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Three-dimensional bin packing with vertical support

Tesi di Laurea Magistrale in
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Abstract

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Contents

1 | Introduction

The three-dimensional bin packing problem (3D-BPP) involves packing, without any overlap, a set of small items into the minimum number of bins. In its most standard form, each item i is a cuboid of dimensions (w_i, d_i, h_i) and each bin is a cuboid with fixed dimensions across all bins (W, D, H). Items can only be placed with their sides parallel to the sides of the bin and can be rotated 90 degrees along their vertical axis.

The 3D-BPP is part of the Cutting and Packing problems where a set of small items (boxes) needs to be packed inside a set of large ones (bins). Based on the dimensionality of the problem and the number of items with different shapes, different typologies of the problem were identified by [Wäscher et al., 2007]. Versions of the problem with strongly heterogeneous item sets are classified as three-dimensional single bin-size bin packing problems (3D-SBSBPP). When considering the real-world instances of the problem, additional constraints need to be considered. In this thesis, the problem that we address is a variation of the 3D-SBSBPP, the three-dimensional bin packing problem with vertical support (3D-BPPWS). In addition to the standard formulation of the 3D-SBSBPP, each item needs to be supported by other items inside the bin. This constraint ensures that items in the bin will not fall in a real-world scenario. The support constraint is usually defined based on the amount of area of an item that is supported by others or the number of corners of an item that rest on top of others as, for example, in [Gzara et al., 2020; Kurpel et al., 2020; Paquay et al., 2016].

The standard 3D-SBSBPP is strongly NP-hard since it is the generalization of the one-dimensional bin packing problem [Martello et al., 2000]. Since our problem is a generalization of the 3D-SBSBPP, exact solutions are only able to solve small instances of the problem, which means that heuristic approaches need to be used for larger instances. Heuristics designed to solve the 3D-SBSBPP do not address the concept of static stability and allow solutions with unsupported items. The concept of static stability has received most of its contributions in Pallet Loading Problems and Container Loading Problems [Calzavara et al., 2021; Kurpel et al., 2020]. In these publications, other practical constraints are also accounted for, which derive from each particular use case of each approach. Most of them implicitly address the concept of support by explicitly building layers or walls of items with a high density, which allows them to reduce the problem to a one-dimensional packing problem. When considering mixed palletization cases, layer-based solutions stack layers ordered by density until the density of the generated layers falls below a certain threshold. Once no more layers can be generated, more simple techniques are employed to pack the remaining items (as for ex. [Elhedhli et al., 2019]) or the use of filler boxes is employed to increase the layer's density (as for ex. [Calzavara et al., 2021]).

In this thesis, we approach the constraint of support in the mixed case palletizing setting by introducing a constructive heuristic that fills bins without explicitly building layered solutions or filler boxes. We then introduce a beam search heuristic that expands the constructive heuristic's solution space by exploring different orders of item palletization. We also provide a formulation for the 3D-BPPWS with a discretized version of the support constraint that we use to validate our heuristic on small instances of the problem. We then provide a generated dataset based on real-world instances that we use to benchmark our heuristic.

1.1. Case study

The work of this thesis stems from the case study of a logistic company in northern Italy. The company manages large warehouses where automated lines bring boxes to different stations where they are loaded onto pallets of standard size. Each box is loaded by a robot or an operator. Each pallet is wrapped as soon as a certain palletization height has been reached. This wrapping improves the stability of the pallet while boxes are still being loaded at the top. To avoid uneven surfaces during the wrapping, the palletized height of each bin needs to have a high fill rate. Boxes that need to be stored in the warehouse are strongly heterogeneous, so layer-based pallet loading solutions have sub-optimal results. The company measures this level of fill rate with a metric called cage ratio. To increase the efficiency of the wrapping and to allow for the stacking of pallets, solutions with a high cage ratio are then required. The cage ratio of commercial solutions currently implemented by the company is around 60%, and a target cage ratio for our case study was set at 70%.

1.2. Overview

The thesis is structured as follows. In chapter 2 we review the relevant literature on the three-dimensional bin packing problem and the cutting and packing problems dealing with vertical support. In chapter 3 we give a formal definition of the problem and formulate a mixed-integer linear programming model that we'll use to validate the proposed solutions of the thesis. Since the model can not be used to solve real-world instances in chapter 4 we proposed a heuristic algorithm that addresses the problem. Results from comparing our heuristic to relevant heuristics from the literature, validation with a direct comparison to the solutions from the proposed MILP model, and solutions to real-world instances are described in chapter 5, together with sources to new benchmark instances generated. In chapter 6 we give final remarks and list further developments about the work of the thesis.

2 | Literature review

In this section, we review the relevant literature to our problem. In section 2.1, we do a brief review of the most relevant two-dimensional bin packing placement heuristics in the context of this thesis. In section 2.2, we review the literature on the 3D-BPP, focusing on the heuristics used to solve the single bin-size bin packing problem. Finally, in section 2.3, we do a brief review of the literature on practical constraints in cutting and packing problems, focusing on the vertical support constraint.

2.1. Two-Dimensional Bin Packing Problem

In the two-dimensional bin packing problem, two major tasks constitute the area of study relevant to constructive heuristics. The first task is to identify the smallest set of points where placements can be made. The second task consists in evaluating which point to select for placements. The main strategies used in selecting points are divided into first-fit and best-fit approaches. The first valid positions are selected in first-fit approaches, while in best-fit approaches, positions are selected based on a metric. The most common classical algorithm to select placements inside a two-dimensional bin given a set of points is the bottom-most left-most algorithm introduced in Baker et al. [1980]. It packs items in the lowest possible position closest to the bottom-left corner of the free area. This algorithm serves as the base of many heuristics that address the two-dimensional bin packing problem (2D-BPP). In Burke et al. [2004], a best-fit algorithm is introduced where placements of items that fit the lowest available area are made first. In Lodi et al. [1999], a maximum touching perimeter approach is used instead.

Considering the identification of possible packing positions, in Martello and Vigo [1998], a branch-and-bound algorithm was proposed to solve the two-dimensional orthogonal packing problem (2D-OPP). The selection of the positions is made in a left-most downwards strategy. Items are placed so that their left and bottom edges touch either other items or the bin. The algorithm is based on a tree search that packs items in every possible position. A set of 10 classical instances to benchmark heuristics against were also presented. In Martello et al. [2003], a branch-and-bound algorithm for the two-dimensional strip packing problem (2D-SPP) was proposed, based on the idea of staircase placements introduced in scheithauer1995equivalence. In staircase placements, a boundary is identified which separates the already packed items from the area of the bin that is still free. The boundary is an envelope composed of segments that touch either the side of an item or the bin. The resulting envelope has a staircase-like shape; the corner points are points where the envelope changes from horizontal to vertical (as noted in [Martello et al., 2000]). A similar approach was used in Crainic et al. [2008], where an extension to the staircase approach was introduced. In the proposed method, each packed item introduces a fixed number of extreme points, which are the projections of his corner points along the orthogonal axis of the bin onto the sides of either the bin or its closest packed neighbor. New niches were identified that were previously discarded by the staircase method. Both the extreme point and corner point strategies were adapted to the three-dimensional bin packing case as seen in section 2.2.

For a recent review of the literature related to the 2D-BPP, we refer the reader to Iori et al. [2021], which surveyed two-dimensional packing problems with mathematical formulations, heuristic and exact methods, relaxations, and open problems.

2.2. Three-Dimensional Single Bin-Size Bin Packing Problem

An exact approach to the 3D-SBSBPP was proposed in Martello et al. [2000] through a two-level branch-and-bound search and the use of a staircase placement method derived from the 2D-BPP field. The algorithm was initially limited to robot packable solutions, and was then extended to the general problem in Martello et al. [2007]. Faroe et al. [2003] proposed a Guided Local Search for both the 2D-SBSBPP and 3D-SBSBPP. Starting from a upper bound on the number of bins calulated through a greedy heuristic it iteratively improved the solutions by searching for new feasible solutions thanks to the proposed GLS method. The process terminated when either a computed lower bound was reached or a certain time limit had expired. In Lodi et al. [2002] a tabu search procedure that addresses the two-dimensional and three-dimensional bin packing case was proposed. The search was based on two steps, starting with one item per bin a certain number of bins were merged together. Two constructive heuristics were then used to create layers in each bin. Between each step a 1D-BPP was solved to stack the generated layers into bins. Lodi et al. [2004] later provided code for a unified tabu search addressing the multi-dimensional bin packing problem. A two-level tabu search for the multi-dimensional bin packing problem was provided in Crainic et al. [2009]. The algorithm started from a greedy feasible solution based on the extreme point heuristic introduced in Crainic et al. [2008]. In the first step of the algorithm they built a neighbourhood by swapping or moving items between bins while relaxing the bin constraints. In the second step they searched for feasible solutions by changing the relative positions of items inside of the bin. In Fekete and Schepers [2004] a new model for bin packing problem based on interval graphs was introduced. A packing was represented as an interval graph of the overlaps of items along each axis. A GRASP-based algorithm for the 3D-SBSBPP and 2D-SBSBPP was proposed by Parreño et al. [2010], the algorithm uses a maximal-space heuristich designed for container loading problems during its constructive phase. Serveral moves were then designed and combined in an improvement phase with a variable neighbourhood descent approach. In Wu et al. [2010] a genetic algorithm was presented which varied the relative positions of items in a mixed-integer linear programming model. The chromosomes represented the order of items to be packed and their orientation. Hifi et al. [2014] proposed a hybrid greedy heuristic which solves the 3D-SBSBPP in two phases; a selection phase where a subset of items to pack as a solution to a knapsack problem and a positioning phase the fixes the position of each item inside the bins. In both phases an integer linear programming model is employed. An additional optimization phase can also introduced at the cost of computational times. Gonçalves and Resende [2013] presented a biased random-key genetic algorithm for the 3D-SBSBPP (BRKGA). The chromosomes reppresented the encoding for the sequence of packing of each item. The packing was done with an underlying heuristic that uses the same maximal-space rapresentation as Parreño et al. [2010]. Zudio et al. [2018] later proposed a variable neighbourhood descent variation of BRKGA which improved the evolving process of BRKGA by finding high quality individuals in earlier generations.

2.3. Vertical Support

Vertical support (or static stability) received most of its contributions from the fields of Pallet Loading Problems (PLP) and Cargo Loading Problems (CLP). In recent years there has been lots of pubblications addressing various practical constraints dictated by the industry needs. In this section we focus on pubblications related to these two problems that dealt with the concept of vertical support.

As noted in Bortfeldt and Wäscher [2013], static stability is one of the most important constraints in Cargo Loading Problems but it is usually implicitly enforced as a consequence of load compactness or explicitly guaranteed by using filler material as a postprocessing step. A MIP formulation was proposed in Paquay et al. [2016] with the inclusion of various practical constraints like vertical support through vertex support, containers of different size and shape, weight distribution, item rotations and load bearing. Since the proposed model was complex, only small instances were solved to optimality in a reasonable time frame. The work was extended in Paquay et al. [2018] where three meta-heuristics were provided to reduce the solve time. In Galrão Ramos et al. [2016] the single container CLP is solved with static mechanical stability by combining a multi-polulation random key genetic algorithm (BRKGA) with a constructive heuristich which determins a two-dimensional box placement strategy. The pubbication also proposed a procedure to fill maximal-spaces based on mechanichal equilibrium conditions applyed to rigidbodies. In Kurpel et al. [2020] several new formulations of CLP are presented with various extensions for practical constraints such as box orientations, stability (including vertical support) and the separation of boxes. Vertical Support is formulated by a discretization of space along each axis and with the help of an overlap matrix to encode the ammount of area support each item can give to the others. The work also presents heuristic approaches and upper and lower bound techniques. In Alonso et al. [2020] a multi container loading problem is solved by using a GRASP meta-heuristich where pallets are built from a set of layers and then positioned inside a container. Practical constraints are considered like weight limits, weight distribution, dynamic stability, delivery dates. The constraint of static stability is implicitly ensured by building dense layers. In Gajda et al. [2022] a constructive randomized heuristic for solving the CLP is proposed with constraints including vertical support ensured by area support, customer priorities, load balancing, stacking constraints, and positioning constraints. In the proposed constructive heuristic a subset of extreme points are evaluated starting from two corners of the cargo to ensure a better weight distribution.

Considering Pallet Loading Problems, in Elhedhli et al. [2019] a column-generation framework and a branch-and-price solution to the mixed-case pallet loading problem was proposed with a two-dimensional layer generation problem as the pricing subproblem. The subproblem was then solved with exact methods and heuristically with additions including item groupings, item replacement the reoganization of layers and spacing. A new instance generator for instances that better reppresent industry instances was provided. The paper didn't directly address vertical support although the layering approach used implicitly favored solutions with support. The work was later extended by Gzara et al. [2020] to explicitly address practical constraints such as vertical support, load bearing, pallet weight limits and planogram sequencing. A second-order cone programming formulation was provided as a solution to a spacing problem between layers of a pallet and further extensions to the previously introduced instance generator were made. In Calzavara et al. [2021] a mathematical formulation for a layer and a pallet generation problem are defined together with heuristics and metaheuristics algorithms designed to solve the PLP with constraints on item groupings, layering, and visibility of items. The work is based on previous papers on PLP by the same authors Iori. et al. [2020]; Iori et al. [2020, 2021] that proposed a raective GRASP metaheuristic to solve the general problem. Stability of the solutions is implicitly ensured with layering and the use of filler boxes to increase the density of problematic layers.

3 | Problem description and mathematical formulation

In this thesis, we address the 3D single bin-size bin packing problem (3D-SBSBPP) with the addition of a few practical constraints. Starting from a set of items of different sizes the goal is to arrange them in the least amount of bins of a given fixed size without any overlap between each other. In addition to the standard formulation of the problem, three practical constraints need to be taken into account:

- each item inside a bin should have static stability, meaning that every item should be supported either by the ground or by other items in the same bin,
- the cage ratio of each used bin should be maximized,
- each item can be rotated orthogonally along its vertical axis.

Given a certain placement of items inside a bin of base $W \times D$ with the top of the highest item being at $z_{\max}$ and the sum of the volume of each item being V, the bin's cage ratio is defined as eq. (3.1).

$$\mathrm{CR} = \frac{V}{W \cdot D \cdot z_{\max}} \tag{3.1}$$

A high cage ratio means that even if a bin isn't fully occupied, it could be used as a base for other structures. This property is desirable in some industrial settings. It is also noted that in a single bin configuration, maximizing cage ratio is equivalent to minimizing $z_{\max}$. Finally, a visual representation of the cage ratio metric is provided in fig. 3.1.

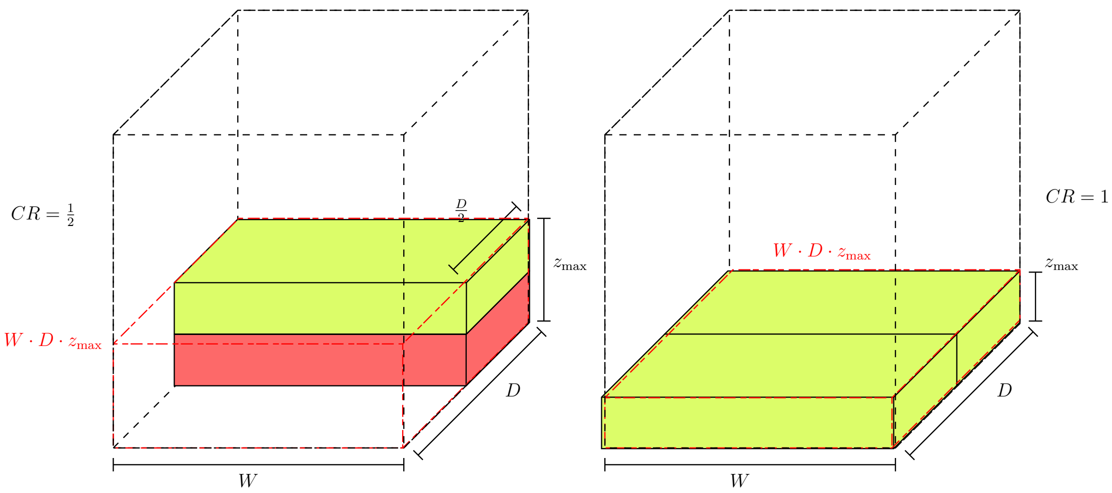


Figure 3.1: Cage ratio of two different bin configurations

Our notion of vertical support stems from rules imposed by the industry and from the literature on Pallet Loading Problems and Container Loading Problems. Vertical stability is usually ensured between horizontal or vertical slices of items as a constraint on the minimum amount of area which rests on other items (as for ex. [Gzara et al., 2020; Kurpel et al., 2020; Paquay et al., 2016]). Given a support area threshold α_s and a vertical tolerance β_s we can define an item as supported if one of the following holds

1. the sum of the overlap area over the XY-plane with every other item on which it is resting is greater than α_s times its base area. (area support)

2. the number of its corners resting on another item is greater than 3, and condition 1 holds with a lower threshold $\alpha'_s < \alpha_s$. (vertex support)

A visual representation of the condition of support is illustrated in fig. 3.2.

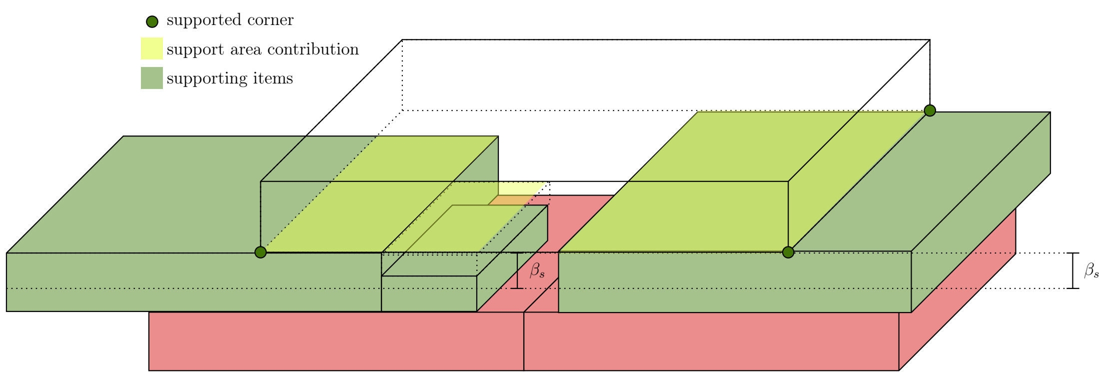


Figure 3.2: Representation of an item with vertical support given $\alpha_s = 0.5, \beta_s$

Given the definitions of our practical constraints, a conceptual formulation of our model would be

minimize	number of used bins
	the unused volume of each bin under $z_{\max}$
subject to	all items assigned to one and only one bin
	all items within the bin dimensions
	no overlaps between items in the same bin
	all items with vertical support

In section 3.1 a mixed-integer linear programming model for the 3D-SBSBPP is presented and it's later extended to include orthogonal rotations in section 3.1.1 and vertical support constraints limited to condition 1 (area support) in section 3.1.2. Cage ratio isn't directly included in the proposed MILP formulation since the evaluation of the heuristic with the model was done in a single bin configuration where minimizing the maximum height of the bin is equivalent to minimizing the cage ratio.

3.1. 3D single bin-size bin packing problem

Let $I = \{1, \dots, n\}$ be the set of items that need to be packed, $B = \{1, \dots, m\}$ the set of bins to evaluate of fixed dimensions $W \times D \times H$. Each item $i \in I$ is characterized by a given width, depth, and height (w_i, d_i, h_i). Let us introduce three continuous variables that identify the position of an item's bottom front left corner (x_i, y_i, h_i) as seen in fig. 3.3. We can now introduce a set of integer variables v_b which will be 1 if bin $b \in B$ will be used in the solution and 0 otherwise. A set of integer variables u_{ib} which will be 1 if item $i \in I$ will be placed in bin $b \in B$ and 0 otherwise. To check for overlaps, we introduce three sets of integer variables for each axis of possible overlap to determine if there is a clear order of precedence on at least one axis. This formulation is also usually used in scheduling problems. One of the introduced sets is the set x^p_{ij}, which will take the value of 1 if item $i \in I$ precedes item $j \in I$ over axis x and 0 otherwise. This condition is verified if $x_i + w_i \leq x_j$. The other two sets are defined in a similar way over the remaining axis y^p_{ij} and z^p_{ij}. An additional set of continuous variables $z^{\max}_b$ is introduced which will assume the value of the maximum $x_i + h_i$ of the items $i \in I$ placed in bin $b \in B$.

The 3D-SBSBPP can then be formulated as a mixed-integer linear programming problem:

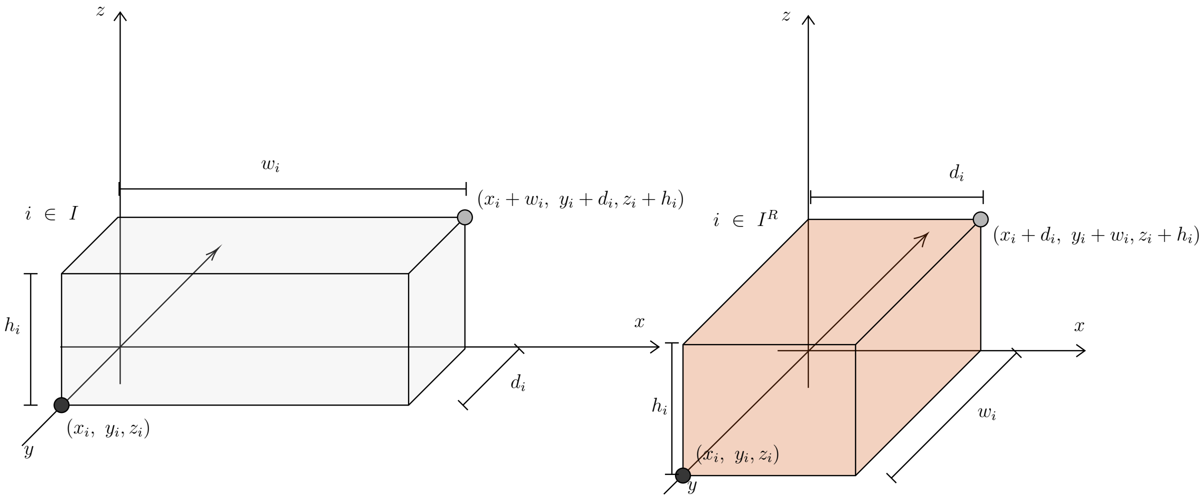


Figure 3.3: Coordinate system representation for a generic item i and its rotated clone $i \in I^R$

$$\min \quad \sum_{b \in B} (Hv_b + z_b^{max}) \tag{3.2}$$

$$\text{s.t.} \quad \sum_{b \in B} u_{ib} = 1 \qquad \forall i \in I \tag{3.3}$$

$$u_{ib} \leq v_b \qquad \forall i \in I, \forall b \in B \tag{3.4}$$

$$v_b \geq v_c \qquad \forall (b,c) \in B : b < c \tag{3.5}$$

$$x_i + w_i \leq W \qquad \forall i \in I \tag{3.6}$$

$$y_i + d_i \leq D \qquad \forall i \in I \tag{3.7}$$

$$z_i + h_i \leq H \qquad \forall i \in I \tag{3.8}$$

$$z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) \qquad \forall i \in I, \forall b \in B \tag{3.9}$$

$$(x_i + w_i) - x_j \leq W(1 - x_{ij}^p) \qquad \forall i,j \in I \tag{3.10}$$

$$x_j - (x_i + w_i) + 1 \leq W x_{ij}^p \qquad \forall i,j \in I \tag{3.11}$$

$$(y_i + d_i) - y_j \leq D(1 - y_{ij}^p) \qquad \forall i,j \in I \tag{3.12}$$

$$y_j - (y_i + d_i) + 1 \leq D y_{ij}^p \qquad \forall i,j \in I \tag{3.13}$$

$$(z_i + h_i) - z_j \leq H(1 - z_{ij}^p) \qquad \forall i,j \in I \tag{3.14}$$

$$z_j - (z_i + h_i) + 1 \leq H z_{ij}^p \qquad \forall i,j \in I \tag{3.15}$$

$$x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 \qquad \forall i,j \in I, \forall b \in B \tag{3.16}$$

The objective function 3.2 seeks to minimize the number of opened bins and the maximum height at which items were placed across bins. In a single bin configuration, since all the volume mass is concentrated inside one bin, it maximizes the cage ratio. Constraint 3.3

ensures that each item is packed in one and only one bin, while constraint 3.4 ensures that items are only packed in bins used in the solution. Since the solution has lots of symmetries with respect to the number of bins, a symmetry-breaking constraint 3.5 can be added on the opening of bins to improve solve times. Each item is also ensured to be placed inside the bin thanks to eqs. (3.6) to (3.8). The value of z_b^{max} is forced to converge to the maximum height of a given bin thanks to constraint 3.9. Constraints from 3.10 to 3.15 are used to define the precedence binary variables x_{ij}^p, y_{ij}^p, z_{ij}^p over each axis as described in the problem formulation. Constraint 3.16 then ensures that if two items are in the same bin, there needs to be at least one axis with a clear order of precedence; otherwise, the two items would overlap.

3.1.1. Orthogonal rotations

Let us extend the definition of the bin packing problem without rotations with a new formulation that allows 90 degrees rotations of each item. Let $I = I^O \cup I^R$ be the new set of items where I^O is the set of original non-rotated items, and I^R is the set of items rotated by 90 degrees. Given the set of tuples $(i, j) \in I^{OR}$ where i is the original item with dimensions (w_i, d_i, h_i) and j is the corresponding rotated clone with dimensions $(w_j, d_j, h_j) = (d_i, w_i, h_i)$, we can now rewrite constraint 3.3 as 3.17 to force only one of them to be part of the solution.

$$\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \qquad \forall (i, j) \in I^{OR} \tag{3.17}$$

3.1.2. Discrete vertical support formulation

We now extend the model to address the constraint of vertical support. In the literature, some mathematical formulations tackle the concept of area support and, in some cases, vertex support. For example, in [Gzara et al., 2020] a second-order cone programming formulation of the support constraint was used but was limited to the problem of spacing between layers with one of them being fixed in position relative to the other. A similar formulation would lead to a non-linear support constraint in our case. By introducing a discretization over the XY-plane a linear version of the constraint can be formulated similar to the one proposed in [Kurpel et al., 2020] without the need to discretize the z-axis as well.

Let us introduce some additional parameters to the model, let $0 \leq \alpha_s \leq 1$ be the amount of area that an item needs to have supported by other items. Let β_s be the tolerance to consider one item as being close enough to support another item (as seen in fig. 3.2). In addition to the support parameters, a parameter δ, which represents the discretization unit used to partition the XY-plane, is given. Let I^B be the set of all the tuples (i, j, b) such that $(i, j) \in I \land i \neq j$ and $b \in B$. We can now compute a few additional parameters that we will use to reduce the number of constraints evaluated by the model. Let γ be the maximum size over a dimension on the XY-plane between all the items as eq. (3.18), and let Δ be the set of all possible distances between the origins of two items along one discretized axis as eq. (3.19).

$$\gamma = \max_{\forall i \in I} \{w_i, d_i\} \tag{3.18}$$

$$\Delta = \left[- \left\lfloor \frac{\gamma}{\delta} \right\rfloor, \left\lfloor \frac{\gamma}{\delta} \right\rfloor \right] \tag{3.19}$$

Let $O(i, j, h, k) \to \mathbb{R}^+$ be a function that computes the amount of overlap between two items $(i, j) \in I$ given the discretized distance between each other $(h, k) \in \Delta$ such that $x_j = x_i + \delta h$ and $y_j = y_i + \delta k$ which returns the area of overlap or 0 otherwise.

Additional new variables need to be added to the ones of the original model, let s_{ij} be a set of binary variables which will assume value 1 if item $i \in I$ can offer support to item $j \in I$ and 0 otherwise. A new set of binary variables z^c_{ij} will be 1 if item $i \in I$ is close w.r.t. the z-axis to item $j \in I$, which would mean that $z_j - (z_i + h_i) \leq \beta_s$, and 0 otherwise. Let us then introduce a new set of binary variables g_i which will assume value 1 if item $i \in I$ will be on the ground or 0 otherwise. A set of binary variables s^{kh}_{ijb} will assume value 1 if item $i \in I$ will receive support from item $j \in I$ and both items will be placed in bin $b \in B$ with a discretized distance of $(k, h) \in \Delta$ between each other and 0 otherwise.

Given all the additional parameters and variables introduced, we can give a new formulation of the model with the same objective function 3.2 and the constraints in section 3.1

with the addition of the following constraints:

$$z_j - (z_i + h_i) \leq \beta_s + H(1 - z^c_{ij}) \qquad \forall (i,j) \in I : i \neq j \tag{3.20}$$
$$z_j - (z_i + h_i) \geq -\beta_s - H(1 - z^c_{ij}) \qquad \forall (i,j) \in I : i \neq j \tag{3.21}$$
$$s_{ij} \leq z^p_{ij} \qquad \forall (i,j) \in I \tag{3.22}$$
$$s_{ij} \leq z^c_{ij} \qquad \forall (i,j) \in I \tag{3.23}$$
$$s_{ij} \geq z^p_{ij} + z^c_{ij} - 2 \qquad \forall (i,j) \in I : i \neq j \tag{3.24}$$
$$\sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} \qquad \forall i \in I \tag{3.25}$$
$$z_i \leq H(1 - g_i) \qquad \forall i \in I \tag{3.26}$$
$$\sum_{(k,h) \in \Delta, b \in B : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq s_{ij} \qquad \forall (i,j) \in I \tag{3.27}$$
$$\sum_{(k,h) \in \Delta : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq u_{ib} \qquad \forall (i,j,b) \in I^B \tag{3.28}$$
$$\sum_{(k,h) \in \Delta : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq u_{jb} \qquad \forall (i,j,b) \in I^B \tag{3.29}$$

Constraints 3.20 and 3.21 ensure that z^c_{ij} is forced to 1 only when the distance over the z-axis between item i and item j is within the range $[-\beta_s, \beta_s]$. The value of s_{ij} is then assigned to the logical equation $z^p_{ij} \wedge z^c_{ij}$ thanks to constraints from 3.22 to 3.24. Since some items could be left out of the solution due to the formulation of orthogonal rotations, we also ensure that support can only come from placed items thanks to constraint 3.25. Constraint 3.26 ensures that g_i will assume value 1 if item i is on the ground Constraints from 3.27 to 3.29 ensure that if a discretized support decision s^{hk}_{ijb} is 1 then every subscript of that variable must be true in the non-discretized model, so item i can give discretized support to item j in in bin b if both items are are assinged to bin b and if i can give support to item j. They also force the selection of only one possible combination of $(h, k) \in \Delta$ for which i gives support to j in bin b.

We can then define a set of constraints which given a discretized placement s^{kh}_{ijb} limits the distance between i and j to a given continuous region in space delimited by a square of the dimension of our discretization unit δ. Given every tuple of possible discretized distances between items $(k, h) \in \Delta$ and every tuple of different pairs of items in the same bin $(i, j, b) \in I^B$ such they have a non-zero discretized overlap over the XY-plane

$(O(i,j,k,h) \neq 0)$. The resulting constraints are defined in eqs. (3.30) to (3.33).

$$x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) \tag{3.30}$$

$$x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) \tag{3.31}$$

$$y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) \tag{3.32}$$

$$y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) \tag{3.33}$$

We can then introduce a feasibility constraint thet ensures that every item that isn't on the ground is supported by other items placed beneath it by at least α_s times its area, which corresponds to condition 1 of the practical constraint of vertical support.

$$\sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h)s_{jib}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i \quad \forall i \in I \tag{3.34}$$

It is noted that every combination of $(i,j,b) \in I^B$ and $(h,k) \in \Delta$ where $O(i,j,k,h) = 0$ do not contribute to the support constraint. We can omit them from the formulation of the problem to reduce the number of constraints to evaluate.

4 | Solution algorithms

This chapter describes a heuristic algorithm to solve the 3D bin packing problem with vertical support. In section 4.1 we describe the concepts which will be used in the algorithm, like the definition of a state, insertions, and the feasibility of a solution. Since the 3D-BPP is NP-Hard, an exhaustive search for a solution is not practical, so a heuristic search is conducted by combining the beam search algorithm described in section 4.2 with the constructive heuristic described in section 4.3. The proposed algorithm takes in input the empty state (4.2), and outputs the best state based on an ordering function defined in section 4.2.1.

4.1. States

States or packings are partial solutions to the 3D-BPPWS. Since our heuristic is constructive by nature, the main idea of the algorithm is that by starting from a state representing an empty solution, we'll iteratively build new states that are always closer to a complete solution to the problem. Given the formal definition of the problem (3.1) we introduce a few new definitions to facilitate the algorithm's definition. First, it is helpful to define a collection of items that still need to be assigned to a particular bin; this collection would then be used to track how many items still need to be placed. Let us define the concept of unpacked items in relation to our MILP formulation.

Definition 4.1 (Unpacked item). *An item $i \in I$ is unpacked* ***iff***

$$\sum_{b \in B} u_{ib} = 0$$

It is also assumed that variables identifying an item's position are independent between states (changes to their values in state s won't affect state s'). In order to simplify the algorithm representation, rotations are handled by simply swapping the dimensions w_i and d_i of item $i \in I$ when needed. A state s can then be defined as follows:

- U: the set of unpacked items,

- B: the set of used bins,
- $Q = (q_1, q_2, \ldots, q_b)$: the set of supporting structures for each bin $b \in B$,
- p: the insertion pending on this state (described by def. 4.4).

Since the heuristic will open new bins when the already opened ones are full, the number of bins in each state can vary and is not fixed as a parameter to the problem like in the MILP formulation. Thanks to the newly introduced definitions, we can trivially define a function that determines if a state is a final state.

Definition 4.2. *A state s is final if there are no more items to pack*

$$IsFinal(s) = \begin{cases} 1, & s.U = \emptyset \\ 0, & otherwise \end{cases} \tag{4.1}$$

The proposed heuristic also stores additional data for each opened bin, which will then be used by the constructive heuristic described in section 4.3. This additional information is stored in the set $s.Q$ so that each bin $b \in B$ has an associated supporting structure $q_b \in s.Q$. The collection of items placed inside a bin, for example, is one piece of data that we store in this structure. Let us then introduce the concept of packed items inside a bin.

Definition 4.3 (Packed item)**.** *Given a state s and a bin $b \in s.B$, we say that item*

$$i \in I \ \text{ is packed in } b \ \textbf{\textit{iff}} \quad u_{ib} = 1$$

In addition to the set of packed items, other supporting structures are needed to facilitate checks of the problem's constraints. Given a bin $b \in s.B$ we can then define structure q_b as follows.

- J: the set of items that are packed inside b,
- Z: the set of planes inside b (section 4.3),
- T: the AABB Tree (section 4.1.1) representing the items inside b.

Both sets $q_b.J$ and $q_b.T$ contain the items packed in b but adding and accessing items in $q_b.J$ has a time complexity of $O(1)$ given an implementation as hash set while maintaining $q_b.T$ usually has a time complexity of $O(log(|q_b.J|))$.

4.1.1. AABB Tree

To determine the feasibility of a given state, checking for overlaps with items already placed is needed. Since every item is a cuboid and our problem formulation only allows for 90 deg rotations over the z-axis, each item is contained inside a bounding box, which is axis-aligned. An adequate structure to compute overlaps is then an Axis-Aligned Bounding Box Tree (AABB Tree) [van den Bergen, 1997].

AABB Trees are bounding volume hierarchies typically used for fast collision detection, and they usually offer a few operations:

- $AABBInsert(i)$: which allows inserting an axis-aligned box i in the tree
- $AABBOverlaps(i)$: which allows determining if an axis-aligned box i overlaps an element in the tree
- $AABBClosest(i, d)$: which, given an axis-aligned box i and an axis-aligned direction d, returns the closest element following that direction starting from the box i

If the tree is appropriately balanced, each operation, on average, has a time complexity of $O(log(n))$ where n is the number of elements in the tree. Maintaining an AABB Tree in the state allows us to do checks for feasibility during the construction of a solution (as detailed in 4.3.1) and feasibility checks on the final states to allow for error detection.

An additional opertation $AABBGetSupporting(i, \beta_s)$ was added to compute the set of supporting boxes of item i given a vertical tolerance β_s. This was possible by only checking intersections over the XY-plane similarly to the $AABBOverlaps$ implementation and filtering each item by the distance with tolerance.

4.1.2. Feasibility

A state s is feasible if the currently packed items in each bin $b \in s.B$ aren't overlapping any other item if they are all contained inside their bin and if each item is either on the ground or satisfies at least one of the support conditions (cond. 1, cond. 2). Since the proposed heuristic is constructive, it is more convenient to define the concept of feasibility relative to a change in the state. In the heuristic, we generate new states by applying insertions starting from an initial feasible one. Let us define the concept of insertion and how an insertion is feasible.

Definition 4.4 (Insertion). *Given a state s, we define an insertion p as a tuple (b, I) where $b \in s.B$, and I is a set of non-overlapping unpacked items such that they are* *inserted at the same z.*

$$I \subseteq s.U \land \exists z(z \in \mathbb{Z} \land \forall i(i \in I \land z_i = z))$$

Observation 4.1. *Given a state s and an insertion $p = (b, \emptyset)$ where $b \notin s.B$, p is an insertion that opens a new bin b in s.*

Definition 4.5 (Next)**.** *Let p be an insertion over a state s, we define $s' = Next(s, p)$ as a "copy" of state s where p is the pending insertion ($s'.p = p$).*

We evaluate the changes to the score of a state based on its pending insertion. In this way, we don't have to update all the structures for every evaluated state. In addition, this property let us do fewer memory clones of states that would have been discarded either way (as seen in section 4.2). We can then define an algorithm that applies a pending insertion p on a given state s with the help of a function $OpenBin(b)$ which initializes a new structure q_b with every element at its empty value. The proposed algorithm is shown in 1.

Algorithm 1: Commit

input : s
output: s'
$(b, I) \leftarrow s.p$
$s' \leftarrow Clone(s)$ `//Memory clone of` s
if $b \in s'.B$ **then**
  $q_b \leftarrow (q_i \in s'.Q : i = b)$
  $q_b.J \leftarrow q_b.J \cup I$
  $s'.U \leftarrow s'.U \setminus I$
end
else Open a new bin
  $s'.B \leftarrow s'.B \cup b$
  $s'.Q \leftarrow s'.Q \cup OpenBin(b)$
end
$s'.p \leftarrow$ none
return s'

Insertion feasibility An insertion $p = (b, I)$ that is pending on a given state s is feasible if every inserted item $i \in p.I$ satisfies the constraint of non-overlap (3.16), the constraint of support (3.34) and if it is placed within the bin. Given an item from the set of the inserted items $i \in p.I$, and the AABB tree for bin $p.b$ in the current state $q_b.T$ as T. Let I_{support} be the set of items that could support item i when placed in the bin, which could be saved in an appropriate structure or computed through the AABB tree as defined in section 4.1.1.

Let $HasSupport(i, I_{\text{support}})$ be the function that returns true if the considered item would verify at least one of the conditions of support (cond. 1 or cond. 2) or false otherwise. We can define a function $IsFeasible(i, I_{\text{support}}, T)$ which returns true if the insertion of i in bin b for state s is feasible and false otherwise. If every item $i \in p.I$ is feasible and every item in I is not overlapping the others, then insertion p is feasible. In case some items in p aren't feasible we can always define a function $RemoveInfeasibleItems(p, I_{\text{support}}, T)$ which removes every unfeasible item and returns a new insertion $p' = (b, I')$ where $I' = p.I \setminus \{i \in p.I : \neg IsFeasible(i, I_{\text{support}}, T)\}$.

Checking if a state is feasible can then be done by iteratively applying all the insertions ordered by z and updating the proper trees, or starting from an already built tree and computing the set I_{support} for each item through the tree as defined in 4.1.1.

Proposition 4.1. *A state s' derived by committing a feasible insertion p to a feasible state s is feasible.*

Observation 4.2. *We can always define the empty state s_e where*

$$\begin{cases} s_e.U = I \\ s_e.Q = \emptyset \\ s_e.B = \emptyset \end{cases}$$

and it is always feasible

4.1.3. State Hashing

From a given state, it's possible to apply two different sequences of insertions and end up with two states that have all the items in the same positions. This undesirable behavior was observed during our computational experiments. A hashing mechanism needs to be introduced to enable checking if two states are likely the same in constant time. In a state s we can identify a packed item $i \in I$ in a given position (x_i, y_i, z_i) with its given dimensions (w_i, d_i, z_i) in a given bin $b \in s.B$ with a non-commutative hashing function $hash\_nc$. The resulting hash $hash_{ib} = hash\_nc(b, x_i, y_i, z_i, w_i, d_i, h_i)$ can identify every similar packing of an item of the same shape in that specific bin spot. Since $hash_{ib}$ identifies one item with the shape of i in the same spot as i, we can use a commutative function to combine every hash for every packed item in every bin to ignore the order with which items were added to the solution. The combined hash can then be saved inside the state structure as follows.

$$s.hash = \sum_{b \in s.B} \sum_{i \in q_b.J} hash_{ib} \tag{4.2}$$

In our tests, by filtering states with the proposed hash as seen in algo. 2, with a simple 64-bit hashing function, this mechanism allowed us to filter out all equal states between iterations with a low amount of collisions. Since the combining of hashes is a simple sum with modulus, the hashing of the state can also be kept updated in constant time at each iteration by simply adding the inserted hashes in the *Commit* function (algo. 1).

4.2. Beam Search

Beam Search (BS) is a heuristic tree search algorithm designed for systems with limited memory where expanding every possible node is unfeasible. The idea behind BS is to conduct an iterative truncated breadth-first search where, at each iteration, only a limited number of k nodes is expanded. After the expansion, every new node needs to be evaluated and sorted to prune the number of nodes down to the k best ones. The algorithm keeps exploring until no further node can be expanded.

To perform BS one must define the node structure, an expansion function to generate new nodes from existing ones, a ranking between nodes, and a function to determine if a node is final.

A node in the tree can be represented as the state in section 4.1 and eq. (4.1) can be used to determine if a state is final. We also know that a new state s' derived by s by applying a feasible insertion p can be computed as in definition 4.5. This state expansion procedure, with the exception of empty insertions, will generate new states in our tree which will add a positive number of bins or packed items to the solution so, eventually, it will generate a final state.

If the starting state for the search is feasible every new state generated will be feasible and if a final state is found it will be feasible (proposition 4.1). We also note that starting from state s the time complexity to compute feasible insertions can be lower than the complexity required to update the structures that will be used for further expansions (AABB Tree insertion and balancing, memory cloning, etc.) so we modified the standard BS algorithm to separate the expansion phase from the commit phase. As noted in section 4.1.3, since by evaluating different insertions on different states it is possible to end up having two equal states, a filtering mechanism should be introduced. During each iteration, it is possible to keep the hashes of the best selected states in a hash set and discard new states with the same hash.

Given a set of initial states S^0 and the number of best states to expand at each iteration k, the described BS can be represented by algorithm 2. As observed in observation 4.2, it's possible to start the search from $S^0 = \{s_e\}$.

Algorithm 2: Beam search

```
input  : S^0, k
output: s_best
S^t ← S^0
S_final ← ∅
repeat
    S^{t+1} ← Expand(S^t) (algo. 3)
    S_final ← S_final ∪ {s ∈ S^{t+1} : IsFinal(s)} (def. 4.2)
    S^{t+1} ← S^{t+1} \ S_final
    S^{t+1} ← Sort(S^{t+1}) (sec. 4.2.1)
    S^t ← ∅
    i ← 0
    seen ← ∅
    forall s ∈ S^{t+1} do
        if s.hash ∈ seen then
            continue
        end
        S^t ← S^t ∪ Commit(s) (algo. 1)
        seen ← seen ∪ {s.hash}
        i ← i + 1
        if i > k then
            break
        end
    end
until S^t ≠ ∅
S_final ← Sort(S_final)
return first element of S_final
```

State Expansion An expansion of a state s can be seen as a new set of states S_{new} derived by a set of feasible insertions. In order to determine these insertions, an underlying heuristic is used (described in section 4.3).

The main idea in this phase of the algorithm is to find feasible insertions in all the bins at the lowest possible height for items that still need to be packed. To reduce the number of possible expansions to evaluate we limit the search to insertions of items with unique shapes. With a similar concept to the one used in section 4.1.3, an hash for each item's dimensions can be computed on the fly or pre-computed as a problem's parameter. Given each item's hash we can then group items that have the same shape. The evaluation of new insertions can then be done with two different approaches:

- **PS**: where we evaluate only the possible insertion of a single item per item type, which would generate insertions of at most 1 item,
- **PM**: where we evaluate the biggest possible insertion of a group of items of the same shape, which would generate insertions of at most the size of the group of items with the same shape.

Creating insertions of groups of similar items is usually used in Pallet Loading Problems (as for ex. [Elhedhli et al., 2019]) to create better bases of support for upper layers. With a similar intuition, the idea of placing groups of items of the same shape is to facilitate the creation of uniform planes (not necessarily layers) to use for further insertions.

Given a set of items I and a tolerance β_s we can introduce an algorithm to group them by their shape and produce a set G of tuples (h, I') where h is the hash summurizing the shape of the group and I' is the set of items grouped as in algo. 4. Once items are grouped by shape the best insertion for each class of items can be computed for each open bin. If no insertion is possible in any bin, then the only viable insertion is the bin opening insertion (observation 4.1). The described procedure is detailed in algo. 3, which can be modified with minor changes to limit the number of items to consider when in PS mode.

Algorithm 3: Expand

```
input  : S
output: S_new
forall s ∈ S do
    S_new ← ∅
    G ← GroupByHash(s.U) (algo. 4)
    placed ← false
    forall (h, I) ∈ G do
        forall q_b ∈ s.Q do
            P ← SPBestInsertion(q_b.Z, I, q_b.T) (section 4.3)
            if P ≠ ∅ then
                placed ← true
                forall p ∈ P do
                    S_new ← S_new ∪ Next(s, p) (def. 4.5)
                end
            end
        end
    end
    if placed = false then
        // Open a new bin with index |s.B| (oss. 4.1)
        S_new ← S_new ∪ Next(s, (|s.B|, ∅))
    end
end
return S_new
```

4.2.1. Sorting States

In order to sort states, an ordering needs to be defined over them. Since the selection of a state over an other is what will influence the final solution the most, parameters that are directly related to minimizing the objective function are used.

In the proposed solution to handle multiple objective functions, lexicographic ordering is used.

Definition 4.6. *Let $f_1(s), f_2(s), f_i(s), \ldots, f_n(s)$ be objective functions ordered by prece-*

Algorithm 4: Group By Hash

```
input  : I
output: G
G ← ∅
forall i ∈ I do
    generate ← true
    forall (h, I') ∈ G do
        if h = hash(w_i, d_i, h_i) then
            generate ← false
            I' ← I' ∪ i
            break
        end
    end
    if generate = true then
        G ← G ∪ (hash(w_i, d_i, h_i), {i})
    end
end
return G
```

dence based on index $i \in \mathbb{Z}$, then

$$s < s' \text{ iff } \exists j \in \mathbb{Z} : \begin{cases} f_j(s) < f_j(s') \\ f_k(s) = f_k(s'), \quad \forall k \in \mathbb{Z} : 0 \leq k < j \end{cases}$$

Scoring metrics for each state s that we want to evaluate can then be computed in the $Next$ algorithm by considering the contents of the pending insertions and updating each parameter differentially.

The defined ordering utilized is the following:

- $f_1(s) = -|s.B|$: we prefer states that opened fewer bins.
- $f_2(s) = \text{avgvol}(s)$: we prefer states that have packed more average volume between bins.
- $f_3(s) = \text{avgcageratio}(s)$: we prefer states that have better average cage ratio (eq. (3.1)) between bins.

4.3. Support Planes

Support Planes (SP) is a constructive heuristic based on an underlying 2D-BPP heuristic which is used to generate feasible insertions inside a bin starting from a set of items to pack. Since insertions must be feasible, SP maintains an internal structure to facilitate feasibility checks. The idea at the base of SP is to build a solution to the 3D-BPP by filling 2D planes called *support planes*.

Each support plane is characterized by the tuple $(z, I_{support}, I_{upper})$ where

- z: is the height of the plane,
- $I_{support}$: the set of the items that can offer support to items placed on the plane,
- I_{upper}: the set of items that will be obstacles to potential new items placed on the plane.

Every item placed in the bin can either generate a new support plane or be part of the supporting items of other planes. Items placed above a particular plane, such that $z_i + h_i > z$, are considered obstacles and are added to the I_{upper} set. When evaluating a new possible insertion, given a set of items to place I, SP selects the first feasible insertion starting from the lowest plane by using a modified version of Extreme Point in two dimensions (introduced in [Crainic et al., 2008]). Once no more insertions can be made on the lowest available support plane, it's removed from the set of planes. Since insertions always happen in the lowest possible planes, the set of obstacles of those planes is composed of items that have only their top face above the z of the evaluated plane, such that $z_i \leq z < z_i + h_i$.

The Extreme Point (EP) heuristic evaluates the placement of rectangles in a plane based on a set of reference points with a best-fit approach. Each rectangle placement generates a new set of reference points which are usually introduced based on the projection of its corner points along the orthogonal axis of the plane. The corner points of an added rectangle r placed in (x_r, y_r) of dimensions (w_r, d_r) are the top left corner $(x_r, y_r + d_r)$ and the bottom right corner $(x_r + w_r, y_r)$. In our version of the algorithm, the corner points of each item are introduced without projecting them to increase the likelihood of evaluating placements that verify the support constraint. Placements follow a first-fit approach where the algorithm selects the first point closest to the origin where a rectangle can fit with or without rotations. In order to facilitate the evaluation of reference points with support, we also generate a reference point in the bottom left corner of each item that belongs to the set of supporting items $I_{support}$. When a reference point is used for a placement, it is then removed from the pool of reference points. Before evaluating placements, the item to place are ordered based on their area. New planes have the origin of space $(0, 0)$ as an available reference point.

Since reference points are usually ordered based on the euclidean distance from the bottom left corner of the plane and the corner points are usually generated and projected towards the origin of each axis, the placements over one plane are usually biased towards the bottom left corner. To address the problem, we evaluate four instances of EP where each has a different coordinate change applied to every item that moves the plane's origin to each corner of the bin. This addition is based on similar approaches from the literature where it is usually used to more uniformly distribute weight across a surface (ex. [Gajda et al., 2022]) and was verified to yield better cage ratio results in our internal testing.

The EP procedure is called for each item to pack on a given plane. In order to produce a valid insertion, every item in the insertion shouldn't overlap each other (as stated in definition 4.4). Since the AABB tree for a given state is shared by each evaluation of a possible insertion, it can't be modified to account for temporary placements of items. This means that we need to keep a temporary AABB tree updated composed of the items that are part of the current insertion T'. We can then define a function that uses the temporary tree and the feasibility function defined in section 4.1.2 to ensure that we are producing a feasible insertion as eq. (4.3).

$$EPCanPack(i, I_{support}, T, T') = IsFeasible(i, I_{support}, T) \land \neg AABBOverlaps(i, T') \tag{4.3}$$

A graphical representation of a support plane is shown in fig. 4.1 with the reference points available. In fig. 4.2 the state of two extreme point instances for the bottom left and top left coordinate changes are shown. When a bin is opened the only support plane available is the one on the ground. In the figure different coordinate changes are marked with different colors.

Given eq. (4.3) to check if a considered placement would lead to a feasible insertion, the set of items to pack I, the AABB tree of the currently packed items T and the set of currently available support planes Z. The heuristic that will output the new best possible feasible insertion for the given set of items or an empty object can be summarized in alg. 5.

Commit Extension We now describe an extension to *Commit* (algo. 1) to update the structures needed by SP.

When a plane is filled, new insertions become less likely to be feasible. To avoid evaluating planes where no insertion is possible a mechanism to prune dead planes can be introduced.

Since best insertions for a bin are always evaluated by considering lower planes first, if all the insertions in *Expand* (algo. 3) happened over a z_{min} then we can safely remove the

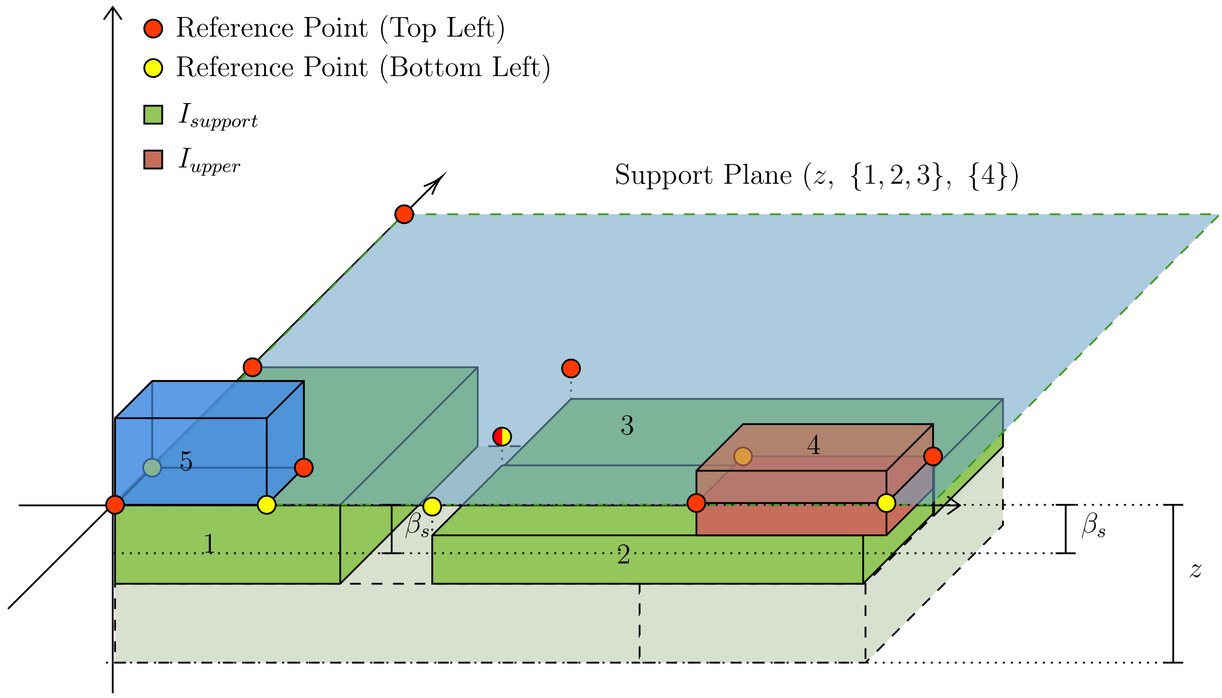


Figure 4.1: Representation of a generic support plane with a placed item

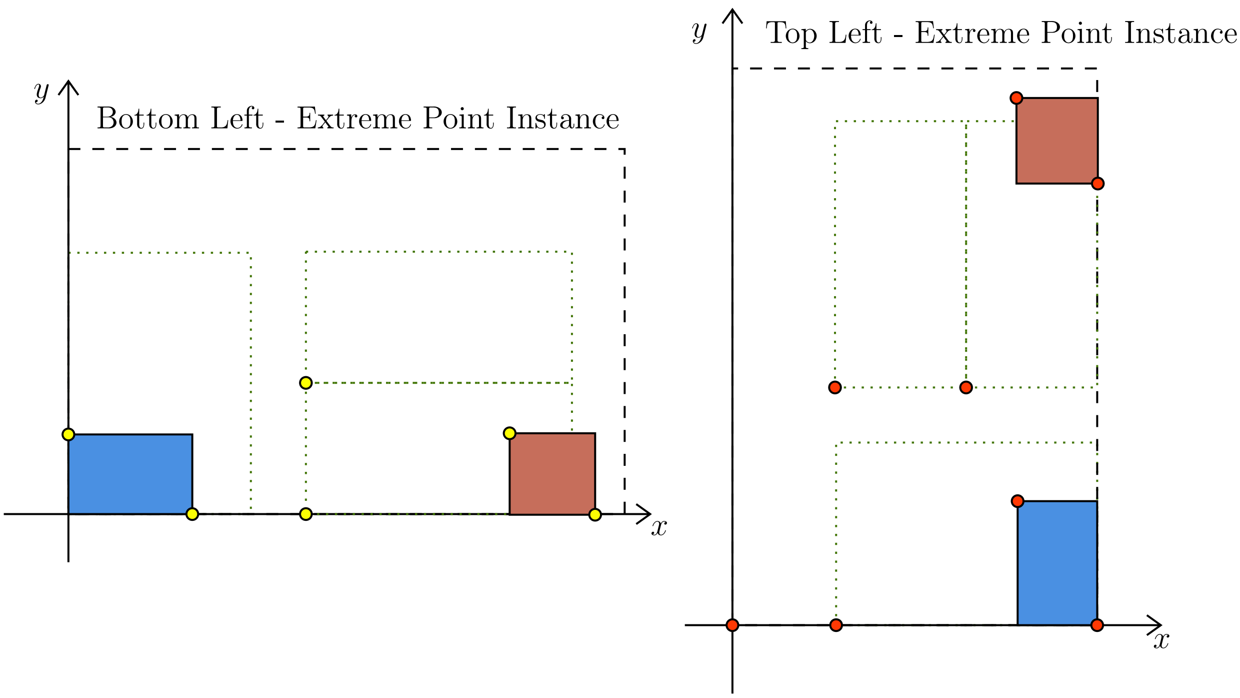


Figure 4.2: Extreme Point instances for some coordinate changes of fig. 4.1

Algorithm 5: SP Best Insertion

```
input  : Z, I, T
output: p
forall (z, I_support, I_upper) ∈ Z do
    P ← ∅
    forall possible coordinate changes do
        p ← (z, ∅)
        T' ← empty AABB tree
        //Initialize reference points
        refPoints ← (0, 0)
        forall i ∈ I_support do
            refPoints ← refPoints ∪ {(x_i, y_i)}
        end
        forall i ∈ I_upper do
            refPoints ← refPoints ∪ {(x_i + w_i, y_i), (x_i, y_i + d_i)}
        end
        sort(refPoints) // Based on euclidean distance from (0, 0)
        //Create a feasible insertion for the given items
        forall i ∈ I do
            //Evaluate first possible placement
            forall (x, y) ∈ refPoints do
                (x_i, y_i, z_i) ← (x, y, z)
                if EPCanPlace(i, T, T') then
                    EPInsertRect(p, i, T', refPoints) // alg.  6
                    break
                end
                (w_i, d_i) ← (d_i, w_i) //Try rotating i
                if EPCanPlace(i, T, T') then
                    EPInsertRect(p, i, T', refPoints) // alg.  6
                    break
                end
                (w_i, d_i) ← (d_i, w_i) //Restore original i rotation
            end
        end
        if p ≠ (z, ∅) then
            P ← P ∪ {p}
        end
    end
    sort(P) //Sorted as in section 4.3.1
    if P ≠ ∅ then
        return first element of P
    end
end
return none
```

Algorithm 6: EP Insert Rect

input : $p, i, T, refPoints$
$refPoints \leftarrow refPoints \setminus \{(x_i, y_i)\}$
$refPoints \leftarrow refPoints \cup \{(x_i + w_i, y_i), (x_i, y_i + d_i)\}$
$sort(refPoints)$ `// Based on euclidean distance from` $(0, 0)$
$p.I \leftarrow p.I \cup \{i\}$
$AABBInsert(i, T')$ `//section 4.1.1`
return

opened planes with $z < z_{min}$ for that bin. Let us introduce a z_{min} variable carried over in q_b for each bin, which is updated during the *Expand* phase with the minimum z of all the insertions on bin b. Once the best states are computed and *Commit* is called we can then use its value to prune planes in each q_b. Other operations are also necessary in the *Commit* algorithm to allow SP to update its data structures accordingly to the insertion.

Given a state s and an insertion p where each packed item $i \in p.I$ in bin b has z_i within tolerance of z and the minimum height for the considered bin $q_b.z_{min}$. The algorithm which updates the structures for a given bin b is represented by algorithm 7. This new algorithm can be used as the last step of the *Commit* algorithm for each $b \in s'.B$.

Algorithm 7: SP Apply and Filter

input : $s, p, z, z_{min}, \beta_s$
output: s
$q_b \leftarrow (q_i \in s.Q : i = p.b)$
`//Filter bad planes`
$q_b.Z \leftarrow q_b.Z \setminus \{(z', I_{support}, I_{upper}) \in q_b.Z : z' < z_{min}\}$
`//Apply insertion`
forall $i \in p.I$ **do**
    $q_b.T \leftarrow InsertAABB(i, q_b.T)$
    $generate \leftarrow true$
    forall $(z', I_{support}, I_{upper}) \in q_b.Z$ **do**
        `//Based on the distance from the top of the item`
        $dz \leftarrow z' - (z_i + h_i)$
        if $0 \leq dz \leq \beta_s$ **then**
            $generate \leftarrow false$
            $I_{support} \leftarrow I_{support} \cup i$
        end
        else if $dz < 0$ **then**
            $I_{upper} \leftarrow I_{upper} \cup i$
        end
    end
    if $generate$ **then**
        $q_b.Z \leftarrow q_b.Z \cup (z_i + h_i, \{i\}, \emptyset)$
    end
end
return s

4.3.1. Sorting Insertions

Similarly to the sorting of states (section 4.2.1), an ordering function is also needed to evaluate different insertions for the same set of items. Given the lexicographic ordering formulation in definition 4.6 a few new statistics can be calculated and stored inside an insertion to help in the evaluation. By using the AABB tree which represents the bin where the insertion is going to happen T it is also possible, given one of the inserted items $i \in p.I$, to define functions that use the tree to calculate usefull statistics:

- $CloseItems(i, T)$: which returns the number of packed items close to i,
- $CloseSameHeight(i, T)$: which returns the number of packed items in the tree close to i and with its same height,
- $CloseSameShape(i, T)$: which returns the number of packed items in the tree close to i and with its same shape,
- $TotalSupportedArea(i, T)$: which returns total base area of i which is supported by other items.

We can then sort an insertion p, given the AABB tree of the bin where the insertion will happen T, with a lexicographic ordering as follows:

- $f_1(p) = -\sum_{i \in p.I} CloseSameShape(i, T) - |p.I|$: maximize number of items inserted (of the same shape) that are close to already packed items of the same shape,
- $f_2(p) = -\sum_{i \in p.I} (w_i d_i + w_i d_i h_i)$: maximize the sum of the area and volume of each packed item,
- $f_3(p) = \max_{i \in p.I}(z_i + h_i)$: minimize the maximum height of the inserted items,
- $f_4(p) = \sum_{i \in p.I} TotalSupportedArea(i, T)$: minimize the support area available to the inserted items,
- $f_5(p) = -\sum_{i \in p.I} CloseSameHeight(i, T) - |p.I|$: maximize the number of items inserted (of the same height) that are close to already packed items of the same height,
- $f_5(p) = -\sum_{i \in p.I} CloseItems(i, T) - |p.I|$: maximize the number of items inserted that are close to already packed items.

It is noted that prefering feasible insertions that minimize the supported area of each item as in f_4 is inspired by other works on spacing from the literature. As shown in [Elhedhli et al., 2019], overly satisfying the support constraint can lead to unbalanced bins. Minimizing the supported area of each item leads to minimizing the perimeter of overlap between items which in turn results in more balanced bins and with better spacing between items.

5 | Computational results

In this chapter, in section 5.1, we evaluate the proposed heuristic against the MILP model (3.1), and in section 5.2 against other heuristics from the literature. We then show the effectiveness of our approach for our case study in section 5.3. All the tests were run on a desktop computer with an AMD Ryzen-7 5800x processor with 8 cores at 3.8 GHz and 32GB of DDR4 system RAM with Windows 10. The algorithm was implemented in Java 11, and the model was run using the python APIs from CPLEX Optimization Studio 22.1.0. In every test, CPLEX was used with a maximum runtime of 1 hour. Each evaluation against the heuristic lists both operational modes described in section 4.2 listed as "PM" and "PS". All the instances used in each section of this chapter are available at `https://github.com/artumino/BinPackingThesis/tree/main/tests/instances`. Out of the 100 instances used for our case study experiments, only 80 were freely sharable with the generation procedure also described in section 5.3.

5.1. Model validation

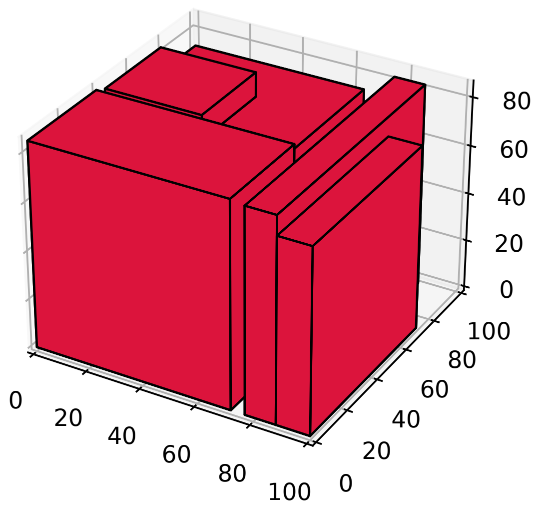
We compared our heuristic to the proposed MILP model of section 3.1 with a single bin and with no limit on the height of the bin (also referred to as the 3D strip packing problem). The heuristic was configured to run without vertex support, using only area support rules for its feasibility checks, and k was set to 200. The configured parameters for the test were $\alpha_s = 0.7$, $\beta_s = 5$, and the discretization unit for the model was $\delta = 10$. Tests were run on the first generated instance of the class 1 problems from [Martello et al., 2000] which we used for literature tests. These classes of problem have a bin base of 100×100. The test were run with an iterative approach by selecting only a limited amount of items from the selected instance, starting from the first item and increasing the number of items to pack by one at each iteration. The problem created with each iteration was saved as a test instance in the same format as the one used for literature tests. All the generated instances are available at `https://github.com/artumino/BinPackingThesis/tree/main/tests/instances/model`. A python script then loaded each generated instance sequentially and evaluated the solutions from the MILP problem and the heuristic. Each instance was run with a time limit of 1 hour. All instances with a MIP gap lower than 4% were accepted. All instances resolved to optimality, except for instance 8, which terminated with a mip gap of 0.02%.

Table 5.1 shows the obtained $z_{\max}$ value of the heuristic and the MILP solution, the runtime in seconds, and the number of items. Since the underlying problem is NP-Hard, it is shown that starting from instances of size bigger than 8 items; the MILP model becomes too slow for practical use while our heuristic maintains negligible execution time. Due to discretization errors, some of the model instances gave solutions that didn't have the expected amount of support and were marked with an asterisk. The solution to instance number 5 and instance number 7 is also shown in fig. 5.1.

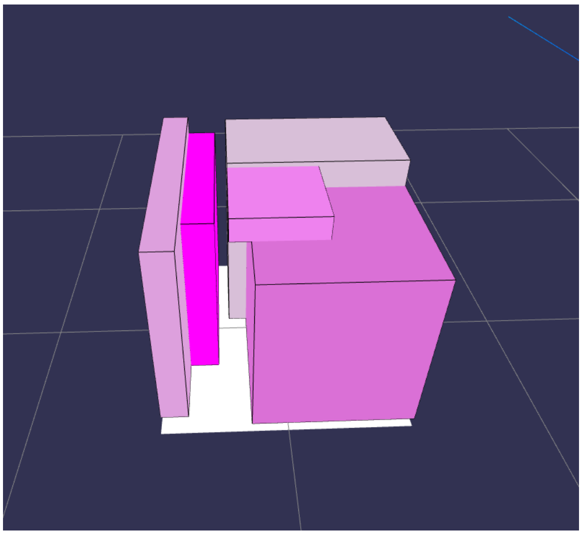
Table 5.1: Comparison with MILP model on limited set of boxes

	MILP Model			**PM**		**PS**	
n	**Max Z**	**TT(s)**	**Gap(%)**	**Max Z**	**TT(s)**	**Max Z**	**TT(s)**
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	0.02	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

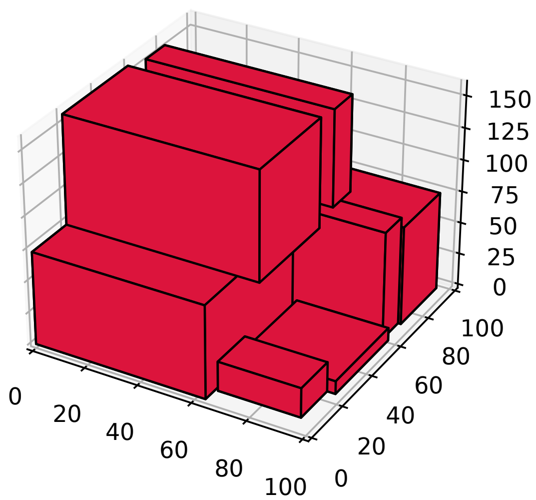
* Some boxes had lower support than expected due to discretization errors within the $0.65 \leq \alpha_s \leq 0.7$ range.



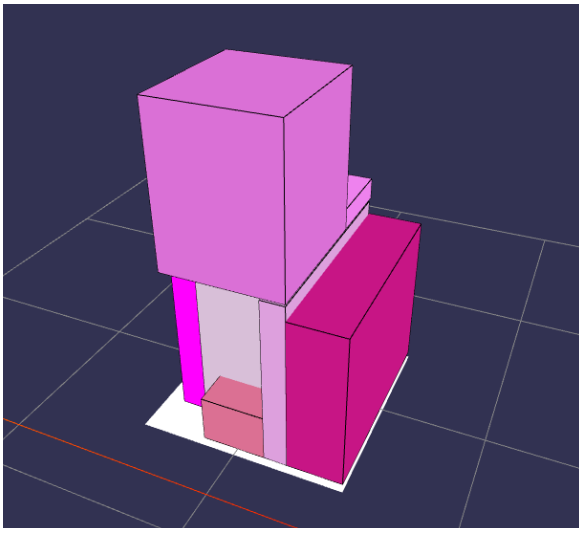
(a) MILP, Instance 5



(b) Heuristic $k = 200$, Instance 5



(c) MILP, Instance 7



(d) Heuristic $k = 200$, Instance 7

Figure 5.1: Graphical comparison between solutions from the heuristic and from the MILP model

5.2. Literature results

The heuristic was also evaluated against instances from the literature defined by [Martello et al., 2000]. Since these instances were designed for heuristics without the vertical support constraint and orthogonal rotations, we ran the experiments with a relaxed version of our heuristic. The heuristic was configured to ignore the support constraint with $\alpha_s = 0$ and $\beta_s = 1$. We also disabled orthogonal rotations and stopped scoring insertions based on the support area available (as described in section 4.3.1).

The literature instances are divided into classes from 1 to 8, with each class having a different bin size and various distributions of types of items. Instances were generated with the C++ instance generator provided by [Martello et al., 2000] at `http://hjemmesider.diku.dk/~pisinger/new3dbpp/test3dbpp.c` which allows the generation of problem instances with a given problem class and number of items to use. We generated 10 instances for each pair of problem class and number of items $n \in \{50, 100, 150, 200\}$ for a total of 320 instances.

In table 5.2 we compare the average number of opened bins across 10 instances for each problem class and n number of items combinations. The results are then compared to the most effective methods from the literature ordered by publishing date and listed as TS3 [Lodi et al., 2002], GLS [Faroe et al., 2003], GASP [Crainic et al., 2009], GVND [Parreño et al., 2010], EHGH2 [Hifi et al., 2014], BRKGA [Gonçalves and Resende, 2013], BRKGA-VD [Zudio et al., 2018]. It is noted that values for other heuristics are reported as in their publications, and our generated instances weren't the same ones which, as indicated in [Hifi et al., 2014], could have different optimal values. The best values of all the heuristics are marked in bold. Best scoring values across different configurations of our heuristic are marked in italic instead. Results show an average gap of 4.1% compared to the average value across the other heuristics and an average gap of 5.32% with respect to the best performing one.

In table 5.3 we give an approximate comparison between the average execution time of our heuristic with respect to BRKGA-VD. Execution times for BRKGA-VD were normalized by comparing directly the floating-point operations per second of the processors used, which resulted in dividing BRKGA-VD execution times by a normalization term of 9.3. The values presented are the times averaged across 8 classes of problems divided according to the size of the instance and the heuristic configuration. In the last column, we also included the average gap of each configuration of the heuristic with respect to the values of BRKGA-VD.

Table 5.2: Literature results for $k = 50$

Class	**n**	**PM** $k=50$	**PS** $k=50$	**TS3**	**GLS**	**GASP**	**EHGH2**	**GVN**	**BRKGA**	**BRKGA-VD**
1	50	14.10	*14*	**13.4**	**13.4**	**13.4**	13.8	**13.4**	**13.4**	**13.4**
	100	28	*27.7*	**26.6**	**26.6**	26.9	27.6	**26.6**	**26.6**	**26.6**
	150	38.4	*37.9*	36.7	37	37	39.8	36.4	36.4	**36.3**
	200	53	52.7	51.2	51.2	51.6	**50.6**	50.9	50.8	50.8
2	50	*14.6*	14.8	**13.8**	-	-	-	**13.8**	**13.8**	**13.8**
	100	26.6	26.7	25.7	-	-	-	25.7	25.6	**25.5**
	150	38.3	39	37.2	-	-	-	36.9	**36.6**	**36.6**
	200	*51*	51.7	50.1	-	-	-	**49.4**	**49.4**	**49.4**
3	50	13.9	13.9	**13.3**	-	-	-	**13.3**	**13.3**	**13.3**
	100	27.8	27.3	26	-	-	-	26	**25.9**	**25.9**
	150	39.2	39	37.7	-	-	-	37.6	**37.5**	**37.5**
	200	51.8	*51.2*	50.5	-	-	-	50	**49.8**	**49.8**
4	50	*29.7*	*29.7*	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**
	100	*59.2*	*59.2*	59	59	59	59.5	59	59	**58.9**
	150	*87.6*	87.7	**86.8**	**86.8**	**86.8**	90.4	**86.8**	**86.8**	**86.8**
	200	*119.5*	*119.5*	**118.8**	119	**118.8**	119	**118.8**	**118.8**	**118.8**
5	50	*8.6*	*8.6*	8.4	8.3	8.4	**7.9**	8.3	8.3	8.3
	100	16	*15.6*	15	15.1	15.1	**14.6**	15	15	15
	150	21.7	21.4	20.4	20.2	20.6	21.5	20.4	20.1	**19.9**
	200	29	28.4	27.6	27.2	27.7	29.6	**27.1**	**27.1**	**27.1**
6	50	*10*	10.3	9.9	9.8	9.9	11.8	9.8	**9.7**	**9.7**
	100	19.8	19.7	19.1	19.1	19.1	19.2	19	**18.9**	**18.9**
	150	30.3	30.2	29.4	29.4	29.5	29.8	29.2	**29**	**29**
	200	38.9	*38.5*	37.7	37.7	38	38.7	37.4	**37.3**	**37.3**
7	50	7.8	*7.6*	7.5	**7.4**	7.5	**7.4**	**7.4**	**7.4**	**7.4**
	100	*13.2*	*13.2*	12.5	12.3	12.7	13.5	12.5	**12.2**	**12.2**
	150	17.1	16.8	16.1	15.8	16.6	18.2	16	15.3	**15.2**
	200	24.9	24.7	23.9	23.5	24.2	24.1	23.5	**23.4**	**23.4**
8	50	9.9	9.7	9.3	**9.2**	9.3	9.4	**9.2**	**9.2**	**9.2**
	100	*19.6*	20	18.9	18.9	19	18.9	18.9	18.9	**18.8**
	150	25.7	25.8	24.1	23.9	24.8	26	24.1	**23.6**	**23.6**
	200	31.6	*31.2*	30.3	29.9	31.1	35.8	29.8	**29.3**	**29.3**

Table 5.3: Average execution time of literature results with bin gap

Heuristic		Execution Time (s)				Bin Gap %
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.03	0.11	0.28	0.54	5.82
	$k = 5$	0.08	0.38	1.00	2.09	5.56
	$k = 10$	0.15	0.73	1.93	4.00	5.54
	$k = 20$	0.29	1.40	3.77	7.71	5.30
	$k = 50$	0.70	3.50	9.39	19.59	5.19
PS	$k = 1$	0.05	0.18	0.50	1.05	5.61
	$k = 5$	0.12	0.72	2.10	4.62	5.26
	$k = 10$	0.23	1.38	4.11	8.95	5.19
	$k = 20$	0.46	2.67	8.21	17.64	4.98
	$k = 50$	1.12	6.45	20.39	43.50	4.75
BRKGA-VD		1.85	8.69	20.53	39.85	0.00

5.3. Case study results

Case study experiments were conducted on a series of problem instances that were divided between 20 real-world instances and 80 generated instances composed of items sampled from a population of real-world products. Each instance was anonymized and converted to a format similar to the one used for the literature tests thanks to a Rust program available at `https://github.com/artumino/BinPackingThesis/tree/main/additional/testConverter`. Support parameters for the heuristic were set to $\alpha_s = 0.7$ and $\beta_s = 10$ with both area and vertex support enabled. All dimensions of the bin, items, and tolerances are assumed to be in millimeters. Different values of $k \in \{1, 5, 10, 20, 50, 100, 200\}$ were tested as well as both placement modes.

Each generated instance is composed of a random number of n items sampled from a given range of possible instance sizes. All generated instances had a bin of standard size $800 \times 1200 \times 2000$. We identified four ranges of interest and generated 20 instances for each range as follows:

- **Class 1-20**: a class of instances with the target sizes for our case-study $n \in [70, 100]$
- **Class 21-40**: a class of small sized instances with number of items $n \in [50, 70]$
- **Class 41-60**: a class of medium sized instances with number of items $n \in [70, 120]$
- **Class 61-80**: a class of big instances with number of items $n \in [120, 200]$

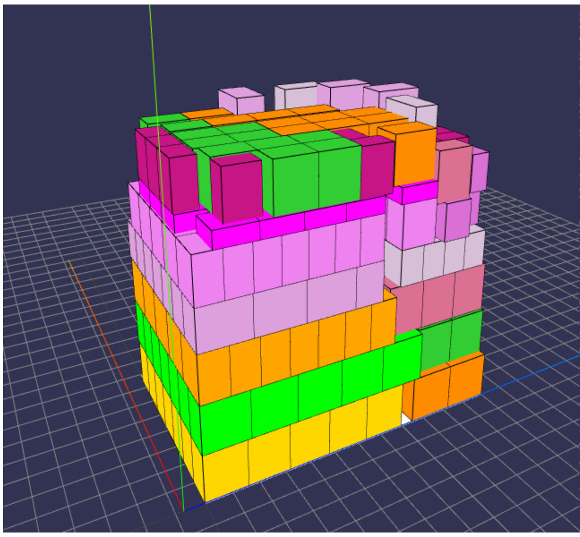
Given an input n (the size of the test instance), the generation procedure uniformly sampled an item type from a population of real-world products. The quantity of items of that type to add to the test instance was then sampled from a normal distribution $\mathcal{N}(\mu = 4.6, \sigma = 1.8)$ with parameters calculated from the real-world instances. The sampled quantity was then floored to be an integer value and clamped to avoid generating more items than n. This uniform sampling of item types was done until the instance was composed of n items.

Real-world instances are listed as **Class 81-100** and have a variable number of items between $[25, 345]$, a variable bin size (although similar to the one used for the generated instances), and a variable number of items of the same type. Some instances were homogeneous with only a few unique items, and some were heterogeneous with every item of a different type. An example of real-world instances is shown in fig. 5.2 where items of the same shape are marked with the same color.

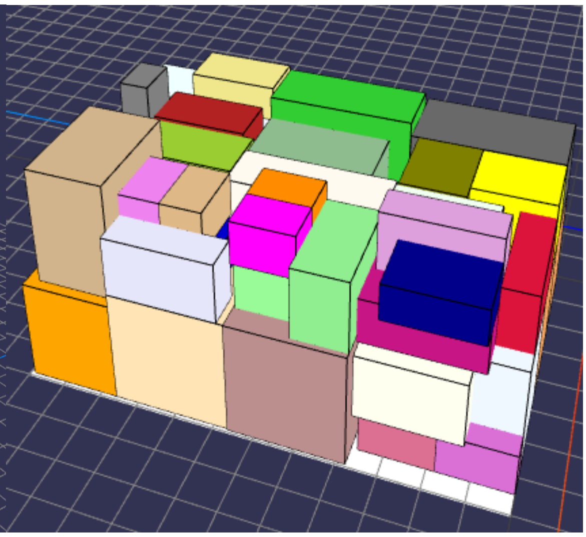
Table 5.3 shows the average results over the 20 instances per class, divided by each configuration of the heuristic with different values of k. The results shown include the total execution time in milliseconds (TT), the number of opened bins (B), and the average cage ratio between the opened bins (CR). It is clear that although the "PS" method had better results when dealing with a relaxed version of the problem, grouping items by type shows considerable improvements under all measured metrics when taking vertical support into account. Most of the configurations lead to an average cage ratio of more than 70%, which was the target value for our case study. It is also possible to see that increasing the value of k improves the quality of the solutions, on average, at the expense of a higher execution time. By doing a case-by-case analysis of each experiment, we discovered that increasing k can temporarily worsen the solution in some instances. A further study of the problematic instances highlighted that the current greedy scoring mechanism of the states leads to cutting out good solutions too early. Further improvements are proposed in chapter 6.

Table 5.4: Summary of case study tests

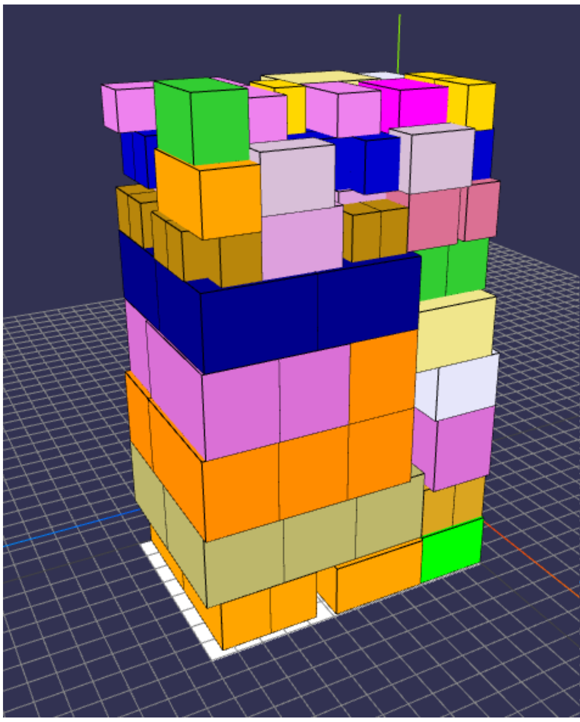
Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
Class 1-20	$k = 1$	187.25	1.15	64.10	54.95	1.05	**70.69**
	$k = 5$	489.40	1.05	70.38	111.75	1.00	**75.36**
	$k = 10$	861.30	1.05	71.94	182.20	1.00	**75.77**
	$k = 20$	1,588.15	1.05	72.04	308.45	1.00	**76.60**
	$k = 50$	3,896.40	1.05	73.07	690.80	1.00	**76.95**
	$k = 100$	7,789.90	1.00	75.45	1,204.35	1.00	**78.46**
	$k = 200$	15,817.20	1.05	74.99	2,192.75	1.00	**78.27**
Class 21-40	$k = 1$	50.90	1.00	68.21	17.80	1.00	**73.66**
$n = [50, 70]$	$k = 5$	138.40	1.00	71.92	39.20	1.00	**74.78**
	$k = 10$	253.10	1.00	73.15	74.95	1.00	**75.28**
	$k = 20$	483.85	1.00	73.86	124.30	1.00	**76.46**
	$k = 50$	1,193.55	1.00	74.77	288.50	1.00	**77.02**
	$k = 100$	2,358.50	1.00	75.08	535.30	1.00	**77.11**
	$k = 200$	4,769.85	1.00	76.69	1,033.00	1.00	**78.64**
Class 41-60	$k = 1$	292.35	1.30	65.62	60.55	1.25	**71.34**
$n = [70, 120]$	$k = 5$	1,025.65	1.30	67.97	172.35	1.30	**72.53**
	$k = 10$	1,910.60	1.30	68.46	304.25	1.25	**72.04**
	$k = 20$	3,666.40	1.30	68.68	571.90	1.25	**74.01**
	$k = 50$	7,649.95	1.25	71.32	1,152.40	1.25	**75.25**
	$k = 100$	15,848.15	1.25	72.90	1,956.55	1.20	**75.67**
	$k = 200$	32,420.40	1.25	73.29	3,472.50	1.20	**76.10**
Class 61-80	$k = 1$	1,371.00	2.20	64.68	158.00	2.05	**69.11**
$n = [120, 200]$	$k = 5$	5,751.95	2.15	66.66	531.80	1.95	**71.31**
	$k = 10$	9,040.85	2.05	68.56	1,033.15	1.90	**72.69**
	$k = 20$	19,116.60	2.15	67.81	1,881.70	1.90	**73.84**
	$k = 50$	52,937.40	2.05	69.94	3,744.70	2.00	**71.25**
	$k = 100$	98,271.55	2.10	70.04	7,010.65	1.90	**73.80**
	$k = 200$	170,191.55	2.00	71.15	13,544.15	1.90	**75.01**
Class 81-100	$k = 1$	217.85	1.20	66.74	34.60	1.20	**68.68**
	$k = 5$	582.30	1.20	69.03	71.00	1.20	**71.41**
	$k = 10$	1,071.75	1.20	69.65	129.95	1.20	**72.00**
	$k = 20$	2,013.95	1.20	71.52	218.40	1.20	**71.97**
	$k = 50$	5,338.20	1.20	71.44	523.40	1.20	**72.57**
	$k = 100$	10,402.95	1.20	**72.68**	995.00	1.20	71.74
	$k = 200$	21,525.50	1.20	73.30	2,086.50	1.15	**73.95**
Global Avg	$k = 1$	423.87	1.37	65.87	65.18	1.31	**70.70**
	$k = 5$	1,597.54	1.34	69.19	185.22	1.29	**73.08**
	$k = 10$	2,627.52	1.32	70.35	344.90	1.27	**73.56**
	$k = 20$	5,373.79	1.34	70.78	620.95	1.27	**74.57**
	$k = 50$	14,203.10	1.31	72.11	1,279.96	1.29	**74.61**
	$k = 100$	26,934.21	1.31	73.23	2,340.37	1.26	**75.36**
	$k = 200$	48,944.90	1.30	73.89	4,465.78	1.25	**76.39**



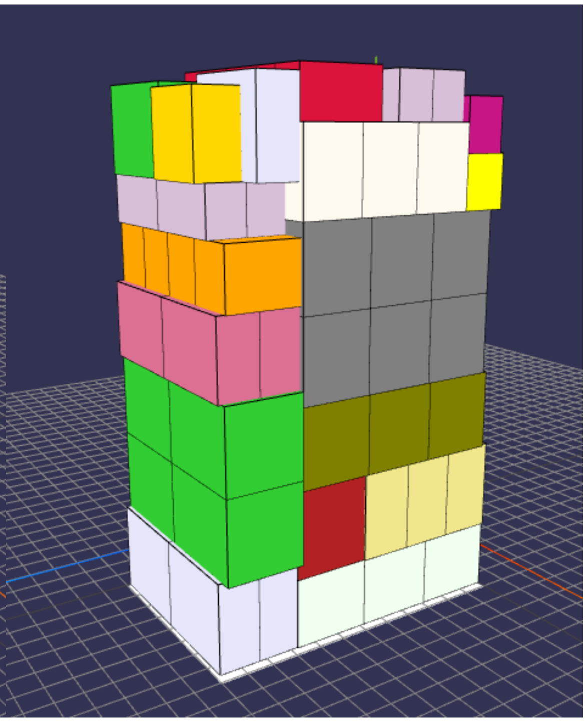
(a) Instance 95



(b) Instance 82



(c) Instance 56



(d) Instance 66, Bin 1

Figure 5.2: Solutions of case study tests with the "PM" placement and $k = 200$

6 | Conclusions and future developments

In this thesis we studied the problem of three-dimensional single bin-size bin packing with support by providing two heuristics. The first contribution is a constructive heuristic that uses a modified version of the first-fit two-dimensional extreme points heuristic of Crainic et al. [2008] which includes the notion of area and vertex support. The heuristic builds solutions to the single bin 3D-BPP by filling planes called support planes that are added to the solution based on the inserted items, these structures are then used to facilitate calculations involving the support constraint. Although layers aren't explicitly built, we evaluate insertions in two different placement modes which allows for placements of groups of similar objects. We also propose a beam-search algorithm which uses multiple instances of the constructive heuristics and evaluates different sequences of insertions to expand the solution space with a hashing mechanism to avoid same packings.

The resulting heuristic is then validated against an equivalent mixed-integer formulation of the 3D-SBSSPP with discretized support which can only solve small instaces of the problem. We then evaluated the results of a relaxed version of our heuristic against other heuristics from the literature on classical benchmark instances from Martello et al. [2000].

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A | Appendix A

Table A.1: Case study results 1-5

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
1	$k = 1$	403	1	69.54	218	1	69.82
	$k = 5$	384	1	70.9	157	1	74.64
	$k = 10$	502	1	71.47	151	1	74.64
	$k = 20$	786	1	71.47	187	1	73.33
	$k = 50$	1732	1	71.47	357	1	74.26
	$k = 100$	3524	1	71.47	613	1	76.54
	$k = 200$	6892	1	74.71	1020	1	74.64
2	$k = 1$	266	2	48.2	67	1	77.19
	$k = 5$	835	1	78.58	196	1	84.69
	$k = 10$	1537	1	78.58	311	1	86.65
	$k = 20$	2045	1	83.22	607	1	87.59
	$k = 50$	5233	1	83.22	1706	1	87.84
	$k = 100$	11422	1	83.22	3226	1	86.94
	$k = 200$	22911	1	83.22	3860	1	85.87
3	$k = 1$	169	1	73.36	62	1	65.48
	$k = 5$	335	1	73.36	104	1	73.31
	$k = 10$	532	1	73.36	141	1	72.73
	$k = 20$	1003	1	73.36	245	1	74.86
	$k = 50$	2621	1	73.46	457	1	74.51
	$k = 100$	5209	1	73.46	897	1	75.02
	$k = 200$	10781	1	74.2	1676	1	78.9
4	$k = 1$	384	1	53.7	57	1	79.2
	$k = 5$	1048	1	59.27	153	1	79.91
	$k = 10$	1934	1	59.27	203	1	76.37
	$k = 20$	3754	1	59.27	313	1	79.44
	$k = 50$	9266	1	65.04	754	1	82.18
	$k = 100$	18445	1	72.44	1467	1	82.18
	$k = 200$	36636	1	72.44	2956	1	82.18
5	$k = 1$	52	1	67.48	25	1	74.44
	$k = 5$	192	1	73.22	75	1	76.16
	$k = 10$	324	1	73.22	104	1	69.76
	$k = 20$	641	1	73.22	144	1	69.18
	$k = 50$	1613	1	73.22	255	1	68.65
	$k = 100$	3466	1	73.22	518	1	68.65
	$k = 200$	7149	1	73.22	1050	1	68.74

Table A.2: Case study results 6-10

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
6	$k = 1$	357	2	35.53	76	1	78.78
	$k = 5$	939	1	65.88	196	1	79.74
	$k = 10$	1638	1	74.23	419	1	82.46
	$k = 20$	3257	1	73.74	624	1	80.73
	$k = 50$	8533	1	73.74	1295	1	80.52
	$k = 100$	16594	1	75.36	2019	1	78.53
	$k = 200$	33658	1	75.36	3925	1	77.35
7	$k = 1$	308	1	62.38	32	1	68.32
	$k = 5$	774	1	62.38	98	1	71.36
	$k = 10$	1052	1	69.06	148	1	81.03
	$k = 20$	2003	1	69.06	299	1	82.35
	$k = 50$	4828	1	71.32	697	1	79.09
	$k = 100$	10009	1	71.32	1138	1	82.12
	$k = 200$	19931	1	71.32	2289	1	82.12
8	$k = 1$	50	1	74.2	36	1	79.27
	$k = 5$	142	1	74.2	46	1	78.78
	$k = 10$	240	1	76.51	66	1	83.85
	$k = 20$	472	1	80.77	126	1	83.85
	$k = 50$	1196	1	82.12	317	1	83.85
	$k = 100$	2410	1	82.12	617	1	83.85
	$k = 200$	4844	1	82.12	1212	1	83.85
9	$k = 1$	188	1	67.28	41	1	69.6
	$k = 5$	580	1	74.36	135	1	73.26
	$k = 10$	989	1	74.36	319	1	81.8
	$k = 20$	1795	1	75.21	364	1	77.87
	$k = 50$	4573	1	78.8	1377	1	80.34
	$k = 100$	8641	1	78.8	1557	1	76.19
	$k = 200$	18028	1	78.8	3058	1	76.07
10	$k = 1$	37	1	75.91	24	1	72.18
	$k = 5$	229	1	76.34	65	1	74.73
	$k = 10$	321	1	76.34	102	1	74.73
	$k = 20$	645	1	76.34	186	1	74.73
	$k = 50$	1641	1	76.34	413	1	80.24
	$k = 100$	3177	1	76.34	685	1	79.76
	$k = 200$	6562	1	76.34	1376	1	79.76

Table A.3: Case study results 11-15

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
11	$k = 1$	83	1	66.82	26	1	69.88
	$k = 5$	227	1	66.82	88	1	73.04
	$k = 10$	437	1	73.04	94	1	73.04
	$k = 20$	901	1	73.04	182	1	73.73
	$k = 50$	2163	1	74.9	374	1	72.55
	$k = 100$	4271	1	75.09	656	1	70.51
	$k = 200$	8965	1	75.09	1338	1	73.61
12	$k = 1$	290	2	54.85	41	2	38.76
	$k = 5$	905	2	49.36	149	1	79.67
	$k = 10$	1746	2	49.36	153	1	77.74
	$k = 20$	3048	2	40.42	304	1	76.43
	$k = 50$	6768	2	40.91	526	1	77.53
	$k = 100$	13867	1	73.25	1054	1	79.32
	$k = 200$	29292	2	42.83	2136	1	79.32
13	$k = 1$	161	1	53.61	23	1	68.76
	$k = 5$	333	1	69.77	62	1	72.32
	$k = 10$	585	1	69.77	120	1	73.12
	$k = 20$	1140	1	70.64	156	1	64.05
	$k = 50$	2821	1	70.64	420	1	65.88
	$k = 100$	5610	1	70.64	833	1	73.76
	$k = 200$	11427	1	75.46	1119	1	72.44
14	$k = 1$	209	1	66.77	30	1	70.43
	$k = 5$	512	1	66.77	71	1	69.51
	$k = 10$	959	1	71.72	229	1	80.74
	$k = 20$	1773	1	71.72	442	1	73.78
	$k = 50$	4093	1	72	993	1	78.03
	$k = 100$	8228	1	72	1915	1	77.62
	$k = 200$	16602	1	75.58	2885	1	74.15
15	$k = 1$	216	1	79.47	74	1	78.65
	$k = 5$	710	1	79.47	161	1	66.78
	$k = 10$	1415	1	80.62	245	1	70.17
	$k = 20$	2661	1	80.62	365	1	77.37
	$k = 50$	6673	1	80.62	865	1	80.52
	$k = 100$	12879	1	80.62	1530	1	85.66
	$k = 200$	25418	1	80.62	3552	1	85.66

Table A.4: Case study results 16-20

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
16	$k = 1$	114	1	68.8	125	1	80.6
	$k = 5$	471	1	71.42	139	1	76.27
	$k = 10$	808	1	71.42	265	1	76.27
	$k = 20$	1529	1	72.13	473	1	78.77
	$k = 50$	3901	1	72.92	1081	1	78.77
	$k = 100$	7624	1	76.33	1654	1	79.27
	$k = 200$	15602	1	76.33	3217	1	78.91
17	$k = 1$	98	1	71.2	27	1	73.37
	$k = 5$	263	1	77.41	61	1	71.09
	$k = 10$	535	1	77.41	148	1	72.96
	$k = 20$	1014	1	77.41	276	1	76
	$k = 50$	2540	1	78.54	616	1	75.31
	$k = 100$	4890	1	78.54	989	1	79.99
	$k = 200$	10395	1	78.54	1790	1	79.92
18	$k = 1$	108	1	60.55	36	1	69.39
	$k = 5$	244	1	75.18	59	1	80.2
	$k = 10$	434	1	75.18	127	1	78
	$k = 20$	801	1	75.18	204	1	78
	$k = 50$	1957	1	77.61	376	1	78.85
	$k = 100$	3807	1	77.61	796	1	78.85
	$k = 200$	7824	1	80	1575	1	77.54
19	$k = 1$	113	1	67.04	52	1	66.58
	$k = 5$	330	1	77.57	133	1	77.82
	$k = 10$	623	1	77.57	172	1	59.29
	$k = 20$	1289	1	77.57	435	1	77.44
	$k = 50$	2977	1	77.57	589	1	74.24
	$k = 100$	6064	1	77.57	1235	1	83.16
	$k = 200$	12258	1	78.13	2461	1	83.16
20	$k = 1$	139	1	65.36	27	1	63
	$k = 5$	335	1	65.36	87	1	73.91
	$k = 10$	615	1	66.36	127	1	70.1
	$k = 20$	1206	1	66.36	237	1	72.51
	$k = 50$	2799	1	66.92	348	1	65.81
	$k = 100$	5661	1	69.53	688	1	71.21
	$k = 200$	11169	1	75.42	1360	1	71.21

Table A.5: Case study results 21-25

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
21	$k = 1$	33	1	67.44	18	1	77.23
	$k = 5$	97	1	75.35	38	1	71.08
	$k = 10$	166	1	75.35	60	1	72.69
	$k = 20$	298	1	76.28	108	1	71.65
	$k = 50$	741	1	76.28	173	1	69.42
	$k = 100$	1399	1	76.28	349	1	69.42
	$k = 200$	2870	1	76.28	692	1	71.4
22	$k = 1$	29	1	72.11	9	1	72.94
	$k = 5$	69	1	73.29	26	1	74.94
	$k = 10$	141	1	74.21	50	1	74.14
	$k = 20$	277	1	74.94	80	1	74.65
	$k = 50$	706	1	74.94	184	1	75.91
	$k = 100$	1385	1	78.81	354	1	75.91
	$k = 200$	2802	1	78.81	716	1	75.91
23	$k = 1$	34	1	63.84	13	1	81.33
	$k = 5$	94	1	73.15	31	1	78.01
	$k = 10$	173	1	75.99	57	1	77.42
	$k = 20$	329	1	75.99	84	1	81.09
	$k = 50$	808	1	76.28	211	1	83.06
	$k = 100$	1594	1	76.28	381	1	83.06
	$k = 200$	3164	1	79.14	743	1	84.51
24	$k = 1$	26	1	73.86	8	1	73.12
	$k = 5$	65	1	75.64	23	1	73.12
	$k = 10$	110	1	79.19	40	1	78.76
	$k = 20$	207	1	79.45	77	1	77
	$k = 50$	511	1	81.14	173	1	70.35
	$k = 100$	1076	1	81.14	349	1	70.35
	$k = 200$	2148	1	81.14	682	1	79.63
25	$k = 1$	82	1	70.7	37	1	71.53
	$k = 5$	229	1	70.7	99	1	69.62
	$k = 10$	431	1	70.7	181	1	69.62
	$k = 20$	816	1	70.7	283	1	78.27
	$k = 50$	2005	1	70.7	462	1	75.37
	$k = 100$	4011	1	71.74	933	1	75.37
	$k = 200$	8134	1	72.09	1759	1	75.37

Table A.6: Case study results 26-30

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
26	$k = 1$	35	1	71.69	16	1	73.01
	$k = 5$	102	1	75.92	26	1	77.99
	$k = 10$	194	1	75.92	49	1	76.17
	$k = 20$	400	1	75.92	68	1	72.31
	$k = 50$	1008	1	75.92	161	1	73.89
	$k = 100$	2126	1	76.05	327	1	73.89
	$k = 200$	4295	1	76.05	644	1	73.89
27	$k = 1$	63	1	66.47	22	1	76.56
	$k = 5$	164	1	70.9	38	1	77.73
	$k = 10$	276	1	70.9	59	1	72.99
	$k = 20$	522	1	70.9	105	1	77.81
	$k = 50$	1291	1	71.76	249	1	77.81
	$k = 100$	2563	1	71.76	483	1	77.81
	$k = 200$	5463	1	77.81	947	1	77.81
28	$k = 1$	55	1	68.54	17	1	77.78
	$k = 5$	136	1	68.54	36	1	78.32
	$k = 10$	236	1	70.2	59	1	79.3
	$k = 20$	429	1	73.5	103	1	83.47
	$k = 50$	1096	1	73.5	272	1	83.95
	$k = 100$	2151	1	73.5	451	1	83.95
	$k = 200$	4486	1	74.86	931	1	86.45
29	$k = 1$	48	1	73.14	17	1	73.85
	$k = 5$	144	1	75.49	41	1	74.03
	$k = 10$	244	1	79.77	68	1	76.69
	$k = 20$	462	1	79.77	132	1	77.94
	$k = 50$	1104	1	81.33	459	1	80.83
	$k = 100$	2230	1	81.33	706	1	84.8
	$k = 200$	4511	1	84.64	1425	1	84.8
30	$k = 1$	25	1	72.31	8	1	75.76
	$k = 5$	111	1	72.31	18	1	75.63
	$k = 10$	199	1	76.44	34	1	75.63
	$k = 20$	349	1	76.44	71	1	75.76
	$k = 50$	946	1	76.44	163	1	80.23
	$k = 100$	1865	1	76.57	317	1	80.23
	$k = 200$	3472	1	79.56	645	1	82.23

Table A.7: Case study results 31-35

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
31	$k = 1$	37	1	59.65	14	1	74.05
	$k = 5$	106	1	71.6	38	1	72.43
	$k = 10$	191	1	72.36	61	1	72.43
	$k = 20$	365	1	72.36	103	1	73.48
	$k = 50$	918	1	75.08	247	1	80.34
	$k = 100$	1745	1	75.08	406	1	77.23
	$k = 200$	3442	1	77.46	798	1	78.02
32	$k = 1$	61	1	65.02	24	1	69.44
	$k = 5$	148	1	72.09	30	1	78.06
	$k = 10$	253	1	73.71	45	1	72.09
	$k = 20$	515	1	73.71	83	1	72.09
	$k = 50$	1229	1	73.71	214	1	72.09
	$k = 100$	2481	1	73.71	425	1	72.09
	$k = 200$	5037	1	73.71	850	1	72.37
33	$k = 1$	106	1	71.14	40	1	79.25
	$k = 5$	239	1	75.74	64	1	78.21
	$k = 10$	462	1	75.74	120	1	78.21
	$k = 20$	854	1	78.73	192	1	78.21
	$k = 50$	2130	1	78.73	412	1	83.68
	$k = 100$	4239	1	78.73	837	1	83.68
	$k = 200$	8519	1	80.04	1685	1	83.68
34	$k = 1$	36	1	61.8	12	1	62.54
	$k = 5$	95	1	64.99	50	1	69.19
	$k = 10$	170	1	64.99	78	1	73.28
	$k = 20$	321	1	67.87	150	1	73.28
	$k = 50$	735	1	73.08	250	1	75.11
	$k = 100$	1390	1	73.08	504	1	75.11
	$k = 200$	2791	1	79.15	868	1	72.67
35	$k = 1$	36	1	69.35	13	1	69.86
	$k = 5$	106	1	69.35	37	1	71.2
	$k = 10$	253	1	72.43	81	1	71.2
	$k = 20$	480	1	74.53	85	1	73.06
	$k = 50$	1277	1	74.58	188	1	71.65
	$k = 100$	2378	1	74.58	373	1	71.65
	$k = 200$	4929	1	74.58	755	1	71.65

Table A.8: Case study results 36-40

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
36	$k = 1$	121	1	66.81	28	1	75
	$k = 5$	343	1	66.81	53	1	73.69
	$k = 10$	618	1	66.81	180	1	78.01
	$k = 20$	1136	1	67.11	291	1	82.68
	$k = 50$	2664	1	71.94	711	1	76.98
	$k = 100$	5253	1	73.12	1223	1	80.92
	$k = 200$	10658	1	75.67	2579	1	81.79
37	$k = 1$	44	1	73.94	13	1	68.3
	$k = 5$	106	1	73.94	36	1	71.96
	$k = 10$	184	1	73.94	92	1	74.65
	$k = 20$	384	1	73.94	124	1	76.03
	$k = 50$	909	1	73.94	270	1	76.34
	$k = 100$	1883	1	73.94	461	1	68.16
	$k = 200$	3836	1	73.94	907	1	79.57
38	$k = 1$	39	1	70.69	17	1	74.76
	$k = 5$	132	1	70.69	43	1	77.39
	$k = 10$	259	1	70.69	83	1	77.39
	$k = 20$	522	1	71.28	160	1	77.39
	$k = 50$	1289	1	72.24	396	1	77.39
	$k = 100$	2606	1	72.24	950	1	78.95
	$k = 200$	5212	1	72.24	1208	1	79.17
39	$k = 1$	24	1	62.25	7	1	76.42
	$k = 5$	71	1	69.91	14	1	79.74
	$k = 10$	121	1	71.76	23	1	79.74
	$k = 20$	230	1	71.76	47	1	80.77
	$k = 50$	559	1	71.76	110	1	80.77
	$k = 100$	1065	1	71.76	216	1	80.77
	$k = 200$	2097	1	74.15	444	1	80.77
40	$k = 1$	84	1	63.4	23	1	70.51
	$k = 5$	211	1	71.97	43	1	73.19
	$k = 10$	381	1	71.97	79	1	75.09
	$k = 20$	781	1	71.97	140	1	72.27
	$k = 50$	1945	1	71.97	465	1	75.22
	$k = 100$	3730	1	71.97	661	1	78.91
	$k = 200$	7531	1	72.51	1382	1	81.13

Table A.9: Case study results 41-45

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
41	$k = 1$	326	1	68.36	38	1	70.62
	$k = 5$	697	1	74.25	104	2	52.93
	$k = 10$	1062	1	75.63	122	1	74.21
	$k = 20$	1948	1	77.68	262	1	78.73
	$k = 50$	4790	1	77.68	668	1	74.45
	$k = 100$	10170	1	79.15	1062	1	77.1
	$k = 200$	20996	1	79.15	2013	1	78.91
42	$k = 1$	149	1	70.28	20	1	67.43
	$k = 5$	479	1	75.83	108	1	68.59
	$k = 10$	849	1	75.83	214	1	73.76
	$k = 20$	1623	1	75.83	374	1	75.83
	$k = 50$	4004	1	75.83	907	1	81.2
	$k = 100$	8093	1	75.83	1923	1	77.36
	$k = 200$	16798	1	75.83	2991	1	77.3
43	$k = 1$	174	2	64.74	96	2	65.13
	$k = 5$	1628	2	71.89	370	2	64.66
	$k = 10$	3028	2	71.89	515	2	58.88
	$k = 20$	6518	2	70.91	1089	2	67.08
	$k = 50$	9427	2	68.02	2335	2	80.84
	$k = 100$	21206	2	71.12	3015	2	76.47
	$k = 200$	67001	2	67.51	5429	2	77.53
44	$k = 1$	367	1	61.69	61	1	71.18
	$k = 5$	868	1	69.72	205	1	83
	$k = 10$	1506	1	69.72	310	1	78.48
	$k = 20$	3062	1	69.72	760	1	76.04
	$k = 50$	7259	1	69.72	1567	1	80.62
	$k = 100$	14367	1	69.72	3204	1	81.5
	$k = 200$	29580	1	69.72	5594	1	80.33
45	$k = 1$	604	2	46.66	65	1	75.62
	$k = 5$	1502	2	44.05	359	1	83.12
	$k = 10$	2890	2	43.78	529	1	83.12
	$k = 20$	4877	2	38.4	810	1	78.72
	$k = 50$	9384	1	76.94	2070	1	83.31
	$k = 100$	22382	1	76.94	4241	1	86.62
	$k = 200$	36854	1	79.15	6044	1	85.35

Table A.10: Case study results 46-50

Instance		PS			PM		
		TT (ms)	B	CR	TT (ms)	B	CR
46	$k = 1$	259	2	77.06	78	2	71.7
	$k = 5$	2706	2	72.32	101	2	66.68
	$k = 10$	4194	2	70.3	280	2	72.68
	$k = 20$	8229	2	70.3	456	2	77.02
	$k = 50$	14847	2	76.33	1201	2	73.15
	$k = 100$	26144	2	76.15	1841	2	74.16
	$k = 200$	45738	2	75.98	4028	2	75.2
47	$k = 1$	395	1	70.04	73	1	72.15
	$k = 5$	1082	1	70.04	188	1	75.28
	$k = 10$	2024	1	70.04	439	1	72.72
	$k = 20$	3741	1	70.04	737	1	78.82
	$k = 50$	8428	1	70.04	1798	1	78.82
	$k = 100$	17165	1	72.83	3143	1	78.82
	$k = 200$	34689	1	72.83	6153	1	78.82
48	$k = 1$	208	2	63.3	75	2	62.24
	$k = 5$	1616	2	69.72	179	2	61.82
	$k = 10$	3395	2	63.35	410	2	54.88
	$k = 20$	6845	2	62.39	749	2	66.88
	$k = 50$	7946	2	65.44	913	2	63.9
	$k = 100$	20583	2	65.15	2056	2	64.02
	$k = 200$	43066	2	70.5	4014	2	66.03
49	$k = 1$	152	1	64.08	36	1	79
	$k = 5$	617	1	69.12	79	1	76.7
	$k = 10$	1140	1	72.53	128	1	76.11
	$k = 20$	2039	1	72.53	252	1	79.88
	$k = 50$	4622	1	74.03	603	1	79.88
	$k = 100$	8767	1	74.03	1170	1	81.7
	$k = 200$	16856	1	74.03	2384	1	81.7
50	$k = 1$	421	1	63.37	63	1	70.85
	$k = 5$	899	1	67.26	279	1	74.19
	$k = 10$	1950	1	70.81	283	1	74.19
	$k = 20$	3077	1	70.81	670	1	67.69
	$k = 50$	6702	1	70.81	1476	1	74.15
	$k = 100$	14720	1	72.07	1272	1	67.3
	$k = 200$	31094	1	72.29	2618	1	67.3

Table A.11: Case study results 51-55

Instance		PS			PM		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
51	$k = 1$	199	1	70.66	62	1	71.17
	$k = 5$	1189	1	71.25	157	1	70.5
	$k = 10$	2375	1	71.25	297	1	69.81
	$k = 20$	4786	1	71.25	1206	1	71.6
	$k = 50$	11267	1	71.25	879	1	64.67
	$k = 100$	20929	1	71.25	1761	1	64.67
	$k = 200$	41807	1	73.87	3514	1	64.67
52	$k = 1$	54	1	71.22	22	1	77.21
	$k = 5$	170	1	74.69	51	1	80.62
	$k = 10$	332	1	74.69	85	1	76.17
	$k = 20$	589	1	76.01	169	1	78.28
	$k = 50$	1589	1	76.01	424	1	80.14
	$k = 100$	3017	1	77.1	517	1	81.3
	$k = 200$	6764	1	77.1	1030	1	81.3
53	$k = 1$	434	1	63.73	54	1	70.09
	$k = 5$	1510	1	63.73	217	1	75.61
	$k = 10$	2843	1	63.73	365	1	69.2
	$k = 20$	5270	1	63.73	544	1	75.31
	$k = 50$	12437	1	68.54	1198	1	77.54
	$k = 100$	22733	1	69.71	1561	1	72.15
	$k = 200$	46766	1	70.13	3089	1	71.75
54	$k = 1$	889	2	71.9	75	2	71.76
	$k = 5$	1611	2	74.52	186	2	77.16
	$k = 10$	3338	2	74.52	452	2	70.11
	$k = 20$	7081	2	72.86	771	2	77.55
	$k = 50$	17401	2	72.55	1609	2	77.67
	$k = 100$	36080	2	77.76	2773	2	71.73
	$k = 200$	69866	2	71.95	5305	2	75.45
55	$k = 1$	136	1	58.5	46	1	69.04
	$k = 5$	455	1	70.35	89	1	71.56
	$k = 10$	786	1	74.32	168	1	71.98
	$k = 20$	1501	1	78.76	303	1	72.31
	$k = 50$	3475	1	78.76	748	1	72.31
	$k = 100$	7132	1	78.76	1376	1	72.53
	$k = 200$	14863	1	79.02	1909	1	79.6

Table A.12: Case study results 56-60

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
56	$k = 1$	319	1	63.92	141	1	79.42
	$k = 5$	828	1	71.46	236	1	77.95
	$k = 10$	1476	1	74.87	420	1	80.12
	$k = 20$	2840	1	74.87	550	1	76.63
	$k = 50$	7310	1	74.87	1478	1	81.85
	$k = 100$	15072	1	74.87	2233	1	75.5
	$k = 200$	30574	1	78.73	3703	1	73.29
57	$k = 1$	249	2	64.23	53	2	55.11
	$k = 5$	743	2	47.8	137	2	58.35
	$k = 10$	1343	2	45.86	295	2	48
	$k = 20$	2089	2	42.54	398	2	48.64
	$k = 50$	5008	2	44.26	1116	2	54.39
	$k = 100$	13920	2	53.72	2074	1	82.3
	$k = 200$	26749	2	51.17	2753	1	82.01
58	$k = 1$	106	1	65	34	1	77.03
	$k = 5$	809	1	65	135	1	76.3
	$k = 10$	1525	1	67.19	195	1	76.3
	$k = 20$	3043	1	67.99	328	1	74.82
	$k = 50$	7037	1	67.99	539	1	76.89
	$k = 100$	14984	1	71.01	1032	1	76.84
	$k = 200$	29794	1	71.48	2064	1	77.89
59	$k = 1$	64	1	69.12	54	1	78.11
	$k = 5$	278	1	69.66	131	1	78.2
	$k = 10$	658	1	69.66	250	1	77.18
	$k = 20$	1229	1	75.21	440	1	76.43
	$k = 50$	3024	1	75.21	643	1	77.6
	$k = 100$	5983	1	75.21	1248	1	77.6
	$k = 200$	11839	1	75.21	2491	1	77.6
60	$k = 1$	342	1	64.54	65	1	71.84
	$k = 5$	826	1	66.78	136	1	77.47
	$k = 10$	1498	1	69.14	328	1	82.87
	$k = 20$	2941	1	71.75	570	1	81.86
	$k = 50$	7042	1	72.18	876	1	71.62
	$k = 100$	13516	1	75.64	1629	1	73.71
	$k = 200$	26714	1	80.19	2324	1	69.89

Table A.13: Case study results 61-65

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
61	$k = 1$	2470	3	61.14	194	2	75.03
	$k = 5$	7566	3	61.46	458	2	79.58
	$k = 10$	7705	2	71.35	882	2	74.15
	$k = 20$	30277	3	57.73	2574	2	78.49
	$k = 50$	41450	2	71.61	4022	2	79.87
	$k = 100$	69808	3	58.95	6114	2	79.77
	$k = 200$	197317	2	71.15	13254	2	77.35
62	$k = 1$	2572	2	66.32	179	2	73.37
	$k = 5$	10605	2	69.85	507	2	73.55
	$k = 10$	17406	2	73.78	1380	2	73.49
	$k = 20$	45291	2	66.46	2006	2	72.71
	$k = 50$	44703	2	72.57	6606	2	77.29
	$k = 100$	147556	2	69.57	7603	2	76.6
	$k = 200$	103674	2	73.59	20459	2	76.31
63	$k = 1$	1665	2	64.66	269	2	75.39
	$k = 5$	11233	2	63.95	956	2	77.32
	$k = 10$	13900	2	68.69	1656	2	77.37
	$k = 20$	28438	2	72.53	2944	2	75.38
	$k = 50$	99439	2	72.61	9805	2	75.62
	$k = 100$	71230	2	72.6	15467	2	77.67
	$k = 200$	137283	2	73.18	12330	2	78.89
64	$k = 1$	2452	2	66.08	156	2	76.31
	$k = 5$	5607	2	71.17	392	2	75.41
	$k = 10$	4290	2	67.29	713	2	72.47
	$k = 20$	7240	2	65.83	1005	2	76.43
	$k = 50$	60548	2	69.7	2428	2	77.74
	$k = 100$	70161	2	73.64	6411	2	79.81
	$k = 200$	144912	2	73.64	10006	2	80.02
65	$k = 1$	1391	2	61.23	126	2	62.06
	$k = 5$	8206	2	61.29	482	2	62.92
	$k = 10$	17117	2	64.36	1116	2	61.67
	$k = 20$	15367	2	69.79	2358	2	65.05
	$k = 50$	78715	2	63.81	3023	2	60.12
	$k = 100$	146277	2	72.45	7925	2	61.3
	$k = 200$	406884	2	63.72	15111	2	64.17

Table A.14: Case study results 66-70

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
66	$k = 1$	1904	3	58.8	175	2	81.55
	$k = 5$	10568	3	67.91	485	2	79.54
	$k = 10$	5507	3	65.99	1231	2	77.61
	$k = 20$	11187	3	65.99	1804	2	78.44
	$k = 50$	119767	3	56.88	3802	2	81.5
	$k = 100$	256247	3	68.01	6740	2	80.58
	$k = 200$	222528	2	77.34	13746	2	82.84
67	$k = 1$	925	2	70.81	122	2	68.56
	$k = 5$	5226	2	69.78	439	2	66.14
	$k = 10$	10700	2	65.29	901	2	66.67
	$k = 20$	10307	2	70.82	1418	2	75.99
	$k = 50$	48933	2	69.91	2782	2	68.27
	$k = 100$	72109	2	68.5	6219	2	70.04
	$k = 200$	80496	2	72.21	13593	2	75.07
68	$k = 1$	662	2	67.89	266	2	74.12
	$k = 5$	2206	2	70.69	485	2	71.13
	$k = 10$	3601	2	70.59	933	2	72.56
	$k = 20$	6176	2	69.41	3152	2	75.02
	$k = 50$	21427	2	72.94	3732	2	75.85
	$k = 100$	46567	2	74.75	5835	2	73.27
	$k = 200$	136385	2	72.67	13022	2	72.26
69	$k = 1$	533	2	67.01	164	2	70.51
	$k = 5$	2336	2	66.89	645	2	56.89
	$k = 10$	3697	2	69.5	906	2	57.8
	$k = 20$	34601	2	68.7	1629	2	63.27
	$k = 50$	25976	2	68.5	2795	2	55.69
	$k = 100$	45713	2	68.21	7779	2	57.47
	$k = 200$	95060	2	69.61	11836	2	67.42
70	$k = 1$	706	2	66.92	114	2	48.99
	$k = 5$	3006	2	66.92	228	2	50.21
	$k = 10$	9862	2	67.94	923	2	49.66
	$k = 20$	18979	2	67.95	1259	2	47.25
	$k = 50$	47210	2	67.95	3296	2	45.48
	$k = 100$	93287	2	67.94	5671	2	50.66
	$k = 200$	175227	2	69.96	10166	2	46.7

Table A.15: Case study results 71-75

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
71	$k = 1$	1937	2	68.46	275	2	78.92
	$k = 5$	6324	2	67.23	923	2	79.06
	$k = 10$	12072	2	60.76	1845	2	77.33
	$k = 20$	43571	2	69.7	2488	2	77.21
	$k = 50$	107498	2	69.7	4843	2	79.95
	$k = 100$	196533	2	74.01	9836	2	80.56
	$k = 200$	165421	2	70.67	30888	2	77.79
72	$k = 1$	3135	2	69.36	180	2	75.19
	$k = 5$	12714	2	70.89	841	2	78.12
	$k = 10$	13866	2	73.13	1489	2	75.76
	$k = 20$	17588	2	74.99	2531	2	80.41
	$k = 50$	63019	2	72.49	4930	2	82.07
	$k = 100$	99912	2	73.73	8868	2	76.87
	$k = 200$	202486	2	73.73	15071	2	80.41
73	$k = 1$	543	2	62.87	106	2	59.24
	$k = 5$	1435	2	61.18	434	2	49.78
	$k = 10$	2961	2	61.18	912	1	81.44
	$k = 20$	5741	2	61.18	1491	1	78.06
	$k = 50$	14454	2	59.95	2519	2	61
	$k = 100$	48160	2	53.75	3573	1	81.19
	$k = 200$	55897	2	56.57	16018	1	81.52
74	$k = 1$	1141	3	59.03	110	3	51.47
	$k = 5$	2047	2	74.12	573	2	78.35
	$k = 10$	11023	2	75.26	647	2	81.52
	$k = 20$	21326	2	78.24	1739	2	78.18
	$k = 50$	22942	2	73.59	3399	3	52.15
	$k = 100$	99298	2	76.12	4867	2	80.29
	$k = 200$	187444	2	76.64	5707	2	78.08
75	$k = 1$	1626	3	67.53	146	3	56.95
	$k = 5$	7600	3	63.87	548	2	79.31
	$k = 10$	14729	3	63.87	1095	2	82.81
	$k = 20$	22788	3	68.43	1881	2	79.28
	$k = 50$	124694	3	76.24	3366	2	78.77
	$k = 100$	205410	3	66.02	7725	2	80.9
	$k = 200$	537763	3	65.18	24478	2	84.28

Table A.16: Case study results 76-80

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
76	$k = 1$	1376	2	63.96	177	2	74.41
	$k = 5$	8926	2	74.31	519	2	72.4
	$k = 10$	17086	2	74.31	751	2	78.52
	$k = 20$	33521	2	74.31	1159	2	78.81
	$k = 50$	60053	2	73.48	3003	2	84.23
	$k = 100$	135048	2	73.09	7363	2	78.8
	$k = 200$	231300	2	71.65	16926	2	80.54
77	$k = 1$	1023	2	49.84	120	1	71.99
	$k = 5$	4338	2	39.27	651	1	82.65
	$k = 10$	5616	1	71.67	1016	1	76.75
	$k = 20$	10976	2	40.22	2028	1	80.46
	$k = 50$	29232	1	72.87	3568	1	75.93
	$k = 100$	41278	1	77.2	8318	1	73.59
	$k = 200$	88763	1	77.2	5868	1	80.87
78	$k = 1$	265	2	69.04	49	2	63.24
	$k = 5$	1194	2	71.41	262	2	67.07
	$k = 10$	2336	2	65.69	571	2	68.79
	$k = 20$	4516	2	69.82	902	2	68.92
	$k = 50$	12694	2	69.79	2142	2	68.06
	$k = 100$	37343	2	70.11	3265	2	71.6
	$k = 200$	68154	2	69.01	5888	2	68.1
79	$k = 1$	619	2	65.18	142	2	68.24
	$k = 5$	2508	2	68	521	2	73.19
	$k = 10$	4839	2	68	912	2	73.2
	$k = 20$	9629	2	68	1748	2	73.2
	$k = 50$	24561	2	68	2860	2	69.99
	$k = 100$	59050	2	70.69	6388	2	68.51
	$k = 200$	119607	2	68.91	8898	2	70.83
80	$k = 1$	475	2	67.45	90	2	76.64
	$k = 5$	1394	2	73	287	2	73.64
	$k = 10$	2504	2	72.45	784	2	74.19
	$k = 20$	4813	2	76.09	1518	2	74.19
	$k = 50$	11433	2	76.26	1973	2	75.36
	$k = 100$	24444	2	71.36	4246	2	76.51
	$k = 200$	47230	2	76.39	7618	2	76.66

Table A.17: Case study results 81-85

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
81	$k = 1$	8	1	71.74	9	1	73.14
	$k = 5$	32	1	71.74	17	1	71.74
	$k = 10$	49	1	73.14	27	1	71.74
	$k = 20$	94	1	73.14	49	1	71.74
	$k = 50$	226	1	73.14	99	1	71.74
	$k = 100$	451	1	74.59	185	1	71.74
	$k = 200$	916	1	74.59	345	1	74.59
82	$k = 1$	30	1	62.99	67	1	73.56
	$k = 5$	128	1	75.39	124	1	77.7
	$k = 10$	242	1	75.39	244	1	77.7
	$k = 20$	460	1	75.89	577	1	75.39
	$k = 50$	1150	1	75.89	1658	1	76.65
	$k = 100$	2312	1	75.89	3109	1	76.14
	$k = 200$	4703	1	75.89	4769	1	76.65
83	$k = 1$	6	1	63.86	4	1	59.07
	$k = 5$	16	1	63.86	11	1	65.64
	$k = 10$	31	1	63.86	19	1	65.64
	$k = 20$	52	1	64.13	39	1	65.64
	$k = 50$	125	1	65.64	87	1	65.64
	$k = 100$	245	1	65.64	173	1	65.64
	$k = 200$	489	1	65.64	347	1	65.64
84	$k = 1$	3	1	66.1	2	1	73.29
	$k = 5$	9	1	73.29	5	1	74.06
	$k = 10$	17	1	73.29	8	1	74.06
	$k = 20$	32	1	73.29	11	1	74.06
	$k = 50$	75	1	73.29	10	1	74.06
	$k = 100$	150	1	73.29	10	1	74.06
	$k = 200$	307	1	73.29	10	1	74.06
85	$k = 1$	24	1	69.56	21	1	66.42
	$k = 5$	75	1	69.56	33	1	79.68
	$k = 10$	140	1	69.56	54	1	79.68
	$k = 20$	267	1	69.56	94	1	80.26
	$k = 50$	670	1	69.56	208	1	80.26
	$k = 100$	1311	1	69.56	378	1	80.26
	$k = 200$	2645	1	69.56	798	1	80.26

Table A.18: Case study results 86-90

Instance		**PS**			**PM**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
86	$k = 1$	34	1	82.3	17	1	75.61
	$k = 5$	80	1	82.3	20	1	82.3
	$k = 10$	133	1	82.3	28	1	82.3
	$k = 20$	259	1	82.3	53	1	82.3
	$k = 50$	604	1	82.3	104	1	82.3
	$k = 100$	1174	1	82.3	180	1	82.3
	$k = 200$	2400	1	82.3	359	1	82.3
87	$k = 1$	250	1	68.79	111	1	71.46
	$k = 5$	380	1	70.77	189	1	76.67
	$k = 10$	617	1	70.77	138	1	70.1
	$k = 20$	1072	1	72.52	274	1	75.49
	$k = 50$	2504	1	76.27	771	1	78.3
	$k = 100$	4903	1	76.27	1531	1	78.3
	$k = 200$	9892	1	77.89	2544	1	77.07
88	$k = 1$	109	1	63.76	53	1	77.67
	$k = 5$	244	1	67.29	98	1	77.67
	$k = 10$	375	1	69.53	145	1	76.85
	$k = 20$	690	1	73.38	310	1	73.38
	$k = 50$	1581	1	75.66	752	1	76.45
	$k = 100$	3019	1	77.26	1447	1	76.45
	$k = 200$	6146	1	77.26	2800	1	76.45
89	$k = 1$	32	1	67.75	22	1	75.37
	$k = 5$	210	1	75.72	24	1	66.46
	$k = 10$	376	1	75.72	41	1	66.46
	$k = 20$	601	1	75.72	76	1	75.37
	$k = 50$	1500	1	75.72	176	1	75.37
	$k = 100$	3224	1	75.72	321	1	75.37
	$k = 200$	5858	1	76.95	619	1	76.59
90	$k = 1$	12	1	80.24	10	1	80.24
	$k = 5$	41	1	80.24	18	1	80.24
	$k = 10$	76	1	80.24	20	1	80.24
	$k = 20$	148	1	80.24	23	1	80.24
	$k = 50$	372	1	80.24	20	1	80.24
	$k = 100$	738	1	80.24	20	1	80.24
	$k = 200$	1517	1	80.24	20	1	80.24

Table A.19: Case study results 91-95

Instance		PS			PM		
		TT (ms)	B	CR	TT (ms)	B	CR
91	$k = 1$	27	2	61.65	23	2	62.87
	$k = 5$	167	2	60.72	51	2	67.36
	$k = 10$	285	2	65.81	99	2	69.17
	$k = 20$	596	2	67.21	189	2	70.48
	$k = 50$	1239	2	64.03	422	2	73.88
	$k = 100$	2370	2	71.91	631	2	64.22
	$k = 200$	4631	2	71.91	1244	2	63.94
92	$k = 1$	41	2	68.98	17	2	64.7
	$k = 5$	102	2	68.5	46	2	60.91
	$k = 10$	180	2	68.5	87	2	63.43
	$k = 20$	343	2	68.5	200	2	64.45
	$k = 50$	1081	2	68.5	481	2	64.45
	$k = 100$	2487	2	75.09	848	2	62.44
	$k = 200$	4615	2	74.62	1428	2	61.01
93	$k = 1$	11	2	53.18	6	2	49.39
	$k = 5$	29	2	60.58	13	2	63.43
	$k = 10$	62	2	60.75	23	2	63.43
	$k = 20$	114	2	60.97	44	2	62.4
	$k = 50$	251	2	53.59	102	2	62.4
	$k = 100$	480	2	53.59	207	1	70.47
	$k = 200$	1141	2	53.59	408	1	70.85
94	$k = 1$	7	2	60.41	6	2	61.33
	$k = 5$	18	2	60.41	10	2	62.49
	$k = 10$	32	2	60.41	17	2	64.19
	$k = 20$	65	2	72.49	30	2	64.19
	$k = 50$	166	2	72.49	72	2	64.19
	$k = 100$	332	2	72.49	146	2	74.4
	$k = 200$	675	2	75.2	247	2	71.17
95	$k = 1$	2596	1	71.15	217	1	80.4
	$k = 5$	5493	1	71.15	345	1	81.11
	$k = 10$	10208	1	71.15	859	1	78.75
	$k = 20$	19066	1	71.15	1031	1	76.3
	$k = 50$	53469	1	71.15	2902	1	82.38
	$k = 100$	101264	1	71.15	5778	1	82.38
	$k = 200$	198827	1	71.15	11145	1	82.38

Table A.20: Case study results 96-100

Instance		PS			PM		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
96	$k = 1$	447	1	60.54	29	1	64.78
	$k = 5$	1417	1	60.54	108	1	67.96
	$k = 10$	2778	1	60.54	191	1	68.33
	$k = 20$	5405	1	66.23	685	1	71.57
	$k = 50$	13724	1	66.23	1133	1	68.28
	$k = 100$	26596	1	67	2122	1	68.28
	$k = 200$	55258	1	67	3725	1	69.27
97	$k = 1$	625	1	56.36	40	1	70.47
	$k = 5$	2977	1	61.41	228	1	69.38
	$k = 10$	5436	1	61.41	459	1	68.66
	$k = 20$	10287	1	62.02	458	1	66.34
	$k = 50$	26357	1	63.42	988	1	58.96
	$k = 100$	53737	1	65.76	1873	2	36.16
	$k = 200$	123811	1	72.94	9152	1	74.29
98	$k = 1$	78	1	79.05	23	1	64.42
	$k = 5$	172	1	79.05	51	1	64.42
	$k = 10$	288	1	81.04	94	1	77.16
	$k = 20$	530	1	81.04	137	1	67.57
	$k = 50$	1193	1	81.04	319	1	73.63
	$k = 100$	2307	1	81.04	621	1	73.75
	$k = 200$	4741	1	81.04	1182	1	77.16
99	$k = 1$	8	1	71.74	9	1	73.14
	$k = 5$	26	1	71.74	17	1	71.74
	$k = 10$	55	1	73.14	27	1	71.74
	$k = 20$	97	1	73.14	53	1	71.74
	$k = 50$	234	1	73.14	93	1	71.74
	$k = 100$	464	1	74.59	186	1	71.74
	$k = 200$	921	1	74.59	338	1	74.59
100	$k = 1$	9	1	54.55	6	1	56.35
	$k = 5$	30	1	56.35	12	1	67.25
	$k = 10$	55	1	56.35	19	1	70.44
	$k = 20$	101	1	67.45	35	1	70.44
	$k = 50$	243	1	67.45	71	1	70.44
	$k = 100$	495	1	70.22	134	1	70.44
	$k = 200$	1017	1	70.44	250	1	70.44

Table A.21: Summary of opened bins by heuristic

Class		PM					PS					TS3	GLS	GASP	EHGH2	GVN	BRKGA	BRKGA-VD
Instance	n	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$							
1	50	14.5	14.5	14.5	14.5	14.10	14.4	14.3	14.3	14.4	14	**13.4**	**13.4**	**13.4**	13.8	**13.4**	**13.4**	**13.4**
	100	28.4	28.3	28.3	28.4	28	27.7	27.8	27.8	27.9	27.7	**26.6**	**26.6**	26.9	27.6	**26.6**	**26.6**	**26.6**
	150	38.5	38.5	38.4	38.3	38.4	38.4	38.5	38.5	38.3	37.9	36.7	37	37	39.8	36.4	36.4	**36.3**
	200	53.1	53.2	53	52.9	53	52.7	52.7	52.6	52.6	52.7	51.2	51.2	51.6	**50.6**	50.9	50.8	50.8
2	50	14.8	14.7	14.7	14.7	14.6	14.9	14.8	14.8	14.9	14.8	**13.8**	-	-	-	**13.8**	**13.8**	**13.8**
	100	26.5	26.6	26.6	26.6	26.6	27.1	26.9	26.9	26.7	26.7	25.7	-	-	-	25.7	25.6	**25.5**
	150	38.4	38.3	38.2	38.3	38.3	39.2	39.2	39.1	39.1	39	37.2	-	-	-	36.9	**36.6**	**36.6**
	200	51.3	51.2	51.1	51.3	51	51.9	51.8	51.9	51.8	51.7	50.1	-	-	-	**49.4**	**49.4**	**49.4**
3	50	14.2	14.1	13.9	13.8	13.9	14	14.1	13.9	14	13.9	**13.3**	-	-	-	**13.3**	**13.3**	**13.3**
	100	27.7	27.7	27.7	27.5	27.8	27.6	27.5	27.2	27.3	27.3	26	-	-	-	26	**25.9**	**25.9**
	150	39.4	39.4	39.5	39.5	39.2	39.1	38.9	38.9	39	39	37.7	-	-	-	37.6	**37.5**	**37.5**
	200	51.8	51.8	51.7	51.6	51.8	51.4	51.4	51.4	51.3	51.2	50.5	-	-	-	50	**49.8**	**49.8**
4	50	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	29.7	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**
	100	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59.2	59	59	59	59.5	59	59	**58.9**
	150	87.6	87.6	87.6	87.6	87.6	87.6	87.7	87.6	87.6	87.7	**86.8**	**86.8**	**86.8**	90.4	**86.8**	**86.8**	**86.8**
	200	119.5	119.5	119.5	119.5	119.5	119.5	119.5	119.5	119.5	119.5	**118.8**	119	**118.8**	119	**118.8**	**118.8**	**118.8**
5	50	8.7	8.7	8.7	8.7	8.6	8.7	8.7	8.6	8.6	8.6	8.4	8.3	8.4	**7.9**	8.3	8.3	8.3
	100	16	16	16	16	16	15.7	15.6	15.6	15.6	15.6	15	15.1	15.1	**14.6**	15	15	15
	150	21.6	21.6	21.6	21.7	21.7	21.3	21.3	21.3	21.3	21.4	20.4	20.2	20.6	21.5	20.4	20.1	**19.9**
	200	28.9	28.9	28.9	29	29	28.4	28.2	28.3	28.4	28.4	27.6	27.2	27.7	29.6	**27.1**	**27.1**	**27.1**
6	50	10.5	10.3	10.1	10.2	10	10.7	10.5	10.4	10.2	10.3	9.9	9.8	9.9	11.8	9.8	**9.7**	**9.7**
	100	19.8	19.8	20	19.5	19.8	19.8	19.7	19.8	19.7	19.7	19.1	19.1	19.1	19.2	19	**18.9**	**18.9**
	150	30.4	30.4	30.3	30.3	30.3	30.4	30.3	30.3	30	30.2	29.4	29.4	29.5	29.8	29.2	**29**	**29**
	200	39.1	39.2	39.2	38.6	38.9	39.1	38.6	38.8	38.6	38.5	37.7	37.7	38	38.7	37.4	**37.3**	**37.3**
7	50	7.9	7.7	7.9	7.8	7.8	7.7	7.9	7.9	7.7	7.6	7.5	**7.4**	7.5	**7.4**	**7.4**	**7.4**	**7.4**
	100	13.5	13.2	13.4	13.3	13.2	13.5	13.5	13.5	13.4	13.2	12.5	12.3	12.7	13.5	12.5	**12.2**	**12.2**
	150	16.9	16.9	17	16.9	17.1	17.1	16.7	16.9	16.8	16.8	16.1	15.8	16.6	18.2	16	15.3	**15.2**
	200	25.1	25	25	25.2	24.9	24.9	24.5	24.8	24.7	24.7	23.9	23.5	24.2	24.1	23.5	**23.4**	**23.4**
8	50	10	10.1	9.9	9.8	9.9	9.7	9.6	9.6	9.6	9.7	9.3	**9.2**	9.3	9.4	**9.2**	**9.2**	**9.2**
	100	19.6	19.7	19.7	19.6	19.6	20	20	19.9	20	20	18.9	18.9	19	18.9	18.9	18.9	**18.8**
	150	25.8	25.6	25.6	25.7	25.7	26	25.8	25.7	25.8	25.8	24.1	23.9	24.8	26	24.1	**23.6**	**23.6**
	200	31.5	31.5	31.6	31.3	31.6	31.6	31.6	31.6	31.4	31.2	30.3	29.9	31.1	35.8	29.8	**29.3**	**29.3**

Table A.22: Detailed execution times for literature tests in milliseconds

Class		PM					PS					BRKGA-VD
Instance	n	$k = 1$	$k = 5$	$k = 10$	$k = 20$	$k = 50$	$k = 1$	$k = 5$	$k = 10$	$k = 20$	$k = 50$	
1	50	80.50	85.20	150.70	307.00	820.30	85.00	109.40	192.80	390.60	968.20	1,293.25
	100	137.90	439.10	819.60	1,500.60	3,737.60	177.20	634.80	1,181.50	2,409.10	5,540.10	7,005.10
	150	341.50	1,237.10	2,478.60	4,677.30	12,143.30	494.10	1,865.90	3,817.80	7,625.60	18,751.70	17,674.42
	200	697.40	2,808.20	5,144.30	10,319.20	26,679.40	1,087.30	4,629.80	9,134.60	18,256.50	44,792.70	34,702.21
2	50	24.70	72.80	126.50	220.10	555.70	35.80	125.60	243.20	510.80	1,190.30	1,293.25
	100	105.80	342.50	647.50	1,528.70	3,921.30	194.00	869.00	1,724.50	3,391.50	8,489.30	7,112.88
	150	266.00	974.60	1,842.70	3,399.20	8,160.60	590.30	2,599.20	4,943.70	10,118.50	24,492.20	17,782.19
	200	532.40	1,971.00	3,817.30	7,228.50	17,694.90	1,320.50	5,792.70	11,790.20	22,535.60	53,024.40	34,486.67
3	50	28.10	82.70	156.30	302.50	710.20	32.50	112.00	204.00	401.30	982.40	1,293.25
	100	120.10	489.30	910.70	1,625.50	4,313.90	162.20	621.70	1,231.80	2,234.10	6,042.70	6,897.33
	150	341.20	1,263.80	2,455.60	4,719.40	11,596.80	461.40	1,979.50	4,027.40	8,214.90	20,657.80	18,428.81
	200	690.00	2,694.00	5,307.70	10,418.30	25,865.20	1,018.20	4,318.50	8,788.70	17,801.60	44,903.80	35,672.15
4	50	45.10	129.40	255.60	506.10	1,200.90	95.60	150.20	280.40	532.50	1,377.00	1,185.48
	100	220.70	817.40	1,582.80	2,973.30	7,125.80	246.00	945.50	1,742.80	3,532.90	8,636.60	6,574.02
	150	559.70	2,074.50	3,944.40	7,778.10	19,574.70	644.20	2,601.60	5,089.70	9,968.20	24,431.50	17,135.56
	200	1,143.20	4,685.50	8,938.80	17,291.40	44,421.60	1,415.90	6,027.40	11,581.60	23,829.20	58,029.70	34,271.13
5	50	23.40	75.40	152.80	275.10	701.90	40.60	155.90	318.80	631.80	1,617.20	3,233.13
	100	69.80	230.80	438.70	861.00	2,229.90	212.40	937.60	1,703.40	3,547.50	7,580.70	13,040.27
	150	164.90	596.50	1,134.80	2,415.50	5,727.40	618.10	2,532.10	4,824.10	9,131.20	27,346.00	29,960.29
	200	283.80	1,055.90	2,071.00	3,899.90	9,916.70	1,210.60	4,988.80	9,872.90	20,611.90	56,078.50	57,226.32
6	50	22.40	73.20	136.10	262.90	568.60	26.90	88.80	177.70	349.40	836.40	969.94
	100	88.50	315.00	583.50	1,139.40	2,684.80	136.40	555.30	1,030.40	1,839.80	4,456.70	5,065.23
	150	223.40	796.90	1,448.70	3,019.10	7,749.50	323.80	1,360.10	2,561.50	5,367.50	11,886.10	13,686.90
	200	401.90	1,555.60	2,820.30	5,229.50	12,899.70	624.40	2,523.20	4,943.60	9,218.90	23,577.80	27,697.11
7	50	18.00	53.70	103.60	190.80	487.40	32.80	113.30	218.20	453.50	1,105.60	2,586.50
	100	68.30	229.50	433.30	846.80	1,993.60	154.30	656.70	1,255.30	2,261.20	5,893.50	10,669.31
	150	153.10	547.60	1,164.10	2,190.90	5,292.60	413.90	1,894.10	3,777.40	7,156.60	17,704.60	24,571.75
	200	270.00	976.90	2,056.00	3,804.10	10,078.00	808.70	4,513.50	7,854.10	14,518.60	32,642.10	46,125.92
8	50	18.60	60.40	119.30	220.30	528.50	29.50	109.00	200.90	406.60	919.30	2,909.81
	100	65.00	208.00	395.90	758.20	1,983.70	160.20	576.30	1,179.10	2,155.50	4,981.50	13,148.04
	150	151.70	516.30	937.80	1,970.20	4,837.30	433.10	1,964.40	3,815.50	8,109.50	17,854.20	25,002.84
	200	296.80	1,004.90	1,835.80	3,471.70	9,174.80	896.00	4,149.50	7,612.00	14,338.20	34,916.60	48,604.65

Table A.23: Cage ratio of literature tests

Class		PM					PS				
Instance	n	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$	$k=1$	$k=5$	$k=10$	$k=20$	$k=50$
1	50	69.36	69.84	69.66	69.74	71.73	70.09	70.5	70.45	70.06	**72.36**
	100	73.01	73.62	73.37	73.44	**74.78**	74.56	74.25	74.26	74.17	74.56
	150	75.66	75.53	75.61	75.91	76.07	75.93	75.93	76.04	76.31	**76.9**
	200	75.33	75.28	75.56	75.62	75.78	75.71	75.8	**75.99**	75.9	75.93
2	50	69.6	69.5	69.48	**69.79**	69.31	67.92	68.71	68.09	68.32	68.78
	100	73.05	72.76	**73.09**	72.7	72.7	71.37	71.74	71.74	72.22	72.54
	150	72.95	73.15	**73.28**	73.11	73.04	71.51	71.28	71.5	71.38	71.66
	200	73.21	73.29	73.34	73.24	**73.42**	72.23	72.36	72.12	72.43	72.55
3	50	71.61	71.81	73.08	72.94	72.87	72.38	71.9	72.63	71.88	**73.4**
	100	73.52	73.62	73.7	74.37	73.73	73.99	74.28	**74.97**	74.79	74.83
	150	74.94	75.1	75.04	75.04	75.78	75.6	75.98	**76.04**	75.83	75.94
	200	76.09	76.01	76.21	76.43	76.08	76.58	76.74	76.82	77.01	**77.03**
4	50	61.49	61.49	61.62	**61.8**	61.79	61.49	61.6	61.7	61.73	61.78
	100	63.48	63.46	63.53	63.53	**63.58**	63.49	63.5	63.52	63.53	63.57
	150	61.91	61.96	61.92	61.95	**61.98**	61.94	61.84	61.94	61.95	61.96
	200	61.83	**61.84**	61.82	**61.84**	61.8	61.81	61.83	61.83	61.82	61.83
5	50	69.08	68.72	69.03	69.44	70.5	69.45	69.54	70.35	70.18	**70.69**
	100	73.49	73.66	73.8	74.1	73.71	74.52	74.94	74.85	**75.02**	75.01
	150	76.38	76.48	76.5	76.14	76.43	77.32	77.28	**77.33**	77.23	77.01
	200	77.12	77.19	77.11	77.06	76.99	77.88	**78.46**	78.14	77.99	77.8
6	50	77.26	78.65	79.52	79.09	**80.12**	75.65	77.23	77.44	79.06	78.44
	100	84.63	84.59	84.17	**85.74**	85.03	84.2	84.69	84.34	84.62	84.91
	150	84.66	85.17	85.33	85.45	85.4	84.97	85.32	85.2	**86.48**	85.93
	200	86.35	86.35	86.36	87.17	86.83	85.99	87.1	86.9	87.27	**87.53**
7	50	64.49	66.32	65.35	66.5	66.69	66.8	65.44	65.53	66.87	68.5
	100	71.22	72.94	71.64	72.64	**73.37**	71.48	71.63	71.8	72.03	73.12
	150	76.65	76.63	76.31	76.69	76.05	75.11	**77.25**	76.11	76.73	76.76
	200	77.48	77.95	77.97	77.08	78.06	77.66	**79**	78.04	78.16	78.44
8	50	69.61	69.48	70.75	70.81	70.62	70.7	71.38	71.38	**71.83**	71.14
	100	74.38	**74.51**	74.38	74.44	74.28	73.16	73.68	73.43	73.13	73.3
	150	77.07	**77.52**	77.19	77.07	77.22	75.89	76.73	76.81	76.4	76.61
	200	79.5	79.51	79.24	**79.86**	79.36	78.47	78.59	78.61	78.99	79.76

Bold values are the best average values

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Variable	Description	SI unit
$\boldsymbol{u}$	solid displacement	m
$\boldsymbol{u}_f$	fluid displacement	m

Acknowledgements

Here you might want to acknowledge someone.