# Calculus, Homework 11

## Problem 1

Construct a closed bounded subset of the metric space that is not compact.

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Take the infinite set of natural numbers $\mathbb{N}$ and define the following metric:

$$d(x, y) = \begin{cases} 0, \text{if } x = y \\ 1, \text{if } x \neq y \end{cases}$$

In this set, all different elements have the same distance between them, so taking any $n \in \mathbb{N}$: $B(n, r), r \geqslant 1$, the ball would include all elements of $\mathbb{N}$, thus making the set bounded and closed (all limit points would be included in the set).

Now, check whether the set is compact by constructing the following cover that would effectively consist of single points:

$$\hat{\mathbb{N}} = \bigcup_{\substack{n \in A \\ 0<r<1}} (n - r, n + r) = \bigcup_{n \in A} \{n\}$$

Since $n \to \infty$, we can't physically create a subcover that would include all points in $\mathbb{N} \implies$ the set is not compact.

## Problem 2

Prove that the $\mathbb{R}$ with a euclidean metric is not a compact space.

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For $\mathbb{R}$ to be compact, we require there to be a finite subcover for any open cover.

Take the following infinite subcover:

$$\mathbb{R} = \bigcup_{r>0}^{\infty} (-r, r)$$

To prove that $\mathbb{R}$ is not compact, we need to show that no finite subset of that subcover covers entire $\mathbb{R}$.

Take some finite number $n$ of intervals $(-r_i, r_i)$:

$$\hat{\mathbb{R}} = \bigcup_{i=1}^{n} (-r_i, r_i)$$

Since there is a finite number of intervals, there is some $\sup \hat{\mathbb{R}} = r_j$ and $\inf \hat{\mathbb{R}} = -r_j$ for any $n$.

Take some $\varepsilon > 0$. We can add/subtract this epsilon to both supremum and infimum, arriving at some values $x_1, x_2 = r_i + \varepsilon, -r_i - \varepsilon \in \mathbb{R}$, but which are $\notin \hat{\mathbb{R}}$, which means it's impossible to choose a finite subset of an open cover for all case $\implies$ $\mathbb{R}$ is not a compact space.

## Problem 3

Let $GL_n(\mathbb{R})$ be $\{M \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \colon\ \det(M) \neq 0\}$. Show that $GL_n(\mathbb{R})$ is open in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ and for any matrix $X \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ find $(dF)_A(X)$, where $F(A)$ is $A^{-1}$. Consider a euclidean metric.

## Problem 4

Prove that if $(n+1)$th derivative of function $f \colon \mathbb{R} \to \mathbb{R}$ is equivalent to $0$, then $f$ is a polynomial of a degree not higher than $n$.

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This is relatively easily proven via induction.

**Induction base**: if $f' = 0$, then the function is a constant, which is a subclass of a polynomial.

**Induction hypothesis**: suppose that $f^{(n)} = 0 \implies f$ is a polynomial of a degree not higher than $n$.

**Induction step**: now suppose that $f^{(n+1)} = 0$. This implies that

$$f^{(n)} = c_n$$

$$f^{(n-1)} = c_n x + c_{n-1}$$

$$f^{(n-2)} = c_n x^2 + c_{n-1} x + c_{n-2}$$

$$\vdots$$

$$f' = c_1 + c_2 x + \cdots + c_n x^n$$

And it is certain that

$$f = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

which implies that for $n+1$ the function is a polynomial $\implies$ it is also a polynomial for $n$, q. e. d.

## Problem 5

How to calculate first $10$ digits of $e$?

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We need to calculate the Taylor series of the function $e^x$ until the next monomial would be less than $10^{-10}$.

Taylor series formula for $e^x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

To calculate $e$, we simply substitute $x = 1$ in the equation above. Let's estimate when the required accuracy would be achieved:

$$\frac{x^n}{n!} = \frac{1}{n!} < 10^{-10} \implies n! > 10^{10} \implies n < 14$$

Therefore, we need to calculate the sum of monomials until degree $n = 13$

```
from math import factorial

total = 0
for n in range(0, 14):
    total += 1 / factorial(n)
print(round(total, 10))
```

$$e = \sum_{n=0}^{13} \frac{1}{n!} = 2.7182818284$$

# Problem 6

Study the following functions for extermities using higher derivatives and schematically plot the graphs.

## Subproblem A

$$f(x) = \sin^3 x + \cos^2 x$$

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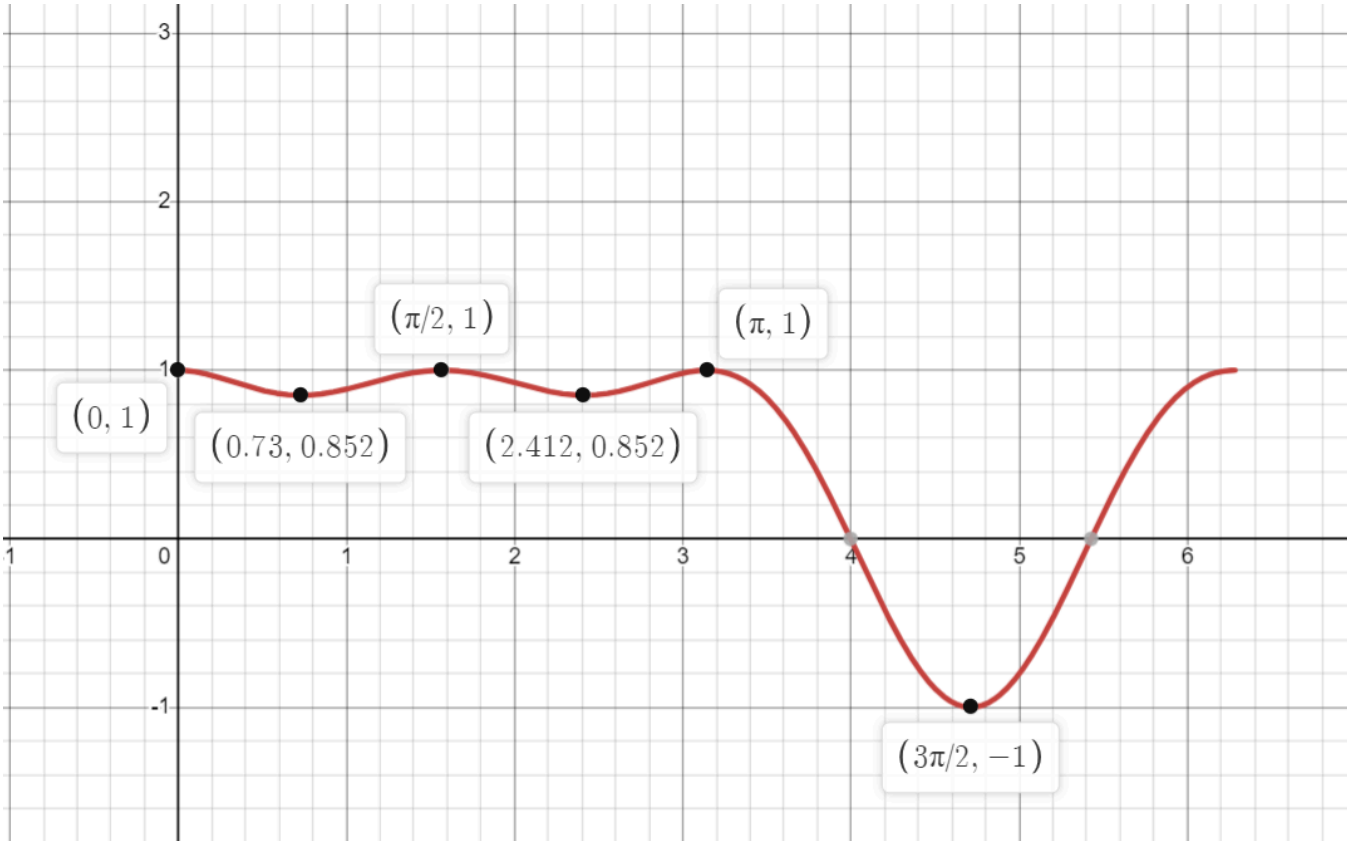
$$f'(x) = 3\sin^2 x \cos x - 2\cos x \sin x = \sin 2x \left(\frac{3}{2}\sin x - 1\right)$$

Since the function is periodic, only consider the extermities on a semi-interval $[0, 2\pi)$:

$$\left[\begin{array}{l} \sin 2x = 0 \\ \sin x = \frac{2}{3} \end{array}\right. \implies \left[\begin{array}{l} x = \frac{1}{2}\pi k_1 \\ x = \arcsin\frac{2}{3} + 2\pi k_2 \\ x = \pi - \arcsin\frac{2}{3} + 2\pi k_3 \end{array}\right. \implies \left[\begin{array}{l} x = 0 \\ x = \arcsin\frac{2}{3} \\ x = \frac{1}{2}\pi \\ x = \pi - \arcsin\frac{2}{3} \\ x = \pi \\ x = \frac{3}{2}\pi \end{array}\right.$$

Since the function is an oscillating wave, the maximums/minimums would be alternating starting from the maximum $\sup_1 f = f(0) = 0 + 1 = 1$ and minimum $\inf_1 f = f(\frac{1}{2}\pi) = \sin^3 x + 1 - \sin^2 x = 1 + \frac{8}{27} - \frac{4}{9} = \frac{23}{27}$ and so on: $\sup_2 f = 1, \sup_3 f = 1, \inf_2 f = \frac{23}{27}, \inf_3 f = -1.$

I will not torture myself with plotting the graph, so here we go:




> In this subproblem I didn't find it necessary to study the function in terms of whether it's convex or concave since it's a basic sine wave.

## Subproblem B

$$f(x) = x^{\frac{2}{3}} - (x^2 - 1)^{\frac{1}{3}}$$

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Firstly, the function is symmetric along the y-axis since we square the argument (the function value would be the same for $-x$ and $x$), so I'll only consider the function on a semi-interval $[0, +\infty)$.

$$f'(x) = \frac{2}{3}\frac{1}{\sqrt[3]{x}} - \frac{2}{3}\frac{x}{\sqrt[3]{x^2 - 1}}$$

The function is non-differentiable at $0$, nor at $1$, since the denominator cannot be equal to $0$. $x = 0$ is an edge case where the function is not smooth, and $x = 1$ is effectively a vertical tangent line to the function (which maintains smoothness, actually, but I'll check this a bit later).

The only solution in $\mathbb{R}^+$ for $f'(x) = 0$ exists if:

$$\frac{1}{\sqrt[3]{x}} = \frac{x}{\sqrt[3]{x^2-1}} \implies \sqrt[3]{x^2-1} = x\sqrt[3]{x} \implies x^2 - 1 = x^4$$

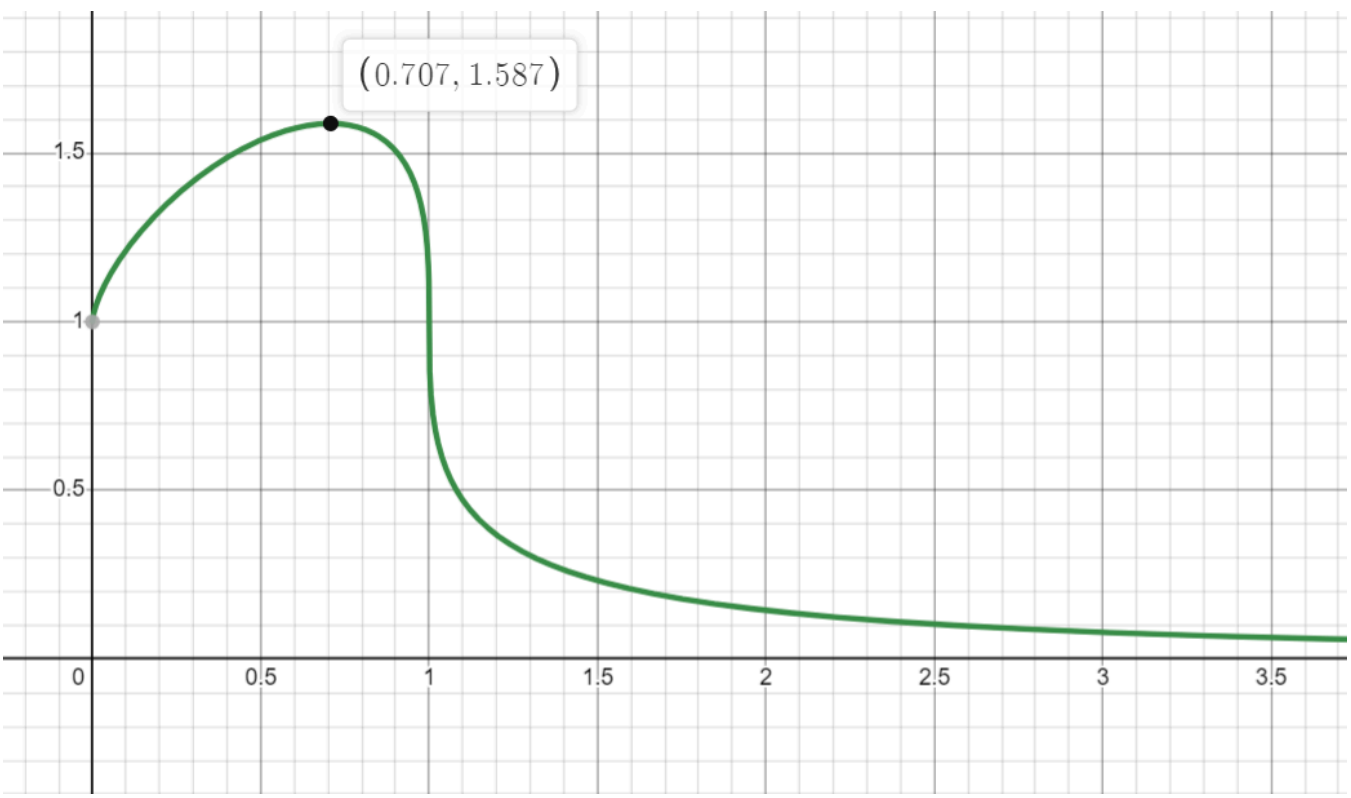
$$x^2 = \pm\frac{1}{2} \implies x = \frac{1}{\sqrt{2}}$$

Since for $x' > \frac{1}{\sqrt{2}}$ $f'(x')$, the value is negative, then

- function is descending on $(\frac{1}{\sqrt{2}}, +\infty)$
- function is ascending on $(0, \frac{1}{\sqrt{2}})$

which implies $\sup_1 f$ at $x = \frac{1}{\sqrt{2}}$

The second derivative does not have any zeros and it's too terrifying to calculate/check, but it has a really fun point $(1, 1)$, where the function is not differentiable but is smooth and continuous, which implies that the entire function consist of a single concave section succeeded by a convex section. Geometrically, the graph of the second derivative goes to $-\infty$ and then emerges from $+\infty$.




## Subproblem C

$$f(x) = \frac{x^2 - 5x + 6}{x^2 + 1}$$

$$f'(x) = \frac{(2x-5)(x^2+1) - 2x(x^2-5x+6)}{(x^2+1)} = \frac{5(x^2-2x-1)}{(x^2+1)^2}$$

Find the extremums:

$$f'(x) = 0 \implies x^2 - 2x - 1 = 0 \implies \left[\begin{array}{l} x = 1+\sqrt{2} \\ x = 1-\sqrt{2} \end{array}\right.$$

$$f''(x) = \frac{5(2x-2)(x^2+1)^2 - 20x(x^2-2x-1)(x^2+1)}{(x^2+1)^4} =$$

$$= \frac{-10(x+1)(x^2-4x+1)}{(x^2+1)^3}$$

$$f''(x) = 0 \implies (x+1)(x^2-4x+1) = 0 \implies \left[\begin{array}{l} x = 2+\sqrt{3} \\ x = 2-\sqrt{3} \\ x = -1 \end{array}\right.$$

Since we have $-10$ in the numerator of the second derivative, we start alternating intervals from the right starting from a concave interval:

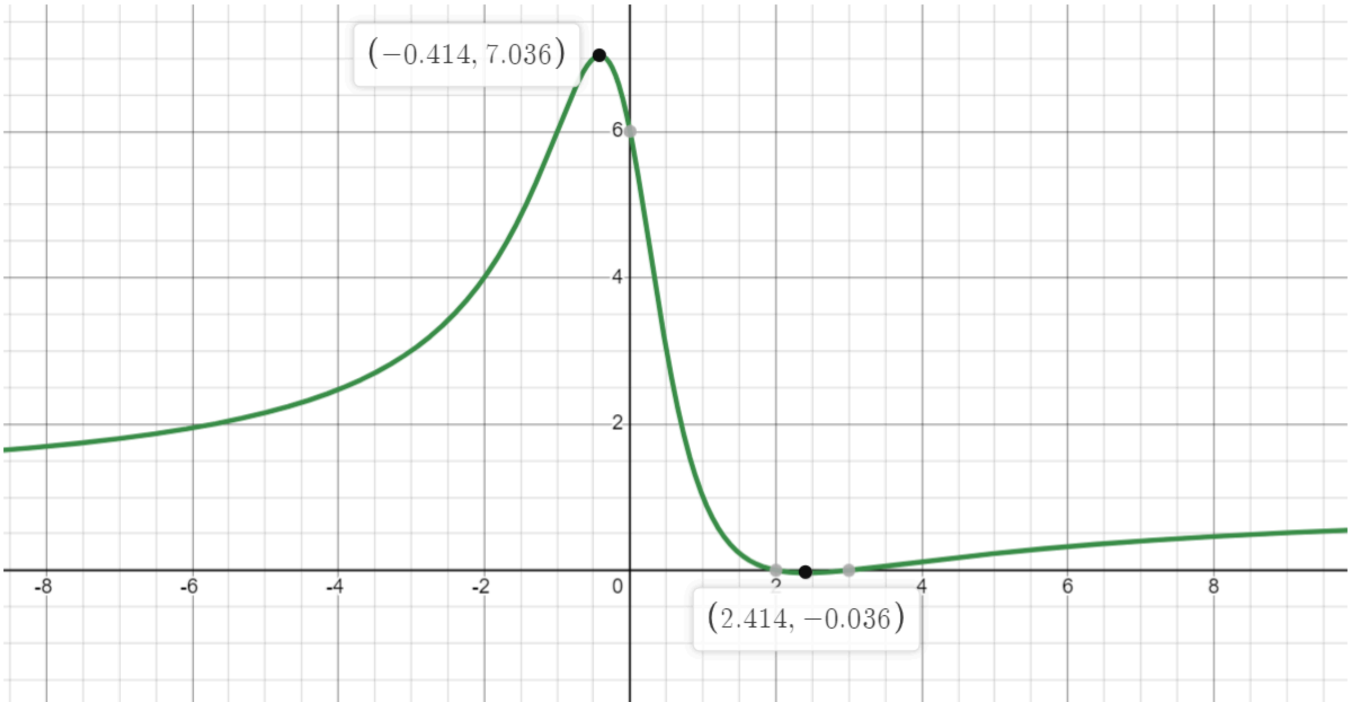
- concave on $(2+\sqrt{3}, +\infty)$
- convex on $(2-\sqrt{3}, 2+\sqrt{3})$
- concave on $(-1, 2-\sqrt{3})$
- convex on $(-\infty, -1)$

As for extremums, since the numberator of the first derivative is positive, we alternate intervals from the right starting from an ascending one:

- function is ascending on $(1+\sqrt{2}, +\infty)$
- function is descending on $(1-\sqrt{2}, 1+\sqrt{2})$
- function is ascending on $(-\infty, 1-\sqrt{2})$

which implies the extremums are:

- $\sup_1 f$ at $1-\sqrt{2}$
- $\inf_1 f$ at $1+\sqrt{2}$



(−0.414, 7.036)
(2.414, −0.036)
6
4
2
-8
-6
-4
-2
0
2
4
6
8