



**POLITECNICO**  
MILANO 1863

# Three-Dimensional Bin Packing with Vertical Support

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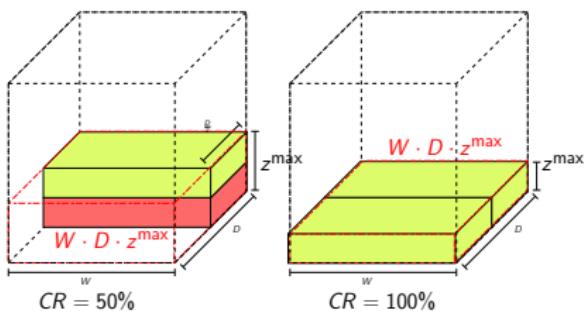


# Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure



Example of pit palletization



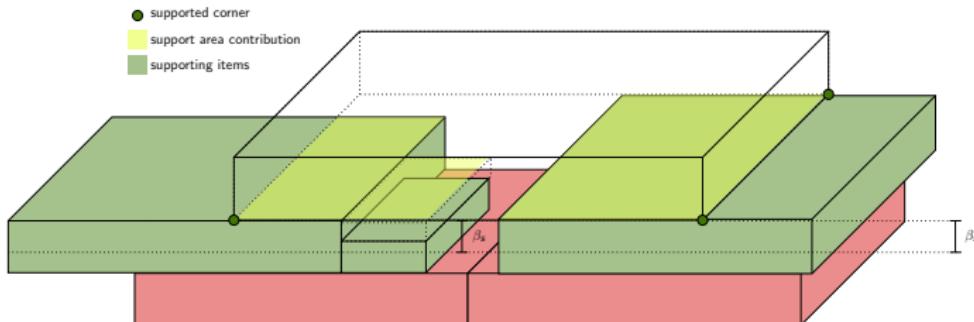
Cage ratio of two different bin configurations

# Vertical Support

## Definition

An item has vertical support if one of the following conditions holds:

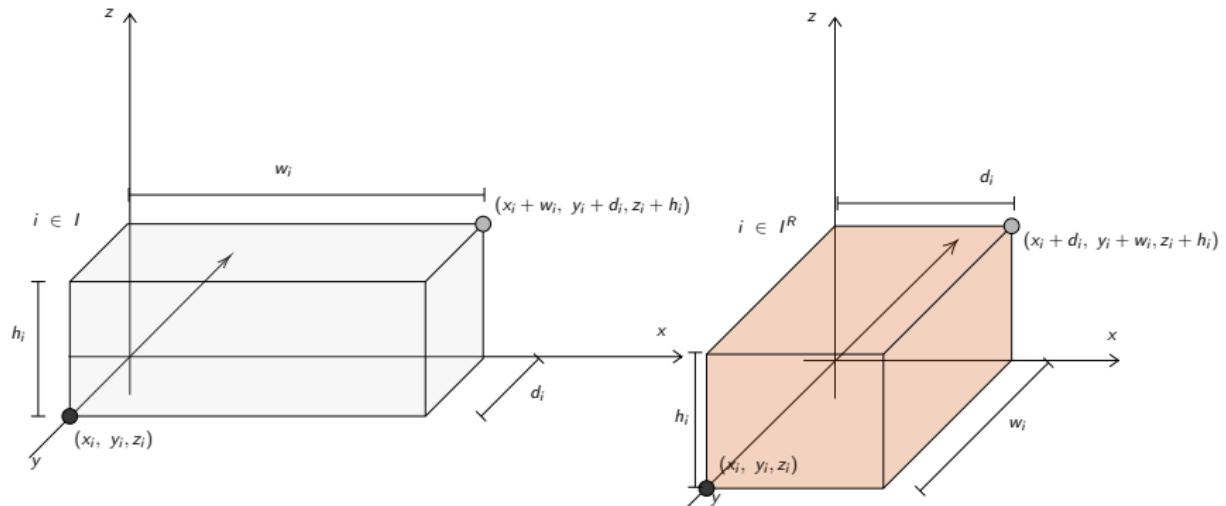
- **Condition 1:** at least a percentage  $\alpha_s$  of its base area is resting on other items
- **Condition 2:** at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage



**Problem:** pack a set of cuboids into the minimum amount of bins, maximizing the cage ratio of used bins, without overlaps, and with every item having vertical support

- The problem is NP-Hard
- Exact methods may lead to optimal solutions only in fairly small instances
- Existing 3D-BPP heuristics don't consider vertical support
- Solutions for container loading and pallet loading problems are layer based

## MILP Model - Coordinate System



Coordinate system representation for a generic item  $i$  and its rotated clone  $i \in I^R$

**minimize** number of used bins

**then, maximize** average cage ratio of the used bins

**subject to** all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

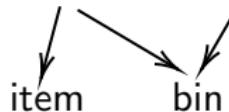
all items have vertical support

## MILP Model - Objective Function

**minimize**                    number of used bins  
**then, maximize**                average cage ratio of the used bins  
**subject to**                 all items are assigned to one and only one bin  
                                  all items are inside the bin's bounds  
                                  no overlaps between items in the same bin  
                                  all items have vertical support

**Assign      Use**

$U_{ib}$  ,  $V_b$

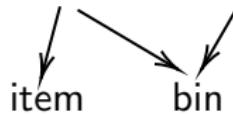


$$\begin{aligned}
 & \text{min} && \sum_{b \in B} (Hv_b + z_b^{\max}) \\
 & \text{s.t.} && \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR} \\
 & && u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B \\
 & && v_b \geq v_c \quad \forall (b, c) \in B : b < c \\
 & && x_i + w_i \leq W \quad \forall i \in I \\
 & && y_i + d_i \leq D \quad \forall i \in I \\
 & && z_i + h_i \leq H \quad \forall i \in I \\
 & && (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I \\
 & && x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I \\
 & && (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I \\
 & && y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I \\
 & && (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I \\
 & && z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I \\
 & && x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B \\
 & && z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B
 \end{aligned}$$

## MILP Model - Geometric Constraints 1

<b>minimize</b> then, <b>maximize</b> <b>subject to</b>	number of used bins average cage ratio of the used bins all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$\sum_{b \in B} (Hv_b + z_b^{max})$ $\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1$ $u_{ib} \leq v_b$ $v_b \geq v_c$ $x_i + w_i \leq W$ $y_i + d_i \leq D$ $z_i + h_i \leq H$ $(x_i + w_i) - x_j \leq W(1 - x_{ij}^p)$ $x_j - (x_i + w_i) + 1 \leq Wx_{ij}^p$ $(y_i + d_i) - y_j \leq D(1 - y_{ij}^p)$ $y_j - (y_i + d_i) + 1 \leq Dy_{ij}^p$ $(z_i + h_i) - z_j \leq H(1 - z_{ij}^p)$ $z_j - (z_i + h_i) + 1 \leq Hz_{ij}^p$ $x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1$ $z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib})$
		$\forall(i,j) \in I^{OR}$ $\forall i \in I, \forall b \in B$ $\forall(b,c) \in B : b < c$ $\forall i \in I$ $\forall i \in I$ $\forall i \in I$ $\forall i, j \in I$

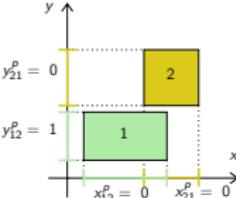
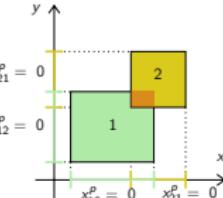
**Assign      Use**  
 $U_{ib}$  ,  $V_b$



## MILP Model - Geometric Constraints 2

<b>minimize</b> then, <b>maximize</b> <b>subject to</b>	number of used bins average cage ratio of the used bins all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$\min \quad \sum_{b \in B} (Hv_b + z_b^{\max})$
		<b>s.t.</b> $\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR}$
		$u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B$
		$v_b \geq v_c \quad \forall (b, c) \in B : b < c$
		$x_i + w_i \leq W \quad \forall i \in I$
		$y_i + d_i \leq D \quad \forall i \in I$
		$z_i + h_i \leq H \quad \forall i \in I$
		$(x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I$
		$x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I$
		$(y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I$
		$y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I$
		$(z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I$
		$z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I$
		$x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B$
		$z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B$

## MILP Model - Geometric Constraints 3

		$\min \quad \sum_{b \in B} (Hv_b + z_b^{max})$	
		$\text{s.t.} \quad \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR}$	
minimize then, maximize	number of used bins average cage ratio of the used bins	$u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B$	
<b>subject to</b>	all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$v_b \geq v_c \quad \forall (b, c) \in B : b < c$ $x_i + w_i \leq W \quad \forall i \in I$ $y_i + d_i \leq D \quad \forall i \in I$ $z_i + h_i \leq H \quad \forall i \in I$ $(x_i + w_i) - x_j \leq W(1 - x_{ij}^p) \quad \forall i, j \in I$ $x_j - (x_i + w_i) + 1 \leq Wx_{ij}^p \quad \forall i, j \in I$ $(y_i + d_i) - y_j \leq D(1 - y_{ij}^p) \quad \forall i, j \in I$ $y_j - (y_i + d_i) + 1 \leq Dy_{ij}^p \quad \forall i, j \in I$ $(z_i + h_i) - z_j \leq H(1 - z_{ij}^p) \quad \forall i, j \in I$ $z_j - (z_i + h_i) + 1 \leq Hz_{ij}^p \quad \forall i, j \in I$ $x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B$ $z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B$	
			

Precedences variables (2D case)

**minimize** number of used bins

**then, maximize** average cage ratio of the used bins

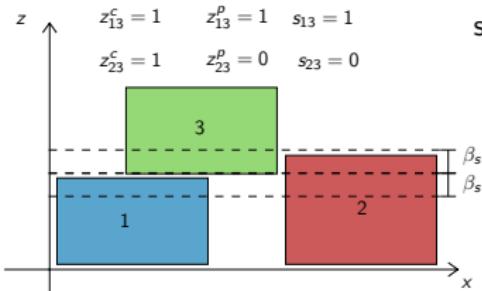
**subject to** all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

## MILP Model - Closeness



$$z_j - (z_i + h_i) \leq \beta_s + H(1 - z_{ij}^c) \quad \forall (i,j) \in I : i \neq j$$

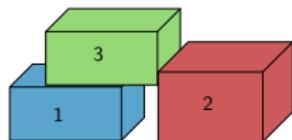
$$z_j - (z_i + h_i) \geq -\beta_s - H(1 - z_{ij}^c) \quad \forall (i,j) \in I : i \neq j$$

$$s_{ij} \leq z_{ij}^p \quad \forall (i,j) \in I$$

$$s_{ij} \leq z_{ij}^c \quad \forall (i,j) \in I$$

$$s_{ij} \geq z_{ij}^p + z_{ij}^c - 2 \quad \forall (i,j) \in I : i \neq j$$

$$\sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} \quad \forall i \in I$$



**Can Support** = **Close**  $\wedge$  **Precedes**

Closeness variables example

$$s_{ij} = z_{ij}^c \wedge z_{ij}^p$$

Pre-Computed Parameter

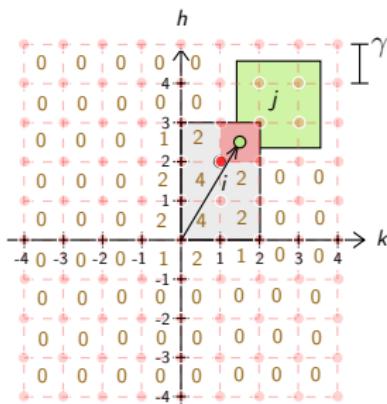
$$O(i, j, k, h)$$

$$O(i, j, 1, 2) = 1$$

Variables

$$s_{ijb}^{12} = 1$$

$$(x_j - x_i, y_j - y_i)$$



Space discretization

$$\begin{aligned}
 & \text{s.t.} && z_i \leq H(1 - g_i) && \forall i \in I \\
 & && \sum_{(k,h) \in \Delta, b \in B: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq s_{ij} && \forall (i,j) \in I \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{ib} && \forall (i,j,b) \in I^B \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{jb} && \forall (i,j,b) \in I^B \\
 & && x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && \sum_{(k,h) \in \Delta, b \in B, j \in I: i \neq j \wedge O(i,j,k,h) \neq 0} O(i,j,k,h) s_{ijb}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
 \end{aligned}$$

# MILP Model - Results

MILP Model			
$n$	Max Z	TT(s)	Gap(%)
1	85	0.01	0.00
2	85	0.07	0.00
3	85	0.13	0.00
4	85	0.20	0.00
5	85	2.02	0.00
6	158	90.58	0.00
7	158	1,369.24	0.00
8	161*	3,600.00	1.86
9	-	-	-

\* Some boxes had lower support than expected due to discretization errors.

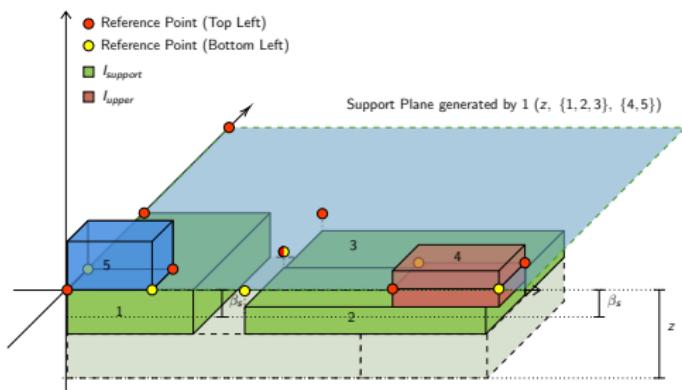
## Overview

Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

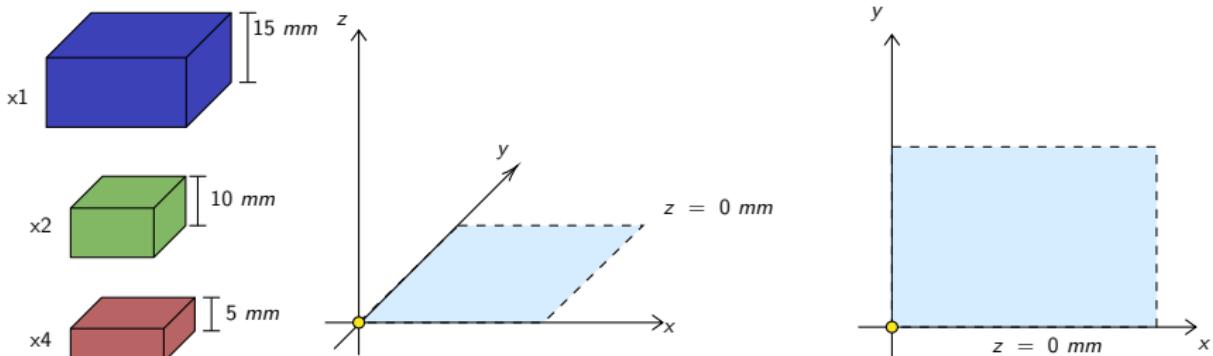
# Support Planes

- Operates on a single bin
- Exploits a modified 2D-BPP heuristic
  - ▶ Lowest plane
  - ▶ First fit
- No explicit layers
- Guarantees vertical support

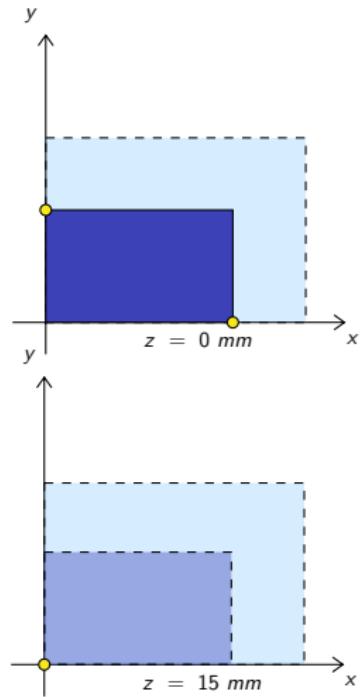
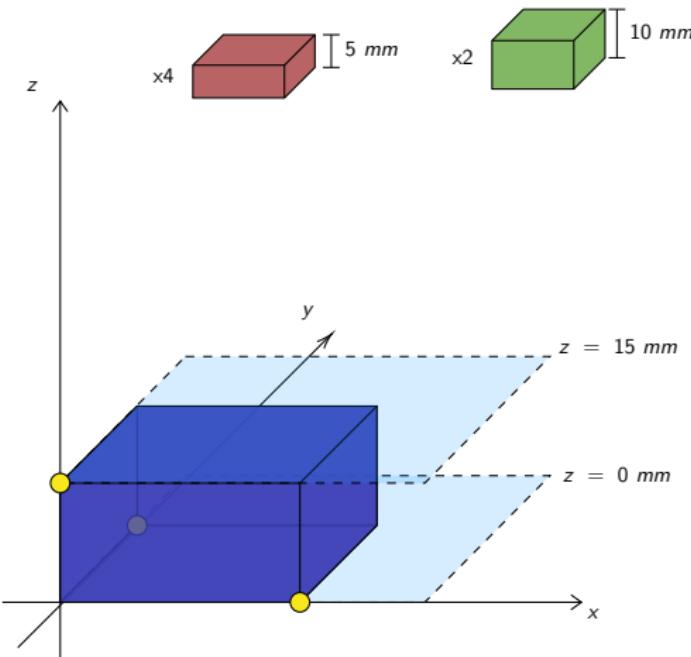


An example of a support plane generated by item 1

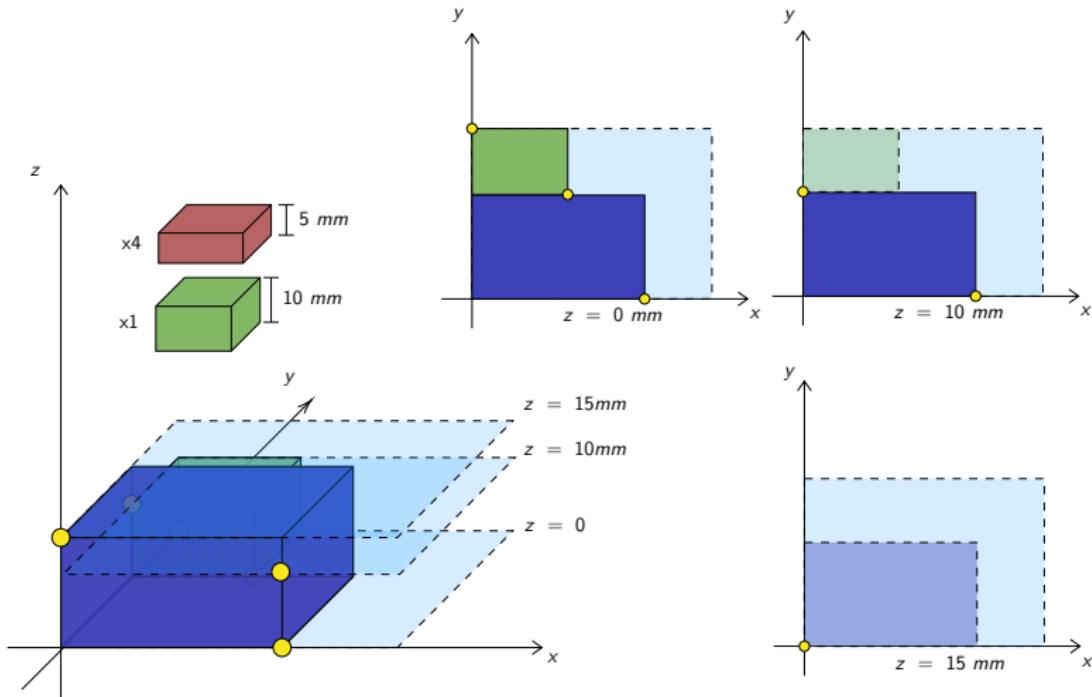
## Support Planes



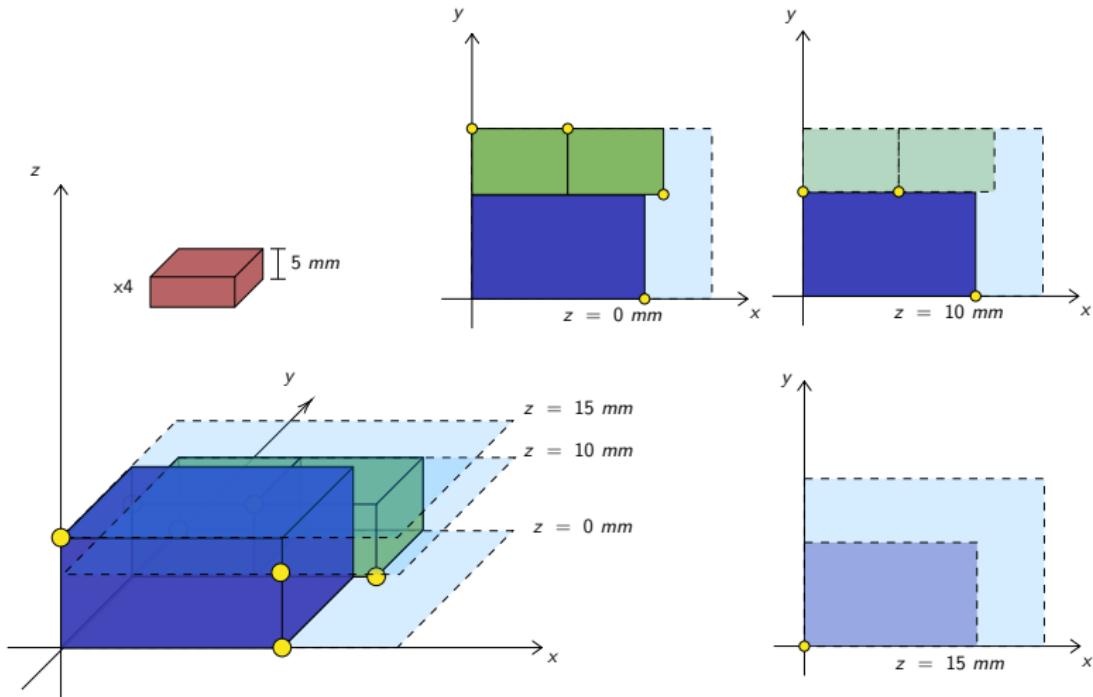
## Support Planes



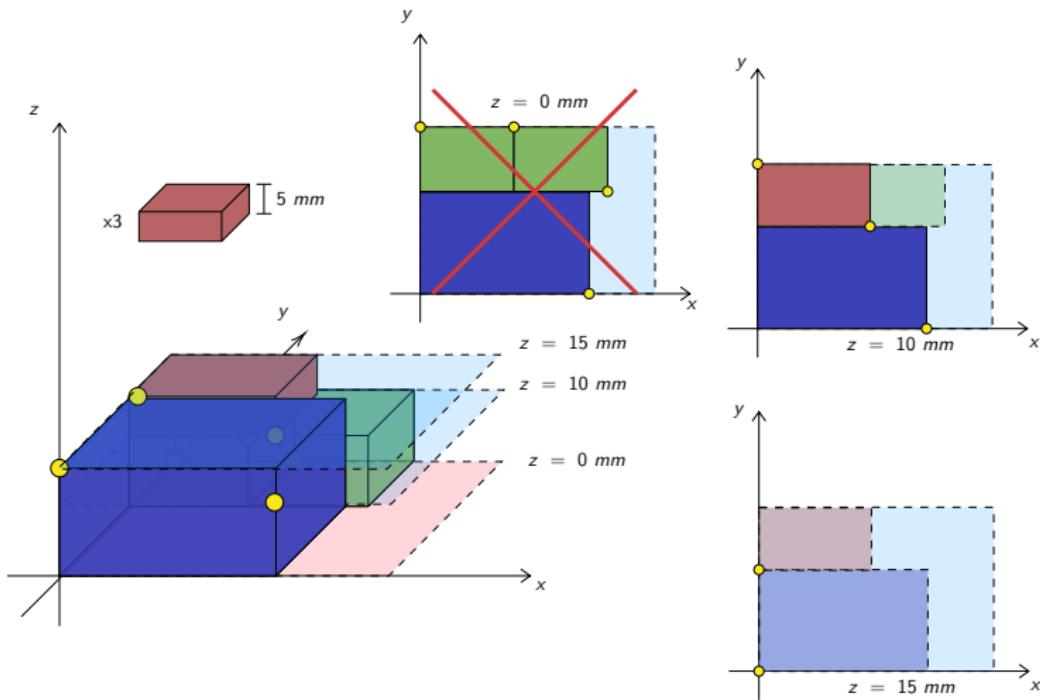
## Support Planes



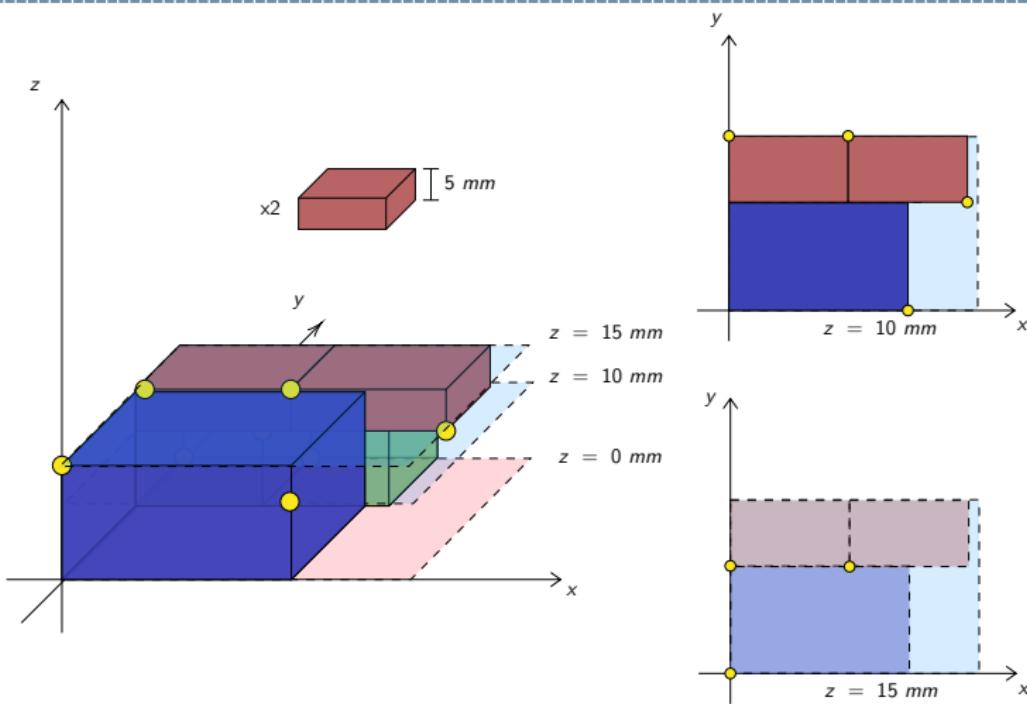
## Support Planes



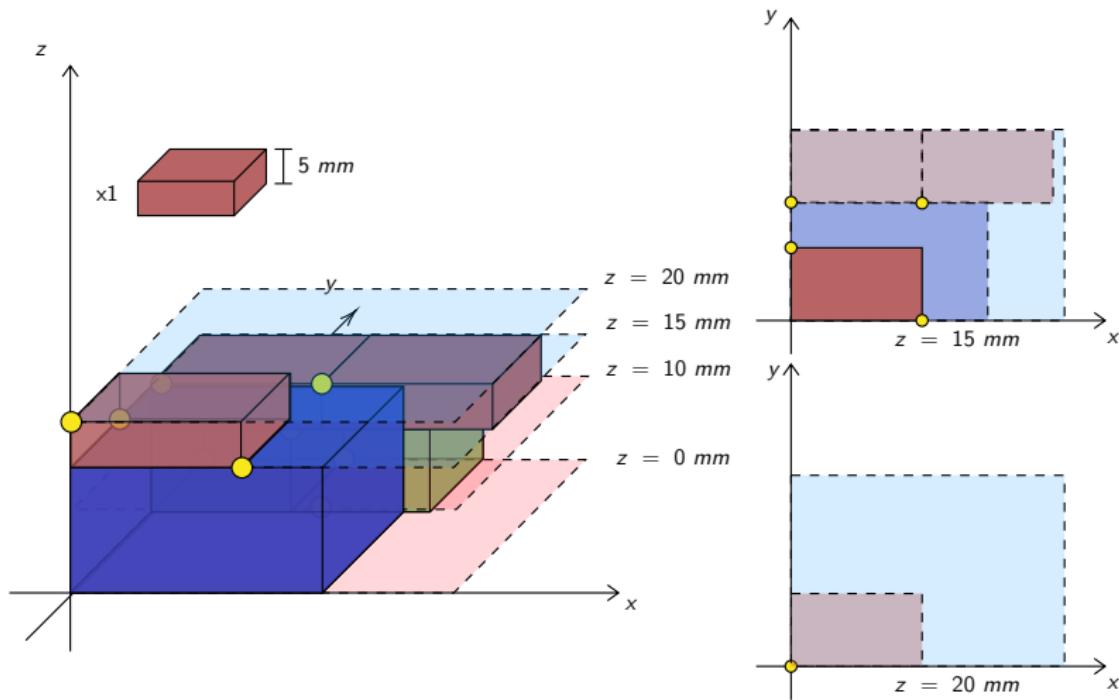
## Support Planes



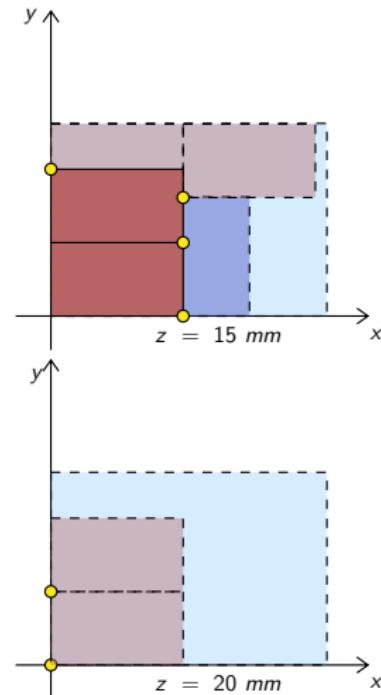
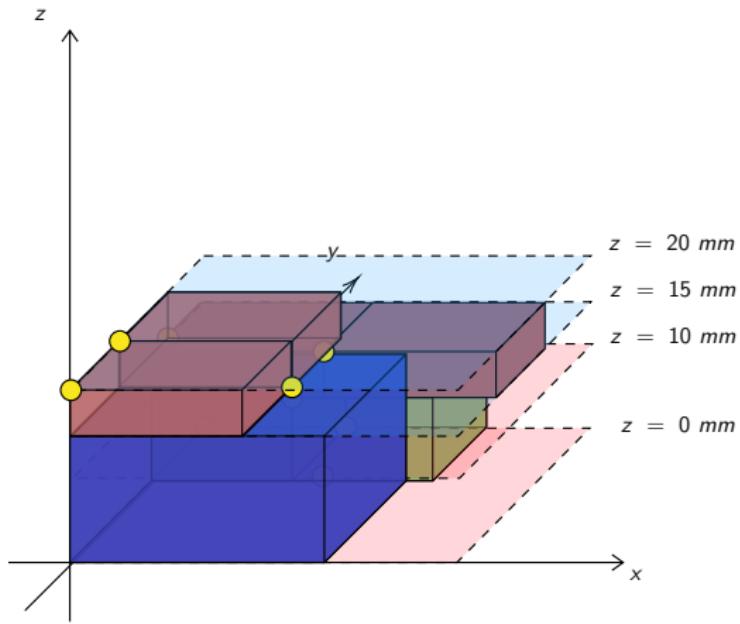
## Support Planes



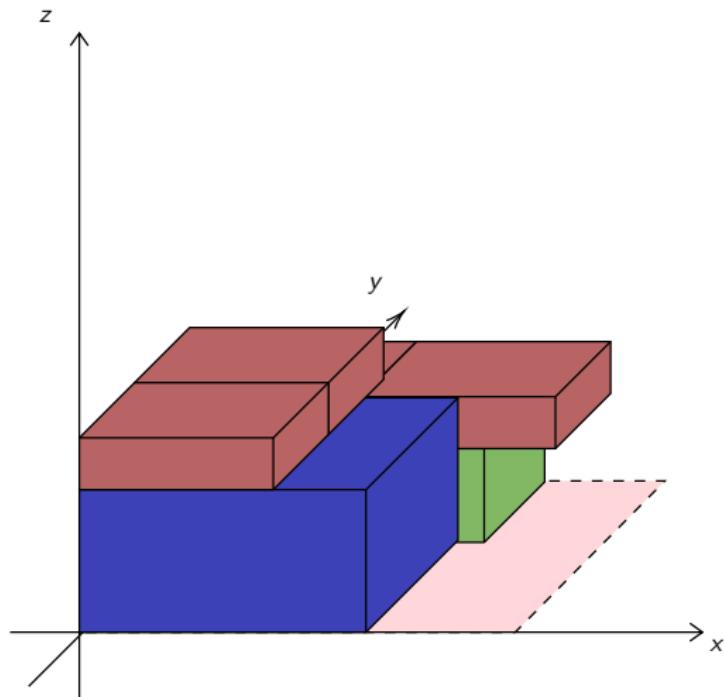
## Support Planes



## Support Planes



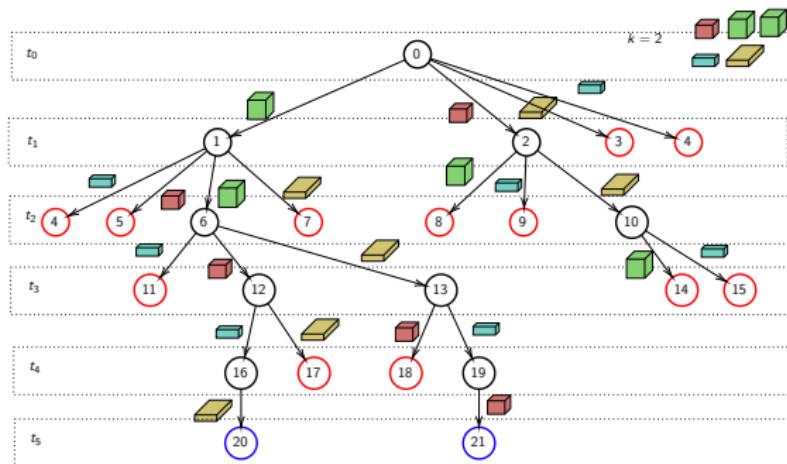
## Support Planes



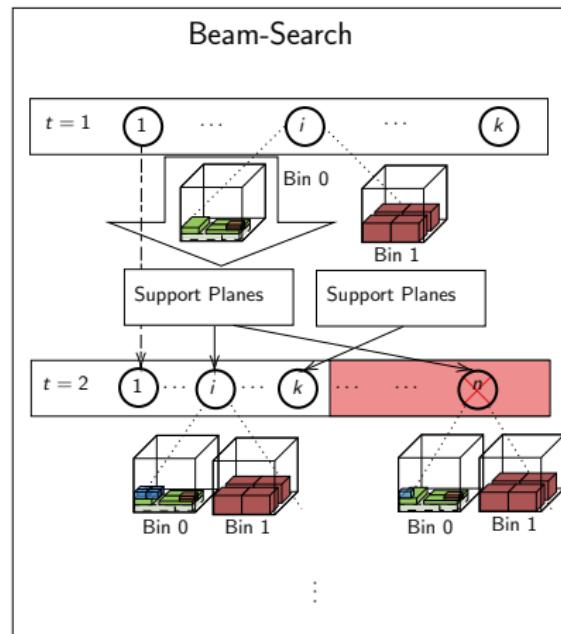
## Beam Search

- Exploits support planes
- Branches on the type of item to place next

**Beam-Search:** truncated breadth-first tree search, limited to  $k$  best nodes

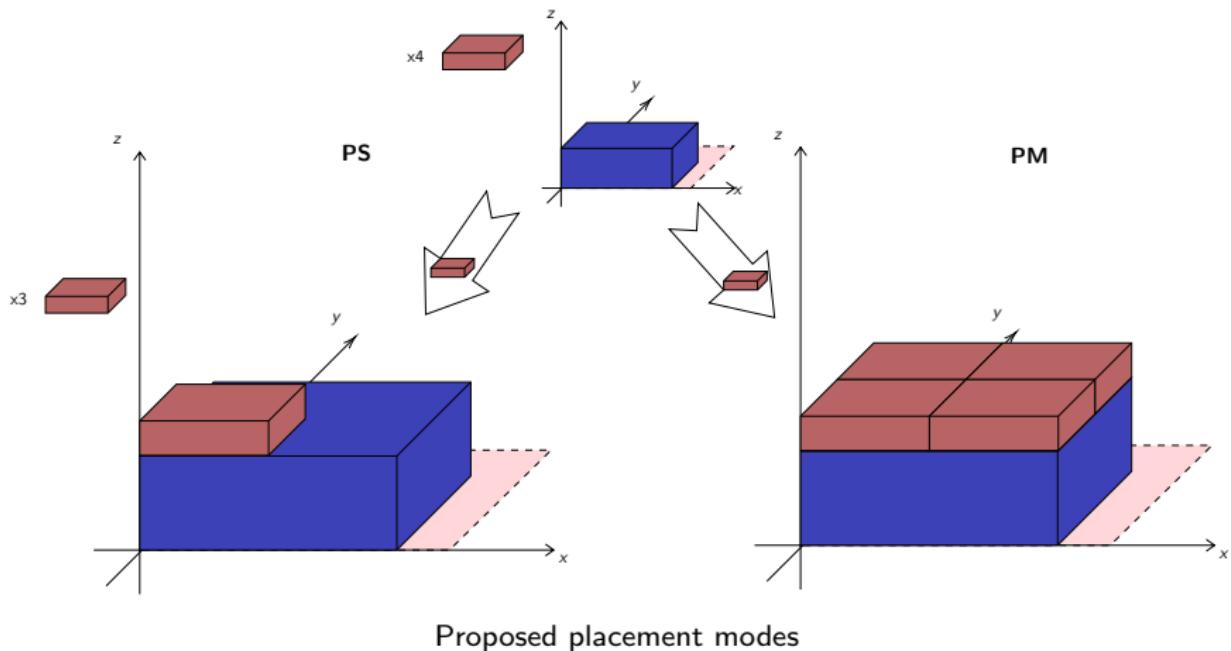


## Beam Search



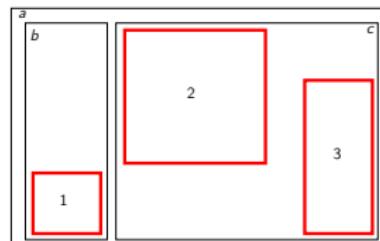
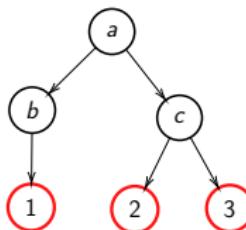
Conceptual representation of the proposed heuristic

## Beam Search



Implemented as a Java program

- Prune duplicate nodes
- Fast overlap checks
- Lazy node updates



AABB Tree, self-balancing tree of axis-aligned  
bounding boxes

# Computational Experiments

Comparison with MILP model on limited set of boxes

<i>n</i>	MILP Model			PM		PS	
	Max Z	TT(s)	Gap(%)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	1.86	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

\* Some boxes had lower support than expected due to discretization errors.

# Computational Experiments

Class	$n$	PM $k = 50$	PS $k = 50$	TS3	GLS	GASP	EHGH2	GVN	BRKGA	BRKGA-VD
1	50	14.10	<b>14</b>	<b>13.4</b>	<b>13.4</b>	<b>13.4</b>	13.8	<b>13.4</b>	<b>13.4</b>	<b>13.4</b>
	100	28.3	27.7	<b>26.6</b>	<b>26.6</b>	26.9	27.6	<b>26.6</b>	<b>26.6</b>	<b>26.6</b>
	150	38.1	38.1	36.7	37	37	39.8	36.4	36.4	<b>36.3</b>
	200	52.9	52.6	51.2	51.2	51.6	<b>50.6</b>	50.9	50.8	50.8
2	50	14.7	14.7	<b>13.8</b>	-	-	-	<b>13.8</b>	<b>13.8</b>	<b>13.8</b>
	100	26.6	26.6	25.7	-	-	-	25.7	25.6	<b>25.5</b>
	150	38.3	38.7	37.2	-	-	-	36.9	<b>36.6</b>	<b>36.6</b>
	200	51.1	51.6	50.1	-	-	-	<b>49.4</b>	<b>49.4</b>	<b>49.4</b>
3	50	13.7	13.8	<b>13.3</b>	-	-	-	<b>13.3</b>	<b>13.3</b>	<b>13.3</b>
	100	27.7	27.3	26	-	-	-	26	<b>25.9</b>	<b>25.9</b>
	150	39.4	39	37.7	-	-	-	37.6	<b>37.5</b>	<b>37.5</b>
	200	51.6	51.3	50.5	-	-	-	50	<b>49.8</b>	<b>49.8</b>
4	50	29.7	29.7	<b>29.4</b>	<b>29.4</b>	<b>29.4</b>	<b>29.4</b>	<b>29.4</b>	<b>29.4</b>	<b>29.4</b>
	100	59.2	59.2	59	59	59	59.5	59	59	<b>58.9</b>
	150	87.7	87.6	<b>86.8</b>	<b>86.8</b>	<b>86.8</b>	90.4	<b>86.8</b>	<b>86.8</b>	<b>86.8</b>
	200	119.5	119.5	<b>118.8</b>	119	<b>118.8</b>	119	<b>118.8</b>	<b>118.8</b>	<b>118.8</b>
5	50	<b>8.6</b>	<b>8.6</b>	8.4	8.3	8.4	<b>7.9</b>	8.3	8.3	8.3
	100	16.1	15.6	15	15.1	15.1	<b>14.6</b>	15	15	15
	150	21.8	21.3	20.4	20.2	20.6	21.5	20.4	20.1	<b>19.9</b>
	200	29.1	28.3	27.6	27.2	27.7	29.6	<b>27.1</b>	<b>27.1</b>	<b>27.1</b>
6	50	10.3	10.3	9.9	9.8	9.9	11.8	9.8	<b>9.7</b>	<b>9.7</b>
	100	19.7	19.7	19.1	19.1	19.1	19.2	19	<b>18.9</b>	<b>18.9</b>
	150	30.2	<b>30.1</b>	29.4	29.4	29.5	29.8	29.2	<b>29</b>	<b>29</b>
	200	39	38.4	37.7	37.7	38	38.7	37.4	<b>37.3</b>	<b>37.3</b>
7	50	7.7	7.8	7.5	<b>7.4</b>	7.5	<b>7.4</b>	<b>7.4</b>	<b>7.4</b>	<b>7.4</b>
	100	13.3	<b>13.1</b>	12.5	12.3	12.7	13.5	12.5	<b>12.2</b>	<b>12.2</b>
	150	17.1	16.8	16.1	15.8	16.6	18.2	16	15.3	<b>15.2</b>
	200	24.8	24.7	23.9	23.5	24.2	24.1	23.5	<b>23.4</b>	<b>23.4</b>
8	50	9.9	9.7	9.3	<b>9.2</b>	9.3	9.4	<b>9.2</b>	<b>9.2</b>	<b>9.2</b>
	100	19.6	19.9	18.9	18.9	19	18.9	18.9	18.9	<b>18.8</b>
	150	25.7	25.6	24.1	23.9	24.8	26	24.1	<b>23.6</b>	<b>23.6</b>
	200	31.6	31.1	30.3	29.9	31.1	35.8	29.8	<b>29.3</b>	<b>29.3</b>

# Computational Experiments

Average execution time of literature results with bin gap

Heuristic		Execution Time (s)				Bin Gap (%)
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
<b>PM</b>	$k = 1$	0.05	0.11	0.28	0.55	4.57
	$k = 5$	0.08	0.39	1.02	2.16	4.32
	$k = 10$	0.15	0.74	1.98	4.12	4.29
	$k = 20$	0.29	1.45	3.89	8.07	4.05
	$k = 50$	0.72	3.63	9.72	20.47	3.95
<b>PS</b>	$k = 1$	0.04	0.18	0.51	1.08	4.35
	$k = 5$	0.12	0.74	2.19	4.79	4.01
	$k = 10$	0.23	1.43	4.19	9.39	3.94
	$k = 20$	0.47	2.81	8.48	18.93	3.74
	$k = 50$	1.15	6.74	21.03	45.78	3.52
<b>BRKGA-VD</b>		17.13	80.63	190.50	369.75	0.00

# Computational Experiments

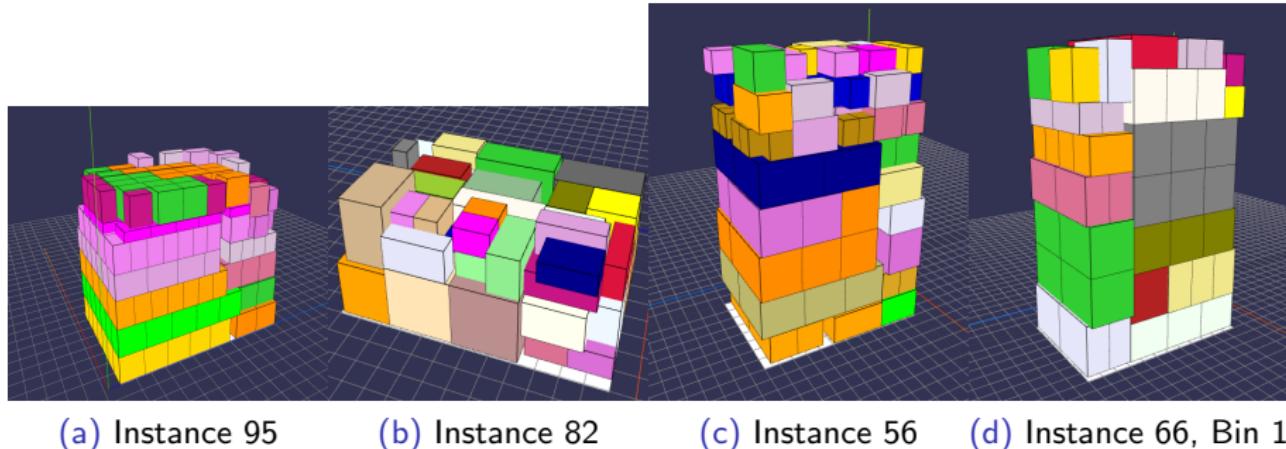
Summary of case study tests

k	PS			PM		
	TT (ms)	B	CR (%)	TT (ms)	B	CR (%)
1	423.87	1.37	65.87	65.18	1.31	<b>70.70</b>
5	1,597.54	1.34	69.19	185.22	1.29	<b>73.08</b>
10	2,627.52	1.32	70.35	344.90	1.27	<b>73.56</b>
20	5,373.79	1.34	70.78	620.95	1.27	<b>74.57</b>
50	14,203.10	1.31	72.11	1,279.96	1.29	<b>74.61</b>
100	26,934.21	1.31	73.23	2,340.37	1.26	<b>75.36</b>
200	48,944.90	1.30	73.89	4,465.78	1.25	<b>76.39</b>

# Computational Experiments

Case study experiments trade off between average execution times and average cage ratio

$k$	PS		PM	
	CR* – CR (%)	TT – TT* (ms)	CR* – CR (%)	TT – TT* (ms)
1	10.56	358.69	5.73	0.00
5	7.24	1,532.36	3.35	120.04
10	6.08	2,562.34	2.87	279.72
<b>20</b>	5.65	5,308.61	<b>1.85</b>	<b>555.77</b>
<b>50</b>	4.32	14,137.92	<b>1.82</b>	<b>1,214.78</b>
100	3.20	26,869.03	1.07	2,275.19
200	2.54	48,879.72	0.04	4,400.60



(a) Instance 95

(b) Instance 82

(c) Instance 56

(d) Instance 66, Bin 1

Solutions of case study tests with the "PM" placement and  $k = 200$

## Future Developments

- More practical constraints
- Solution optimization heuristics
- Improvements to the underlying 2D-BPP heuristic

Thank you for your attention!  
Questions?