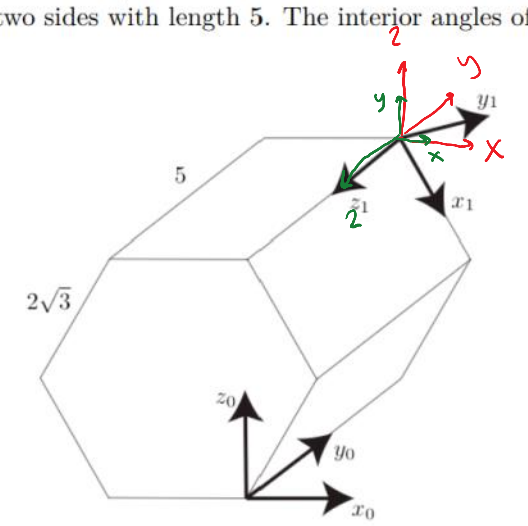
Question 1

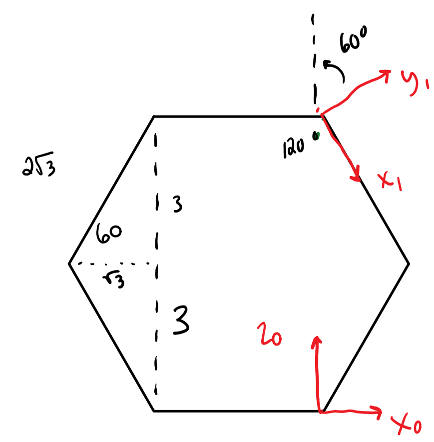
Monday, November 28, 2022 12:46 PM

1. (10 points) Consider the 3D shape pictured below. Find H_1^0. The front and back faces are regular hexagons, the length of each side is $2\sqrt{3}$. The six side faces are all identical rectangles that have two sides with length $2\sqrt{3}$ and two sides with length 5. The interior angles of a regular hexagon are 120^o.



Diagram

Used to derive the angle of rotation and translation needed when performing linked transforms, there will need to be a vertical displacement of 6 units and a rotational transform of 60 degrees. Not pictured is a translation of 5 units that will need to be made along the length of the prism.



Transform 1:

Perform a translation of 5 units along the 0 frame y axis and 6 units along the 0 frame z axis, resulting in the red frame above.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform 2:

Perform a positive rotation of 90 degrees around the relative x axis, yielding the green frame.

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c90 & -s90 & 0 \\ 0 & s90 & c90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transform 3:

Perform a negative rotation of 60 degrees around the relative z axis, yielding the desired frame.

$$G_3 = \begin{bmatrix} c(-60) & -s(-60) & 0 & 0 \\ s(-60) & c(-60) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Combine:

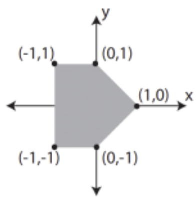
Combine all three relative motions by multiplying on the right successively: H0_1 = G1 * G2 * G3. The multiplication was performed on a MATLAB code that will be attached at the end of the document.

```
The Transformation H0_1:
    0.5000    0.8660         0         0
   -0.0000    0.0000   -1.0000    5.0000
   -0.8660    0.5000    0.0000    6.0000
         0         0         0    1.0000
```

Question 2

Sunday, December 4, 2022 1:03 AM

2. (15 points) **Fun With SE(2):** Consider the planar rigid body defined below. It is possible to "move" this body by transforming all of the corner points according to a 2D homogeneous transformation matrix and plotting the result.



Consider the two transformation matrices:

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0.866 & 0.500 & 0 \\ -0.500 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Move the rigid body according to the motions described below and plot the results. (It's probably best to write some sort of program to do this – we recommend using MATLAB, but you can use any language you want. Please include the source file of your program with your solution.)

(a) A, relative to the fixed frame.

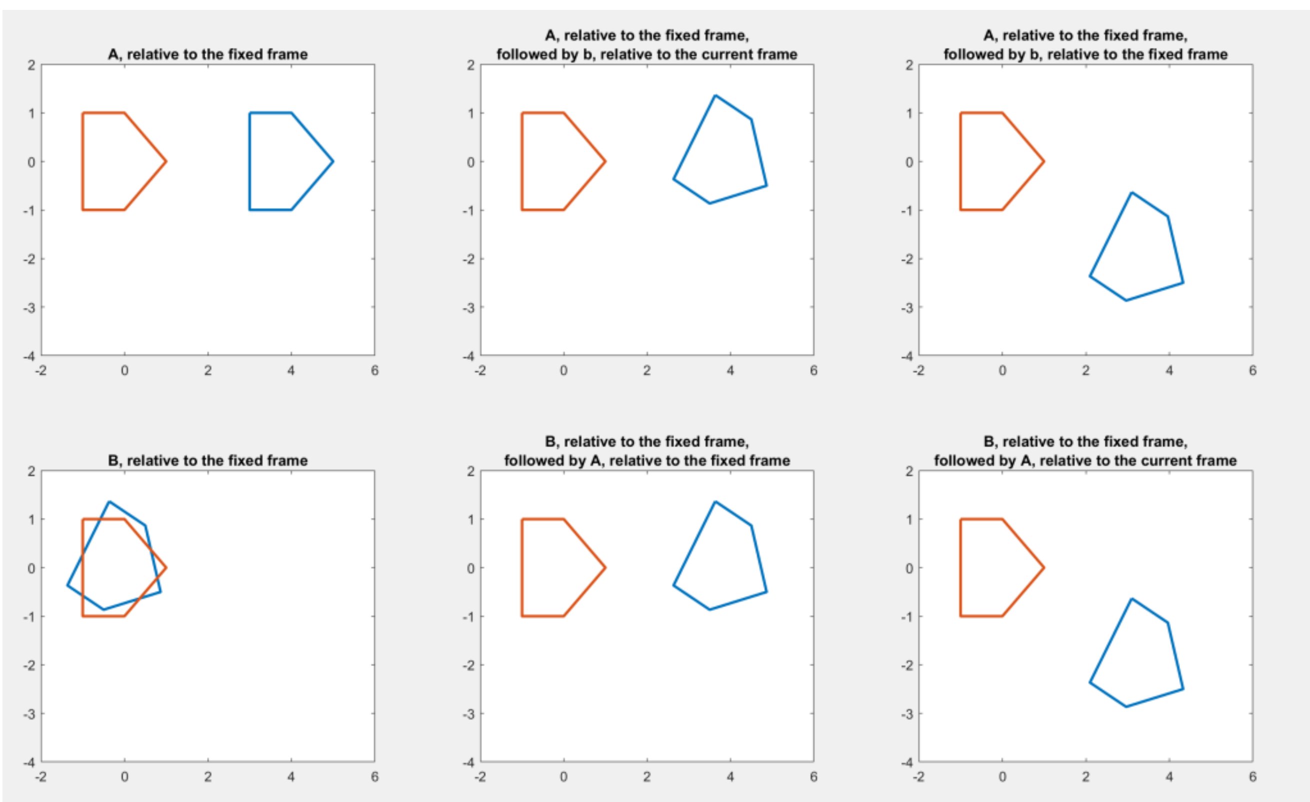
(b) A, relative to the fixed frame, followed by B, relative to the current frame.

(c) A, relative to the fixed frame, followed by B, relative to the fixed frame.

(d) B, relative to the fixed frame.

(e) B, relative to the fixed frame, followed by A, relative to the fixed frame.

(f) B, relative to the fixed frame, followed by A, relative to the current frame.

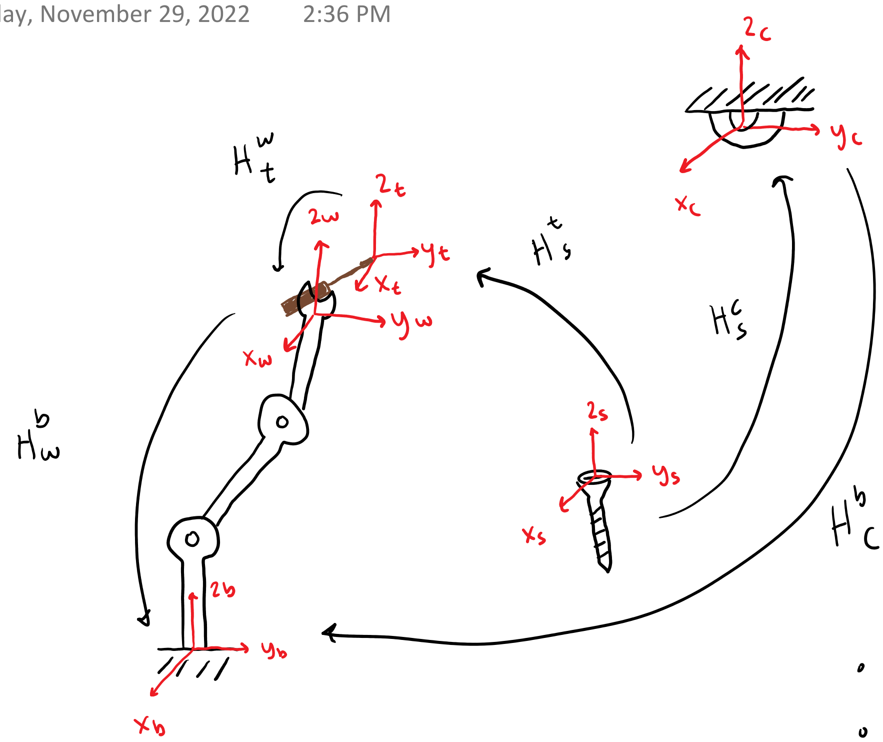


MATLAB Code

Please refer to the attached MATLAB code at the end of this document for any code relating to this question.

Question 3

Tuesday, November 29, 2022   2:36 PM



NOTE:

- $H^t_s = I$
- THIS IS MEANT TO EXPRESS THAT THE SCREW AND SCREWDRIVER SHOULD INTERFACE

SCREW → SCREWDRIVER → WRIST → BASE

$$\begin{bmatrix} p_b \\ 1 \end{bmatrix} = H^b_s \begin{bmatrix} p_s \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p_b \\ 1 \end{bmatrix} = H^b_w H^w_t H^t_s \begin{bmatrix} p_s \\ 1 \end{bmatrix}$$

SCREW → CAMERA → BASE

$$\begin{bmatrix} p_b \\ 1 \end{bmatrix} = H^b_s \begin{bmatrix} p_s \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p_b \\ 1 \end{bmatrix} = H^b_c H^c_s \begin{bmatrix} p_s \\ 1 \end{bmatrix}$$

COMPARE EQUIVALENT TRANSFORMS:

$$H^b_w H^w_t H^t_s = H^b_c H^c_s$$

$H^t_s = I$:

$$H^b_w H^w_t = H^b_c H^c_s$$

ISOLATE H^b_w:

$$H^b_w (H^w_t)(H^w_t)^{-1} = H^b_c H^c_s (H^w_t)^{-1}$$

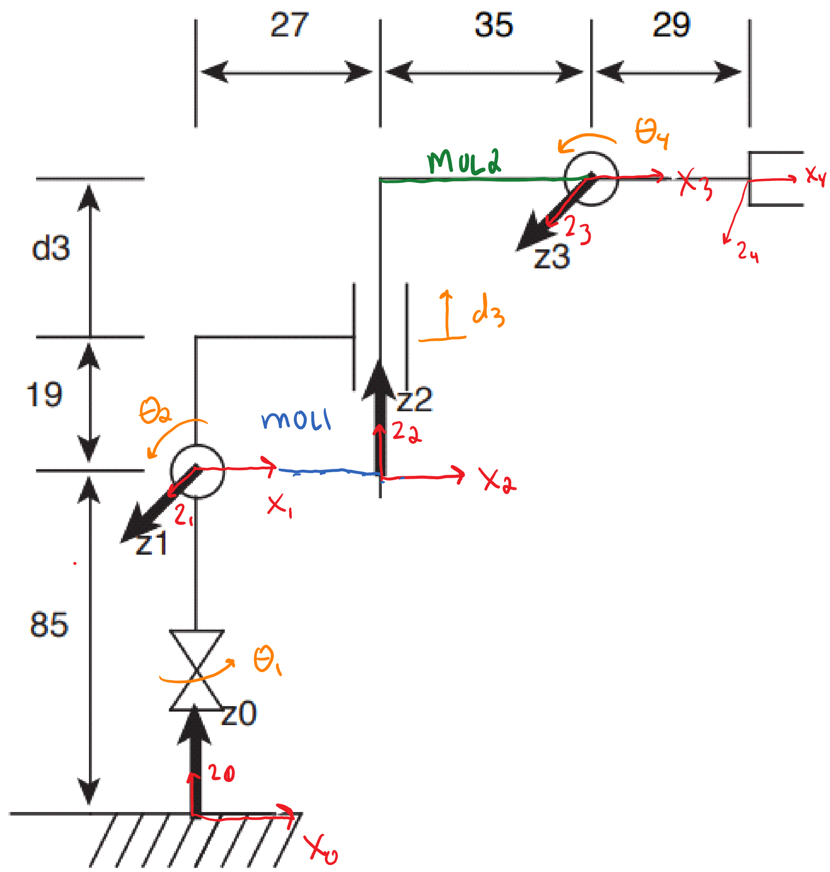
$$\boxed{H^b_w = H^b_c H^c_s (H^w_t)^{-1}}$$

General Approach

- Obtain the transform the of the points on the screw through the camera fame to the base frame
- Obtain the transform of the points on the screw through the screwdriver and wrist to the base frame
- Since this will yield the same screw points expressed in the base frame make the transforms equivalent
- Solve for the desired transform

Question 4

Tuesday, November 29, 2022 4:25 PM



i	θ_i	d_i	a_i	α_i
1	θ_1	85	0	90
2	θ_2	0	27	-90
3	0	d_3+19	35	90
4	θ_4	0	29	0

Choices in Defining Frames:

- The choice in the base frame x-axis was for convenience, and ended up working out nicely with the configuration of frame one such that there is no additional rotation between the x0 and x1 axis outside of the joint angle theta1.
- The x1 axis was also chosen for convenience and allowed there to be no extra terms associated with aligning the x1 and x2 axes outside of the joint angle theta2.
- No choice in selection for frame 2 and frame 3, they are both uniquely defined.
- Frame 4, the end-effector frame, was selected with the common convention of putting o4 between the fingers, with y4 being aligned with the direction the fingers move, z4 coming out of the page, and x4 chosen with the right hand rule.

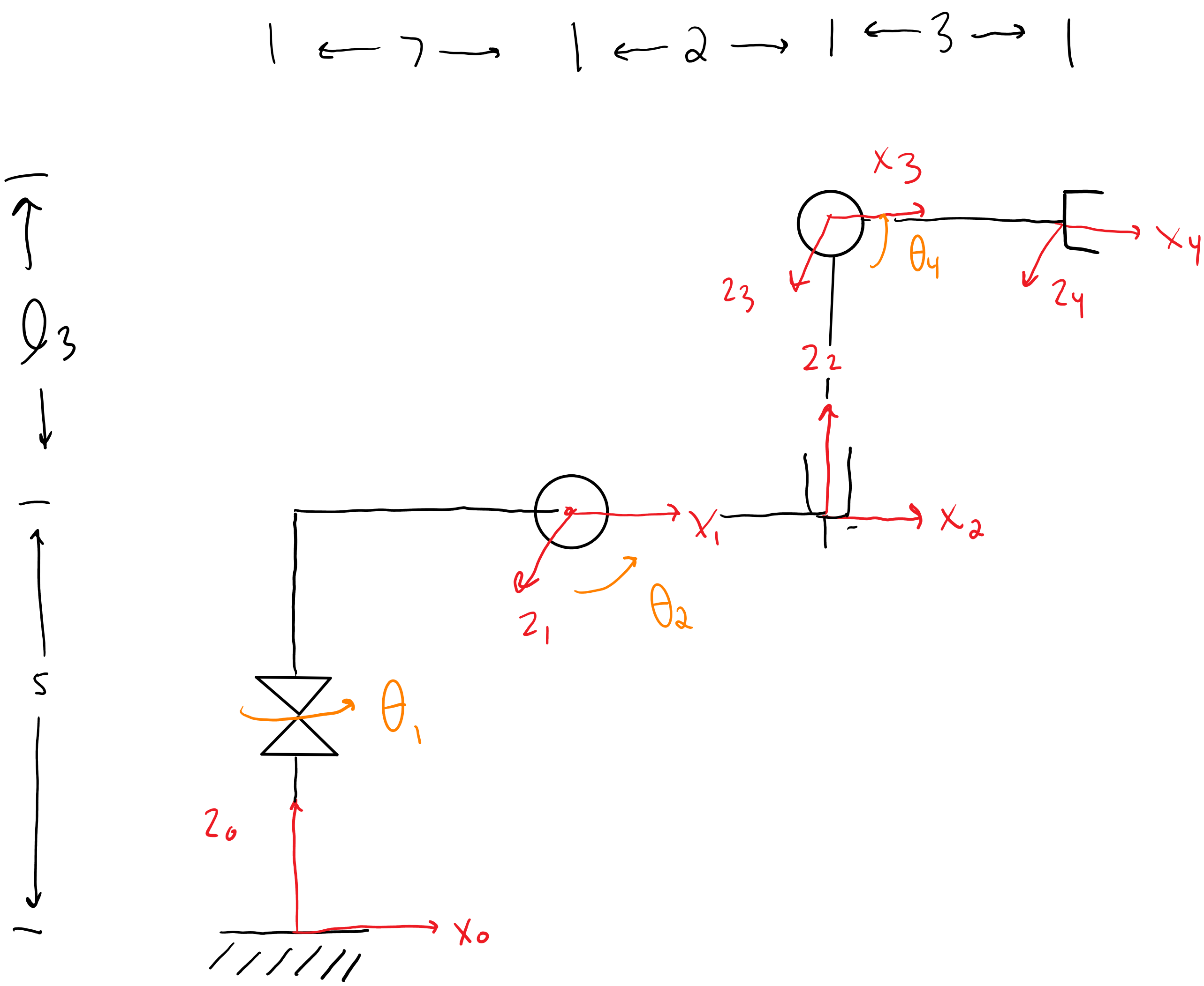
Question 5

Tuesday, November 29, 2022 6:16 PM

5. (15 points) Given the DH parameters in the table below, draw the manipulator that they describe. Assume that frame 0 is oriented with the z-axis pointing up, x-axis pointing right, and y-axis pointing into the page.

i	θ_i	d_i	a_i	α_i
1	θ_1	5	7	90^o
2	θ_2	0	2	-90^o
3	0	ℓ_3	0	90^o
4	θ_4	0	3	0

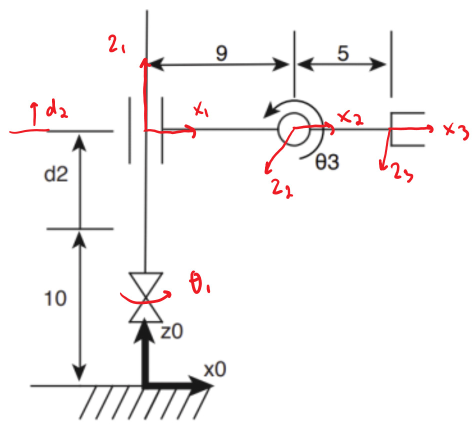
hint: First use the parameters to determine where all of the frames should be, then determine where the joints should go, then draw in the links connecting them.



Question 6

Thursday, December 1, 2022 7:41 PM

6. Consider the manipulator drawn below in a configuration where $\theta_1 = \theta_3 = 0$ and the task space is assumed to be the position only of the end effector (i.e., $\mathbb{X} = \mathbb{R}^3$). The positive direction of the first joint is given by the z_0 axis.



- (10 points) Find the Jacobian $J(\Theta)$ using the direct differentiation method.
- (10 points) Find the Jacobian $J(\Theta)$ using the column-by-column building method. Make sure to explain your answer.
- (10 points) Are there any singular configurations? If so list them. You can use whatever method you want to find them, but make sure to explain your answer.

DH Table:

	d_i	θ_i	a_i	α_i
1	d_2+10	θ_1	0	0
2	0	0	9	90
3	0	θ_3	5	0

Question 6

Monday, December 5, 2022 8:20 PM

Process

Using the DH table from the prior page, I created a MATLAB script that takes in a row of the DH table and generates a Homogeneous Transformation Matrix that results from the given parameters.

$$H_i^{i-1} = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

```
function HTM = DH2HTM(theta_i, d_i, alpha_i, a_i)

    HTM = [cos(theta_i), -sin(theta_i) * cos(alpha_i), sin(theta_i) * sin(alpha_i), a_i * cos(theta_i);
           sin(theta_i), cos(theta_i) * cos(alpha_i), -cos(theta_i) * sin(alpha_i), a_i * sin(theta_i);
           0, sin(alpha_i), cos(alpha_i), d_i;
           0, 0, 0, 1];
end
```

By symbolically asserting values for d2, theta1, and theta 3 I was able to construct the intermediate transforms H0_1, H1_2, and H2_3. Multiplying them in a relative manner (sequentially on the right) I was able to obtain a generic form of the transform H0_3 (given on the next page).

```
syms d2;
syms theta1;
syms theta3;

theta2 = pi/2;

H0_1 = DH2HTM(theta1, d2+10, 0, 0)

H1_2 = DH2HTM(0, 0, theta2, 9)

H2_3 = DH2HTM(theta3, 0, 0, 5)

H0_3 = vpa((H0_1 * H1_2 * H2_3), 2)
```

Question 6

Saturday, December 3, 2022 11:53 PM

H0_3 Given from MATLAB script

$$H_3^0 = \begin{bmatrix} c\theta_1 c\theta_3 & -c\theta_1 s\theta_3 & s\theta_1 & 9c\theta_1 + 5c\theta_1 c\theta_3 \\ c\theta_3 s\theta_1 & -s\theta_1 s\theta_3 & -c\theta_1 & 9s\theta_1 + 5c\theta_3 s\theta_1 \\ s\theta_3 & c\theta_3 & 0 & d_2 + 5s\theta_3 + 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After obtaining the proper transform, the displacement vector d0_3 was extracted from the first three rows of the last column, for which a partial derivative was taken for each parameter of the joint space.

$$J_v = \frac{\partial O_n^0}{\partial \Theta} :$$

$$\boxed{J_v = \begin{bmatrix} -9s\theta_1 - 5s\theta_1 c\theta_3 & 0 & -5c\theta_1 s\theta_3 \\ 9c\theta_1 + 5c\theta_3 c\theta_1 & 0 & -5s\theta_1 s\theta_3 \\ 0 & 1 & 5c\theta_3 \end{bmatrix}}$$

Question 6

Monday, December 5, 2022 9:42 PM

Column by Column Method

In performing the column by column configuration of the Jacobian the two equations listed below were used and abstracted into a MATLAB code. The equation on the left corresponds to prismatic joints while the equation on the right corresponds to revolute joints.

$$J_{vi} = R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \qquad J_{vi} = R^0_{i-1}\begin{bmatrix}0\\0\\1\end{bmatrix} \times \left(o^0_n - o^0_{i-1}\right).$$

Column 1:

This the first revolute joint, which requires that I take the rotation matrix R0_0 (the identity matrix) and a displacement of all zeros for d0_0.

```
d0_3 = H0_3(1:3, 4);

% Column 1
R0_0 = diag([1;1;1]);
d0_0 = [0;0;0];
col1 = cross(R0_0 * [0; 0; 1], d0_3 - d0_0);
```

Column 2:

The next joint is prismatic, meaning there is no need to cross with a displacement vector, so the only thing that is needed is to extract the rotation matrix R0_2. This was extracted from the transform H0_1 that was generated during the construction of the H0_3 matrix needed for the direct differentiation method.

```
% Column 2
R0_1 = H0_1(1:3, 1:3);
col2 = R0_1 * [0;0;1];
```

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Monday, December 5, 2022 9:54 PM

Column 3:

Joint three is revolute, meaning that I needed to obtain the rotation matrix R0_2 and the displacement vector d0_2. Again, from the direct differentiation method I had access to the transforms H0_1 and H0_2. Multiplying these together I was able to obtain H0_2, from which I extracted the appropriate rotation matrix and displacement vector. Applying these to the generalized formula was how I obtained the third column of the Jacobian.

```
% Column 3
H0_2 = H0_1 * H1_2;
d0_2 = H0_2(1:3, 4);
R0_2 = H0_2(1:3, 1:3);
col3 = cross(R0_2 * [0;0;1], d0_3 - d0_2);
```

Result

Below are the resultant columns of the Jacobian, which can be horizontally stacked to generate a final Jacobian. MATLABs symbolic solver was unable to drop zero terms or simplify them, so for completeness and coherency, I will show the outputs by row and show how they simplify.

Column 1

```
col1 =

- 9.0*sin(theta1) - 3.1e-16*cos(theta1)*sin(theta3) - 5.0*cos(theta3)*sin(theta1)
  9.0*cos(theta1) + 5.0*cos(theta1)*cos(theta3) - 3.1e-16*sin(theta1)*sin(theta3)
                                                                                 0
```

Simplifications

- In row one and three terms associate with 3e-16 were dropped

$$col1 = \begin{bmatrix} -9s\theta_1 - 5c\theta_3 s\theta_1 \\ 9c\theta_1 + 5c\theta_1 c\theta_3 \\ 0 \end{bmatrix}$$

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Column 2

```
col2 =

0
0
1
```

Column 3

```
col3 =

                                          - 5.0*cos(theta1)*sin(theta3) - 3.1e-16*cos(theta3)*sin(theta1)
                                            3.1e-16*cos(theta1)*cos(theta3) - 5.0*sin(theta1)*sin(theta3)
cos(theta1)*(5.0*cos(theta1)*cos(theta3) - 3.1e-16*sin(theta1)*sin(theta3)) + sin(theta1)*(3.1e-16*cos(theta1)*sin(theta3) + 5.0*cos(theta3)*sin(theta1))
```

Simplifications

- All terms associated with 3.1e-16 were dropped
- In row three the trigonometric identity $\cos^2 + \sin^2 = 1$ is used to drop out all theta 1 terms
  - See below for the work

- AFTER DROPPING ALL 0 TERMS, ROW THREE BECOMES

$$5c\theta_3 c\theta_1 c\theta_1 + 5c\theta_3 s\theta_1 s\theta_1$$

- PULL OUT $5c\theta_3$

$$5c\theta_3 \left(c^2\theta_1 + s^2\theta_1 \right)$$

- USE $s^2\theta + c^2\theta = 1$

$$5c\theta_3$$

$$col3 = \begin{bmatrix} -5c\theta_1 s\theta_3 \\ -5s\theta_1 s\theta_3 \\ 5\theta_3 \end{bmatrix}$$

Result

$$\boxed{J_v = \begin{bmatrix} -9s\theta_1 - 5c\theta_3 s\theta_1 & 0 & -5c\theta_1 s\theta_3 \\ 9c\theta_1 + 5c\theta_1 c\theta_3 & 0 & -5s\theta_1 s\theta_3 \\ 0 & 1 & 5\theta_3 \end{bmatrix}}$$

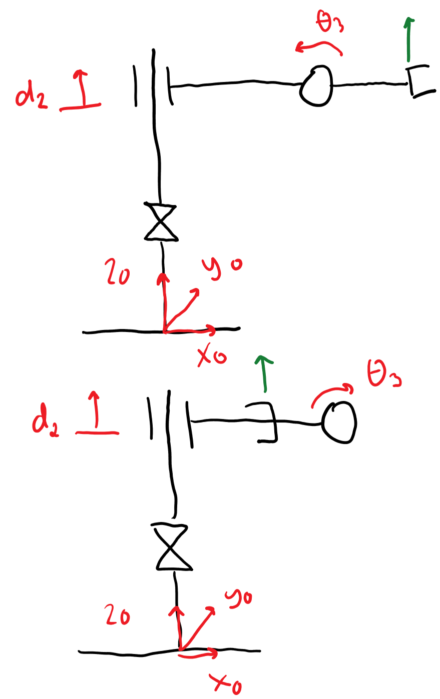
As one can see, this Jacobian derived by column by column construction is equivalent to that of the Jacobian derived from direct differentiation performed in the first part of this problem.

Question 6

Sunday, December 4, 2022 12:27 AM

Singularities

Two singularities of this design are when the value of theta 3 is zero, or when it is pi radians (or some multiple of that if you do not consider motor restrictions). Physically, this means that link 3 and link 4 are colinear, either with the arm extended or folded on top of itself (see right). In both these configurations, you lose a rank of the Jacobian because either actuating the prismatic joint two or the revolute joint three would result in the same outcome in terms of end effector motion.



For example, in the above configuration one could move the prismatic joint vertically, an output that could also be achieved by rotating the third joint counterclockwise. Similarly, in the bottom configuration one could move the prismatic second joint vertically, whose result could be achieved by rotating the third revolute joint clockwise. Another important aspect is that this configuration has no was to generate a velocity component in the base frame x axis. It can generate vertical movement by going up and down along the z0 axis with either joint three rotation of joint two translation. Or, it can generation motion in the base frame y with a rotation of joint one. However it can never generate motion along the base x0 axis.

A good logic test to perform is to plug these two singularities into the parameterized Jacobian from direct differentiation:

$$J_v = \begin{bmatrix} -9s\theta_1 - 5s\theta_1 c\theta_3 & 0 & -5c\theta_1 s\theta_3 \\ 9c\theta_1 + 5c\theta_3 c\theta_1 & 0 & -5s\theta_1 s\theta_3 \\ 0 & 1 & 5c\theta_3 \end{bmatrix}$$

$$\xrightarrow{\theta_1 = 0,\ \theta_3 = 0} \begin{bmatrix} 0 & 0 & 0 \\ 14 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\xrightarrow{\theta_1 = 0,\ \theta_3 = \pi} \begin{bmatrix} 0 & 0 & 0 \\ -14 & 0 & 0 \\ 0 & 1 & -5 \end{bmatrix}$$

From inspection one can see that the third column is a multiple of the second column, meaning that the Jacobian has dropped rank - leading to a singularity.