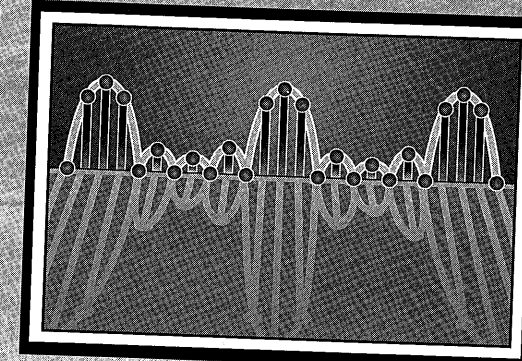
(e) It is also possible to find a 1^{st}-order all-pass system for $G(z^{-1})$ such that the prototype lowpass filter is transformed to a discrete-time highpass filter with cutoff ω_p. Note that such a transformation must map $Z^{-1} = e^{j\theta_p} \rightarrow z^{-1} = e^{j\omega_p}$ and also map $Z^{-1} = 1 \rightarrow z^{-1} = -1$; i.e., $\theta = 0$ maps to $\omega = \pi$. Find $G(z^{-1})$ for this transformation, and also, find an expression for α in terms of θ_p and ω_p.

(f) Using the same prototype filter and values for α as in part (d), sketch the frequency responses for the highpass filters resulting from the transformation you specified in part (e).

Similar, but more complicated, transformations can be used to convert the prototype lowpass filter $H_{lp}(Z)$ into bandpass and bandstop filters. Constantinides (1970) describes these transformations in more detail.

8 The Discrete Fourier Transform



8.0 INTRODUCTION

In Chapters 2 and 3, we discussed the representation of sequences and LTI systems in terms of the discrete-time Fourier and z-transforms, respectively. For finite-duration sequences, there is an alternative discrete-time Fourier representation, referred to as the *discrete Fourier transform* (DFT). The DFT is itself a sequence rather than a function of a continuous variable, and it corresponds to samples, equally spaced in frequency, of the DTFT of the signal. In addition to its theoretical importance as a Fourier representation of sequences, the DFT plays a central role in the implementation of a variety of digital signal-processing algorithms. This is because efficient algorithms exist for the computation of the DFT. These algorithms will be discussed in detail in Chapter 9. The application of the DFT to spectrum analysis will be described in Chapter 10.

Although several points of view can be taken toward the derivation and interpretation of the DFT representation of a finite-duration sequence, we have chosen to base our presentation on the relationship between periodic sequences and finite-length sequences. We begin by considering the Fourier series representation of periodic sequences. Although this representation is important in its own right, we are most often interested in the application of Fourier series results to the representation of finite-length sequences. We accomplish this by constructing a periodic sequence for which each period is identical to the finite-length sequence. The Fourier series representation of the periodic sequence then corresponds to the DFT of the finite-length sequence. Thus, our approach is to define the Fourier series representation for periodic sequences and to study the properties of such representations. Then, we repeat essentially the same derivations, assuming that the sequence to be represented is a finite-length sequence.

This approach to the DFT emphasizes the fundamental inherent periodicity of the DFT representation and ensures that this periodicity is not overlooked in applications of the DFT.

8.1 REPRESENTATION OF PERIODIC SEQUENCES: THE DISCRETE FOURIER SERIES

Consider a sequence $\tilde{x}[n]$ that is periodic[1] with period N, so that $\tilde{x}[n] = \tilde{x}[n + rN]$ for any integer values of n and r. As with continuous-time periodic signals, such a sequence can be represented by a Fourier series corresponding to a sum of harmonically related complex exponential sequences, i.e., complex exponentials with frequencies that are integer multiples of the fundamental frequency $(2\pi/N)$ associated with the periodic sequence $\tilde{x}[n]$. These periodic complex exponentials are of the form

$$e_k[n] = e^{j(2\pi/N)kn} = e_k[n + rN], \tag{8.1}$$

where k is any integer, and the Fourier series representation then has the form[2]

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k]e^{j(2\pi/N)kn}. \tag{8.2}$$

The Fourier series representation of a continuous-time periodic signal generally requires infinitely many harmonically related complex exponentials, whereas the Fourier series for any discrete-time signal with period N requires only N harmonically related complex exponentials. To see this, note that the harmonically related complex exponentials $e_k[n]$ in Eq. (8.1) are identical for values of k separated by N; i.e., $e_0[n] = e_N[n]$, $e_1[n] = e_{N+1}[n]$, and, in general,

$$e_{k+\ell N}[n] = e^{j(2\pi/N)(k+\ell N)n} = e^{j(2\pi/N)kn}e^{j2\pi\ell n} = e^{j(2\pi/N)kn} = e_k[n], \tag{8.3}$$

where ℓ is any integer. Consequently, the set of N periodic complex exponentials $e_0[n]$, $e_1[n], \ldots, e_{N-1}[n]$ defines all the distinct periodic complex exponentials with frequencies that are integer multiples of $(2\pi/N)$. Thus, the Fourier series representation of a periodic sequence $\tilde{x}[n]$ need contain only N of these complex exponentials. For notational convenience, we choose k in the range of 0 to $N-1$; hence, Eq. (8.2) has the form

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)kn}. \tag{8.4}$$

However, choosing k to range over any full period of $\tilde{X}[k]$ would be equally valid.

To obtain the sequence of Fourier series coefficients $\tilde{X}[k]$ from the periodic sequence $\tilde{x}[n]$, we exploit the orthogonality of the set of complex exponential sequences.

[1]Henceforth, we will use the tilde (˜) to denote periodic sequences whenever it is important to clearly distinguish between periodic and aperiodic sequences.

[2]The multiplicative constant $1/N$ is included in Eq. (8.2) for convenience. It could also be absorbed into the definition of $\tilde{X}[k]$.

After multiplying both sides of Eq. (8.4) by $e^{-j(2\pi/N)rn}$ ar
$n = N-1$, we obtain

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2$$

After interchanging the order of summation on the right-h

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{k=0}^{N-1}\tilde{X}[k]\left[\frac{1}{N}\sum_{n=0}^{N-1}e^{j(2\right.$$

The following identity expresses the orthogonality of the co

$$\frac{1}{N}\sum_{n=0}^{N-1}e^{j(2\pi/N)(k-r)n} = \begin{cases}1, & k-r = mN,\\ 0, & \text{otherwise.}\end{cases}$$

This identity can easily be proved (see Problem 8.54), an
summation in brackets in Eq. (8.6), the result is

$$\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)rn} = \tilde{X}[r].$$

Thus, the Fourier series coefficients $\tilde{X}[k]$ in Eq. (8.4) are
relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)kn}.$$

Note that the sequence $\tilde{X}[k]$ defined in Eq. (8.9) is also perio
is evaluated outside the range $0 \le k \le N-1$; i.e., $\tilde{X}[0] =$
and, more generally,

$$\tilde{X}[k+N] = \sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)(k+N)n}$$

$$= \left(\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j(2\pi/N)kn}\right)e^{-j2\pi}$$

for any integer k.

The Fourier series coefficients can be interpreted to be
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After multiplying both sides of Eq. (8.4) by $e^{-j(2\pi/N)rn}$ and summing from $n = 0$ to $n = N-1$, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]e^{j(2\pi/N)(k-r)n}. \tag{8.5}$$

After interchanging the order of summation on the right-hand side, Eq. (8.5) becomes

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \sum_{k=0}^{N-1} \tilde{X}[k]\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n}\right]. \tag{8.6}$$

The following identity expresses the orthogonality of the complex exponentials:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n} = \begin{cases} 1, & k - r = mN, \quad m \text{ an integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.7}$$

This identity can easily be proved (see Problem 8.54), and when it is applied to the summation in brackets in Eq. (8.6), the result is

$$\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)rn} = \tilde{X}[r]. \tag{8.8}$$

Thus, the Fourier series coefficients $\tilde{X}[k]$ in Eq. (8.4) are obtained from $\tilde{x}[n]$ by the relation

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}. \tag{8.9}$$

Note that the sequence $\tilde{X}[k]$ defined in Eq. (8.9) is also periodic with period N if Eq. (8.9) is evaluated outside the range $0 \le k \le N-1$; i.e., $\tilde{X}[0] = \tilde{X}[N]$, $\tilde{X}[1] = \tilde{X}[N+1]$, and, more generally,

$$\begin{aligned}\tilde{X}[k+N] &= \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)(k+N)n} \\ &= \left(\sum_{n=0}^{N-1} \tilde{x}[n]e^{-j(2\pi/N)kn}\right)e^{-j2\pi n} = \tilde{X}[k],\end{aligned}$$

for any integer k.

The Fourier series coefficients can be interpreted to be a sequence of finite length, given by Eq. (8.9) for $k = 0, \ldots, (N-1)$, and zero otherwise, or as a periodic sequence defined for all k by Eq. (8.9). Clearly, both of these interpretations are acceptable, since in Eq. (8.4) we use only the values of $\tilde{X}[k]$ for $0 \le k \le (N-1)$. An advantage to interpreting the Fourier series coefficients $\tilde{X}[k]$ as a periodic sequence is that there is then a duality between the time and frequency domains for the Fourier series representation of periodic sequences. Equations (8.9) and (8.4) together are an analysis–synthesis pair and will be referred to as the *discrete Fourier series* (DFS) representation of a periodic sequence.

For convenience in notation, these equations are often written in terms of the complex quantity

$$W_N = e^{-j(2\pi/N)}. \tag{8.10}$$

With this notation, the DFS analysis–synthesis pair is expressed as follows:

$$\text{Analysis equation:} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}. \tag{8.11}$$

$$\text{Synthesis equation:} \quad \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{8.12}$$

In both of these equations, $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences. We will sometimes find it convenient to use the notation

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \tag{8.13}$$

to signify the relationships of Eqs. (8.11) and (8.12). The following examples illustrate the use of those equations.

Example 8.1 DFS of a Periodic Impulse Train

We consider the periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \begin{cases} 1, & n = rN, \quad r \text{ any integer}, \\ 0, & \text{otherwise}. \end{cases} \tag{8.14}$$

Since $\tilde{x}[n] = \delta[n]$ for $0 \le n \le N-1$, the DFS coefficients are found, using Eq. (8.11), to be

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1. \tag{8.15}$$

In this case, $\tilde{X}[k] = 1$ for all k. Thus, substituting Eq. (8.15) into Eq. (8.12) leads to the representation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}. \tag{8.16}$$

Example 8.1 produced a useful representation of a periodic impulse train in terms of a sum of complex exponentials, wherein all the complex exponentials have the same magnitude and phase and add to unity at integer multiples of N and to zero for all other integers. If we look closely at Eqs. (8.11) and (8.12), we see that the two equations are very similar, differing only in a constant multiplier and the sign of the exponents. This duality between the periodic sequence $\tilde{x}[n]$ and its DFS coefficients $\tilde{X}[k]$ is illustrated in the following example.

Example 8.2 Duality in the

In this example, the DFS coefficients

$$\tilde{Y}[k] = \sum_{r=-\infty}^{\infty}$$

Substituting $\tilde{Y}[k]$ into Eq. (8.12) gives

$$\tilde{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} N\delta[k$$

In this case, $\tilde{y}[n] = 1$ for all n. Compar
of Example 8.1, we see that $\tilde{Y}[k] = N$
show that this example is a special cas

If the sequence $\tilde{x}[n]$ is equal to unity ov
tain a closed-form expression for the DFS coe
example.

Example 8.3 The DFS of a Pe

For this example, $\tilde{x}[n]$ is the sequence
From Eq. (8.11),

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum$$

This finite sum has the closed form

$$\tilde{X}[k] = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4}$$

The magnitude and phase of the periodic

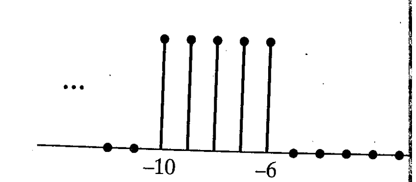

Figure 8.1 Periodic sequence with pe
representation is to be computed.

With this notation, the DFS analysis–synthesis pair is expressed as follows:

$$\text{Analysis equation:} \quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}. \tag{8.11}$$

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In both of these equations, $\tilde{X}[k]$ and $\tilde{x}[n]$ are periodic sequences. We will sometimes find it convenient to use the notation

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k] \tag{8.13}$$

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Since $\tilde{x}[n] = \delta[n]$ for $0 \le n \le N-1$, the DFS coefficients are found, using Eq. (8.11), to be

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = W_N^0 = 1. \tag{8.15}$$

In this case, $\tilde{X}[k] = 1$ for all k. Thus, substituting Eq. (8.15) into Eq. (8.12) leads to the representation

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN] = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N} \sum_{k=0}^{N-1} e^{j(2\pi/N)kn}. \tag{8.16}$$

Example 8.1 produced a useful representation of a periodic impulse train in terms of a sum of complex exponentials, wherein all the complex exponentials have the same magnitude and phase and add to unity at integer multiples of N and to zero for all other integers. If we look closely at Eqs. (8.11) and (8.12), we see that the two equations are very similar, differing only in a constant multiplier and the sign of the exponents. This duality between the periodic sequence $\tilde{x}[n]$ and its DFS coefficients $\tilde{X}[k]$ is illustrated in the following example.

Example 8.2 Duality in the DFS

In this example, the DFS coefficients are a periodic impulse train:

$$\tilde{Y}[k] = \sum_{r=-\infty}^{\infty} N\delta[k - rN].$$

Substituting $\tilde{Y}[k]$ into Eq. (8.12) gives

$$\tilde{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} N\delta[k] W_N^{-kn} = W_N^{-0} = 1.$$

In this case, $\tilde{y}[n] = 1$ for all n. Comparing this result with the results for $\tilde{x}[n]$ and $\tilde{X}[k]$ of Example 8.1, we see that $\tilde{Y}[k] = N\tilde{x}[k]$ and $\tilde{y}[n] = \tilde{X}[n]$. In Section 8.2.3, we will show that this example is a special case of a more general duality property.

If the sequence $\tilde{x}[n]$ is equal to unity over only part of one period, we can also obtain a closed-form expression for the DFS coefficients. This is illustrated by the following example.

Example 8.3 The DFS of a Periodic Rectangular Pulse Train

For this example, $\tilde{x}[n]$ is the sequence shown in Figure 8.1, whose period is $N = 10$. From Eq. (8.11),

$$\tilde{X}[k] = \sum_{n=0}^{4} W_{10}^{kn} = \sum_{n=0}^{4} e^{-j(2\pi/10)kn}. \tag{8.17}$$

This finite sum has the closed form

$$\tilde{X}[k] = \frac{1 - W_{10}^{5k}}{1 - W_{10}^{k}} = e^{-j(4\pi k/10)} \frac{\sin(\pi k/2)}{\sin(\pi k/10)}. \tag{8.18}$$

The magnitude and phase of the periodic sequence $\tilde{X}[k]$ are shown in Figure 8.2.

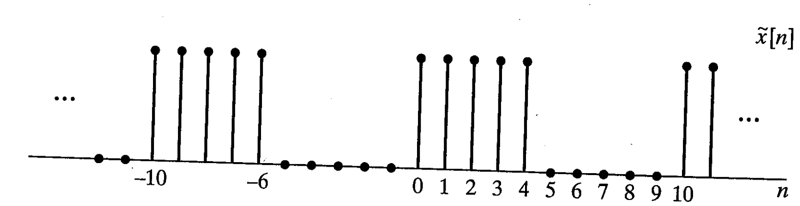


Figure 8.1 Periodic sequence with period $N = 10$ for which the Fourier series representation is to be computed.

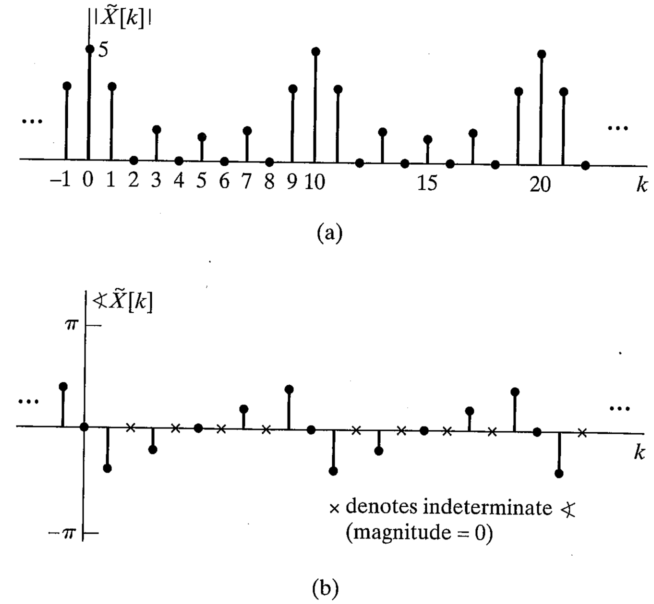


Figure 8.2 Magnitude and phase of the Fourier series coefficients of the sequence of Figure 8.1.

We have shown that any periodic sequence can be represented as a sum of complex exponential sequences. The key results are summarized in Eqs. (8.11) and (8.12). As we will see, these relationships are the basis for the DFT, which focuses on finite-length sequences. Before discussing the DFT, however, we will consider some of the basic properties of the DFS representation of periodic sequences in Section 8.2, and then, in Section 8.3, we will show how we can use the DFS representation to obtain a DTFT representation of periodic signals.

8.2 PROPERTIES OF THE DFS

Just as with Fourier series and Fourier and Laplace transforms for continuous-time signals, and with discrete-time Fourier and z-transforms for nonperiodic sequences, certain properties of the DFS are of fundamental importance to its successful use in signal-processing problems. In this section, we summarize these important properties. It is not surprising that many of the basic properties are analogous to properties of the z-transform and DTFT. However, we will be careful to point out where the periodicity of both $\tilde{x}[n]$ and $\tilde{X}[k]$ results in some important distinctions. Furthermore, an exact

8.2.1 Linearity

Consider two periodic sequences $\tilde{x}$

and

Then

$a\tilde{x}_1[n] + b\tilde{x}$

This linearity property follows imm

8.2.2 Shift of a Sequence

If a periodic sequence $\tilde{x}[n]$ has Fo
version of $\tilde{x}[n]$, and

$\tilde{x}[n -$

The proof of this property is consid
greater than or equal to the period
domain from a shorter shift m_1 such
and $0 \le m_1 \le N-1$. (Another w
equivalently, m_1 is the remainder wh
this representation of m, $W_N^{km} = W_N^{k}$
the time domain is also manifest in t

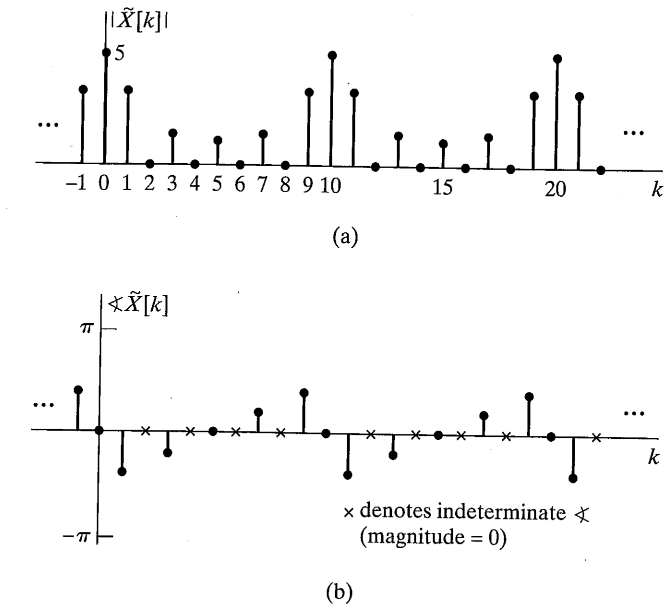
Because the sequence of Four
periodic sequence, a similar result a
integer ℓ. Specifically,

$W_N^{-n\ell}\tilde{x}$

Note the difference in the sign of the

8.2.3 Duality

Because of the strong similarity betw
in continuous time, there is a duality
However, for the DTFT of aperiodic
signals and their Fourier transforms
discrete-time signals are, of course, a
ways periodic functions of a continuo



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8.2.1 Linearity

Consider two periodic sequences $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$, both with period N, such that

$$\tilde{x}_1[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k], \tag{8.19a}$$

and

$$\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_2[k]. \tag{8.19b}$$

Then

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \overset{\mathcal{DFS}}{\longleftrightarrow} a\tilde{X}_1[k] + b\tilde{X}_2[k]. \tag{8.20}$$

This linearity property follows immediately from the form of Eqs. (8.11) and (8.12).

8.2.2 Shift of a Sequence

If a periodic sequence $\tilde{x}[n]$ has Fourier coefficients $\tilde{X}[k]$, then $\tilde{x}[n-m]$ is a shifted version of $\tilde{x}[n]$, and

$$\tilde{x}[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} W_N^{km}\tilde{X}[k]. \tag{8.21}$$

The proof of this property is considered in Problem 8.55. Note that any shift that is greater than or equal to the period (i.e., $m \geq N$) cannot be distinguished in the time domain from a shorter shift m_1 such that $m = m_1 + m_2N$, where m_1 and m_2 are integers and $0 \leq m_1 \leq N-1$. (Another way of stating this is that $m_1 = m$ modulo N or, equivalently, m_1 is the remainder when m is divided by N.) It is easily shown that with this representation of m, $W_N^{km} = W_N^{km_1}$; i.e., as it must be, the ambiguity of the shift in the time domain is also manifest in the frequency-domain representation.

Because the sequence of Fourier series coefficients of a periodic sequence is a periodic sequence, a similar result applies to a shift in the Fourier coefficients by an integer ℓ. Specifically,

$$W_N^{-n\ell}\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k-\ell]. \tag{8.22}$$

Note the difference in the sign of the exponents in Eqs. (8.21) and (8.22).

8.2.3 Duality

Because of the strong similarity between the Fourier analysis and synthesis equations in continuous time, there is a duality between the time domain and frequency domain. However, for the DTFT of aperiodic signals, no similar duality exists, since aperiodic signals and their Fourier transforms are very different kinds of functions: Aperiodic discrete-time signals are, of course, aperiodic sequences, whereas their DTFTs are always periodic functions of a continuous frequency variable.

From Eqs. (8.11) and (8.12), we see that the DFS analysis and synthesis equations differ only in a factor of $1/N$ and in the sign of the exponent of W_N. Furthermore, a periodic sequence and its DFS coefficients are the same kinds of functions; they are both periodic sequences. Specifically, taking account of the factor $1/N$ and the difference in sign in the exponent between Eqs. (8.11) and (8.12), it follows from Eq. (8.12) that

$$N\tilde{x}[-n] = \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{kn} \tag{8.23}$$

or, interchanging the roles of n and k in Eq. (8.23),

$$N\tilde{x}[-k] = \sum_{n=0}^{N-1} \tilde{X}[n] W_N^{nk}. \tag{8.24}$$

We see that Eq. (8.24) is similar to Eq. (8.11). In other words, the sequence of DFS coefficients of the periodic sequence $\tilde{X}[n]$ is $N\tilde{x}[-k]$, i.e., the original periodic sequence in reverse order and multiplied by N. This duality property is summarized as follows: If

$$\tilde{x}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}[k], \tag{8.25a}$$

then

$$\tilde{X}[n] \overset{\mathcal{DFS}}{\longleftrightarrow} N\tilde{x}[-k]. \tag{8.25b}$$

8.2.4 Symmetry Properties

As we discussed in Section 2.8, the Fourier transform of an aperiodic sequence has a number of useful symmetry properties. The same basic properties also hold for the DFS representation of a periodic sequence. The derivation of these properties, which is similar in style to the derivations in Chapter 2, is left as an exercise. (See Problem 8.56.) The resulting properties are summarized for reference as properties 9–17 in Table 8.1 in Section 8.2.6.

8.2.5 Periodic Convolution

Let $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ be two periodic sequences, each with period N and with DFS coefficients denoted by $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$, respectively. If we form the product

$$\tilde{X}_3[k] = \tilde{X}_1[k]\tilde{X}_2[k], \tag{8.26}$$

then the periodic sequence $\tilde{x}_3[n]$ with Fourier series coefficients $\tilde{X}_3[k]$ is

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]. \tag{8.27}$$

This result is not surprising, since our previous experience with transforms suggests that multiplication of frequency-domain functions corresponds to convolution of time-domain functions and Eq. (8.27) looks very much like a convolution sum. Equation (8.27) involves the summation of values of the product of $\tilde{x}_1[m]$ with $\tilde{x}_2[n-m]$, which is a time-reversed and time-shifted version of $\tilde{x}_2[m]$, just as in aperiodic discrete convo-

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to as a *periodic convolution*. Just as with aperiodic convolution, periodic convolution is commutative; i.e.,

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_2[m]\tilde{x}_1[n-m]. \tag{8.28}$$

To demonstrate that $\tilde{X}_3[k]$, given by Eq. (8.26), is the sequence of Fourier coefficients corresponding to $\tilde{x}_3[n]$ given by Eq. (8.27), let us first apply Eq. (8.11), the DFS analysis equation, to Eq. (8.27) to obtain

$$\tilde{X}_3[k] = \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \right) W_N^{kn}, \tag{8.29}$$

which, after we interchange the order of summation, becomes

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \left(\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} \right). \tag{8.30}$$

The inner sum on the index n is the DFS for the shifted sequence $\tilde{x}_2[n-m]$. Therefore, from the shifting property of Section 8.2.2, we obtain

$$\sum_{n=0}^{N-1} \tilde{x}_2[n-m] W_N^{kn} = W_N^{km} \tilde{X}_2[k],$$

which can be substituted into Eq. (8.30) to yield

$$\tilde{X}_3[k] = \sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \tilde{X}_2[k] = \left(\sum_{m=0}^{N-1} \tilde{x}_1[m] W_N^{km} \right) \tilde{X}_2[k] = \tilde{X}_1[k]\tilde{X}_2[k]. \tag{8.31}$$

In summary,

$$\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \overset{\mathcal{DFS}}{\longleftrightarrow} \tilde{X}_1[k]\tilde{X}_2[k]. \tag{8.32}$$

The periodic convolution of periodic sequences thus corresponds to multiplication of the corresponding periodic sequences of Fourier series coefficients.

Since periodic convolutions are somewhat different from aperiodic convolutions, it is worthwhile to consider the mechanics of evaluating Eq. (8.27). First, note that Eq. (8.27) calls for the product of sequences $\tilde{x}_1[m]$ and $\tilde{x}_2[n-m] = \tilde{x}_2[-(m-n)]$ viewed as functions of m with n fixed. This is the same as for an aperiodic convolution, but with the following two major differences:

1. The sum is over the finite interval $0 \le m \le N-1$.
2. The values of $\tilde{x}_2[n-m]$ in the interval $0 \le m \le N-1$ repeat periodically for m outside of that interval.

These details are illustrated by the following example.

Example 8.4 Periodic Convolution

An illustration of the procedure for forming the periodic convolution of two periodic sequences corresponding to Eq. (8.27) is given in Figure 8.3, wherein we have illustrated the sequences $\tilde{x}_2[m]$, $\tilde{x}_1[m]$, $\tilde{x}_2[-m]$, $\tilde{x}_2[1-m] = \tilde{x}_2[-(m-1)]$, and $\tilde{x}_2[2-m] = \tilde{x}_2[-(m-2)]$. To evaluate $\tilde{x}_3[n]$ in Eq. (8.27) for $n = 2$, for example, we multiply $\tilde{x}_1[m]$ by $\tilde{x}_2[2-m]$ and then sum the product terms $\tilde{x}_1[m]\tilde{x}_2[2-m]$ for $0 \le m \le N-1$, obtaining $\tilde{x}_3[2]$. As n changes, the sequence $\tilde{x}_2[n-m]$ shifts appropriately, and Eq. (8.27) is evaluated for each value of $0 \le n \le N-1$. Note that as the sequence $\tilde{x}_2[n-m]$ shifts to the right or left, values that leave the interval between the dotted lines at one end reappear at the other end because of the periodicity. Because of the periodicity of $\tilde{x}_3[n]$, there is no need to continue to evaluate Eq. (8.27) outside the interval $0 \le n \le N-1$.

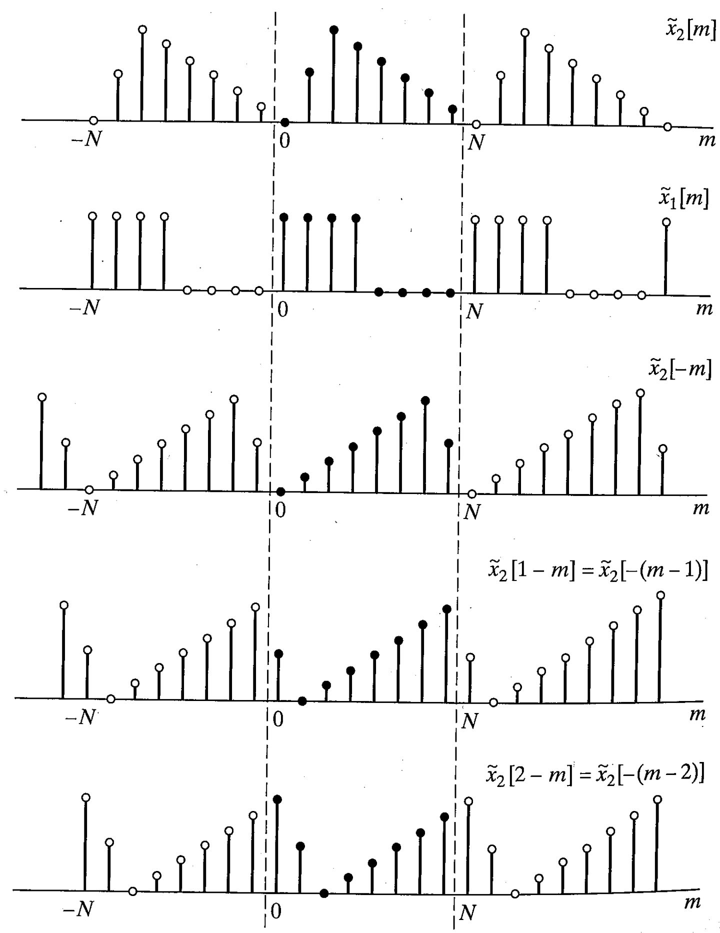

Figure 8.3 Procedure for forming the periodic convolution of two periodic sequences.

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8.3 THE FOURIER TRANSFORM

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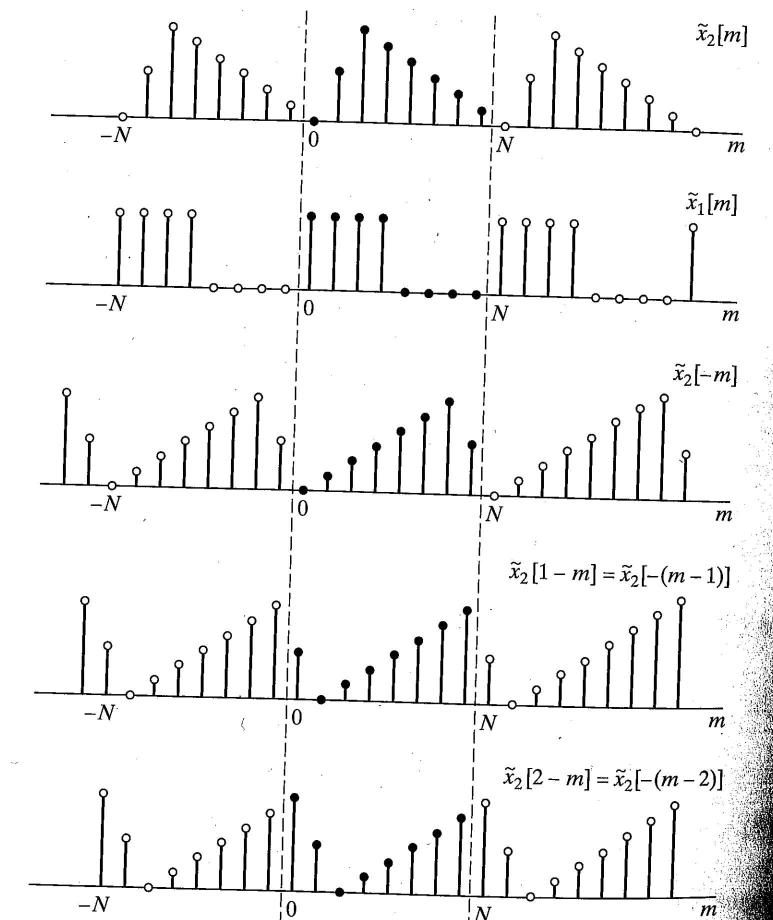

Figure 8.3 Procedure for forming the periodic convolution of two periodic sequences.

The duality theorem in Section 8.2.3 suggests that if the roles of time and frequency are interchanged, we will obtain a result almost identical to the previous result. That is, the periodic sequence

$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n], \tag{8.33}$$

where $\tilde{x}_1[n]$ and $\tilde{x}_2[n]$ are periodic sequences, each with period N, has the DFS coefficients given by

$$\tilde{X}_3[k] = \frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell], \tag{8.34}$$

corresponding to $1/N$ times the periodic convolution of $\tilde{X}_1[k]$ and $\tilde{X}_2[k]$. This result can also be verified by substituting $\tilde{X}_3[k]$, given by Eq. (8.34), into the Fourier series relation of Eq. (8.12) to obtain $\tilde{x}_3[n]$.

8.2.6 Summary of Properties of the DFS Representation of Periodic Sequences

The properties of the DFS representation discussed in this section are summarized in Table 8.1.

8.3 THE FOURIER TRANSFORM OF PERIODIC SIGNALS

As discussed in Section 2.7, uniform convergence of the Fourier transform of a sequence requires that the sequence be absolutely summable, and mean-square convergence requires that the sequence be square summable. Periodic sequences satisfy neither condition. However, as we discussed briefly in Section 2.7, sequences that can be expressed as a sum of complex exponentials can be considered to have a Fourier transform representation in the form of Eq. (2.147), i.e., as a train of impulses. Similarly, it is often useful to incorporate the DFS representation of periodic signals within the framework of the discrete-time Fourier transform. This can be done by interpreting the discrete-time Fourier transform of a periodic signal to be an impulse train in the frequency domain with the impulse values proportional to the DFS coefficients for the sequence. Specifically, if $\tilde{x}[n]$ is periodic with period N and the corresponding DFS coefficients are $\tilde{X}[k]$, then the Fourier transform of $\tilde{x}[n]$ is defined to be the impulse train

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right). \tag{8.35}$$

Note that $\tilde{X}(e^{j\omega})$ has the necessary periodicity with period 2π since $\tilde{X}[k]$ is periodic with period N, and the impulses are spaced at integer multiples of $2\pi/N$, where N is an

TABLE 8.1 SUMMARY OF PROPERTIES OF THE DFS

Periodic Sequence (Period N)	DFS Coefficients (Period N)
1. $\tilde{x}[n]$	$\tilde{X}[k]$ periodic with period N
2. $\tilde{x}_1[n], \tilde{x}_2[n]$	$\tilde{X}_1[k], \tilde{X}_2[k]$ periodic with period N
3. $a\tilde{x}_1[n] + b\tilde{x}_2[n]$	$a\tilde{X}_1[k] + b\tilde{X}_2[k]$
4. $\tilde{X}[n]$	$N\tilde{x}[-k]$
5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1}\tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
9. $\tilde{x}^*[n]$	$\tilde{X}^*[-k]$
10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
12. $j\mathcal{I}m\{\tilde{x}[n]\}$	$\tilde{X}_o[k] = \frac{1}{2}(\tilde{X}[k] - \tilde{X}^*[-k])$
13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties for $\tilde{x}[n]$ real.	$\begin{cases} \tilde{X}[k] = \tilde{X}^*[-k] \\ \mathcal{R}e\{\tilde{X}[k]\} = \mathcal{R}e\{\tilde{X}[-k]\} \\ \mathcal{I}m\{\tilde{X}[k]\} = -\mathcal{I}m\{\tilde{X}[-k]\} \\ \lvert\tilde{X}[k]\rvert = \lvert\tilde{X}[-k]\rvert \\ \angle\tilde{X}[k] = -\angle\tilde{X}[-k] \end{cases}$
16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
17. $\tilde{x}_0[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$

integer. To show that $\tilde{X}(e^{j\omega})$ as defined in Eq. (8.35) is a Fourier transform representation of the periodic sequence $\tilde{x}[n]$, we substitute Eq. (8.35) into the inverse Fourier transform Eq. (2.130); i.e.,

$$\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon}\tilde{X}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon}\sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\tilde{X}[k]\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega, \quad (8.36)$$

where ϵ satisfies the inequality $0 < \epsilon < (2\pi/N)$. Recall that in evaluating the inverse Fourier transform, we can integrate over any interval of length 2π, since the integrand $\tilde{X}(e^{j\omega})e^{j\omega n}$ is periodic with period 2π. In Eq. (8.36) the integration limits are denoted $0-\epsilon$ and $2\pi-\epsilon$, which means that the integration is from just before ...

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$$\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon}\tilde{X}(e^{j\omega})e^{j\omega n}$$

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5. $\tilde{x}[n-m]$	$W_N^{km}\tilde{X}[k]$
6. $W_N^{-\ell n}\tilde{x}[n]$	$\tilde{X}[k-\ell]$
7. $\sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m]$ (periodic convolution)	$\tilde{X}_1[k]\tilde{X}_2[k]$
8. $\tilde{x}_1[n]\tilde{x}_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} \tilde{X}_1[\ell]\tilde{X}_2[k-\ell]$ (periodic convolution)
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10. $\tilde{x}^*[-n]$	$\tilde{X}^*[k]$
11. $\mathcal{R}e\{\tilde{x}[n]\}$	$\tilde{X}_e[k] = \frac{1}{2}(\tilde{X}[k] + \tilde{X}^*[-k])$
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13. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}^*[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
14. $\tilde{x}_o[n] = \frac{1}{2}(\tilde{x}[n] - \tilde{x}^*[-n])$	$j\mathcal{I}m\{\tilde{X}[k]\}$
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16. $\tilde{x}_e[n] = \frac{1}{2}(\tilde{x}[n] + \tilde{x}[-n])$	$\mathcal{R}e\{\tilde{X}[k]\}$
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where ϵ satisfies the inequality $0 < \epsilon < (2\pi/N)$. Recall that in evaluating the inverse Fourier transform, we can integrate over any interval of length 2π, since the integrand $\tilde{X}(e^{j\omega})e^{j\omega n}$ is periodic with period 2π. In Eq. (8.36) the integration limits are denoted $0-\epsilon$ and $2\pi-\epsilon$, which means that the integration is from just before $\omega = 0$ to just before $\omega = 2\pi$. These limits are convenient, because they include the impulse at $\omega = 0$ and

exclude the impulse at $\omega = 2\pi$.[3] Interchanging the order of integration and summation leads to

$$\begin{aligned}\frac{1}{2\pi}\int_{0-\epsilon}^{2\pi-\epsilon} \tilde{X}(e^{j\omega})e^{j\omega n}d\omega &= \frac{1}{N}\sum_{k=-\infty}^{\infty}\tilde{X}[k]\int_{0-\epsilon}^{2\pi-\epsilon}\delta\left(\omega - \frac{2\pi k}{N}\right)e^{j\omega n}d\omega \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{j(2\pi/N)kn}.\end{aligned} \tag{8.37}$$

The final form of Eq. (8.37) results because only the impulses corresponding to $k = 0, 1, \ldots, (N-1)$ are included in the interval between $\omega = 0 - \epsilon$ and $\omega = 2\pi - \epsilon$.

Comparing Eq. (8.37) and Eq. (8.12), we see that the final right-hand side of Eq. (8.37) is exactly equal to the Fourier series representation for $\tilde{x}[n]$, as specified by Eq. (8.12). Consequently, the inverse Fourier transform of the impulse train in Eq. (8.35) is the periodic signal $\tilde{x}[n]$, as desired.

Although the Fourier transform of a periodic sequence does not converge in the normal sense, the introduction of impulses permits us to include periodic sequences formally within the framework of Fourier transform analysis. This approach was also used in Chapter 2 to obtain a Fourier transform representation of other nonsummable sequences, such as the two-sided constant sequence (Example 2.19) or the complex exponential sequence (Example 2.20). Although the DFS representation is adequate for most purposes, the Fourier transform representation of Eq. (8.35) sometimes leads to simpler or more compact expressions and simplified analysis.

Example 8.5 The Fourier Transform of a Periodic Discrete-Time Impulse Train

Consider the periodic discrete-time impulse train

$$\tilde{p}[n] = \sum_{r=-\infty}^{\infty}\delta[n - rN], \tag{8.38}$$

which is the same as the periodic sequence $\tilde{x}[n]$ considered in Example 8.1. From the results of that example, it follows that

$$\tilde{P}[k] = 1, \qquad \text{for all } k. \tag{8.39}$$

Therefore, the DTFT of $\tilde{p}[n]$ is

$$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right). \tag{8.40}$$

The result of Example 8.5 is the basis for a useful interpretation of the relation between a periodic signal and a finite-length signal. Consider a finite-length signal $x[n]$ such that $x[n] = 0$ except in the interval $0 \le n \le N-1$, and consider the convolution

[3] The limits 0 to 2π would present a problem since the impulses at both 0 and 2π would require special handling.

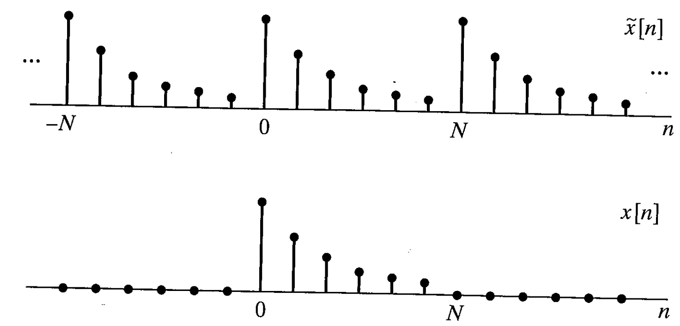

Figure 8.4 Periodic sequence $\tilde{x}[n]$ formed by repeating a finite-length sequence, $x[n]$, periodically. Alternatively, $x[n] = \tilde{x}[n]$ over one period and is zero otherwise.

of $x[n]$ with the periodic impulse train $\tilde{p}[n]$ of Example 8.5:

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} x[n - rN]. \tag{8.41}$$

Equation (8.41) states that $\tilde{x}[n]$ consists of a set of periodically repeated copies of the finite-length sequence $x[n]$. Figure 8.4 illustrates how a periodic sequence $\tilde{x}[n]$ can be formed from a finite-length sequence $x[n]$ through Eq. (8.41). The Fourier transform of $x[n]$ is $X(e^{j\omega})$, and the Fourier transform of $\tilde{x}[n]$ is

$$\begin{aligned}\tilde{X}(e^{j\omega}) &= X(e^{j\omega})\tilde{P}(e^{j\omega}) \\ &= X(e^{j\omega}) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right) \qquad (8.42) \\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k})\delta\left(\omega - \frac{2\pi k}{N}\right).\end{aligned}$$

Comparing Eq. (8.42) with Eq. (8.35), we conclude that

$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\Big|_{\omega=(2\pi/N)k}. \tag{8.43}$$

In other words, the periodic sequence $\tilde{X}[k]$ of DFS coefficients in Eq. (8.11) has an discrete-time interpretation as equally spaced samples of the DTFT of the finite-length sequence obtained by extracting one period of $\tilde{x}[n]$; i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.44}$$

This is also consistent with Figure 8.4, where it is clear that $x[n]$ can be obtained from $\tilde{x}[n]$ using Eq. (8.44). We can verify Eq. (8.43) in yet another way. Since $x[n] = \tilde{x}[n]$ for $0 \le n \le N-1$ and $x[n] = 0$ otherwise,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\omega n}. \tag{8.45}$$

Comparing Eq. (8.45) and Eq. (8.11), we see again that

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=2\pi k/N}. \tag{8.46}$$

This corresponds to sampling the Fourier transform at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spacing of $2\pi/N$.

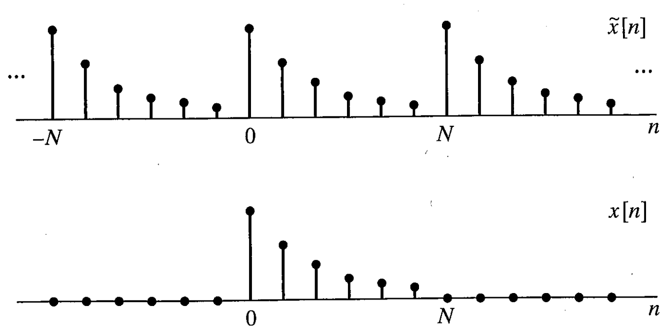

Figure 8.4 Periodic sequence $\tilde{x}[n]$ formed by repeating a finite-length sequence, $x[n]$, periodically. Alternatively, $x[n] = \tilde{x}[n]$ over one period and is zero otherwise.

of $x[n]$ with the periodic impulse train $\tilde{p}[n]$ of Example 8.5:

$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta[n - rN] = \sum_{r=-\infty}^{\infty} x[n - rN]. \qquad (8.41)$$

Equation (8.41) states that $\tilde{x}[n]$ consists of a set of periodically repeated copies of the finite-length sequence $x[n]$. Figure 8.4 illustrates how a periodic sequence $\tilde{x}[n]$ can be formed from a finite-length sequence $x[n]$ through Eq. (8.41). The Fourier transform of $x[n]$ is $X(e^{j\omega})$, and the Fourier transform of $\tilde{x}[n]$ is

$$\begin{aligned} \tilde{X}(e^{j\omega}) &= X(e^{j\omega})\tilde{P}(e^{j\omega}) \\ &= X(e^{j\omega}) \sum_{k=-\infty}^{\infty} \frac{2\pi}{N}\delta\left(\omega - \frac{2\pi k}{N}\right) \qquad (8.42) \\ &= \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k})\delta\left(\omega - \frac{2\pi k}{N}\right). \end{aligned}$$

Comparing Eq. (8.42) with Eq. (8.35), we conclude that

$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})\Big|_{\omega=(2\pi/N)k}. \qquad (8.43)$$

In other words, the periodic sequence $\tilde{X}[k]$ of DFS coefficients in Eq. (8.11) has an discrete-time interpretation as equally spaced samples of the DTFT of the finite-length sequence obtained by extracting one period of $\tilde{x}[n]$; i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \qquad (8.44)$$

This is also consistent with Figure 8.4, where it is clear that $x[n]$ can be obtained from $\tilde{x}[n]$ using Eq. (8.44). We can verify Eq. (8.43) in yet another way. Since $x[n] = \tilde{x}[n]$ for $0 \le n \le N-1$ and $x[n] = 0$ otherwise,

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\omega n}. \qquad (8.45)$$

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This corresponds to sampling the Fourier transform at N equally spaced frequencies between $\omega = 0$ and $\omega = 2\pi$ with a frequency spacing of $2\pi/N$.

Example 8.6 Relationship Between the Fourier Series Coefficients and the Fourier Transform of One Period

We again consider the sequence $\tilde{x}[n]$ of Example 8.3, which is shown in Figure 8.1. One period of $\tilde{x}[n]$ for the sequence in Figure 8.1 is

$$x[n] = \begin{cases} 1, & 0 \le n \le 4, \\ 0, & \text{otherwise}. \end{cases} \qquad (8.47)$$

The Fourier transform of one period of $\tilde{x}[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=0}^{4} e^{-j\omega n} = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}. \qquad (8.48)$$

Equation (8.46) can be shown to be satisfied for this example by substituting $\omega = 2\pi k/10$ into Eq. (8.48), giving

$$\tilde{X}[k] = e^{-j(4\pi k/10)}\frac{\sin(\pi k/2)}{\sin(\pi k/10)},$$

which is identical to the result in Eq. (8.18). The magnitude and phase of $X(e^{j\omega})$ are sketched in Figure 8.5. Note that the phase is discontinuous at the frequencies where $X(e^{j\omega}) = 0$. That the sequences in Figures 8.2(a) and (b) correspond to samples of Figures 8.5(a) and (b), respectively, is demonstrated in Figure 8.6, where Figures 8.2 and 8.5 have been superimposed.

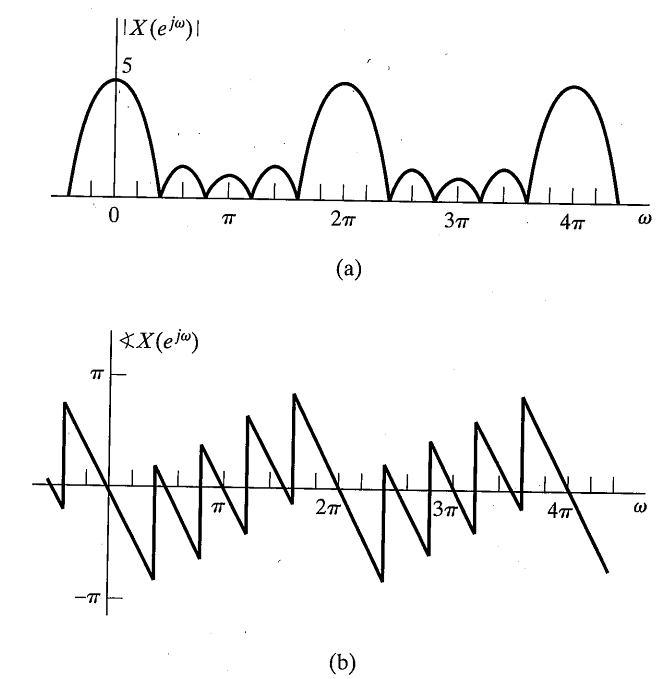


Figure 8.5 Magnitude and phase of the Fourier transform of one period of the sequence in Figure 8.1.

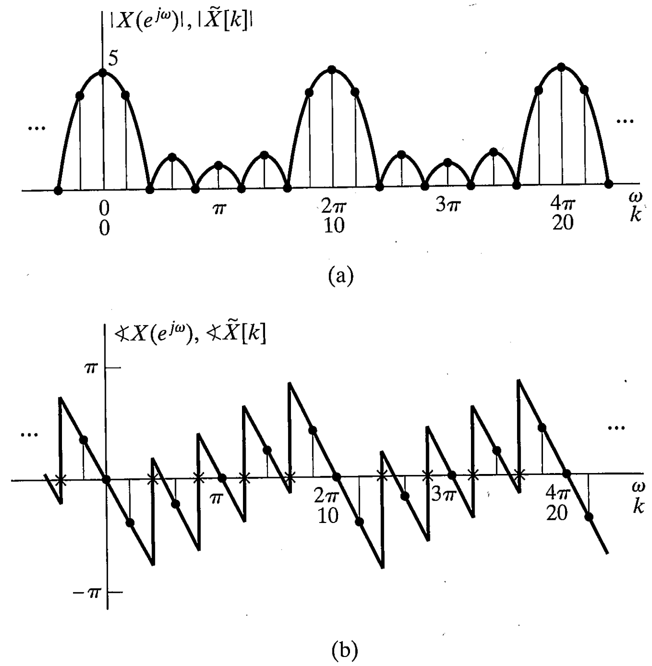


Figure 8.6 Overlay of Figures 8.2 and 8.5 illustrating the DFS coefficients of a periodic sequence as samples of the Fourier transform of one period.

8.4 SAMPLING THE FOURIER TRANSFORM

In this section, we discuss with more generality the relationship between an aperiodic sequence with Fourier transform $X(e^{j\omega})$ and the periodic sequence for which the DFS coefficients correspond to samples of $X(e^{j\omega})$ equally spaced in frequency. We will find this relationship to be particularly important when we discuss the discrete Fourier transform and its properties later in the chapter.

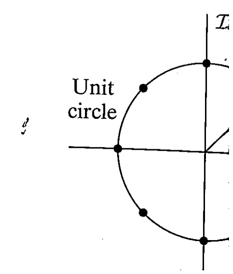
Consider an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, and assume that a sequence $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/N$; i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k}). \tag{8.49}$$

Since the Fourier transform is periodic in ω with period 2π, the resulting sequence is periodic in k with period N. Also, since the Fourier transform is equal to the z-transform evaluated on the unit circle, it follows that $\tilde{X}[k]$ can also be obtained by sampling $X(z)$ at N equally spaced points on the unit circle. Thus,

$$\tilde{X}[k] = X(z)|_{z=e^{j(2\pi/N)k}} = X(e^{j(2\pi/N)k}). \tag{8.50}$$

These sampling points are depicted in Figure 8.7 for $N = 8$. The figure makes it clear that the sequence of samples is periodic, since the N points are equally spaced starting with zero angle. Therefore, the same sequence repeats as k varies outside the range $0 \le k \le N-1$ since we simply continue around the unit circle visiting the same set of



Note that the sequence of sam
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simply substitute $\tilde{X}[k]$ obtained by s

$\tilde{x}[n] =$

Since we have made no assumption
exists, we can use infinite limits to in

$X(e^{j\omega})$

is over all nonzero values of $x[m]$.

Substituting Eq. (8.52) into Eq.
sion for $\tilde{X}[k]$ into Eq. (8.51) gives

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\Bigg[$$

which, after we interchange the orde

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\Bigg[\frac{1}{N}\sum$$

The term in brackets in Eq. (8.5
to be the Fourier series representati
and 8.2. Specifically,

$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1}$$

and therefore,

$$\tilde{x}[n] = x[n] * \sum_{r=-\infty}^{\infty}$$

where $*$ denotes aperiodic convolutio

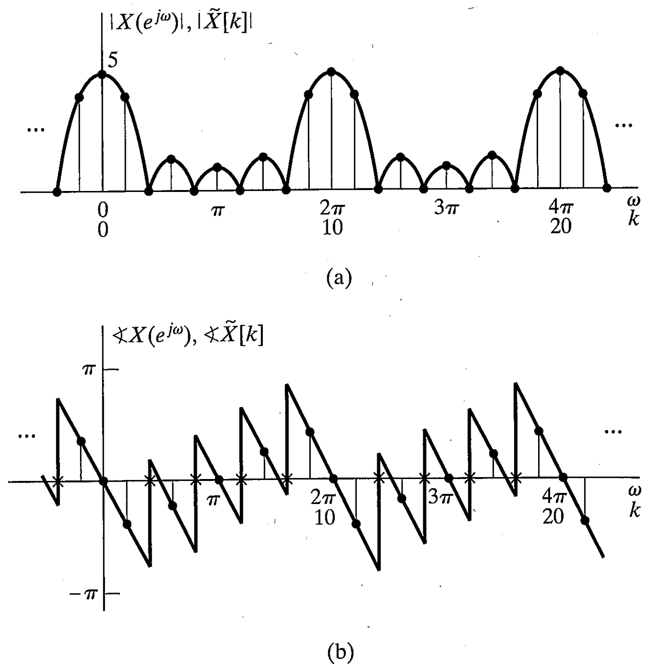

Figure 8.6 Overlay of Figures 8.2 and 8.5 illustrating the DFS coefficients of a periodic sequence as samples of the Fourier transform of one period.

8.4 SAMPLING THE FOURIER TRANSFORM

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Consider an aperiodic sequence $x[n]$ with Fourier transform $X(e^{j\omega})$, and assume that a sequence $\tilde{X}[k]$ is obtained by sampling $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/N$; i.e.,

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k}). \tag{8.49}$$

Since the Fourier transform is periodic in ω with period 2π, the resulting sequence is periodic in k with period N. Also, since the Fourier transform is equal to the z-transform evaluated on the unit circle, it follows that $\tilde{X}[k]$ can also be obtained by sampling $X(z)$ at N equally spaced points on the unit circle. Thus,

$$\tilde{X}[k] = X(z)|_{z=e^{j(2\pi/N)k}} = X(e^{j(2\pi/N)k}). \tag{8.50}$$

These sampling points are depicted in Figure 8.7 for $N = 8$. The figure makes it clear that the sequence of samples is periodic, since the N points are equally spaced starting with zero angle. Therefore, the same sequence repeats as k varies outside the range $0 \leq k \leq N-1$ since we simply continue around the unit circle visiting the same set of N points.

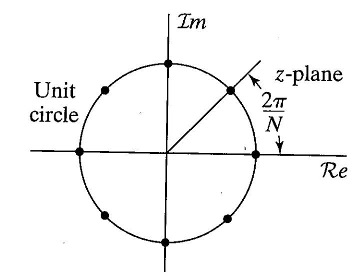

Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Note that the sequence of samples $\tilde{X}[k]$, being periodic with period N, could be the sequence of DFS coefficients of a sequence $\tilde{x}[n]$. To obtain that sequence, we can simply substitute $\tilde{X}[k]$ obtained by sampling into Eq. (8.12):

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}. \tag{8.51}$$

Since we have made no assumption about $x[n]$ other than that the Fourier transform exists, we can use infinite limits to indicate that the sum is

$$X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \tag{8.52}$$

is over all nonzero values of $x[m]$.

Substituting Eq. (8.52) into Eq. (8.49) and then substituting the resulting expression for $\tilde{X}[k]$ into Eq. (8.51) gives

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{\infty} x[m]e^{-j(2\pi/N)km}\right]W_N^{-kn}, \tag{8.53}$$

which, after we interchange the order of summation, becomes

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{\infty} x[m]\tilde{p}[n-m]. \tag{8.54}$$

The term in brackets in Eq. (8.54) can be seen from either Eq. (8.7) or Eq. (8.16) to be the Fourier series representation of the periodic impulse train of Examples 8.1 and 8.2. Specifically,

$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \sum_{r=-\infty}^{\infty}\delta[n-m-rN] \tag{8.55}$$

and therefore,

$$\tilde{x}[n] = x[n] * \sum_{r=-\infty}^{\infty}\delta[n-rN] = \sum_{r=-\infty}^{\infty} x[n-rN], \tag{8.56}$$

where $*$ denotes aperiodic convolution. That is, $\tilde{x}[n]$ is the periodic sequence that results from the aperiodic convolution of $x[n]$ with a periodic unit-impulse train. Thus, the

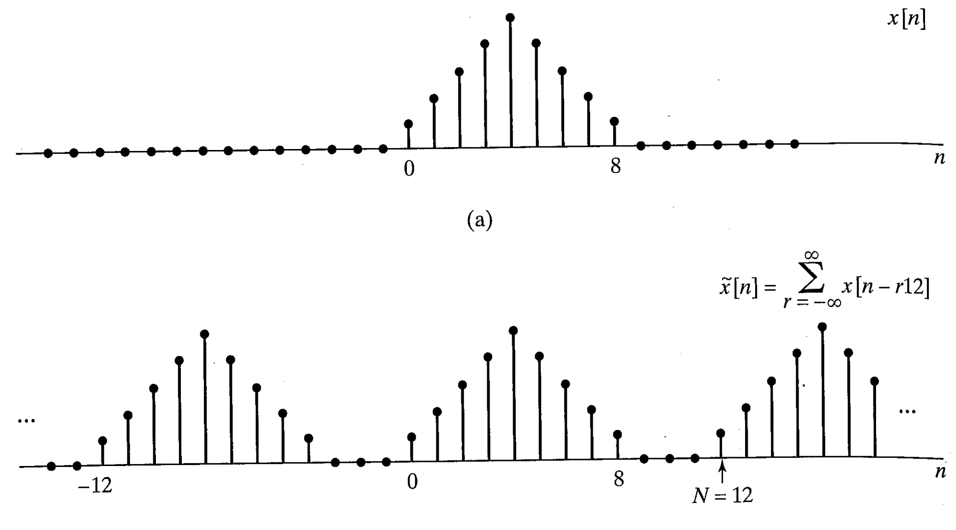

Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

periodic sequence $\tilde{x}[n]$, corresponding to $\tilde{X}[k]$ obtained by sampling $X(e^{j\omega})$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all the positive and negative integer multiples of N, the period of the sequence $\tilde{X}[k]$. This is illustrated in Figure 8.8, where the sequence $x[n]$ is of length 9 and the value of N in Eq. (8.56) is $N = 12$. Consequently, the delayed replications of $x[n]$ do not overlap, and one period of the periodic sequence $\tilde{x}[n]$ is recognizable as $x[n]$. This is consistent with the discussion in Section 8.3 and Example 8.6, wherein we showed that the Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period. In Figure 8.9 the same sequence $x[n]$ is used, but the value of N is now $N = 7$. In this case, the replicas of $x[n]$ overlap and one period of $\tilde{x}[n]$ is no longer identical to $x[n]$. In both cases, however, Eq. (8.49) still holds; i.e., in both cases, the DFS coefficients of $\tilde{x}[n]$ are samples of the Fourier transform of $x[n]$ spaced

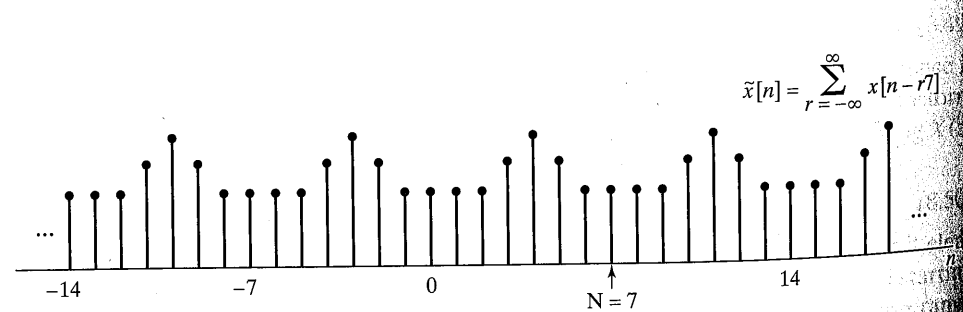

Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier trans-

in frequency at integer multiples
discussion of sampling in Chapte
frequency domain rather than in
mathematical representations ar
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This discussion highlights se
in the remainder of the chapter. W
an aperiodic sequence $x[n]$ can be
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$x[n] =$

If the interval of support of $x[n]$
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A direct relationship betwee
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$X(e^{j\omega})$ at all frequencies if $x[n]$ ha
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a DFS. Alternatively, given the seq
and then use Eq. (8.57) to obtain
represent finite-length sequences,
developing, discussing, and applyin
the representation through samples

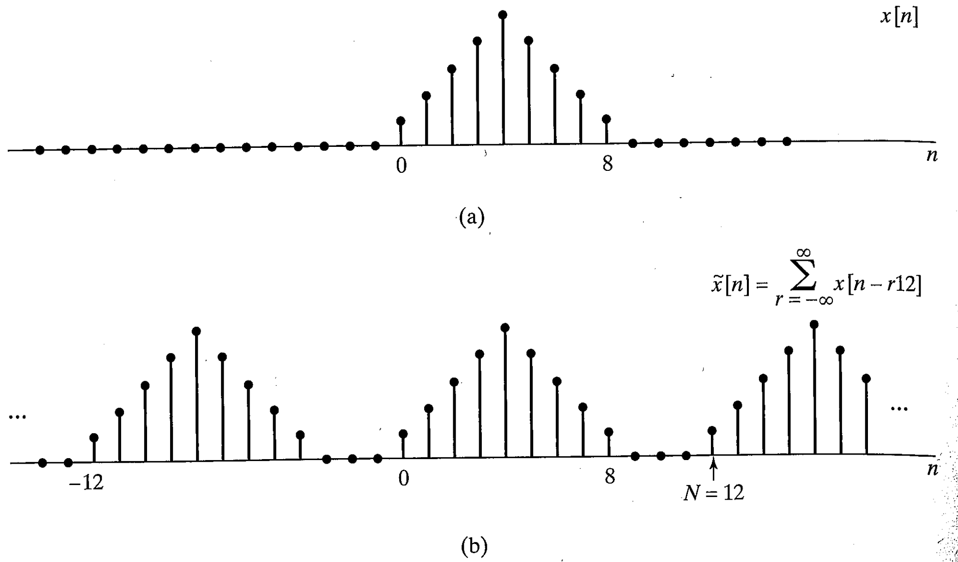

Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

periodic sequence $\tilde{x}[n]$, corresponding to $\tilde{X}[k]$ obtained by sampling $X(e^{j\omega})$, is formed from $x[n]$ by adding together an infinite number of shifted replicas of $x[n]$. The shifts are all the positive and negative integer multiples of N, the period of the sequence $\tilde{X}[k]$. This is illustrated in Figure 8.8, where the sequence $x[n]$ is of length 9 and the value of N in Eq. (8.56) is $N = 12$. Consequently, the delayed replications of $x[n]$ do not overlap, and one period of the periodic sequence $\tilde{x}[n]$ is recognizable as $x[n]$. This is consistent with the discussion in Section 8.3 and Example 8.6, wherein we showed that the Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period. In Figure 8.9 the same sequence $x[n]$ is used, but the value of N is now $N = 7$. In this case, the replicas of $x[n]$ overlap and one period of $\tilde{x}[n]$ is no longer identical to $x[n]$. In both cases, however, Eq. (8.49) still holds; i.e., in both cases, the DFS coefficients of $\tilde{x}[n]$ are samples of the Fourier transform of $x[n]$ spaced

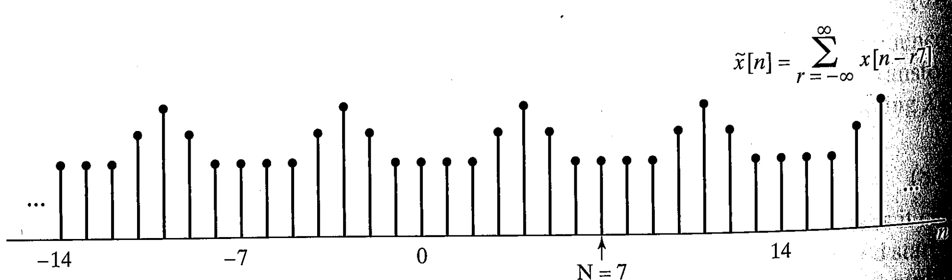

Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8.8(a) with $N = 7$.

in frequency at integer multiples of $2\pi/N$. This discussion should be reminiscent of our discussion of sampling in Chapter 4. The difference is that here we are sampling in the frequency domain rather than in the time domain. However, the general outlines of the mathematical representations are very similar.

For the example in Figure 8.8, the original sequence $x[n]$ can be recovered from $\tilde{x}[n]$ by extracting one period. Equivalently, the Fourier transform $X(e^{j\omega})$ can be recovered from the samples spaced in frequency by $2\pi/12$. In contrast, in Figure 8.9, $x[n]$ cannot be recovered by extracting one period of $\tilde{x}[n]$, and, equivalently, $X(e^{j\omega})$ cannot be recovered from its samples if the sample spacing is only $2\pi/7$. In effect, for the case illustrated in Figure 8.8, the Fourier transform of $x[n]$ has been sampled at a sufficiently small spacing (in frequency) to be able to recover it from these samples, whereas Figure 8.9 represents a case for which the Fourier transform has been undersampled. The relationship between $x[n]$ and one period of $\tilde{x}[n]$ in the undersampled case can be thought of as a form of aliasing in the time domain, essentially identical to the frequency-domain aliasing (discussed in Chapter 4) that results from undersampling in the time domain. Obviously, time-domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency-domain aliasing can be avoided only for signals that have bandlimited Fourier transforms.

This discussion highlights several important concepts that will play a central role in the remainder of the chapter. We have seen that samples of the Fourier transform of an aperiodic sequence $x[n]$ can be thought of as DFS coefficients of a periodic sequence $\tilde{x}[n]$ obtained through summing periodic replicas of $x[n]$. If $x[n]$ is finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, and, equivalently, $x[n]$ is recoverable from the corresponding periodic sequence $\tilde{x}[n]$. Specifically, if $x[n] = 0$ outside the interval $n = 0$, $n = N - 1$, then

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.57}$$

If the interval of support of $x[n]$ is different than $0, N-1$ then Eq. (8.57) would be appropriately modified.

A direct relationship between $X(e^{j\omega})$ and its samples $\tilde{X}[k]$, i.e., an interpolation formula for $X(e^{j\omega})$, can be derived (see Problem 8.57). However, the essence of our previous discussion is that to represent or to recover $x[n]$, it is not necessary to know $X(e^{j\omega})$ at all frequencies if $x[n]$ has finite length. Given a finite-length sequence $x[n]$, we can form a periodic sequence using Eq. (8.56), which in turn can be represented by a DFS. Alternatively, given the sequence of Fourier coefficients $\tilde{X}[k]$, we can find $\tilde{x}[n]$ and then use Eq. (8.57) to obtain $x[n]$. When the Fourier series is used in this way to represent finite-length sequences, it is called the discrete Fourier transform or DFT. In developing, discussing, and applying the DFT, it is always important to remember that the representation through samples of the Fourier transform is in effect a representation of the finite-duration sequence by a periodic sequence, one period of which is the finite-duration sequence that we wish to represent.

8.5 FOURIER REPRESENTATION OF FINITE-DURATION SEQUENCES: THE DFT

In this section, we formalize the point of view suggested at the end of the previous section. We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \le n \le N-1$. In many instances, we will want to assume that a sequence has length N, even if its length is $M \le N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]. \tag{8.58a}$$

The finite-length sequence $x[n]$ can be recovered from $\tilde{x}[n]$ through Eq. (8.57), i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise}. \end{cases} \tag{8.58b}$$

Recall from Section 8.4 that the DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $2\pi/N$) of the Fourier transform of $x[n]$. Since $x[n]$ is assumed to have finite length N, there is no overlap between the terms $x[n - rN]$ for different values of r. Thus, Eq. (8.58a) can alternatively be written as

$$\tilde{x}[n] = x[(n \text{ modulo } N)]. \tag{8.59}$$

For convenience, we will use the notation $((n))_N$ to denote $(n$ modulo $N)$; with this notation, Eq. (8.59) is expressed as

$$\tilde{x}[n] = x[((n))_N]. \tag{8.60}$$

Note that Eq. (8.60) is equivalent to Eq. (8.58a) only when $x[n]$ has length less than or equal to N. The finite-duration sequence $x[n]$ is obtained from $\tilde{x}[n]$ by extracting one period, as in Eq. (8.58b).

One informal and useful way of visualizing Eq. (8.59) is to think of wrapping a plot of the finite-duration sequence $x[n]$ around a cylinder with a circumference equal to the length of the sequence. As we repeatedly traverse the circumference of the cylinder, we see the finite-length sequence periodically repeated. With this interpretation, representation of the finite-length sequence by a periodic sequence corresponds to wrapping the sequence around the cylinder; recovering the finite-length sequence from the periodic sequence using Eq. (8.58b) can be visualized as unwrapping the cylinder and laying it flat ... the sequence is displayed on a linear time axis rather than a circular (modulo N)

As defined in Section 8.1, the
sequence $\tilde{x}[n]$ is itself a periodic seq
the time and frequency domains, we
with a finite-duration sequence to b
period of $\tilde{X}[k]$. This finite-duration s
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and

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From Section 8.1, $\tilde{X}[k]$ and $\tilde{x}[n]$ are

$\tilde{X}[k] =$

$\tilde{x}[n] =$

where $W_N = e^{-j(2\pi/N)}$.

Since the summations in Eqs. (
zero and $(N-1)$, it follows from Eq

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n \ldots \\ 0, \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} \ldots \\ 0, \end{cases}$$

8.5 FOURIER REPRESENTATION OF FINITE-DURATION SEQUENCES: THE DFT

In this section, we formalize the point of view suggested at the end of the previous section. We begin by considering a finite-length sequence $x[n]$ of length N samples such that $x[n] = 0$ outside the range $0 \le n \le N-1$. In many instances, we will want to assume that a sequence has length N, even if its length is $M \le N$. In such cases, we simply recognize that the last $(N-M)$ samples are zero. To each finite-length sequence of length N, we can always associate a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n-rN]. \tag{8.58a}$$

The finite-length sequence $x[n]$ can be recovered from $\tilde{x}[n]$ through Eq. (8.57), i.e.,

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.58b}$$

Recall from Section 8.4 that the DFS coefficients of $\tilde{x}[n]$ are samples (spaced in frequency by $2\pi/N$) of the Fourier transform of $x[n]$. Since $x[n]$ is assumed to have finite length N, there is no overlap between the terms $x[n-rN]$ for different values of r. Thus, Eq. (8.58a) can alternatively be written as

$$\tilde{x}[n] = x[(n \text{ modulo } N)]. \tag{8.59}$$

For convenience, we will use the notation $((n))_N$ to denote $(n \text{ modulo } N)$; with this notation, Eq. (8.59) is expressed as

$$\tilde{x}[n] = x[((n))_N]. \tag{8.60}$$

Note that Eq. (8.60) is equivalent to Eq. (8.58a) only when $x[n]$ has length less than or equal to N. The finite-duration sequence $x[n]$ is obtained from $\tilde{x}[n]$ by extracting one period, as in Eq. (8.58b).

One informal and useful way of visualizing Eq. (8.59) is to think of wrapping a plot of the finite-duration sequence $x[n]$ around a cylinder with a circumference equal to the length of the sequence. As we repeatedly traverse the circumference of the cylinder, we see the finite-length sequence periodically repeated. With this interpretation, representation of the finite-length sequence by a periodic sequence corresponds to wrapping the sequence around the cylinder; recovering the finite-length sequence from the periodic sequence using Eq. (8.58b) can be visualized as unwrapping the cylinder and laying it flat so that the sequence is displayed on a linear time axis rather than a circular (modulo N) time axis.

As defined in Section 8.1, the sequence of DFS coefficients $\tilde{X}[k]$ of the periodic sequence $\tilde{x}[n]$ is itself a periodic sequence with period N. To maintain a duality between the time and frequency domains, we will choose the Fourier coefficients that we associate with a finite-duration sequence to be a finite-duration sequence corresponding to one period of $\tilde{X}[k]$. This finite-duration sequence, $X[k]$, will be referred to as the DFT. Thus, the DFT, $X[k]$, is related to the DFS coefficients, $\tilde{X}[k]$, by

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1, \\ 0, & \text{otherwise,} \end{cases} \tag{8.61}$$

and

$$\tilde{X}[k] = X[(k \text{ modulo } N)] = X[((k))_N]. \tag{8.62}$$

From Section 8.1, $\tilde{X}[k]$ and $\tilde{x}[n]$ are related by

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}, \tag{8.63}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}. \tag{8.64}$$

where $W_N = e^{-j(2\pi/N)}$.

Since the summations in Eqs. (8.63) and (8.64) involve only the interval between zero and $(N-1)$, it follows from Eqs. (8.58b) to (8.64) that

$$X[k] = \begin{cases} \displaystyle\sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1, \\ 0, & \text{otherwise,} \end{cases} \tag{8.65}$$

$$x[n] = \begin{cases} \displaystyle\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases} \tag{8.66}$$

Generally, the DFT analysis and synthesis equations are written as follows:

$$\text{Analysis equation:} \quad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad 0 \le k \le N-1, \tag{8.67}$$

$$\text{Synthesis equation:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad 0 \le n \le N-1. \tag{8.68}$$

That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly. The relationship between $x[n]$ and $X[k]$ implied by Eqs. (8.67) and (8.68) will sometimes be denoted as

$$x[n] \overset{\mathcal{DFT}}{\longleftrightarrow} X[k]. \tag{8.69}$$

In recasting Eqs. (8.11) and (8.12) in the form of Eqs. (8.67) and (8.68) for finite-duration sequences, we have not eliminated the inherent periodicity. As with the DFS, the DFT $X[k]$ is equal to samples of the periodic Fourier transform $X(e^{j\omega})$, and if Eq. (8.68) is evaluated for values of n outside the interval $0 \le n \le N-1$, the result will not be zero, but rather a periodic extension of $x[n]$. The inherent periodicity is always present. Sometimes, it causes us difficulty, and sometimes we can exploit it, but to totally ignore it is to invite trouble. In defining the DFT representation, we are simply recognizing that we are interested in values of $x[n]$ only in the interval $0 \le n \le N-1$, because $x[n]$ is really zero outside that interval, and we are interested in values of $X[k]$ only in the interval $0 \le k \le N-1$ because these are the only values needed in Eq. (8.68) to reconstruct $X[n]$.

Example 8.7 The DFT of a Rectangular Pulse

To illustrate the DFT of a finite-duration sequence, consider $x[n]$ shown in Figure 8.10(a). In determining the DFT, we can consider $x[n]$ as a finite-duration sequence with any length greater than or equal to $N = 5$. Considered as a sequence of length $N = 5$, the periodic sequence $\tilde{x}[n]$ whose DFS corresponds to the DFT of $x[n]$ is shown in Figure 8.10(b). Since the sequence in Figure 8.10(b) is constant over the interval $0 \le n \le 4$, it follows that

$$\tilde{X}[k] = \sum_{n=0}^{4} e^{-j(2\pi k/5)n} = \frac{1 - e^{-j2\pi k}}{1 - e^{-j(2\pi k/5)}} \tag{8.70}$$

$$= \begin{cases} 5, & k = 0, \pm 5, \pm 10, \ldots, \\ 0, & \text{otherwise;} \end{cases}$$

i.e., the only nonzero DFS coefficients $\tilde{X}[k]$ are at $k = 0$ and integer multiples of $k = 5$ (all of which represent the same complex exponential frequency). The DFS coefficients are shown in Figure 8.10(c). Also shown is the magnitude of the DTFT, $|X(e^{j\omega})|$. Clearly, $\tilde{X}[k]$ is a sequence of samples of $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/5$. According to Eq. (8.61), the five-point DFT of $x[n]$ corresponds to the finite-length sequence obtained by extracting one period of $\tilde{X}[k]$. Consequently, the five-point DFT

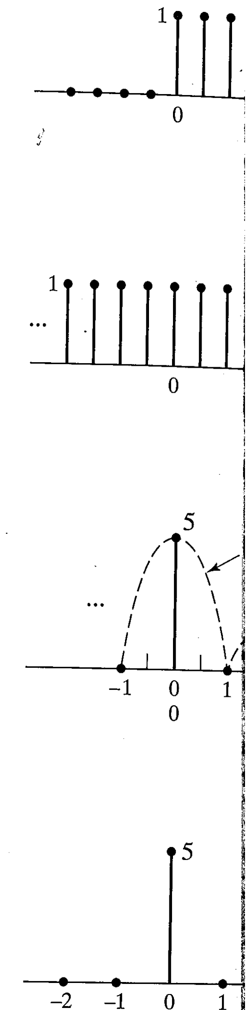

Figure 8.10 Illustra… sequence $\tilde{x}[n]$ formed… $\tilde{X}[k]$ for $\tilde{x}[n]$. To emp… Fourier transform, $|X($…

If, instead, we consi… sequence is that shown in… in Example 8.3. Therefor… DFT $X[k]$ shown in Figur…

Generally, the DFT analysis and synthesis equations are written as follows:

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That is, the fact that $X[k] = 0$ for k outside the interval $0 \le k \le N-1$ and that $x[n] = 0$ for n outside the interval $0 \le n \le N-1$ is implied, but not always stated explicitly. The relationship between $x[n]$ and $X[k]$ implied by Eqs. (8.67) and (8.68) will sometimes be denoted as

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$$= \begin{cases} 5, & k = 0, \pm 5, \pm 10, \ldots, \\ 0, & \text{otherwise;} \end{cases} \tag{8.70}$$

i.e., the only nonzero DFS coefficients $\tilde{X}[k]$ are at $k = 0$ and integer multiples of $k = 5$ (all of which represent the same complex exponential frequency). The DFS coefficients are shown in Figure 8.10(c). Also shown is the magnitude of the DTFT, $|X(e^{j\omega})|$. Clearly, $\tilde{X}[k]$ is a sequence of samples of $X(e^{j\omega})$ at frequencies $\omega_k = 2\pi k/5$. According to Eq. (8.61), the five-point DFT of $x[n]$ corresponds to the finite-length sequence obtained by extracting one period of $\tilde{X}[k]$. Consequently, the five-point DFT of $x[n]$ is shown in Figure 8.10(d).

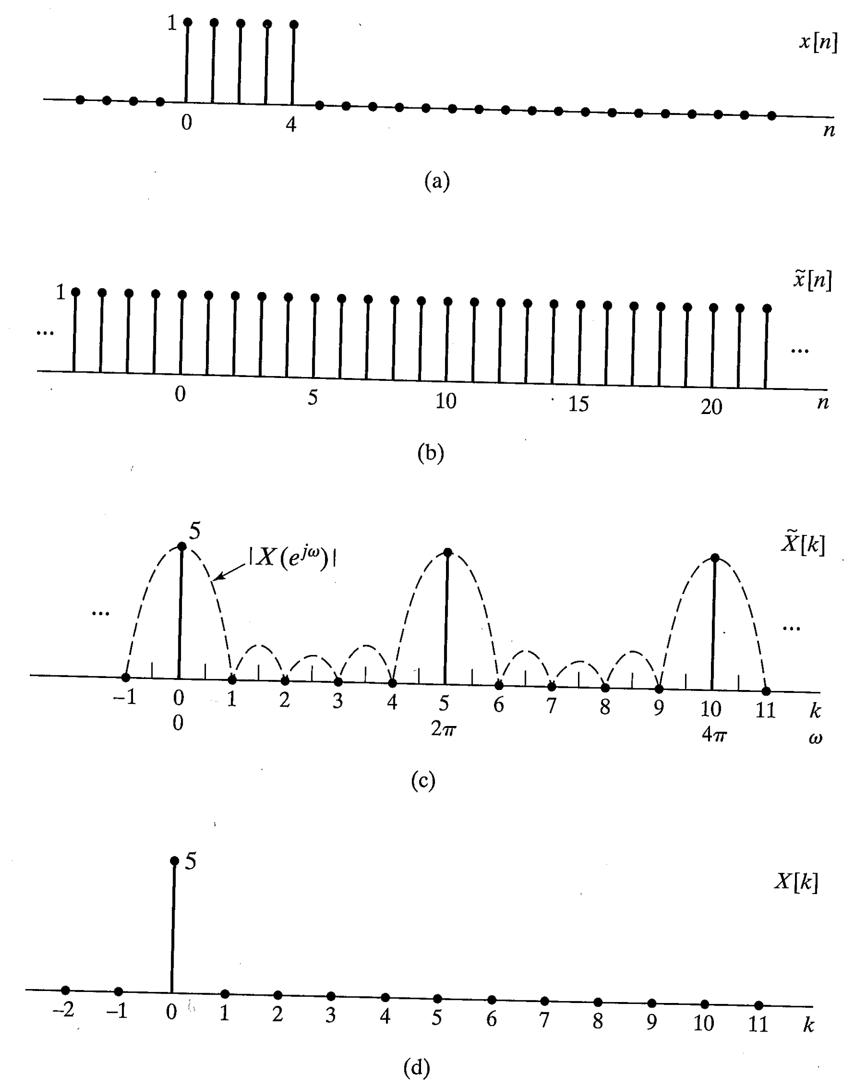

Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

If, instead, we consider $x[n]$ to be of length $N = 10$, then the underlying periodic sequence is that shown in Figure 8.11(b), which is the periodic sequence considered in Example 8.3. Therefore, $\tilde{X}[k]$ is as shown in Figures 8.2 and 8.6, and the 10-point DFT $X[k]$ shown in Figures 8.11(c) and 8.11(d) is one period of $\tilde{X}[k]$.

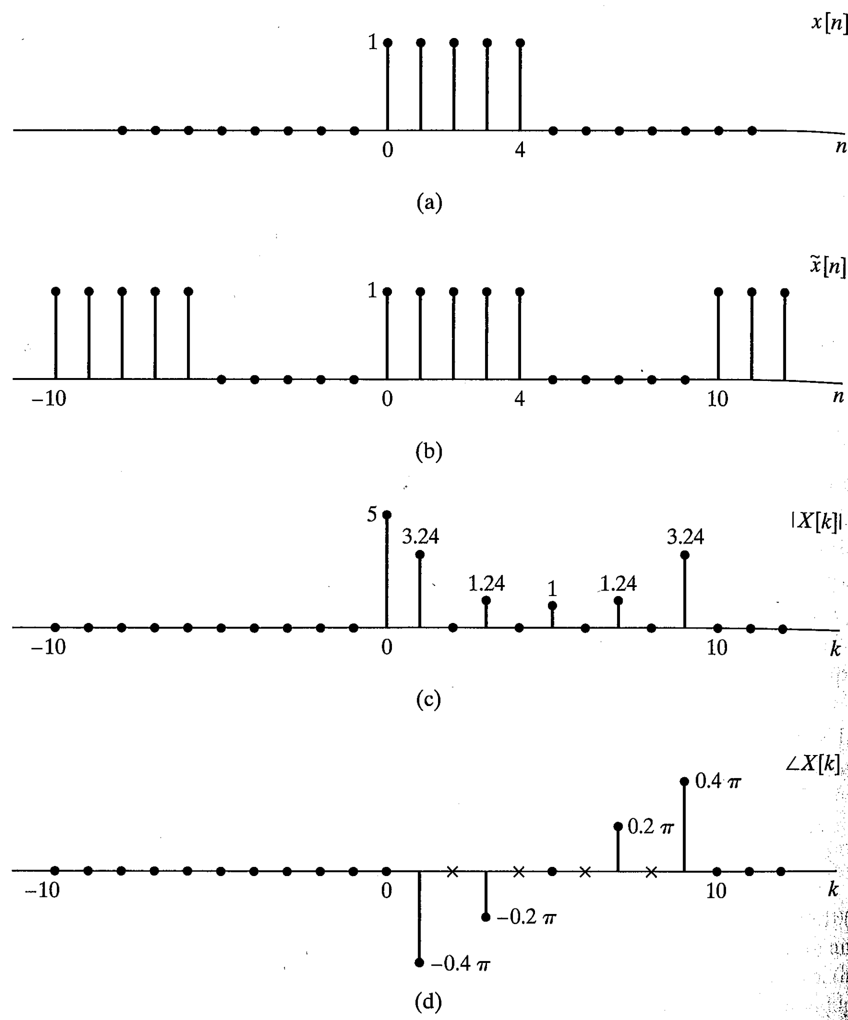

Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

The distinction between the finite-duration sequence $x[n]$ and the periodic sequence $\tilde{x}[n]$ related through Eqs. (8.57) and (8.60) may seem minor, since, by using these equations, it is straightforward to construct one from the other. However, the distinction becomes important in considering properties of the DFT and in considering the effect on $x[n]$ of modifications to $X[k]$. This will become evident in the next section,

8.6 PROPERTIES OF THE DFT

In this section, we consider
sequences. Our discussion pa
However, particular attentio
and the implicit periodicity o

8.6.1 Linearity

If two finite-duration sequen

then the DFT of $x_3[n]$ is

Clearly, if $x_1[n]$ has length N
$x_3[n]$ will be $N_3 = \max(N_1,$
DFTs must be computed with
$X_1[k]$ is the DFT of the seq
N_2-point DFT of $x_1[n]$ is

$X_1[k] =$

and the N_2-point DFT of $x_2[$

$X_2[k] =$

In summary, if

and