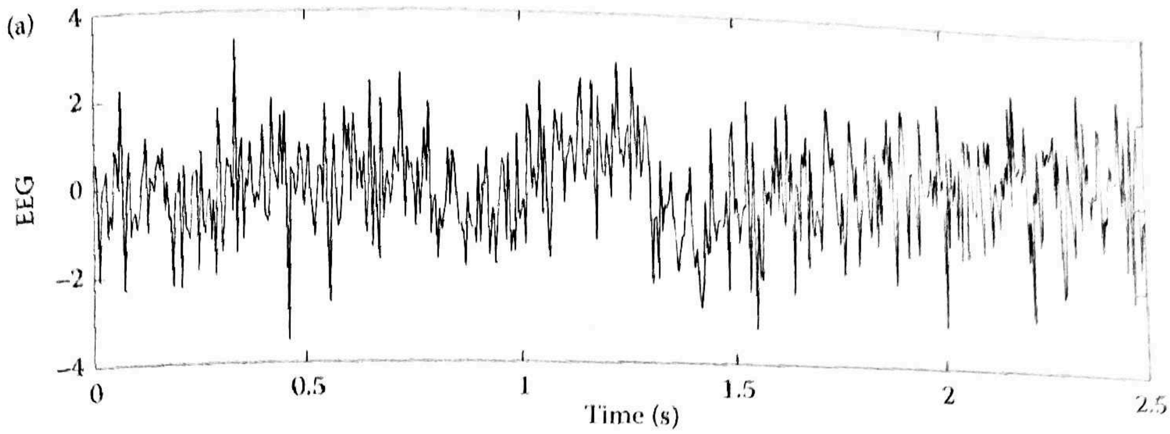

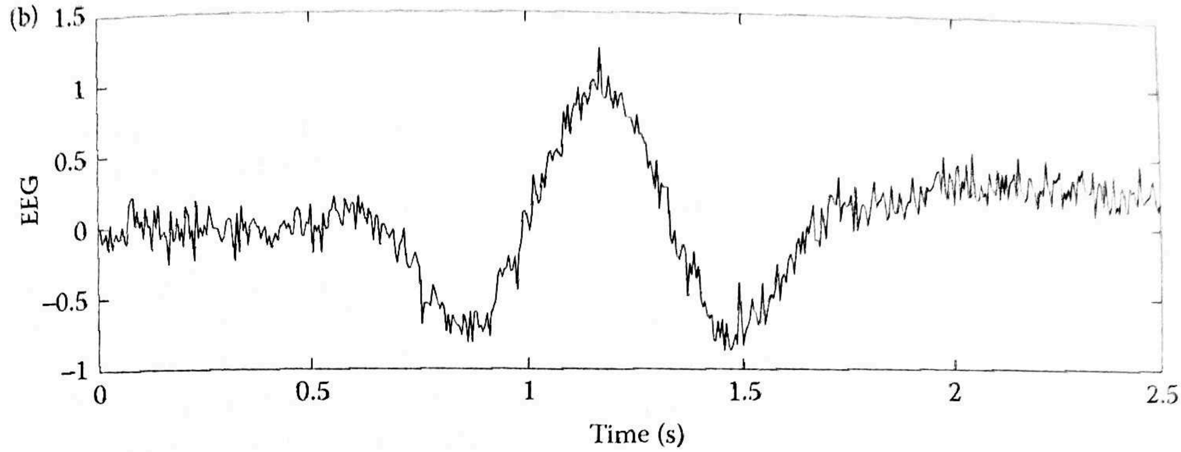

Figure 4.2 (a) The raw EEG signal showing a single response to the stimulus. (b) The ensemble average of 100 individual responses such as in graph (a) with the VER now clearly visible.

4.3 Z-Transform

The frequency-based analysis introduced in the last chapter is the most useful tool for analyzing systems or responses in which the waveforms are periodic or aperiodic, but cannot be applied to transient responses of infinite length, such as step functions, or systems with nonzero initial conditions. These shortcomings motivated the development of the *Laplace transform* in the analog domain. Laplace analysis uses the complex variable s, where $s = \sigma + j\omega$ as a representation of complex frequency in place of $j\omega$ in the Fourier transform.

$$X(\sigma,\omega) = \int_{0}^{\infty} x(t)e^{-\sigma t}\, e^{-j\omega t}\, dt = \int_{0}^{\infty} x(t)e^{-st}\, dt \tag{4.4}$$

The addition of the $e^{-\sigma t}$ term in the Laplace transform insures convergence for a wide range of input functions, $x(t)$, provided σ has the proper value. Like the Fourier transform, the Laplace transform is bilateral, but unlike the Fourier transform, it is not determined numerically: transformation tables and analytical calculations are used.

The *Z-transform* is a modification of the Laplace transform that is more suited to the digital domain; it is much easier to make the transformation between the discrete time domain and the Z-domain. Since we need a discrete equation, the first step in modifying the Laplace transform is to convert it to a discrete form by replacing the integral with a summation and substituting sample number n for the time variable t. (Actually, $t = nT_s$, but we assume that T_s is normalized to 1.0 for this development.) These modifications give

$$X(\sigma,\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-\sigma n}e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]r^{n}e^{-j\omega n} \tag{4.5}$$

where $r = e^{\sigma}$.

Equation 4.5 is a valid equation for the Z-transform, but in the usual format, the equation is simplified by defining another new, complex variable. This is essentially the idea used in the Laplace transform, where the complex variable s is introduced to represent $\sigma + j\omega$. The new variable, z, is defined as $z = re^{-j\omega} = |z|\ e^{-j\omega}$, and Equation 4.5 becomes

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = Z\left[x[n]\right] \tag{4.6}$$

This is the defining equation for the Z-transform, notated as $Z[x[n]]$. The Z-transform follows the format of the general transform equation involving projection on a basis. In this case, the basis is z^{-n}. In any real application, the limit of the summation in Equation 4.6 is finite, usually the length of $x[n]$.

As with the Laplace transform, the Z-transform is based around the complex variable, in this case, the arbitrary complex number z, which equals $|z|\ e^{j\omega}$. Like the analogous variable s in the Laplace transform, z is termed the *complex frequency*. As with the Laplace variable s, it is possible to substitute $e^{j\omega}$ for z to perform a strictly sinusoidal analysis.* This is a very useful property as it allows us to easily determine the frequency characteristic of a Z-transform function.

While the substitutions made to convert the Laplace transform to the Z-transform may seem arbitrary, they greatly simplify conversion between time and frequency in the digital domain. Unlike the Laplace transform, which requires a table and often considerable algebra, the conversion between discrete complex frequency representation and the discrete time function is easy as shown in the next section. Once we know the Z-transform of a filter, we can easily find the spectral characteristics of the filter: simply substitute $e^{j\omega}$ for z and apply standard techniques like the Fourier transform.

Equation 4.6 indicates that every data sample in the sequence $x[n]$ is associated with a unique power of z; that is, a unique value of n. The value of n defines a sample's position in the sequence. If $x[n]$ is a time sequence, the higher the value of n in the Z-transform, the further along in the time sequence a given data sample is located. In other words, in the Z-transform, the value of n indicates a time shift for the associated input waveform, $x[n]$. This time-shifting property of z^{-n} can be formally stated as

$$Z[x(n-k)] = z^{-k}Z[x(n)] \tag{4.7}$$

This time-shifting property of the Z-transform makes it very easy to implement Z-transform conversation, either from a data sequence to the Z-transform representation or vice versa.

4.3.1 Digital Transfer Function

As in Laplace transform analysis, the most useful applications of the Z-transform lies in its ability to define the digital equivalent of a transfer function. By analogy to linear system analysis, the digital transfer function is defined as

$$H[z] = \frac{Y[z]}{x[z]} \tag{4.8}$$

* If $|z|$ is set to 1, then $z = e^{j\omega}$. This is called evaluating z on the unit circle. See Smith (1997) or Bruce (2001) for a clear and detailed discussion of the properties of z and the Z-transform.

be obtained from Equation 4.10 by applying the time-shift interpretation (i.e., Equation 4.7) to the term z^{-n}:

$$y[n] = \sum_{k=0}^{K-1} b[k]x[n-k] - \sum_{\lambda=1}^{L} a[\ell]y[n-\lambda] \tag{4.12}$$

This equation is derived assuming that $a[0] = 1$ as specified in Equation 4.9. In addition to its application to filters, Equation 4.12 can be used to implement any linear process given the a and b coefficients. We will find it can be used to represent IIR filter later in this chapter and the autoregressive moving average (ARMA) model described in Chapter 5.

Equation 4.12 is the fundamental equation for implementation of all linear digital filters. Given the a and b weights (or coefficients), it is easy to write a code that is based on this equation. Such an exercise is unnecessary, as MATLAB has again beaten us to it. The MATLAB routine `filter` uses Equation 4.12 to implement a wide range of digital filters. This routine can also be used to realize any linear process given its Z-transform transfer function as shown in Example 4.2. The calling structure is

```
y = filter(b,a,x)
```

where `x` is the input, `y` the output, and `b` and `a` are the coefficients in Equation 4.12. If `a` = 1, all higher coefficients are of a in Equation 4.12 are zero and this equation reduces to the convolution equation (Equation 2.55); hence, this routine can also be used to perform the convolution of `b` and `x`.

Designing a digital filter is just a matter of finding coefficients, $a[\ell]$ and $b[k]$, that provide the desired spectral shaping. This design process is aided by MATLAB routines that generate the a and b coefficients that produce a desired frequency response.

In the Laplace domain, the frequency spectra of a linear system can be obtained by substituting the real frequency, $j\omega$, for the complex frequency, s, into the transfer function. This is acceptable as long as the input signals are restricted to sinusoids and only sinusoids are needed to determine a spectrum. A similar technique can be used to determine the frequency spectrum of the Z-transfer function $H(z)$ by substituting $e^{j\omega}$ for z. With this substitution, Equation 4.12 becomes

$$H[m] = \frac{\sum_{k=0}^{K} b[k]e^{-j\omega k}}{\sum_{\lambda=0}^{L} a[\ell]e^{-j\omega\lambda}} = \frac{\sum_{k=0}^{K} b(k)e^{-2\pi mn/N}}{\sum_{\lambda=0}^{L} a[\ell]e^{-2\pi mn/N}} = \frac{FT(b[k])}{FT(a[\ell])} \tag{4.13}$$

where FT indicates the Fourier transform. As with all Fourier transforms, the actual frequency can be obtained from the harmonic number m after multiplying by f_s/N or $1/(NT_s)$.

EXAMPLE 4.2

A system is defined by the digital transfer function below. Plot its frequency spectrum: magnitude in dB and phase in degrees. Also plot the time-domain response of the system to an impulse input (i.e., the system's impulse response). Assume a sampling frequency of $f_s = 1$ kHz and make $N = 512$. These are both optional: f_s is provided so that the frequency curve can be plotted in Hz instead of relative frequency, and N is chosen to be large enough to produce a smooth frequency curve. (In this example, it is actually much larger than needed for a smooth curve: $N = 128$ would be sufficient).

$$H[z] = \frac{0.2 + 0.5z^{-1}}{1 - 0.2z^{-1} + 0.8z^{-2}}$$