2.3.1.3 Basis Functions

A transform can be thought of as a re-mapping of the original data into something that provides more information.* The Fourier transform described in Chapter 3 is a classic example, as it converts the original time-domain data into frequency-domain data that often provide greater insight into the nature and/or origin of the signal. Many of the transforms described in this book achieved by comparing the signal of interest with some sort of probing function or a whole family of probing functions termed a *basis*. Usually the probing function or basis is simpler than the signal, for example, a sine wave or series of sine waves. (As shown in Chapter 3, a sine wave is about as simple as a waveform gets.) A quantitative comparison can tell you how much your complicated signal is like a simpler basis or reference family. If enough comparisons are made with a well-chosen basis, these comparisons taken together can provide an alternative representation of the signal. We hope the new representation is more informative or enlightening than the original.

To compare a waveform with a number of different functions that form the basis requires modification of the basic correlation equation (Equation 2.29) so that one of the functions becomes a family of functions, $f_m[n]$. If the comparison is made with a family of functions, a series of correlation values is produced, one for each family member:

$$X[m] = \sum_{n=1}^{N} x[n] f_m[n] \tag{2.36}$$

The multiple correlations over the entire basis result in the series $X[m]$, where m indicates the specific basis member. This analysis can also be applied to continuous functions:

$$X(m) = \int_{-\infty}^{\infty} x(t) f_m(t) dt \tag{2.37}$$

where $x(t)$ is now a continuous signal and $f_m(t)$ is the set of continuous basis functions.

Equations 2.36 and 2.37 assume that the signal and basis are the same length. If the length of the basis, $f_m[n]$, is shorter than the waveform, then the comparison can only be carried out on a portion of $x[n]$. The signal can be segmented by truncation, cutting out the desired portion, or by multiplying the signal by yet another function that is zero outside the desired portion. A function used to segment a waveform is termed a *window* function and its application is illustrated in Figure 2.11. Note that simple truncation can be viewed as multiplying the function by a *rectangular window*, a function whose value is 1 for the excised portion of the waveform and 0 elsewhere. The influence of different window functions is discussed in Chapter 3. If a window function is used, Equation 2.37 becomes:

$$X[m] = \sum_{n=1}^{N} x[n] f_m[n]\, W[n] \tag{2.38}$$

where $W[n]$ is the window function. In this equation, all the vectors are assumed to be the same length. Alternatively the summation can be limited to the length of the shortest function; that is, N = length of the shortest function. If $W[n]$ is a rectangular function, then $W[n] = 1$ for $1 \geq n \geq N$ and it can be omitted from the equation: a rectangular window is implemented implicitly by limits on the summation.

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* Some definitions would be more restrictive and require that a transform be *bilateral*, that is, it must be possible to recover the original data from the transformed data. We use the looser definition and reserve the term *bilateral transform* to describe reversible transformations.

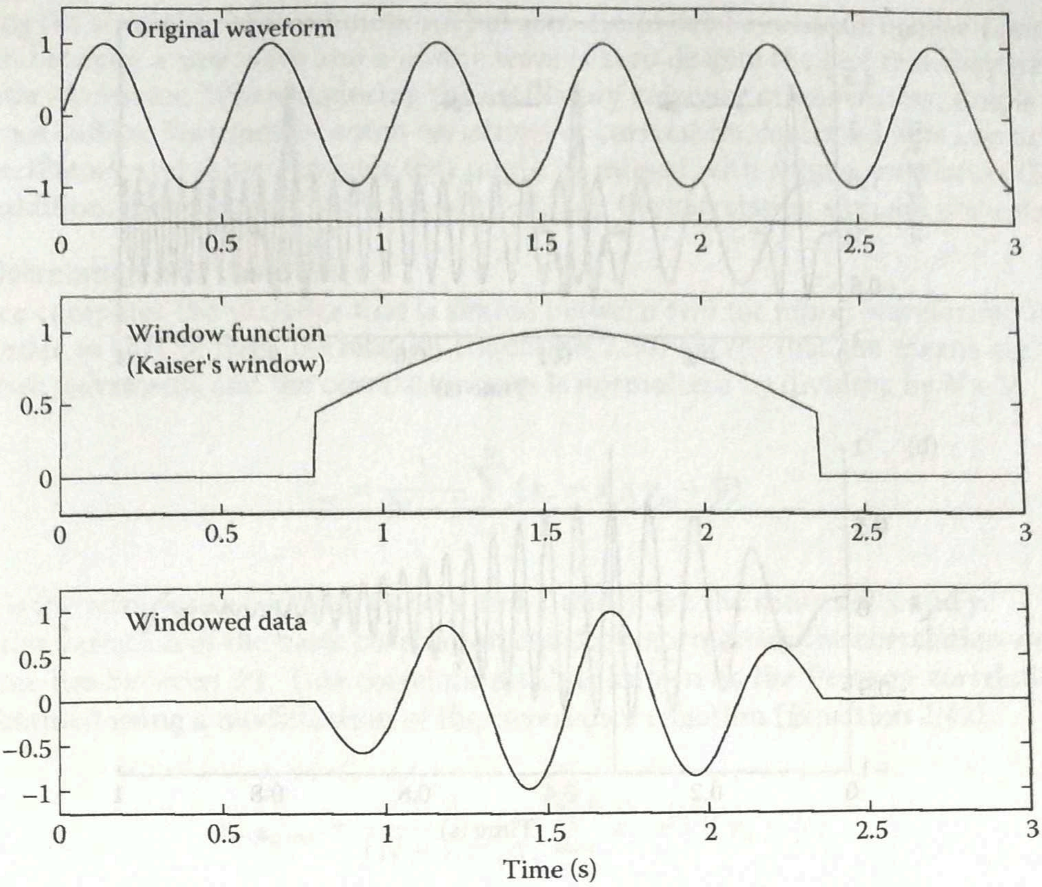

Figure 2.11 A waveform (upper plot) is multiplied by a window function (middle plot) to create a truncated version (lower plot) of the original waveform. The window function is shown in the middle plot. This particular window function is called the Kaiser window, one of many popular window functions described in Chapter 3.

If the probing function or basis is shorter than the waveform, then it might be appropriate to translate or slide it over the waveform. Then the correlation operation can take place at various relative positions along the waveform. Such a sliding correlation is shown in Figure 2.11 where a single probing function slides along a longer signal. At each position, a single correlation value is calculated:

$$X[k] = \sum_{n=1}^{N} x[n] f[n+k] \tag{2.39}$$

The output is a series of correlation values as a function of k which defines the relative position of the probing function. If the probing function is a basis (i.e., a family of functions), then the correlation values are a function of both k and family member m:

$$X[m,k] = \sum_{n=1}^{N} x[n]\, f_m[n+k] \tag{2.40}$$

where the variable k indicates the relative position between the two functions and m defines the specific family member of the basis. Equation 2.40 is used to implement the continuous wavelet transform described in Chapter 7 (Figure 2.12).

A variation of this approach can be used for long, or even infinite, probing functions, provided the probing function itself is shortened by windowing to a length that is less than the

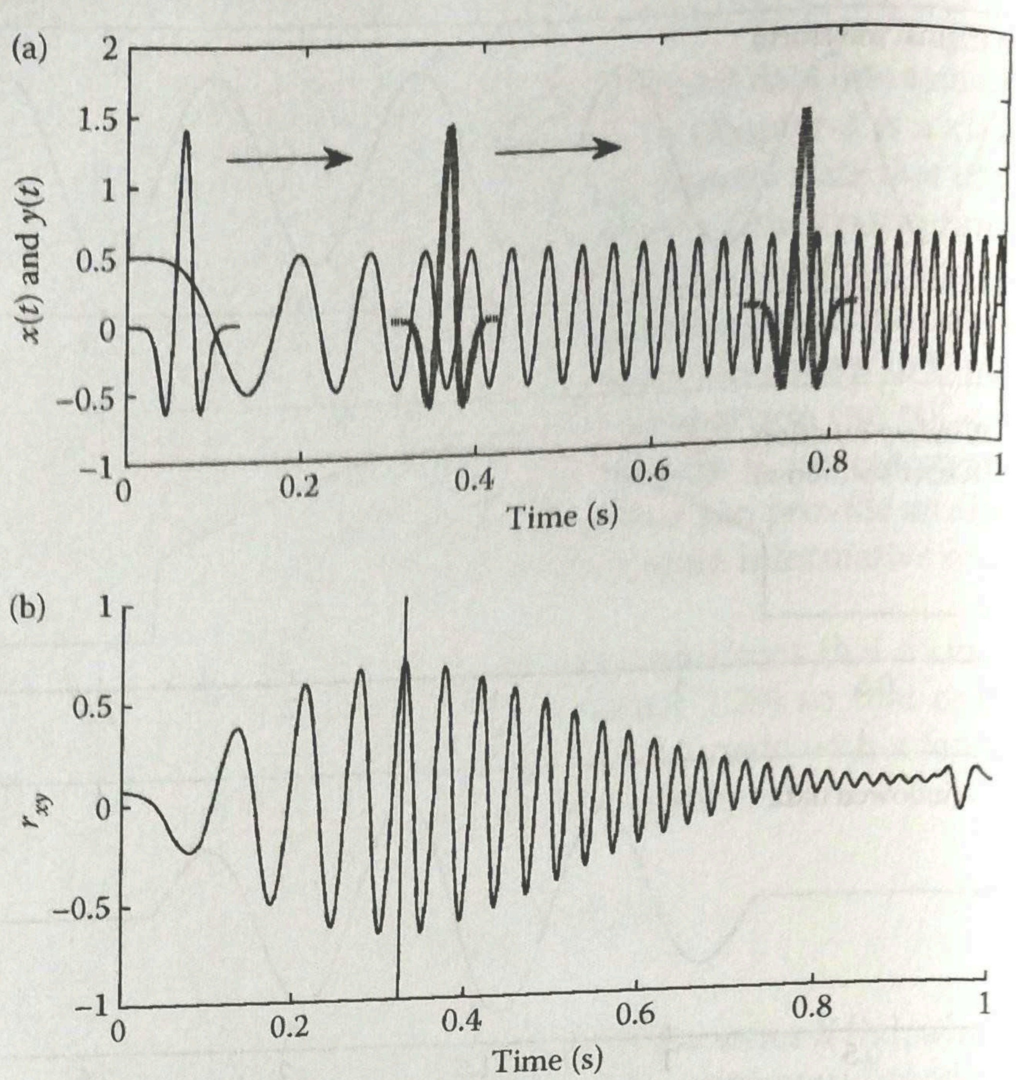

Figure 2.12 (a) The probing function slides over the signal and at each position the corre between probe and signal is calculated using Equation 2.39. In this example, the probing fu is one member of the *Mexican Hat* family (see Chapter 7) and the signal is a sinusoid that increa its frequency linearly over time known as a *chirp*. (b) The result shows the correlation betwe waveform and the probing function as it slides across the waveform. Note that this correlation sinusoidally as the phase between the two functions varies, but reaches a maximum aroun the time when the signal is most like the probing function.

waveform. Then the shortened probing function can be moved across the waveform in th manner as a short probing function. The equation for this condition becomes

$$X[m,k] = \sum_{n=1}^{N} x[n]\big(W[n+k] \quad f_m[n]\big)$$

where $f_m[n]$ are basis functions shortened or restricted by the sliding window function. It The variables n and k have the same meaning as in Equation 2.40: basis family membe relative position. This is the approach taken in the short-term Fourier transform describ Chapter 6.

Note that Equations 2.36 through 2.41 all involve correlation between the signal and a or sliding probing function or basis. A fixed or sliding window function may also be invo A series of correlations is generated, possibly even a two-dimensional series. We will encou each of these equations again in different signal analyses, illustrating the importance and fu damental nature of the basic correlation equation (Equation 2.29).