



POLITECNICO
MILANO 1863

Three-Dimensional Bin Packing with Vertical Support

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Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure



Figure: Example of pit palletization
(Schäfer Case Picking — SSI SCHÄFER)

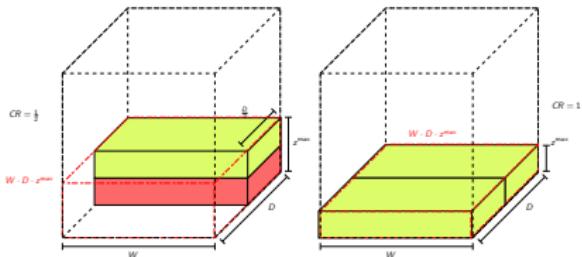


Figure: Cage ratio of two different bin configurations

Definition

An item has vertical support if one of the following conditions hold:

- **Condition 1:** at least a percentage α_s of its base area is resting on other items
- **Condition 2:** at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

Vertical Support

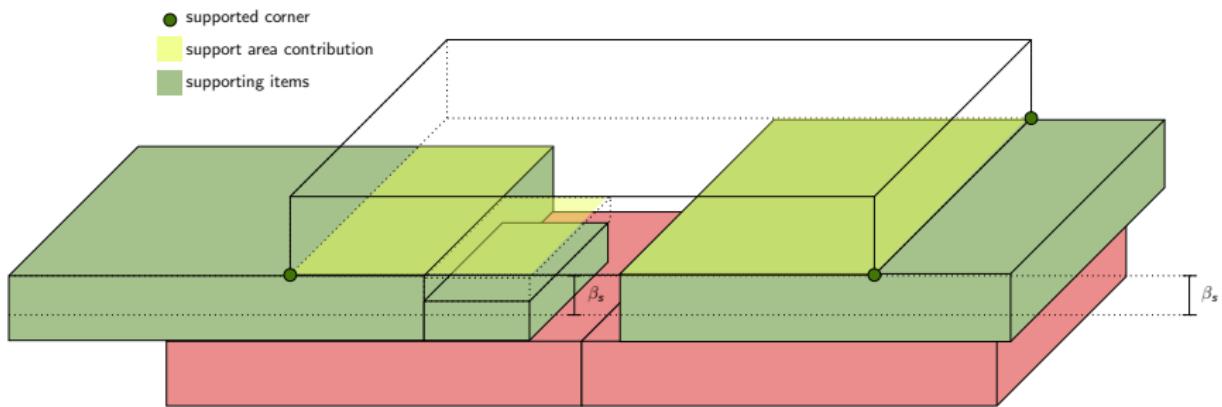


Figure: Representation of an item with conditions 1 and 2 of vertical support given
 $\alpha_s = 0.5, \beta_s$

- The problem is NP-Hard
- Exact methods only for small instances
- Existing 3D-BPP heuristics don't consider practical constraints
- Solutions for container loading and pallet loading problems are layer based

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Proxy Model - Objective Function

$$\begin{array}{ll}
 \min & \sum_{b \in B} (Hv_b + z_b^{\max}) \\
 \text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR} \\
 & u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B \\
 & v_b \geq v_c \quad \forall (b, c) \in B : b < c \\
 & x_i + w_i \leq W \quad \forall i \in I \\
 & y_i + d_i \leq D \quad \forall i \in I \\
 & z_i + h_i \leq H \quad \forall i \in I \\
 & (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I \\
 & x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I \\
 & (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I \\
 & y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I \\
 & (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I \\
 & z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I \\
 & x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B \\
 & z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B
 \end{array}$$

minimize number of used bins
then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
 all items are inside the bin's bounds
 no overlaps between items in the same bin
 all items have vertical support

MILP Proxy Model - Geometric Constraints 1

$$\begin{array}{ll}
 \min & \sum_{b \in B} (Hv_b + z_b^{\max}) \\
 \text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR} \\
 & u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B \\
 & v_b \geq v_c \quad \forall (b, c) \in B : b < c \\
 & x_i + w_i \leq W \quad \forall i \in I \\
 & y_i + d_i \leq D \quad \forall i \in I \\
 & z_i + h_i \leq H \quad \forall i \in I \\
 & (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I \\
 & x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I \\
 & (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I \\
 & y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I \\
 & (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I \\
 & z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I \\
 & x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B \\
 & z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B
 \end{array}$$

minimize number of used bins
 then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
 all items are inside the bin's bounds
 no overlaps between items in the same bin
 all items have vertical support

MILP Proxy Model - Geometric Constraints 2

minimize then, maximize subject to	number of used bins average cage ratio of the used bins all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	
		$\sum_{b \in B} (Hv_b + z_b^{\max})$
		s.t. $\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR}$
		$u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B$
		$v_b \geq v_c \quad \forall (b, c) \in B : b < c$
		$x_i + w_i \leq W \quad \forall i \in I$
		$y_i + d_i \leq D \quad \forall i \in I$
		$z_i + h_i \leq H \quad \forall i \in I$
		$(x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I$
		$x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I$
		$(y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I$
		$y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I$
		$(z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I$
		$z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I$
		$x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B$
		$z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B$

		$\min \quad \sum_{b \in B} (Hv_b + z_b^{max})$	
		$\text{s.t.} \quad \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR}$	
minimize then, maximize	number of used bins average cage ratio of the used bins	$u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B$	
subject to	all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$v_b \geq v_c \quad \forall (b, c) \in B : b < c$ $x_i + w_i \leq W \quad \forall i \in I$ $y_i + d_i \leq D \quad \forall i \in I$ $z_i + h_i \leq H \quad \forall i \in I$ $(x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I$ $x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I$ $(y_i + d_i) - y_j \leq D(y_{ij}^P) \quad \forall i, j \in I$ $y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I$ $(z_i + h_i) - z_j \leq H(z_{ij}^P) \quad \forall i, j \in I$ $z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I$ $x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B$ $z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B$	

Figure: Precedences variables (2D case)

minimize number of used bins

then, maximize average cage ratio of the used bins

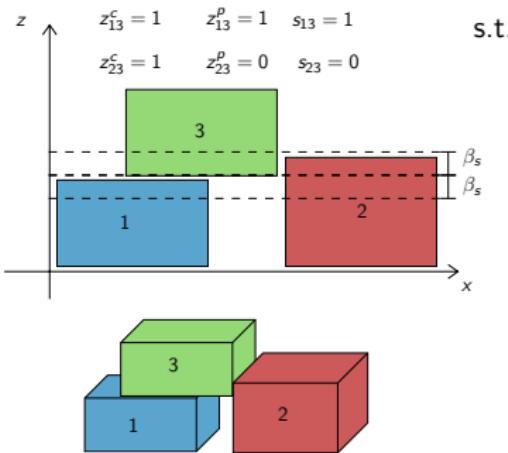
subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Proxy Model - Closeness



$$\begin{aligned}
 & z \\
 & z_{13}^c = 1 \quad z_{13}^p = 1 \quad s_{13} = 1 \\
 & z_{23}^c = 1 \quad z_{23}^p = 0 \quad s_{23} = 0 \\
 & \text{s.t.} \quad z_j - (z_i + h_i) \leq \beta_s + H(1 - z_{ij}^c) \quad \forall (i, j) \in I : i \neq j \\
 & z_j - (z_i + h_i) \geq -\beta_s - H(1 - z_{ij}^c) \quad \forall (i, j) \in I : i \neq j \\
 & s_{ij} \leq z_{ij}^p \quad \forall (i, j) \in I \\
 & s_{ij} \leq z_{ij}^c \quad \forall (i, j) \in I \\
 & s_{ij} \geq z_{ij}^p + z_{ij}^c - 2 \quad \forall (i, j) \in I : i \neq j \\
 & \sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} \quad \forall i \in I
 \end{aligned}$$

Figure: Closeness variables example

MILP Proxy Model - Discretized Vertical Support

Pre-Computed Parameter

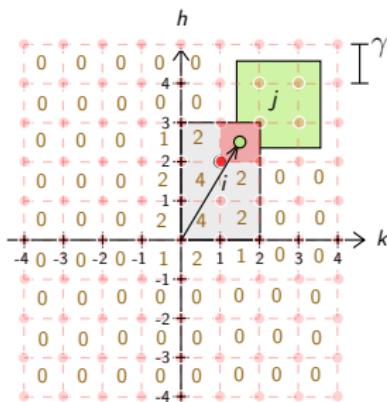
$$O(i, j, k, h)$$

$$O(i, j, 1, 2) = 1$$

Variables

$$s_{ijb}^{12} = 1$$

$$(x_j - x_i, y_j - y_i)$$



$$\begin{aligned}
 & \text{s.t.} && z_i \leq H(1 - g_i) && \forall i \in I \\
 & && \sum s_{ijb}^{kh} \leq s_{ij} && \forall (i, j) \in I \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{ib} && \forall (i, j, b) \in I^B \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{jb} && \forall (i, j, b) \in I^B \\
 & && x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k, h) \in \Delta, \forall (i, j, b) \in I^B : O(i, j, k, h) \neq 0 \\
 & && x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k, h) \in \Delta, \forall (i, j, b) \in I^B : O(i, j, k, h) \neq 0 \\
 & && y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k, h) \in \Delta, \forall (i, j, b) \in I^B : O(i, j, k, h) \neq 0 \\
 & && y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k, h) \in \Delta, \forall (i, j, b) \in I^B : O(i, j, k, h) \neq 0 \\
 & && \sum_{(k,h) \in \Delta, b \in B, j \in I: i \neq j \wedge O(i,j,k,h) \neq 0} O(i, j, k, h) s_{ijb}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
 \end{aligned}$$

Figure: Space discretization

Proposed Heuristic Overview

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Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

Each node is a partial solution, starts from the empty solution

Support Planes

- Operates on a single bin
- Exploits a modified 2D-BPP heuristic
- No explicit layers
- Guarantees vertical support

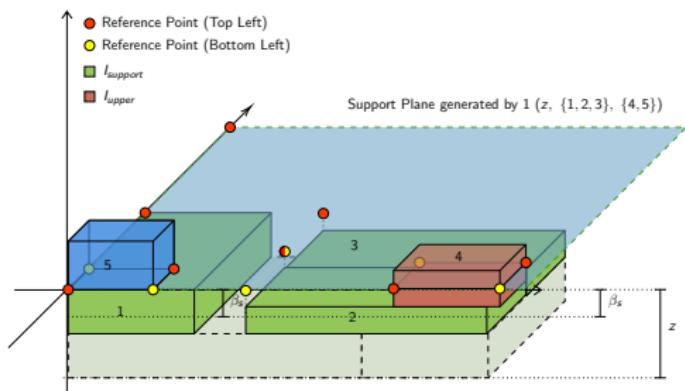
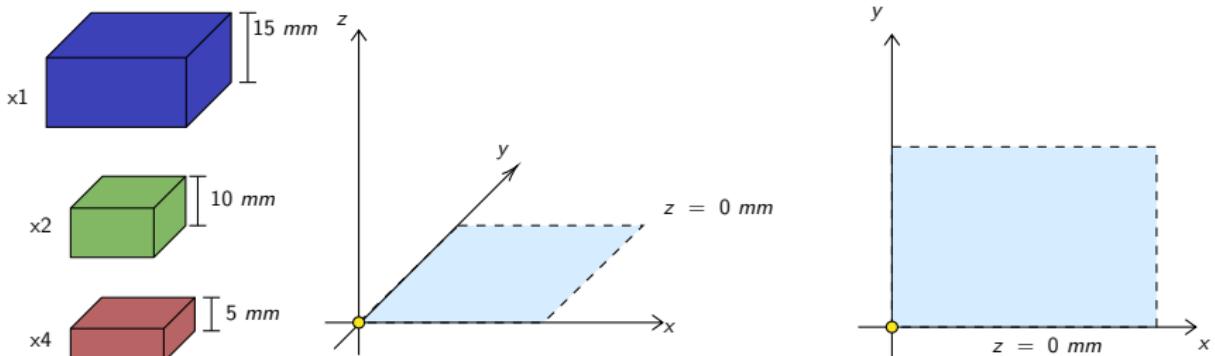
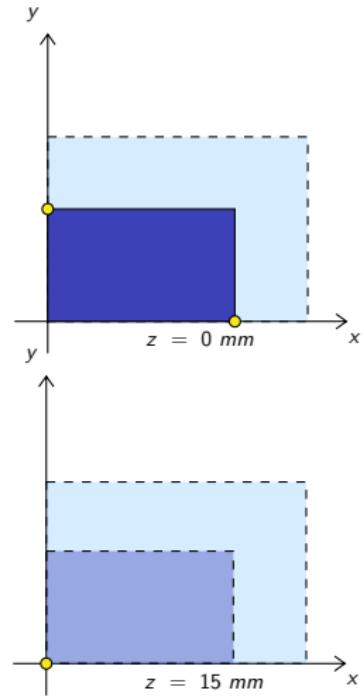
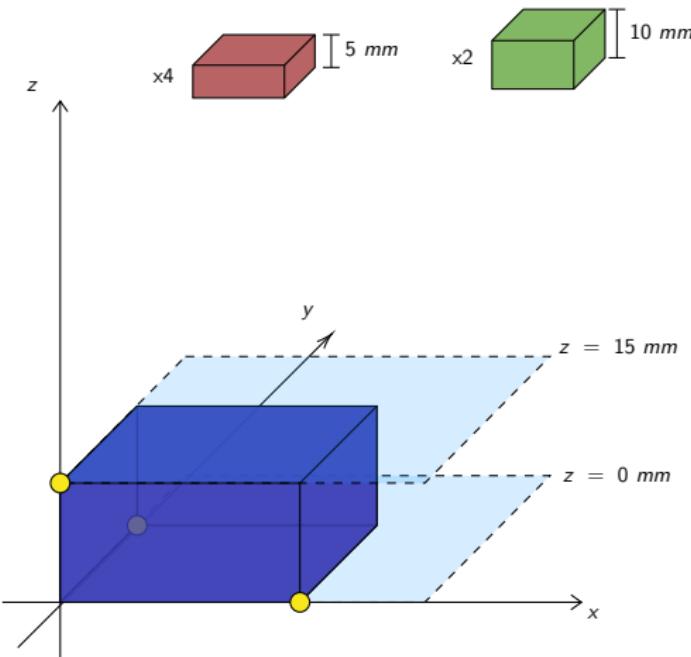


Figure: An example of a support plane generated by item 1

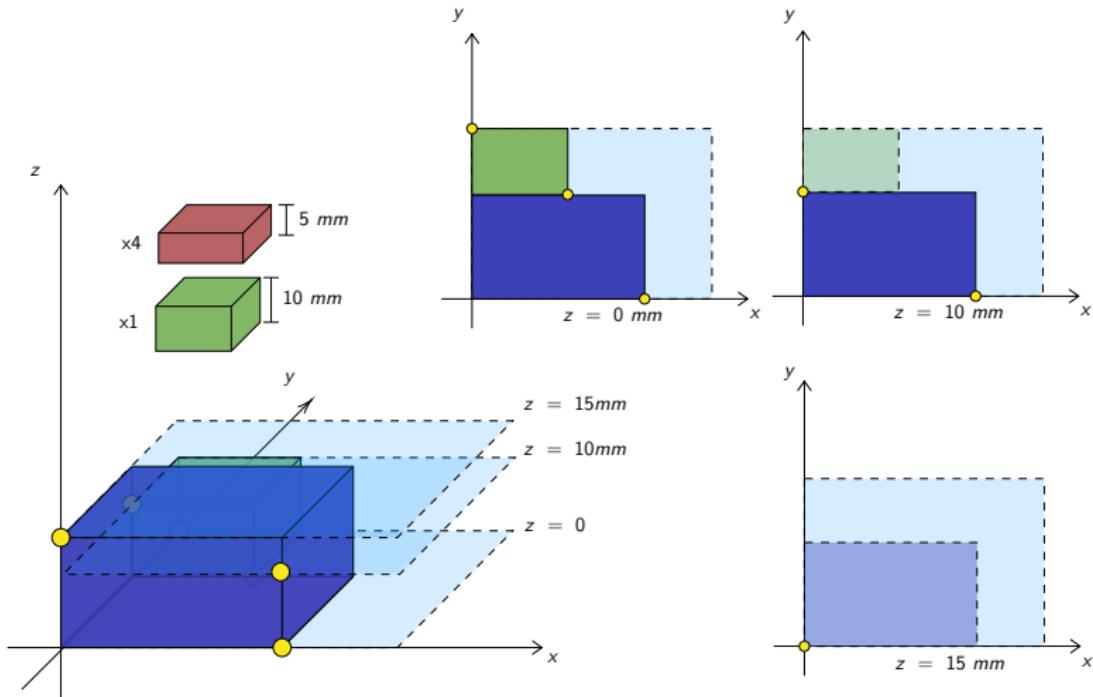
Support Planes



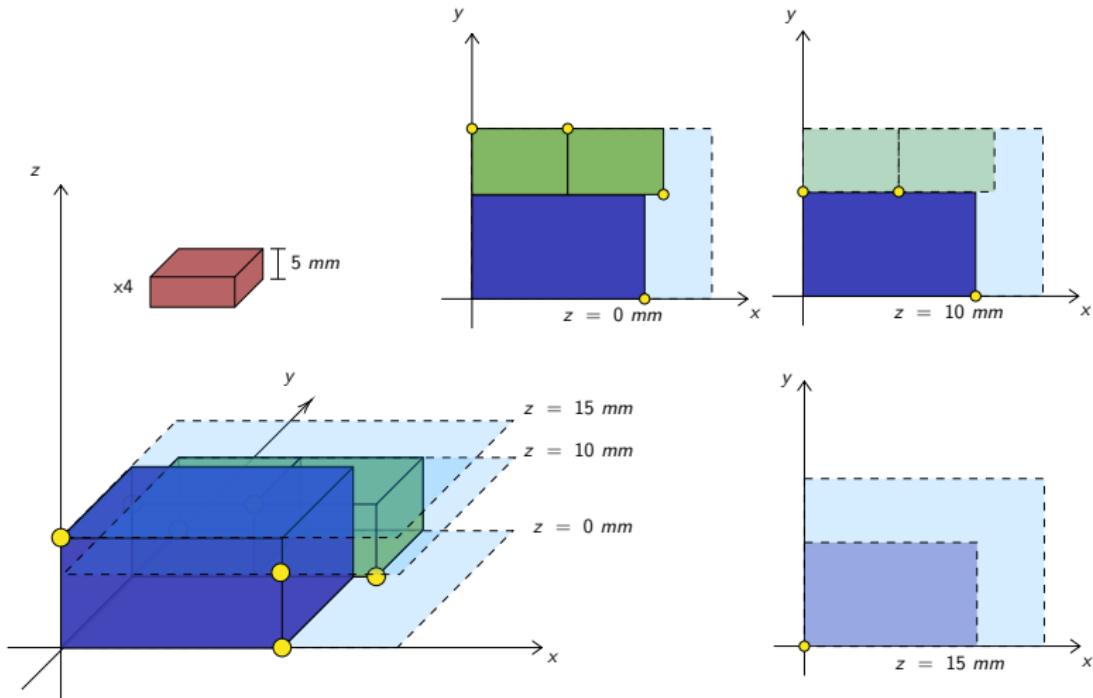
Support Planes



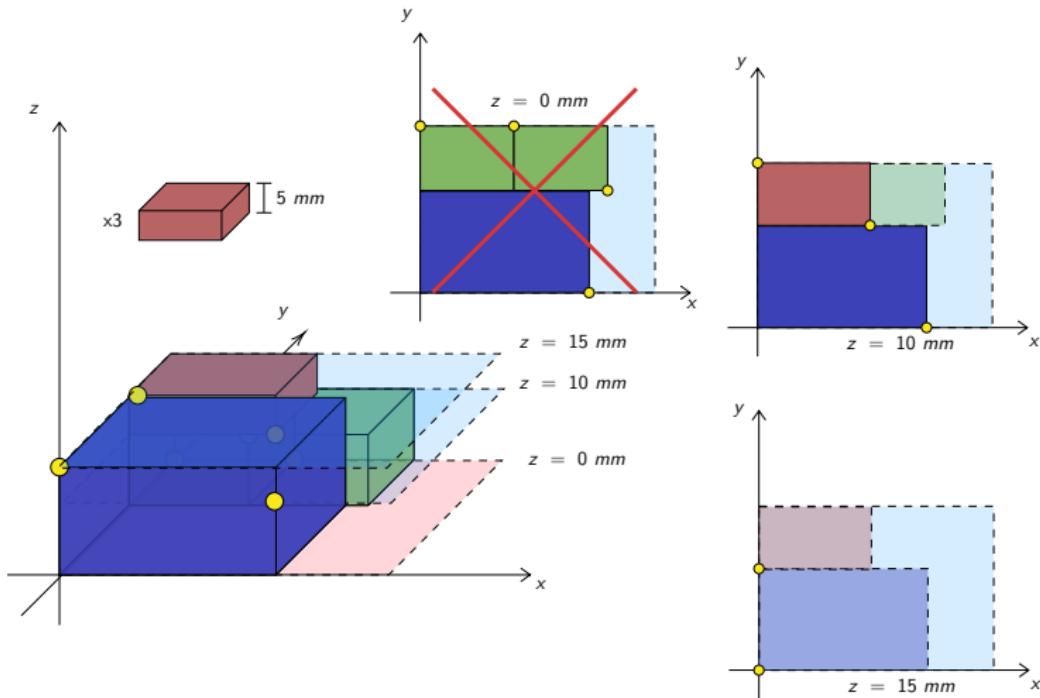
Support Planes



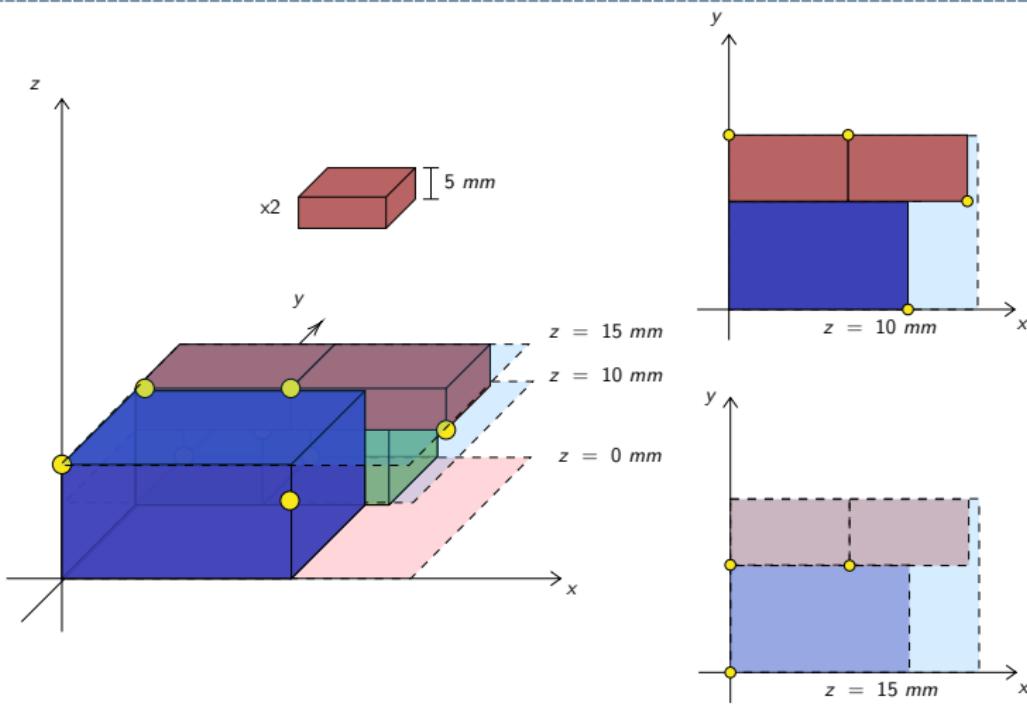
Support Planes



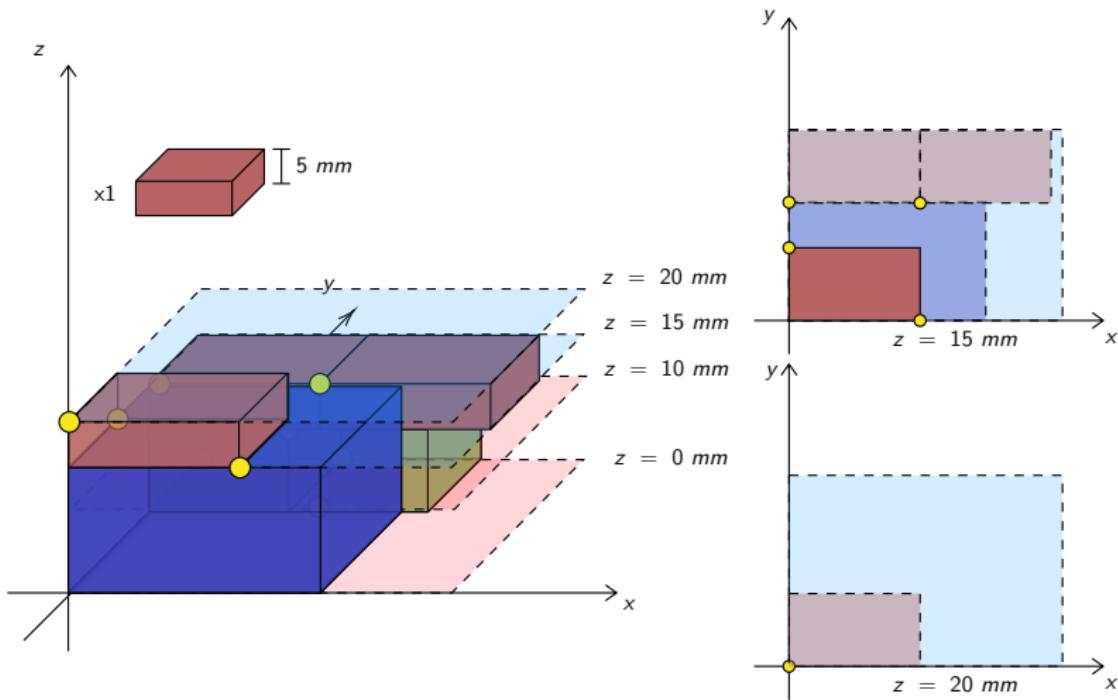
Support Planes



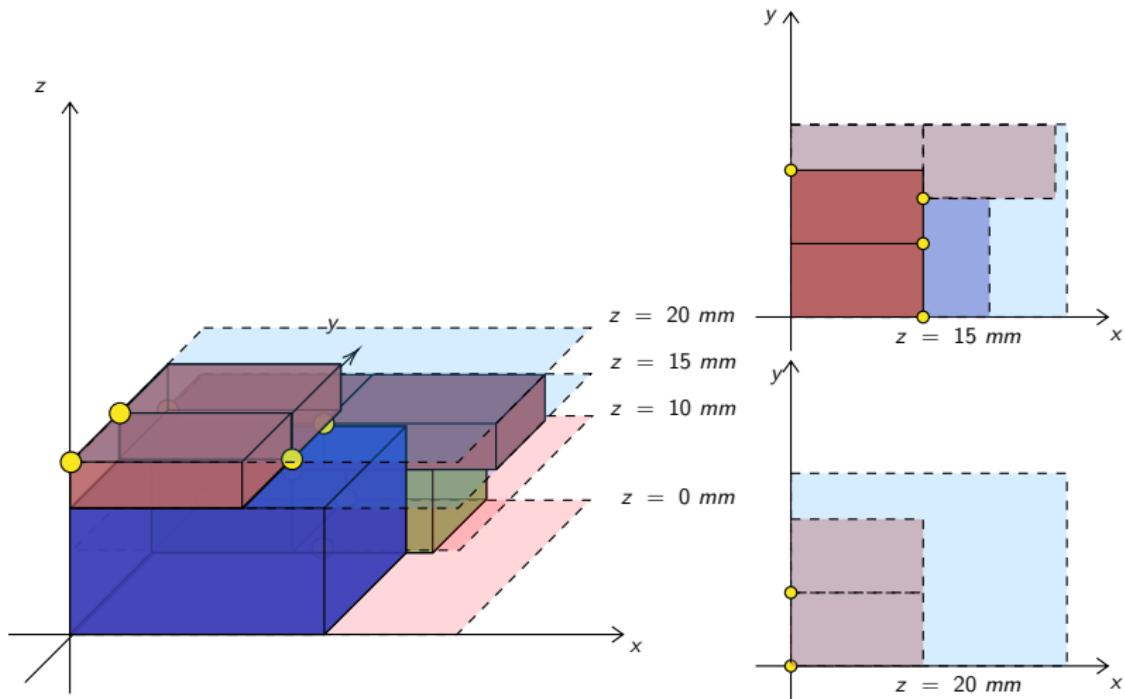
Support Planes



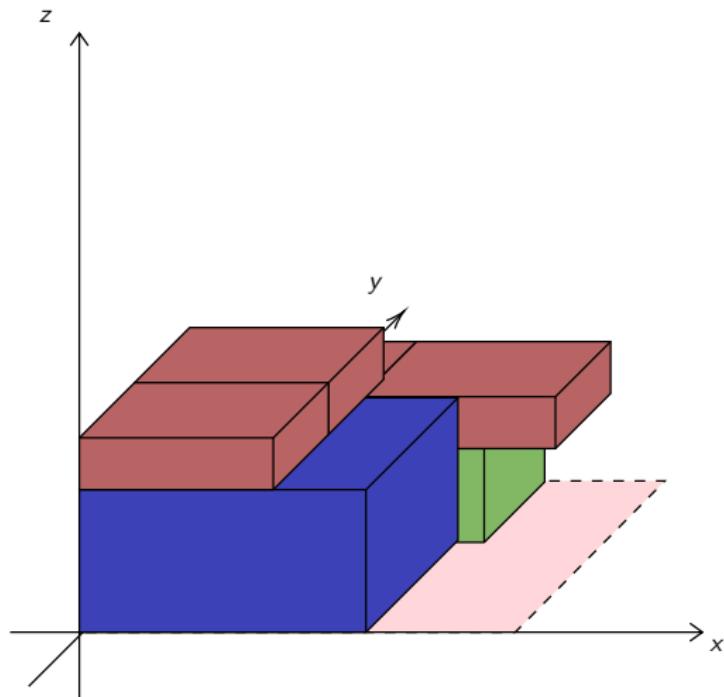
Support Planes



Support Planes

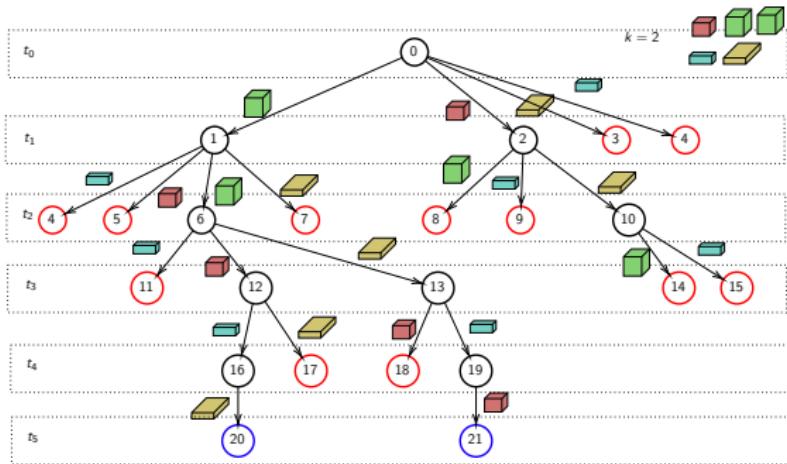


Support Planes



Beam Search

- Exploits support planes
- Explores different sequences of placements
- Limited to k best placements



Beam Search

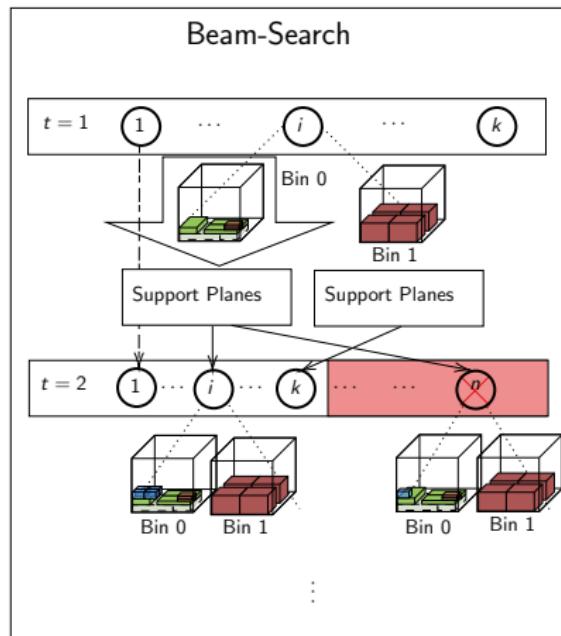


Figure: Conceptual representation of the proposed heuristic

Beam Search

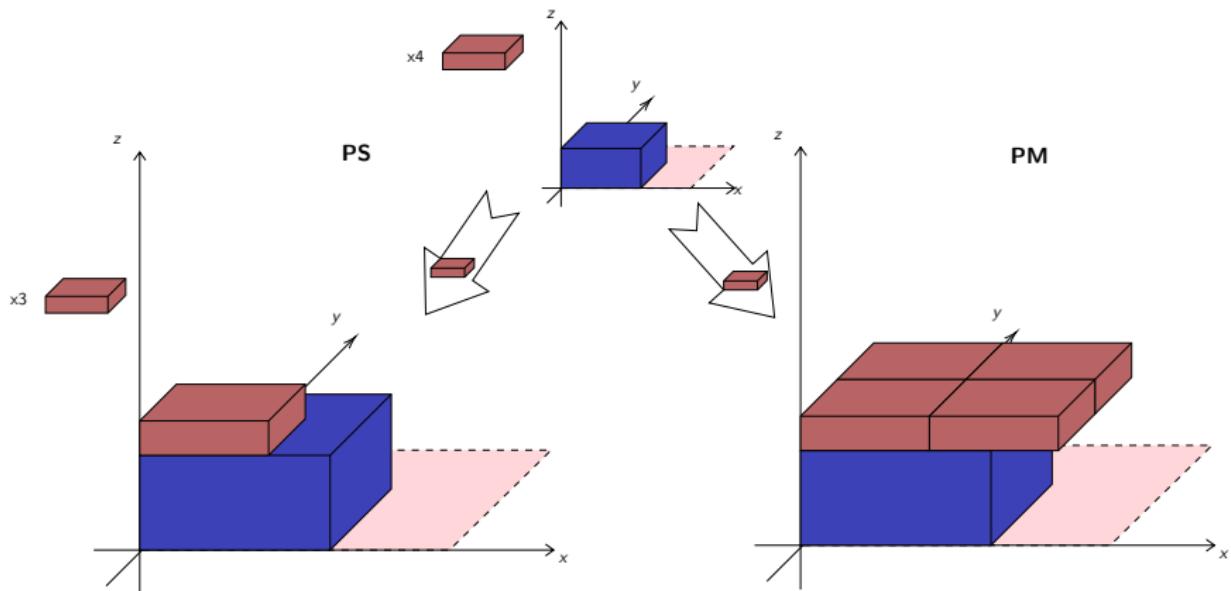


Figure: Proposed placement modes

Proposed Heuristic Optimizations

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- Duplicates removal
- Fast overlap checks
- Memory management

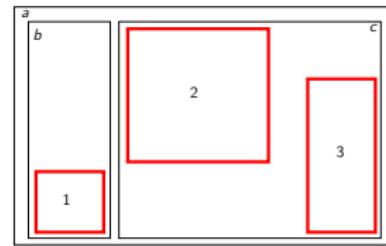
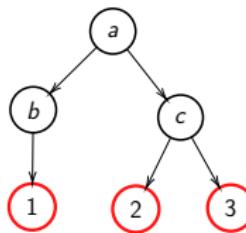


Figure: AABB Tree

Computational Experiments

Table: Comparison with MILP model on limited set of boxes

<i>n</i>	MILP Model			PM		PS	
	Max Z	TT(s)	Gap(%)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	1.86	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

* Some boxes had lower support than expected due to discretization errors.

Computational Experiments

Table: Average execution time of literature results with bin gap

Heuristic		Execution Time (s)				Bin Gap (%)
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.05	0.11	0.28	0.55	4.57
	$k = 5$	0.08	0.39	1.02	2.16	4.32
	$k = 10$	0.15	0.74	1.98	4.12	4.29
	$k = 20$	0.29	1.45	3.89	8.07	4.05
	$k = 50$	0.72	3.63	9.72	20.47	3.95
PS	$k = 1$	0.04	0.18	0.51	1.08	4.35
	$k = 5$	0.12	0.74	2.19	4.79	4.01
	$k = 10$	0.23	1.43	4.19	9.39	3.94
	$k = 20$	0.47	2.81	8.48	18.93	3.74
	$k = 50$	1.15	6.74	21.03	45.78	3.52
BRKGA-VD		17.13	80.63	190.50	369.75	0.00

Computational Experiments

Table: Summary of case study tests

k	PS			PM		
	TT (ms)	B	CR (%)	TT (ms)	B	CR (%)
1	423.87	1.37	65.87	65.18	1.31	70.70
5	1,597.54	1.34	69.19	185.22	1.29	73.08
10	2,627.52	1.32	70.35	344.90	1.27	73.56
20	5,373.79	1.34	70.78	620.95	1.27	74.57
50	14,203.10	1.31	72.11	1,279.96	1.29	74.61
100	26,934.21	1.31	73.23	2,340.37	1.26	75.36
200	48,944.90	1.30	73.89	4,465.78	1.25	76.39

Computational Experiments

Table: Case study experiments trade off between average execution times and average cage ratio

k	PS		PM	
	$\mathbf{CR^* - CR (\%)} \quad \mathbf{TT - TT^* (ms)}$	$\mathbf{CR^* - CR (\%)} \quad \mathbf{TT - TT^* (ms)}$	$\mathbf{CR^* - CR (\%)} \quad \mathbf{TT - TT^* (ms)}$	$\mathbf{CR^* - CR (\%)} \quad \mathbf{TT - TT^* (ms)}$
1	10.56	358.69	5.73	0.00
5	7.24	1,532.36	3.35	120.04
10	6.08	2,562.34	2.87	279.72
20	5.65	5,308.61	1.85	555.77
50	4.32	14,137.92	1.82	1,214.78
100	3.20	26,869.03	1.07	2,275.19
200	2.54	48,879.72	0.04	4,400.60

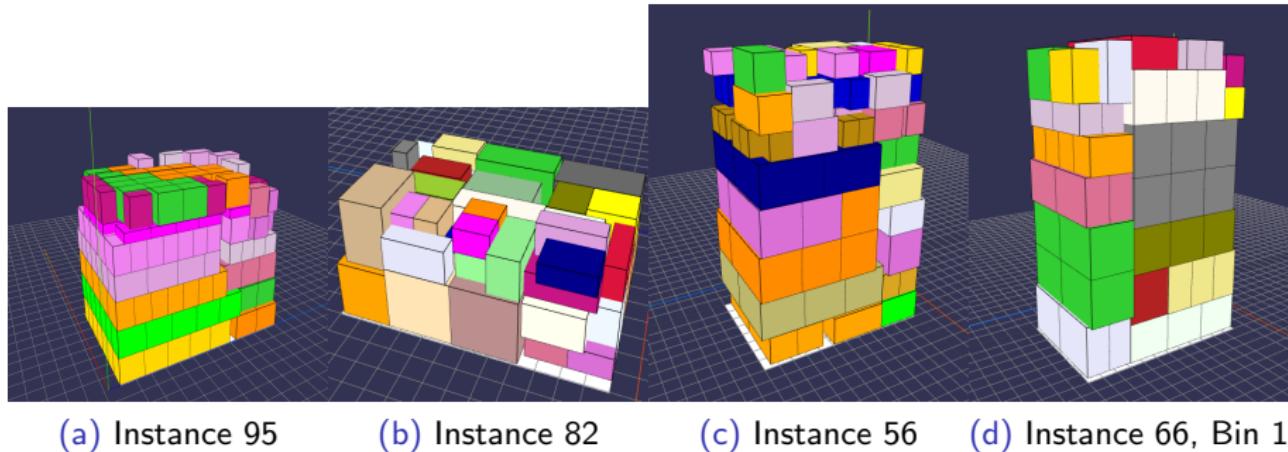


Figure: Solutions of case study tests with the "PM" placement and $k = 200$

Future Developments

- More practical constraints
- Solution optimization heuristics
- Improvements to the underlying 2D-BPP heuristic

Thank you for your attention!
Questions?