Three-Dimensional Bin Packing with Vertical Support

Jacopo Libè
952914

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Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure

Figure: Example of pit pallettization

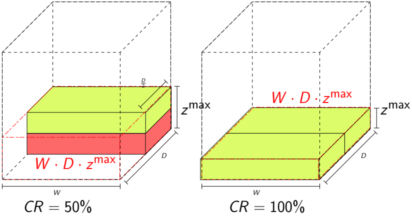


Figure: Cage ratio of two different bin configurations

Vertical Support

Definition

An item has vertical support if one of the following conditions hold:

- **Condition 1**: at least a percentage α_s of its base area is resting on other items
- **Condition 2**: at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

Vertical Support

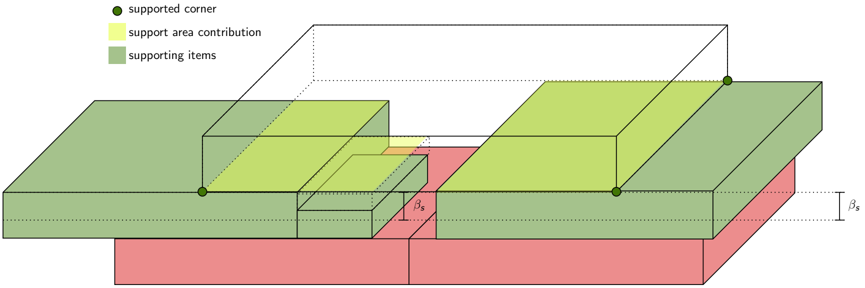


Figure: Representation of an item with conditions 1 and 2 of vertical support given $\alpha_s = 0.5, \beta_s$

Literature Review

Problem: pack a set of cuboids into the minimum amount of bins, maximizing the cage ratio of used bins, without overlaps, and with every item having vertical support

- The problem is NP-Hard
- Exact methods only for small instances
- Existing 3D-BPP heuristics don't consider practical constraints
- Solutions for container loading and pallet loading problems are layer based

Conceptual Formulation

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

MILP Model - Coordinate System

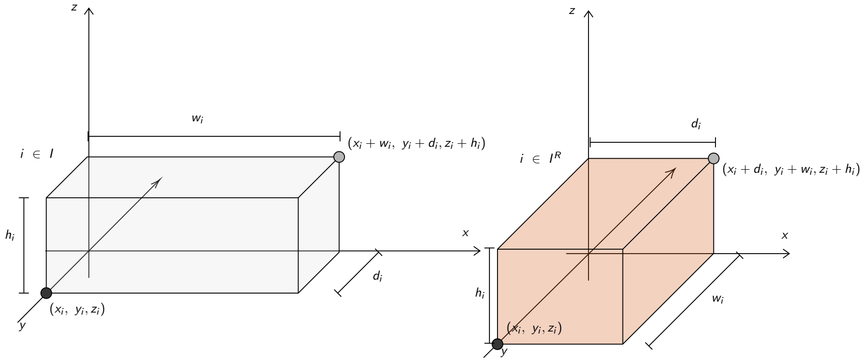
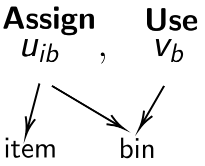


Figure: Coordinate system representation for a generic item i and its rotated clone $i \in I^R$

MILP Model - Objective Function

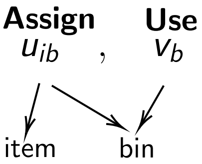
minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support



$$
\begin{array}{lll}
\min & \sum_{b \in B} (H v_b + z_b^{max}) & \\
\text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
& x_i + w_i \leq W & \forall i \in I \\
& y_i + d_i \leq D & \forall i \in I \\
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}
$$

MILP Model - Geometric Constraints 1

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support



$$
\begin{aligned}
\text{min} \quad & \sum_{b \in B} (H v_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
& x_i + w_i \leq W & \forall i \in I \\
& y_i + d_i \leq D & \forall i \in I \\
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

MILP Model - Geometric Constraints 2

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

$$
\begin{aligned}
\min \quad & \sum_{b\in B}(Hv_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b\in B} u_{ib} + \sum_{b\in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \le v_b & \forall i \in I, \forall b \in B \\
& v_b \ge v_c & \forall (b,c) \in B : b < c \\
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& y_i + d_i \le D & \forall i \in I \\
& z_i + h_i \le H & \forall i \in I \\
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& (y_i + d_i) - y_j \le D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \le D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \le H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \le H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \ge u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \ge (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

MILP Model - Geometric Constraints 3

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

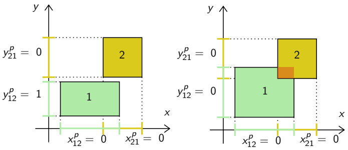

Figure: Precedences variables (2D case)

$$
\begin{aligned}
\min \quad & \sum_{b \in B} (H v_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
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& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

MILP Model - Vertical Support

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Model - Closeness

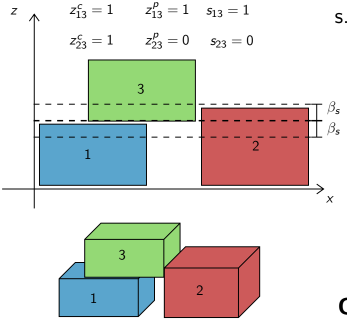


Figure: Closeness variables example

s.t.

$$
\begin{aligned}
& z_j - (z_i + h_i) \leq \beta_s + H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& z_j - (z_i + h_i) \geq -\beta_s - H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& s_{ij} \leq z^p_{ij} && \forall (i,j) \in I \\
& s_{ij} \leq z^c_{ij} && \forall (i,j) \in I \\
& s_{ij} \geq z^p_{ij} + z^c_{ij} - 2 && \forall (i,j) \in I : i \neq j \\
& \sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} && \forall i \in I
\end{aligned}
$$

Can Support $=$ **Close** $\wedge$ **Precedes**

$$s_{ij} = z^c_{ij} \wedge z^p_{ij}$$

MILP Model - Discretized Vertical Support

Pre-Computed Parameter

$O(i, j, k, h)$

$O(i, j, 1, 2) = 1$

Variables

$s_{ijb}^{12} = 1$

$(x_j - x_i, y_j - y_i)$

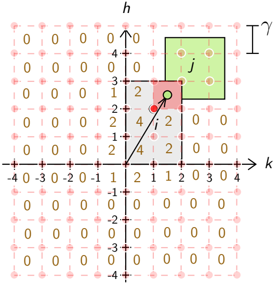


Figure: Space discretization

$$
\begin{aligned}
\text{s.t.} \quad & z_i \leq H(1 - g_i) && \forall i \in I \\
& \sum_{(k,h)\in\Delta, b\in B: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \leq s_{ij} && \forall (i,j) \in I \\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \leq u_{ib} && \forall (i,j,b) \in I^B \\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \leq u_{jb} && \forall (i,j,b) \in I^B \\
& x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& x_j - x_i \leq \gamma (k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& y_j - y_i \leq \gamma (h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& \sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h) s_{jib}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
\end{aligned}
$$

MILP Model - Results

	MILP Model		
n	**Max Z**	**TT(s)**	**Gap(%)**
1	85	0.01	0.00
2	85	0.07	0.00
3	85	0.13	0.00
4	85	0.20	0.00
5	85	2.02	0.00
6	158	90.58	0.00
7	158	1,369.24	0.00
8	161*	3,600.00	1.86
9	-	-	-

* Some boxes had lower support than expected due to discretization errors.

Overview

Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

Support Planes

- Operates on a single bin
- Exploits a modified 2D-BPP heuristic
- No explicit layers
- Guarantees vertical support

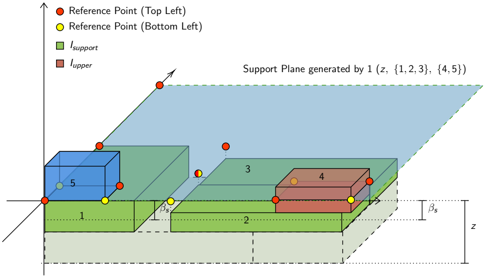
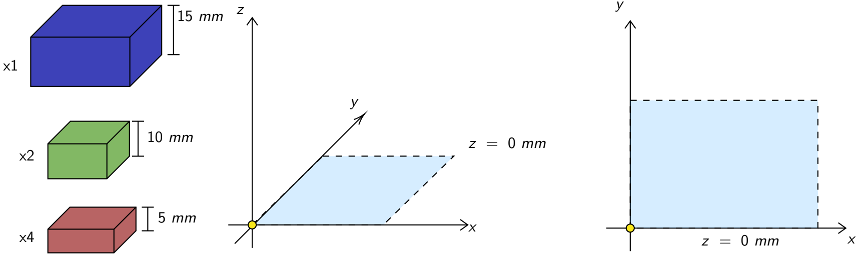
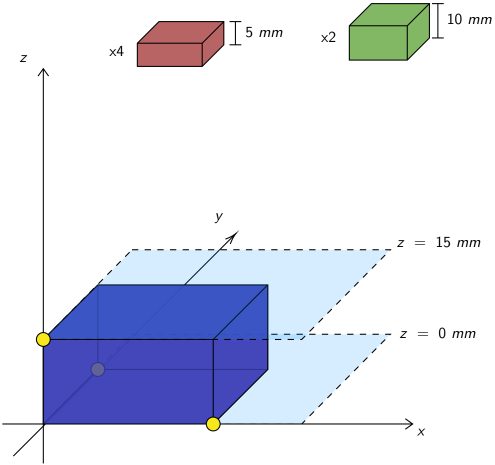


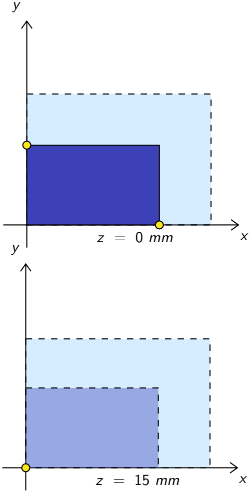
Figure: An example of a support plane generated by item 1

Support Planes

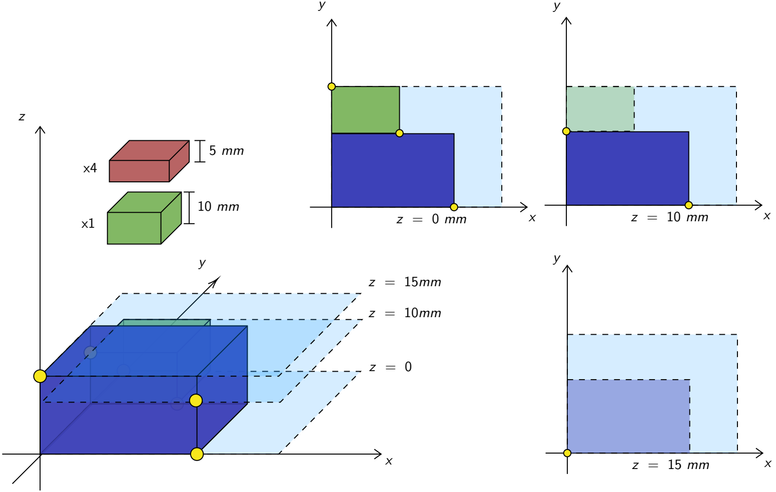


Support Planes

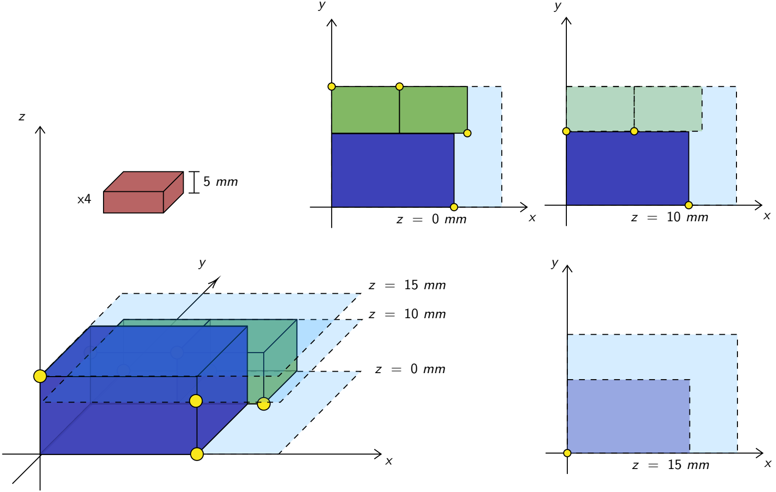




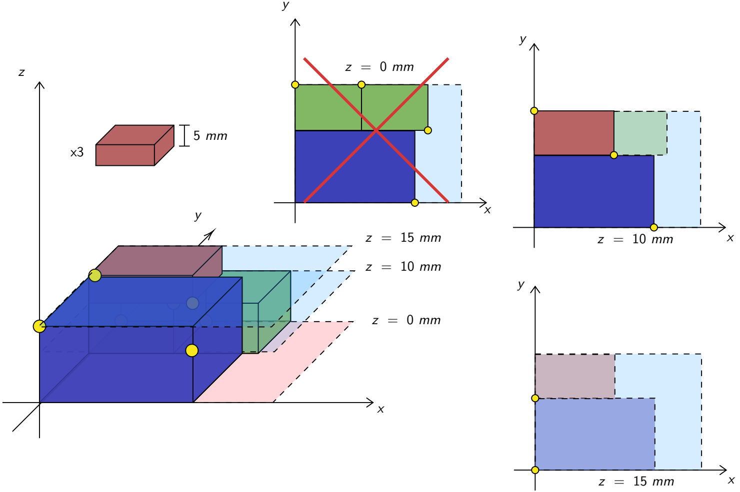
Support Planes



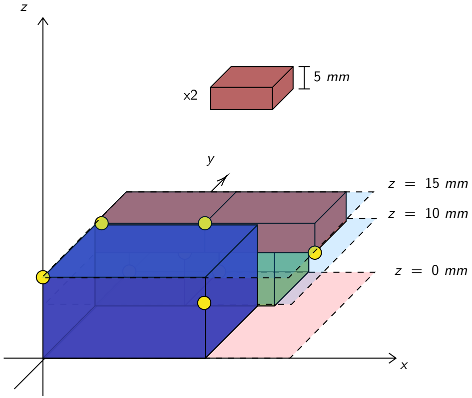
Support Planes

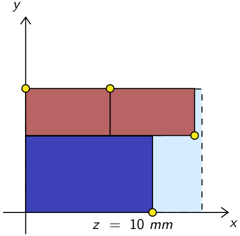


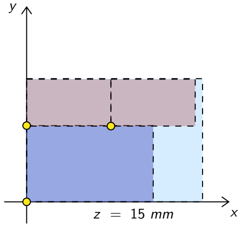
Support Planes



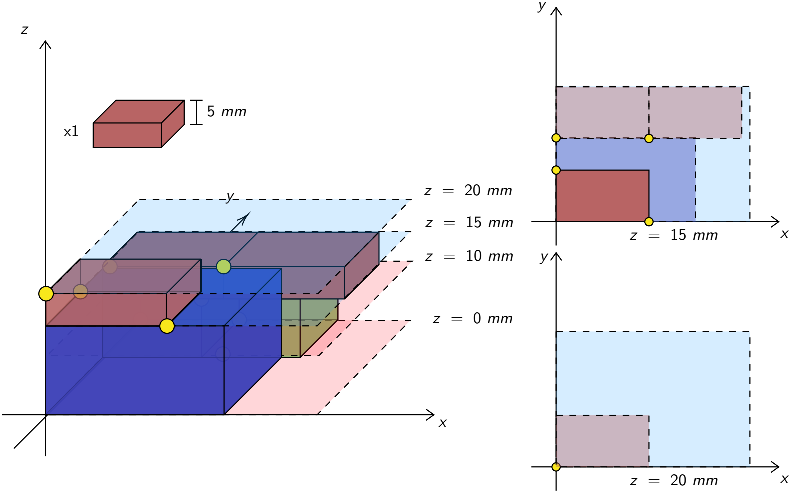
Support Planes







Support Planes



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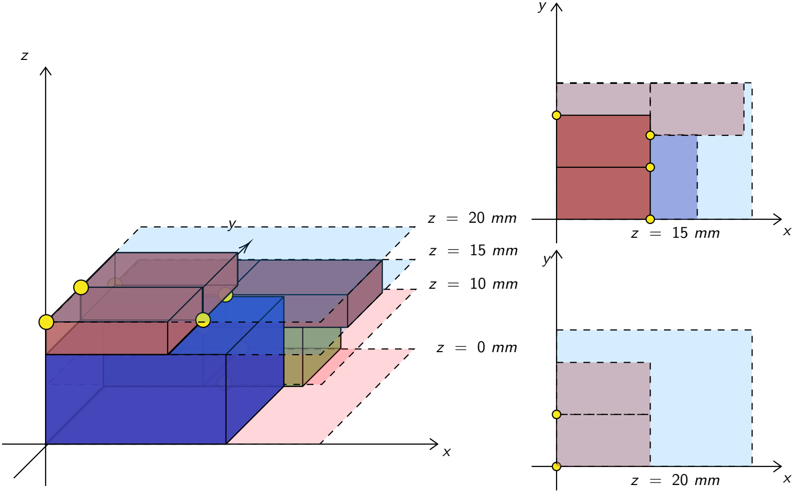
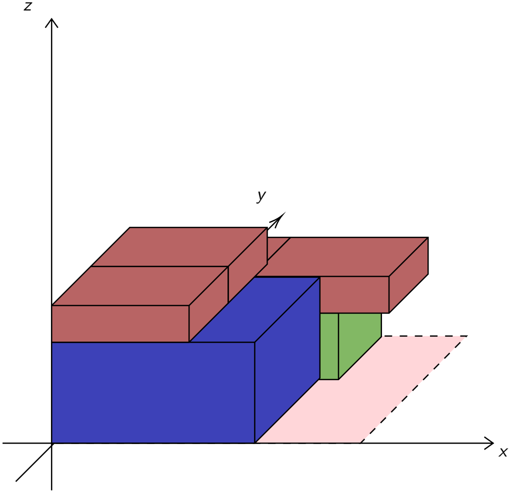


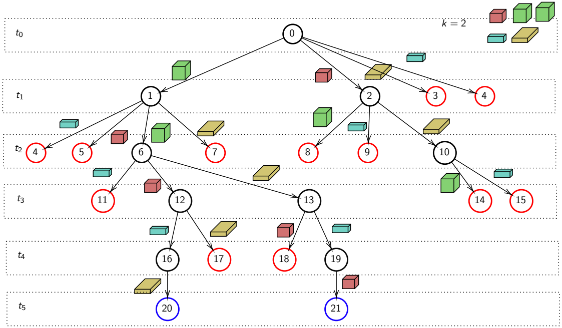
Support Planes



Beam Search

- Exploits support planes
- Branches on the type of item to place next

Beam-Search: truncated breadth-first tree search, limited to k best nodes



Beam Search

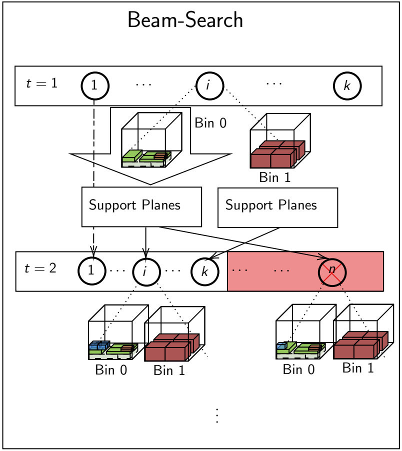


Figure: Conceptual representation of the proposed heuristic

Beam Search

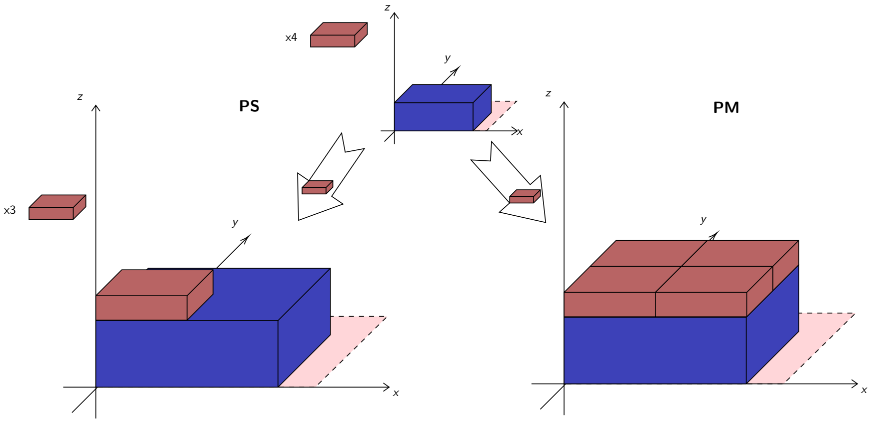


Figure: Proposed placement modes

Optimizations

- Prune duplicate nodes
- Fast overlap checks
- Lazy node updates

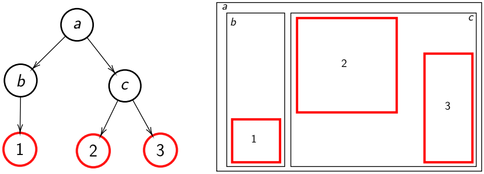


Figure: AABB Tree, self-balancing tree of axis-aligned bounding boxes

Computational Experiments

Table: Comparison with MILP model on limited set of boxes

	MILP Model			PM		PS	
n	Max Z	TT(s)	Gap(%)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	1.86	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

* Some boxes had lower support than expected due to discretization errors.

Computational Experiments

Table: Average execution time of literature results with bin gap

Heuristic		**Execution Time (s)**				**Bin Gap (%)**
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.05	0.11	0.28	0.55	4.57
	$k = 5$	0.08	0.39	1.02	2.16	4.32
	$k = 10$	0.15	0.74	1.98	4.12	4.29
	$k = 20$	0.29	1.45	3.89	8.07	4.05
	$k = 50$	0.72	3.63	9.72	20.47	3.95
PS	$k = 1$	0.04	0.18	0.51	1.08	4.35
	$k = 5$	0.12	0.74	2.19	4.79	4.01
	$k = 10$	0.23	1.43	4.19	9.39	3.94
	$k = 20$	0.47	2.81	8.48	18.93	3.74
	$k = 50$	1.15	6.74	21.03	45.78	3.52
BRKGA-VD		17.13	80.63	190.50	369.75	0.00

Computational Experiments

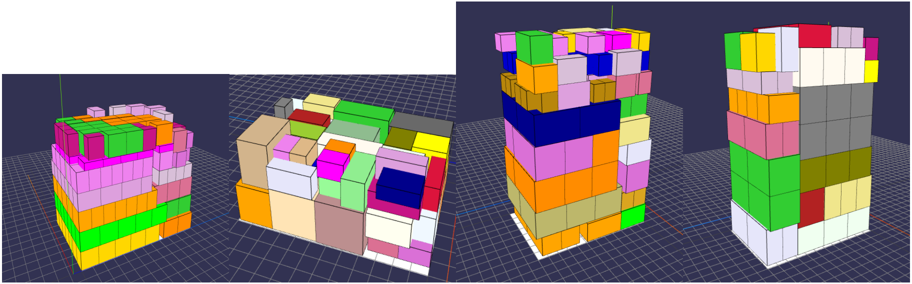
Table: Summary of case study tests

	PS			PM		
k	**TT (ms)**	**B**	**CR (%)**	**TT (ms)**	**B**	**CR (%)**
1	423.87	1.37	65.87	65.18	1.31	**70.70**
5	1,597.54	1.34	69.19	185.22	1.29	**73.08**
10	2,627.52	1.32	70.35	344.90	1.27	**73.56**
20	5,373.79	1.34	70.78	620.95	1.27	**74.57**
50	14,203.10	1.31	72.11	1,279.96	1.29	**74.61**
100	26,934.21	1.31	73.23	2,340.37	1.26	**75.36**
200	48,944.90	1.30	73.89	4,465.78	1.25	**76.39**

Computational Experiments

Table: Case study experiments trade off between average execution times and average cage ratio

k	PS		PM	
	CR* – CR (%)	**TT – TT* (ms)**	**CR* – CR (%)**	**TT – TT* (ms)**
1	10.56	358.69	5.73	0.00
5	7.24	1,532.36	3.35	120.04
10	6.08	2,562.34	2.87	279.72
20	5.65	5,308.61	**1.85**	**555.77**
50	4.32	14,137.92	**1.82**	**1,214.78**
100	3.20	26,869.03	1.07	2,275.19
200	2.54	48,879.72	0.04	4,400.60



(a) Instance 95 (b) Instance 82 (c) Instance 56 (d) Instance 66, Bin 1

Figure: Solutions of case study tests with the "PM" placement and $k = 200$

Future Developments

- More practical constraints
- Solution optimization heuristics
- Improvements to the underlying 2D-BPP heuristic

Thank you for your attention!
Questions?