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Three-dimensional bin packing with vertical support

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Abstract

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Contents

1 | Introduction

The three-dimensional bin packing problem (3D-BPP) involves packing, without any overlap, a set of small items into the minimum number of bins. In its most standard form, each item i is a cuboid of dimensions (w_i, d_i, h_i) and each bin is a cuboid with fixed dimensions (W, D, H). Items can only be placed with edges that are parallel to the sides of the bin and can be rotated by 90 degrees along their vertical axis.

The 3D-BPP is part of the family of Cutting and Packing problems, where a set of small items (boxes) needs to be packed inside a set of large ones (bins). Based on the dimensionality of the problem and on the number of items with different shapes, Wäscher et al. [2007] identifier proposed typology of cutting and packing problems. Versions of the 3D-BPP with strongly heterogeneous items are classified as three-dimensional single bin-size bin packing problems (3D-SBSBPP).

In this thesis, we address a variation of the 3D-SBSBPP stemming from a real case study of mixed-case palletization: the Three-Dimensional Bin Packing Problem With Vertical Support (3D-BPPWS). Modeling real-world case studies with the 3D-SBSBPP requires additional constraints to be considered. We extend the standard formulation of the 3D-SBSBPP by ensuring that all items that are packed inside a bin will not fall, and we refer to this property as the vertical support constraint. Support constraints are usually defined based on the amount of area of an item that lies on top of other items, or by the number of corners of an item that rest on top of other items (Gzara et al. [2020]; Kurpel et al. [2020]; Paquay et al. [2016]).

The standard 3D-SBSBPP is strongly NP-hard since it is a generalization of the one-dimensional bin packing problem [Martello et al., 2000]. Since our problem is a generalization of the 3D-SBSBPP, exact solution algorithms are only able to solve small instances of the problem, therefore heuristic approaches need to be used to solve larger instances. Heuristics designed to solve the 3D-SBSBPP do not address the concept of static stability and allow solutions with unsupported items. The concept of static stability has received most of its contributions in Pallet Loading Problems and Container Loading Problems (Calzavara et al. [2021]; Kurpel et al. [2020]). In these publications, the concept of support is addressed explicitly by building layers or walls of items with high density, which allows them to reduce the problem to a one-dimensional packing problem (Bortfeldt and Wäscher [2013]). In mixed-case palletization scenarios, layer-based solutions represent the majority of work in the literature. They usually work by stacking layers ordered by density until the density of the last generated layers falls below a certain threshold. Once no more layers can be generated, simpler techniques are employed to pack the remaining items (Elhedhli et al. [2019]), or the use of filler boxes is employed to increase the layer's density (Calzavara et al. [2021]).

In this thesis, we approach the constraint of support by developing a constructive heuristic that fills bins without explicitly building layered solutions or using filler boxes. We then introduce a beam search heuristic that expands the constructive heuristic's solution space by exploring different orders of item palletization. We also provide a formulation for the 3D-BPPWS with a discretized version of the support constraint. Such a formulation is used to validate our heuristic on small instances of the problem. Finally, we provide a generated data set based on real-world instances that we use to benchmark our heuristic.

1.1. Case study

The work of this thesis stems from the case study of a logistics company in northern Italy. The company manages large warehouses where automated lines bring boxes to different packing stations, and then they are loaded onto pallets of standard size. Each box is loaded manually by an operator, and soon as the height of the pallet reaches a certain threshold, the packing station lowers it and wraps it with an elastic material that guarantees its stability. This wrapping improves the stability of the pallet while boxes are still being loaded on the top. To avoid uneven surfaces during the wrapping phase, pallets should not have empty spaces inside. When dealing directly with customers' orders, boxes have very different sizes and they are usually packed in smaller quantities. This implies that layer-based pallet loading solutions are impractical in these cases, since it is usually not possible to build full layers of boxes of the same height. The company is interested in building pallets that do not have empty spaces inside, and they measure this property with a metric called cage ratio. A formal definition of this metric is reported in eq. (3.1). To increase the efficiency of the wrapping and to allow for the stacking of pallets, solutions with a high cage ratio are required. The cage ratio of commercial solutions currently implemented by the company is around 60%, and a target cage ratio for our case study was set at 70%.

1.2. Overview

In chapter 2 we review the relevant literature on the three-dimensional bin packing problem and the cutting and packing problems dealing with vertical support. In chapter 3 we give a formal definition of the problem and formulate a mixed-integer linear programming model that we will use to validate the proposed solution algorithm. Since the model can not be used to solve real-world instances, in chapter 4 we propose a heuristic algorithm that is able to solve larger problem instances. In chapter chapter 5 we present our computational experiments. We compare our heuristic algorithm against relevant heuristics from the literature, against the solutions from our MILP model, and against solutions from our real-world case study. We also describe the process that we used to generate new instances. Finally, in chapter 6 we give final remarks and list possible further developments of this research.

2 | Literature review

In this section, we review the relevant literature to our problem. In section 2.1, we do a brief review of the most relevant two-dimensional bin packing placement heuristics in the context of this thesis. In section 2.2, we review the literature on the 3D-BPP, focusing on the heuristics used to solve the single bin-size bin packing problem. Finally, in section 2.3, we do a brief review of the literature on practical constraints in cutting and packing problems, focusing on the vertical support constraint.

2.1. Two-Dimensional Bin Packing Problem

In the two-dimensional bin packing problem, two major tasks constitute the area of study relevant to constructive heuristics. The first task is to identify the smallest set of points where placements can be made. The second task consists in evaluating which point to select for placements. The main strategies used in selecting points are divided into first-fit and best-fit approaches. The first valid positions are selected in first-fit approaches, while in best-fit approaches, positions are selected based on a metric. The most common classical algorithm to select placements inside a two-dimensional bin given a set of points is the bottom-most left-most algorithm introduced in Baker et al. [1980]. It packs items in the lowest possible position closest to the bottom-left corner of the free area. This algorithm serves as the base of many heuristics that address the two-dimensional bin packing problem (2D-BPP). In Burke et al. [2004], a best-fit algorithm is introduced where placements of items that fit the lowest available area are made first. In Lodi et al. [1999], a maximum touching perimeter approach is used instead.

Considering the identification of possible packing positions, in Martello and Vigo [1998], a branch-and-bound algorithm was proposed to solve the two-dimensional orthogonal packing problem (2D-OPP). The selection of the positions is made in a left-most downwards strategy. Items are placed so that their left and bottom edges touch either other items or the bin. The algorithm is based on a tree search that packs items in every possible position. A set of 10 classical instances to benchmark heuristics against were also presented. In Martello et al. [2003], a branch-and-bound algorithm for the two-dimensional strip packing problem (2D-SPP) was proposed, based on the idea of staircase placements introduced in scheithauer1995equivalence. In staircase placements, a boundary is identified which separates the already packed items from the area of the bin that is still free. The boundary is an envelope composed of segments that touch either the side of an item or the bin. The resulting envelope has a staircase-like shape; the corner points are points where the envelope changes from horizontal to vertical (as noted in [Martello et al., 2000]). A similar approach was used in Crainic et al. [2008], where an extension to the staircase approach was introduced. In the proposed method, each packed item introduces a fixed number of extreme points, which are the projections of his corner points along the orthogonal axis of the bin onto the sides of either the bin or its closest packed neighbor. New niches were identified that were previously discarded by the staircase method. Both the extreme point and corner point strategies were adapted to the three-dimensional bin packing case as seen in section 2.2.

For a recent review of the literature related to the 2D-BPP, we refer the reader to Iori et al. [2021], which surveyed two-dimensional packing problems with mathematical formulations, heuristic and exact methods, relaxations, and open problems.

2.2. Three-Dimensional Single Bin-Size Bin Packing Problem

An exact approach to the 3D-SBSBPP was proposed in Martello et al. [2000] through a two-level branch-and-bound search and a staircase placement approach derived from the 2D-BPP field. The algorithm was initially limited to robot packable solutions and later extended to the general problem in Martello et al. [2007]. Faroe et al. [2003] proposed a Guided Local Search for 2D-SBSBPP and 3D-SBSBPP. The algorithm started from an upper bound on the number of bins calculated through a greedy heuristic and iteratively improved the solutions by searching for new feasible solutions thanks to the proposed GLS method. The process terminated when it reached a computed lower bound or a specific time limit had expired. In Lodi et al. [2002], a tabu search procedure was proposed to address the two-dimensional and three-dimensional bin packing case. The search was based on two steps, starting with one item per bin. In the first phase, it merged bins, and then two constructive heuristics were used to create layers in each bin. Between each step, a 1D-BPP was solved to stack the generated layers into bins. Lodi et al. [2004] later provided code for a unified tabu search addressing the multi-dimensional bin packing problem. A two-level tabu search for the multi-dimensional bin packing problem was also provided in Crainic et al. [2009]. The algorithm started from a greedy feasible solution based on the extreme point heuristic introduced in Crainic et al. [2008]. In the algorithm's first step, they built a neighborhood by swapping or moving items between bins while relaxing the bin constraints. In the second step, they searched for feasible solutions by changing the relative positions of items inside the bin. In Fekete and Schepers [2004] a new model for bin packing problems based on interval graphs was introduced. Each packing was represented as an interval graph derived from the overlaps of items along each axis. A GRASP-based algorithm for the 3D-SBSBPP and 2D-SBSBPP was proposed by Parreño et al. [2010]. During its constructive phase, the algorithm used a maximal-space heuristic designed for container loading problems. Then, several moves were designed and combined in an improvement phase with a variable neighborhood descent approach. In Wu et al. [2010], a genetic algorithm was presented that varied the relative positions of items in a mixed-integer linear programming model. The chromosomes represented the order of items to be packed and their orientation. Hifi et al. [2014] proposed a hybrid greedy heuristic that solves the 3D-SBSBPP in two phases. A selection phase identifies a subset of items to pack by solving a knapsack problem. Subsequently, a positioning phase fixes each item's position inside the bins. In both phases, an integer linear programming model is employed. An additional optimization phase can also be introduced at the cost of computational times. Gonçalves and Resende [2013] presented a biased random-key genetic algorithm for the 3D-SBSBPP (BRKGA). The chromosomes represented the encoding for the sequence of items to pack in the solution. The packing was done with an underlying heuristic that uses the same maximal-space representation as Parreño et al. [2010]. Zudio et al. [2018] later proposed a variable neighborhood descent variation of BRKGA that improved the evolving process of BRKGA by finding high-quality individuals in earlier generations.

2.3. Vertical Support

In recent years, many publications have addressed various practical constraints dictated by industry needs. Vertical support (or static stability) received most of its contributions from the fields of Pallet Loading Problems (PLP) and Cargo Loading Problems (CLP). In this section, we focus on publications related to these two problems that dealt with the concept of vertical support.

As noted in Bortfeldt and Wäscher [2013], static stability is one of the most critical constraints in Cargo Loading Problems, but it is usually implicitly enforced as a consequence of load compactness or explicitly guaranteed by using filler material as a postprocessing step. A MIP formulation was proposed in Paquay et al. [2016] with the inclusion of various practical constraints like vertical support through vertex support, containers of different sizes and shapes, weight distribution, item rotations, and load-bearing. Since the proposed model was complex, only small instances were solved to optimality in a reasonable time frame. Therefore, the work was extended in Paquay et al. [2018] where three meta-heuristics were provided to reduce the solve time. In Galrão Ramos et al. [2016] the single container CLP is solved with static mechanical stability by combining a multi-population random key genetic algorithm (BRKGA) with a constructive heuristic that determines a two-dimensional box placement strategy. The publication also proposed a procedure to fill maximal-spaces based on mechanical equilibrium conditions applied to rigid bodies. In Kurpel et al. [2020] several new formulations of CLP are presented with various extensions for practical constraints such as box orientations, stability (including vertical support), and the separation of boxes. Vertical support is formulated through the discretization of space along each axis and with the help of an overlap matrix that encodes the amount of area support each item can give to the others. The work also presents heuristic approaches and upper and lower bounding techniques. In Alonso et al. [2020] a multi-container loading problem is solved using a GRASP meta-heuristic where pallets are built from a set of layers and then positioned inside a container. Practical constraints are considered like weight limits, weight distribution, dynamic stability, delivery dates. The constraint of static stability is implicitly ensured by building dense layers. In Gajda et al. [2022] a constructive randomized heuristic for solving the CLP is proposed with constraints including vertical support ensured by area support, customer priorities, load balancing, stacking constraints, and positioning constraints. In the proposed constructive heuristic, a subset of extreme points is evaluated starting from two corners of the cargo to ensure a better weight distribution.

Considering Pallet Loading Problems, in Elhedhli et al. [2019] a column-generation framework and a branch-and-price solution to the mixed-case pallet loading problem was proposed with a two-dimensional layer generation problem as the pricing subproblem. The subproblem was then solved with exact methods and heuristically with additions, including item groupings, item replacement, layers' reorganization, and spacing. A new instance generator for instances that better represent industry instances was provided. Although the layering approach used implicitly favored solutions with support, the paper did not directly address vertical support. The work was later extended by Gzara et al. [2020] to explicitly address practical constraints such as vertical support, load-bearing, pallet weight limits, and planogram sequencing. A second-order cone programming formulation was provided as a solution to a spacing problem pallet layers, and further extensions to the previously introduced instance generator were made. In Calzavara et al. [2021] a mathematical formulation for a layer and a pallet generation problem is defined together with heuristics and metaheuristics algorithms designed to solve the PLP with constraints on item groupings, layering, and visibility of items. The work is based on previous papers on PLP by the same authors (Iori. et al. [2020]; Iori et al. [2020, 2021]) that proposed a reactive GRASP metaheuristic to solve the general problem. The stability of the solutions is implicitly ensured with layering and the use of filler boxes to increase the density of problematic layers.

3 | Problem description and mathematical formulation

In this thesis, we address the 3D single bin-size bin packing problem (3D-SBSBPP) with the addition of a few practical constraints. Starting from a set of items of different sizes the goal is to arrange them in the least amount of bins of a given fixed size without any overlap between each other. In addition to the standard formulation of the problem, three practical constraints need to be taken into account:

- each item inside a bin should have static stability, meaning that every item should be supported either by the ground or by other items in the same bin,
- the cage ratio of each used bin should be maximized,
- each item can be rotated orthogonally along its vertical axis.

Given a certain placement of items inside a bin of base $W \times D$ with the top of the highest item being at $z_{\max}$ and the sum of the volume of each item being V, the bin's cage ratio is defined as eq. (3.1).

$$\mathrm{CR} = \frac{V}{W \cdot D \cdot z_{\max}} \tag{3.1}$$

A high cage ratio means that even if a bin isn't fully occupied, it could be used as a base for other structures. This property is desirable in some industrial settings. It is also noted that in a single bin configuration, maximizing cage ratio is equivalent to minimizing $z_{\max}$. Finally, a visual representation of the cage ratio metric is provided in fig. 3.1.

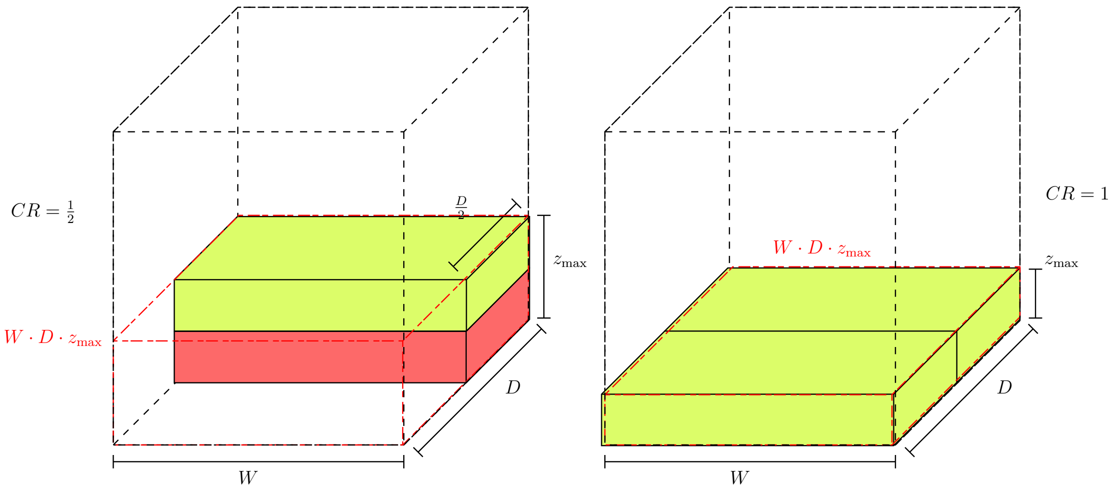


Figure 3.1: Cage ratio of two different bin configurations

Our notion of vertical support stems from rules imposed by the industry and from the literature on Pallet Loading Problems and Container Loading Problems. Vertical stability is usually ensured between horizontal or vertical slices of items as a constraint on the minimum amount of area which rests on other items (as for ex. [Gzara et al., 2020; Kurpel et al., 2020; Paquay et al., 2016]). Given a support area threshold α_s and a vertical tolerance β_s we can define an item as supported if one of the following holds

1. the sum of the overlap area over the XY-plane with every other item on which it is resting is greater than α_s times its base area. (area support)
2. the number of its corners resting on another item is greater than 3, and condition 1 holds with a lower threshold $\alpha'_s < \alpha_s$. (vertex support)

A visual representation of the condition of support is illustrated in fig. 3.2.

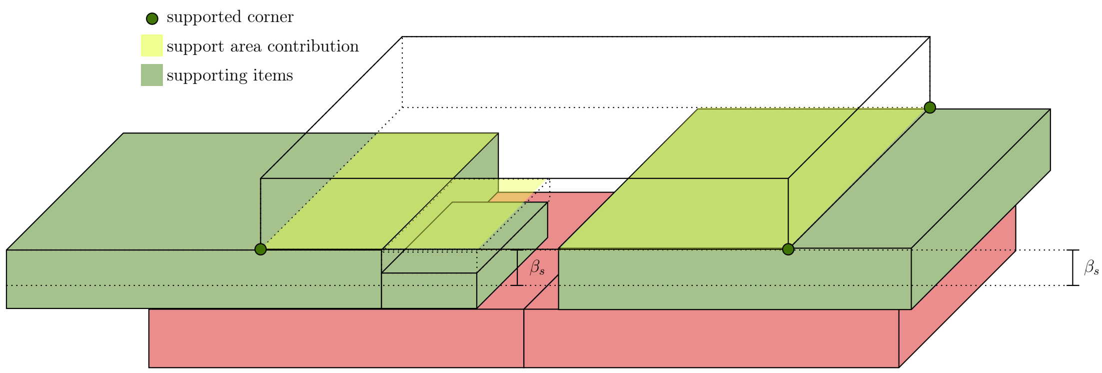


Figure 3.2: Representation of an item with vertical support given $\alpha_s = 0.5, \beta_s$

Given the definitions of our practical constraints, a conceptual formulation of our model would be

minimize	number of used bins
	the unused volume of each bin under $z_{\max}$
subject to	all items assigned to one and only one bin
	all items within the bin dimensions
	no overlaps between items in the same bin
	all items with vertical support

In section 3.1 a mixed-integer linear programming model for the 3D-SBSBPP is presented and it's later extended to include orthogonal rotations in section 3.1.1 and vertical support constraints limited to condition 1 (area support) in section 3.1.2. Cage ratio isn't directly included in the proposed MILP formulation since the evaluation of the heuristic with the model was done in a single bin configuration where minimizing the maximum height of the bin is equivalent to minimizing the cage ratio.

3.1. 3D single bin-size bin packing problem

Let $I = \{1, \dots, n\}$ be the set of items that need to be packed, $B = \{1, \dots, m\}$ the set of bins to evaluate of fixed dimensions $W \times D \times H$. Each item $i \in I$ is characterized by a given width, depth, and height (w_i, d_i, h_i). Let us introduce three continuous variables that identify the position of an item's bottom front left corner (x_i, y_i, h_i) as seen in fig. 3.3. We can now introduce a set of integer variables v_b which will be 1 if bin $b \in B$ will be used in the solution and 0 otherwise. A set of integer variables u_{ib} which will be 1 if item $i \in I$ will be placed in bin $b \in B$ and 0 otherwise. To check for overlaps, we introduce three sets of integer variables for each axis of possible overlap to determine if there is a clear order of precedence on at least one axis. This formulation is also usually used in scheduling problems. One of the introduced sets is the set x^p_{ij}, which will take the value of 1 if item $i \in I$ precedes item $j \in I$ over axis x and 0 otherwise. This condition is verified if $x_i + w_i \leq x_j$. The other two sets are defined in a similar way over the remaining axis y^p_{ij} and z^p_{ij}. An additional set of continuous variables $z^{\max}_b$ is introduced which will assume the value of the maximum $x_i + h_i$ of the items $i \in I$ placed in bin $b \in B$.

The 3D-SBSBPP can then be formulated as a mixed-integer linear programming problem:

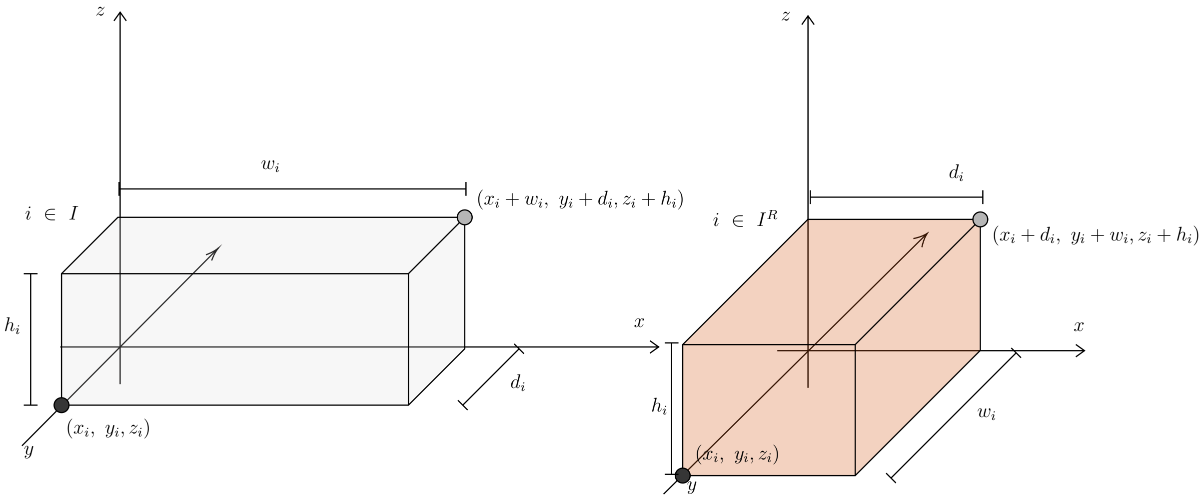


Figure 3.3: Coordinate system representation for a generic item i and its rotated clone $i \in I^R$

$$\min \quad \sum_{b \in B} (Hv_b + z_b^{max}) \tag{3.2}$$

$$\text{s.t.} \quad \sum_{b \in B} u_{ib} = 1 \qquad \forall i \in I \tag{3.3}$$

$$u_{ib} \leq v_b \qquad \forall i \in I, \forall b \in B \tag{3.4}$$

$$v_b \geq v_c \qquad \forall (b,c) \in B : b < c \tag{3.5}$$

$$x_i + w_i \leq W \qquad \forall i \in I \tag{3.6}$$

$$y_i + d_i \leq D \qquad \forall i \in I \tag{3.7}$$

$$z_i + h_i \leq H \qquad \forall i \in I \tag{3.8}$$

$$z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) \qquad \forall i \in I, \forall b \in B \tag{3.9}$$

$$(x_i + w_i) - x_j \leq W(1 - x_{ij}^p) \qquad \forall i,j \in I \tag{3.10}$$

$$x_j - (x_i + w_i) + 1 \leq W x_{ij}^p \qquad \forall i,j \in I \tag{3.11}$$

$$(y_i + d_i) - y_j \leq D(1 - y_{ij}^p) \qquad \forall i,j \in I \tag{3.12}$$

$$y_j - (y_i + d_i) + 1 \leq D y_{ij}^p \qquad \forall i,j \in I \tag{3.13}$$

$$(z_i + h_i) - z_j \leq H(1 - z_{ij}^p) \qquad \forall i,j \in I \tag{3.14}$$

$$z_j - (z_i + h_i) + 1 \leq H z_{ij}^p \qquad \forall i,j \in I \tag{3.15}$$

$$x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 \qquad \forall i,j \in I, \forall b \in B \tag{3.16}$$

The objective function 3.2 seeks to minimize the number of opened bins and the maximum height at which items were placed across bins. In a single bin configuration, since all the volume mass is concentrated inside one bin, it maximizes the cage ratio. Constraint 3.3

ensures that each item is packed in one and only one bin, while constraint 3.4 ensures that items are only packed in bins used in the solution. Since the solution has lots of symmetries with respect to the number of bins, a symmetry-breaking constraint 3.5 can be added on the opening of bins to improve solve times. Each item is also ensured to be placed inside the bin thanks to eqs. (3.6) to (3.8). The value of z_b^{max} is forced to converge to the maximum height of a given bin thanks to constraint 3.9. Constraints from 3.10 to 3.15 are used to define the precedence binary variables x_{ij}^p, y_{ij}^p, z_{ij}^p over each axis as described in the problem formulation. Constraint 3.16 then ensures that if two items are in the same bin, there needs to be at least one axis with a clear order of precedence; otherwise, the two items would overlap.

3.1.1. Orthogonal rotations

Let us extend the definition of the bin packing problem without rotations with a new formulation that allows 90 degrees rotations of each item. Let $I = I^O \cup I^R$ be the new set of items where I^O is the set of original non-rotated items, and I^R is the set of items rotated by 90 degrees. Given the set of tuples $(i, j) \in I^{OR}$ where i is the original item with dimensions (w_i, d_i, h_i) and j is the corresponding rotated clone with dimensions $(w_j, d_j, h_j) = (d_i, w_i, h_i)$, we can now rewrite constraint 3.3 as 3.17 to force only one of them to be part of the solution.

$$\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \qquad \forall (i, j) \in I^{OR} \tag{3.17}$$

3.1.2. Discrete vertical support formulation

We now extend the model to address the constraint of vertical support. In the literature, some mathematical formulations tackle the concept of area support and, in some cases, vertex support. For example, in [Gzara et al., 2020] a second-order cone programming formulation of the support constraint was used but was limited to the problem of spacing between layers with one of them being fixed in position relative to the other. A similar formulation would lead to a non-linear support constraint in our case. By introducing a discretization over the XY-plane a linear version of the constraint can be formulated similar to the one proposed in [Kurpel et al., 2020] without the need to discretize the z-axis as well.

Let us introduce some additional parameters to the model, let $0 \leq \alpha_s \leq 1$ be the amount of area that an item needs to have supported by other items. Let β_s be the tolerance to consider one item as being close enough to support another item (as seen in fig. 3.2). In addition to the support parameters, a parameter δ, which represents the discretization unit used to partition the XY-plane, is given. Let I^B be the set of all the tuples (i, j, b) such that $(i, j) \in I \wedge i \neq j$ and $b \in B$. We can now compute a few additional parameters that we will use to reduce the number of constraints evaluated by the model. Let γ be the maximum size over a dimension on the XY-plane between all the items as eq. (3.18), and let Δ be the set of all possible distances between the origins of two items along one discretized axis as eq. (3.19).

$$\gamma = \max_{\forall i \in I} \{w_i, d_i\} \tag{3.18}$$

$$\Delta = \left[-\left\lfloor \frac{\gamma}{\delta} \right\rfloor, \left\lfloor \frac{\gamma}{\delta} \right\rfloor \right] \tag{3.19}$$

Let $O(i, j, h, k) \rightarrow \mathbb{R}^+$ be a function that computes the amount of overlap between two items $(i, j) \in I$ given the discretized distance between each other $(h, k) \in \Delta$ such that $x_j = x_i + \delta h$ and $y_j = y_i + \delta k$ which returns the area of overlap or 0 otherwise.

Additional new variables need to be added to the ones of the original model, let s_{ij} be a set of binary variables which will assume value 1 if item $i \in I$ can offer support to item $j \in I$ and 0 otherwise. A new set of binary variables z^c_{ij} will be 1 if item $i \in I$ is close w.r.t. the z-axis to item $j \in I$, which would mean that $z_j - (z_i + h_i) \leq \beta_s$, and 0 otherwise. Let us then introduce a new set of binary variables g_i which will assume value 1 if item $i \in I$ will be on the ground or 0 otherwise. A set of binary variables s^{kh}_{ijb} will assume value 1 if item $i \in I$ will receive support from item $j \in I$ and both items will be placed in bin $b \in B$ with a discretized distance of $(k, h) \in \Delta$ between each other and 0 otherwise.

Given all the additional parameters and variables introduced, we can give a new formulation of the model with the same objective function 3.2 and the constraints in section 3.1

with the addition of the following constraints:

$$z_j - (z_i + h_i) \leq \beta_s + H(1 - z^c_{ij}) \qquad \forall (i,j) \in I : i \neq j \tag{3.20}$$

$$z_j - (z_i + h_i) \geq -\beta_s - H(1 - z^c_{ij}) \qquad \forall (i,j) \in I : i \neq j \tag{3.21}$$

$$s_{ij} \leq z^p_{ij} \qquad \forall (i,j) \in I \tag{3.22}$$

$$s_{ij} \leq z^c_{ij} \qquad \forall (i,j) \in I \tag{3.23}$$

$$s_{ij} \geq z^p_{ij} + z^c_{ij} - 2 \qquad \forall (i,j) \in I : i \neq j \tag{3.24}$$

$$\sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} \qquad \forall i \in I \tag{3.25}$$

$$z_i \leq H(1 - g_i) \qquad \forall i \in I \tag{3.26}$$

$$\sum_{(k,h) \in \Delta, b \in B : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq s_{ij} \qquad \forall (i,j) \in I \tag{3.27}$$

$$\sum_{(k,h) \in \Delta : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq u_{ib} \qquad \forall (i,j,b) \in I^B \tag{3.28}$$

$$\sum_{(k,h) \in \Delta : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq u_{jb} \qquad \forall (i,j,b) \in I^B \tag{3.29}$$

Constraints 3.20 and 3.21 ensure that z^c_{ij} is forced to 1 only when the distance over the z-axis between item i and item j is within the range $[-\beta_s, \beta_s]$. The value of s_{ij} is then assigned to the logical equation $z^p_{ij} \wedge z^c_{ij}$ thanks to constraints from 3.22 to 3.24. Since some items could be left out of the solution due to the formulation of orthogonal rotations, we also ensure that support can only come from placed items thanks to constraint 3.25. Constraint 3.26 ensures that g_i will assume value 1 if item i is on the ground Constraints from 3.27 to 3.29 ensure that if a discretized support decision s^{hk}_{ijb} is 1 then every subscript of that variable must be true in the non-discretized model, so item i can give discretized support to item j in in bin b if both items are are assinged to bin b and if i can give support to item j. They also force the selection of only one possible combination of $(h,k) \in \Delta$ for which i gives support to j in bin b.

We can then define a set of constraints which given a discretized placement s^{kh}_{ijb} limits the distance between i and j to a given continuous region in space delimited by a square of the dimension of our discretization unit δ. Given every tuple of possible discretized distances between items $(k,h) \in \Delta$ and every tuple of different pairs of items in the same bin $(i,j,b) \in I^B$ such they have a non-zero discretized overlap over the XY-plane

$(O(i,j,k,h) \neq 0)$. The resulting constraints are defined in eqs. (3.30) to (3.33).

$$x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) \tag{3.30}$$

$$x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) \tag{3.31}$$

$$y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) \tag{3.32}$$

$$y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) \tag{3.33}$$

We can then introduce a feasibility constraint thet ensures that every item that isn't on the ground is supported by other items placed beneath it by at least α_s times its area, which corresponds to condition 1 of the practical constraint of vertical support.

$$\sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h) s_{jib}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i \quad \forall i \in I \tag{3.34}$$

It is noted that every combination of $(i,j,b) \in I^B$ and $(h,k) \in \Delta$ where $O(i,j,k,h) = 0$ do not contribute to the support constraint. We can omit them from the formulation of the problem to reduce the number of constraints to evaluate.

4 | Solution algorithms

In this chapter we present our heuristic algorithm to solve the 3D-BPPWS. Since the 3D-BPP is NP-Hard, an exhaustive search for a solution is not practical, therefore a heuristic search is conducted by combining the beam search algorithm described in section 4.2 with the constructive heuristic described in section 4.3. In section 4.1 we define the preliminary concepts that will be used in the algorithm: states, insertions and solution's feasibility.

4.1. States

States are partial solutions of the 3D-BPPWS. Since our heuristic is based on a constructive method, starting from a state representing an empty solution we'll iteratively build new states that are always closer to a complete solution of the problem. Being a partial solution, a state s can be represented as the set of all variables present in the MILP model (3.1), where some values have been fixed. We move closer to a complete solution of the problem by packing more items and opening new bins, i.e. fixing more variables.

In order to simplify the algorithm's description, we introduce a few new definitions.

Definition 4.1 (Open bin). *A bin $b \in B$ is open in state s* ***iff***

$$v_b^s = 1$$

Definition 4.2 (Set of open bins). *Let s be a state, we define B^s as the set of bins that are open in s.*

$$B^s = \{b \in B \mid v_b^s = 1\}$$

Definition 4.3 (Unpacked item). *An item $i \in I$ is unpacked in state s* ***iff***

$$\sum_{b \in B} u_{ib}^s = 0$$

i.e. it has yet to be assigned to an open bin.

Definition 4.4 (Set of unpacked items). *Let s be a state, we define U^s as the set of unpacked items in s.*

$$U^s = \{i \in I \mid \sum_{b \in B} u^s_{ib} = 0\}$$

Definition 4.5 (Set of packed items). *Let s be a state and let $b \in B^s$, we define J^s_b as the set of items that are packed inside b.*

$$J^s_b = \{i \in I \mid u^s_{ib} = 1\}$$

Thanks to these new definitions, we can define a function that determines if a state is a final state.

Definition 4.6. *A state s is final if there are no more items to pack.*

$$IsFinal(s) = \begin{cases} 1, & U^s = \emptyset \\ 0, & otherwise \end{cases} \tag{4.1}$$

We can also define the empty state, which will be used as a starting point for our algorithm.

Definition 4.7 (Empty state). *A state s is empty if it contains a single open bin without any item packed inside.*

$$U^s = I \wedge |B^s| = 1$$

By problem definition, the first expression implies that $J^s_b = \varnothing \; \forall b \in B^s$.

Given a state s, for each item $i \in I$ packed in $b \in B^s$ ($i \in J^s_b$), we let (x^s_i, y^s_i, z^s_i) be the coordinates of its bottom front left corner. In order to simplify the algorithm representation, rotations are handled implicitly by swapping the dimensions w_i and d_i of item $i \in I$ when needed. An item can have different rotations if packed in different states, therefore we use its horizontal dimensions as variables and refer to them with w^s_i and d^s_i.

The proposed heuristic also stores additional data for each state, which will be used in the constructive heuristic described in section 4.3. This additional information includes:

- Z^s_b: the set of planes inside b in state s. This will be described in detail in section 4.3.
- T^s_b: the AABB Tree of items packed inside b in s. This will be described in detail in section 4.1.1.

Given a well formed state s and a bin $b \in B^s$, both J_b^s and T_b^s contain the items packed in b, however adding and accessing items in J_b^s has a time complexity of $O(1)$ (supposing an implementation as a hash set) while maintaining T_b^s usually has a time complexity of $O(log(|J_b^s|))$. The cost of maintaining T_b^s is repaid by the gain in performing checks on overlapping items, as described in the next section.

4.1.1. AABB Tree

To determine the feasibility of a given state, one needs to check if no placed items overlap. Since every item is a cuboid and our problem formulation only allows for 90 deg rotations over the z-axis, each item is contained inside a bounding box, which is axis-aligned. An adequate structure to compute overlaps is then an Axis-Aligned Bounding Box Tree (AABB Tree) [van den Bergen, 1997].

AABB Trees are bounding volume hierarchies typically used for fast collision detection, and they usually offer a few operations:

- $AABBInsert(i)$: which allows inserting an axis-aligned box i in the tree,
- $AABBOverlaps(i)$: which allows determining if an axis-aligned box i overlaps an element in the tree,
- $AABBClosest(i, d)$: which, given an axis-aligned box i and an axis-aligned direction d, returns the closest element following that direction starting from the box i.

If the tree is appropriately balanced, each operation has a worst-case time complexity of $O(log(n))$, where n is the number of elements in the tree. Given a state s, maintaining an AABB Tree T_b^s for each bin $b \in B^s$ allows us to do fast checks for feasibility during the construction of a solution (as detailed in 4.3.1) and fast feasibility checks on the final states for error detection.

An additional opertation $AABBGetSupporting(i, \beta_s)$ was added to compute the set of supporting boxes of item i given a vertical tolerance β_s. This was possible by checking intersections over the XY-plane, similarly to the $AABBOverlaps$ implementation, and keeping items that are below i with a distance within tolerance.

4.1.2. Insertions

Our algorithm is based on the constructive heuristic described in section 4.3. Starting from an empty bin, it places items inside the open bins until no more items are available. We model the concept of placing a set of unpacked items inside an open bin as an insertion.

Definition 4.8 (Insertion). *Let s be a state, we define an insertion p as a tuple (b_p, I_p) where $b_p \in B^s$, and $I_p \subseteq U^s$. Such a tuple represents the placement of items from I_p in bin b_p at coordinates (x_i^s, y_i^s, z_i^s) $\forall i \in I_p$.*

Observation 4.1. *Given a state s and an insertion $p = (b, \emptyset)$ where $b \notin B^s$, p is an insertion that opens a new bin b in s.*

Observation 4.2 (Same z insertion). *In our algorithm we will always insert items on the same "plane", that is with the same z coordinate. Let $p = (b_p, I_p)$ be an insertion, then:*

$$\exists z (z \in \{0, .., H\} \wedge \forall i (i \in I_p \implies z_i^s = z)) \tag{4.2}$$

To perform an insertion $p = (b_p, I_p)$ means placing all the items from I_p inside the bin b_p. Performing p on a state s generates a new state s'. This, however, is an expensive operation: it requires cloning the state, a heavy time-consuming and memory-intensive task, and updating all the related data structures with a time complexity of $O(|I_p| log(|J^s_{b_p}|))$ (dominated by the update of the AABB Tree $T^s_{b_p}$).

In our algorithm, starting from a feasible state s, we generate multiple new states s' by performing different insertions on s. These new states are then evaluated (as described in section 4.2.1) and only some of the best ones are retained for the rest of the process. To avoid performing insertions on states that will be discarded, we divide the insertion process in two phases: the first phase enables us to compute the score of a state without updating all its data structures, while the second phase actually performs the updates but only on the retained states.

Let us define the concept of pending insertion.

Definition 4.9 (Pending insertion). *We define p^s as the insertion that is pending on state s. A pending insertion is an insertion that in future will be applied to its state.*

The first phase of the insertion process consists in the application of the $Next$ operator.

Definition 4.10 (Next). *Let p be an insertion over a state s, we define $s' = Next(s, p)$ as a shallow copy of s where p is the pending insertion ($p^{s'} = p$).*

Creating a shallow copy of a state means creating a copy of such a state without cloning it in memory. For each new state s' obtained in this way, we compute its score by considering the effects of a future application of $p_{s'}$. After evaluating all states and selecting the best ones, we proceed with the second phase of the insertion process, which is the *Commit* of pending insertions. This operation copies the states in memory and updates all the relevant data structures. The *Commit* scheme is shown in Algorithm 1.

Algorithm 1: Commit

input : s
output: s'
$s' \leftarrow Clone(s)$ `//Memory clone of` s
$(b, I) \leftarrow p^{s'}$
if $b \in B^{s'}$ **then**
  $J_b^{s'} \leftarrow J_b^s \cup I$
  $T_b^{s'} \leftarrow T_b^s \cup I$
  $U^{s'} \leftarrow U^s \setminus I$
else
  `// Open a new bin`
  $B^{s'} \leftarrow B^s \cup b$
  $J_b^{s'} \leftarrow I$
  $T_b^{s'} \leftarrow I$
  $U^{s'} \leftarrow U^s \setminus I$
end
$p^{s'} \leftarrow \varnothing$
return s'

4.1.3. Feasibility

A state s is feasible if, for each bin $b \in B^s$:

- items in J_b^s do not overlap among themselves,
- all items in J_b^s are placed within the bin's bounds,
- each item in J_b^s is either on the ground or satisfies at least one of the support conditions (cond. 1, cond. 2).

Since the proposed heuristic is constructive, we start with an initial feasible state and generate new states by applying insertions that maintain feasibility. It is therefore more convenient to define the concept of feasibility relatively to an insertion.

Insertion feasibility An insertion $p = (b_p, I_p)$ that is pending on a given state s is feasible if every inserted item $i \in I_p$ satisfies the constraint of non-overlap (3.16, both with items placed in $J_{b_p}^s$ and with other items in I_p), the constraint of support (3.34) and if it is placed within the bin. Let I_{support} be the set of items that could support item i, computed through the AABB tree $T^s_{b_p}$ as defined in section 4.1.1. Let $HasSupport(i, I_{\text{support}})$ be a function that returns true if the considered item would verify at least one of the support conditions (cond. 1 or cond. 2) and false otherwise. We define a function $IsFeasible(i, I_{\text{support}}, T^s_{b_p})$ which returns true if the insertion of i in bin b_p for state s is feasible, and false otherwise. If every item $i \in I_p$ is feasible then insertion p is feasible. In case the insertions of some items in I_p aren't feasible, we can always define a function $RemoveInfeasibleItems(p, I_{\text{support}}, T^s_{b_p})$ which removes every unfeasible item from I_p and returns a new insertion $p' = (b_p, I_{p'})$ where:

$$I_{p'} = I_p \setminus \{i \in I_p : \neg IsFeasible(i, I_{\text{support}}, T^s_{b_p})\}.$$

Checking if a state is feasible can then be done by iteratively applying all the insertions ordered by z and updating the proper data structures.

Proposition 4.1. *A state s' derived by committing a feasible insertion p to a feasible state s is always feasible.*

This proposition is true by construction of insertions p, and combined with observation 4.3 it proves that our constructive heuristic always maintains feasible solutions.

Observation 4.3. *Let s_e be an empty state as stated in definition 4.7, then it is feasible.*

4.1.4. State Hashing

From a given state, it is possible to apply two different sequences of insertions and end up with two states that have the same items in the same positions. This undesirable behavior was observed during our computational experiments. We develop a hashing mechanism that enables checking if two states are likely the same in constant time. In a state s, we can uniquely identify a packed item $i \in J^s_b$ in a given position (x^s_i, y^s_i, z^s_i) with its given dimensions (w^s_i, d^s_i, z_i) with a non-commutative hashing function $hash\_nc$. The resulting hash $hash_{ib} = hash\_nc(b, x^s_i, y^s_i, z^s_i, w^s_i, d^s_i, h_i)$ can identify every equivalent packing of an item of the same shape in that specific bin spot. Since $hash_{ib}$ identifies one item with the shape of i in the same spot as i, we can use a commutative function to combine every hash for every packed item in every bin to ignore the order with which items were added to the solution. The combined hash can then be saved inside our state structures as follows.

$$hash^s = \sum_{b \in B^s} \sum_{i \in J^s_b} hash_{ib} \tag{4.3}$$

In our tests, by filtering states with the proposed hash as seen in Algorithm 2 with a simple 64-bit hashing function, we were able to filter out all equal states between iterations with a low amount of collisions. Since the combination of hashes is a simple sum with modulus, the hashing of a state can also be kept updated in constant time at each iteration by simply adding the inserted hashes in the *Commit* function (Algorithm 1).

4.2. Beam Search

Beam Search (BS) is a heuristic tree search algorithm designed for systems with limited memory, where expanding every possible node is unfeasible. The idea behind BS is to conduct an iterative truncated breadth-first search where, at each iteration, only a limited number k of nodes is expanded. After the expansion, every new node is evaluated and the k best nodes are retained for the next iteration. The algorithm keeps exploring the solution tree until no further node can be expanded.

To perform BS one must define the node structure, an expansion function to generate new nodes from existing ones, a ranking between nodes, and a function to determine if a node is final.

A node in the tree can be represented as a state, as described in section 4.1, and eq. (4.1) can be used to determine if a state is final. We also know that a new state s' derived by s by applying a feasible insertion p can be computed as in section 4.1.2. This state expansion procedure, with the exception of empty insertions, will generate new states in our tree which will add a positive number of bins or packed items to the solution. This procedure, eventually, will converge and generate a final state.

If the starting state for the search is feasible, every new generated state will be feasible and if a final state is found it will be feasible (proposition 4.1). States are expanded by generating insertions and applying such insertions to them, following the two phase procedure outlined in section 4.1.2. In our BS, the first phase is performed before the evaluation of each new state while the second phase is performed only after the selection of the k best states. As noted in section 4.1.4, since by evaluating different insertions on different states it is possible to end up having two equal states, a filtering mechanism based on hashing is introduced. During each iteration, it is possible to keep the hashes of the best selected states in a hash set and discard new states with the same hash.

Given a set of initial states S^0 and the number of best states to expand at each iteration k, the described BS is described in Algorithm 2. As observed in definition 4.7, it's possible to start the search from $S^0 = \{s_e\}$.

Algorithm 2: Beam search

```
input  : S^0, k
output: s_best
S^t ← S^0
S_final ← ∅
repeat
    S^{t+1} ← Expand(S^t) (algo. 3)
    S_final ← S_final ∪ {s ∈ S^{t+1} : IsFinal(s)} (def. 4.6)
    S^{t+1} ← S^{t+1} \ S_final
    S^{t+1} ← Sort(S^{t+1}) (sec. 4.2.1)
    S^t ← ∅
    i ← 0
    seen ← ∅
    forall s ∈ S^{t+1} do
        if hash^s ∈ seen then
            continue
        end
        S^t ← S^t ∪ Commit(s) (algo. 1)
        seen ← seen ∪ {hash^s}
        i ← i + 1
        if i > k then
            break
        end
    end
until S^t ≠ ∅
S_final ← Sort(S_final)
return best element of S_final
```

State Expansion An expansion of a state s is a new set of states S_{new} obtained by applying a set of feasible insertions to s. In order to determine these insertions, an underlying heuristic is used (described in section 4.3).

The main idea in this phase of the algorithm is to find feasible insertions in all the bins in B^s at the lowest possible height, for each item in U^s. To reduce the number of possible expansions to evaluate, we limit the search only to insertions of items with unique shapes. With a similar concept to the one used in section 4.1.4, we compute an hash for each item's dimensions and then use it to group items that have the same shape. The evaluation of new insertions can then be done with two different approaches:

- **PS**: (single placement) where we evaluate only the insertion of a single item per item type. This generates insertions of at most 1 item.
- **PM**: (multiple placement) where we evaluate the biggest possible insertion of a group of items of the same shape. This generate insertions of at most the size of the considered group of items with the same shape.

Creating insertions of groups of similar items is a common strategy in Pallet Loading Problems (e.g. Elhedhli et al. [2019]) to create better bases of support for upper layers. With a similar intuition, the idea of placing groups of items of the same shape is to facilitate the creation of uniform planes to be used to support future insertions.

Given a set of items I and a tolerance β_s, we can introduce an algorithm to group the items by their shape and produce a set G of tuples (h, I'), where h is the hash summarizing the shape of the group and I' is the set of items grouped. This procedure is described in Algo. 4. Once items are grouped by shape, the best insertion for each class of items is computed for each open bin. If no insertion is possible in any bin, then the only viable insertion is the bin opening insertion (observation 4.1). The state expansion procedure is detailed in Algo. 3. The algorithm is described in PM mode, however minor modifications are needed to switch to PS mode.

Algorithm 3: Expand

input : S

output: S_{new}

forall $s \in S$ **do**

  $S_{new} \leftarrow \emptyset$

  $G \leftarrow GroupByHash(U^s)$ (Algo. 4)

  $placed \leftarrow false$

  forall $(h, I) \in G$ **do**

    forall $b \in B^s$ **do**

      $P \leftarrow SPBestInsertion(Z_b^s, I, T_b^s)$ (section 4.3)

      if $P \neq \emptyset$ **then**

        $placed \leftarrow true$

        forall $p \in P$ **do**

          $S_{new} \leftarrow S_{new} \cup Next(s, p)$ (def. 4.10)

        end

      end

    end

  end

  if $placed = false$ **then**

    `Open a new bin` $b' \notin B^s$ `(oss. 4.1)`

    $S_{new} \leftarrow S_{new} \cup Next(s, (b', \emptyset))$

  end

end

return S_{new}

4.2.1. Sorting States

In order to sort states, an ordering needs to be defined over them. Since the selection of a state over another is what will influence the final solution the most, parameters that are directly related to minimizing the objective function are used.

In the proposed solution, we use lexicographic ordering to handle multiple objective functions.

Definition 4.11. *Let* $f_1(s), f_2(s), f_i(s), \ldots, f_n(s)$ *be objective functions ordered by prece-*

Algorithm 4: Group By Hash

```
input  : I
output: G
G ← ∅
forall i ∈ I do
    generate ← true
    forall (h, I') ∈ G do
        if h = hash(w_i, d_i, h_i) then
            generate ← false
            I' ← I' ∪ i
            break
        end
    end
    if generate = true then
        G ← G ∪ (hash(w_i, d_i, h_i), {i})
    end
end
return G
```

---

dence based on index $i \in \mathbb{Z}$, then

$$s < s' \textit{ iff } \exists j \in \mathbb{Z} : \begin{cases} f_j(s) < f_j(s') \\ f_k(s) = f_k(s'), \quad \forall k \in \mathbb{Z} : 0 \leq k < j \end{cases}$$

Scoring metrics for each state s that we want to evaluate can then be computed in the $Next$ algorithm by considering the contents of the pending insertions and updating each objective function value differentially.

We use the following objective functions (considering a minimization direction):

- $f_1(s) = |B^s|$: we prefer states that opened fewer bins.
- $f_2(s) = -\text{avgvol}(s)$: we prefer states that have packed more average volume between bins.
- $f_3(s) = -\text{avgcageratio}(s)$: we prefer states that have better average cage ratio (eq. (3.1)) between bins.

4.3. Support Planes

Support Planes (SP) is a constructive heuristic for the 3D-BPPWS which is based on an underlying 2D-BPP heuristic. The latter is used to generate feasible insertions inside a bin starting from a set of items to pack. Since insertions must be feasible, SP maintains an internal data structure to facilitate feasibility checks. The idea at the base of SP is to build a solution to the 3D-BPP by filling 2D planes called *support planes*.

Each support plane is a tuple $(z, I_{support}, I_{upper})$ where

- z: is the height of the plane,
- $I_{support}$: the set of the items that can offer support to items placed on the plane,
- I_{upper}: the set of items that will be obstacles to potential new items placed on the plane.

Every item placed in the bin can either generate a new support plane or be part of the supporting items of other planes. Items placed above a particular plane, such that $z_i + h_i > z$, are considered obstacles and are added to the I_{upper} set. When creating new insertion, given a set of items to place I, SP selects the first feasible insertion starting from the lowest plane by using a modified version of the Extreme Point algorithm (Crainic et al. [2008]) that works in two dimensions. Once no more insertions can be made on the lowest available support plane, it is removed from the set of planes. Since insertions always happen in the lowest possible planes, the set of obstacles of those planes is composed of items that have only their top face above the z of the evaluated plane, such that $z_i \leq z < z_i + h_i$.

The modified Extreme Point (EP) heuristic evaluates the placement of rectangles in a plane based on a set of reference points with a best-fit approach. Each rectangle placement generates a new set of reference points which are usually introduced based on the projection of its corner points along the orthogonal axis of the plane. The corner points of an added rectangle r placed in (x_r, y_r) of dimensions (w_r, d_r) are the top left corner $(x_r, y_r + d_r)$ and the bottom right corner $(x_r + w_r, y_r)$. In our version of the algorithm, however, the corner points of each item are introduced without projecting them to increase the likelihood of generating placements that verify the support constraint. Placements follow a first-fit approach where the algorithm selects the first point closest to the origin where a rectangle can fit with or without rotations. In order to facilitate the evaluation of reference points with support, we also generate a reference point in the bottom left corner of each item that belongs to the set of supporting items $I_{support}$. When a reference point is used for a placement, it is then removed from the pool of reference points. Before evaluating placements, the item to place are ordered based on their area (this is meaningful only in multiple placements PM mode). New planes have always the origin of space $(0, 0)$ as a first reference point.

Since reference points are usually ordered based on the euclidean distance from the bottom left corner of the plane and the corner points are usually generated and projected towards the origin of each axis, the placements over one plane are usually biased towards the bottom left corner. To address the problem, whenever we are generating 2D placements, we evaluate four instances of EP where each one has a different coordinate change that moves the plane's origin to a different corner of the bin. This addition is based on similar approaches from the literature where it is used to distribute weight more uniformly across a surface (ex. Gajda et al. [2022]), and it was proved to yield better cage ratio results in our computational experiments.

The EP procedure is called for each item to pack on a given plane. In order to produce a valid insertion p, every item in I_p should not overlap with other items in I_p (as stated in definition 4.8). Since the AABB tree for a given state is shared by each evaluation of a possible insertion, it cannot be modified to account for temporary placements of items. This means that we need to keep a temporary AABB tree updated composed of the items that are part of the current insertion T'_p. We can then define a function that uses the temporary tree and the feasibility function defined in section 4.1.3 to ensure that we are producing a feasible insertion as eq. (4.4).

$$EPCanPack(i, I_{support}, T^s_{b_p}, T'_p) = IsFeasible(i, I_{support}, T^s_{b_p}) \land \neg AABBOverlaps(i, T'_p) \tag{4.4}$$

A graphical representation of a support plane is shown in fig. 4.1, with the reference points available. In fig. 4.2 the state of two extreme point instances for the bottom left and top left coordinate changes are shown. When a bin is opened, the only support plane available is the one on the ground. In the figure different coordinate changes are marked with different colors.

Given eq. (4.4) to check if a considered placement would lead to a feasible insertion, a set of items to pack I, a set s and a bin $b \in B^s$, the heuristic that will output the new best possible feasible insertion for the given set of items is outlined in Algo. 5.

Commit Extension We now describe an extension to *Commit* (Algo. 1) that updates the structures needed by SP.

When a plane is filled, new insertions become less likely to be feasible. To avoid evaluating planes where no insertion is possible we develop a mechanism to prune dead planes.

Since best insertions for a bin are always evaluated by considering lower planes first, if all the insertions in *Expand* (Algo. 3) happened over a z_{min}, then we can safely remove the

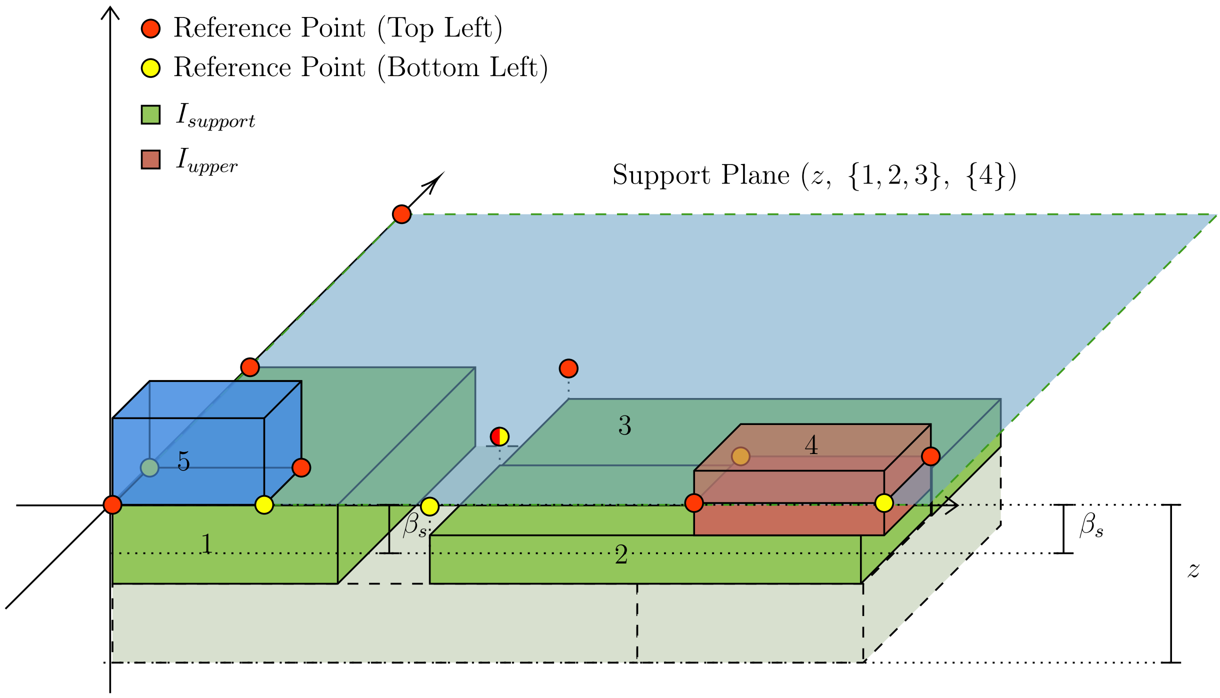


Figure 4.1: Representation of a generic support plane with a placed item

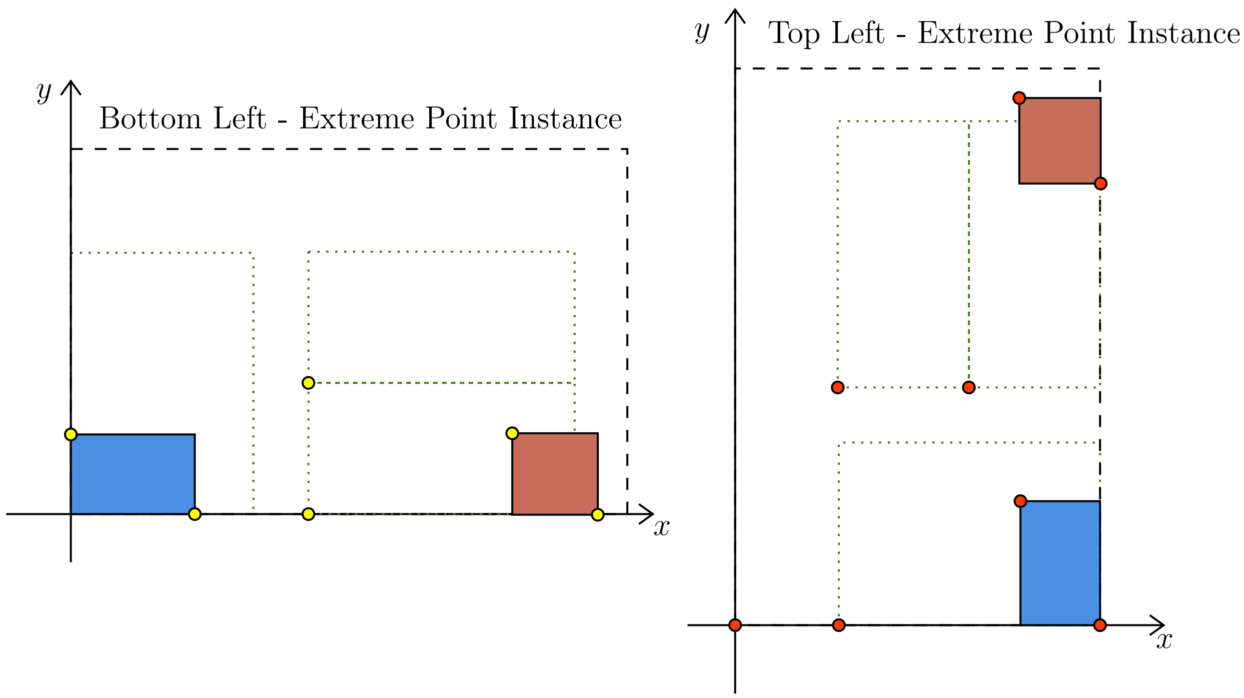


Figure 4.2: Extreme Point instances for some coordinate changes of fig. 4.1

Algorithm 5: SP Best Insertion

```
input  : s, b, I
output: p
forall (z, I_support, I_upper) ∈ Z_b^s do
    P ← ∅
    forall possible coordinate changes do
        p ← (z, ∅)
        T' ← empty AABB tree
        //Initialize reference points
        refPoints ← (0, 0)
        forall i ∈ I_support do
            refPoints ← refPoints ∪ {(x_i^s, y_i^s)}
        end
        forall i ∈ I_upper do
            refPoints ← refPoints ∪ {(x_i^s + w_i^s, y_i^s), (x_i^s, y_i^s + d_i^s)}
        end
        sort(refPoints) // Based on euclidean distance from (0, 0)
        //Create a feasible insertion for the given items
        forall i ∈ I do
            //Evaluate first possible placement
            forall (x, y) ∈ refPoints do
                (x_i, y_i, z_i) ← (x, y, z)
                if EPCanPlace(i, T_b^s, T') then
                    EPInsertRect(p, i, T', refPoints) // Algo. 6
                    break
                end
                (w_i^s, d_i^s) ← (d_i^s, w_i^s) //Try rotating i
                if EPCanPlace(i, T_b^s, T') then
                    EPInsertRect(p, i, T', refPoints) // Algo. 6
                    break
                end
                (w_i^s, d_i^s) ← (d_i^s, w_i^s) //Restore original rotation
            end
        end
        if p ≠ (z, ∅) then
            P ← P ∪ {p}
        end
    end
    sort(P) //Sorted as in section 4.3.1
    if P ≠ ∅ then
        return first element of P
    end
end
return none
```

Algorithm 6: EP Insert Rect

input : $p, i, T, refPoints$
$refPoints \leftarrow refPoints \setminus \{(x_i, y_i)\}$
$refPoints \leftarrow refPoints \cup \{(x_i + w_i, y_i), (x_i, y_i + d_i)\}$
$sort(refPoints)$ `// Based on euclidean distance from` $(0, 0)$
$p.I \leftarrow p.I \cup \{i\}$
$AABBInsert(i, T')$ `//section 4.1.1`
return

opened planes with $z < z_{min}$ for that bin. Let us introduce a z^s_{min} variable that is updated during the *Expand* phase with the minimum z of all the insertions on bin b. Once the best states are computed and *Commit* is called, we can use z^s_{min} to prune planes in each $b \in B^s$. Other operations are also necessary in the *Commit* algorithm to allow SP to update its data structures accordingly to the insertion.

Let s be a state and let p be an insertion where each packed item $i \in I_p$ in bin b_p has z^s_i within tolerance of z. The algorithm which updates the structures for a given bin b is represented by algorithm 7. This new algorithm can be used as the last step of the

Commit algorithm for each $b \in B^{s'}$.

Algorithm 7: SP Apply and Filter

input : s, p, z, β_s

output: s

```
//Filter dead planes
```
$Z^s_{b_p} \leftarrow Z^s_{b_p} \setminus \{(z', I_{support}, I_{upper}) \in Z^s_{b_p} \mid z' < z^s_{min}\}$
```
//Apply insertion
```
forall $i \in I_p$ **do**

  $T^s_{b_p} \leftarrow InsertAABB(i, T^s_{b_p})$

  $generate \leftarrow true$

  forall $(z', I_{support}, I_{upper}) \in Z^s_{b_p}$ **do**

    `//Based on the distance from the top of the item`

    $dz \leftarrow z' - (z^s_i + h_i)$

    if $0 \leq dz \leq \beta_s$ **then**

      $generate \leftarrow false$

      $I_{support} \leftarrow I_{support} \cup i$

    end

    else if $dz < 0$ **then**

      $I_{upper} \leftarrow I_{upper} \cup i$

    end

  end

  if $generate$ **then**

    $Z^s_{b_p} \leftarrow Z^s_{b_p} \cup (z^s_i + h_i, \{i\}, \emptyset)$

  end

end

return s

4.3.1. Sorting Insertions

Similarly to the sorting of states (section 4.2.1), an ordering function is also needed to evaluate different insertions for the same set of items. Given the lexicographic ordering formulation in definition 4.11, a few new functions can be calculated and stored inside an insertion to help in the evaluation. Given T as the AABB Tree that represents the bin where the insertion is going to happen, and given one of the inserted items $i \in I_p$, we define functions that use the tree to calculate useful metrics:

- $CloseItems(i, T)$: which returns the number of packed items that are close to i,

- $CloseSameHeight(i, T)$: which returns the number of packed items in the tree that are close to i and with its same height,
- $CloseSameShape(i, T)$: which returns the number of packed items in the tree that are close to i and with its same shape,
- $TotalSupportedArea(i, T)$: which returns the total base area of i which is supported by other items.

We can then sort insertions p, given T as the AABB tree of the bin where the insertion will happen, with a lexicographic ordering as follows:

- $f_1(p) = -\sum_{i \in I_p} CloseSameShape(i, T) - |I_p|$: maximize number of items inserted (of the same shape) that are close to already packed items of the same shape,
- $f_2(p) = -\sum_{i \in I_p} (w_i^s d_i^s + w_i^s d_i^s h_i^s)$: maximize the sum of the area and volume of each packed item,
- $f_3(p) = \max_{i \in I_p}(z_i^s + h_i)$: minimize the maximum height of the inserted items,
- $f_4(p) = \sum_{i \in I_p} TotalSupportedArea(i, T)$: minimize the support area available to the inserted items,
- $f_5(p) = -\sum_{i \in I_p} CloseSameHeight(i, T) - |I_p|$: maximize the number of items inserted (of the same height) that are close to already packed items of the same height,
- $f_5(p) = -\sum_{i \in I_p} CloseItems(i, T) - |I_p|$: maximize the number of items inserted that are close to already packed items.

It is noted that preferring feasible insertions that minimize the supported area of each item as in f_4 is inspired by other works on spacing from the literature. As shown in Elhedhli et al. [2019], overly satisfying the support constraint can lead to unbalanced bins. Minimizing the supported area of each item leads to minimizing the perimeter of overlap between items which in turn results in more balanced bins that have better spacing between items.

5 | Computational results

In this chapter, in section 5.1, we evaluate the proposed heuristic against the MILP model (3.1), and in section 5.2 against other heuristics from the literature. We then show the effectiveness of our approach for our case study in section 5.3. All the tests were run on a desktop computer with an AMD Ryzen-7 5800x processor with 8 cores at 3.8 GHz and 32GB of DDR4 system RAM with Windows 10. The algorithm was implemented in Java 11, and the model was run using the python APIs from CPLEX Optimization Studio 22.1.0. In every test, CPLEX was used with a maximum runtime of 1 hour. Each evaluation against the heuristic lists both operational modes described in section 4.2 listed as "PM" and "PS". All the instances used in each section of this chapter are available at `https://github.com/artumino/BinPackingThesis/tree/main/tests/instances`. Out of the 100 instances used for our case study experiments, only 80 were freely sharable with the generation procedure also described in section 5.3.

5.1. Model validation

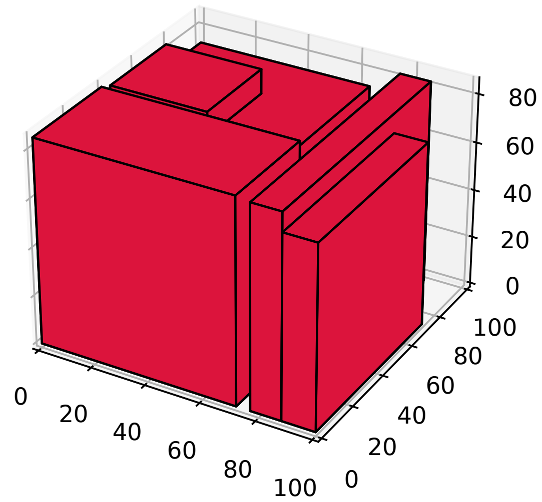
We compared our heuristic to the proposed MILP model of section 3.1 with a single bin and with no limit on the height of the bin (also referred to as the 3D strip packing problem). The heuristic was configured to run without vertex support, using only area support rules for its feasibility checks, and k was set to 200. The configured parameters for the test were $\alpha_s = 0.7$, $\beta_s = 5$, and the discretization unit for the model was $\delta = 10$. Tests were run on the first generated instance of the class 1 problems from [Martello et al., 2000] which we used for literature tests. These classes of problem have a bin base of 100×100. The test were run with an iterative approach by selecting only a limited amount of items from the selected instance, starting from the first item and increasing the number of items to pack by one at each iteration. The problem created with each iteration was saved as a test instance in the same format as the one used for literature tests. All the generated instances are available at `https://github.com/artumino/BinPackingThesis/tree/main/tests/instances/model`. A python script then loaded each generated instance sequentially and evaluated the solutions from the MILP problem and the heuristic. Each instance was run with a time limit of 1 hour. All instances with a MIP gap lower than 4% were accepted. All instances resolved to optimality, except for instance 8, which terminated with a mip gap of 0.02%.

Table 5.1 shows the obtained $z_{\max}$ value of the heuristic and the MILP solution, the runtime in seconds, and the number of items. Since the underlying problem is NP-Hard, it is shown that starting from instances of size bigger than 8 items; the MILP model becomes too slow for practical use while our heuristic maintains negligible execution time. Due to discretization errors, some of the model instances gave solutions that didn't have the expected amount of support and were marked with an asterisk. The solution to instance number 5 and instance number 7 is also shown in fig. 5.1.

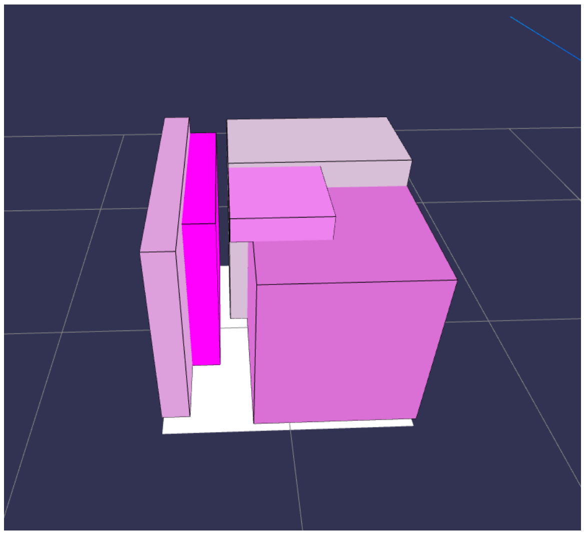
Table 5.1: Comparison with MILP model on limited set of boxes

	MILP Model			**PM**		**PS**	
n	**Max Z**	**TT(s)**	**Gap(%)**	**Max Z**	**TT(s)**	**Max Z**	**TT(s)**
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	0.02	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

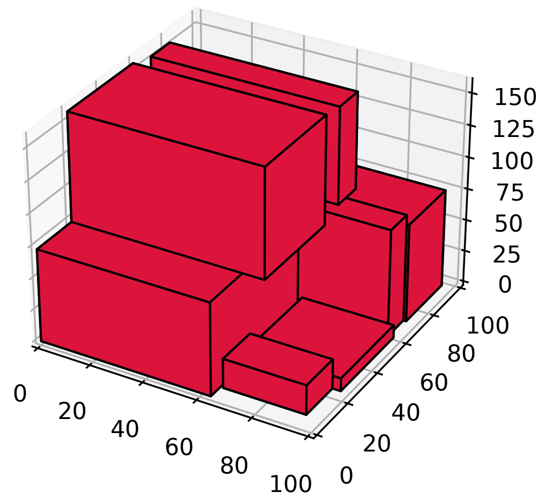
* Some boxes had lower support than expected due to discretization errors within the $0.65 \leq \alpha_s \leq 0.7$ range.



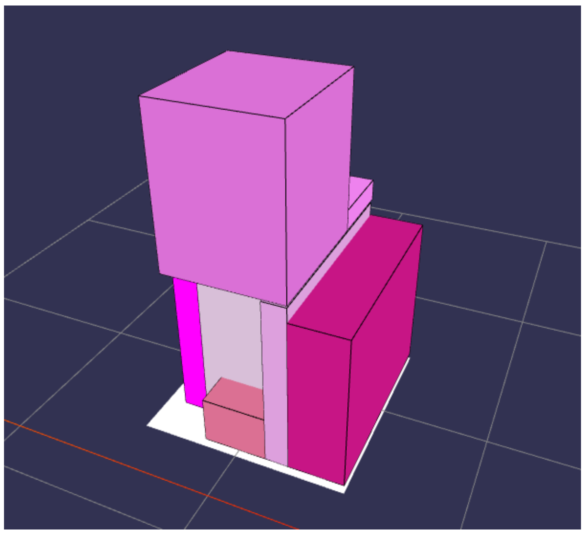
(a) MILP, Instance 5



(b) Heuristic $k = 200$, Instance 5



(c) MILP, Instance 7



(d) Heuristic $k = 200$, Instance 7

Figure 5.1: Graphical comparison between solutions from the heuristic and from the MILP model

5.2. Literature results

The heuristic was also evaluated against instances from the literature defined by [Martello et al., 2000]. Since these instances were designed for heuristics without the vertical support constraint and orthogonal rotations, we ran the experiments with a relaxed version of our heuristic. The heuristic was configured to ignore the support constraint with $\alpha_s = 0$ and $\beta_s = 1$. We also disabled orthogonal rotations and stopped scoring insertions based on the support area available (as described in section 4.3.1).

The literature instances are divided into classes from 1 to 8, with each class having a different bin size and various distributions of types of items. Instances were generated with the C++ instance generator provided by [Martello et al., 2000] at `http://hjemmesider.diku.dk/~pisinger/new3dbpp/test3dbpp.c` which allows the generation of problem instances with a given problem class and number of items to use. We generated 10 instances for each pair of problem class and number of items $n \in \{50, 100, 150, 200\}$ for a total of 320 instances.

In table 5.2 we compare the average number of opened bins across 10 instances for each problem class and n number of items combinations. The results are then compared to the most effective methods from the literature ordered by publishing date and listed as TS3 [Lodi et al., 2002], GLS [Faroe et al., 2003], GASP [Crainic et al., 2009], GVND [Parreño et al., 2010], EHGH2 [Hifi et al., 2014], BRKGA [Gonçalves and Resende, 2013], BRKGA-VD [Zudio et al., 2018]. It is noted that values for other heuristics are reported as in their publications, and our generated instances weren't the same ones which, as indicated in [Hifi et al., 2014], could have different optimal values. The best values of all the heuristics are marked in bold. Best scoring values across different configurations of our heuristic are marked in italic instead. Results show an average gap of 4.1% compared to the average value across the other heuristics and an average gap of 5.32% with respect to the best performing one.

In table 5.3 we give an approximate comparison between the average execution time of our heuristic with respect to BRKGA-VD. Execution times for BRKGA-VD were normalized by comparing directly the floating-point operations per second of the processors used, which resulted in dividing BRKGA-VD execution times by a normalization term of 9.3. The values presented are the times averaged across 8 classes of problems divided according to the size of the instance and the heuristic configuration. In the last column, we also included the average gap of each configuration of the heuristic with respect to the values of BRKGA-VD.

Table 5.2: Literature results for $k = 50$

Class	n	PM $k = 50$	PS $k = 50$	TS3	GLS	GASP	EHGH2	GVN	BRKGA	BRKGA-VD
1	50	14.10	*14*	**13.4**	**13.4**	**13.4**	13.8	**13.4**	**13.4**	**13.4**
	100	28	*27.7*	**26.6**	**26.6**	26.9	27.6	**26.6**	**26.6**	**26.6**
	150	38.4	*37.9*	36.7	37	37	39.8	36.4	36.4	**36.3**
	200	53	52.7	51.2	51.2	51.6	**50.6**	50.9	50.8	50.8
2	50	*14.6*	14.8	**13.8**	-	-	-	**13.8**	**13.8**	**13.8**
	100	26.6	26.7	25.7	-	-	-	25.7	25.6	**25.5**
	150	38.3	39	37.2	-	-	-	36.9	**36.6**	**36.6**
	200	*51*	51.7	50.1	-	-	-	**49.4**	**49.4**	**49.4**
3	50	13.9	13.9	**13.3**	-	-	-	**13.3**	**13.3**	**13.3**
	100	27.8	27.3	26	-	-	-	26	**25.9**	**25.9**
	150	39.2	39	37.7	-	-	-	37.6	**37.5**	**37.5**
	200	51.8	*51.2*	50.5	-	-	-	50	**49.8**	**49.8**
4	50	*29.7*	*29.7*	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**
	100	*59.2*	*59.2*	59	59	59	59.5	59	59	**58.9**
	150	*87.6*	87.7	**86.8**	**86.8**	**86.8**	90.4	**86.8**	**86.8**	**86.8**
	200	*119.5*	*119.5*	**118.8**	119	**118.8**	119	**118.8**	**118.8**	**118.8**
5	50	*8.6*	*8.6*	8.4	8.3	8.4	**7.9**	8.3	8.3	8.3
	100	16	*15.6*	15	15.1	15.1	**14.6**	15	15	15
	150	21.7	21.4	20.4	20.2	20.6	21.5	20.4	20.1	**19.9**
	200	29	28.4	27.6	27.2	27.7	29.6	**27.1**	**27.1**	**27.1**
6	50	*10*	10.3	9.9	9.8	9.9	11.8	9.8	**9.7**	**9.7**
	100	19.8	19.7	19.1	19.1	19.1	19.2	19	**18.9**	**18.9**
	150	30.3	30.2	29.4	29.4	29.5	29.8	29.2	**29**	**29**
	200	38.9	*38.5*	37.7	37.7	38	38.7	37.4	**37.3**	**37.3**
7	50	7.8	*7.6*	7.5	**7.4**	7.5	**7.4**	**7.4**	**7.4**	**7.4**
	100	*13.2*	*13.2*	12.5	12.3	12.7	13.5	12.5	**12.2**	**12.2**
	150	17.1	16.8	16.1	15.8	16.6	18.2	16	15.3	**15.2**
	200	24.9	24.7	23.9	23.5	24.2	24.1	23.5	**23.4**	**23.4**
8	50	9.9	9.7	9.3	**9.2**	9.3	9.4	**9.2**	**9.2**	**9.2**
	100	*19.6*	20	18.9	18.9	19	18.9	18.9	18.9	**18.8**
	150	25.7	25.8	24.1	23.9	24.8	26	24.1	**23.6**	**23.6**
	200	31.6	*31.2*	30.3	29.9	31.1	35.8	29.8	**29.3**	**29.3**

Table 5.3: Average execution time of literature results with bin gap

Heuristic		**Execution Time (s)**				**Bin Gap %**
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.03	0.11	0.28	0.54	5.82
	$k = 5$	0.08	0.38	1.00	2.09	5.56
	$k = 10$	0.15	0.73	1.93	4.00	5.54
	$k = 20$	0.29	1.40	3.77	7.71	5.30
	$k = 50$	0.70	3.50	9.39	19.59	5.19
PS	$k = 1$	0.05	0.18	0.50	1.05	5.61
	$k = 5$	0.12	0.72	2.10	4.62	5.26
	$k = 10$	0.23	1.38	4.11	8.95	5.19
	$k = 20$	0.46	2.67	8.21	17.64	4.98
	$k = 50$	1.12	6.45	20.39	43.50	4.75
BRKGA-VD		1.85	8.69	20.53	39.85	0.00

5.3. Case study results

Case study experiments were conducted on a series of problem instances that were divided between 20 real-world instances and 80 generated instances composed of items sampled from a population of real-world products. Each instance was anonymized and converted to a format similar to the one used for the literature tests thanks to a Rust program available at `https://github.com/artumino/BinPackingThesis/tree/main/additional/testConverter`. Support parameters for the heuristic were set to $\alpha_s = 0.7$ and $\beta_s = 10$ with both area and vertex support enabled. All dimensions of the bin, items, and tolerances are assumed to be in millimeters. Different values of $k \in \{1, 5, 10, 20, 50, 100, 200\}$ were tested as well as both placement modes.

Each generated instance is composed of a random number of n items sampled from a given range of possible instance sizes. All generated instances had a bin of standard size $800 \times 1200 \times 2000$. We identified four ranges of interest and generated 20 instances for each range as follows:

- **Class 1-20**: a class of instances with the target sizes for our case-study $n \in [70, 100]$
- **Class 21-40**: a class of small sized instances with number of items $n \in [50, 70]$
- **Class 41-60**: a class of medium sized instances with number of items $n \in [70, 120]$
- **Class 61-80**: a class of big instances with number of items $n \in [120, 200]$

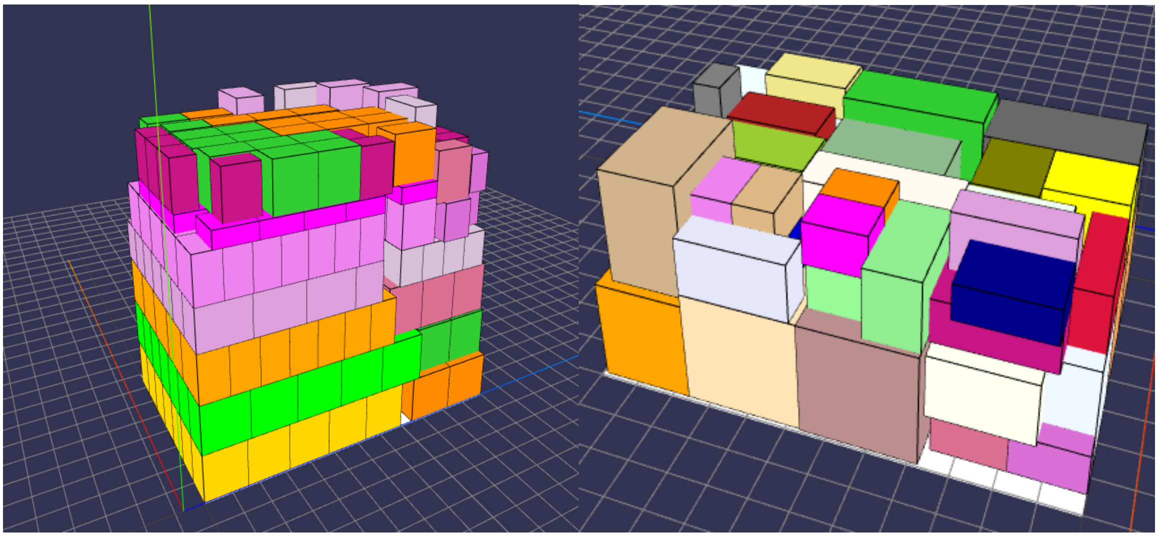
Given an input n (the size of the test instance), the generation procedure uniformly sampled an item type from a population of real-world products. The quantity of items of that type to add to the test instance was then sampled from a normal distribution $\mathcal{N}(\mu = 4.6, \sigma = 1.8)$ with parameters calculated from the real-world instances. The sampled quantity was then floored to be an integer value and clamped to avoid generating more items than n. This uniform sampling of item types was done until the instance was composed of n items.

Real-world instances are listed as **Class 81-100** and have a variable number of items between $[25, 345]$, a variable bin size (although similar to the one used for the generated instances), and a variable number of items of the same type. Some instances were homogeneous with only a few unique items, and some were heterogeneous with every item of a different type. An example of real-world instances is shown in fig. 5.2 where items of the same shape are marked with the same color.

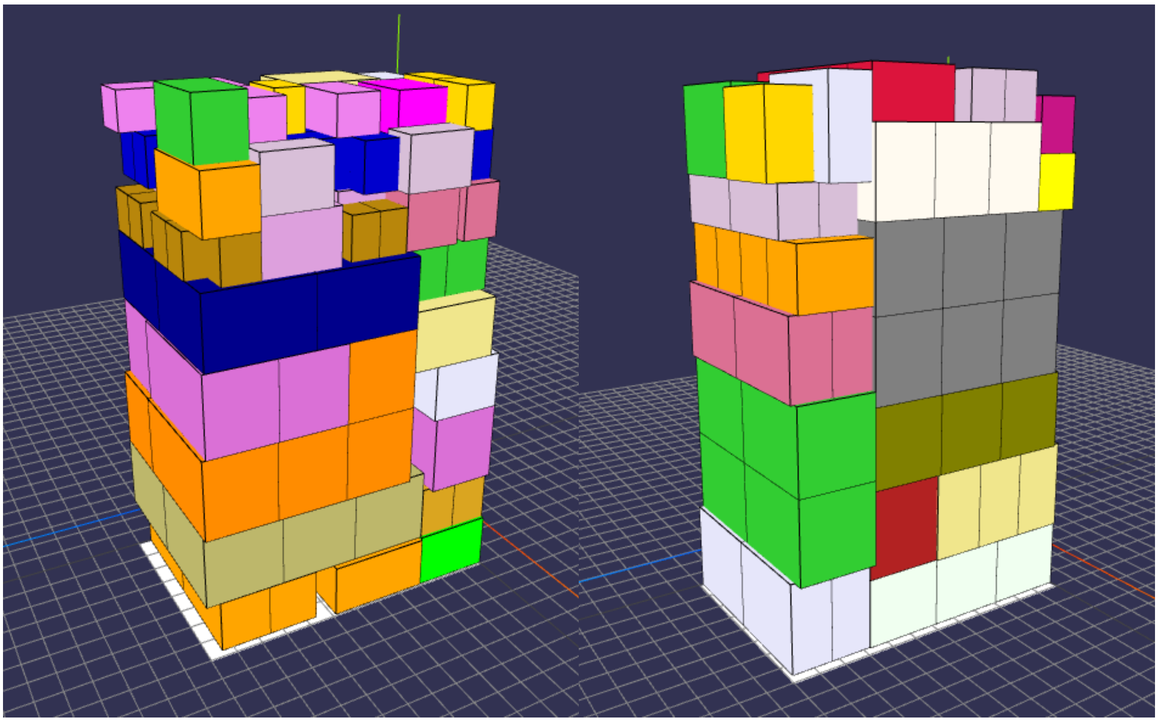
Table 5.3 shows the average results over the 20 instances per class, divided by each configuration of the heuristic with different values of k. The results shown include the total execution time in milliseconds (TT), the number of opened bins (B), and the average cage ratio between the opened bins (CR). It is clear that although the "PS" method had better results when dealing with a relaxed version of the problem, grouping items by type shows considerable improvements under all measured metrics when taking vertical support into account. Most of the configurations lead to an average cage ratio of more than 70%, which was the target value for our case study. It is also possible to see that increasing the value of k improves the quality of the solutions, on average, at the expense of a higher execution time. By doing a case-by-case analysis of each experiment, we discovered that increasing k can temporarily worsen the solution in some instances. A further study of the problematic instances highlighted that the current greedy scoring mechanism of the states leads to cutting out good solutions too early. Further improvements are proposed in chapter 6.

Table 5.4: Summary of case study tests

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
Class 1-20	$k = 1$	187.25	1.15	64.10	54.95	1.05	**70.69**
	$k = 5$	489.40	1.05	70.38	111.75	1.00	**75.36**
	$k = 10$	861.30	1.05	71.94	182.20	1.00	**75.77**
	$k = 20$	1,588.15	1.05	72.04	308.45	1.00	**76.60**
	$k = 50$	3,896.40	1.05	73.07	690.80	1.00	**76.95**
	$k = 100$	7,789.90	1.00	75.45	1,204.35	1.00	**78.46**
	$k = 200$	15,817.20	1.05	74.99	2,192.75	1.00	**78.27**
Class 21-40 $n = [50, 70]$	$k = 1$	50.90	1.00	68.21	17.80	1.00	**73.66**
	$k = 5$	138.40	1.00	71.92	39.20	1.00	**74.78**
	$k = 10$	253.10	1.00	73.15	74.95	1.00	**75.28**
	$k = 20$	483.85	1.00	73.86	124.30	1.00	**76.46**
	$k = 50$	1,193.55	1.00	74.77	288.50	1.00	**77.02**
	$k = 100$	2,358.50	1.00	75.08	535.30	1.00	**77.11**
	$k = 200$	4,769.85	1.00	76.69	1,033.00	1.00	**78.64**
Class 41-60 $n = [70, 120]$	$k = 1$	292.35	1.30	65.62	60.55	1.25	**71.34**
	$k = 5$	1,025.65	1.30	67.97	172.35	1.30	**72.53**
	$k = 10$	1,910.60	1.30	68.46	304.25	1.25	**72.04**
	$k = 20$	3,666.40	1.30	68.68	571.90	1.25	**74.01**
	$k = 50$	7,649.95	1.25	71.32	1,152.40	1.25	**75.25**
	$k = 100$	15,848.15	1.25	72.90	1,956.55	1.20	**75.67**
	$k = 200$	32,420.40	1.25	73.29	3,472.50	1.20	**76.10**
Class 61-80 $n = [120, 200]$	$k = 1$	1,371.00	2.20	64.68	158.00	2.05	**69.11**
	$k = 5$	5,751.95	2.15	66.66	531.80	1.95	**71.31**
	$k = 10$	9,040.85	2.05	68.56	1,033.15	1.90	**72.69**
	$k = 20$	19,116.60	2.15	67.81	1,881.70	1.90	**73.84**
	$k = 50$	52,937.40	2.05	69.94	3,744.70	2.00	**71.25**
	$k = 100$	98,271.55	2.10	70.04	7,010.65	1.90	**73.80**
	$k = 200$	170,191.55	2.00	71.15	13,544.15	1.90	**75.01**
Class 81-100	$k = 1$	217.85	1.20	66.74	34.60	1.20	**68.68**
	$k = 5$	582.30	1.20	69.03	71.00	1.20	**71.41**
	$k = 10$	1,071.75	1.20	69.65	129.95	1.20	**72.00**
	$k = 20$	2,013.95	1.20	71.52	218.40	1.20	**71.97**
	$k = 50$	5,338.20	1.20	71.44	523.40	1.20	**72.57**
	$k = 100$	10,402.95	1.20	**72.68**	995.00	1.20	71.74
	$k = 200$	21,525.50	1.20	73.30	2,086.50	1.15	**73.95**
Global Avg	$k = 1$	423.87	1.37	65.87	65.18	1.31	**70.70**
	$k = 5$	1,597.54	1.34	69.19	185.22	1.29	**73.08**
	$k = 10$	2,627.52	1.32	70.35	344.90	1.27	**73.56**
	$k = 20$	5,373.79	1.34	70.78	620.95	1.27	**74.57**
	$k = 50$	14,203.10	1.31	72.11	1,279.96	1.29	**74.61**
	$k = 100$	26,934.21	1.31	73.23	2,340.37	1.26	**75.36**
	$k = 200$	48,944.90	1.30	73.89	4,465.78	1.25	**76.39**



(a) Instance 95 (b) Instance 82



(c) Instance 56 (d) Instance 66, Bin 1

Figure 5.2: Solutions of case study tests with the "PM" placement and $k = 200$

6 | Conclusions and future developments

In this thesis we studied the problem of three-dimensional single bin-size bin packing with support by providing two heuristics. The first contribution is a constructive heuristic that uses a modified version of the first-fit two-dimensional extreme points heuristic of Crainic et al. [2008] that includes the notion of area and vertex support. The heuristic builds solutions to the single bin 3D-BPP by filling planes called support planes that are added to the solution based on the inserted items, these structures are then used to facilitate calculations involving the support constraint. Although layers aren't explicitly built, we evaluate insertions in two different placement modes which allows for placements of groups of similar objects. We also propose a beam-search algorithm which uses multiple instances of the constructive heuristics and evaluates different sequences of insertions to expand the solution space with a hashing mechanism to avoid same packings.

The resulting heuristic was then validated against small instances solved with our MILP formulation to optimality. Since the underlying problem is NP-Hard the MILP solve time quickly We then evaluated the results of a relaxed version of our heuristic against other heuristics from the literature on classical benchmark instances from Martello et al. [2000].

Further developments in

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A | Appendix A

Table A.1: Case study results 1-5

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
1	$k=1$	403	1	69.54	218	1	69.82
	$k=5$	384	1	70.9	157	1	74.64
	$k=10$	502	1	71.47	151	1	74.64
	$k=20$	786	1	71.47	187	1	73.33
	$k=50$	1732	1	71.47	357	1	74.26
	$k=100$	3524	1	71.47	613	1	76.54
	$k=200$	6892	1	74.71	1020	1	74.64
2	$k=1$	266	2	48.2	67	1	77.19
	$k=5$	835	1	78.58	196	1	84.69
	$k=10$	1537	1	78.58	311	1	86.65
	$k=20$	2045	1	83.22	607	1	87.59
	$k=50$	5233	1	83.22	1706	1	87.84
	$k=100$	11422	1	83.22	3226	1	86.94
	$k=200$	22911	1	83.22	3860	1	85.87
3	$k=1$	169	1	73.36	62	1	65.48
	$k=5$	335	1	73.36	104	1	73.31
	$k=10$	532	1	73.36	141	1	72.73
	$k=20$	1003	1	73.36	245	1	74.86
	$k=50$	2621	1	73.46	457	1	74.51
	$k=100$	5209	1	73.46	897	1	75.02
	$k=200$	10781	1	74.2	1676	1	78.9
4	$k=1$	384	1	53.7	57	1	79.2
	$k=5$	1048	1	59.27	153	1	79.91
	$k=10$	1934	1	59.27	203	1	76.37
	$k=20$	3754	1	59.27	313	1	79.44
	$k=50$	9266	1	65.04	754	1	82.18
	$k=100$	18445	1	72.44	1467	1	82.18
	$k=200$	36636	1	72.44	2956	1	82.18
5	$k=1$	52	1	67.48	25	1	74.44
	$k=5$	192	1	73.22	75	1	76.16
	$k=10$	324	1	73.22	104	1	69.76
	$k=20$	641	1	73.22	144	1	69.18
	$k=50$	1613	1	73.22	255	1	68.65
	$k=100$	3466	1	73.22	518	1	68.65
	$k=200$	7149	1	73.22	1050	1	68.74

Table A.2: Case study results 6-10

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
6	$k = 1$	357	2	35.53	76	1	78.78
	$k = 5$	939	1	65.88	196	1	79.74
	$k = 10$	1638	1	74.23	419	1	82.46
	$k = 20$	3257	1	73.74	624	1	80.73
	$k = 50$	8533	1	73.74	1295	1	80.52
	$k = 100$	16594	1	75.36	2019	1	78.53
	$k = 200$	33658	1	75.36	3925	1	77.35
7	$k = 1$	308	1	62.38	32	1	68.32
	$k = 5$	774	1	62.38	98	1	71.36
	$k = 10$	1052	1	69.06	148	1	81.03
	$k = 20$	2003	1	69.06	299	1	82.35
	$k = 50$	4828	1	71.32	697	1	79.09
	$k = 100$	10009	1	71.32	1138	1	82.12
	$k = 200$	19931	1	71.32	2289	1	82.12
8	$k = 1$	50	1	74.2	36	1	79.27
	$k = 5$	142	1	74.2	46	1	78.78
	$k = 10$	240	1	76.51	66	1	83.85
	$k = 20$	472	1	80.77	126	1	83.85
	$k = 50$	1196	1	82.12	317	1	83.85
	$k = 100$	2410	1	82.12	617	1	83.85
	$k = 200$	4844	1	82.12	1212	1	83.85
9	$k = 1$	188	1	67.28	41	1	69.6
	$k = 5$	580	1	74.36	135	1	73.26
	$k = 10$	989	1	74.36	319	1	81.8
	$k = 20$	1795	1	75.21	364	1	77.87
	$k = 50$	4573	1	78.8	1377	1	80.34
	$k = 100$	8641	1	78.8	1557	1	76.19
	$k = 200$	18028	1	78.8	3058	1	76.07
10	$k = 1$	37	1	75.91	24	1	72.18
	$k = 5$	229	1	76.34	65	1	74.73
	$k = 10$	321	1	76.34	102	1	74.73
	$k = 20$	645	1	76.34	186	1	74.73
	$k = 50$	1641	1	76.34	413	1	80.24
	$k = 100$	3177	1	76.34	685	1	79.76
	$k = 200$	6562	1	76.34	1376	1	79.76

Table A.3: Case study results 11-15

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
11	$k = 1$	83	1	66.82	26	1	69.88
	$k = 5$	227	1	66.82	88	1	73.04
	$k = 10$	437	1	73.04	94	1	73.04
	$k = 20$	901	1	73.04	182	1	73.73
	$k = 50$	2163	1	74.9	374	1	72.55
	$k = 100$	4271	1	75.09	656	1	70.51
	$k = 200$	8965	1	75.09	1338	1	73.61
12	$k = 1$	290	2	54.85	41	2	38.76
	$k = 5$	905	2	49.36	149	1	79.67
	$k = 10$	1746	2	49.36	153	1	77.74
	$k = 20$	3048	2	40.42	304	1	76.43
	$k = 50$	6768	2	40.91	526	1	77.53
	$k = 100$	13867	1	73.25	1054	1	79.32
	$k = 200$	29292	2	42.83	2136	1	79.32
13	$k = 1$	161	1	53.61	23	1	68.76
	$k = 5$	333	1	69.77	62	1	72.32
	$k = 10$	585	1	69.77	120	1	73.12
	$k = 20$	1140	1	70.64	156	1	64.05
	$k = 50$	2821	1	70.64	420	1	65.88
	$k = 100$	5610	1	70.64	833	1	73.76
	$k = 200$	11427	1	75.46	1119	1	72.44
14	$k = 1$	209	1	66.77	30	1	70.43
	$k = 5$	512	1	66.77	71	1	69.51
	$k = 10$	959	1	71.72	229	1	80.74
	$k = 20$	1773	1	71.72	442	1	73.78
	$k = 50$	4093	1	72	993	1	78.03
	$k = 100$	8228	1	72	1915	1	77.62
	$k = 200$	16602	1	75.58	2885	1	74.15
15	$k = 1$	216	1	79.47	74	1	78.65
	$k = 5$	710	1	79.47	161	1	66.78
	$k = 10$	1415	1	80.62	245	1	70.17
	$k = 20$	2661	1	80.62	365	1	77.37
	$k = 50$	6673	1	80.62	865	1	80.52
	$k = 100$	12879	1	80.62	1530	1	85.66
	$k = 200$	25418	1	80.62	3552	1	85.66

Table A.4: Case study results 16-20

Instance		**PS**			**PM**		
		TT (ms)	B	CR	TT (ms)	B	CR
16	$k = 1$	114	1	68.8	125	1	80.6
	$k = 5$	471	1	71.42	139	1	76.27
	$k = 10$	808	1	71.42	265	1	76.27
	$k = 20$	1529	1	72.13	473	1	78.77
	$k = 50$	3901	1	72.92	1081	1	78.77
	$k = 100$	7624	1	76.33	1654	1	79.27
	$k = 200$	15602	1	76.33	3217	1	78.91
17	$k = 1$	98	1	71.2	27	1	73.37
	$k = 5$	263	1	77.41	61	1	71.09
	$k = 10$	535	1	77.41	148	1	72.96
	$k = 20$	1014	1	77.41	276	1	76
	$k = 50$	2540	1	78.54	616	1	75.31
	$k = 100$	4890	1	78.54	989	1	79.99
	$k = 200$	10395	1	78.54	1790	1	79.92
18	$k = 1$	108	1	60.55	36	1	69.39
	$k = 5$	244	1	75.18	59	1	80.2
	$k = 10$	434	1	75.18	127	1	78
	$k = 20$	801	1	75.18	204	1	78
	$k = 50$	1957	1	77.61	376	1	78.85
	$k = 100$	3807	1	77.61	796	1	78.85
	$k = 200$	7824	1	80	1575	1	77.54
19	$k = 1$	113	1	67.04	52	1	66.58
	$k = 5$	330	1	77.57	133	1	77.82
	$k = 10$	623	1	77.57	172	1	59.29
	$k = 20$	1289	1	77.57	435	1	77.44
	$k = 50$	2977	1	77.57	589	1	74.24
	$k = 100$	6064	1	77.57	1235	1	83.16
	$k = 200$	12258	1	78.13	2461	1	83.16
20	$k = 1$	139	1	65.36	27	1	63
	$k = 5$	335	1	65.36	87	1	73.91
	$k = 10$	615	1	66.36	127	1	70.1
	$k = 20$	1206	1	66.36	237	1	72.51
	$k = 50$	2799	1	66.92	348	1	65.81
	$k = 100$	5661	1	69.53	688	1	71.21
	$k = 200$	11169	1	75.42	1360	1	71.21

Table A.5: Case study results 21-25

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
21	$k = 1$	33	1	67.44	18	1	77.23
	$k = 5$	97	1	75.35	38	1	71.08
	$k = 10$	166	1	75.35	60	1	72.69
	$k = 20$	298	1	76.28	108	1	71.65
	$k = 50$	741	1	76.28	173	1	69.42
	$k = 100$	1399	1	76.28	349	1	69.42
	$k = 200$	2870	1	76.28	692	1	71.4
22	$k = 1$	29	1	72.11	9	1	72.94
	$k = 5$	69	1	73.29	26	1	74.94
	$k = 10$	141	1	74.21	50	1	74.14
	$k = 20$	277	1	74.94	80	1	74.65
	$k = 50$	706	1	74.94	184	1	75.91
	$k = 100$	1385	1	78.81	354	1	75.91
	$k = 200$	2802	1	78.81	716	1	75.91
23	$k = 1$	34	1	63.84	13	1	81.33
	$k = 5$	94	1	73.15	31	1	78.01
	$k = 10$	173	1	75.99	57	1	77.42
	$k = 20$	329	1	75.99	84	1	81.09
	$k = 50$	808	1	76.28	211	1	83.06
	$k = 100$	1594	1	76.28	381	1	83.06
	$k = 200$	3164	1	79.14	743	1	84.51
24	$k = 1$	26	1	73.86	8	1	73.12
	$k = 5$	65	1	75.64	23	1	73.12
	$k = 10$	110	1	79.19	40	1	78.76
	$k = 20$	207	1	79.45	77	1	77
	$k = 50$	511	1	81.14	173	1	70.35
	$k = 100$	1076	1	81.14	349	1	70.35
	$k = 200$	2148	1	81.14	682	1	79.63
25	$k = 1$	82	1	70.7	37	1	71.53
	$k = 5$	229	1	70.7	99	1	69.62
	$k = 10$	431	1	70.7	181	1	69.62
	$k = 20$	816	1	70.7	283	1	78.27
	$k = 50$	2005	1	70.7	462	1	75.37
	$k = 100$	4011	1	71.74	933	1	75.37
	$k = 200$	8134	1	72.09	1759	1	75.37

Table A.6: Case study results 26-30

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
26	$k = 1$	35	1	71.69	16	1	73.01
	$k = 5$	102	1	75.92	26	1	77.99
	$k = 10$	194	1	75.92	49	1	76.17
	$k = 20$	400	1	75.92	68	1	72.31
	$k = 50$	1008	1	75.92	161	1	73.89
	$k = 100$	2126	1	76.05	327	1	73.89
	$k = 200$	4295	1	76.05	644	1	73.89
27	$k = 1$	63	1	66.47	22	1	76.56
	$k = 5$	164	1	70.9	38	1	77.73
	$k = 10$	276	1	70.9	59	1	72.99
	$k = 20$	522	1	70.9	105	1	77.81
	$k = 50$	1291	1	71.76	249	1	77.81
	$k = 100$	2563	1	71.76	483	1	77.81
	$k = 200$	5463	1	77.81	947	1	77.81
28	$k = 1$	55	1	68.54	17	1	77.78
	$k = 5$	136	1	68.54	36	1	78.32
	$k = 10$	236	1	70.2	59	1	79.3
	$k = 20$	429	1	73.5	103	1	83.47
	$k = 50$	1096	1	73.5	272	1	83.95
	$k = 100$	2151	1	73.5	451	1	83.95
	$k = 200$	4486	1	74.86	931	1	86.45
29	$k = 1$	48	1	73.14	17	1	73.85
	$k = 5$	144	1	75.49	41	1	74.03
	$k = 10$	244	1	79.77	68	1	76.69
	$k = 20$	462	1	79.77	132	1	77.94
	$k = 50$	1104	1	81.33	459	1	80.83
	$k = 100$	2230	1	81.33	706	1	84.8
	$k = 200$	4511	1	84.64	1425	1	84.8
30	$k = 1$	25	1	72.31	8	1	75.76
	$k = 5$	111	1	72.31	18	1	75.63
	$k = 10$	199	1	76.44	34	1	75.63
	$k = 20$	349	1	76.44	71	1	75.76
	$k = 50$	946	1	76.44	163	1	80.23
	$k = 100$	1865	1	76.57	317	1	80.23
	$k = 200$	3472	1	79.56	645	1	82.23

Table A.7: Case study results 31-35

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
31	$k = 1$	37	1	59.65	14	1	74.05
	$k = 5$	106	1	71.6	38	1	72.43
	$k = 10$	191	1	72.36	61	1	72.43
	$k = 20$	365	1	72.36	103	1	73.48
	$k = 50$	918	1	75.08	247	1	80.34
	$k = 100$	1745	1	75.08	406	1	77.23
	$k = 200$	3442	1	77.46	798	1	78.02
32	$k = 1$	61	1	65.02	24	1	69.44
	$k = 5$	148	1	72.09	30	1	78.06
	$k = 10$	253	1	73.71	45	1	72.09
	$k = 20$	515	1	73.71	83	1	72.09
	$k = 50$	1229	1	73.71	214	1	72.09
	$k = 100$	2481	1	73.71	425	1	72.09
	$k = 200$	5037	1	73.71	850	1	72.37
33	$k = 1$	106	1	71.14	40	1	79.25
	$k = 5$	239	1	75.74	64	1	78.21
	$k = 10$	462	1	75.74	120	1	78.21
	$k = 20$	854	1	78.73	192	1	78.21
	$k = 50$	2130	1	78.73	412	1	83.68
	$k = 100$	4239	1	78.73	837	1	83.68
	$k = 200$	8519	1	80.04	1685	1	83.68
34	$k = 1$	36	1	61.8	12	1	62.54
	$k = 5$	95	1	64.99	50	1	69.19
	$k = 10$	170	1	64.99	78	1	73.28
	$k = 20$	321	1	67.87	150	1	73.28
	$k = 50$	735	1	73.08	250	1	75.11
	$k = 100$	1390	1	73.08	504	1	75.11
	$k = 200$	2791	1	79.15	868	1	72.67
35	$k = 1$	36	1	69.35	13	1	69.86
	$k = 5$	106	1	69.35	37	1	71.2
	$k = 10$	253	1	72.43	81	1	71.2
	$k = 20$	480	1	74.53	85	1	73.06
	$k = 50$	1277	1	74.58	188	1	71.65
	$k = 100$	2378	1	74.58	373	1	71.65
	$k = 200$	4929	1	74.58	755	1	71.65

Table A.8: Case study results 36-40

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
36	$k = 1$	121	1	66.81	28	1	75
	$k = 5$	343	1	66.81	53	1	73.69
	$k = 10$	618	1	66.81	180	1	78.01
	$k = 20$	1136	1	67.11	291	1	82.68
	$k = 50$	2664	1	71.94	711	1	76.98
	$k = 100$	5253	1	73.12	1223	1	80.92
	$k = 200$	10658	1	75.67	2579	1	81.79
37	$k = 1$	44	1	73.94	13	1	68.3
	$k = 5$	106	1	73.94	36	1	71.96
	$k = 10$	184	1	73.94	92	1	74.65
	$k = 20$	384	1	73.94	124	1	76.03
	$k = 50$	909	1	73.94	270	1	76.34
	$k = 100$	1883	1	73.94	461	1	68.16
	$k = 200$	3836	1	73.94	907	1	79.57
38	$k = 1$	39	1	70.69	17	1	74.76
	$k = 5$	132	1	70.69	43	1	77.39
	$k = 10$	259	1	70.69	83	1	77.39
	$k = 20$	522	1	71.28	160	1	77.39
	$k = 50$	1289	1	72.24	396	1	77.39
	$k = 100$	2606	1	72.24	950	1	78.95
	$k = 200$	5212	1	72.24	1208	1	79.17
39	$k = 1$	24	1	62.25	7	1	76.42
	$k = 5$	71	1	69.91	14	1	79.74
	$k = 10$	121	1	71.76	23	1	79.74
	$k = 20$	230	1	71.76	47	1	80.77
	$k = 50$	559	1	71.76	110	1	80.77
	$k = 100$	1065	1	71.76	216	1	80.77
	$k = 200$	2097	1	74.15	444	1	80.77
40	$k = 1$	84	1	63.4	23	1	70.51
	$k = 5$	211	1	71.97	43	1	73.19
	$k = 10$	381	1	71.97	79	1	75.09
	$k = 20$	781	1	71.97	140	1	72.27
	$k = 50$	1945	1	71.97	465	1	75.22
	$k = 100$	3730	1	71.97	661	1	78.91
	$k = 200$	7531	1	72.51	1382	1	81.13

Table A.9: Case study results 41-45

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
41	$k = 1$	326	1	68.36	38	1	70.62
	$k = 5$	697	1	74.25	104	2	52.93
	$k = 10$	1062	1	75.63	122	1	74.21
	$k = 20$	1948	1	77.68	262	1	78.73
	$k = 50$	4790	1	77.68	668	1	74.45
	$k = 100$	10170	1	79.15	1062	1	77.1
	$k = 200$	20996	1	79.15	2013	1	78.91
42	$k = 1$	149	1	70.28	20	1	67.43
	$k = 5$	479	1	75.83	108	1	68.59
	$k = 10$	849	1	75.83	214	1	73.76
	$k = 20$	1623	1	75.83	374	1	75.83
	$k = 50$	4004	1	75.83	907	1	81.2
	$k = 100$	8093	1	75.83	1923	1	77.36
	$k = 200$	16798	1	75.83	2991	1	77.3
43	$k = 1$	174	2	64.74	96	2	65.13
	$k = 5$	1628	2	71.89	370	2	64.66
	$k = 10$	3028	2	71.89	515	2	58.88
	$k = 20$	6518	2	70.91	1089	2	67.08
	$k = 50$	9427	2	68.02	2335	2	80.84
	$k = 100$	21206	2	71.12	3015	2	76.47
	$k = 200$	67001	2	67.51	5429	2	77.53
44	$k = 1$	367	1	61.69	61	1	71.18
	$k = 5$	868	1	69.72	205	1	83
	$k = 10$	1506	1	69.72	310	1	78.48
	$k = 20$	3062	1	69.72	760	1	76.04
	$k = 50$	7259	1	69.72	1567	1	80.62
	$k = 100$	14367	1	69.72	3204	1	81.5
	$k = 200$	29580	1	69.72	5594	1	80.33
45	$k = 1$	604	2	46.66	65	1	75.62
	$k = 5$	1502	2	44.05	359	1	83.12
	$k = 10$	2890	2	43.78	529	1	83.12
	$k = 20$	4877	2	38.4	810	1	78.72
	$k = 50$	9384	1	76.94	2070	1	83.31
	$k = 100$	22382	1	76.94	4241	1	86.62
	$k = 200$	36854	1	79.15	6044	1	85.35

Table A.10: Case study results 46-50

Instance		PS			PM		
		TT (ms)	B	CR	TT (ms)	B	CR
46	$k = 1$	259	2	77.06	78	2	71.7
	$k = 5$	2706	2	72.32	101	2	66.68
	$k = 10$	4194	2	70.3	280	2	72.68
	$k = 20$	8229	2	70.3	456	2	77.02
	$k = 50$	14847	2	76.33	1201	2	73.15
	$k = 100$	26144	2	76.15	1841	2	74.16
	$k = 200$	45738	2	75.98	4028	2	75.2
47	$k = 1$	395	1	70.04	73	1	72.15
	$k = 5$	1082	1	70.04	188	1	75.28
	$k = 10$	2024	1	70.04	439	1	72.72
	$k = 20$	3741	1	70.04	737	1	78.82
	$k = 50$	8428	1	70.04	1798	1	78.82
	$k = 100$	17165	1	72.83	3143	1	78.82
	$k = 200$	34689	1	72.83	6153	1	78.82
48	$k = 1$	208	2	63.3	75	2	62.24
	$k = 5$	1616	2	69.72	179	2	61.82
	$k = 10$	3395	2	63.35	410	2	54.88
	$k = 20$	6845	2	62.39	749	2	66.88
	$k = 50$	7946	2	65.44	913	2	63.9
	$k = 100$	20583	2	65.15	2056	2	64.02
	$k = 200$	43066	2	70.5	4014	2	66.03
49	$k = 1$	152	1	64.08	36	1	79
	$k = 5$	617	1	69.12	79	1	76.7
	$k = 10$	1140	1	72.53	128	1	76.11
	$k = 20$	2039	1	72.53	252	1	79.88
	$k = 50$	4622	1	74.03	603	1	79.88
	$k = 100$	8767	1	74.03	1170	1	81.7
	$k = 200$	16856	1	74.03	2384	1	81.7
50	$k = 1$	421	1	63.37	63	1	70.85
	$k = 5$	899	1	67.26	279	1	74.19
	$k = 10$	1950	1	70.81	283	1	74.19
	$k = 20$	3077	1	70.81	670	1	67.69
	$k = 50$	6702	1	70.81	1476	1	74.15
	$k = 100$	14720	1	72.07	1272	1	67.3
	$k = 200$	31094	1	72.29	2618	1	67.3

Table A.11: Case study results 51-55

Instance		PS			PM		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
51	$k = 1$	199	1	70.66	62	1	71.17
	$k = 5$	1189	1	71.25	157	1	70.5
	$k = 10$	2375	1	71.25	297	1	69.81
	$k = 20$	4786	1	71.25	1206	1	71.6
	$k = 50$	11267	1	71.25	879	1	64.67
	$k = 100$	20929	1	71.25	1761	1	64.67
	$k = 200$	41807	1	73.87	3514	1	64.67
52	$k = 1$	54	1	71.22	22	1	77.21
	$k = 5$	170	1	74.69	51	1	80.62
	$k = 10$	332	1	74.69	85	1	76.17
	$k = 20$	589	1	76.01	169	1	78.28
	$k = 50$	1589	1	76.01	424	1	80.14
	$k = 100$	3017	1	77.1	517	1	81.3
	$k = 200$	6764	1	77.1	1030	1	81.3
53	$k = 1$	434	1	63.73	54	1	70.09
	$k = 5$	1510	1	63.73	217	1	75.61
	$k = 10$	2843	1	63.73	365	1	69.2
	$k = 20$	5270	1	63.73	544	1	75.31
	$k = 50$	12437	1	68.54	1198	1	77.54
	$k = 100$	22733	1	69.71	1561	1	72.15
	$k = 200$	46766	1	70.13	3089	1	71.75
54	$k = 1$	889	2	71.9	75	2	71.76
	$k = 5$	1611	2	74.52	186	2	77.16
	$k = 10$	3338	2	74.52	452	2	70.11
	$k = 20$	7081	2	72.86	771	2	77.55
	$k = 50$	17401	2	72.55	1609	2	77.67
	$k = 100$	36080	2	77.76	2773	2	71.73
	$k = 200$	69866	2	71.95	5305	2	75.45
55	$k = 1$	136	1	58.5	46	1	69.04
	$k = 5$	455	1	70.35	89	1	71.56
	$k = 10$	786	1	74.32	168	1	71.98
	$k = 20$	1501	1	78.76	303	1	72.31
	$k = 50$	3475	1	78.76	748	1	72.31
	$k = 100$	7132	1	78.76	1376	1	72.53
	$k = 200$	14863	1	79.02	1909	1	79.6

Table A.12: Case study results 56-60

Instance		**PS**			**PM**		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
56	$k = 1$	319	1	63.92	141	1	79.42
	$k = 5$	828	1	71.46	236	1	77.95
	$k = 10$	1476	1	74.87	420	1	80.12
	$k = 20$	2840	1	74.87	550	1	76.63
	$k = 50$	7310	1	74.87	1478	1	81.85
	$k = 100$	15072	1	74.87	2233	1	75.5
	$k = 200$	30574	1	78.73	3703	1	73.29
57	$k = 1$	249	2	64.23	53	2	55.11
	$k = 5$	743	2	47.8	137	2	58.35
	$k = 10$	1343	2	45.86	295	2	48
	$k = 20$	2089	2	42.54	398	2	48.64
	$k = 50$	5008	2	44.26	1116	2	54.39
	$k = 100$	13920	2	53.72	2074	1	82.3
	$k = 200$	26749	2	51.17	2753	1	82.01
58	$k = 1$	106	1	65	34	1	77.03
	$k = 5$	809	1	65	135	1	76.3
	$k = 10$	1525	1	67.19	195	1	76.3
	$k = 20$	3043	1	67.99	328	1	74.82
	$k = 50$	7037	1	67.99	539	1	76.89
	$k = 100$	14984	1	71.01	1032	1	76.84
	$k = 200$	29794	1	71.48	2064	1	77.89
59	$k = 1$	64	1	69.12	54	1	78.11
	$k = 5$	278	1	69.66	131	1	78.2
	$k = 10$	658	1	69.66	250	1	77.18
	$k = 20$	1229	1	75.21	440	1	76.43
	$k = 50$	3024	1	75.21	643	1	77.6
	$k = 100$	5983	1	75.21	1248	1	77.6
	$k = 200$	11839	1	75.21	2491	1	77.6
60	$k = 1$	342	1	64.54	65	1	71.84
	$k = 5$	826	1	66.78	136	1	77.47
	$k = 10$	1498	1	69.14	328	1	82.87
	$k = 20$	2941	1	71.75	570	1	81.86
	$k = 50$	7042	1	72.18	876	1	71.62
	$k = 100$	13516	1	75.64	1629	1	73.71
	$k = 200$	26714	1	80.19	2324	1	69.89

Table A.13: Case study results 61-65

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
61	$k = 1$	2470	3	61.14	194	2	75.03
	$k = 5$	7566	3	61.46	458	2	79.58
	$k = 10$	7705	2	71.35	882	2	74.15
	$k = 20$	30277	3	57.73	2574	2	78.49
	$k = 50$	41450	2	71.61	4022	2	79.87
	$k = 100$	69808	3	58.95	6114	2	79.77
	$k = 200$	197317	2	71.15	13254	2	77.35
62	$k = 1$	2572	2	66.32	179	2	73.37
	$k = 5$	10605	2	69.85	507	2	73.55
	$k = 10$	17406	2	73.78	1380	2	73.49
	$k = 20$	45291	2	66.46	2006	2	72.71
	$k = 50$	44703	2	72.57	6606	2	77.29
	$k = 100$	147556	2	69.57	7603	2	76.6
	$k = 200$	103674	2	73.59	20459	2	76.31
63	$k = 1$	1665	2	64.66	269	2	75.39
	$k = 5$	11233	2	63.95	956	2	77.32
	$k = 10$	13900	2	68.69	1656	2	77.37
	$k = 20$	28438	2	72.53	2944	2	75.38
	$k = 50$	99439	2	72.61	9805	2	75.62
	$k = 100$	71230	2	72.6	15467	2	77.67
	$k = 200$	137283	2	73.18	12330	2	78.89
64	$k = 1$	2452	2	66.08	156	2	76.31
	$k = 5$	5607	2	71.17	392	2	75.41
	$k = 10$	4290	2	67.29	713	2	72.47
	$k = 20$	7240	2	65.83	1005	2	76.43
	$k = 50$	60548	2	69.7	2428	2	77.74
	$k = 100$	70161	2	73.64	6411	2	79.81
	$k = 200$	144912	2	73.64	10006	2	80.02
65	$k = 1$	1391	2	61.23	126	2	62.06
	$k = 5$	8206	2	61.29	482	2	62.92
	$k = 10$	17117	2	64.36	1116	2	61.67
	$k = 20$	15367	2	69.79	2358	2	65.05
	$k = 50$	78715	2	63.81	3023	2	60.12
	$k = 100$	146277	2	72.45	7925	2	61.3
	$k = 200$	406884	2	63.72	15111	2	64.17

Table A.14: Case study results 66-70

Instance		PS			PM		
		TT (ms)	B	CR	TT (ms)	B	CR
66	$k = 1$	1904	3	58.8	175	2	81.55
	$k = 5$	10568	3	67.91	485	2	79.54
	$k = 10$	5507	3	65.99	1231	2	77.61
	$k = 20$	11187	3	65.99	1804	2	78.44
	$k = 50$	119767	3	56.88	3802	2	81.5
	$k = 100$	256247	3	68.01	6740	2	80.58
	$k = 200$	222528	2	77.34	13746	2	82.84
67	$k = 1$	925	2	70.81	122	2	68.56
	$k = 5$	5226	2	69.78	439	2	66.14
	$k = 10$	10700	2	65.29	901	2	66.67
	$k = 20$	10307	2	70.82	1418	2	75.99
	$k = 50$	48933	2	69.91	2782	2	68.27
	$k = 100$	72109	2	68.5	6219	2	70.04
	$k = 200$	80496	2	72.21	13593	2	75.07
68	$k = 1$	662	2	67.89	266	2	74.12
	$k = 5$	2206	2	70.69	485	2	71.13
	$k = 10$	3601	2	70.59	933	2	72.56
	$k = 20$	6176	2	69.41	3152	2	75.02
	$k = 50$	21427	2	72.94	3732	2	75.85
	$k = 100$	46567	2	74.75	5835	2	73.27
	$k = 200$	136385	2	72.67	13022	2	72.26
69	$k = 1$	533	2	67.01	164	2	70.51
	$k = 5$	2336	2	66.89	645	2	56.89
	$k = 10$	3697	2	69.5	906	2	57.8
	$k = 20$	34601	2	68.7	1629	2	63.27
	$k = 50$	25976	2	68.5	2795	2	55.69
	$k = 100$	45713	2	68.21	7779	2	57.47
	$k = 200$	95060	2	69.61	11836	2	67.42
70	$k = 1$	706	2	66.92	114	2	48.99
	$k = 5$	3006	2	66.92	228	2	50.21
	$k = 10$	9862	2	67.94	923	2	49.66
	$k = 20$	18979	2	67.95	1259	2	47.25
	$k = 50$	47210	2	67.95	3296	2	45.48
	$k = 100$	93287	2	67.94	5671	2	50.66
	$k = 200$	175227	2	69.96	10166	2	46.7

Table A.15: Case study results 71-75

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
71	$k = 1$	1937	2	68.46	275	2	78.92
	$k = 5$	6324	2	67.23	923	2	79.06
	$k = 10$	12072	2	60.76	1845	2	77.33
	$k = 20$	43571	2	69.7	2488	2	77.21
	$k = 50$	107498	2	69.7	4843	2	79.95
	$k = 100$	196533	2	74.01	9836	2	80.56
	$k = 200$	165421	2	70.67	30888	2	77.79
72	$k = 1$	3135	2	69.36	180	2	75.19
	$k = 5$	12714	2	70.89	841	2	78.12
	$k = 10$	13866	2	73.13	1489	2	75.76
	$k = 20$	17588	2	74.99	2531	2	80.41
	$k = 50$	63019	2	72.49	4930	2	82.07
	$k = 100$	99912	2	73.73	8868	2	76.87
	$k = 200$	202486	2	73.73	15071	2	80.41
73	$k = 1$	543	2	62.87	106	2	59.24
	$k = 5$	1435	2	61.18	434	2	49.78
	$k = 10$	2961	2	61.18	912	1	81.44
	$k = 20$	5741	2	61.18	1491	1	78.06
	$k = 50$	14454	2	59.95	2519	2	61
	$k = 100$	48160	2	53.75	3573	1	81.19
	$k = 200$	55897	2	56.57	16018	1	81.52
74	$k = 1$	1141	3	59.03	110	3	51.47
	$k = 5$	2047	2	74.12	573	2	78.35
	$k = 10$	11023	2	75.26	647	2	81.52
	$k = 20$	21326	2	78.24	1739	2	78.18
	$k = 50$	22942	2	73.59	3399	3	52.15
	$k = 100$	99298	2	76.12	4867	2	80.29
	$k = 200$	187444	2	76.64	5707	2	78.08
75	$k = 1$	1626	3	67.53	146	3	56.95
	$k = 5$	7600	3	63.87	548	2	79.31
	$k = 10$	14729	3	63.87	1095	2	82.81
	$k = 20$	22788	3	68.43	1881	2	79.28
	$k = 50$	124694	3	76.24	3366	2	78.77
	$k = 100$	205410	3	66.02	7725	2	80.9
	$k = 200$	537763	3	65.18	24478	2	84.28

Table A.16: Case study results 76-80

Instance		**PS**			**PM**		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
76	$k = 1$	1376	2	63.96	177	2	74.41
	$k = 5$	8926	2	74.31	519	2	72.4
	$k = 10$	17086	2	74.31	751	2	78.52
	$k = 20$	33521	2	74.31	1159	2	78.81
	$k = 50$	60053	2	73.48	3003	2	84.23
	$k = 100$	135048	2	73.09	7363	2	78.8
	$k = 200$	231300	2	71.65	16926	2	80.54
77	$k = 1$	1023	2	49.84	120	1	71.99
	$k = 5$	4338	2	39.27	651	1	82.65
	$k = 10$	5616	1	71.67	1016	1	76.75
	$k = 20$	10976	2	40.22	2028	1	80.46
	$k = 50$	29232	1	72.87	3568	1	75.93
	$k = 100$	41278	1	77.2	8318	1	73.59
	$k = 200$	88763	1	77.2	5868	1	80.87
78	$k = 1$	265	2	69.04	49	2	63.24
	$k = 5$	1194	2	71.41	262	2	67.07
	$k = 10$	2336	2	65.69	571	2	68.79
	$k = 20$	4516	2	69.82	902	2	68.92
	$k = 50$	12694	2	69.79	2142	2	68.06
	$k = 100$	37343	2	70.11	3265	2	71.6
	$k = 200$	68154	2	69.01	5888	2	68.1
79	$k = 1$	619	2	65.18	142	2	68.24
	$k = 5$	2508	2	68	521	2	73.19
	$k = 10$	4839	2	68	912	2	73.2
	$k = 20$	9629	2	68	1748	2	73.2
	$k = 50$	24561	2	68	2860	2	69.99
	$k = 100$	59050	2	70.69	6388	2	68.51
	$k = 200$	119607	2	68.91	8898	2	70.83
80	$k = 1$	475	2	67.45	90	2	76.64
	$k = 5$	1394	2	73	287	2	73.64
	$k = 10$	2504	2	72.45	784	2	74.19
	$k = 20$	4813	2	76.09	1518	2	74.19
	$k = 50$	11433	2	76.26	1973	2	75.36
	$k = 100$	24444	2	71.36	4246	2	76.51
	$k = 200$	47230	2	76.39	7618	2	76.66

Table A.17: Case study results 81-85

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
81	$k = 1$	8	1	71.74	9	1	73.14
	$k = 5$	32	1	71.74	17	1	71.74
	$k = 10$	49	1	73.14	27	1	71.74
	$k = 20$	94	1	73.14	49	1	71.74
	$k = 50$	226	1	73.14	99	1	71.74
	$k = 100$	451	1	74.59	185	1	71.74
	$k = 200$	916	1	74.59	345	1	74.59
82	$k = 1$	30	1	62.99	67	1	73.56
	$k = 5$	128	1	75.39	124	1	77.7
	$k = 10$	242	1	75.39	244	1	77.7
	$k = 20$	460	1	75.89	577	1	75.39
	$k = 50$	1150	1	75.89	1658	1	76.65
	$k = 100$	2312	1	75.89	3109	1	76.14
	$k = 200$	4703	1	75.89	4769	1	76.65
83	$k = 1$	6	1	63.86	4	1	59.07
	$k = 5$	16	1	63.86	11	1	65.64
	$k = 10$	31	1	63.86	19	1	65.64
	$k = 20$	52	1	64.13	39	1	65.64
	$k = 50$	125	1	65.64	87	1	65.64
	$k = 100$	245	1	65.64	173	1	65.64
	$k = 200$	489	1	65.64	347	1	65.64
84	$k = 1$	3	1	66.1	2	1	73.29
	$k = 5$	9	1	73.29	5	1	74.06
	$k = 10$	17	1	73.29	8	1	74.06
	$k = 20$	32	1	73.29	11	1	74.06
	$k = 50$	75	1	73.29	10	1	74.06
	$k = 100$	150	1	73.29	10	1	74.06
	$k = 200$	307	1	73.29	10	1	74.06
85	$k = 1$	24	1	69.56	21	1	66.42
	$k = 5$	75	1	69.56	33	1	79.68
	$k = 10$	140	1	69.56	54	1	79.68
	$k = 20$	267	1	69.56	94	1	80.26
	$k = 50$	670	1	69.56	208	1	80.26
	$k = 100$	1311	1	69.56	378	1	80.26
	$k = 200$	2645	1	69.56	798	1	80.26

Table A.18: Case study results 86-90

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
86	$k = 1$	34	1	82.3	17	1	75.61
	$k = 5$	80	1	82.3	20	1	82.3
	$k = 10$	133	1	82.3	28	1	82.3
	$k = 20$	259	1	82.3	53	1	82.3
	$k = 50$	604	1	82.3	104	1	82.3
	$k = 100$	1174	1	82.3	180	1	82.3
	$k = 200$	2400	1	82.3	359	1	82.3
87	$k = 1$	250	1	68.79	111	1	71.46
	$k = 5$	380	1	70.77	189	1	76.67
	$k = 10$	617	1	70.77	138	1	70.1
	$k = 20$	1072	1	72.52	274	1	75.49
	$k = 50$	2504	1	76.27	771	1	78.3
	$k = 100$	4903	1	76.27	1531	1	78.3
	$k = 200$	9892	1	77.89	2544	1	77.07
88	$k = 1$	109	1	63.76	53	1	77.67
	$k = 5$	244	1	67.29	98	1	77.67
	$k = 10$	375	1	69.53	145	1	76.85
	$k = 20$	690	1	73.38	310	1	73.38
	$k = 50$	1581	1	75.66	752	1	76.45
	$k = 100$	3019	1	77.26	1447	1	76.45
	$k = 200$	6146	1	77.26	2800	1	76.45
89	$k = 1$	32	1	67.75	22	1	75.37
	$k = 5$	210	1	75.72	24	1	66.46
	$k = 10$	376	1	75.72	41	1	66.46
	$k = 20$	601	1	75.72	76	1	75.37
	$k = 50$	1500	1	75.72	176	1	75.37
	$k = 100$	3224	1	75.72	321	1	75.37
	$k = 200$	5858	1	76.95	619	1	76.59
90	$k = 1$	12	1	80.24	10	1	80.24
	$k = 5$	41	1	80.24	18	1	80.24
	$k = 10$	76	1	80.24	20	1	80.24
	$k = 20$	148	1	80.24	23	1	80.24
	$k = 50$	372	1	80.24	20	1	80.24
	$k = 100$	738	1	80.24	20	1	80.24
	$k = 200$	1517	1	80.24	20	1	80.24

Table A.19: Case study results 91-95

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
91	$k = 1$	27	2	61.65	23	2	62.87
	$k = 5$	167	2	60.72	51	2	67.36
	$k = 10$	285	2	65.81	99	2	69.17
	$k = 20$	596	2	67.21	189	2	70.48
	$k = 50$	1239	2	64.03	422	2	73.88
	$k = 100$	2370	2	71.91	631	2	64.22
	$k = 200$	4631	2	71.91	1244	2	63.94
92	$k = 1$	41	2	68.98	17	2	64.7
	$k = 5$	102	2	68.5	46	2	60.91
	$k = 10$	180	2	68.5	87	2	63.43
	$k = 20$	343	2	68.5	200	2	64.45
	$k = 50$	1081	2	68.5	481	2	64.45
	$k = 100$	2487	2	75.09	848	2	62.44
	$k = 200$	4615	2	74.62	1428	2	61.01
93	$k = 1$	11	2	53.18	6	2	49.39
	$k = 5$	29	2	60.58	13	2	63.43
	$k = 10$	62	2	60.75	23	2	63.43
	$k = 20$	114	2	60.97	44	2	62.4
	$k = 50$	251	2	53.59	102	2	62.4
	$k = 100$	480	2	53.59	207	1	70.47
	$k = 200$	1141	2	53.59	408	1	70.85
94	$k = 1$	7	2	60.41	6	2	61.33
	$k = 5$	18	2	60.41	10	2	62.49
	$k = 10$	32	2	60.41	17	2	64.19
	$k = 20$	65	2	72.49	30	2	64.19
	$k = 50$	166	2	72.49	72	2	64.19
	$k = 100$	332	2	72.49	146	2	74.4
	$k = 200$	675	2	75.2	247	2	71.17
95	$k = 1$	2596	1	71.15	217	1	80.4
	$k = 5$	5493	1	71.15	345	1	81.11
	$k = 10$	10208	1	71.15	859	1	78.75
	$k = 20$	19066	1	71.15	1031	1	76.3
	$k = 50$	53469	1	71.15	2902	1	82.38
	$k = 100$	101264	1	71.15	5778	1	82.38
	$k = 200$	198827	1	71.15	11145	1	82.38

Table A.20: Case study results 96-100

Instance		**PS**			**PM**		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
96	$k = 1$	447	1	60.54	29	1	64.78
	$k = 5$	1417	1	60.54	108	1	67.96
	$k = 10$	2778	1	60.54	191	1	68.33
	$k = 20$	5405	1	66.23	685	1	71.57
	$k = 50$	13724	1	66.23	1133	1	68.28
	$k = 100$	26596	1	67	2122	1	68.28
	$k = 200$	55258	1	67	3725	1	69.27
97	$k = 1$	625	1	56.36	40	1	70.47
	$k = 5$	2977	1	61.41	228	1	69.38
	$k = 10$	5436	1	61.41	459	1	68.66
	$k = 20$	10287	1	62.02	458	1	66.34
	$k = 50$	26357	1	63.42	988	1	58.96
	$k = 100$	53737	1	65.76	1873	2	36.16
	$k = 200$	123811	1	72.94	9152	1	74.29
98	$k = 1$	78	1	79.05	23	1	64.42
	$k = 5$	172	1	79.05	51	1	64.42
	$k = 10$	288	1	81.04	94	1	77.16
	$k = 20$	530	1	81.04	137	1	67.57
	$k = 50$	1193	1	81.04	319	1	73.63
	$k = 100$	2307	1	81.04	621	1	73.75
	$k = 200$	4741	1	81.04	1182	1	77.16
99	$k = 1$	8	1	71.74	9	1	73.14
	$k = 5$	26	1	71.74	17	1	71.74
	$k = 10$	55	1	73.14	27	1	71.74
	$k = 20$	97	1	73.14	53	1	71.74
	$k = 50$	234	1	73.14	93	1	71.74
	$k = 100$	464	1	74.59	186	1	71.74
	$k = 200$	921	1	74.59	338	1	74.59
100	$k = 1$	9	1	54.55	6	1	56.35
	$k = 5$	30	1	56.35	12	1	67.25
	$k = 10$	55	1	56.35	19	1	70.44
	$k = 20$	101	1	67.45	35	1	70.44
	$k = 50$	243	1	67.45	71	1	70.44
	$k = 100$	495	1	70.22	134	1	70.44
	$k = 200$	1017	1	70.44	250	1	70.44

Table A.21: Summary of opened bins by heuristic

Class		PM					PS					TS3	GLS	GASP	EHGH2	GVN	BRKGA	BRKGA-VD
Instance	n	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$							
1	50	14.5	14.5	14.5	14.5	14.10	14.4	14.3	14.3	14.4	14	**13.4**	**13.4**	**13.4**	13.8	**13.4**	**13.4**	**13.4**
	100	28.4	28.3	28.3	28.4	28	27.7	27.8	27.8	27.9	27.7	**26.6**	**26.6**	26.9	27.6	**26.6**	**26.6**	**26.6**
	150	38.5	38.5	38.4	38.3	38.4	38.4	38.5	38.5	38.3	37.9	36.7	37	37	39.8	36.4	36.4	**36.3**
	200	53.1	53.2	53	52.9	53	52.7	52.7	52.6	52.6	52.7	51.2	51.2	51.6	**50.6**	50.9	50.8	50.8
2	50	14.8	14.7	14.7	14.7	14.6	14.9	14.8	14.8	14.9	14.8	**13.8**	-	-	-	**13.8**	**13.8**	**13.8**
	100	26.5	26.6	26.6	26.6	26.6	27.1	26.9	26.9	26.7	26.7	25.7	-	-	-	25.7	25.6	**25.5**
	150	38.4	38.3	38.2	38.3	38.3	39.2	39.2	39.1	39.1	39	37.2	-	-	-	36.9	**36.6**	**36.6**
	200	51.3	51.2	51.1	51.3	51	51.9	51.8	51.9	51.8	51.7	50.1	-	-	-	**49.4**	**49.4**	**49.4**
3	50	14.2	14.1	13.9	13.8	13.9	14	14.1	13.9	14	13.9	**13.3**	-	-	-	**13.3**	**13.3**	**13.3**
	100	27.7	27.7	27.7	27.5	27.8	27.6	27.5	27.2	27.3	27.3	26	-	-	-	26	**25.9**	**25.9**
	150	39.4	39.4	39.5	39.5	39.2	39.1	38.9	38.9	39	39	37.7	-	-	-	37.6	**37.5**	**37.5**
	200	51.8	51.8	51.7	51.6	51.8	51.4	51.4	51.4	51.3	51.2	50.5	-	-	-	50	**49.8**	**49.8**
4	50	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**
	100	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59	59	59	59.5	59	59	**58.9**
	150	87.6	87.6	87.6	87.6	87.6	87.6	87.7	87.6	87.6	87.7	**86.8**	**86.8**	**86.8**	90.4	**86.8**	**86.8**	**86.8**
	200	119.5	119.5	119.5	119.5	119.5	119.5	119.5	119.5	119.5	119.5	**118.8**	119	**118.8**	119	**118.8**	**118.8**	**118.8**
5	50	8.7	8.7	8.7	8.7	8.6	8.7	8.7	8.6	8.6	8.6	8.4	8.3	8.4	**7.9**	8.3	8.3	8.3
	100	16	16	16	16	16	15.7	15.6	15.6	15.6	15.6	15	15.1	15.1	**14.6**	15	15	15
	150	21.6	21.6	21.6	21.7	21.7	21.3	21.3	21.3	21.3	21.4	20.4	20.2	20.6	21.5	20.4	20.1	**19.9**
	200	28.9	28.9	28.9	29	29	28.4	28.2	28.3	28.4	28.4	27.6	27.2	27.7	29.6	**27.1**	**27.1**	**27.1**
6	50	10.5	10.3	10.1	10.2	10	10.7	10.5	10.4	10.2	10.3	9.9	9.8	9.9	11.8	9.8	**9.7**	**9.7**
	100	19.8	19.8	20	19.5	19.8	19.8	19.7	19.8	19.7	19.7	19.1	19.1	19.1	19.2	19	**18.9**	**18.9**
	150	30.4	30.4	30.3	30.3	30.3	30.4	30.3	30.3	30	30.2	29.4	29.4	29.5	29.8	29.2	**29**	**29**
	200	39.1	39.2	39.2	38.6	38.9	39.1	38.6	38.8	38.6	38.5	37.7	37.7	38	38.7	37.4	**37.3**	**37.3**
7	50	7.9	7.7	7.9	7.8	7.8	7.7	7.9	7.9	7.7	7.6	7.5	**7.4**	7.5	**7.4**	**7.4**	**7.4**	**7.4**
	100	13.5	13.2	13.4	13.3	13.2	13.5	13.5	13.5	13.4	13.2	12.5	12.3	12.7	13.5	12.5	**12.2**	**12.2**
	150	16.9	16.9	17	16.9	17.1	17.1	16.7	16.9	16.8	16.8	16.1	15.8	16.6	18.2	16	15.3	**15.2**
	200	25.1	25	25	25.2	24.9	24.9	24.5	24.8	24.7	24.7	23.9	23.5	24.2	24.1	23.5	**23.4**	**23.4**
8	50	10	10.1	9.9	9.8	9.9	9.7	9.6	9.6	9.6	9.7	9.3	**9.2**	9.3	9.4	**9.2**	**9.2**	**9.2**
	100	19.6	19.7	19.7	19.6	19.6	20	20	19.9	20	20	18.9	18.9	19	18.9	18.9	18.9	**18.8**
	150	25.8	25.6	25.6	25.7	25.7	26	25.8	25.7	25.8	25.8	24.1	23.9	24.8	26	24.1	**23.6**	**23.6**
	200	31.5	31.5	31.6	31.3	31.6	31.6	31.6	31.6	31.4	31.2	30.3	29.9	31.1	35.8	29.8	**29.3**	**29.3**

Table A.22: Detailed execution times for literature tests in milliseconds

Class		PM					PS					BRKGA-VD
Instance	n	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$	
1	50	80.50	85.20	150.70	307.00	820.30	85.00	109.40	192.80	390.60	968.20	1,293.25
	100	137.90	439.10	819.60	1,500.60	3,737.60	177.20	634.80	1,181.50	2,409.10	5,540.10	7,005.10
	150	341.50	1,237.10	2,478.60	4,677.30	12,143.30	494.10	1,865.90	3,817.80	7,625.60	18,751.70	17,674.42
	200	697.40	2,808.20	5,144.30	10,319.20	26,679.40	1,087.30	4,629.80	9,134.60	18,256.50	44,792.70	34,702.21
2	50	24.70	72.80	126.50	220.10	555.70	35.80	125.60	243.20	510.80	1,190.30	1,293.25
	100	105.80	342.50	647.50	1,528.70	3,921.30	194.00	869.00	1,724.50	3,391.50	8,489.30	7,112.88
	150	266.00	974.60	1,842.70	3,399.20	8,160.60	590.30	2,599.20	4,943.70	10,118.50	24,492.20	17,782.19
	200	532.40	1,971.00	3,817.30	7,228.50	17,694.90	1,320.50	5,792.70	11,790.20	22,535.60	53,024.40	34,486.67
3	50	28.10	82.70	156.30	302.50	710.20	32.50	112.00	204.00	401.30	982.40	1,293.25
	100	120.10	489.30	910.70	1,625.50	4,313.90	162.20	621.70	1,231.80	2,234.10	6,042.70	6,897.33
	150	341.20	1,263.80	2,455.60	4,719.40	11,596.80	461.40	1,979.50	4,027.40	8,214.90	20,657.80	18,428.81
	200	690.00	2,694.00	5,307.70	10,418.30	25,865.20	1,018.20	4,318.50	8,788.70	17,801.60	44,903.80	35,672.15
4	50	45.10	129.40	255.60	506.10	1,200.90	95.60	150.20	280.40	532.50	1,377.00	1,185.48
	100	220.70	817.40	1,582.80	2,973.30	7,125.80	246.00	945.50	1,742.80	3,532.90	8,636.60	6,574.02
	150	559.70	2,074.50	3,944.40	7,778.10	19,574.70	644.20	2,601.60	5,089.70	9,968.20	24,431.50	17,135.56
	200	1,143.20	4,685.50	8,938.80	17,291.40	44,421.60	1,415.90	6,027.40	11,581.60	23,829.20	58,029.70	34,271.13
5	50	23.40	75.40	152.80	275.10	701.90	40.60	155.90	318.80	631.80	1,617.20	3,233.13
	100	69.80	230.80	438.70	861.00	2,229.90	212.40	937.60	1,703.40	3,547.50	7,580.70	13,040.27
	150	164.90	596.50	1,134.80	2,415.50	5,727.40	618.10	2,532.10	4,824.10	9,131.20	27,346.00	29,960.29
	200	283.80	1,055.90	2,071.00	3,899.90	9,916.70	1,210.60	4,988.80	9,872.90	20,611.90	56,078.50	57,226.32
6	50	22.40	73.20	136.10	262.90	568.60	26.90	88.80	177.70	349.40	836.40	969.94
	100	88.50	315.00	583.50	1,139.40	2,684.80	136.40	555.30	1,030.40	1,839.80	4,456.70	5,065.23
	150	223.40	796.90	1,448.70	3,019.10	7,749.50	323.80	1,360.10	2,561.50	5,367.50	11,886.10	13,686.90
	200	401.90	1,555.60	2,820.30	5,229.50	12,899.70	624.40	2,523.20	4,943.60	9,218.90	23,577.80	27,697.11
7	50	18.00	53.70	103.60	190.80	487.40	32.80	113.30	218.20	453.50	1,105.60	2,586.50
	100	68.30	229.50	433.30	846.80	1,993.60	154.30	656.70	1,255.30	2,261.20	5,893.50	10,669.31
	150	153.10	547.60	1,164.10	2,190.90	5,292.60	413.90	1,894.10	3,777.40	7,156.60	17,704.60	24,571.75
	200	270.00	976.90	2,056.00	3,804.10	10,078.00	808.70	4,513.50	7,854.10	14,518.60	32,642.10	46,125.92
8	50	18.60	60.40	119.30	220.30	528.50	29.50	109.00	200.90	406.60	919.30	2,909.81
	100	65.00	208.00	395.90	758.20	1,983.70	160.20	576.30	1,179.10	2,155.50	4,981.50	13,148.04
	150	151.70	516.30	937.80	1,970.20	4,837.30	433.10	1,964.40	3,815.50	8,109.50	17,854.20	25,002.84
	200	296.80	1,004.90	1,835.80	3,471.70	9,174.80	896.00	4,149.50	7,612.00	14,338.20	34,916.60	48,604.65

Table A.23: Cage ratio of literature tests

Class		PM					PS				
Instance	n	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$
1	50	69.36	69.84	69.66	69.74	71.73	70.09	70.5	70.45	70.06	**72.36**
	100	73.01	73.62	73.37	73.44	**74.78**	74.56	74.25	74.26	74.17	74.56
	150	75.66	75.53	75.61	75.91	76.07	75.93	75.93	76.04	76.31	**76.9**
	200	75.33	75.28	75.56	75.62	75.78	75.71	75.8	**75.99**	75.9	75.93
2	50	69.6	69.5	69.48	**69.79**	69.31	67.92	68.71	68.09	68.32	68.78
	100	73.05	72.76	**73.09**	72.7	72.7	71.37	71.74	71.74	72.22	72.54
	150	72.95	73.15	**73.28**	73.11	73.04	71.51	71.28	71.5	71.38	71.66
	200	73.21	73.29	73.34	73.24	**73.42**	72.23	72.36	72.12	72.43	72.55
3	50	71.61	71.81	73.08	72.94	72.87	72.38	71.9	72.63	71.88	**73.4**
	100	73.52	73.62	73.7	74.37	73.73	73.99	74.28	**74.97**	74.79	74.83
	150	74.94	75.1	75.04	75.04	75.78	75.6	75.98	**76.04**	75.83	75.94
	200	76.09	76.01	76.21	76.43	76.08	76.58	76.74	76.82	77.01	**77.03**
4	50	61.49	61.49	61.62	**61.8**	61.79	61.49	61.6	61.7	61.73	61.78
	100	63.48	63.46	63.53	63.53	**63.58**	63.49	63.5	63.52	63.53	63.57
	150	61.91	61.96	61.92	61.95	**61.98**	61.94	61.84	61.94	61.95	61.96
	200	61.83	**61.84**	61.82	**61.84**	61.8	61.81	61.83	61.83	61.82	61.83
5	50	69.08	68.72	69.03	69.44	70.5	69.45	69.54	70.35	70.18	**70.69**
	100	73.49	73.66	73.8	74.1	73.71	74.52	74.94	74.85	**75.02**	75.01
	150	76.38	76.48	76.5	76.14	76.43	77.32	77.28	**77.33**	77.23	77.01
	200	77.12	77.19	77.11	77.06	76.99	77.88	**78.46**	78.14	77.99	77.8
6	50	77.26	78.65	79.52	79.09	**80.12**	75.65	77.23	77.44	79.06	78.44
	100	84.63	84.59	84.17	**85.74**	85.03	84.2	84.69	84.34	84.62	84.91
	150	84.66	85.17	85.33	85.45	85.4	84.97	85.32	85.2	**86.48**	85.93
	200	86.35	86.35	86.36	87.17	86.83	85.99	87.1	86.9	87.27	**87.53**
7	50	64.49	66.32	65.35	66.5	66.69	66.8	65.44	65.53	66.87	68.5
	100	71.22	72.94	71.64	72.64	**73.37**	71.48	71.63	71.8	72.03	73.12
	150	76.65	76.63	76.31	76.69	76.05	75.11	**77.25**	76.11	76.73	76.76
	200	77.48	77.95	77.97	77.08	78.06	77.66	**79**	78.04	78.16	78.44
8	50	69.61	69.48	70.75	70.81	70.62	70.7	71.38	71.38	**71.83**	71.14
	100	74.38	**74.51**	74.38	74.44	74.28	73.16	73.68	73.43	73.13	73.3
	150	77.07	**77.52**	77.19	77.07	77.22	75.89	76.73	76.81	76.4	76.61
	200	79.5	79.51	79.24	**79.86**	79.36	78.47	78.59	78.61	78.99	79.76

Bold values are the best average values

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Variable	**Description**	**SI unit**
$\boldsymbol{u}$	solid displacement	m
$\boldsymbol{u}_f$	fluid displacement	m

Acknowledgements

Here you might want to acknowledge someone.