Part 1a - 1b

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$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 5 & 7 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} x(t).$$

Figure 1. *LTI used to solve question one.*

(a) Is the system stable? Explain your answer.

No, the system is not currently stable because one of its eigenvalues has a positive real part. The eigenvalues are as follows, rounded for ease of viewing: [7.669, -0.334 + 0.136i, -0.334 - 0.136i].

(b) Is the system controllable? Explain your answer.

Yes, the system is controllable because the matrix Q ([B | A | A^2B]) is full row rank: i.e. it has n linearly dependent rows.

(c) Let the initial state vector be x0 = x(0) = [0; 1; 0] Using the MATLAB expm command, plot the output of the unforced system for t ∈[0,2].

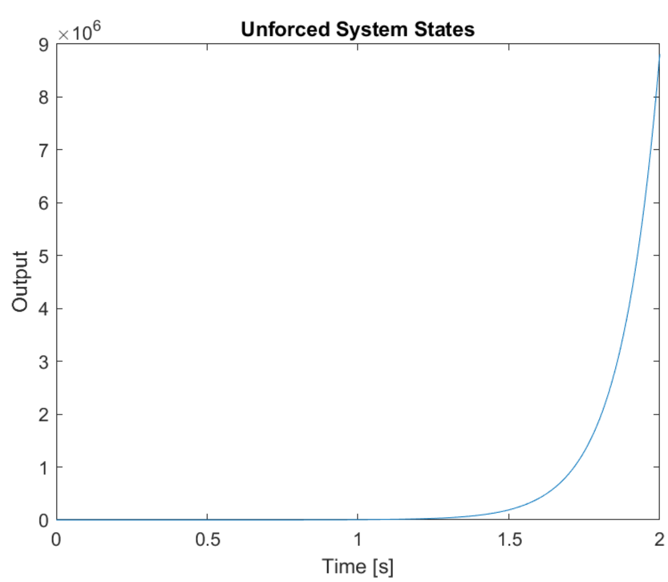

Figure 2. *Unforced system output for the LTI for t ∈ [0,2].*

Part 1.d - 1.e

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(d) Use the MATLAB place command to find the matrix K such that the matrix A—bK contains the following set of eigenvalues: {—1 + i, —1 —i, —2}.

```
Check for correct poles for 1.d:
  -2.0000 + 0.0000i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
```

Figure 3. *Print statement for the adjusted eigenvalues , after applying K, in MATLAB.*

(e) Let the initial state vector be $x_0 = x(0) = [0; 1; 0]$. Use the MATLAB expm command to plot the output of the system under the feedback law $u(t) = -Kx(t)$ for $t \in [0,10]$. Use the K found in the previous problem.

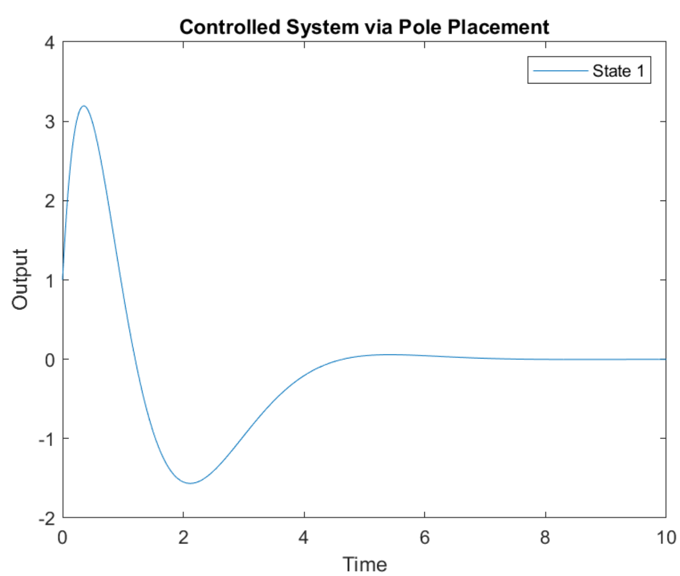

Figure 4. *Controlled system, via pole placement, after applying K found in 1.d.*

Part 2.a

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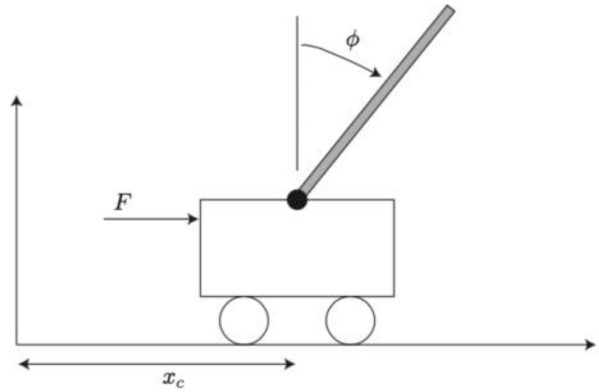
2. Consider the "pendulum on a cart" system:



The equations of motion for this system are

$$\gamma \ddot{x}_c - \beta \ddot{\phi} \cos\phi + \beta \dot{\phi}^2 \sin\phi + \mu \dot{x}_c = F$$

$$\alpha \ddot{\phi} - \beta \ddot{x}_c \cos\phi - D \sin\phi = 0,$$

where α, β, γ, D, and μ are physical constants determined by the masses of the cart and pendulum, pendulum length, and friction, and F is an externally applied force. All units are mks, and angles are expressed in radians.

(a) (10 points) Define the state vector

$$x = \begin{bmatrix} x_c \\ \phi \\ \dot{x}_c \\ \dot{\phi} \end{bmatrix}$$

and the input $u = F$. Write the cart/pendulum equations of motion as a nonlinear state space equation. *hint:* It may help to convert the system into standard mechanical form as an intermediate step, and note that the matrix

$$M = \begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix}$$

is always invertible.

STANDARD MECHANICAL FORM:

$$M(q)\ddot{q} + C(q,\dot{q}) + G(\dot{q}) = \tau$$

$$\begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} \beta\dot{\phi}^2 \sin & \mu\dot{x}_c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -D\sin\phi \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Part 2.a Cont.

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REARRANGE:

$$\begin{bmatrix} \gamma & -\beta\cos\phi \\ -\beta\cos\phi & \alpha \end{bmatrix} \begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} - \begin{bmatrix} \beta\dot{\phi}^2 \sin & \mu\dot{x}_c \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -D\sin\phi \end{bmatrix}$$

FIND INVERSE OF $m(q)$:

$$m(q)^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\gamma\alpha - \beta^2\cos^2\phi} \begin{bmatrix} \alpha & \beta\cos\phi \\ \beta\cos\phi & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha}{\gamma\alpha - \beta^2\cos^2\phi} & \frac{\beta\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi} \\ \frac{\beta\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi} & \frac{\gamma}{\gamma\alpha - \beta^2\cos^2\phi} \end{bmatrix}$$

APPLY INVERSE TO ORIGINAL EQ.:

(SKIP INITIAL MULTIPLICATION)

$$\begin{bmatrix} \ddot{x}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{F\alpha}{\gamma\alpha - \beta^2\cos^2\phi} \\ \frac{F\beta\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi} \end{bmatrix} - \begin{bmatrix} \frac{\alpha\beta\dot{\phi}^2\sin\phi}{\gamma\alpha - \beta^2\cos^2\phi} & \frac{\alpha\mu\dot{x}_c}{\gamma\alpha - \beta^2\cos^2\phi} \\ \frac{\beta^2\dot{\phi}^2\sin\phi\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi} & \frac{\mu\dot{x}_c\beta\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi} \end{bmatrix} - \begin{bmatrix} \frac{-D\beta\sin\phi\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi} \\ \frac{-D\gamma\sin\phi}{\gamma\alpha - \beta^2\cos^2\phi} \end{bmatrix}$$

$$\ddot{x}_c = \frac{F\alpha - \alpha\beta\dot{\phi}^2\sin\phi - \alpha\mu\dot{x}_c + D\beta\sin\phi\cos\phi}{\gamma\alpha - \beta^2\cos^2\phi}$$

$$\ddot{\phi} = \frac{F\beta\cos\phi - \beta^2\dot{\phi}^2\sin\phi\cos\phi - \mu\dot{x}_c\beta\cos\phi + D\gamma\sin\phi}{\gamma\alpha - \beta^2\cos^2\phi}$$

Part 2.a Cont. & 2.b

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$$\dot{X} = \begin{bmatrix} \dot{X}_c \\ \dot{\phi} \\ \ddot{X}_c \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} X_3 \\ X_4 \\ \dfrac{u\alpha - \alpha\beta X_4^2 \sin X_2 - \alpha\mu X_3 + D\beta \sin X_2 \cos X_2}{\gamma\alpha - \beta^2\cos^2 X_2} \\ \dfrac{u\beta\cos X_2 - \beta^2 X_4^2 \sin X_2 \cos X_2 - \mu X_3 \beta\cos X_2 - D\gamma \sin X_2}{\gamma\alpha - \beta^2\cos^2 X_2} \end{bmatrix}$$

(b) Describe the set of equilibrium points for the system. Describe them mathematically (i.e., with equations) and in English (i.e., what do they physically mean).

Mathematical Description

$u = X_3 = X_4 = 0$

① $\dfrac{D\beta \sin X_2 \cos X_2}{\gamma\alpha - \beta^2\cos^2 X_2} = 0$ ② $\dfrac{D\gamma \sin X_2}{\gamma\alpha - \beta^2\cos^2 X_2} = 0$

BOTH ①/② ARE ZERO WHEN $\sin X_2 = 0$:

- $\sin X_2 = 0$ AT $\ldots -\pi, 0, \pi, 2\pi, k\pi$ WHERE k IS AN INTEGER

- $\{ k\pi \mid k \in \mathbb{Z} \}$

Physical Description

Physically, this set of angles for x_2, or phi, describe the scenario where the pendulum is either vertical upwards or vertical downwards.

Part 2.c - 2.d

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(c) (5 points) Let $\gamma = 2$, $\alpha = 1$, $\beta = 1$, $D = 1$, and $\mu = 3$. The linearized system about the equilibrium point at $x = 0$ is

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -3 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u(t).$$

Compute the eigenvalues of A, and use them to say whatever you can about the stability of the equilibrium point at $x = 0$ for both the linearized system and original nonlinear system.

- The eigenvalues of the linearized system are as follows:
    - 0, -3.33, 1.13, -0.80
    - One eigenvalue has a positive real part, so the system is unstable
    - Rounded to two decimal points for readability
- Since this linearized system (near the equilibrium point x=0) is unstable, we can say that the original nonlinear system is unstable

(d) (15 points) Assume that the entire state can be measured directly (i.e., there is no need for an observer). Letting $R = 10$ and

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix},$$

use the MATLAB lqr command to find the corresponding optimal feedback control $u(t) = -K_c x(t)$.

Using a timestep of $T = 0.01$ seconds and a final time of $t_f = 30$ seconds and an initial state of $x_0 = [0,\ 0.1,\ 0,\ 0]^T$, calculate and plot the state of the *linearized system* under the feedback control law above. You may either write your own 4th order Runge-Kutta routine to solve the state equation, or use the MATLAB function `ode45`. Repeat for $x_0 = [0,\ 0.5,\ 0,\ 0]^T$, $x_0 = [0\ 1.0886\ 0\ 0]^T$, and $x_0 = [0,\ 1.1,\ 0,\ 0]^T$.

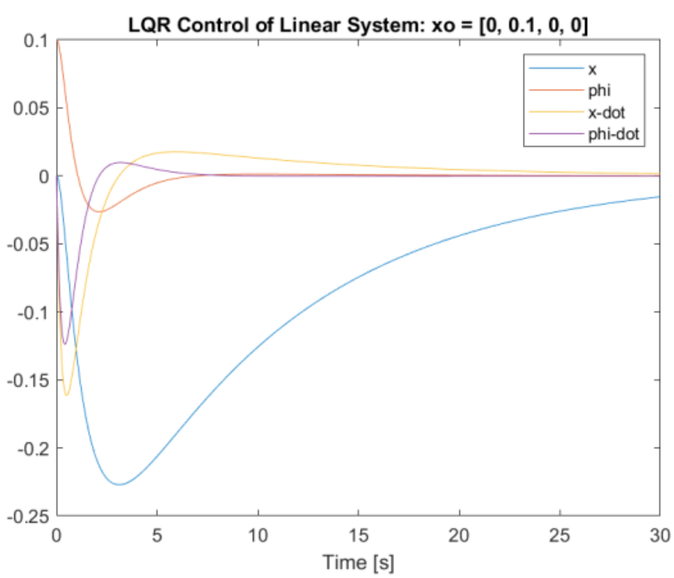

Figure 5. *Controlled Linearized Pendulum System, with initial conditions [0, 0.1, 0, 0], over t = [0, 30].*

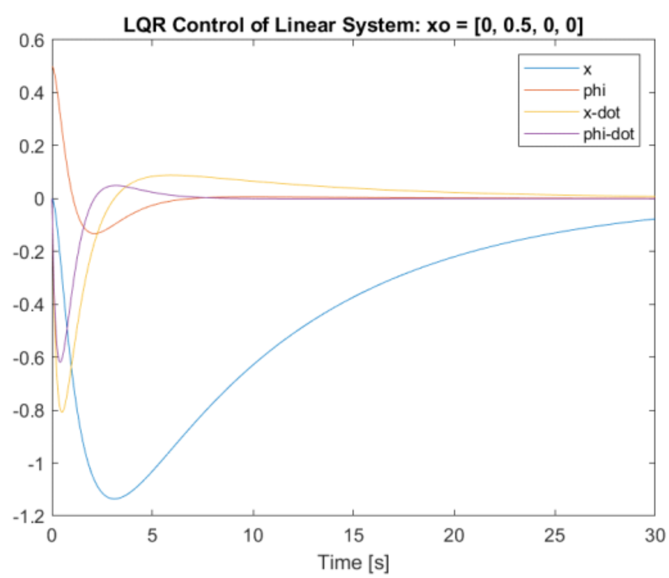

Figure 6. *Controlled Linearized Pendulum System, with initial conditions [0, 0.5, 0, 0], over t = [0, 30].*

Part 2d Cont. - 2.e

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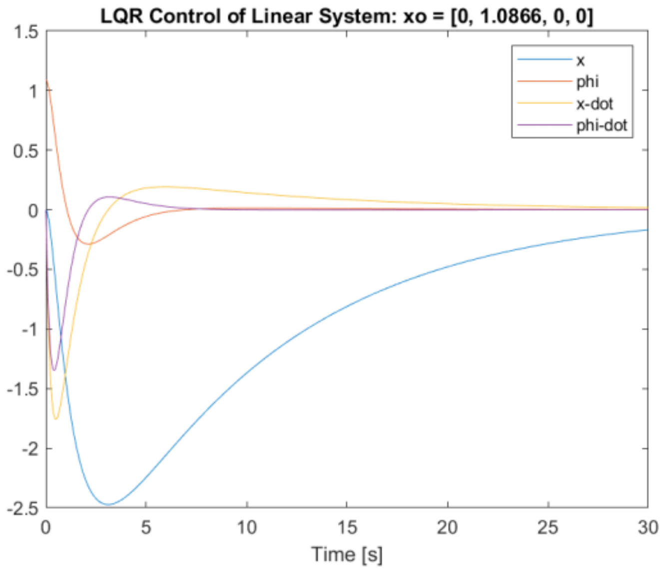

Figure 5. Controlled Linearized Pendulum System, with initial conditions [0, 1.0866, 0, 0].

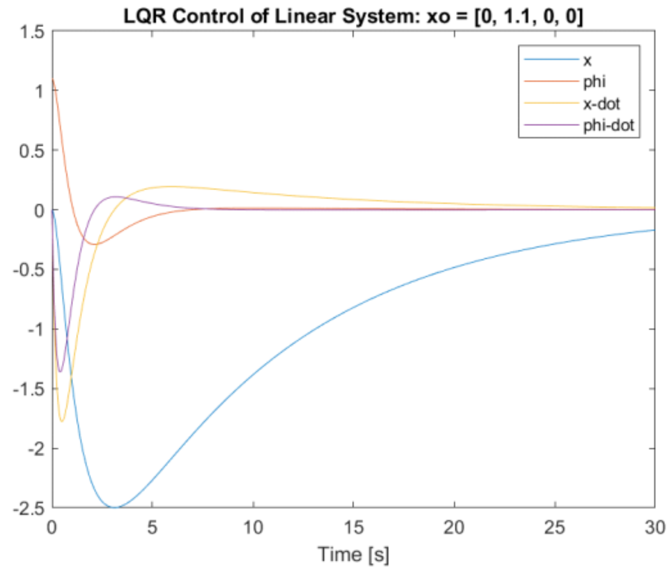

Figure 8. Controlled Linearized Pendulum System, with initial conditions [0, 1.1, 0, 0].

(e) Repeat part 2d using the full nonlinear state equations in the simulation (as opposed to the linearized state equations). Explain any differences you see in the results.

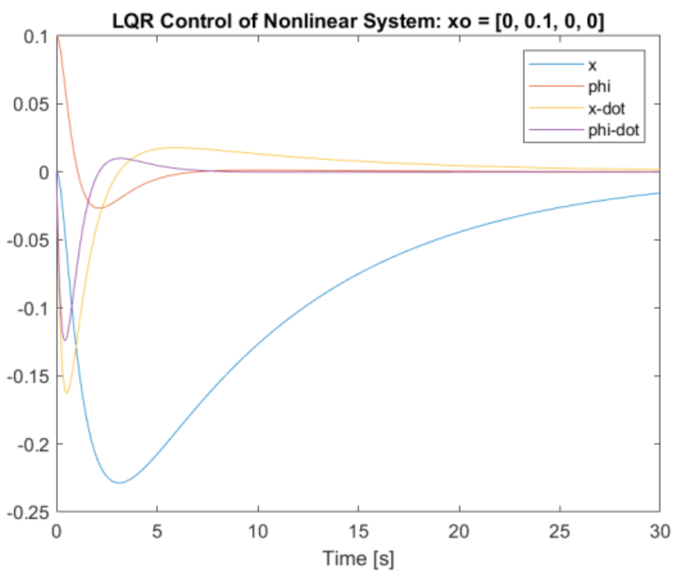

Figure 9. Controlled Nonlinear Pendulum System, with initial conditions [0, 0.1, 0, 0].

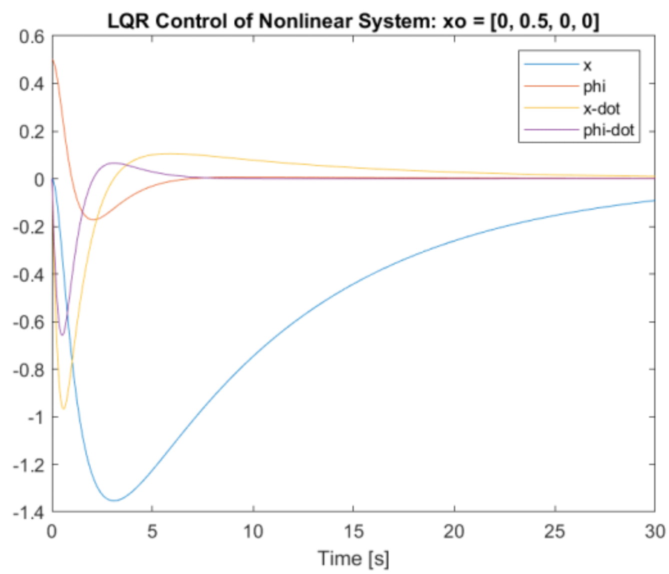

Figure 10. Controlled Nonlinear Pendulum System, with initial conditions [0, 0.5, 0, 0].

Part 2.e Cont. - 2.f

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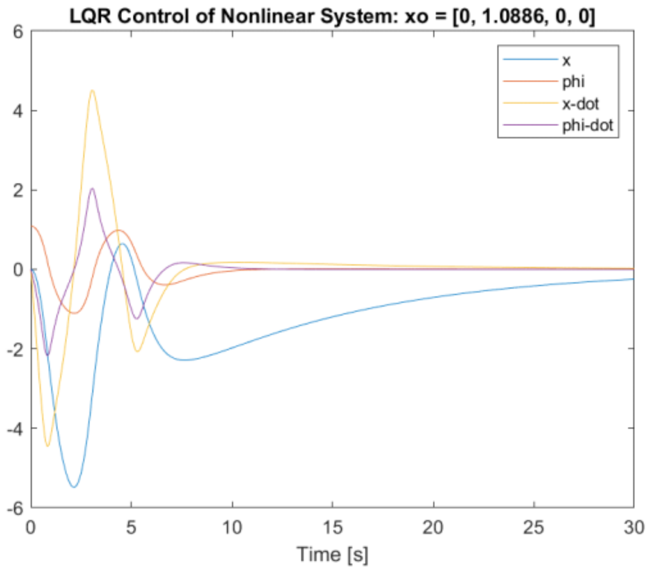

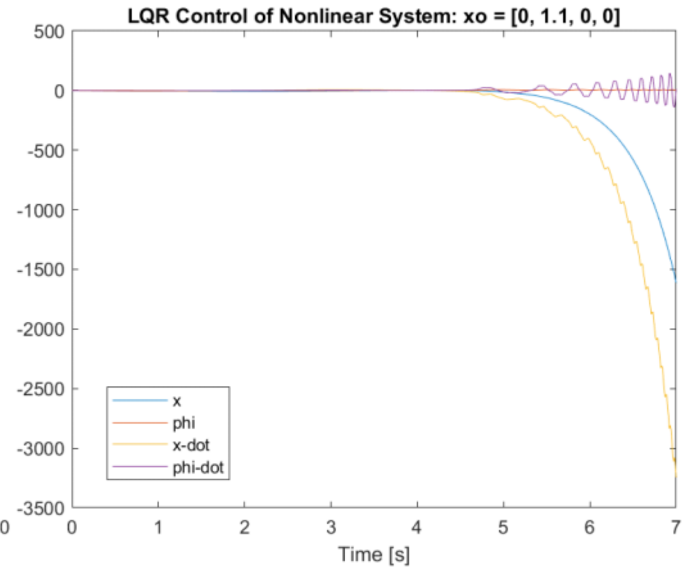

Figure 11. *Controlled Nonlinear Pendulum System, with initial conditions [0, 1.0866, 0, 0], over t = [0, 30].*

Figure 12. *Controlled Nonlinear Pendulum System, with initial conditions [0, 1.1, 0, 0], over t = [0, 7].*

Observable Differences

As the initial conditions for phi gets larger, the system becomes progressively more unstable. It appears to be leaving the "basin of attraction" for the stabilized linearized form near the equilibrium point. As the initial state of phi increases, the nonlinear system becomes progressively more unstable, until it can no longer maintain stability. When the initial condition for phi is 1.1, we see instability ensues around 4.5 seconds. This plot is bounded [0, 7], as under the 30 second time interval MATLAB is unable to plot the states

(f) Assume you have an output y given by a sensor that measures the cart position in inches. Find the matrix C so that y = Cx.

C = [39.3701 0 0 0]

2.g

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(g) (15 points) Using the LQR controller designed above, create a tracking controller that allows you to specify a desired cart position trajectory. Test this tracking controller for a desired output y_d that is a square wave as shown in the picture below. Simulate the full nonlinear dynamics using $T = 0.01$ seconds, $t_f = 200$ seconds, and $x_0 = [0, 0, 0, 0]^T$. Plot the state vs. time. On a separate graph, make a plot that overlays the desired and actual outputs. Explain what is going on.

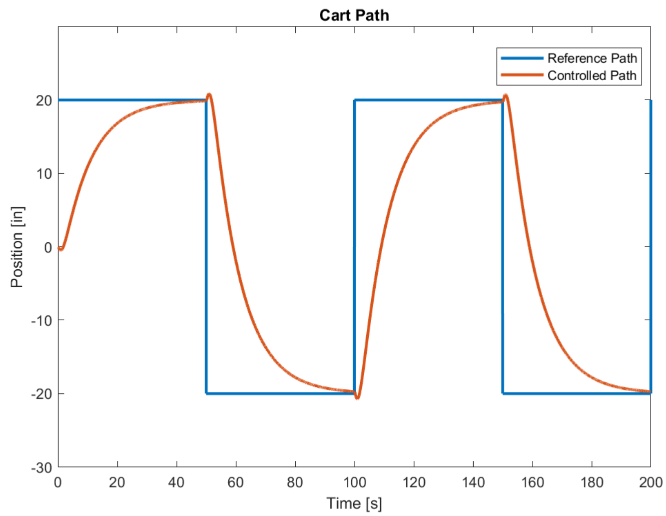

Figure 13. *Cart position overlayed with the desired position, with initial conditions [0, 0, 0, 0], over t = [0, 200].*

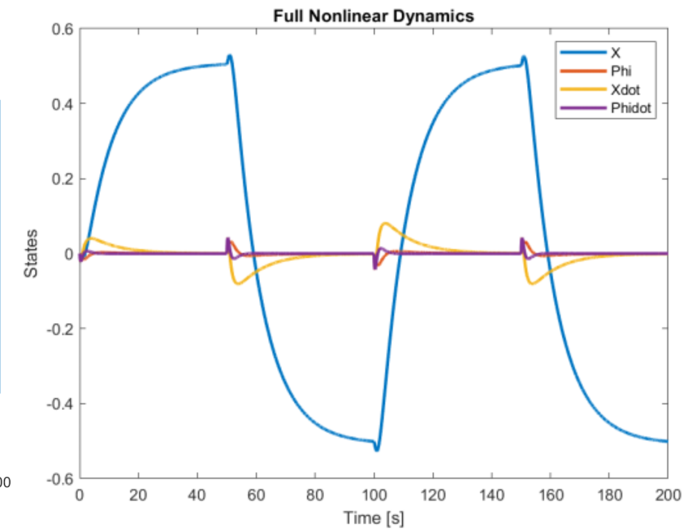

Figure 14. *Cart Path Following states, with initial conditions [0, 0, 0, 0], over t = [0, 200].*

Explain:

What we are seeing here is that we controlled for position, but these states are coupled so you see an effect in all states. Upon each turn to follow the path you also have a spike in velocity to attempt to continue on the path, and changes in phi and phidot to stabilize the system.

The spike in velocity is a result of the cart attempting to get up to 20m within the 50 sec window set by the reference path. This requires a degree of acceleration to achieve.

The changes in phi and phi dot are more of a result in the oscillation of the pendulum required to keep the pendulum stable and upright (as the initial conditions imply).

2.h

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(h) Choose new Q and R for the LQR controller to make the tracking controller better. Explain what you mean by “better”, and describe your reasoning for how you changed Q and R. Simulate the system using the same parameters as above, and demonstrate your improvement by plotting the desired and actual outputs.

The first attempt at LQR (Figure 13) was not a bad one, it was able to reach the peak value for the path within the 50 second interval before the desired change to the opposite extreme. Also, it only had minor "overshoot" behavior.

In order to tune the Q and R matrices, I decided that "better" meant a higher accuracy in reference to path following, while not generating an erratic system. To me, that meant to achieve a better path I should focus on penalizing bad performance on the states related to the position of the cart and the input force to the system. Phi and phi dot have no direct influence on the movement of the cart just maintaining the pendulum position.

The final Q and R matrices were, respectively; [15 0 0 0; 0 1 0 0; 0 0 15 0; 0 0 0 1] & 25. After playing with the values for position states and actuation effort, it was obvious there was a tradeoff between "overshoot" at corners of the reference path and a strict following of the path. You could follow the path very closely if you penalized just position, but this would come at a cost of high actuation and high velocity of the cart. This is why I choose to slightly improve the path following capabilities: to both minimize potential input requirements to the system as well as decrease the erratic maneuvers required during the "overshoot" periods.

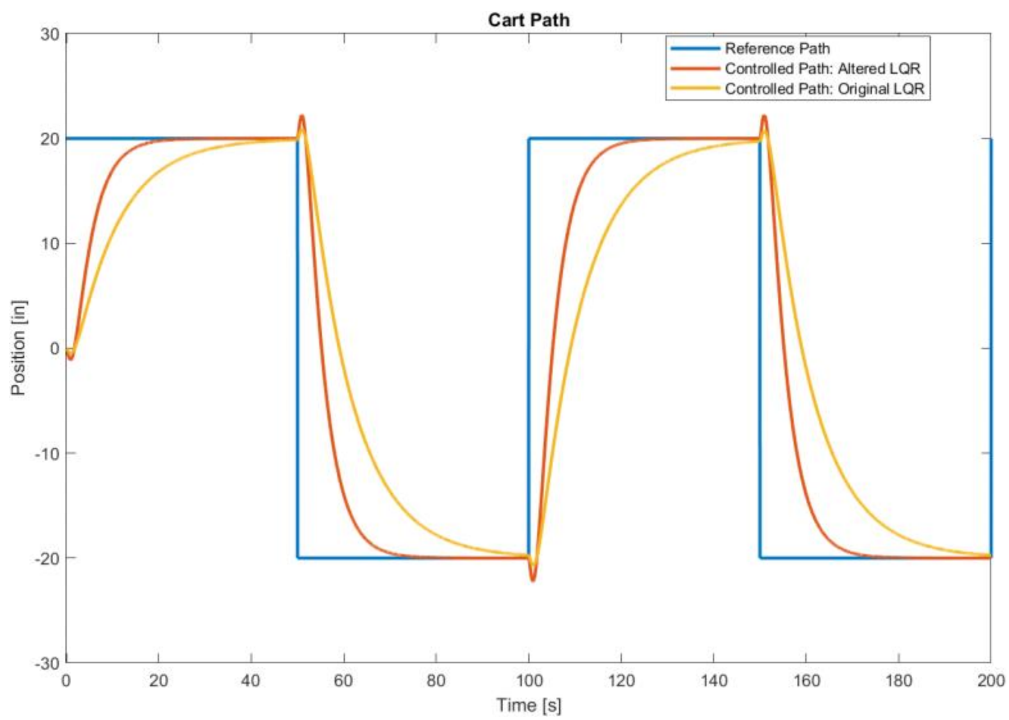

Figure 15. *Reference path for the cart, as well as the Original LQR controlled path given in Figure 13, and the final LQR controlled path designed in path (h).*