Calculus, Homework 14

Problem 1

Find extremums of function $z = x + 2y$ given that $x^2 + y^2 = 5$.

If $x^2 + y^2 = 5$, then $x = \pm\sqrt{5 - y^2}$.

$$z^+ = \sqrt{5 - y^2} + 2y, \quad z^- = -\sqrt{5 - y^2} + 2y$$

$$z_y'^+ = -\frac{y}{\sqrt{5 - y^2}} + 2 = 0, \quad z_y'^- = \frac{y}{\sqrt{5 - y^2}} + 2 = 0$$

$$2\sqrt{5 - y^2} = y, \quad 2\sqrt{5 - y^2} = -y$$

$$4(5 - y^2) = y^2$$

$$20 - 4y^2 = y^2$$

$$y^2 = 4 \implies y_1 = 2, y_2 = -2$$

Find corresponding pairs: $(x_1, y_1) = (2, 1), (x_1, y_2) = (-2, -1)$.

Obviously, $z(x_1, y_1) = 2 + 2 \times 1 = 5$ and $z(x_2, y_2) = -2 + 2 \times (-1) = -5$ and the maximum and the minimum respectively.

Answer: $\min z$ is at $(-2, -1)$ and $\max z$ is at $(2, 1)$.

Problem 2

Find the minimum and maximum values of function $z = x^2 - 2y^2 + 4xy - 6x + 5$ in area bounded by lines $x = 0, y = 0, x + y = 3$.

Find stationary points:

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2x + 4y - 6 = 0 \\ \\ \dfrac{\partial z}{\partial y} = -4y + 4x = 0 \end{cases} \implies \begin{cases} x = 3 - 2y \\ x = y \end{cases} \implies \begin{cases} x = 1 \\ y = 1 \end{cases}$$

Hesse matrix:

$$\mathbb{H}_{(1,1)} = \begin{pmatrix} \frac{\partial z}{\partial^2 x} & \frac{\partial z}{\partial x \partial y} \\ \frac{\partial z}{\partial x \partial y} & \frac{\partial z}{\partial^2 y} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 \\ 0 & -12 \end{pmatrix}$$

Thus, this is not an extremum.

Consider extremum on each of the lines.

First, along $x = 0$, find the extremum along $z = -2y^2 + 5$. For $y > 0$, this function is descending, which means that the maximum here is $z(0,0)_1 = 5$, and the minimum is $z(0,3)_2 = -13$

Second, along $y = 0$, find the extremum along $z = x^2 - 6x + 5$. This is a parabola facing up with its vertex being at $x = 3$, thus the function is descending within our limitations until $x = 3$. Thus, the maximum here is also $z(0,0)_1 = 5$ and the minimum is $z(3,0)_3 = -4$.

Third, for $x = 3 - y$ find the extremum along $z = (3-y)^2 - 2y^2 + 4(3-y)y - 6(3-y) + 5 = 9 - 6y + y^2 - 2y^2 + 12y - 4y^2 - 18 + 6y + 5 = -5y^2 + 12y - 4$. The maximum is at $z'_y = 0$: $-10y + 12 = 0 \implies y = \frac{6}{5}$. The corresponding maximum is $z(\frac{9}{5}, \frac{6}{5})_4 = \frac{16}{5}$.

Therefore, the minimum is -13 and it is at $(0,3)$ and the maximum is 5 and it is at $(0,0)$.

Answer: $\min z$ is -13 at $(0,3)$ and $\max z$ is 5 at $(0,0)$.

Problem 3

Find the maximum and minimum values of function $z = x^2y(4 - x - y)$ in area bounded by lines $x = 0, y = 0, x + y = 6$.

Seek stationary points:

$$\begin{cases} \dfrac{\partial z}{\partial x} = 2xy(4 - x - y) - x^2y \\ \\ \dfrac{\partial z}{\partial y} = x^2(4 - x - y) - x^2y \end{cases} \implies \begin{cases} 2xy(4 - x - y) - x^2y = 0 \\ x^2(4 - x - y) - x^2y = 0 \end{cases} \implies$$

Since $x, y > 0$:

$$\begin{cases} 2(4-x-y)-x=0 \\ (4-x-y)-y=0 \end{cases} \implies \begin{cases} 8-3x-2y=0 \\ 4-x-2y=0 \end{cases} \implies$$

$$\begin{cases} 4y=4 \\ x=4-2y \end{cases} \implies \begin{cases} x=2 \\ y=1 \end{cases}$$

Hesse matrix to find whether this is the maximum or the minimum:

$$\mathbb{H}_{(2,1)} = \begin{pmatrix} \frac{\partial z}{\partial^2 x} & \frac{\partial z}{\partial x \partial y} \\ \frac{\partial z}{\partial x \partial y} & \frac{\partial z}{\partial^2 y} \end{pmatrix} = \begin{pmatrix} 8y-6xy-2y^2 & 8x-3x^2-4xy \\ 8x-3x^2-4xy & -2x^2 \end{pmatrix} =$$

$$= \begin{pmatrix} 8-12-2 & 16-12-4 \\ 16-12-8 & -8 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 0 & -8 \end{pmatrix}$$

Thus, this is the maximum equal to $2^2(4-2-1)=4$. Check other key points along the lines in the given conditions to find the minumum. Values z for $x=0, y=0$ are all equal to 0, so check the values along $y=6-x$:

$$z = x^2(6-x)(4-x-6+x) = -2x^2(6-x)$$

$$z'_x = 0 \implies -4x + x^2 = 0$$

Divide by x since we already have $x=0$ accounted for and $x>0$:

$$x = 4 \implies y = 2$$

since it's a parabola with its branches down, then it's the minimum and the value at this point is $z = 4^2 \times 2(4-4-2) = -64$, which is our sought-for minimum at point $(4, 2)$

Answer: $\max z$ is 4 at $(2,1)$ and $\min z$ is -64 at $(4,2)$.

Problem 4

Find the dimensions of a right parallelepiped of given volume V such that its surface area is minimal.

We have

$$S(a,b,c) = 2(ab+bc+ac)$$

and

$$V = abc \implies c = \frac{V}{ab}$$

$$S(a,b,c) = 2\left(ab + b\frac{V}{ab} + a\frac{V}{ab}\right) = 2\left(ab + \frac{V}{a} + \frac{V}{b}\right)$$

Simply find the minimum of this function (we know that this function has a minimum since all a, b, c are positive):

$$\begin{cases} \dfrac{\partial S}{\partial a} = 2b - \dfrac{2V}{a^2} = 0 \\ \dfrac{\partial S}{\partial b} = 2a - \dfrac{2V}{b^2} = 0 \end{cases} \implies \begin{cases} a^2 b = V \\ ab^2 = V \end{cases} \implies$$

$$\begin{cases} b = \dfrac{V}{a^2} \\ a\dfrac{V^2}{a^4} = V \end{cases} \implies \begin{cases} b = \dfrac{V}{2a^2} \\ \dfrac{V}{a^3} = 1 \end{cases} \implies \begin{cases} a = \sqrt[3]{V} \\ b = \sqrt[3]{V} \end{cases}$$

Thus, $c = \dfrac{V}{\sqrt[3]{V^2}} = \sqrt[3]{V}$ and $a = b = c = \sqrt[3]{V}$.

Problem 5

Find the local extremums of function $u = x + y + z$ given that $xyz = 8$, $\frac{xy}{z} = 8$.

$$xyz = 8, \frac{xy}{z} = 8 \implies xy = 8, z = 1 \text{ or } xy = -8, z = -1$$

Therefore, we need to find local extremums of functions $u^+ = x + y + 1$ and $u^- = x + y - 1$ given that $xy = 8 \implies y = \dfrac{8}{x}$ and $xy = -8 \implies y = -\dfrac{8}{x}$:

$$u^+ = x + \frac{8}{x} + 1, \quad u^- = x - \frac{8}{x} - 1$$

Find stationary points of these two:

$$\frac{\partial u^+}{\partial x} = 1 - \frac{8}{x^2}, \quad \frac{\partial u^-}{\partial x} = 1 + \frac{8}{x^2}$$

$$1 - \frac{8}{x^2} = 0, \quad \varnothing$$

$$x^2 = 8 \implies x_1 = \sqrt{8}, \quad x_2 = -\sqrt{8}$$

only when $z = 1$.

Therefore, the extremums of the function are $(\sqrt{8}, \sqrt{8}, 1)$ and $(-\sqrt{8}, -\sqrt{8}, 1)$, and since $\frac{\partial u^+}{\partial^2 x} = \frac{8}{x^3}$, then Hasse matrices are:

$$\mathbb{H}_{(\sqrt{8},\sqrt{8},1)} = \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{H}_{(-\sqrt{8},-\sqrt{8},1)} = \begin{pmatrix} -\frac{1}{\sqrt{8}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

thus, the first point is a local minimum and the second point is the local maximum.

Problem 6

On an ellipse $x^2 + 4y^2 = 4$ there are two points, $A(-\sqrt{3}, \frac{1}{2})$ and $B(1, \frac{\sqrt{3}}{2})$. Find a third point on this ellipse such that the triangle ABC will have the largest area.

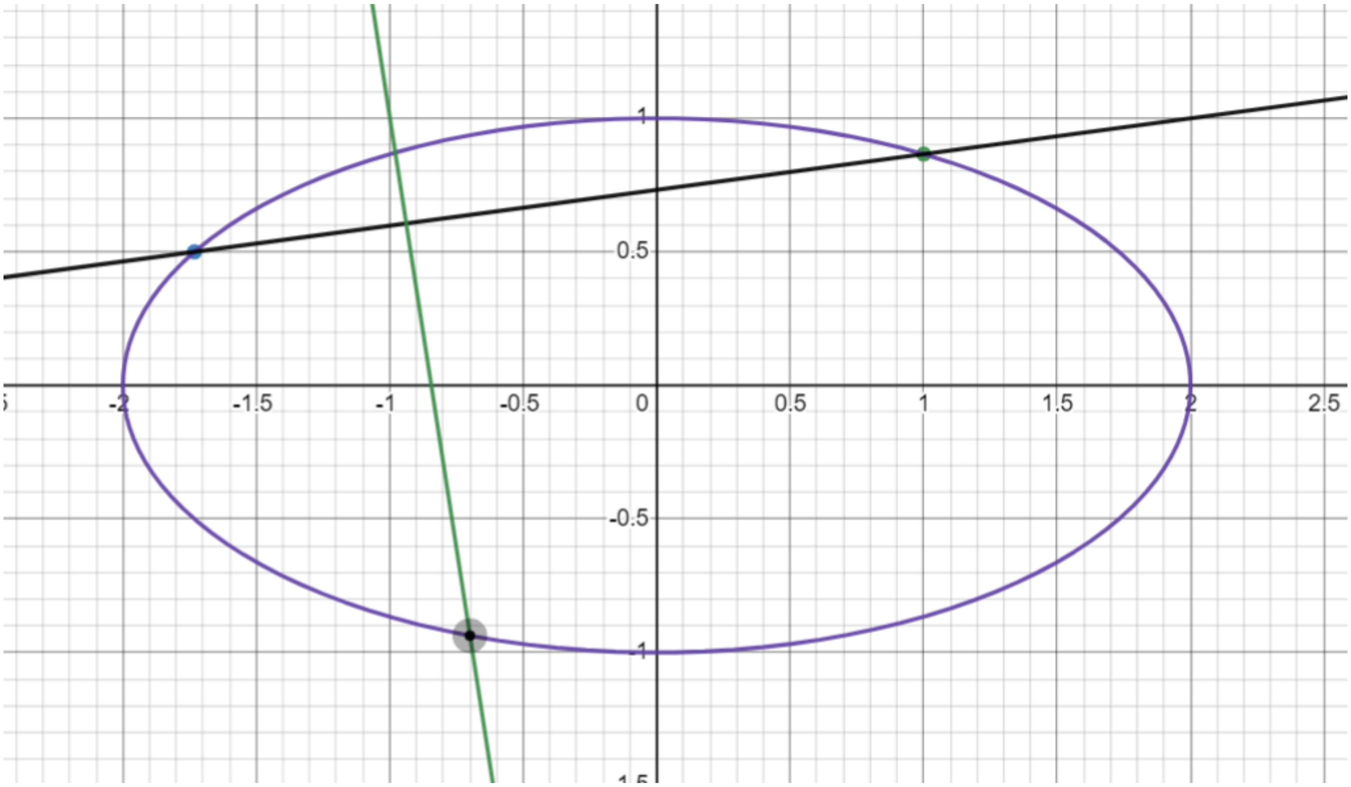
We need maximize the function $S(a, h)$, where a is the base of the triangle and h is its height formed by a perpendicular line to it passing through point C.

Write the linear equation of the that passes through points A, B, $y = kx + b$:

$$\begin{cases} \frac{1}{2} = -\sqrt{3}k + b \\ \frac{\sqrt{3}}{2} = k + b \end{cases} \implies \begin{cases} k = 1 - \frac{\sqrt{3}}{2} \\ b = \sqrt{3} - 1 \end{cases}$$

$$y = \left(1 - \frac{\sqrt{3}}{2}\right) x + \sqrt{3} - 1$$

The linear equation of the line perpendicular to it depending on point a over x taking into account that the optimal point is certainly in the half of the ellipse that is below oX (see figure below):



$$\begin{cases} k_2 = 1 - \frac{1}{k} = -3 - 2\sqrt{3} \\ b_2 = \frac{-\sqrt{4-a^2}}{2} - k_2 a, \quad a \in [-2, 2] \end{cases} \implies \begin{cases} k_2 = -3 - 2\sqrt{3} \\ b_2 = \frac{-\sqrt{4-a^2}}{2} + 3a + 2\sqrt{3}a \end{cases}$$

$$y = (-3 - 2\sqrt{3})x + \frac{-\sqrt{4-a^2}}{2} + 3a + 2\sqrt{3}a$$

Find coordinates of the intersection of those lines to find the distance between that intersection and point $\left(a, \dfrac{-\sqrt{4-a^2}}{2}\right)$. Then, we would just have to maximize over this distance since it would be the height of the triangle and its maximum height would give us maximum distance since its base has a constant length.

$$(-3 - 2\sqrt{3})x + \frac{-\sqrt{4-a^2}}{2} + 3a + 2\sqrt{3}a = \left(1 - \frac{\sqrt{3}}{2}\right)x + \sqrt{3} - 1$$

Coordinates

$$x = \frac{-\sqrt{4-a^2} + 4\sqrt{3}a + 6a - 2\sqrt{3} + 2}{8 + 3\sqrt{3}}$$

$$y = \left(1 - \frac{\sqrt{3}}{2}\right)\frac{-\sqrt{4-a^2} + 4\sqrt{3}a + 6a - 2\sqrt{3} + 2}{8 + 3\sqrt{3}} + \sqrt{3} - 1$$

Now, calculating the distance between that intersection point and point C, we get:

$$d_x = \frac{-\sqrt{4-a^2} + 4\sqrt{3}a + 6a - 2\sqrt{3} + 2}{8 + 3\sqrt{3}} - a$$

$$d_y = \left(1 - \frac{\sqrt{3}}{2}\right) \frac{-\sqrt{4-a^2} + 4\sqrt{3}a + 6a - 2\sqrt{3} + 2}{8 + 3\sqrt{3}} + \sqrt{3} - 1 - \frac{-\sqrt{4-a^2}}{2}$$

$$h = \sqrt{d_x^2 + d_y^2}$$

Maximizing this value (through wolfram because there's no way in hell I'm doing this by hand correctly first time), we magically get $a = \sqrt{2-\sqrt{3}}$, which means that the optimal point is

$$(x, y) = \left(\sqrt{2-\sqrt{3}}, -\frac{\sqrt{2+\sqrt{3}}}{2}\right)$$

Some visual proofs that everything is calculated correctly: