Three-Dimensional Bin Packing with Vertical Support

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Case study

- Large warehouses
- Mixed-case palletization
- No control over items’ shape (strongly heterogeneous)
- Pallets wrapped during loading procedure

Figure: Example of pit pallettization (Schäfer Case Picking — SSI SCHÄFER)

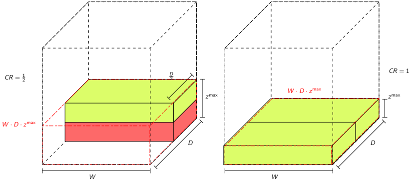

Figure: Cage ratio of two different bin configurations

Vertical Support

Definition

An item has vertical support if one of the following conditions hold:

- **Condition 1**: at least a percentage α_s of its base area is resting on other items
- **Condition 2**: at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

Vertical Support

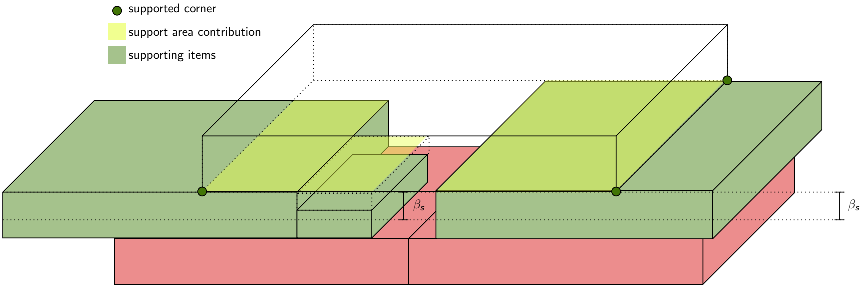

Figure: Representation of an item with conditions 1 and 2 of vertical support given $\alpha_s = 0.5$, β_s

Literature Review

- The problem is NP-Hard
- Exact methods only for small instances
- Existing 3D-BPP heuristics don't consider practical constraints
- Solutions for container loading and pallet loading problems are layer based

Conceptual Formulation

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

MILP Proxy Model - Objective Function

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

$$
\begin{array}{lll}
\min & \sum_{b \in B} (H v_b + z_b^{max}) & \\
\text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
& x_i + w_i \leq W & \forall i \in I \\
& y_i + d_i \leq D & \forall i \in I \\
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}
$$

MILP Proxy Model - Geometric Constraints 1

$$
\begin{aligned}
\min \quad & \sum_{b \in B} (Hv_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
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& x_i + w_i \leq W & \forall i \in I \\
& y_i + d_i \leq D & \forall i \in I \\
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq Wx_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq Dy_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq Hz_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

MILP Proxy Model - Geometric Constraints 2

$$
\begin{aligned}
\min \quad & \sum_{b \in B} (Hv_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
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& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
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& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
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MILP Proxy Model - Geometric Constraints 3

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$$
\begin{aligned}
\min \quad & \sum_{b\in B}(Hv_b + z_b^{max}) & \\
\text{s.t.} \quad & \sum_{b\in B} u_{ib} + \sum_{b\in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \leq v_b & \forall i \in I, \forall b \in B \\
& v_b \geq v_c & \forall (b,c) \in B : b < c \\
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& y_i + d_i \leq D & \forall i \in I \\
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& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
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& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{aligned}
$$

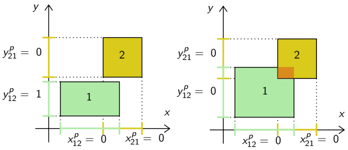

Figure: Precedences variables (2D case)

MILP Proxy Model - Vertical Support

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then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
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MILP Proxy Model - Closeness

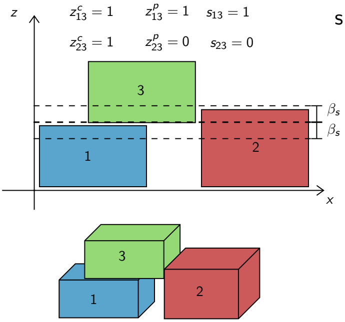


Figure: Closeness variables example

$$
\text{s.t.} \quad
\begin{aligned}
& z_j - (z_i + h_i) \le \beta_s + H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& z_j - (z_i + h_i) \ge -\beta_s - H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& s_{ij} \le z^p_{ij} && \forall (i,j) \in I \\
& s_{ij} \le z^c_{ij} && \forall (i,j) \in I \\
& s_{ij} \ge z^p_{ij} + z^c_{ij} - 2 && \forall (i,j) \in I : i \neq j \\
& \sum_{j \in I} s_{ij} \le \sum_{b \in B} u_{ib} && \forall i \in I
\end{aligned}
$$

MILP Proxy Model - Discretized Vertical Support

Pre-Computed Parameter

$O(i,j,k,h)$

$O(i,j,1,2) = 1$

Variables

$s_{ijb}^{12} = 1$

$(x_j - x_i, y_j - y_i)$

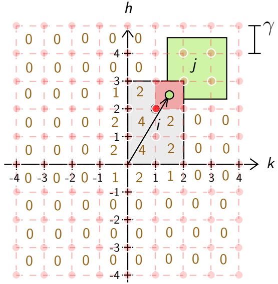


Figure: Space discretization

$$
\begin{aligned}
\text{s.t.}\quad & z_i \le H(1-g_i) && \forall i \in I \\
& \sum_{(k,h)\in\Delta, b\in B: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le s_{ij} && \forall (i,j) \in I \\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le u_{ib} && \forall (i,j,b) \in I^B \\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le u_{jb} && \forall (i,j,b) \in I^B \\
& x_j - x_i \ge \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& x_j - x_i \le \gamma (k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& y_j - y_i \ge \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& y_j - y_i \le \gamma (h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& \sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h) s_{jib}^{kh} \ge \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
\end{aligned}
$$

Overview

Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

Each node is a partial solution, starts from the empty solution

- Operates on a single bin
- Exploits a modified 2D-BPP heuristic
- No explicit layers
- Guarantees vertical support

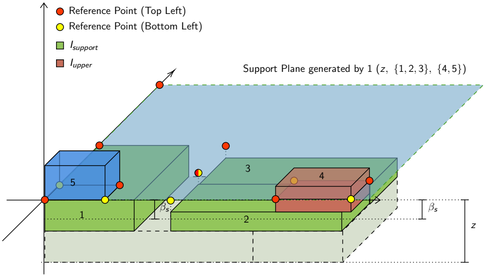
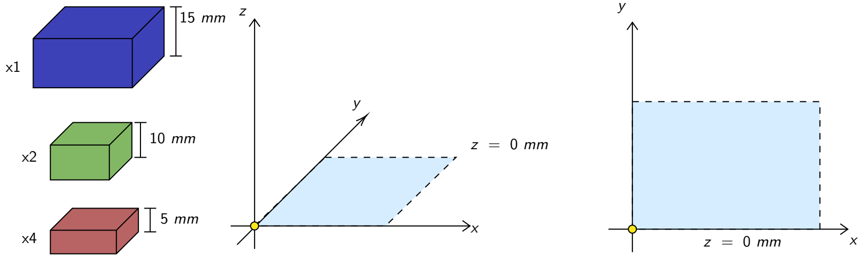
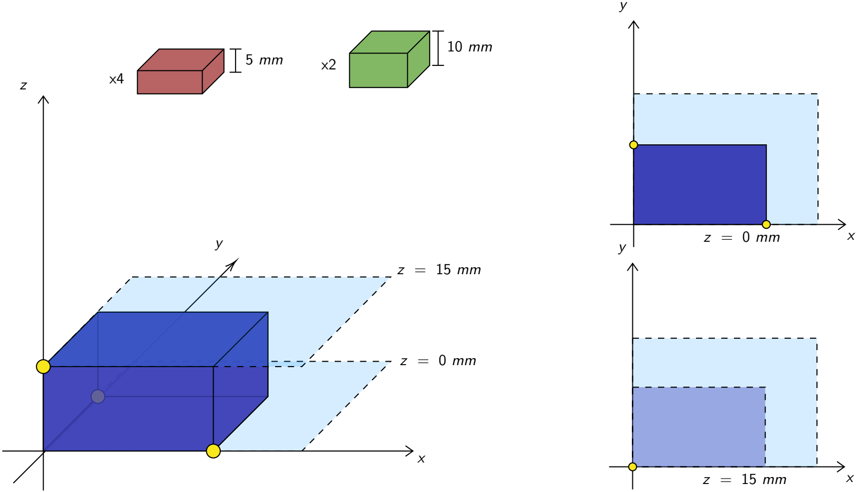


Figure: An example of a support plane generated by item 1

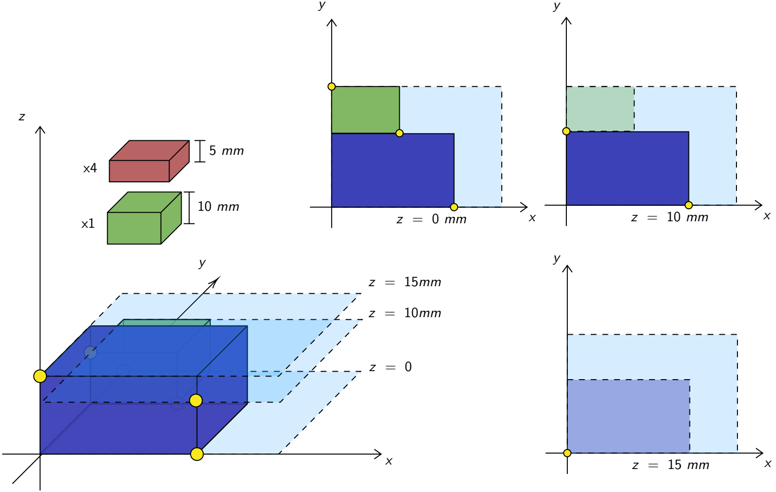
Support Planes



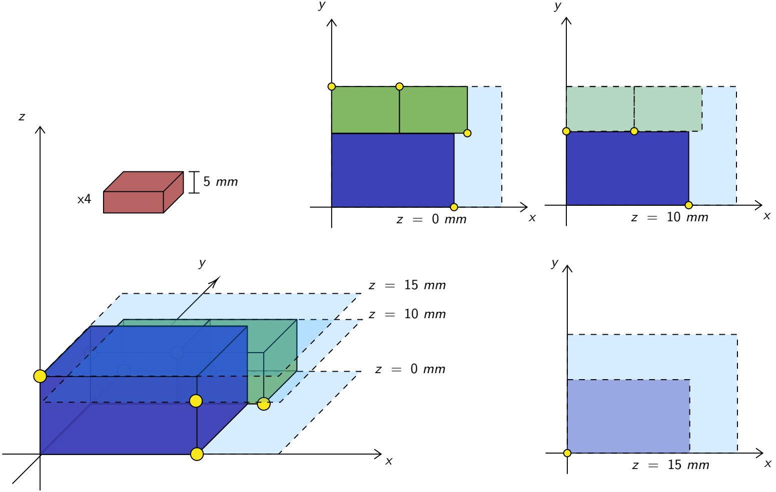
Support Planes



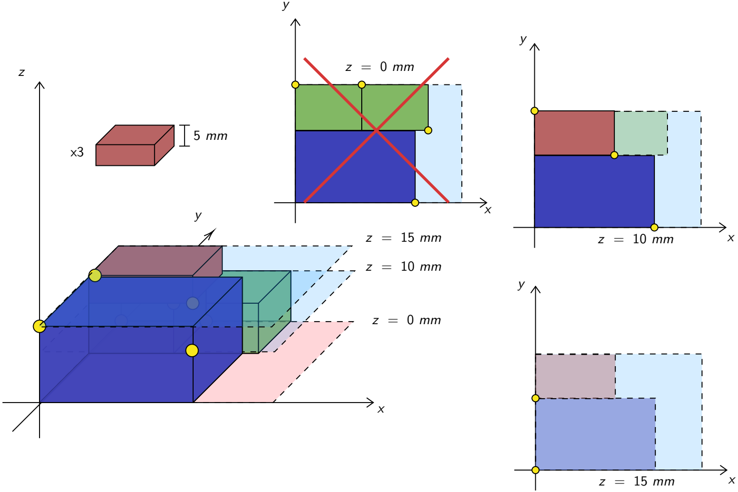
Support Planes



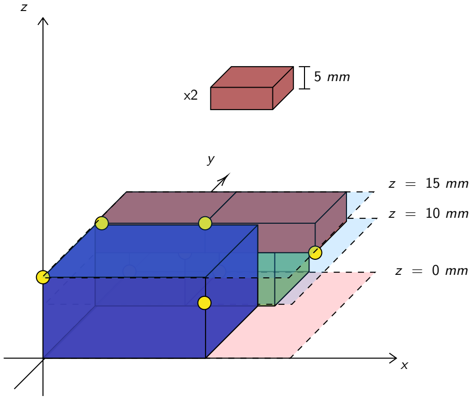
Support Planes

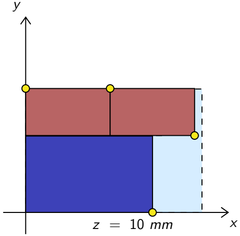


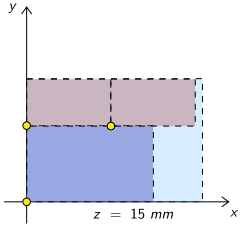
Support Planes



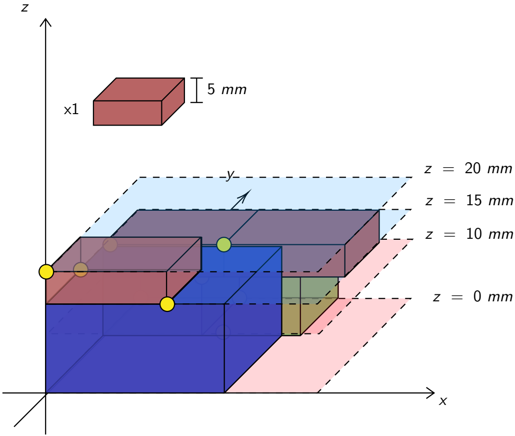
Support Planes

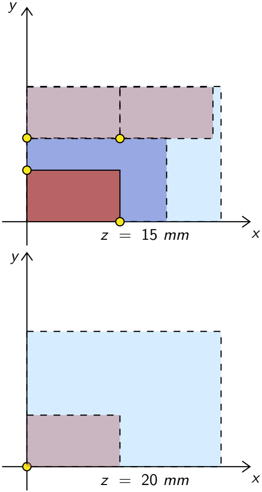




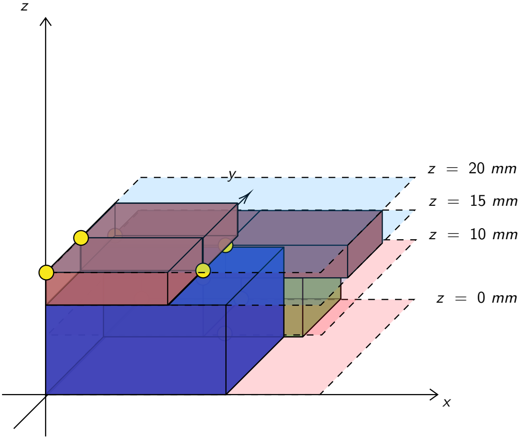


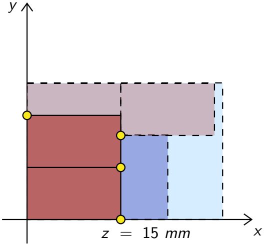
Support Planes

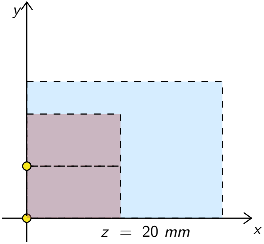




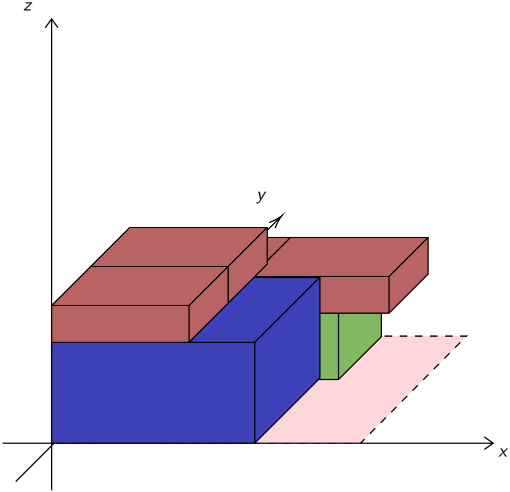
Support Planes





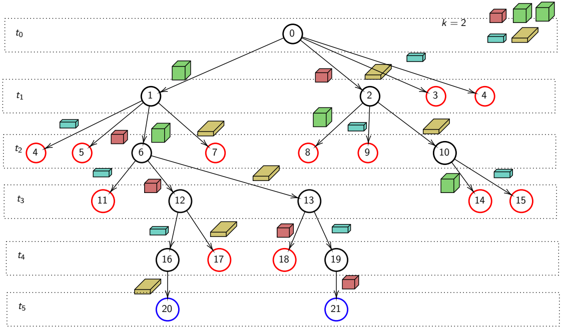


Support Planes



Beam Search

- Exploits support planes
- Explores different sequences of placements
- Limited to k best placements



Beam Search

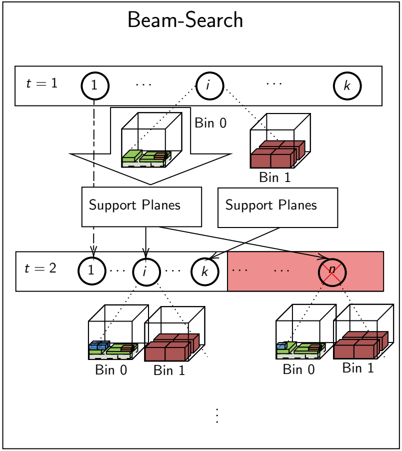


Figure: Conceptual representation of the proposed heuristic

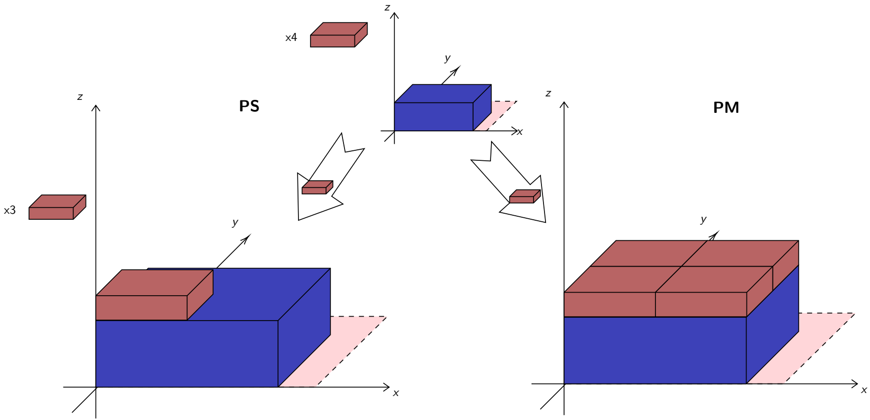


Figure: Proposed placement modes

Optimizations

- Duplicates removal
- Fast overlap checks
- Memory management

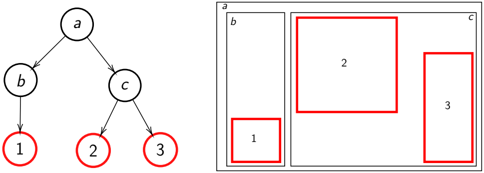


Figure: AABB Tree

Computational Experiments

Table: Comparison with MILP model on limited set of boxes

	MILP Model			PM		PS	
n	Max Z	TT(s)	Gap(%)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	1.86	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

* Some boxes had lower support than expected due to discretization errors.

Computational Experiments

Table: Average execution time of literature results with bin gap

Heuristic		**Execution Time (s)**				**Bin Gap (%)**
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.05	0.11	0.28	0.55	4.57
	$k = 5$	0.08	0.39	1.02	2.16	4.32
	$k = 10$	0.15	0.74	1.98	4.12	4.29
	$k = 20$	0.29	1.45	3.89	8.07	4.05
	$k = 50$	0.72	3.63	9.72	20.47	3.95
PS	$k = 1$	0.04	0.18	0.51	1.08	4.35
	$k = 5$	0.12	0.74	2.19	4.79	4.01
	$k = 10$	0.23	1.43	4.19	9.39	3.94
	$k = 20$	0.47	2.81	8.48	18.93	3.74
	$k = 50$	1.15	6.74	21.03	45.78	3.52
BRKGA-VD		17.13	80.63	190.50	369.75	0.00

Computational Experiments

Table: Summary of case study tests

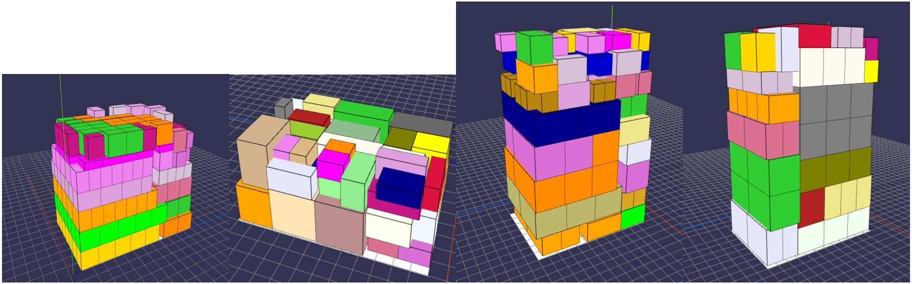
	PS			PM		
k	**TT (ms)**	**B**	**CR (%)**	**TT (ms)**	**B**	**CR (%)**
1	423.87	1.37	65.87	65.18	1.31	**70.70**
5	1,597.54	1.34	69.19	185.22	1.29	**73.08**
10	2,627.52	1.32	70.35	344.90	1.27	**73.56**
20	5,373.79	1.34	70.78	620.95	1.27	**74.57**
50	14,203.10	1.31	72.11	1,279.96	1.29	**74.61**
100	26,934.21	1.31	73.23	2,340.37	1.26	**75.36**
200	48,944.90	1.30	73.89	4,465.78	1.25	**76.39**

Computational Experiments

Table: Case study experiments trade off between average execution times and average cage ratio

k	PS		PM	
	CR* − CR (%)	**TT − TT* (ms)**	**CR* − CR (%)**	**TT − TT* (ms)**
1	10.56	358.69	5.73	0.00
5	7.24	1,532.36	3.35	120.04
10	6.08	2,562.34	2.87	279.72
20	5.65	5,308.61	**1.85**	**555.77**
50	4.32	14,137.92	**1.82**	**1,214.78**
100	3.20	26,869.03	1.07	2,275.19
200	2.54	48,879.72	0.04	4,400.60

Results



(a) Instance 95 (b) Instance 82 (c) Instance 56 (d) Instance 66, Bin 1

Figure: Solutions of case study tests with the "PM" placement and $k = 200$

Future Developments

- More practical constraints
- Solution optimization heuristics
- Improvements to the underlying 2D-BPP heuristic

Thank you for your attention!
Questions?