Three-Dimensional Bin Packing with Vertical Support

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Table of contents

Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure

ERMES-X

Figure: Example of pit pallettization (Schäfer Case Picking — SSI SCHÄFER)

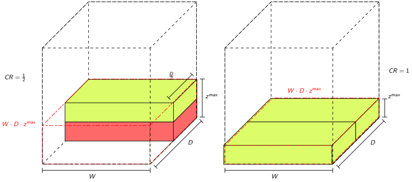

Figure: Cage ratio of two different bin configurations

Vertical Support

Definition

An item has vertical support if one of the following conditions hold:

- **Condition 1**: at least a percentage α_s of its base area is resting on other items
- **Condition 2**: at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

Vertical Support

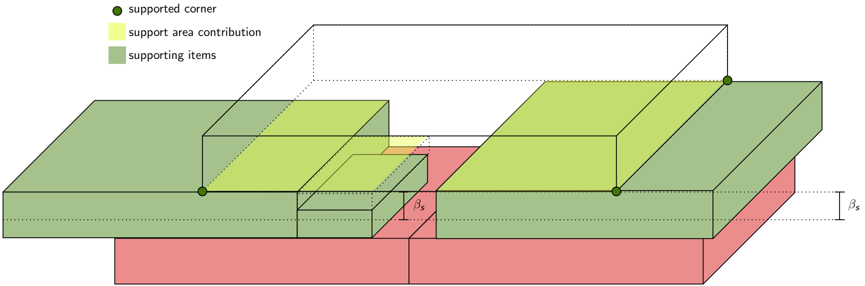

Figure: Representation of an item with conditions 1 and 2 of vertical support given $\alpha_s = 0.5$, β_s

Literature Review

- The problem is NP-Hard
- Exact methods only for small instances
- Existing 3D-BPP heuristics don't consider practical constraints
- Solutions for container loading and pallet loading problems are layer based

Conceptual Formulation

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

MILP Proxy Model - Objective Function

minimize number of used bins
then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
all items are inside the bin's bounds
no overlaps between items in the same bin
all items have vertical support

$$
\begin{aligned}
\min \quad & \sum_{b \in B} (Hv_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
& x_i + w_i \leq W & \forall i \in I \\
& y_i + d_i \leq D & \forall i \in I \\
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

MILP Proxy Model - Geometric Constraints 1

minimize number of used bins
then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
all items are inside the bin's bounds
no overlaps between items in the same bin
all items have vertical support

$$
\begin{array}{lll}
\min & \sum_{b \in B} (Hv_b + z_b^{max}) & \\
\text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
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& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}
$$

MILP Proxy Model - Geometric Constraints 2

minimize number of used bins
then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
all items are inside the bin's bounds
no overlaps between items in the same bin
all items have vertical support

$$
\begin{aligned}
\min \quad & \sum_{b \in B} (H v_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
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& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

MILP Proxy Model - Geometric Constraints 3

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

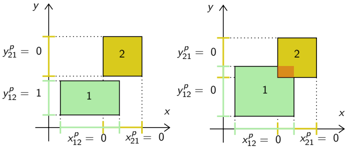


Figure: Precedences variables (2D case)

$$
\begin{aligned}
\min \quad & \sum_{b\in B} (Hv_b + z_b^{max}) && \\
\text{s.t.} \quad & \sum_{b\in B} u_{ib} + \sum_{b\in B} u_{jb} = 1 && \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b && \forall i \in I, \forall b \in B \\
& v_b \geq v_c && \forall (b,c) \in B : b < c \\
& x_i + w_i \leq W && \forall i \in I \\
& y_i + d_i \leq D && \forall i \in I \\
& z_i + h_i \leq H && \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x^p_{ij}) && \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x^p_{ij} && \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y^p_{ij}) && \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y^p_{ij} && \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z^p_{ij}) && \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z^p_{ij} && \forall i,j \in I \\
& x^p_{ij} + x^p_{ji} + y^p_{ij} + y^p_{ji} + z^p_{ij} + z^p_{ji} \geq u_{ib} + u_{jb} - 1 && \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) && \forall i \in I, \forall b \in B
\end{aligned}
$$

MILP Proxy Model - Vertical Support

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

MILP Proxy Model - Closeness

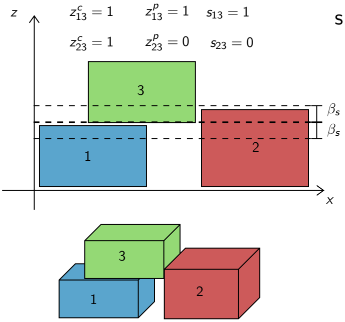


Figure: Closeness variables example

s.t.

$$
\begin{aligned}
& z_j - (z_i + h_i) \leq \beta_s + H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& z_j - (z_i + h_i) \geq -\beta_s - H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& s_{ij} \leq z^p_{ij} && \forall (i,j) \in I \\
& s_{ij} \leq z^c_{ij} && \forall (i,j) \in I \\
& s_{ij} \geq z^p_{ij} + z^c_{ij} - 2 && \forall (i,j) \in I : i \neq j \\
& \sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} && \forall i \in I
\end{aligned}
$$

MILP Proxy Model - Discretized Vertical Support

Pre-Computed Parameter

$O(i,j,k,h)$

$O(i,j,1,2) = 1$

Variables

$s_{ijb}^{12} = 1$

$(x_j - x_i, y_j - y_i)$

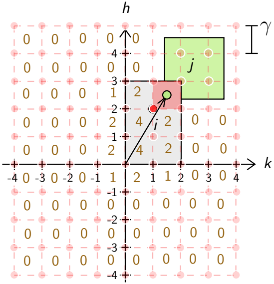


Figure: Space discretization

$$
\begin{aligned}
\text{s.t.}\quad & z_i \le H(1-g_i) && \forall i \in I\\
& \sum_{(k,h)\in\Delta, b\in B: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le s_{ij} && \forall (i,j) \in I\\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le u_{ib} && \forall (i,j,b) \in I^B\\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le u_{jb} && \forall (i,j,b) \in I^B\\
& x_j - x_i \ge \gamma k - 2W(1-s_{ijb}^{kh}) && \forall (k,h)\in\Delta, \forall (i,j,b)\in I^B : O(i,j,k,h)\neq 0\\
& x_j - x_i \le \gamma (k+1) + 2W(1-s_{ijb}^{kh}) && \forall (k,h)\in\Delta, \forall (i,j,b)\in I^B : O(i,j,k,h)\neq 0\\
& y_j - y_i \ge \gamma h - 2D(1-s_{ijb}^{kh}) && \forall (k,h)\in\Delta, \forall (i,j,b)\in I^B : O(i,j,k,h)\neq 0\\
& y_j - y_i \le \gamma (h+1) + 2D(1-s_{ijb}^{kh}) && \forall (k,h)\in\Delta, \forall (i,j,b)\in I^B : O(i,j,k,h)\neq 0\\
& \sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h) s_{jib}^{kh} \ge \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
\end{aligned}
$$

Overview

Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

Each node is a partial solution, starts from the empty solution

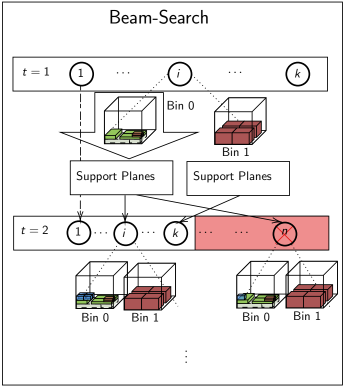


Figure: Conceptual representation of the proposed heuristic

Support Plane

- Operates on a single bin
- Each item inserted generates planes
- Insertions on the lowest possible level
- No explicit layers

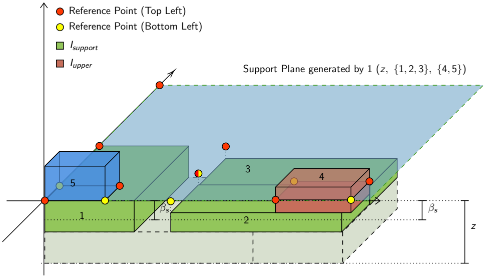
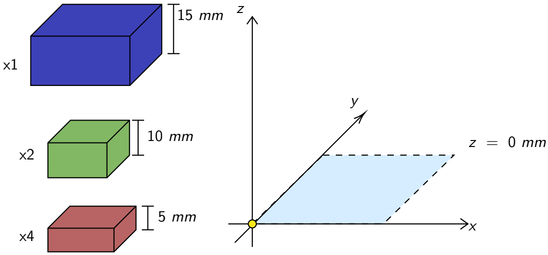
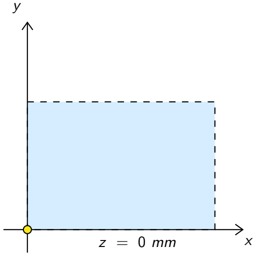


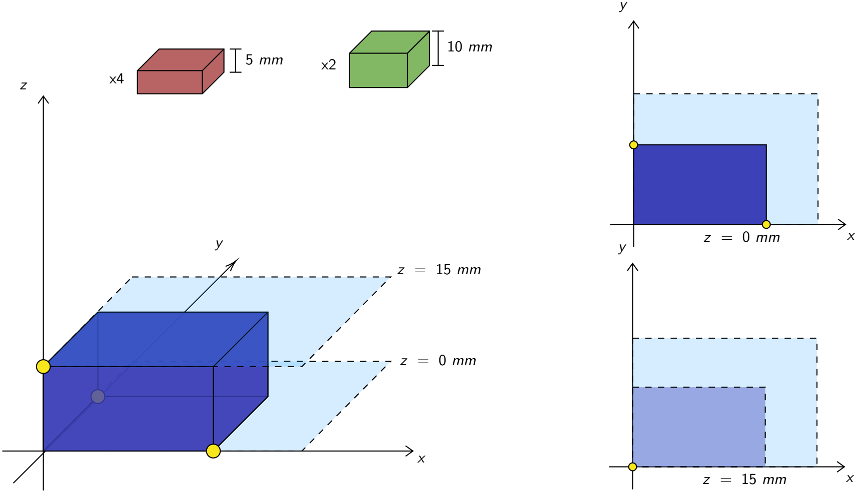
Figure: Conceptual representation of the proposed heuristic

Support Plane

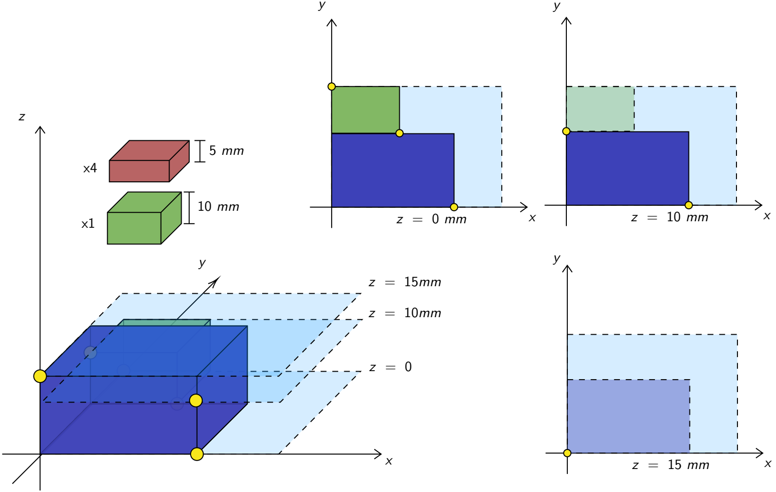




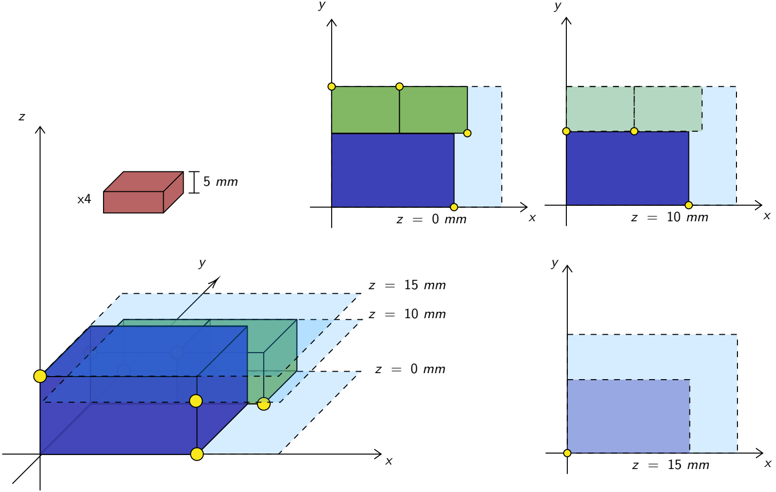
Support Plane



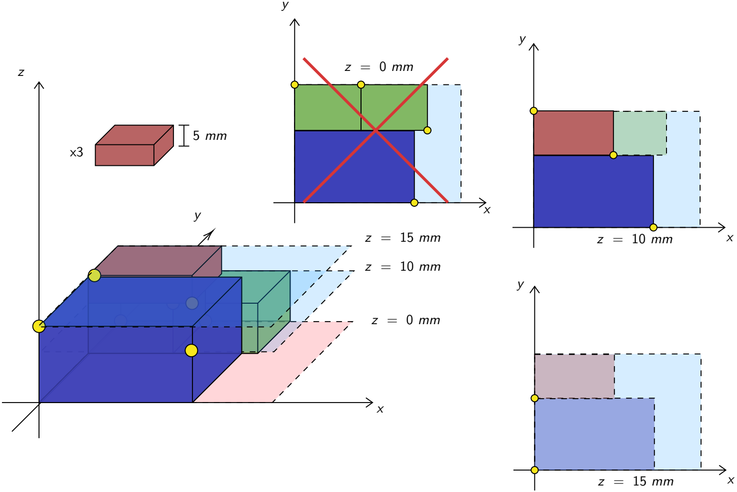
Support Plane



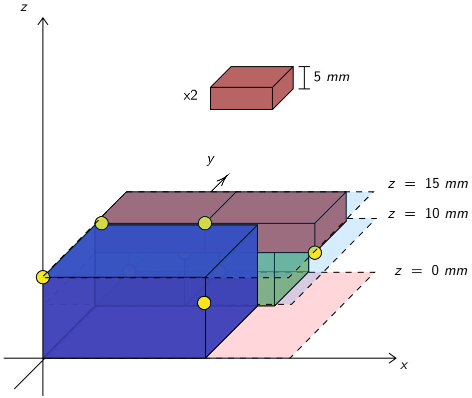
Support Plane

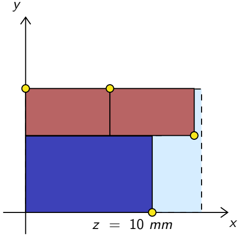


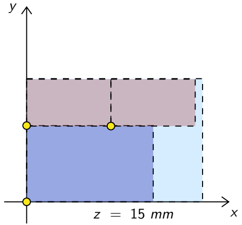
Support Plane



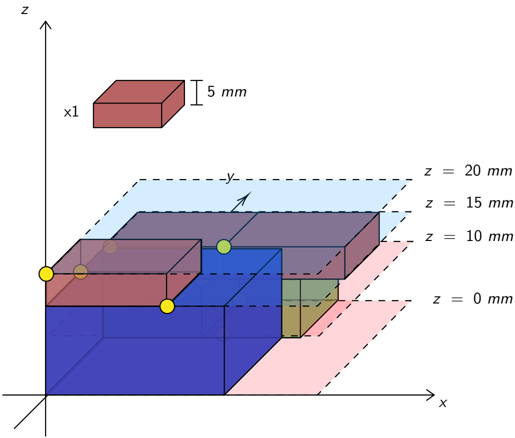
Support Plane

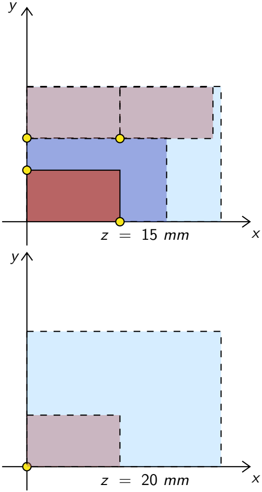






Support Plane





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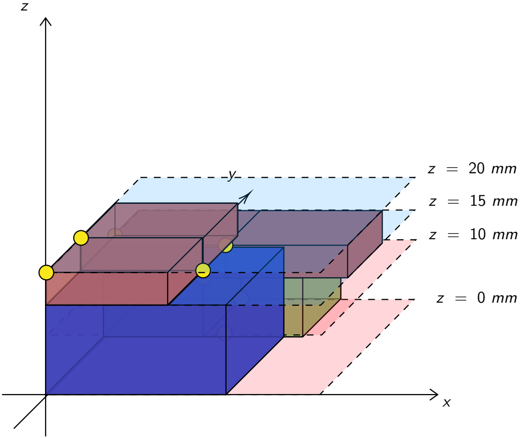
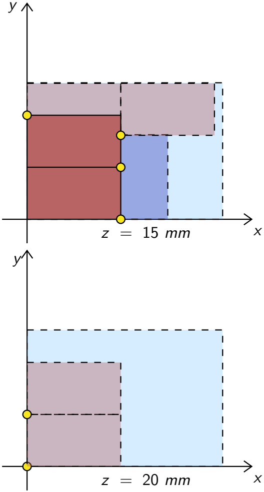
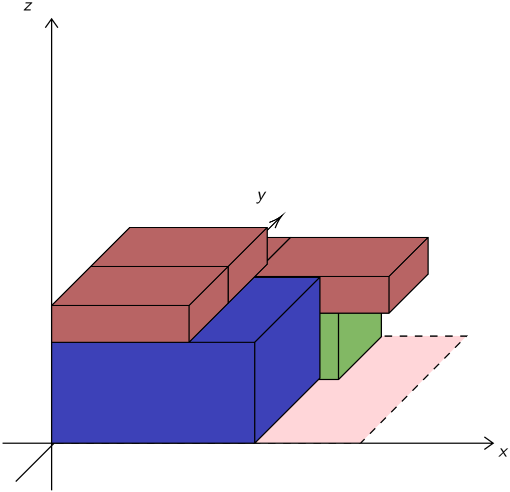




Support Plane



Beam Search

Optimizations

- Duplicates removal
- Fast overlap checks
- Memory management

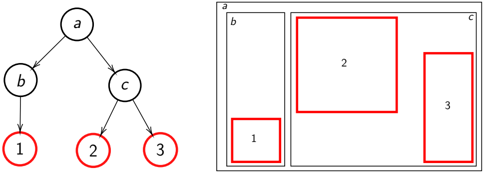


Figure: AABB Tree

Results & Future Developments