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Abstract

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Abstract in lingua italiana

Qui va l'Abstract in lingua italiana della tesi seguito dalla lista di parole chiave.

Parole chiave: qui, vanno, le parole chiave, della tesi

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1 | Introduction

Intro

Case study

Overview

2 | Literature review

3 | Problem description and mathematical formulation

3.1. 3D Bin Packing Problem

3.2. Support

3.3. MILP Formulation

Conceptual model

A conceptual model of the problem we are trying to solve would be:

minimize	unused volume in used bins
subject to	all items assigned to one and only one bin
	all items within the bin dimensions
	no overlaps between items in the same bin
	all items with support

We can now provide the formal definition of the 3DBPP by formulating a mixed integer linear programming problem model.

Formal model

We'll now introduce a MILP model for the standard 3DBPP problem definition and then we'll expand it to address the stability constraint afterwards.

We start by defining the known sets and parameters of the problem.

Sets

$$I = \{1, \dots, n\} : \text{ set of items}$$

$$B = \{1, \dots, m\} : \text{ set of bins}$$

Parameters

$$W \times D \times H \quad \text{width} \times \text{depth} \times \text{height of a bin}$$

$$V \quad \text{bin volume}$$

$$w_i \times d_i \times h_i \quad \text{width} \times \text{depth} \times \text{height of item } i \quad \forall i \in I \quad (3.1)$$

Variables We can now introduce the following sets of integer variables

$$(x_i, y_i, z_i) \quad \text{bottom front left corner of an item} \quad \forall i \in I \quad (3.2)$$

$$(x'_i, y'_i) \quad \text{back right corner of an item} \quad \forall i \in I \quad (3.3)$$

$$r_i \quad \begin{cases} 1, \text{ if item } i \text{ is rotated } 90^\circ \text{ over its z-axis} \\ 0, \text{ otherwise} \end{cases} \quad \forall i \in I \quad (3.4)$$

$$u_{ib} \quad \begin{cases} 1, \text{ if item } i \text{ is placed in bin } b \\ 0, \text{ otherwise} \end{cases} \quad \forall i \in I, \forall b \in B$$

$$x_{ij}^p \quad \begin{cases} 1, \text{ if } x_i \leq x'_j \\ 0, \text{ otherwise} \end{cases} \quad \forall i, j \in I$$

$$y_{ij}^p \quad \begin{cases} 1, \text{ if } y_i \leq y'_j \\ 0, \text{ otherwise} \end{cases} \quad \forall i, j \in I$$

$$z_{ij}^p \quad \begin{cases} 1, \text{ if } z_i \leq z_j + h_j \\ 0, \text{ otherwise} \end{cases} \quad \forall i, j \in I$$

$$z_b^{\max} \quad \text{maximum height of bin } b \quad \forall b \in B$$

Given a coordinate system, each item i can be represented univocally in 3D space by eqs. (3.1) to (3.4) as seen in figure 3.1

Introducing static stability

We now extend the model introduced in section 3.3 to introduce constraint addressing static stability.

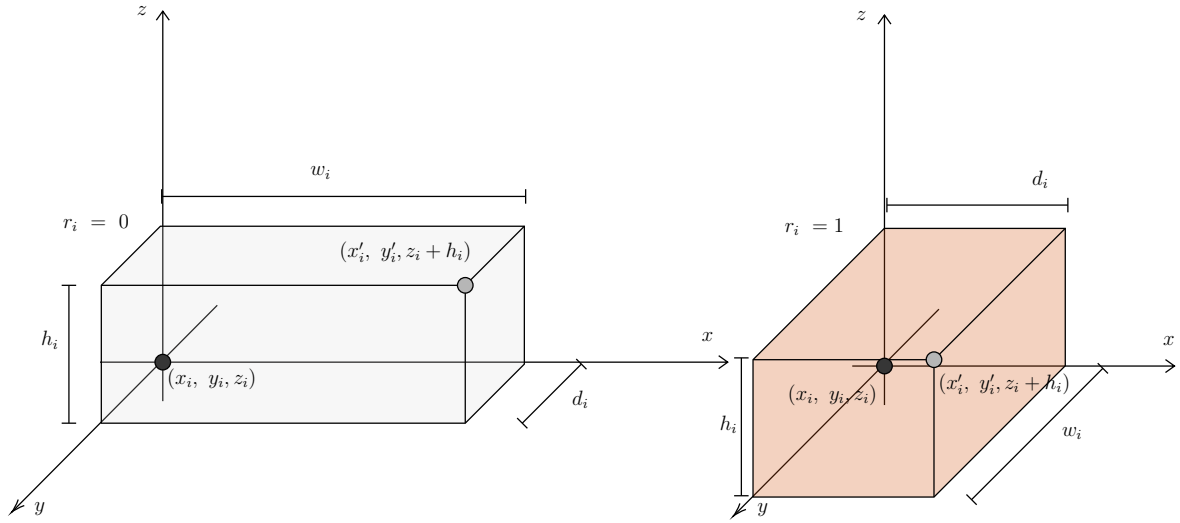


Figure 3.1: Coordinate system representation for a generic item i given its rotation r_i

Additional Parameters

α_s : support area threshold

β_s : support height tolerance

4 | Solution algorithms

In this chapter we describe a solution to the 3D bin packing problem with static stability. A solution candidate to the problem can be found by conducting a search over the tree of possible packings or states. In section 4.1 we describe what a state or packing is and its representation. Since an exhaustive search isn't feasible, an heuristic search is conducted by combining a beam search algorithm described in section 4.2 and constructive heuristic described in section 4.3. The proposed algorithm takes in input an initial feasible state (as defined in section 4.1.2) usually represented by the empty state (4.3) and outputs the best scoring state based on an ordering function defined in section 4.2.1.

4.1. State

States or packings are partial solutions to the 3DBPP. Given the formal definition of the problem (3.3) a few new definitions are introduced to facilitate the algorithm's definition.

Definition 4.1 (Unpacked item). *Given an item $i \in I$ we define it as unpacked iff*

$$\sum_{b \in B} u_{ib} = 0$$

It is also assumed that variables identifying an item's position, and rotation are independent between states (changes to their values in state s won't affect state s').

A state s can then be defined as follows

- U : the set of unpacked items
- B : the set of bins
- (s_1, s_2, \dots, s_b) : the set of supporting structures for each bin $b \in B$
- p : the insertion pending on this state (described by def. 4.4)

Observation 4.1. *Given two states s and s' we can have that $|s.B| \neq |s'.B|$ since the number of bins is also a variable in the proposed heuristic*

We can also trivially define a function which determines if a state is a final state

Definition 4.2. *A state s is final if there are no more items to pack*

$$IsFinal(s) = \begin{cases} 1, & s.U = \emptyset \\ 0, & otherwise \end{cases} \quad (4.1)$$

Each bin b has additional data that is contained in a structure s_b used to facilitate the execution of the algorithm.

Let us introduce the concept of packed items inside a bin:

Definition 4.3 (Packed item). *Given a state s and a bin $b \in s.B$, we say that item*

$$i \in I \text{ is packed in } b \text{ iff } \begin{cases} u_{ib} = 1, \\ \sum_{j \in s.B, j \neq b} u_{ij} = 0 \end{cases}$$

Given a bin $b \in s.B$ we can then define structure s_b as follows

- J : the set of items that are packed inside b
- Z : the set of planes inside b (section 4.3)
- T : the AABB Tree (section 4.1.1) representing the items inside b

Notice that two separate sets containing the items packed in b are present inside s_b but adding and accessing items in $s_b.J$ has time complexity of $O(1)$ given an underlying implementation as hashset while maintaining $s_b.T$ usually has a time complexity of $O(\log(|s_b.J|))$.

The reason to include an AABB Tree inside this structure is further explained in sections 4.1.2 and 4.3.1

4.1.1. AABB Tree

In order to determine the feasibility of a given state, a way of checking for overlaps with items already placed is needed. Since our formulation of the problem only allows for 90 deg rotations over the z-axis. Every item in a solution, by the problem formulation (3.1), is

contained inside a bounding box and this box is axis-aligned. An adequate structure to compute overlaps is then an Axis-Aligned Bounding Box Tree (AABB Tree) [1].

AABB Trees are bounding volume hierarchies typically used for fast collision detection and they usually offer a few operations:

- $AABBI\text{nsert}(i)$: which allows to insert an axis-aligned box i in the tree
- $AABBO\text{verlaps}(i)$: which allows to determine if an axis-aligned box i overlaps an element in the tree
- $AABBC\text{losest}(i, d)$: which given an axis-aligned box i and a direction $d \in \{XP, XN, YP, YN, ZP, ZN\}$ along an axis, returns the closest element inside the tree following that direction starting from the box i

If the tree is properly balanced each operation on average has a time complexity of $O(\log(n))$ where n is the number of elements in the tree.

Maintaining an AABB Tree in the state allows us to do checks for feasibility during the construction of a solution (as detailed in 4.3.1) and feasibility checks on the final states to allow for error detection.

4.1.2. Feasibility

A state s is said to be feasible if the currently packed items for every bin $b \in s.B$ respects the constraints defined in the problem formulation (3.3)

Since the proposed heuristic is constructive it is more convenient to define the concept of feasibility relative to a change in the state.

Insertions Given a state s and $b \in s.B$, an insertion of items is a set of items that are placed in b and have their z_i within tolerance of a certain z .

Definition 4.4 (Insertion). *Given a state s and a tolerance β_s we define an insertion or placement p a tuple (b, I) where b is a bin and I is a set of items that are going to be packed in b such that, $I \subseteq s.U \wedge \exists z(z \in \mathbb{Z} \wedge \forall i(i \in I \wedge |z_i - z| \leq \beta_s))$*

Observation 4.2. *Given s and $p = (b, \emptyset)$ where $b \notin s.B$, p is an insertion which will open bin b in s .*

Definition 4.5 (Next). *Let p be an insertion over a state s we can then define $s' = \text{Next}(s, p)$ as the "copy" of state s with $s'.p = p$. And p is then pending on s' .*

In this way we can evaluate the changes to the score of a state based on its pending insertion without having to update all the structures for every evaluated state. This property will become apparent in section 4.2.

We can then define an algorithm that applies insertions to a given state s with pending insertions with the help of a function $OpenBin(b)$ which initializes a new structure s_b with every element at its empty value. The proposed algorithm is shown in 1.

Algorithm 1: Commit

```

input  :  $s$ 
output:  $s'$ 
 $(b, I) \leftarrow s.p$ 
 $s' \leftarrow Clone(s)$  //Memory clone of  $s$ 
if  $b \in s'.B$  then
  |  $s'.s_b.J \leftarrow s'.s_b.J \cup I$ 
  |  $s'.U \leftarrow s'.U \setminus I$ 
end
else Open a new bin
  |  $s'.B \leftarrow s'.B \cup b$ 
  |  $s'.s_b \leftarrow OpenBin(b)$ 
end
 $s'.p \leftarrow \emptyset$ 
return  $s'$ 

```

Insertion feasibility Describe insertion feasibility givine the sets defined

Algorithm 2: Is Insertion Feasible

```

input  :  $b, I, z, I_{support}, I_{upper}$ 
output:  $isFeasible$ 
return  $true$ 

```

State feasibility Describe how to compute the sets efficiently to use the insertion feasibility logic

Algorithm 3: Is State Feasible

```

input  :  $s$ 
output:  $isFeasible$ 
return  $true$ 

```

Proposition 4.1. A state s' derived by committing a feasible insertion p to a feasible state s is feasible.

Observation 4.3. *We can always define the empty state s_e where*

$$\begin{cases} s_e.U = I \\ s_e.B = \emptyset \end{cases}$$

and it is always feasible

4.2. Beam Search

Beam Search (BS) is an heuristic tree search algorithm designed for systems with limited memory where expanding every possible node is unfeasible. The idea behind BS is to conduct an iterative truncated breadth-first search where, at each iteration, only a limited number of k nodes is expanded. After the expansion every new node needs to be evaluated and sorted in order to prune the number of nodes down to the k best ones. The algorithm keeps exploring until no further node can be expanded.

To perform BS one must define the node structure, an expansion function to generate new nodes from existing ones, a ranking between nodes and a function to determine if a node is final.

A node in the tree can be represented as the state in section 4.1 and eq. (4.1) can be used to determine if a state is final. We also know that a new state s' derived by s by applying a feasible insertion p can be computed as in definition 4.5. This state expansion procedure, with the exception of empty insertions, will generate new states in our tree which will add a positive number of bins or packed items to the solution so, eventually, it will generate a final state.

If the starting state for the search is feasible every new state generated will be feasible and if a final state is found it will be feasible (proposition 4.1). We also note that starting from state s the time complexity to compute feasible insertions can be lower than the complexity required to update the structures that will be used for further expansions (AABB Tree insertion and balancing, memory cloning, etc.) so we modified the standard BS algorithm to separate the expansion phase from the commit phase.

Given S^0 the set of initial states and k the number of best states to expand at each iteration, the described procedure is represented by algorithm 4. As observed in observation 4.3 it's possible to start the search from $S^0 = \{s_e\}$.

Algorithm 4: Beam search

```

input :  $S^0, k$ 
output:  $s_{best}$ 
 $S^t \leftarrow S^0$ 
 $S_{final} \leftarrow \emptyset$ 
repeat
   $S^{t+1} \leftarrow Expand(S^t)$  (algo. 6)
   $S_{final} \leftarrow S_{final} \cup \{s \in S^{t+1} : IsFinal(s)\}$  (def. 4.2)
   $S^{t+1} \leftarrow S^{t+1} \setminus S_{final}$ 
   $S^{t+1} \leftarrow Sort(S^{t+1})$  (sec. 4.2.1)
   $S^t \leftarrow \emptyset$ 
   $i \leftarrow 0$ 
  forall  $s \in S^{t+1}$  do
     $S^t \leftarrow S^t \cup Commit(s)$  (algo. 1)
     $i \leftarrow i + 1$ 
    if  $i > k$  then
      break
    end
  end
until  $S^t \neq \emptyset$ 
 $S_{final} \leftarrow Sort(S_{final})$ 
return  $s_0 \in S_{final}$ 

```

State Expansion An expansion of a state s can be seen as a new set of states S_{new} derived by a set of feasible insertions. In order to determine these insertions an underlying heuristic is used (described in section 4.3).

The main idea in this phase of the algorithm is to find feasible insertions in all the bins for items that still need to be packed and that are of the same height. With this approach the solutions given by the algorithm will start by trying to fill layers with items of the same height if possible.

The underlying heuristic will also use a scoring mechanism to select the best insertions for a given class of heights in order to avoid having too many states.

Given a set of items I and a tolerance β_s we can introduce an algorithm to group them by height and produce a set G of tuples (h, I') where h is the height of the group and I' is the set of items grouped as line 5.

Algorithm 5: Group By Height

input : I, β_s **output:** G $G \leftarrow \emptyset$ **forall** $i \in I$ **do** $generate \leftarrow \text{true}$ **forall** $(h, I') \in G$ **do** **if** $|h_i - h| \leq \beta_s$ **then** $generate \leftarrow \text{false}$ $I' \leftarrow I' \cup i$ **break** **end** **end** **if** $generate = \text{true}$ **then** $G \leftarrow G \cup (h_i, \{i\})$ **end****end****return** G

Once items are grouped by height the best insertion foreach class of items can be computed for each open bin. If no insertion is possible in any bin, then the only viable expansion is the bin opening insertion (observation 4.2). The described procedure is detailed in line 6.

Algorithm 6: Expand

```

input :  $S$ 
output:  $S_{new}$ 
forall  $s \in S$  do
     $S_{new} \leftarrow \emptyset$ 
     $G \leftarrow \text{GroupByHeight}(s.U)$  (algo. 5)
     $placed \leftarrow false$ 
    forall  $(h, I) \in G$  do
        forall  $b \in s.B$  do
             $P \leftarrow \text{SPBestInsertion}(s_b, I)$  (algo. 7)
            if  $P \neq \emptyset$  then
                 $placed \leftarrow true$ 
                forall  $p \in P$  do
                     $S_{new} \leftarrow S_{new} \cup \text{Next}(s, p)$  (def. 4.5)
                end
            end
        end
    end
    if  $placed = false$  then
        Open a new bin with index  $|s.B|$  (oss. 4.2)
         $S_{new} \leftarrow S_{new} \cup \text{Next}(s, (|s.B|, \emptyset))$ 
    end
end
return  $S_{new}$ 

```

4.2.1. Scoring States

In order to sort states, a scoring function needs to be defined over them. Since the scoring of the states is what will influence the final solution the most, parameters that are directly related to minimizing the objective function are selected.

In the proposed solution to handle multiple objective functions a lexicographic ordering is used.

Definition 4.6. Let $f_1(s), f_2(s), f_i(s), \dots, f_n(s)$ be objective functions ordered by precedence based on index i , then $s < s'$ **iff** $f_1(s) < f_1(s')$ or $f_1(s) = f_1(s'), \dots, f_i(s) = f_i(s') \wedge f_{i+1}(s) < f_{i+1}(s')$

Scoring metrics for each state s that we want to evaluate can then be computed in the *Next* algorithm by considering the contents of the pending insertions and updating each parameter differentially.

The defined ordering utilized is the following:

- $f_1(s) = -|s.B|$: we prefer states that opened less bins.
- $f_2(s) = \text{avgvol}(s)$: we prefer states that have packed more average volume between bins.
- $f_3(s) = \text{avgcageratio}(s)$: we prefer states that have better average cage ratio (def. ??) between bins.

4.3. Support Planes

We introduce Support Planes (SP) which is an heuristic introduced in this thesis based on an underlying 2DBPP heuristic which is used to evaluate feasible expansions of a given node in the BS. The proposed heuristic ensures that the constraint of support isn't violated. The idea at the base of SP is to build a solution to the 3DSPP by filling 2D planes called support planes.

Each support plane can be characterized by the triple $S_z = (z, I_{\text{support}}, I_{\text{upper}})$ where

- z : the height of the plane
- I_{support} : the set of the items that can offer support to items placed on the plane
- I_{upper} : the set of items that will be obstacles to potential new items placed on the plane

Let s_b be a data structure containing

- *planes*: the set of triples S_z of support planes to evaluate, ordered in ascending z order
- *aabb*: the AABB Tree of the items placed in the evaluated bin
- (W_b, D_b, H_b) : the dimensions of the bin

Let *coords* be the set of possible coordinate changes which allow for the problem to evaluate placements starting from different corners of the bin.

Given a function $IsFeasible(i, bin, I_{\text{support}}, I_{\text{upper}}, aabb)$ which evaluates if a packing of item i in bin bin is feasible, and the function $ComparePacking(p, p')$ which defines a

ranking over placements in the same plane, the SP algorithm can be written as algorithm 7.

Algorithm 7: SP Best Insertion

input : s_b, I

output: $placement$

$placement \leftarrow \emptyset$

forall $S_z \in planes$ **do**

$I_p \leftarrow I \setminus \{i \in I : z + i.h > H_b\}$

forall $change \in coords$ **do**

$I'_{upper} \leftarrow CoordinateChange(change, I_{upper})$

$I'_p \leftarrow CoordinateChange(change, I_p)$

$P' \leftarrow SP-PackPlane(W_b, D_b, I'_{upper}, I'_p)$ (Algorithm 8)

$P \leftarrow CoordinateChange(change, P')$

$P \leftarrow \{i \in P : IsFeasible(i, bin, I_{support}, I_{upper}, aabb)\}$

if $ComparePacking(placement, P)$ **then**

$placement \leftarrow P$

end

end

if $placement \neq \emptyset$ **then**

return $placement$

end

end

return $placement$

To evaluate a packing on a plane an heuristic to solve the 2DBPP is used with the introduction of fixed placements which represent items on other planes that will be obstacles in the current one.

Given the dimensions of the 2D bin (W_b, D_b) , the set of obstacles I_o and the set of items to pack I_p a new placement can be computed following algorithm 8

Algorithm 8: SP Pack Plane

input : W_b, D_b, I_o, I_p **output:** P $P \leftarrow \emptyset$ $2dPacking \leftarrow \emptyset$ **foreach** $i \in I_o$ **do** //Initialize the 2D bin packing instance with each obstable already
 placed $2DPlaceRect(2dPacking, i)$ **end****repeat**

//Pack untill full

 $p \leftarrow 2DPackRect(2dPacking, W_b, D_b, i)$ $P \leftarrow P \cup \{p\}$ **until** $p \neq \emptyset$ **return** P

Commit Extension We now describe an extension to *Commit* (algo. 1) to update the structures needed by SP.

When a plane is filled, new insertions become less likely to be feasible. To avoid evaluating planes where no insertion is possible a mechanism to prune dead planes can be introduced.

Since best insertions for a bin are always evaluated by considering lower planes first, if all the insertions in *Expand* (algo. 6) happened over a z_{min} then we can safely remove the opened planes with $z < z_{min}$ for that bin. We can easily introduce a z_{min} variable carried over in each s_b for each bin which is updated during the *Expand* phase with the minimum z of the planes that generated an insertion on bin b .

Once the best states are computed and *Commit* is called we can then use that value to prune planes in each s_b . Other operations are also necessary in the *Commit* to allow for SP to update its data structures accordingly to the insertion.

Given a state s and an insertion p where each packed item $i \in p.I$ in bin b has z_i within tolerance of z and the minimum height for the considered bin $s_b.z_{min}$. The algorithm which updates the structures for a given bin b is represented by algorithm 9. This new algorithm can be used as last step of the *Commit* algorithm for each $b \in s'.B$.

Algorithm 9: SP Apply and Filter

input : $s, p, z, z_{min}, \beta_s$
output: s
 $b \leftarrow p.b$
 //Filter bad planes
 $s.s_b.Z \leftarrow s.s_b.Z \setminus \{(z', I_{support}, I_{upper}) \in s.s_b.Z : z' < z_{min}\}$
 //Apply insertion
forall $i \in p.I$ **do**
 $s.s_b.T \leftarrow \text{InsertAABB}(i, s.s_b.T)$ //If balanced $O(\log(n))$
 $generate \leftarrow true$
 forall $(z', I_{support}, I_{upper}) \in s.s_b.Z$ **do**
 //Based on the distance from the top of the item
 $dz \leftarrow z' - (z_i + h_i)$
 if $0 \leq dz \leq \beta_s$ **then**
 $generate \leftarrow false$
 $I_{support} \leftarrow I_{support} \cup i$
 end
 else if $dz < 0$ **then**
 $I_{upper} \leftarrow I_{upper} \cup i$
 end
 end
 if $generate$ **then**
 $s.s_b.Z \leftarrow s.s_b.Z \cup (z_i + h_i, \{i\}, \emptyset)$
 end
end
return s

4.3.1. Scoring Insertions

5 | Computational experiments

6 | Conclusions and future developments

A final chapter containing the main conclusions of your research/study and possible future developments of your work have to be inserted in this chapter.

Bibliography

- [1] G. v. d. Bergen. Efficient collision detection of complex deformable models using aabb trees. *Journal of graphics tools*, 2(4):1–13, 1997.

A | Appendix A

If you need to include an appendix to support the research in your thesis, you can place it at the end of the manuscript. An appendix contains supplementary material (figures, tables, data, codes, mathematical proofs, surveys, ...) which supplement the main results contained in the previous chapters.

B | Appendix B

It may be necessary to include another appendix to better organize the presentation of supplementary material.

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List of Symbols

Variable	Description	SI unit
\boldsymbol{u}	solid displacement	m
\boldsymbol{u}_f	fluid displacement	m

Acknowledgements

Here you might want to acknowledge someone.

