



POLITECNICO
MILANO 1863

Three-Dimensional Bin Packing with Vertical Support

Jacopo Libè
952914

Table of contents

1/37

- ① Introduction
- ② Problem Definition
- ③ Proposed Algorithm
- ④ Conclusions



Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure



Figure: Example of pit palletization

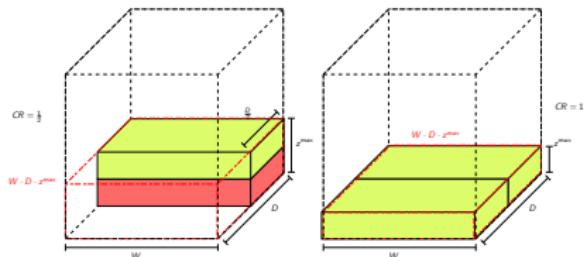


Figure: Cage ratio of two different bin configurations

Definition

An item has vertical support if one of the following conditions hold:

- **Condition 1:** at least a percentage α_s of its base area is resting on other items
- **Condition 2:** at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

Introduction

Vertical Support

4/37

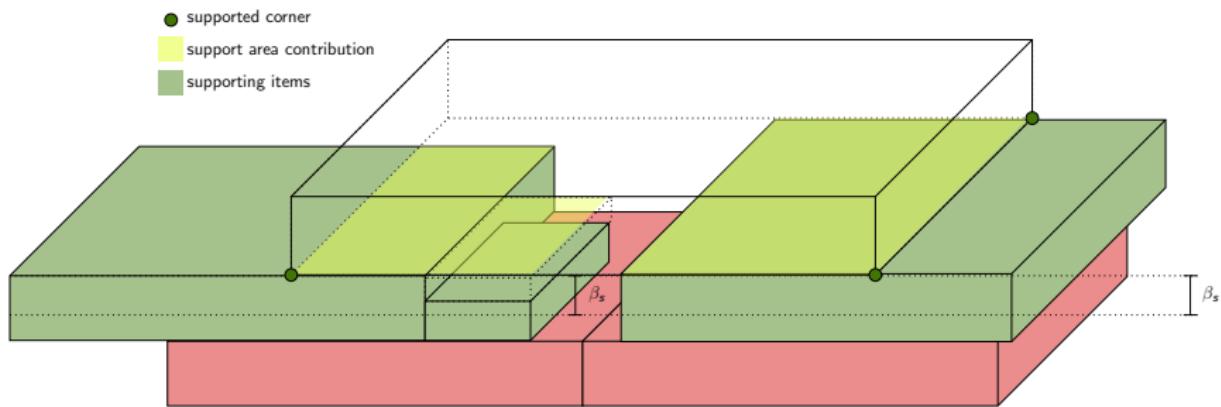


Figure: Representation of an item with conditions 1 and 2 of vertical support given
 $\alpha_s = 0.5, \beta_s$

- The problem is NP-Hard
- Exact methods only for small instances
- Existing 3D-BPP heuristics don't consider practical constraints
- Solutions for container loading and pallet loading problems are layer based

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Model - Coordinate System

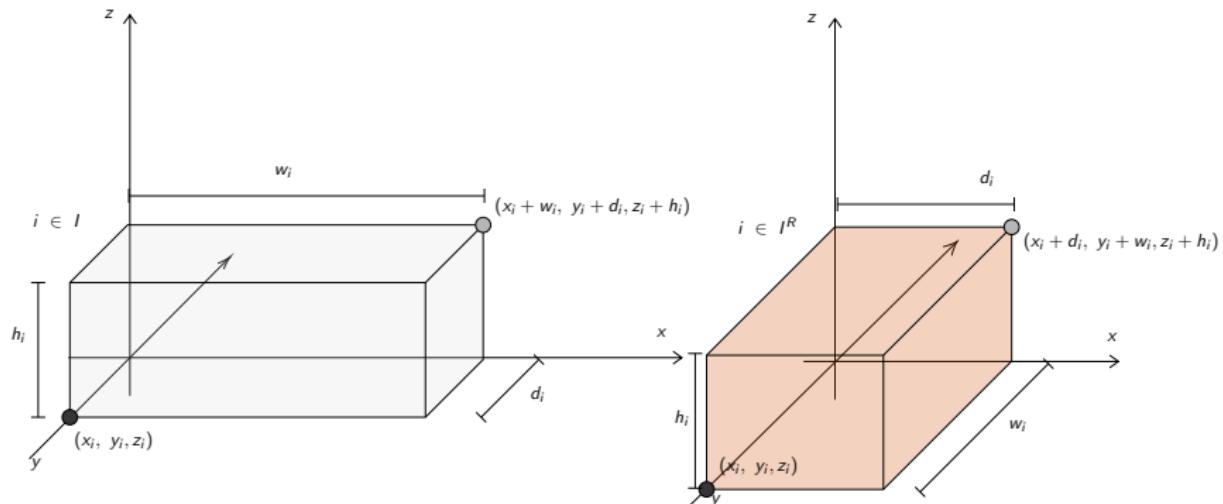


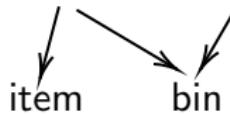
Figure: Coordinate system representation for a generic item i and its rotated clone $i \in I^R$

MILP Model - Objective Function

minimize number of used bins
then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
 all items are inside the bin's bounds
 no overlaps between items in the same bin
 all items have vertical support

Assign Use

U_{ib} , V_b



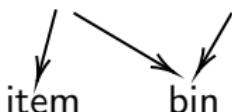
$$\begin{aligned}
 & \text{min} && \sum_{b \in B} (Hv_b + z_b^{\max}) \\
 & \text{s.t.} && \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \quad \forall (i, j) \in I^{OR} \\
 & && u_{ib} \leq v_b \quad \forall i \in I, \forall b \in B \\
 & && v_b \geq v_c \quad \forall (b, c) \in B : b < c \\
 & && x_i + w_i \leq W \quad \forall i \in I \\
 & && y_i + d_i \leq D \quad \forall i \in I \\
 & && z_i + h_i \leq H \quad \forall i \in I \\
 & && (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) \quad \forall i, j \in I \\
 & && x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P \quad \forall i, j \in I \\
 & && (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) \quad \forall i, j \in I \\
 & && y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P \quad \forall i, j \in I \\
 & && (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) \quad \forall i, j \in I \\
 & && z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P \quad \forall i, j \in I \\
 & && x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 \quad \forall i, j \in I, \forall b \in B \\
 & && z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) \quad \forall i \in I, \forall b \in B
 \end{aligned}$$

MILP Model - Geometric Constraints 1

minimize then, maximize subject to	number of used bins average cage ratio of the used bins all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$\sum_{b \in B} (Hv_b + z_b^{max})$ $\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1$ $u_{ib} \leq v_b$ $v_b \geq v_c$ $x_i + w_i \leq W$ $y_i + d_i \leq D$ $z_i + h_i \leq H$ $(x_i + w_i) - x_j \leq W(1 - x_{ij}^p)$ $x_j - (x_i + w_i) + 1 \leq Wx_{ij}^p$ $(y_i + d_i) - y_j \leq D(1 - y_{ij}^p)$ $y_j - (y_i + d_i) + 1 \leq Dy_{ij}^p$ $(z_i + h_i) - z_j \leq H(1 - z_{ij}^p)$ $z_j - (z_i + h_i) + 1 \leq Hz_{ij}^p$ $x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1$ $z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib})$
		$\forall (i, j) \in I^{OR}$ $\forall i \in I, \forall b \in B$ $\forall (b, c) \in B : b < c$ $\forall i \in I$ $\forall i \in I$ $\forall i \in I$ $\forall i, j \in I$

Assign Use

U_{ib} , V_b



MILP Model - Geometric Constraints 2

minimize then, maximize subject to	number of used bins average cage ratio of the used bins all items are assigned to one and only one bin all items are inside the bin's bounds no overlaps between items in the same bin all items have vertical support	$\min \quad \sum_{b \in B} (Hv_b + z_b^{\max})$ $\text{s.t.} \quad \begin{aligned} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 && \forall (i, j) \in I^{OR} \\ & u_{ib} \leq v_b && \forall i \in I, \forall b \in B \\ & v_b \geq v_c && \forall (b, c) \in B : b < c \\ & x_i + w_i \leq W && \forall i \in I \\ & y_i + d_i \leq D && \forall i \in I \\ & z_i + h_i \leq H && \forall i \in I \\ & (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) && \forall i, j \in I \\ & x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P && \forall i, j \in I \\ & (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) && \forall i, j \in I \\ & y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P && \forall i, j \in I \\ & (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) && \forall i, j \in I \\ & z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P && \forall i, j \in I \\ & x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 && \forall i, j \in I, \forall b \in B \\ & z_b^{\max} \geq (z_i + h_i) - H(1 - u_{ib}) && \forall i \in I, \forall b \in B \end{aligned}$
---	---	--

MILP Model - Geometric Constraints 3

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to

- all items are assigned to one and only one bin
- all items are inside the bin's bounds
- no overlaps between items in the same bin**
- all items have vertical support

$$\begin{array}{lll}
\min & \sum_{b \in B} (Hv_b + z_b^{max}) \\
\text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
& x_i + w_i \leq W & \forall i \in I \\
& y_i + d_i \leq D & \forall i \in I \\
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^P) & \forall i, j \in I \\
& x_j - (x_i + w_i) + 1 \leq Wx_{ij}^P & \forall i, j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^P) & \forall i, j \in I \\
& y_j - (y_i + d_i) + 1 \leq Dy_{ij}^P & \forall i, j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^P) & \forall i, j \in I \\
& z_j - (z_i + h_i) + 1 \leq Hz_{ij}^P & \forall i, j \in I \\
& x_{ij}^P + x_{ji}^P + y_{ij}^P + y_{ji}^P + z_{ij}^P + z_{ji}^P \geq u_{ib} + u_{jb} - 1 & \forall i, j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}$$

Figure: Precedences variables (2D case)

minimize number of used bins

then, maximize average cage ratio of the used bins

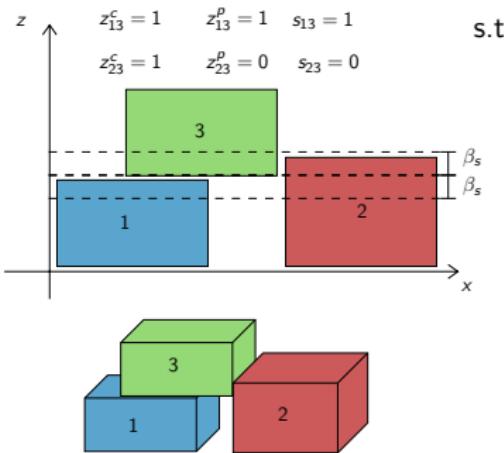
subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Model - Closeness



$$\text{s.t.} \quad \begin{aligned} z_j - (z_i + h_i) &\leq \beta_s + H(1 - z_{ij}^c) & \forall (i,j) \in I : i \neq j \\ z_j - (z_i + h_i) &\geq -\beta_s - H(1 - z_{ij}^c) & \forall (i,j) \in I : i \neq j \\ s_{ij} &\leq z_{ij}^p & \forall (i,j) \in I \\ s_{ij} &\leq z_{ij}^c & \forall (i,j) \in I \\ s_{ij} &\geq z_{ij}^p + z_{ij}^c - 2 & \forall (i,j) \in I : i \neq j \\ \sum_{j \in I} s_{ij} &\leq \sum_{b \in B} u_{ib} & \forall i \in I \end{aligned}$$

Figure: Closeness variables example

Pre-Computed Parameter

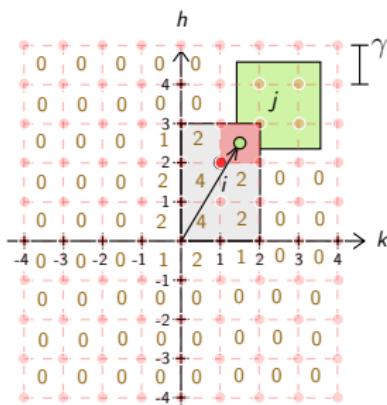
$$O(i, j, k, h)$$

$$O(i, j, 1, 2) = 1$$

Variables

$$s_{ijb}^{12} = 1$$

$$(x_j - x_i, y_j - y_i)$$



$$\begin{aligned}
 & \text{s.t.} && z_i \leq H(1 - g_i) && \forall i \in I \\
 & && \sum_{(k,h) \in \Delta, b \in B: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq s_{ij} && \forall (i,j) \in I \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{ib} && \forall (i,j,b) \in I^B \\
 & && \sum_{(k,h) \in \Delta: O(i,j,k,h) \neq 0} s_{ijb}^{kh} \leq u_{jb} && \forall (i,j,b) \in I^B \\
 & && x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
 & && \sum_{(k,h) \in \Delta, b \in B, j \in I: i \neq j \wedge O(i,j,k,h) \neq 0} O(i,j,k,h) s_{ijb}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
 \end{aligned}$$

Figure: Space discretization

MILP Model - Results

MILP Model			
n	Max Z	TT(s)	Gap(%)
1	85	0.01	0.00
2	85	0.07	0.00
3	85	0.13	0.00
4	85	0.20	0.00
5	85	2.02	0.00
6	158	90.58	0.00
7	158	1,369.24	0.00
8	161*	3,600.00	1.86
9	-	-	-

* Some boxes had lower support than expected due to discretization errors.

Overview

Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

Support Planes

- Operates on a single bin
- Exploits a modified 2D-BPP heuristic
- No explicit layers
- Guarantees vertical support

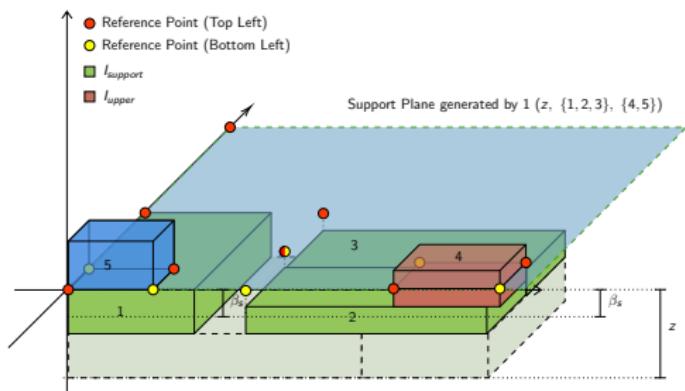
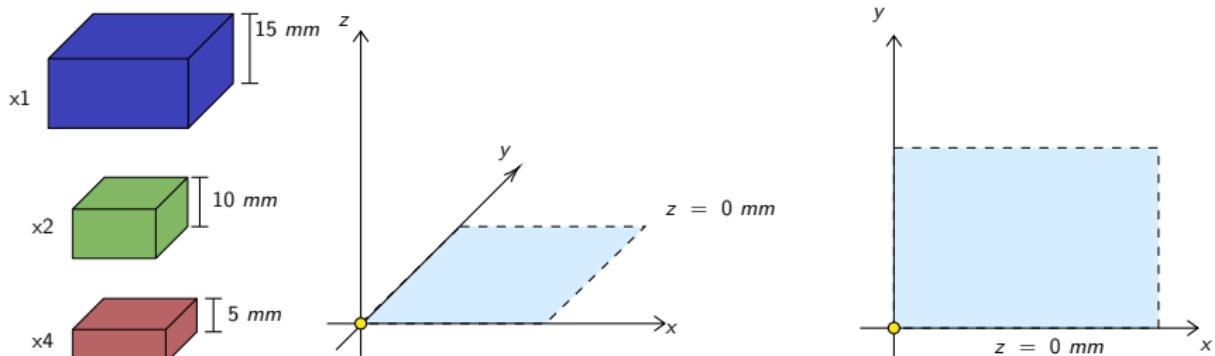
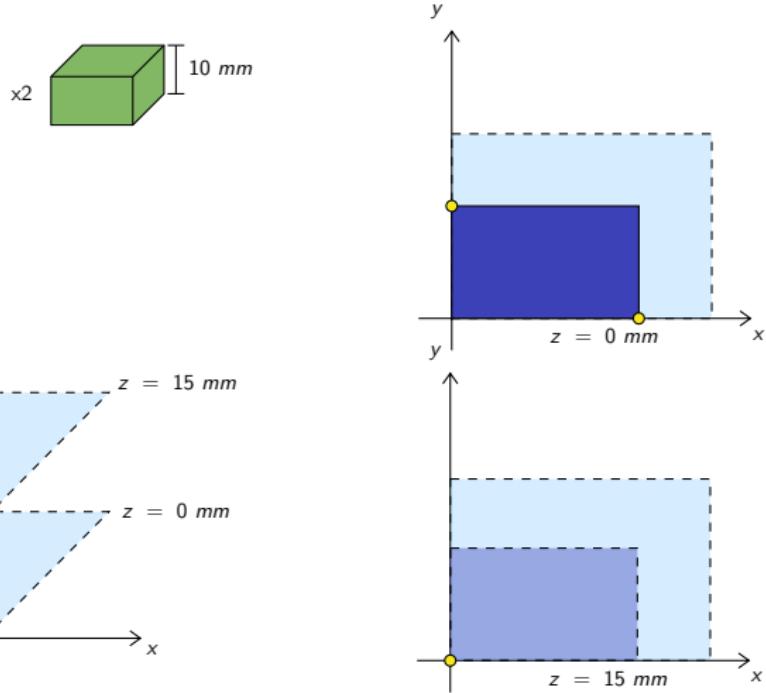
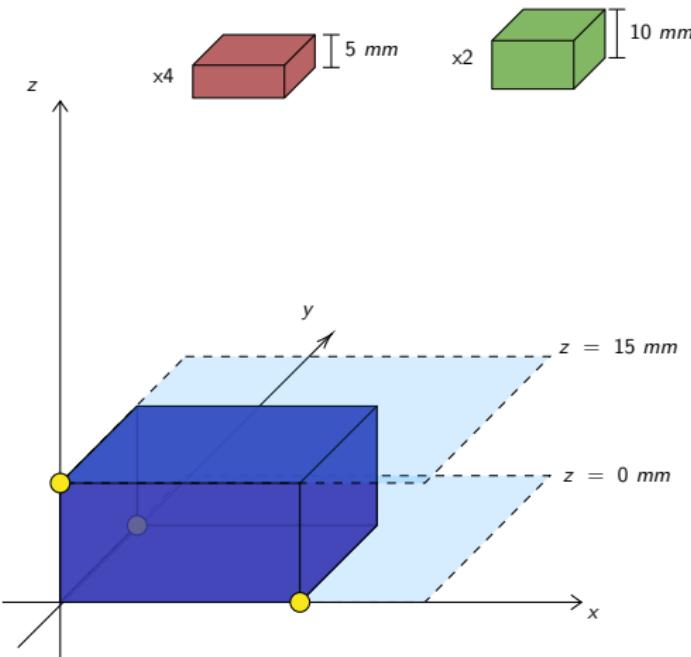


Figure: An example of a support plane generated by item 1

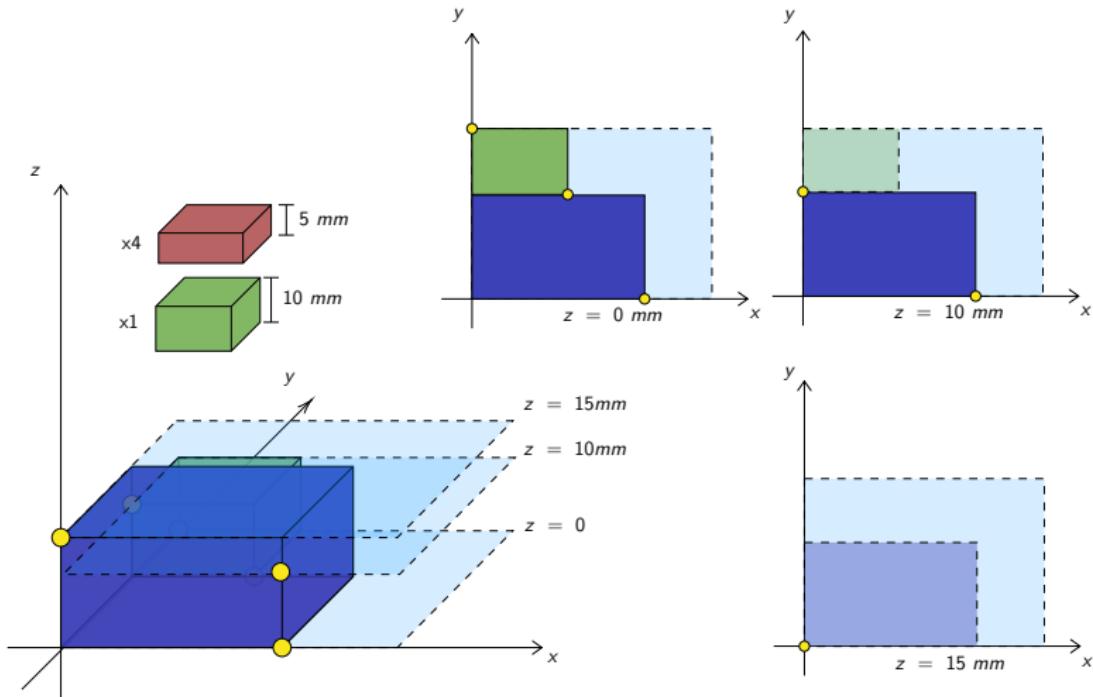
Support Planes



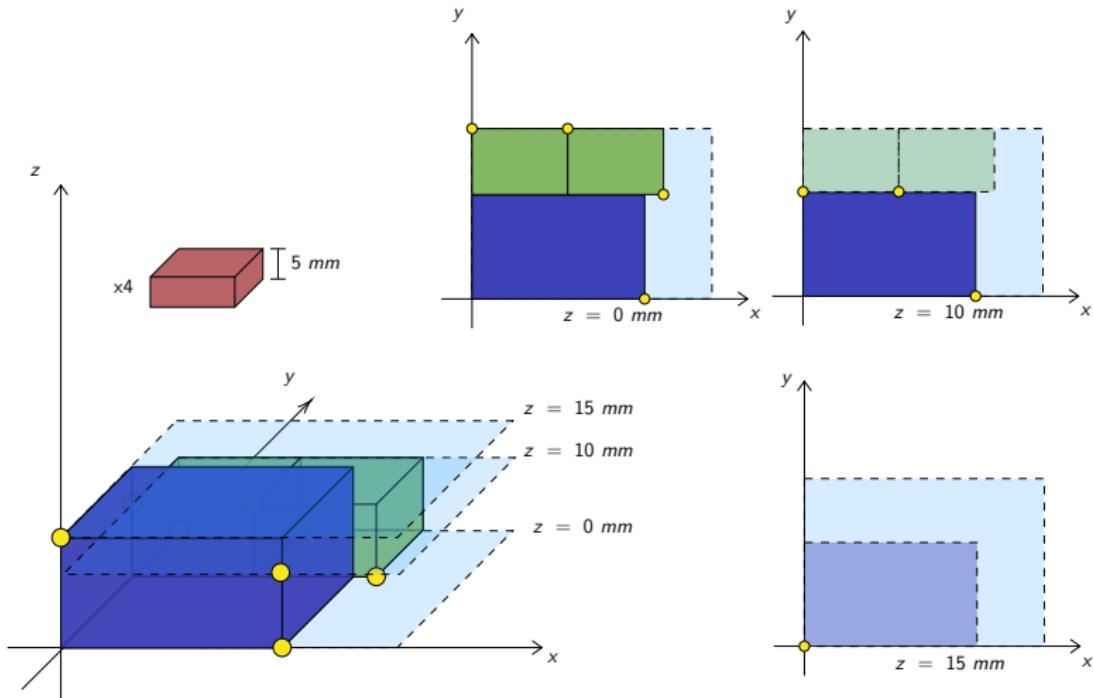
Support Planes



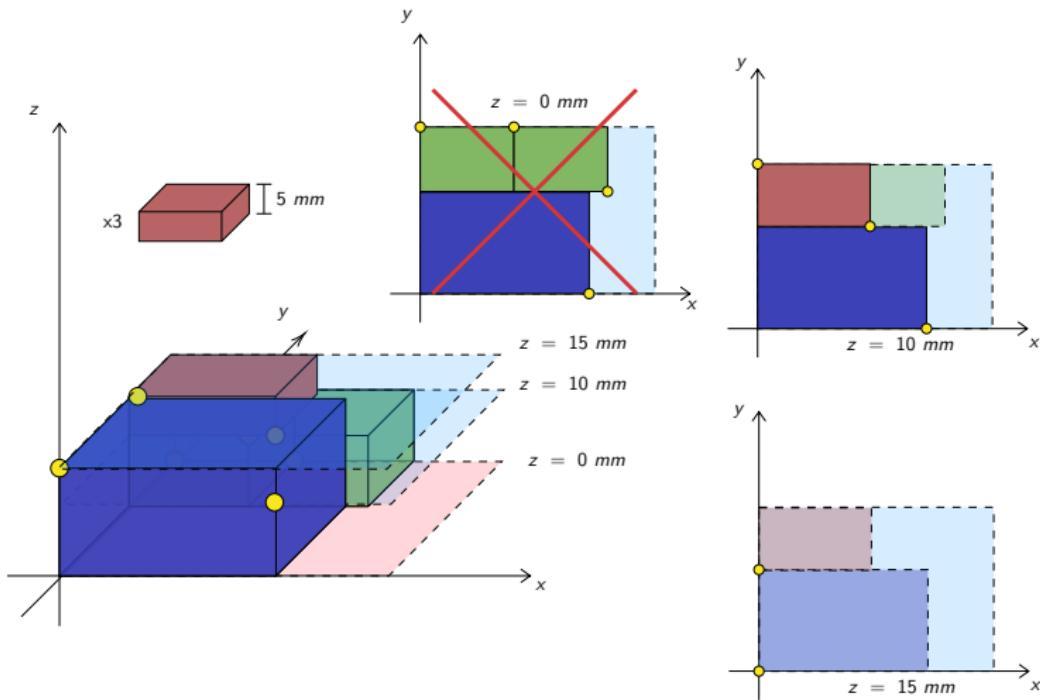
Support Planes



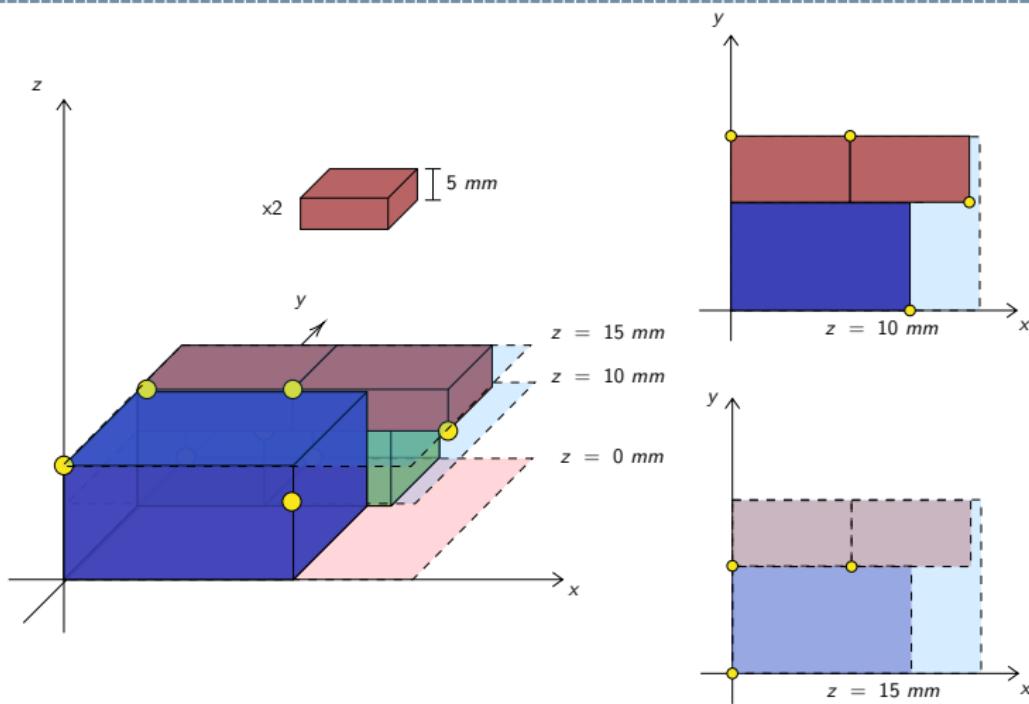
Support Planes



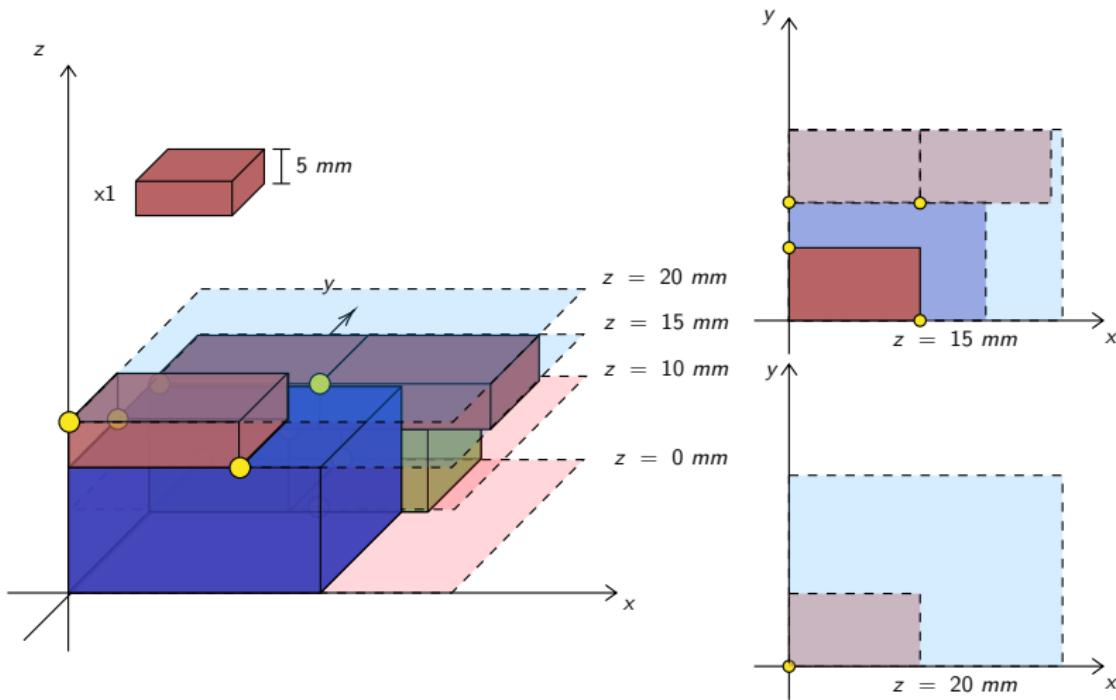
Support Planes



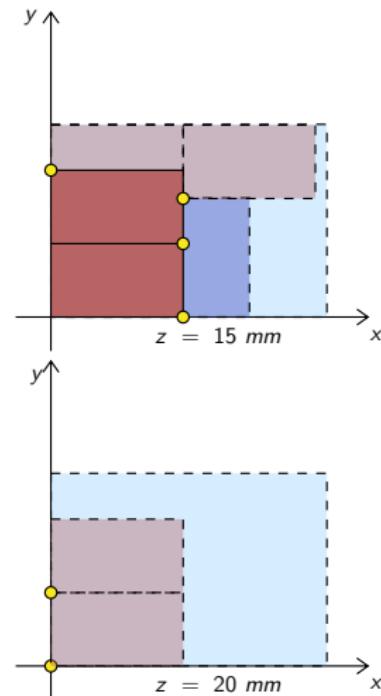
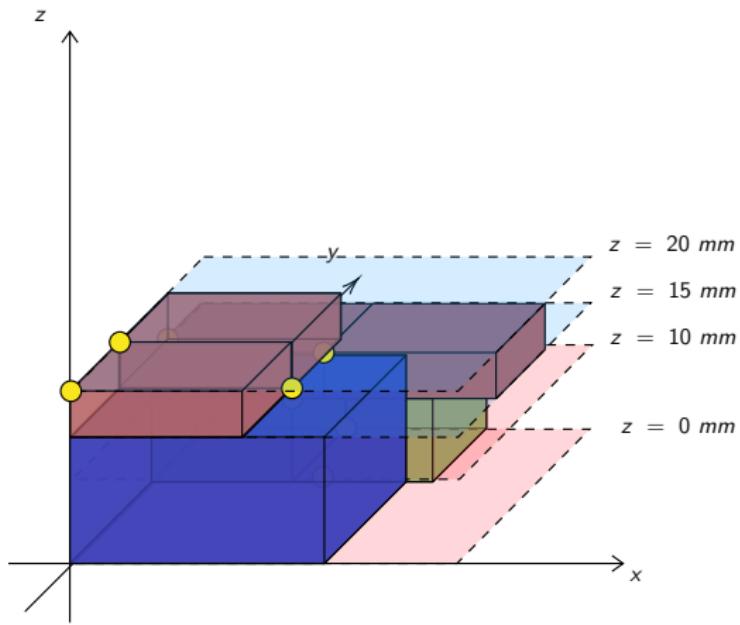
Support Planes



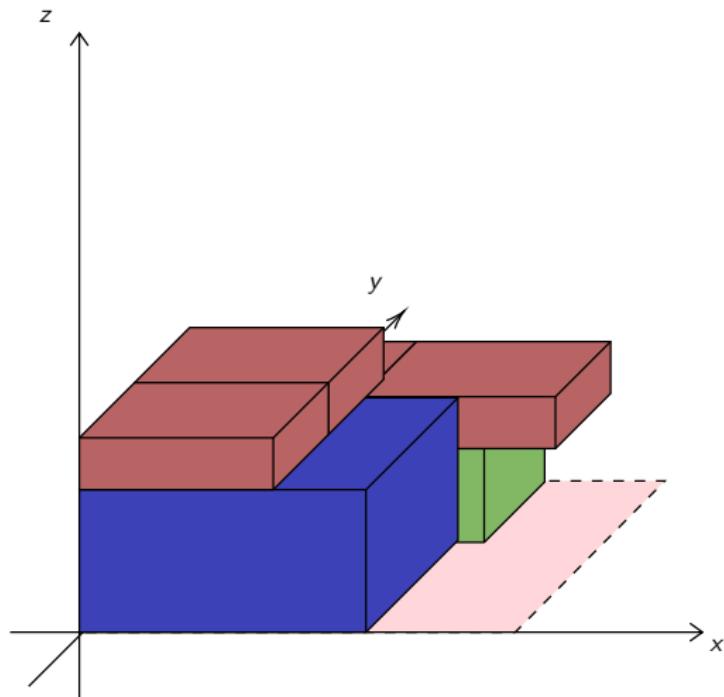
Support Planes



Support Planes

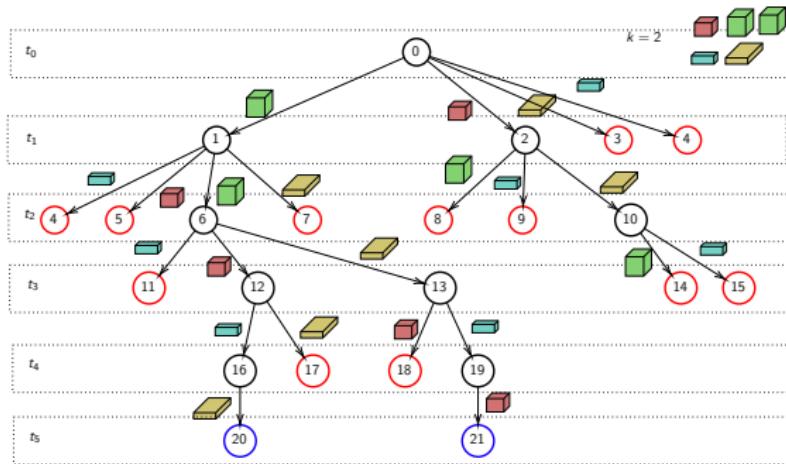


Support Planes



Beam Search

- Exploits support planes
- Branches on the sequence of placements
- Limited to k best placements



Beam Search

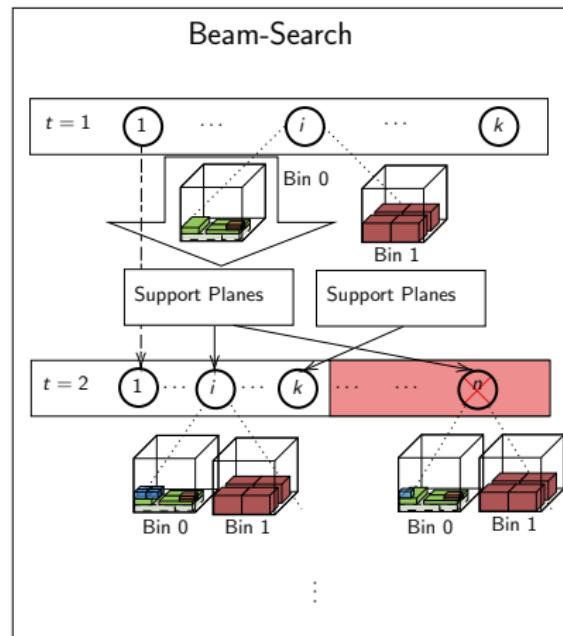


Figure: Conceptual representation of the proposed heuristic

Beam Search

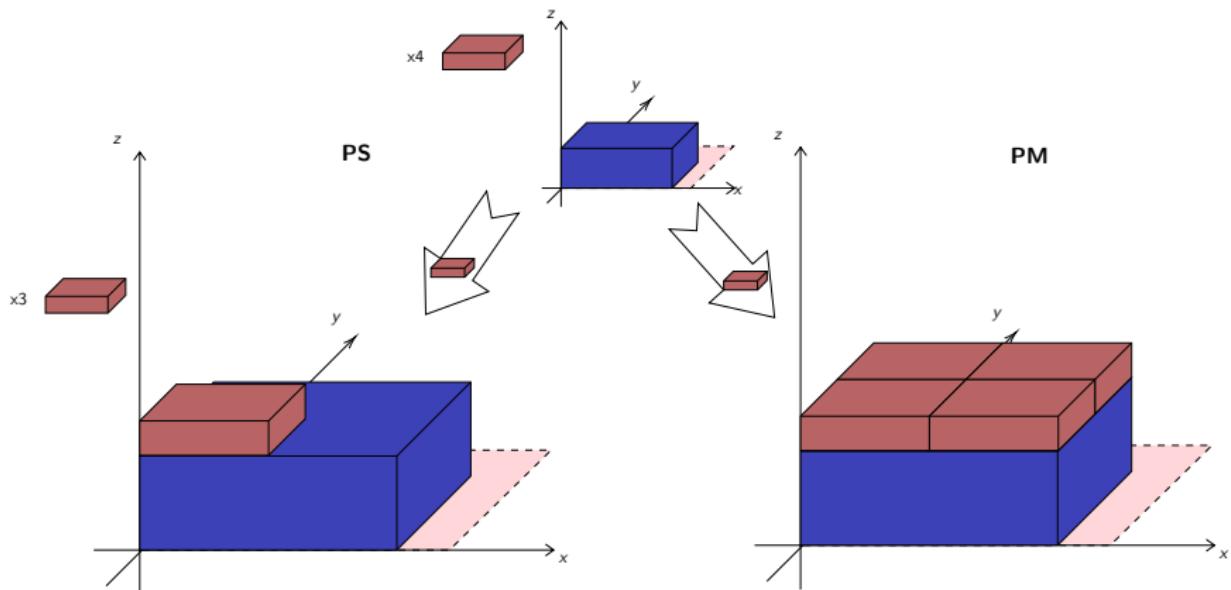


Figure: Proposed placement modes

Proposed Algorithm Optimizations

30/37

- Prune duplicate nodes
- Fast overlap checks
- Lazy node updates

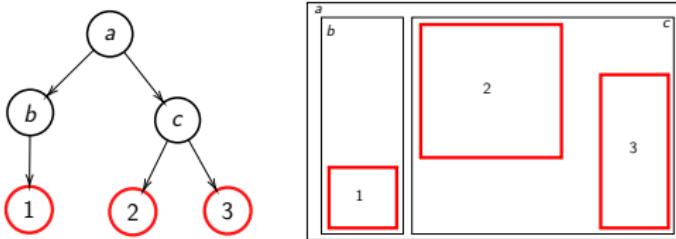


Figure: AABB Tree, self-balancing tree of axis-aligned bounding boxes

Computational Experiments

Table: Comparison with MILP model on limited set of boxes

<i>n</i>	MILP Model			PM		PS	
	Max Z	TT(s)	Gap(%)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	1.86	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

* Some boxes had lower support than expected due to discretization errors.

Computational Experiments

Table: Average execution time of literature results with bin gap

Heuristic		Execution Time (s)				Bin Gap (%)
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.05	0.11	0.28	0.55	4.57
	$k = 5$	0.08	0.39	1.02	2.16	4.32
	$k = 10$	0.15	0.74	1.98	4.12	4.29
	$k = 20$	0.29	1.45	3.89	8.07	4.05
	$k = 50$	0.72	3.63	9.72	20.47	3.95
PS	$k = 1$	0.04	0.18	0.51	1.08	4.35
	$k = 5$	0.12	0.74	2.19	4.79	4.01
	$k = 10$	0.23	1.43	4.19	9.39	3.94
	$k = 20$	0.47	2.81	8.48	18.93	3.74
	$k = 50$	1.15	6.74	21.03	45.78	3.52
BRKGA-VD		17.13	80.63	190.50	369.75	0.00

Computational Experiments

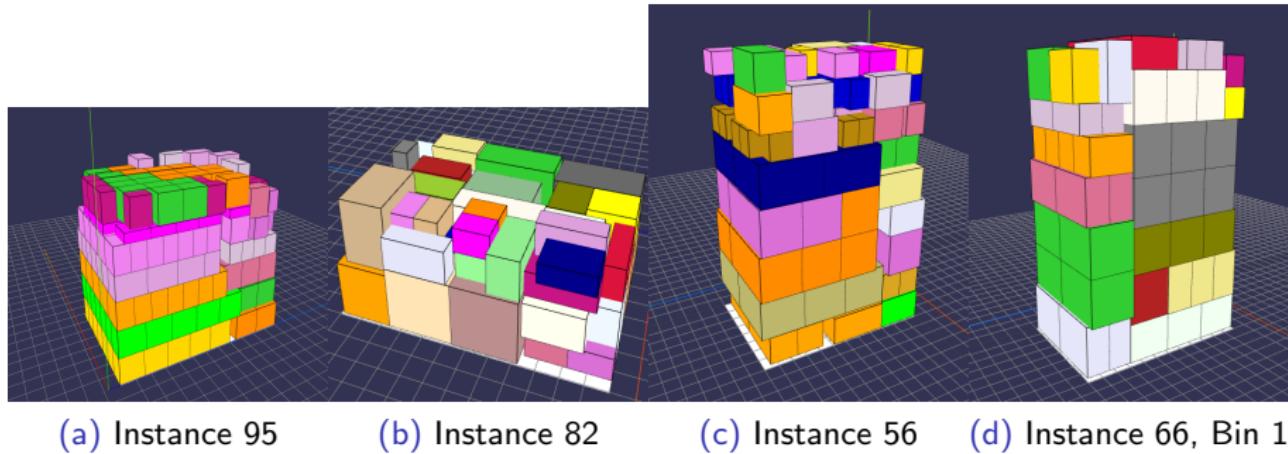
Table: Summary of case study tests

k	PS			PM		
	TT (ms)	B	CR (%)	TT (ms)	B	CR (%)
1	423.87	1.37	65.87	65.18	1.31	70.70
5	1,597.54	1.34	69.19	185.22	1.29	73.08
10	2,627.52	1.32	70.35	344.90	1.27	73.56
20	5,373.79	1.34	70.78	620.95	1.27	74.57
50	14,203.10	1.31	72.11	1,279.96	1.29	74.61
100	26,934.21	1.31	73.23	2,340.37	1.26	75.36
200	48,944.90	1.30	73.89	4,465.78	1.25	76.39

Computational Experiments

Table: Case study experiments trade off between average execution times and average cage ratio

k	PS		PM	
	CR* – CR (%)	TT – TT* (ms)	CR* – CR (%)	TT – TT* (ms)
1	10.56	358.69	5.73	0.00
5	7.24	1,532.36	3.35	120.04
10	6.08	2,562.34	2.87	279.72
20	5.65	5,308.61	1.85	555.77
50	4.32	14,137.92	1.82	1,214.78
100	3.20	26,869.03	1.07	2,275.19
200	2.54	48,879.72	0.04	4,400.60



(a) Instance 95

(b) Instance 82

(c) Instance 56

(d) Instance 66, Bin 1

Figure: Solutions of case study tests with the "PM" placement and $k = 200$

Future Developments

- More practical constraints
- Solution optimization heuristics
- Improvements to the underlying 2D-BPP heuristic

Thank you for your attention!
Questions?