Just Great Tasks

Problem 1

Show that $\sup(A) = 1$, where $A = \left\{\frac{n}{n+1}, n \in \mathbb{N}\right\}$.

Solution

First, show that the upper bound exists. $\frac{n}{n+1} < 1$ since $n < n+1$ for all natural numbers. Therefore, there is some upper bound $= 1$, and we need to prove whether it is the supremum (if it is actually the minimum upper bound).

Assume $\sup(A) = 1$. For some arbitrary $\varepsilon > 0$, try to find some value such that for $a \in A$: $1 - \varepsilon < a < 1$. For instance, for $\varepsilon = 0.01, n = 100$, $\exists a = \frac{n}{n+1} = \frac{100}{101}$. Similarly, for any ε, we could find some n that would lie between $1 - \varepsilon$ and 1. Therefore, the assumption is true, q. e. d.

Problem 2

Show that $\inf(A) = 0$, where $A = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$.

Solution

Similarly to task above, first, show that the lower bound exists. $\frac{1}{n} > 0$ since $n \in \mathbb{N}$. Therefore, there is some lower bound $= 0$, and we need to prove whether it is the infinum (if it is actually the maximum lower bound).

Assume $\inf(A) = 0$. For some arbitrary $\varepsilon > 0$, try to find some value such that for $a \in A$: $0 < a < 0 + \varepsilon$. For instance, for $\varepsilon = 0.01, n = 101$, $\exists a = \frac{1}{n} = \frac{1}{101}$. Similarly, for any ε, we could find some n that would lie between 0 and $0 + \varepsilon$. Therefore, the assumption is true, q. e. d.

Problem 3

Let $A, B \subseteq \mathbb{R}$ be two non-empty subsets in set $\mathbb{R}$. Define *distance* from A to B as a positive number like the following:

$$d(A, B) := \inf_{x \in A, y \in B} |x - y|$$

Is it possible that $A \cap B = \varnothing$, but $d(A, B) = 0$?

Solution

Proof by example. Take some one-element set $A = \{a\}$, where $a \in \mathbb{R}$. Derive the second set as follows: $B = \mathbb{R} \setminus A$. In other words, there are two sets, the first of which is just a single point and the second of which is all points except for that single point. The intersection of these sets would be $A \cap B = \varnothing$.

Separate set B into two intervals: $B_1 = (-\infty, a), B_2 = (a, +\infty)$. These sets are symmetric over the point a and appear to infinitely converge to some value next to a from both sides (and since we define $d(A, B)$ as an absolute value, we could only consider one set, as the other would be identical to it by symmetry).

There is always some $\varepsilon > 0$ such that $a - \varepsilon \in B_2$; therefore, similarly to tasks above, since for $a \in A, a - \varepsilon \in B, |a - (a - \varepsilon)| = \varepsilon$. Since the distance (without the infinum just yet) is equal to ε , then out of all upper bounds, the smallest would be $0 \Rightarrow d(A, B) := \inf_{x \in A, y \in B} |x - y| = 0$. We have found such an example, q. e. d.

Answer: no :(

Problem 4

Examine the following recurrent sequences for convergence and find the limit if they do converge:

Preamble

Per Weierstrass, we need to find either the supremum or the infinum of each sequence.

If the sequence is non-descending and has an upper bound, then $\exists \lim_{n \to \infty} a_n = \sup\{a_n\}$.

If the sequence is non-ascending and has a lower bound, then $\exists \lim_{n \to \infty} a_n = \inf\{a_n\}$

Subproblem A

$$a_{n+1} = 1 - \frac{1}{4a_n}, a_1 = 1$$

Check whether the sequence is non-ascending by comparing two elements of the equation:

$$a_{n+1} - a_n = 1 - \frac{1}{4a_n} - a_n = \frac{4a_n - 4a_n^2 - 1}{4a_n} = \frac{-(2a_n - 1)^2}{4a_n}$$

The numerator of the fraction is always positive as it's a square; therefore, the sign of the equation depends only on a_n. If $a_n > 0$, then the entire sequence will be descending, which is true since $a_1 = 1$.

Great, the sequence has a lower bound (infinum), $\exists a = \lim_{n\to\infty} a_n = \inf\{a_n\}$. Try to evaluate it by assuming $a_n = a_{n+1} = a$, since we want the difference between two successive elements to be minimal (0):

$$a = \lim_{x\to\infty}\left(1 - \frac{1}{4a_n}\right) = 1 - \frac{1}{4a} \Rightarrow 4a^2 - 4a + 1 = 0 \Rightarrow (2a-1)^2 = 0 \Rightarrow a = \frac{1}{2}$$

Awesome, a_n never evaluates to negative numbers, which means that our value is the limit.

Answer: $\frac{1}{2}$

Subproblem B

$$a_{n+1} = \frac{4}{3}a_n - a_n^2, a_1 = \frac{1}{2}$$

Check whether the sequence is non-ascending by comparing two elements of the equation:

$$a_{n-1} - a_n = \frac{4}{3}a_n - a_n^2 - a_n = \frac{1}{3}a_n - a_n^2 = a_n\left(\frac{1}{3} - a_n\right)$$

Since the sign of the equation depends on a_n, then the equation above would evaluate to a negative value if $a_n > \frac{1}{3} \vee a_n < 0$, which means that the sequence is probably non-ascending (check later).

Awesome, the sequence has a lower bound, $\exists a = \lim_{n\to\infty} a_n = \inf\{a_n\}$. Attempt to evaluate it by assuming $a_n = a_{n+1} = a$, since we want the difference between two successive elements to be minimal (0):

$$a = \lim_{n\to\infty}\left(\frac{4}{3}a_n - a_n^2\right) = \frac{4}{3}a - a^2 \Rightarrow a^2 - \frac{1}{3}a = 0 \Rightarrow a\left(a - \frac{1}{3}\right) = 0$$

There appears to be at least two possible infinums ($a = 0 \vee a = \frac{1}{3}$); however, only one of them can exist, so we take the highest, which we approach from the top, i. e. $\inf\{a_n\} = \frac{1}{3} \Rightarrow \lim_{n\to\infty} a_n = \frac{1}{3}$.

Amazing, a_n is never $\in [0, \frac{1}{3}]$ so it is never ascending and is bounded.

Answer: $\frac{1}{3}$

Subproblem C

$$x_{n+1} = \frac{1}{m}\left((m-1)x_n + \frac{a}{x_n^{m-1}}\right), n \geq 1, m \in \mathbb{N}, x_1 > 0$$

Since this task is eerily similar to the last one, we could instantly try to find such values that the difference between them is minimal. Therefore, assume $x_{n+1} = x_n = x$:

$$x = \frac{1}{m}\left((m-1)x + \frac{a}{x^{m-1}}\right) \Rightarrow x = \frac{m-1}{m}x + \frac{a}{mx^{m-1}} \Rightarrow mx^m\left(\frac{m-1}{m} - 1\right) = a \Rightarrow$$

$$x = \sqrt[m]{a}$$

Since this value exists, the sequence has some kind of bound.

As for ascension/descention, if $x_n < \sqrt[m]{a}$, then the two elements would be in ascending order and if $x_n > \sqrt[m]{a}$, then they would be in descending order. This way, if the starting value $x_1 < \sqrt[m]{a}$, it would be ascend once, and starting $n = 2$, it would descend infinitely, approaching $\sqrt[m]{a}$. Otherwise, if $x_n > \sqrt[m]{a}$, the sequence would descend infinitely starting from $n = 1$.

Therefore, we may simply cut off the first element of the sequence and always take some $A_0 \subseteq \{x_n\}$ such that all its elements are descending from the first one. There is an infinite number of these descending subsets and they would be bound by $\inf\{a_n\} = \sqrt[m]{a}$, which is the limit: $\lim_{n\to\infty} x_n = \sqrt[m]{a}$.

Answer: $\sqrt[m]{a}$

Subproblem D

$$x_1 = a, x_2 = b, x_{n+1} = \frac{1}{2}(x_n + x_{n-1}), n \geq 1$$

Tough one, it's pretty obvious from messing around with Python that the limit is $\frac{2b+a}{3}$, but proving it is an entirely different case. Attempt to subtract one sequence element from the next to compare the elements:

$$x_{n+1} - x_n = \frac{1}{2}(x_n + x_{n-1}) - x_n = -\frac{1}{2}x_n + \frac{1}{2}x_{n-1} = \frac{1}{2}(x_{n-1} - x_n) = -\frac{1}{2}(x_n - x_{n-1})$$

Insane, we got the very same expression! Therefore,

$$x_{n+1} - x_n = \underbrace{(-1) \times \cdots \times (-1)}_{n-1} \overbrace{\frac{1}{2} \times \cdots \times \frac{1}{2}}^{n-1} (x_2 - x_1) = (-1)^{n-1} \frac{1}{2^{n-1}} (x_2 - x_1) = (-1)^{n-1} \frac{1}{2^{n-1}} (b - a) = (*)$$

Using some other funny tricks (add $\frac{1}{2} x_n$ to both sides), we notice that once again, it's the very same expression on both sides:

$$x_{n+1} = \frac{1}{2} x_n + \frac{1}{2} x_{n-1} \Rightarrow \underbrace{x_{n+1} + \frac{1}{2} x_n = x_n + \frac{1}{2} x_{n-1} = \cdots = x_2 + \frac{1}{2} x_1}_{n} = b + \frac{1}{2} a = (**)$$

Subtract $(**) - (*)$:

$$(**) = x_{n+1} + \frac{1}{2} x_n = b + \frac{1}{2} a$$

$$(*) = x_{n+1} - x_n = (-1)^{n-1} \frac{1}{2^{n-1}} (b - a)$$

$$\frac{3}{2} x_n = b + \frac{1}{2} a - (-1)^{n-1} \frac{1}{2^{n-1}} (b - a) = b + \frac{1}{2} a - \frac{(-1)^{n-1}}{2^{n-1}} (b - a)$$

$$x_n = \frac{2b + a}{3} - \frac{(-1)^{n-1}}{2^{n-1}} (b - a)$$

Great, we have a non-recursive limit, which we can calculate:

$$\lim_{n \to \infty} x_n = \lim_{x \to \infty} \left(\frac{2b + a}{3} - \frac{(-1)^{n-1}}{2^{n-1}} (b - a) \right) = \frac{2b + a}{3} - \lim_{x \to \infty} \left(\frac{(-1)^{n-1}}{2^{n-1}} (b - a) \right)$$

Per squeeze theorem,

$$0 < \frac{(-1)^{n-1}}{2^{n-1}} (b - a) < \frac{1}{n!}$$

$$\lim_{n \to \infty} 0 = \lim_{n \to \infty} \left(\frac{(-1)^{n-1}}{2^{n-1}} (b - a) \right) = \lim_{n \to \infty} \frac{1}{n!} = 0$$

Therefore,

$$\lim_{n \to \infty} x_n = \frac{2b + a}{3} - \lim_{x \to \infty} \left(\frac{(-1)^{n-1}}{2^{n-1}} (b - a) \right) = \frac{2b + a}{3}$$

Answer:

$$\frac{2b + a}{3}$$

Problem 5

Define sequence $\{a_n\}$ as $a_{n+1} = a_n^2 + a_n$. What should a_1 be in order for this sequence to have a limit?

Solution

Compare two successive elements:

$$a_{n+1} - a_n = a_n^2 + a_n - a_n = a_n^2$$

Since $a_n^2 \geq 0$, this sequence is non-descending. For it to have a limit per Weierstrass, it needs to have an upper bound. Assume that the difference between two elements of the sequence is so small that $a_n = a_{n+1} = a$:

$$a = a^2 + a \Rightarrow a^2 = 0 \Rightarrow a = 0$$

Okay, so if there is a limit, then it is equal to 0. $\lim_{n\to\infty} a_n = 0$. Now consider some options:

- if $a_n > 0$, then the sequence diverges as it infinitely increases and does not have an upper bound;
- if $a_n = 0$, then the sequence consists of a stable sequence of 0-s, and then $\lim_{x\to\infty}\{a_n\} = 0$;
- if $a_n < -1$, then the sequence diverges as it would infinitely increase from $n = 2$ since $a_n^2 - a_n > 0$;
- if $a_n = -1$, then the first element of the sequence is -1 and afterwards all elements are equal to $1 - 1 = 0$, so it suffers the same fate as $a_n = 0$;
- finally, if $-1 < a_n < 0$, then the sequence converges because it has a lower bound of $\inf\{a_n\} = -1$ (the sequence is ascending) and has an upper bound of $\sup\{a_n\} = 0$ per Weierstrass since we have proven that a limit of this sequence exists.

Answer: $\lim_{n\to\infty} a_n = 0, a_n \in [-1, 0]$