



POLITECNICO
MILANO 1863

Three-Dimensional Bin Packing with Vertical Support

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Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure



Figure: Example of pit palletization (Schäfer Case Picking — SSI SCHÄFER)

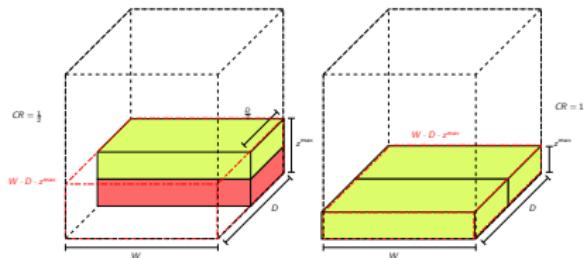


Figure: Cage ratio of two different bin configurations

Definition

An item has vertical support if one of the following conditions hold:

- **Condition 1:** at least a percentage α_s of its base area is resting on other items
- **Condition 2:** at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

Introduction

Vertical Support

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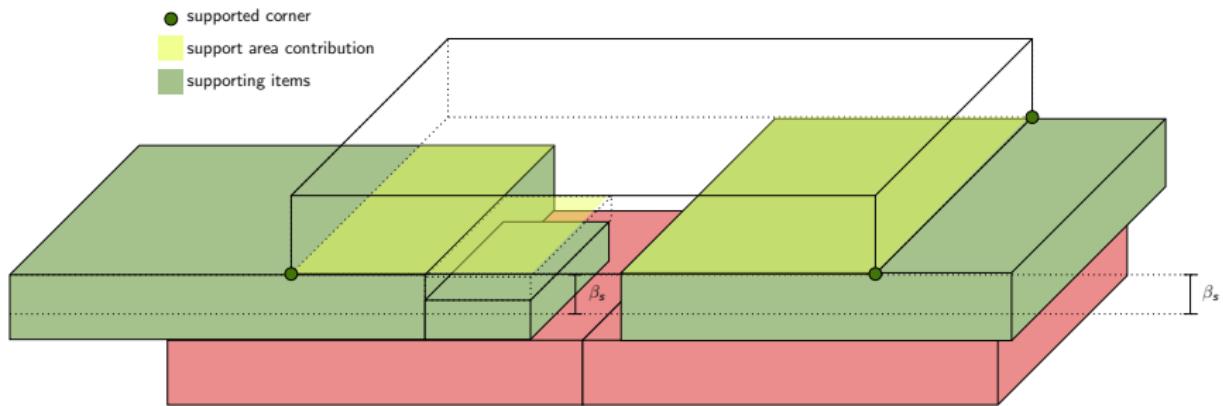


Figure: Representation of an item with conditions 1 and 2 of vertical support given
 $\alpha_s = 0.5, \beta_s$

- The problem is NP-Hard
- Exact methods only for small instances
- Existing 3D-BPP heuristics don't consider practical constraints
- Solutions for container loading and pallet loading problems are layer based

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Proxy Model - Objective Function

$$\begin{array}{ll}
 \min & \sum_{b \in B} (Hv_b + z_b^{\max}) \\
 \text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR} \\
 & u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B \\
 & v_b \geq v_c \quad \forall (b, c) \in B : b < c \\
 & x_i + w_i \leq W \quad \forall i \in I \\
 & y_i + d_i \leq D \quad \forall i \in I \\
 & z_i + h_i \leq H \quad \forall i \in I \\
 & (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I \\
 & x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I \\
 & (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I \\
 & y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I \\
 & (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I \\
 & z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I \\
 & x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B \\
 & z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B
 \end{array}$$

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 all items are inside the bin's bounds
 no overlaps between items in the same bin
 all items have vertical support

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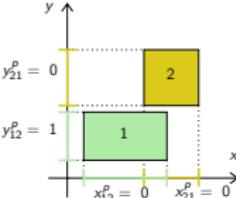
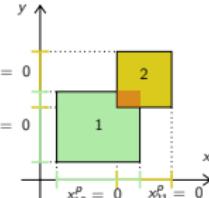
		$\min \quad \sum_{b \in B} (Hv_b + z_b^{max})$	
		$\text{s.t.} \quad \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR}$	
minimize then, maximize	number of used bins average cage ratio of the used bins	$u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B$	
subject to	all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$v_b \geq v_c \quad \forall (b, c) \in B : b < c$ $x_i + w_i \leq W \quad \forall i \in I$ $y_i + d_i \leq D \quad \forall i \in I$ $z_i + h_i \leq H \quad \forall i \in I$ $(x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I$ $x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I$ $(y_i + d_i) - y_j \leq D(y_{ij}^P) \quad \forall i, j \in I$ $y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I$ $(z_i + h_i) - z_j \leq H(z_{ij}^P) \quad \forall i, j \in I$ $z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I$ $x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B$ $z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B$	
			

Figure: Precedences variables (2D case)

minimize number of used bins

then, maximize average cage ratio of the used bins

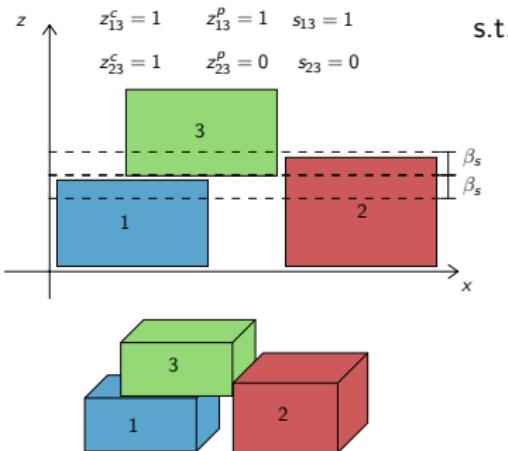
subject to all items are assigned to one and only one bin

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all items have vertical support

MILP Proxy Model - Closeness



s.t.

$$\begin{aligned}
 z_j - (z_i + h_i) &\leq \beta_s + H(1 - z_{ij}^c) & \forall (i, j) \in I : i \neq j \\
 z_j - (z_i + h_i) &\geq -\beta_s - H(1 - z_{ij}^c) & \forall (i, j) \in I : i \neq j \\
 s_{ij} &\leq z_{ij}^p & \forall (i, j) \in I \\
 s_{ij} &\leq z_{ij}^c & \forall (i, j) \in I \\
 s_{ij} &\geq z_{ij}^p + z_{ij}^c - 2 & \forall (i, j) \in I : i \neq j \\
 \sum_{j \in I} s_{ij} &\leq \sum_{b \in B} u_{ib} & \forall i \in I
 \end{aligned}$$

Figure: Closeness variables example

MILP Proxy Model - Discretized Vertical Support

Pre-Computed Parameter

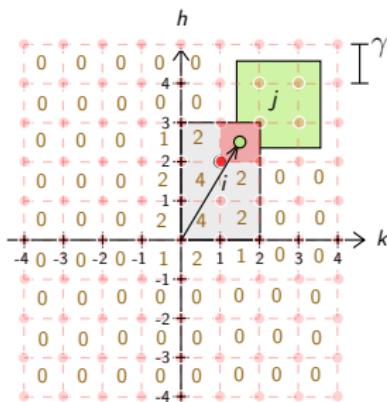
$$O(i, j, k, h)$$

$$O(i, j, 1, 2) = 1$$

Variables

$$s_{ijb}^{12} = 1$$

$$(x_j - x_i, y_j - y_i)$$



$$\begin{aligned}
 & \text{s.t.} && z_i \leq H(1 - g_i) && \forall i \in I \\
 & && \sum_{(k,h) \in \Delta, b \in B: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq s_{ij} && \forall (i,j) \in I \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{ib} && \forall (i,j,b) \in I^B \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{jb} && \forall (i,j,b) \in I^B \\
 & && x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && \sum_{(k,h) \in \Delta, b \in B, j \in I: i \neq j \wedge O(i,j,k,h) \neq 0} O(i,j,k,h) s_{ijb}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
 \end{aligned}$$

Figure: Space discretization

Proposed Heuristic Overview

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Composed of:

- Constructive heuristic
(Support Planes)
- Beam-Search

Each node is a partial solution, starts from the empty solution

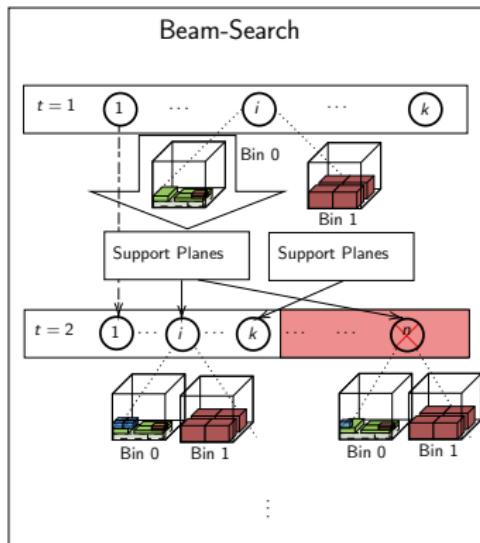


Figure: Conceptual representation of the proposed heuristic

Proposed Heuristic Support Plane

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- Operates on a single bin
- Each item inserted generates planes
- Insertions on the lowest possible level
- Items grouped by their shape
- No explicit layers

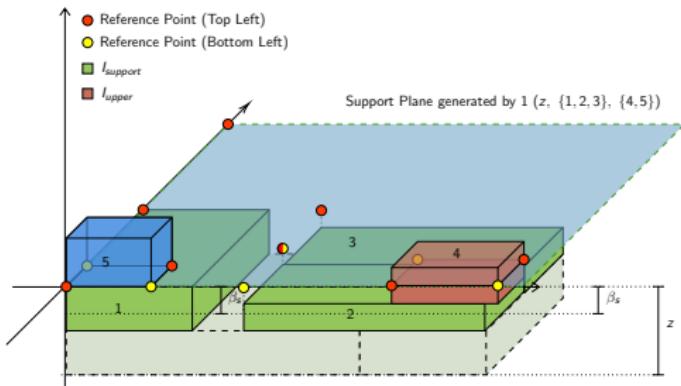


Figure: Conceptual representation of the proposed heuristic

Proposed Heuristic Optimizations

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- Duplicates removal
- Fast overlap checks
- Memory management

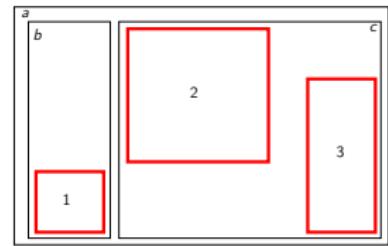
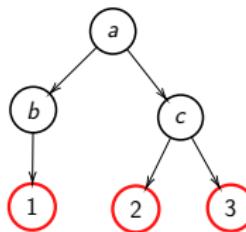


Figure: AABB Tree

Computational Experiments

