 & 0 & 1 & \dots & 0 \ \vdots & \vdots & \vdots & \ddots & \vdots \ 0 & 0 & 0 & \dots & 1 \ 0 & 0 & 0 & \dots & 0 \end{pmatrix}=\begin{pmatrix} 0 & \dots & 0 & 1 & 0 & \dots & 0 \ 0 & \dots & 0 & 0 & 1 & \dots & 0 \

\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \ 0 & \dots & 0 & 0 & 0 & \dots & 1 \ 0 & \dots & 0 & 0 & 0 & \dots & 0 \ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}=H^{n-1}$$

Therefore, the induction step is true, q. e. d.

**NB:** The equation that determines the result matrix is identical to a zero matrix if $n \geq k$.

**Answer:** $H^n = (\, h_{ij}\,)$, where: $h_{ij} = \{1, \mathrm{if}\ j-i=n\ 0, \mathrm{otherwise}$