# EE486: Noise in Biomedical Signals

Instructor: Won
Department of Electrical and Computer Engineering, CSULA

# 1 Noise sources

## 1.1 Electronic

1. thermal: random movement of charge carriers in conductor, "white"
   e.g., for a MOS amplifier input with pmos (M1) in series with an nmos (M2)
   $$N_{thermal} = \frac{16kT}{3g_{m1}}\left(1 + \frac{g_{m2}}{g_{m1}}\right)$$
   Depends on properties of recording site
   - materials
   - homogeneties
   - size
   - electrode impedance
2. shot noise: also white noise. random fluctuations in movement of charge carriers
3. flicker noise
   $$N_{1/f}(f) = \frac{K}{C_{ox}WL}\frac{1}{f^a}$$
   $a \sim 0.7 - 1.2$ (a device characteristic)

## 1.2 Physiological Variability

This essentially refers to the fact that biomedical variables cannot usually be measured under completely controlled circumstances. Some examples include:

1. measuring heart rate: person may have been virtually motionless on one measurement and moving on another
2. measuring neural activity: person may be performing different tasks; their attention may be elsewhere
3. protein levels may change depending on diet
4. because of circadian rhythms, time of day can affect measurements

### 1.3 Environmental/extrinsic noise

1. power line interference
2. mechanical vibrations
3. can be other biological sources: e.g.,
   - mother's heart beat when trying to measure fetal heart beat
   - movement during recording (motion artifact)

## 2 A useful concept

For independent identically distributed (i.i.d.) RVs e.g., separate measurements of a Gaussian voltage.

If $X_1, X_2, \cdots X_n$ are $n$ independent observations of the same voltage in a particular circuit, and

$$Y \equiv X_1 + X_2 + \cdots + X_n$$

i.e., $Y$ is the sum of the independent observations, then:

$$\begin{aligned}
E[Y] &= \int_{x_n} \cdots \int_{x_2} \int_{x_1} (x_1 + x_2 + \cdots + x_n) f_X(x_1) f_X(x_2) \cdots f_X(x_n) dx_1 dx_2 \cdots dx_n \\
&= \int_{x_n} \cdots \int_{x_2} \int_{x_1} x_n f_X(x_1) dx_1 f_X(x_2) dx_2 \cdots f_X(x_n) dx_n \cdots \\
&\quad + \int_{x_n} \cdots \int_{x_2} \int_{x_1} x_2 f_X(x_1) dx_1 f_X(x_2) dx_2 \cdots f_X(x_n) dx_n \\
&\quad + \int_{x_n} \cdots \int_{x_2} \int_{x_1} x_1 f_X(x_1) dx_1 f_X(x_2) dx_2 \cdots f_X(x_n) dx_n \\
&= \int_{x_n} x_n f_X(x_n) dx_n + \cdots + \int_{x_2} x_2 f_X(x_2) dx_2 + \int_{x_1} x_1 f_X(x_1) dx_1 \\
&= E[X_n] + \cdots + E[X_2] + E[X_1]
\end{aligned}$$

Since $X_j$s are i.i.d, then $E[X_1] = E[X_2] = \cdots = E[X_n] \equiv \overline{X}$

$$E[Y] = \mathbf{n\overline{X}}$$

Similarly for $E[Y^2]$

$$E[Y^2] = \mathbf{n\overline{X^2}}$$

Thus,

$$\begin{aligned}\sigma_Y^2 &= n\overline{X^2} - n\overline{X} \\ &= n\left(\overline{X^2} - \overline{X}\right) \\ &= \mathbf{n\sigma_X^2}\end{aligned}$$

Now, we can see how averaging observations increases our "signal-to-noise ratio" [1]. Let's define $S_i$ to be the signal component of the $i^{th}$ observation.

$$\bar{S}_{RMS} = \left(\frac{1}{n}(S_1^2 + S_2^2 + ... + S_n^2)\right)^{1/2} \equiv S_{RMS} = \sigma_S$$

For a single observation,

$$\text{SNR}(S) \equiv \frac{\sigma_S}{\sigma_N}$$

whereas, for the average of the noise observations $Z \equiv \frac{1}{n}Y$,

$$\begin{aligned}\sigma_{\bar{S}} &= \sigma_S \\ \sigma_Z^2 &= \frac{1}{n^2}\sigma_Y^2 = \frac{\sigma_N^2}{n}\end{aligned}$$

$$\text{SNR}(Z) \equiv \frac{\sigma_S}{\frac{\sigma_N}{\sqrt{n}}} = \sqrt{n} \cdot \text{SNR}(S)$$

Making use of this concept is called *"averaging out noise"*.

# 3 Statistical properties of common types of noise

## 3.1 Uniform Distribution

Some noise is modeled with a uniform distribution, such as quantization noise. The uniform distribution is described by the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{x_2 - x_1} & x_1 < x \le x_2 \\ 0 & \text{otherwise} \end{cases}$$

Mean

$$\overline{X} = \frac{x_1 + x_2}{2}$$

[1] Here, we define "signal to noise ratio" as the ratio of the RMS of the signal to the RMS of noise, although technically it should be the ratio of the signal power to noise power.

Mean square value

$$
\begin{aligned}
\overline{X^2} &= \int_{x_1}^{x_2} \frac{x^2}{x_2 - x_1} dx \\
&= \frac{1}{x_2 - x_1} \left[ \frac{x^3}{3} \right]_{x_1}^{x_2} \\
&= \frac{x_2^3 - x_1^3}{3(x_2 - x_1)} \\
&= \frac{b^2 + ab + a^2}{3}
\end{aligned}
$$

Variance

$$
\begin{aligned}
\sigma_X^2 &= \frac{b^2 + ab + a^2}{3} - \left( \frac{x_1 + x_2}{2} \right)^2 \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}
$$

## 3.2 Normal Distribution

Many other types of noise, such as thermal noise, is often modeled as Gaussian, or normally distributed, noise. A Gaussian random variable $X$ has a density function given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x - \overline{X}}{\sigma_X}\right)^2}$$

and

$$P(X \le x_0) = \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{x_0} e^{-\frac{1}{2}\left(\frac{x - \overline{X}}{\sigma_X}\right)^2} dX$$

If we define $T \equiv \frac{x - \overline{X}}{\sigma_X}$, then

$$
\begin{aligned}
\frac{dT}{dX} &= \frac{1}{\sigma_X} \\
\text{or } dT &= \frac{1}{\sigma_X} dX \\
\Rightarrow P(X \le x_0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t(x_0)} e^{-\frac{1}{2}(t)^2} dT \\
&= \Phi(t(x_0)) = \Phi\left( \frac{x - \overline{X}}{\sigma_X} \right)
\end{aligned}
$$

## 3.3 White noise

White noise refers to noise for which the spectrum is flat, analogous to the flat spectrum of white light.

# 4 References


- Semmlow, Biosignal and Medical Image Processing, CRC Press 2009
- Webster, Medical Instrumentation, Wiley, 3rd edition.