Linear Algebra, Individual Homework 7

Problem 1

Find the normalized form and some non-singular coordinate replacement for the following quadratic form over $\mathbb{R}$:

$$Q(x_1, x_2, x_3) = -81x_1^2 - 29x_2^2 - 101x_3^2 - 36x_1x_2 + 162x_1x_3 + 76x_2x_3$$

Corresponding matrix of the quadratic form is

$$\begin{pmatrix} -81 & -\frac{36}{2} & \frac{162}{2} \\ -\frac{36}{2} & -29 & \frac{76}{2} \\ \frac{162}{2} & \frac{76}{2} & -101 \end{pmatrix} = \begin{pmatrix} -81 & -18 & 81 \\ -18 & -29 & 38 \\ 81 & 38 & -101 \end{pmatrix}$$

Now use symmetric Gauss' method to find the normalized form:

$$\begin{pmatrix} -81 & -18 & 81 \\ -18 & -29 & 38 \\ 81 & 38 & -101 \end{pmatrix} \xrightarrow{(3):=(3)+1\times(1)} \begin{pmatrix} -81 & -18 & 0 \\ -18 & -29 & 20 \\ 0 & 20 & 20 \end{pmatrix} \xrightarrow{(2):=(2)-\frac{2}{9}\times(1)}$$

$$\begin{pmatrix} -81 & 0 & 0 \\ 0 & -25 & 20 \\ 0 & 20 & -20 \end{pmatrix} \xrightarrow{(3):=(3)+\frac{4}{5}\times(2)} \begin{pmatrix} -81 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

Since we have tracked all the transformations, we may get the following transformation matrix:

$$\begin{pmatrix} 1 & \frac{2}{9} & -1 \\ 0 & 1 & -\frac{4}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

which corresponds to the following replacement:

$$\begin{cases} y_1' = (x_1 + \frac{2}{9}x_2 - x_3) \\ y_2' = (x_2 - \frac{4}{5}x_3) \\ y_3' = x_3 \end{cases}$$

and we get the form:

$$Q(y_1, y_2, y_3) = -81y_1'^2 - 25y_2'^2 - 4y_3'^2$$

Introduce a replacement

$$\begin{cases} y_1 = 9y_1' \\ y_2 = 5y_2' \\ y_3 = 6y_3' \end{cases} \implies \begin{cases} y_1 = 9(x_1 + \frac{2}{9}x_2 - x_3) \\ y_2 = 5(x_2 - \frac{4}{5}x_3) \\ y_3 = 6x_3 \end{cases}$$

This means that the normalized form would be

$$-y_1^2 - y_2^2 - y_3^2$$

and the replacement is

$$\begin{cases} y_1 = 9(x_1 + \frac{2}{9}x_2 - x_3) \\ y_2 = 5(x_2 - \frac{4}{5}x_3) \\ y_3 = 2x_3 \end{cases} \implies \begin{cases} x_1 = \frac{y_1}{9} - \frac{2y_2}{45} + \frac{37y_3}{90} \\ x_2 = \frac{y_2}{5} + \frac{2y_3}{5} \\ x_3 = \frac{y_3}{2} \end{cases}$$

Problem 2

Determine the normalized form of the quadratic form depending on value b:

$$Q(x_1, x_2, x_3) = -9x_1^2 + (-7b+4)x_2^2 + (-7b-7)x_3^2 - 18x_1x_2 + 14x_1x_3 + 2(7b-1)x_2x_3$$

Firstly, use symmetric Gauss to arrive at some canonized form:

$$\begin{pmatrix} -9 & -9 & 7 \\ -9 & -7b+4 & 7b-1 \\ 7 & 7b-1 & -7b-7 \end{pmatrix} \xRightarrow{(2):=(2)-1\times(1)} \begin{pmatrix} -9 & 0 & 7 \\ 0 & -7b+13 & 7b-8 \\ 7 & 7b-8 & -7b-7 \end{pmatrix}$$

$$\xRightarrow{(3):=(3)+\frac{7}{9}\times(1)} \begin{pmatrix} -9 & 0 & 0 \\ 0 & -7b+13 & 7b-8 \\ 0 & 7b-8 & -7b-\frac{14}{9} \end{pmatrix} \xRightarrow{(3):=(3)+1\times(2)}$$

$$\begin{pmatrix} -9 & 0 & 0 \\ 0 & -7b+13 & 5 \\ 0 & 5 & -\frac{41}{9} \end{pmatrix} \xRightarrow{(2):=(2)+\frac{45}{41}\times(3)} \begin{pmatrix} -9 & 0 & 0 \\ 0 & -7b+\frac{758}{41} & 0 \\ 0 & 0 & -\frac{41}{9} \end{pmatrix}$$

Thus, the canonized quadratic form currently looks like

$$-9y_1'^2 + \left(-7b + \frac{758}{41}\right) y_2'^2 - \frac{41}{9} y_3'^2$$

We can easily normalize the first two coefficients to be -1, so the normalized form depends only on the second coefficient. Thus, the form would have normalized forms of

$$-y_1^2 + y_2^2 - y_3^2, \quad b < \frac{758}{287}$$

$$-y_1^2 - y_3^2, \quad b = \frac{758}{287}$$

$$-y_1^2 - y_2^2 - y_3^2, \quad b > \frac{758}{287}$$

Problem 3

In vector space $V = \mathbb{R}[x]_{\leq 3}$ consider a function

$$Q(f) = \int\limits_{-1}^{1} f^2 dx - \int\limits_{0}^{2} f^2 dx$$

Subproblem A

Prove that Q is a quadratic form in V.

First, write out the intergral using the Newton-Leibnitz formula for some $f = ax^3 + bx^2 + cx + d$:

$$Q(f) = \int_{-1}^{1} (ax^3 + bx^2 + cx + d)^2 dx - \int_{0}^{2} (ax^3 + bx^2 + cx + d)^2 dx =$$

$$\int (ax^3 + bx^2 + cx + d)^2 dx =$$

$$= \frac{a^2x^7}{7} + \frac{x^5(2ac + b^2)}{5} + \frac{x^4(ad + bc)}{2} + \frac{abx^6}{3} + \frac{x^3(2bd + c^2)}{3} + cdx^2 + d^2x$$

Now calculate all this garbage:

$$f(1) = \frac{a^2}{7} + \frac{ab}{3} + \frac{2ac}{5} + \frac{ad}{2} + \frac{b^2}{5} + \frac{bc}{2} + \frac{2bd}{3} + \frac{c^2}{3} + cd + d^2$$

$$f(-1) = -\frac{a^2}{7} + \frac{ab}{3} - \frac{2ac}{5} + \frac{ad}{2} - \frac{b^2}{5} + \frac{bc}{2} - \frac{2bd}{3} - \frac{c^2}{3} + cd - d^2$$

$$f(2) = \frac{128a^2}{7} + \frac{64ab}{3} + \frac{64ac}{5} + 8ad + \frac{32b^2}{5} + 8bc + \frac{16bd}{3} + \frac{8c^2}{3} + 4cd + 2d^2$$

$$f(0) = 0$$

Now,

$$
\begin{aligned}
Q(f) = f(1) - f(-1) - f(2) + f(0) = \\
&\frac{a^2}{7} + \frac{ab}{3} + \frac{2ac}{5} + \frac{ad}{2} + \frac{b^2}{5} + \frac{bc}{2} + \frac{2bd}{3} + \frac{c^2}{3} + cd + d^2 - \\
&(-\frac{a^2}{7} + \frac{ab}{3} - \frac{2ac}{5} + \frac{ad}{2} - \frac{b^2}{5} + \frac{bc}{2} - \frac{2bd}{3} - \frac{c^2}{3} + cd - d^2) - \\
&(\frac{128a^2}{7} + \frac{64ab}{3} + \frac{64ac}{5} + 8ad + \frac{32b^2}{5} + 8bc + \frac{16bd}{3} + \frac{8c^2}{3} + 4cd + 2d^2) \\
&= -18a^2 - \frac{64ab}{3} - 12ac - 8ad - 6b^2 - 8bc - 4bd - 2c^2 - 4cd
\end{aligned}
$$

which is a polynomial of mononomials of degree exactly 2, so this is definitely a quadratic form.

Subproblem B

Does such a basis $\mathbf{e} = (e_1, \dots, e_4)$ exist that quadratic form Q in coordinates $\mathbf{x} = (x_1, \dots, x_4)$ in this basis has a form of

$$
55x_1^2 - 58x_1x_2 + 120x_1x_3 + 46x_1x_4 + 20x_2^2 - 74x_2x_3 - 14x_2x_4 + 3x_3^2 - 10x_3x_4 + 12x_4^2
$$

If this basis exists, present it.

Previous form matrix:

$$
\begin{pmatrix}
-18 & -\frac{32}{3} & -6 & -4 \\
-\frac{32}{3} & -6 & -4 & -2 \\
-6 & -4 & -2 & -4 \\
-4 & -2 & -4 & 0
\end{pmatrix}
$$

Current form matrix:

$$
\begin{pmatrix}
55 & -29 & 60 & 23 \\
-29 & 20 & -37 & -7 \\
60 & -37 & 3 & -5 \\
23 & -7 & -5 & 12
\end{pmatrix}
$$

If their inertia coefficients would be different, then it would prove that it's impossible to find such a basis.

Use Jacobi's approach because it's the fastest one since I can easily force the computer to do it for me. First coefficients:

$$\begin{cases} \Delta_1 = -18 \\ \Delta_2 = -\frac{52}{9} \\ \Delta_3 = \frac{32}{9} \\ \Delta_4 = \frac{256}{9} \end{cases}$$

which implies that this form has a normalized form of

$$-x_1^2 + x_2^2 - x_3^2 + x_4^2$$

Second coefficients

$$\begin{cases} \Delta_1 = 55 \\ \Delta_2 = 259 \\ \Delta_3 = -17758 \\ \Delta_4 = -95481 \end{cases}$$

which implies that this form has a normalized form of

$$x_1^2 + x_2^2 - x_3^2 + x_4^2$$

> Technically this could have been checked by simply looking at the first coefficient, but alas, I have already forced the computer to suffer.

These normalized forms are different (their inertia coefficients are also different), which means that it's impossible to find such a basis that is wanted from us in the given conditions.

Problem 4

Determine all values of a, b when bilinear form

$$\beta(x,y) = (-9b+31)x_1y_1 + (3b-9)x_1y_2 + (-1.5a+5.5)x_1y_3 + (3b-9)x_2y_1 + 2x_2y_2 + (2b-4)x_2y_3 + (-3b+11)x_3y_1 + (2b-4)x_3y_2 + 3x_3y_3$$

defines a dot product in $\mathbb{R}^3$.

For a bilinear form to define a dot product, we need it to be symmetric and positively-defined.

Firstly, let's construct a matrix of this form:

$$\begin{pmatrix} -9b+31 & 3b-9 & -1.5a+5.5 \\ 3b-9 & 2 & 2b-4 \\ -3b+11 & 2b-4 & 3 \end{pmatrix}$$

To check when it is symmetric, solve the following system of equations:

$$\begin{cases} 3b-9=3b-9 \\ -3b+11=-1.5a+5.5 \implies \\ 2b-4=2b-4 \end{cases} \quad \begin{gathered} -3b+11=-1.5a+5.5 \implies \\ -6b+22=-3a+11 \implies 3a-6b=-11 \implies \\ a=2b-\frac{11}{3} \end{gathered}$$

Plug this back into the original matrix:

$$\begin{pmatrix} -9b+31 & 3b-9 & -3b+11 \\ 3b-9 & 2 & 2b-4 \\ -3b+11 & 2b-4 & 3 \end{pmatrix}$$

To check whether it's positively defined, we could use Sylvester's criterion:

$$\begin{cases} \Delta_1>0 \\ \Delta_2>0 \\ \Delta_3>0 \end{cases} \implies \begin{cases} \begin{vmatrix} -9b+31 \end{vmatrix} > 0 \\ \begin{vmatrix} -9b+31 & 3b-9 \\ 3b-9 & 2 \end{vmatrix} > 0 \\ \begin{vmatrix} -9b+31 & 3b-9 & -3b+11 \\ 3b-9 & 2 & 2b-4 \\ -3b+11 & 2b-4 & 3 \end{vmatrix} > 0 \end{cases} \implies$$

$$\begin{cases} 31-9b>0 \\ -9b^2+36b-19>0 \\ -b^2+4b-3>0 \end{cases} \implies \begin{cases} b<\frac{31}{9} \\ b\in(2-\frac{\sqrt{17}}{3}, 2+\frac{\sqrt{17}}{3}) \\ b\in(1,3) \end{cases} \implies b\in(1,3)$$

Thus, all pairs for which this would be a dot product are

$$(a,b)=\left(2b-\frac{11}{3}, b\right), \quad b\in(1,3)$$

Problem 5

Does a system of three vectors in $\mathbb{R}^3$ exist such that its Graham matrix is equal to

$$\begin{pmatrix} 25 & 16 & -82 \\ 16 & 18 & -68 \\ -82 & -68 & 300 \end{pmatrix}$$

exist? If it exists, then present such system. The dot product is standard.

Graham's matrix for a system of vectors $\langle v_1, v_2, v_3 \rangle$ looks like

$$\begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & v_1\cdot v_3\\ v_2\cdot v_1 & v_2\cdot v_2 & v_2\cdot v_3\\ v_3\cdot v_1 & v_3\cdot v_2 & v_3\cdot v_3\end{pmatrix} = \begin{pmatrix} 25 & 16 & -82\\ 16 & 18 & -68\\ -82 & -68 & 300\end{pmatrix}$$

Then this would effectively correspond to some kinda quadratic form.

$$25(v_1\cdot v_1) + 32(v_1\cdot v_2) - 164(v_1\cdot v_3) + 18(v_2\cdot v_2) - 136(v_2\cdot v_3) + 300(v_3\cdot v_3)$$

Use symmetric Gauss once again, tracking the transformations:

$$\begin{pmatrix} 25 & 16 & -82\\ 16 & 18 & -68\\ -82 & -68 & 300\end{pmatrix} \xrightarrow{(2):=(2)-\frac{16}{25}\times(1)} \begin{pmatrix} 25 & 0 & -82\\ 0 & \frac{194}{25} & -\frac{388}{25}\\ -82 & -\frac{388}{25} & 300\end{pmatrix} \xrightarrow{(3):=(3)+\frac{82}{25}\times(1)}$$

$$\begin{pmatrix} 25 & 0 & 0\\ 0 & \frac{194}{25} & -\frac{388}{25}\\ 0 & -\frac{388}{25} & \frac{776}{25}\end{pmatrix} \xrightarrow{(3):=(3)+2\times(2)} \begin{pmatrix} 25 & 0 & 0\\ 0 & \frac{194}{25} & 0\\ 0 & 0 & 0\end{pmatrix}$$

Canonized quadratic form:

$$25y_1^2 + \frac{194}{25}y_2^2$$

Transformation matrix:

$$C = \begin{pmatrix} 1 & -\frac{16}{25} & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$$

We get that this form is positively defined, so it definitely exists. Take its inverse:

$$C^{-1} = \begin{pmatrix} 1 & \frac{16}{25} & -\frac{82}{25}\\ 0 & 1 & -2\\ 0 & 0 & 1\end{pmatrix}$$

Now say that we have some vectors $\langle e_1, e_2, e_3\rangle = \langle (5,0,0), (0, \frac{\sqrt{194}}{5}, 0), (0,0,0)\rangle$

then we could get the required set of vectors by

$$\langle f_1, f_2, f_3\rangle = \langle (5,0,0), (0, \frac{\sqrt{194}}{5}, 0), (0,0,0)\rangle \begin{pmatrix} 1 & -\frac{16}{25} & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{pmatrix} =$$

$$= \langle (5,0,0), \frac{16}{25}(5,0,0) + (0, \frac{\sqrt{194}}{5}, 0), -\frac{82}{25}(5,0,0) - 2(0, \frac{\sqrt{194}}{5}, 0)\rangle =$$

$$= \left\langle (5,0,0), \left(\frac{16}{5}, \frac{\sqrt{194}}{5}, 0\right), \left(-\frac{82}{5}, -\frac{2\sqrt{194}}{5}, 0\right)\right\rangle$$