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Three-dimensional bin packing with vertical support

Tesi di Laurea Magistrale in
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Abstract

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Abstract in lingua italiana

Qui va l'Abstract in lingua italiana della tesi seguito dalla lista di parole chiave.

Parole chiave: qui, vanno, le parole chiave, della tesi

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1 | Introduction

The problem addressed by our contribution is the 3D-SBSBPP (3D Single Bin-Size Bin Packing Problem) which is strongly NP-hard and is the generalization of the one-dimensional bin packing problem [4].

Case study

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2 | Literature review

Static stability

Static stability or vertical support received most of its contributions from the fields of Pallet Loading Problems (PLP) and Cargo Loading Problems (CLP). In [2] a second order cone programming formulation was provided as a solution to a spacing problem between layers of a pallet. The pubblication further described the concept of minimizing the area of overlap between items of different layers to increase spacing between items of the same layer and increase the ammount of potential support for higher layers. In [5] a formulation of the CLP was given with practical constraints like weight distrubtiom, non-regular container shapes and vertical stability through vertex support.

3 | Problem description and mathematical formulation

In this thesis we address the 3D single bin-size bin packing problem (3D-SBSBPP) with the addition of a few practical constraints. Starting from a set of items of different size the goal is to arrange them in the least ammount of bins of a given fixed size without any overlap between eachother. In addition to the standard formulation of the problem three additional practical constraints need to be taken into account to satisfy the requests of our use-case:

- each item inside a bin should have static stability, meaning that every item should be supported either by the ground or by other items in the same bin
- the cage ratio of each used bin should be maximized
- each item can be rotated orthogonally along its vertical axis

Given a certain placement of items inside a bin of base $W \times D$ with the top of the highest item being at $z_{\max}$ and the sum of the volume of each item being V, the bin's cage ratio is defined as eq. (3.1).

$$\mathrm{CR} = \frac{V}{W \cdot D \cdot z_{\max}} \tag{3.1}$$

A high cage ratio means that even if a bin isn't fully occupied it could potentially be used as a base for other structures, a property which is desirable in some industrial setting. It is also noted that in a single bin configuration, maximizing cage ratio is equivalent to minimizing $z_{\max}$. A visual rappresentation of the cage ratio metric is provided in fig. 3.1.

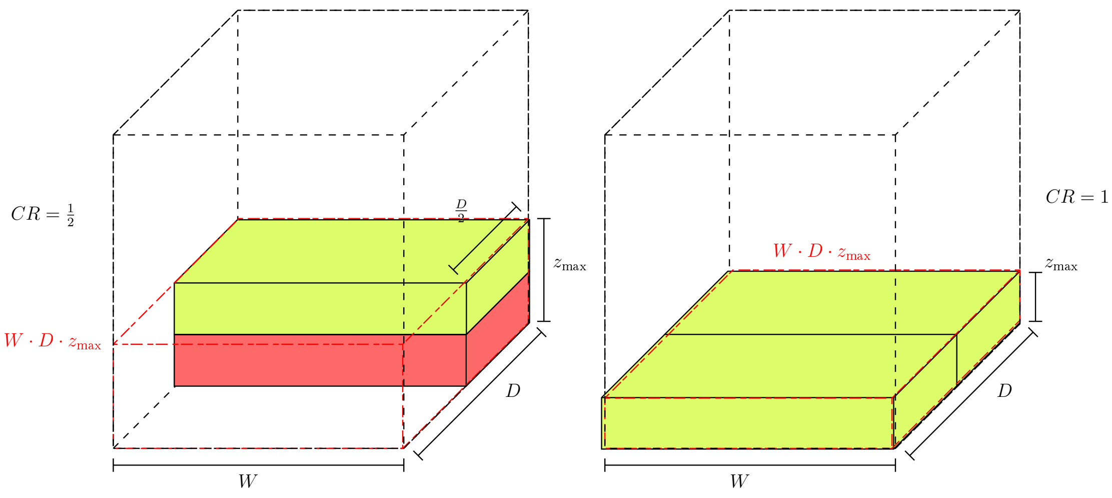


Figure 3.1: Cage ratio of two different bin configurations

Our notion of vertical support stems from rules imposed by the industry and from the literature on Pallet Loading Problems and Container Loading Problems where stability is usually ensured between horizontal or vertical slices of items as a constraint on the minimum ammount of area which rests on other items (as for ex. [2, 3, 5]). Given a support area threshold which is usually equal to $\alpha_s = 0.7$ and a tollerance which in our case is equal to $\beta_s = 1$cm, we can define an item as supported if

1. the sum of the overlap area over the XY-plane with every other item on which it is resting is greater then α_s times it's base area (area support)
2. the number of its corners resting on another item is greater than 3 and condition 1 holds with a lower threshold $\alpha'_s < \alpha_s$ (vertex support)

A visual rappresentation of the condition of support is illustrated in fig. 3.2.

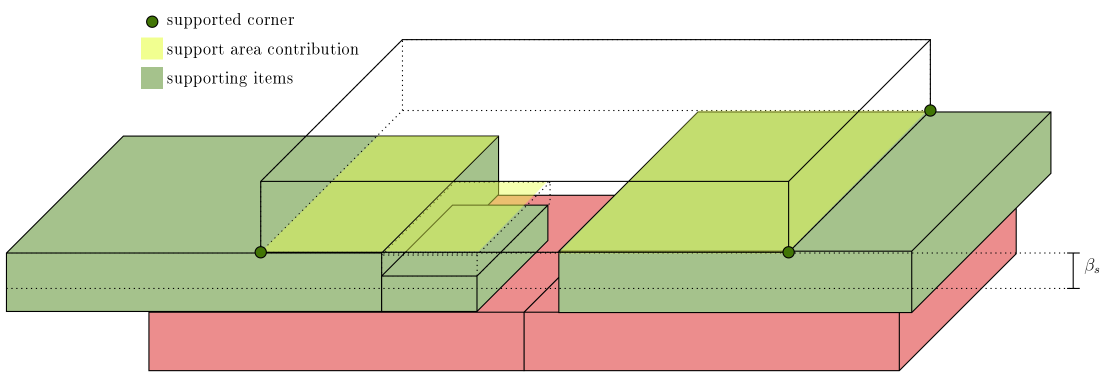


Figure 3.2: Rappresentation of an item with vertical support given $\alpha_s = 0.5, \beta_s$

Given the definitions of our practical constraints, a conceptual formulation of our model would be

minimize	number of used bins
	unused volume of each bin under $z_{\max}$
subject to	all items assigned to one and only one bin
	all items within the bin dimensions
	no overlaps between items in the same bin
	all items with vertical support

In section 3.1 a mixed integer linear programming model for the 3D-SBSBPP is presented and it's later extended to include orthogonal rotations in section 3.1.1 and vertical support constraints limited to condition 1 (area support) in section 3.1.2. Cage ratio isn't directly included in the proposed MILP formulation since the evaluation of the heuristic with the model was done in a single bin configuration where minimizing the maximum height of the bin is equivalent to minimizing the cage ratio.

3.1. 3D single bin-size bin packing problem

Let $I = \{1, \dots, n\}$ be the set of items that needs to be packed, $B = \{1, \dots, m\}$ the set of bins to evaluate of fixed dimensions $W \times D \times H$. Each item $i \in I$ is characterized by a given width, depth and height (w_i, d_i, h_i). Let us introduce three continuous variables that identify the position of the bottom front left corner of an item (x_i, y_i, h_i) as seen in fig. 3.3. We can now introduce a set of integer variables v_b which will be 1 if bin $b \in B$ will be used in the solution and 0 otherwhise. A set of integer variable u_{ib} which will be 1 if item $i \in I$ will be placed in bin $b \in B$ and 0 otherwhise. In order to check for overlaps three sets of integer variables are introduced for each axis of possible overlap that are used to determine if there is a clear order of precedence on at least on axis. This formulation is also usually used in scheduling problems. The three sets of variables are x_{ij}^p which will take the value of one if item $i \in I$ precedes item $j \in I$ over axis x given that item i precedes item j if $x_i + w_i \leq x_j$ and 0 otherwhise. The other two sets are defined in a similar way over the remaining axis y_{ij}^p and z_{ij}^p. An additional set of continuous variables $z_b^{\max}$ is introduced which will assume the value of the maximum $x_i + h_i$ of the items $i \in I$ placed in bin $b \in B$.

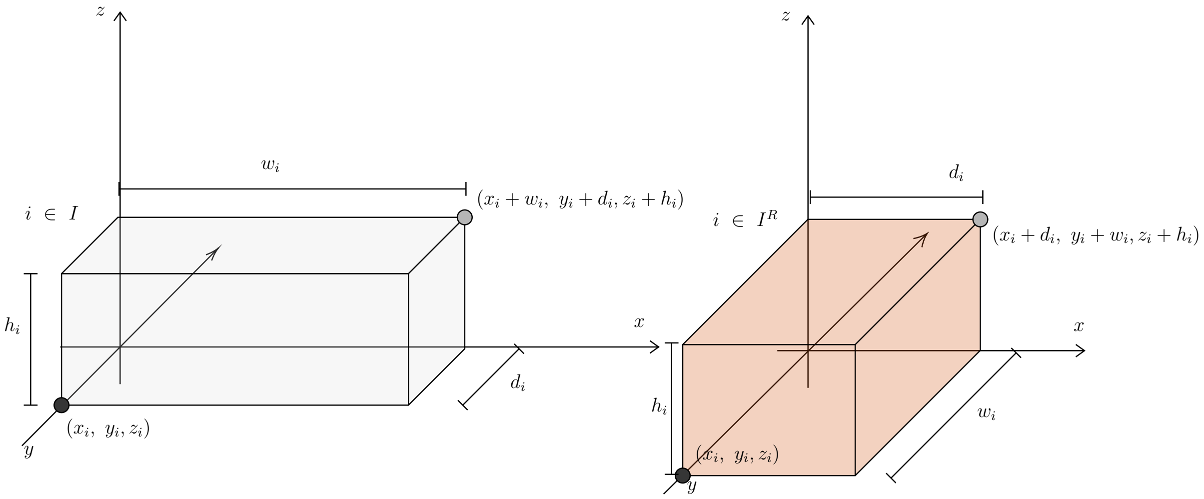


Figure 3.3: Coordinate system representation for a generic item i and it's rotated clone $i \in I^R$

The 3D-SBSBPP can then be formulated as a mixed-integer linear programming problem:

$$
\begin{aligned}
\min \quad & \sum_{b \in B} (Hv_b + z_b^{max}) && && (3.2)\\
\text{s.t.} \quad & \sum_{b \in B} u_{ib} = 1 && \forall i \in I && (3.3)\\
& u_{ib} \leq v_b && \forall i \in I, \forall b \in B && (3.4)\\
& v_b \geq v_c && \forall (b,c) \in B : b < c && (3.5)\\
& x_i + w_i \leq W && \forall i \in I && (3.6)\\
& y_i + d_i \leq D && \forall i \in I && (3.7)\\
& z_i + h_i \leq H && \forall i \in I && (3.8)\\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) && \forall i \in I, \forall b \in B && (3.9)\\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) && \forall i,j \in I && (3.10)\\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p && \forall i,j \in I && (3.11)\\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) && \forall i,j \in I && (3.12)\\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p && \forall i,j \in I && (3.13)\\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) && \forall i,j \in I && (3.14)\\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p && \forall i,j \in I && (3.15)\\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 && \forall i,j \in I, \forall b \in B && (3.16)
\end{aligned}
$$

The objective function 3.2 seeks to minimize the number of opened bins and the maximum heights of the opened bins. In a single bin configuration since all the volume mass is concentrated inside one bin this also means that it maximises the cage ratio. Constraint 3.3 ensures that each item is packed in one and only one bin, while contraint 3.4 ensures that items are only packed in bins that are used in the solution. Since the solution as lots of simmetries with respects to the number of bins a simmetry breaking constraint 3.5 can be added on the opening of bins to improve solve times. Each item is also ensured to be placed inside the bin thanks to eqs. (3.6) to (3.8). The value of z_b^{max} is forced to converge to the maximum height of a given bin thanks to constraint 3.9. Constraints from 3.10 to 3.15 are used to define the precedence binary variables x_{ij}^p, y_{ij}^p, z_{ij}^p over each axis as described in the problem formulation. Constraint 3.16 then ensures that if two items are in the same bin then there needs to be at least one axis with a clear order of precedence, otherwise the two items would be overlapping eachother.

3.1.1. Othogonal rotations

Let us extend the definition of the bin packing problem without rotations with a new formulation which allows 90 degrees rotations of each item. Let $I = I^O \cup I^R$ be the new set of items where I^O is the set of original non-rotated items and I^R is the set of items rotated by 90 degrees. Given the set of tuples $(i, j) \in I^{OR}$ where i is the original item with dimensions (w_i, d_i, h_i) and j is the corresponding rotated clone with dimensions $(w_j, d_j, h_j) = (d_i, w_i, h_i)$, we can now rewrite constraint 3.3 as 3.17 to force only one for the item between the original and rotated to be part of the solution.

$$\sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \qquad \forall (i, j) \in I^{OR} \tag{3.17}$$

3.1.2. Discrete vertical support formulation

We now extend the model to address the constraint of static support. In the literature there are some mathematical formulations that tackle the concept of area support, and in some cases vertex support. In Elhedhli et al. a SOCP formulation of the support constraint was used but was limited to the problem of spacing between layers with one of the layers being fixed in position, in our case a similar formulation would lead to a non-linear support constraint. By introducing a discretization over the XY-plane a linear version of the constraint can be formulated similar to the one proposed in Kurpel et al. without the need to discretize the z-axis as well.

Let us introduce some additional parameters to the model, let $0 \leq \alpha_s \leq 1$ be the ammount of area that an item needs to have supported by other items and let β_s be the height tollerance to consider one item as being close enough to support another item (as seen in fig. 3.2). In addition to the support parameters an additional parameter δ is given which rappresents the discretization unit used to partition the XY-plane. Let I^B be the set of all the tuples (i, j, b) such that $(i, j) \in I \land i \neq j$ and $b \in B$. We can now compute a few additional parameters that will be used to reduce the number of constraints evaluated by the model. Let γ be the maximum size over a dimension on the XY-plane between all the items as eq. (3.18), and let Δ be the set of all possible distances between the origins of two items along one discretized axis as eq. (3.19).

$$\gamma = \max_{\forall i \in I}\{w_i, d_i\} \tag{3.18}$$

$$\Delta = \left[-\left\lfloor\frac{\gamma}{\delta}\right\rfloor, \left\lfloor\frac{\gamma}{\delta}\right\rfloor\right] \tag{3.19}$$

Let $O(i, j, h, k) \to \mathbb{R}^+$ be a function that computes the ammount of overlap between two items $(i, j) \in I$ given the discretized distance between each other $(h, k) \in \Delta$ such that $x_j = x_i + \delta h$ and $y_j = y_i + \delta k$ which returns the area of overlap or 0 otherwhise.

Additional new variables need to be added to the ones of the original model, let s_{ij} be a set of binary variables which will assume value 1 if item $i \in I$ can offer support to item $j \in I$ and 0 otherwhise. A new set of binary variables z^c_{ij} will be 1 if item $i \in I$ is close w.r.t. the z-axis to item $j \in I$, which would mean that $z_j - (z_i + h_i) \leq \beta_s$, and 0 otherwhise. Let us then introduce a new set of binary variables g_i which will assume value 1 if item $i \in I$ will be on the ground or 0 otherwhise and a set of binary variables s^{kh}_{ijb} that will assume value 1 if item $i \in I$ will receive support from item $j \in I$ and both will be placed in bin $b \in B$ with a discretized distance of $(k, h) \in \Delta$ between eachother and 0 otherwise.

Given all the additional parameters and variables introduced, a new formulation of the model can be given with the same objective function 3.2 and the constraints in section 3.1

with the addition of the following constraints:

$$z_j - (z_i + h_i) \leq \beta_s + H(1 - z^c_{ij}) \qquad \forall (i,j) \in I : i \neq j \tag{3.20}$$
$$z_j - (z_i + h_i) \geq -\beta_s - H(1 - z^c_{ij}) \qquad \forall (i,j) \in I : i \neq j \tag{3.21}$$
$$s_{ij} \leq z^p_{ij} \qquad \forall (i,j) \in I \tag{3.22}$$
$$s_{ij} \leq z^c_{ij} \qquad \forall (i,j) \in I \tag{3.23}$$
$$s_{ij} \geq z^p_{ij} + z^c_{ij} - 2 \qquad \forall (i,j) \in I : i \neq j \tag{3.24}$$
$$\sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} \qquad \forall i \in I \tag{3.25}$$
$$z_i \leq H(1 - g_i) \qquad \forall i \in I \tag{3.26}$$
$$\sum_{(k,h) \in \Delta, b \in B : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq s_{ij} \qquad \forall (i,j) \in I \tag{3.27}$$
$$\sum_{(k,h) \in \Delta : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq u_{ib} \qquad \forall (i,j,b) \in IB \tag{3.28}$$
$$\sum_{(k,h) \in \Delta : O(i,j,k,h) \neq 0} s^{kh}_{ijb} \leq u_{jb} \qquad \forall (i,j,b) \in IB \tag{3.29}$$

Constraints 3.20 and 3.21 ensure that z^c_{ij} is forced to 1 only when the distance over the z-axis between item i and item j is within the range $[-\beta_s, \beta_s]$. The value of s_{ij} is then assigned to the logical equation $z^p_{ij} \wedge z^c_{ij}$ thanks to constraints from 3.22 to 3.24. Since some items could be left out of the solution due to the formulation of orthogonal rotations, we also ensure that support can only come from placed items thanks to constraint 3.25. Constraint 3.26 together with the hard constraint of support that will be introduced ensures that g_i assumes value 1 if item i is on the ground. Constraints from 3.27 to 3.29 ensure that if a discretized support decision s^{hk}_{ijb} is 1 then every subscript of that variable must be true in the non-discretized model, so item i can give discretized support to item j in in bin b if both items are are assinged to bin b and if i can give support to item j. They also force the selection of only one possible combination of $(h,k) \in \Delta$ for which i gives support to j in bin b.

We can then define a set of constraints which given a discretized placement s^{kh}_{ijb} limits the distance between i and j to a given continuous region in space delimited by a square of the dimmension of our discretization unit δ. Given every tuple of possible discretized distances between items $(k,h) \in \Delta$ and every tuple of different pairs of items in the same bin $(i,j,b) \in I^B$ such that the items would overlap over the discretized plane XY for a non-null region $(O(i,j,k,h) \neq 0)$ the resulting constraints are defined in eqs. (3.30) to (3.33).

$$x_j - x_i \geq \gamma k - 2W(1 - s_{ijb}^{kh}) \tag{3.30}$$

$$x_j - x_i \leq \gamma(k+1) + 2W(1 - s_{ijb}^{kh}) \tag{3.31}$$

$$y_j - y_i \geq \gamma h - 2D(1 - s_{ijb}^{kh}) \tag{3.32}$$

$$y_j - y_i \leq \gamma(h+1) + 2D(1 - s_{ijb}^{kh}) \tag{3.33}$$

And finally we can introduce a feasibility constraint which ensures that every item that isn't on the ground is supported by other items placed beneath it by at least α_s times its area, which corresponds to condition 1 of the pratical constraint of vertical support.

$$\sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h)s_{jib}^{kh} \geq \alpha_s w_i d_i - w_i d_i g_i \quad \forall i \in I \tag{3.34}$$

It is noted that since every combination of $(i, j, b) \in I^B$ and $(h, k) \in \Delta$ where $O(i, j, k, h) = 0$ do not contribute to the support constraint, then they can be omitted from the formulation of the problem to reduce the number of constraints to evaluate.

4 | Solution algorithms

In this chapter, we describe an heuristic algorithm to solve the 3D bin packing problem with vertical support. In section 4.1 we describe the concepts which will be used in the algorithm, like the definition of a state, insertions, and the feasibility of a solution. Since the 3D-BPP is NP-Hard an exhaustive search for a solution isn't practical so an heuristic search is conducted by combining a beam search algorithm described in section 4.2 and a constructive heuristic described in section 4.3. The proposed algorithm takes in input an initial feasible state (as defined in section 4.1.2) usually represented by the empty state (4.3) and outputs the best scoring state based on an ordering function defined in section 4.2.1.

4.1. State

States or packings are partial solutions to the 3D-BPP. Given the formal definition of the problem (3.1) a few new definitions are introduced to facilitate the algorithm's definition.

Definition 4.1 (Unpacked item). *An item $i \in I$ is unpacked* ***iff***

$$\sum_{b \in B} u_{ib} = 0$$

It is also assumed that variables identifying an item's position are independent between states (changes to their values in state s won't affect state s'). In order to simplify the algorithm rappresentation, rotations are handled by simply swapping the dimensions w_i and d_i of item i when needed.

A state s can then be defined as follows

- U: the set of unpacked items
- B: the set of used bins
- $Q = (q_1, q_2, \dots, q_b)$: the set of supporting structures for each bin $b \in B$

- p: the insertion pending on this state (described by def. 4.4)

Observation 4.1. *Given two states s and s' we can have that $|s.B| \neq |s'.B|$ since the number of bins is also a variable in the proposed heuristic*

We can also trivially define a function that determines if a state is a final state

Definition 4.2. *A state s is final if there are no more items to pack*

$$IsFinal(s) = \begin{cases} 1, & s.U = \emptyset \\ 0, & otherwise \end{cases} \tag{4.1}$$

Each used bin b has additional data associated to it that is contained in $q_b \in s.Q$ which is used to store structures usefull to the constructive heuristic definied in section 4.3. Let us introduce the concept of packed items inside a bin:

Definition 4.3 (Packed item)**.** *Given a state s and a bin $b \in s.B$, we say that item*

$$i \in I \ \textit{ is packed in } b \ \textbf{\textit{iff}} \quad u_{ib} = 1$$

Given a bin $b \in s.B$ we can then define structure q_b as follows

- J: the set of items that are packed inside b
- Z: the set of planes inside b (section 4.3)
- T: the AABB Tree (section 4.1.1) representing the items inside b

Both sets $q_b.J$ and $q_b.T$ contain the items packed in b but adding and accessing items in $q_b.J$ has a time complexity of $O(1)$ given an underlying implementation as HashSet while maintaining $q_b.T$ usually has a time complexity of $O(log(|q_b.J|))$. The AABB Tree inside this structure is included to allow for fast overlap checks between items and is further explained in sections 4.1.2 and 4.3.1.

4.1.1. AABB Tree

In order to determine the feasibility of a given state, a way of checking for overlaps with items already placed is needed. Since our formulation of the problem only allows for 90 deg rotations over the z-axis. Every item in a solution, by the problem formulation (3.3), is contained inside a bounding box and this box is axis-aligned. An adequate structure to compute overlaps is then an Axis-Aligned Bounding Box Tree (AABB Tree) [1].

AABB Trees are bounding volume hierarchies typically used for fast collision detection and they usually offer a few operations:

- $AABBInsert(i)$: which allows inserting an axis-aligned box i in the tree
- $AABBOverlaps(i)$: which allows determining if an axis-aligned box i overlaps an element in the tree
- $AABBClosest(i, d)$: which, given an axis-aligned box i and a direction $d \in \{XP, XN, YP, YN, ZP, ZN\}$ along an axis, returns the closest element following that direction starting from the box i

If the tree is properly balanced each operation on average has a time complexity of $O(log(n))$ where n is the number of elements in the tree. Maintaining an AABB Tree in the state allows us to do checks for feasibility during the construction of a solution (as detailed in 4.3.1) and feasibility checks on the final states to allow for error detection.

4.1.2. Feasibility

A state s is said to be feasible if the currently packed items for every bin $b \in s.B$ aren't overlapping any other item, if they are all contained inside their bin and if each item is either on the ground or satisfies at least one of the support conditions (cond. 1, cond. 2). Since the proposed heuristic is constructive it is more convenient to define the concept of feasibility relative to a change in the state.

Insertions Given a state s and $b \in s.B$, an insertion of items is a set of non-overlapping items that are placed in b and have their z_i within a tolerance from a certain z.

Definition 4.4 (Insertion)**.** *Given a state s and a tolerance β_s we define an insertion or placement p a tuple (b, I) where b is a bin and I is a set of non-overlapping items that are going to be packed in b such that, $I \subseteq s.U \land \exists z(z \in \mathbb{Z} \land \forall i(i \in I \land |z_i - z| \leq \beta_s))$*

Observation 4.2. *Given a state s and an insertion $p = (b, \emptyset)$ where $b \notin s.B$, p is an insertion which will open bin b in s.*

Definition 4.5 (Next)**.** *Let p be an insertion over a state s we can then define $s' = Next(s, p)$ as the "copy" of state s with $s'.p = p$. And p is then a pending insertion on s'.*

We can evaluate the changes to the score of a state based on its pending insertion without having to update all the structures for every evaluated state. This property will become apparent in section 4.2. We can then define an algorithm that applies a pending insertion

p on a given state s with the help of a function $OpenBin(b)$ which initializes a new structure q_b with every element at its empty value. The proposed algorithm is shown in 1.

Algorithm 1: Commit

input : s
output: s'
$(b, I) \leftarrow s.p$
$s' \leftarrow Clone(s)$ `//Memory clone of` s
if $b \in s'.B$ **then**
  $q_b \leftarrow (q_i \in s'.Q : i = b)$
  $q_b.J \leftarrow q_b.J \cup I$
  $s'.U \leftarrow s'.U \setminus I$
end
else Open a new bin
  $s'.B \leftarrow s'.B \cup b$
  $s'.Q \leftarrow s'.Q \cup OpenBin(b)$
end
$s'.p \leftarrow$ none
return s'

Insertion feasibility An insertion $p = (b, I)$ that is pending on a given state s is feasible if every inserted item $i \in p.I$ satisfies the constraint of non-overlap (3.16), the constraint of support (3.34) and if it is placed within the size of the given bin.

Algorithm 2: Is Insertion Feasible

input : $b, I, z, I_{support}, I_{upper}$
output: $isFeasible$
return $true$

State feasibility Describe how to compute the sets efficiently to use the insertion feasibility logic

Algorithm 3: Is State Feasible

input : s
output: $isFeasible$
return $true$

Proposition 4.1. *A state s' derived by committing a feasible insertion p to a feasible state s is feasible.*

Observation 4.3. *We can always define the empty state s_e where*

$$\begin{cases} s_e.U = I \\ s_e.Q = \emptyset \\ s_e.B = \emptyset \end{cases}$$

and it is always feasible

4.2. Beam Search

Beam Search (BS) is a heuristic tree search algorithm designed for systems with limited memory where expanding every possible node is unfeasible. The idea behind BS is to conduct an iterative truncated breadth-first search where, at each iteration, only a limited number of k nodes is expanded. After the expansion, every new node needs to be evaluated and sorted in order to prune the number of nodes down to the k best ones. The algorithm keeps exploring until no further node can be expanded.

To perform BS one must define the node structure, an expansion function to generate new nodes from existing ones, a ranking between nodes, and a function to determine if a node is final.

A node in the tree can be represented as the state in section 4.1 and eq. (4.1) can be used to determine if a state is final. We also know that a new state s' derived by s by applying a feasible insertion p can be computed as in definition 4.5. This state expansion procedure, with the exception of empty insertions, will generate new states in our tree which will add a positive number of bins or packed items to the solution so, eventually, it will generate a final state.

If the starting state for the search is feasible every new state generated will be feasible and if a final state is found it will be feasible (proposition 4.1). We also note that starting from state s the time complexity to compute feasible insertions can be lower than the complexity required to update the structures that will be used for further expansions (AABB Tree insertion and balancing, memory cloning, etc.) so we modified the standard BS algorithm to separate the expansion phase from the commit phase.

Given S^0 the set of initial states and k the number of best states to expand at each iteration, the described procedure is represented by algorithm 4. As observed in observation 4.3 it's possible to start the search from $S^0 = \{s_e\}$.

Algorithm 4: Beam search

input : S^0, k

output: s_{best}

$S^t \leftarrow S^0$

$S_{final} \leftarrow \emptyset$

repeat

  $S^{t+1} \leftarrow Expand(S^t)$ (algo. 5)

  $S_{final} \leftarrow S_{final} \cup \{s \in S^{t+1} : IsFinal(s)\}$ (def. 4.2)

  $S^{t+1} \leftarrow S^{t+1} \setminus S_{final}$

  $S^{t+1} \leftarrow Sort(S^{t+1})$ (sec. 4.2.1)

  $S^t \leftarrow \emptyset$

  $i \leftarrow 0$

  forall $s \in S^{t+1}$ **do**

    $S^t \leftarrow S^t \cup Commit(s)$ (algo. 1)

    $i \leftarrow i + 1$

    if $i > k$ **then**

      break

    end

  end

until $S^t \neq \emptyset$

$S_{final} \leftarrow Sort(S_{final})$

return *first element of* S_{final}

State Expansion An expansion of a state s can be seen as a new set of states S_{new} derived by a set of feasible insertions. In order to determine these insertions, an underlying heuristic is used (described in section 4.3).

The main idea in this phase of the algorithm is to find feasible insertions in all the bins for items that still need to be packed and that are of the same height. With this approach, the solutions given by the algorithm will start by trying to fill lower layers with items of the same height if possible and they'll become more heterogeneous in upper layers where the classes of height will start to mix up. The underlying heuristic will also use a scoring mechanism to select the best insertions for a given class of heights in order to avoid having too many states to sort.

Given a set of items I and a tolerance β_s we can introduce an algorithm to group them by their shap and produce a set G of tuples (h, I') where h is the hash summurizing the shape of the group and I' is the set of items grouped as in algo. 6.

Once items are grouped by shape the best insertion for each class of items can be computed for each open bin. If no insertion is possible in any bin, then the only viable insertion is the bin opening insertion (observation 4.2). The described procedure is detailed in algo. 5.

Algorithm 5: Expand

```
input  : S
output: S_new
forall s ∈ S do
    S_new ← ∅
    G ← GroupByHash(s.U) (algo. 6)
    placed ← false
    forall (h, I) ∈ G do
        forall q_b ∈ s.Q do
            P ← SPBestInsertion(q_b, I) (algo. 7)
            if P ≠ ∅ then
                placed ← true
                forall p ∈ P do
                    S_new ← S_new ∪ Next(s, p) (def. 4.5)
                end
            end
        end
    end
    if placed = false then
        Open a new bin with index |s.B| (oss.  4.2)
        S_new ← S_new ∪ Next(s, (|s.B|, ∅))
    end
end
return S_new
```

4.2.1. Scoring States

In order to sort states, a scoring function needs to be defined over them. Since the scoring of the states is what will influence the final solution the most, parameters that are directly related to minimizing the objective function are selected.

In the proposed solution to handle multiple objective functions, lexicographic ordering is used.

Algorithm 6: Group By Hash

```
input  : I
output: G
G ← ∅
forall i ∈ I do
    generate ← true
    forall (h, I′) ∈ G do
        if h = hash(w_i, d_i, h_i) then
            generate ← false
            I′ ← I′ ∪ i
            break
        end
    end
    if generate = true then
        G ← G ∪ (hash(w_i, d_i, h_i), {i})
    end
end
return G
```

Definition 4.6. *Let $f_1(s), f_2(s), f_i(s), \ldots, f_n(s)$ be objective functions ordered by precedence based on index i, then*

$$s < s' \textit{ iff } \exists j \in \mathbb{Z} : \begin{cases} f_j(s) < f_j(s') \\ f_k(s) = f_k(s'), \quad \forall k \in \mathbb{Z} : 0 \leq k < j \end{cases}$$

Scoring metrics for each state s that we want to evaluate can then be computed in the $Next$ algorithm by considering the contents of the pending insertions and updating each parameter differentially.

The defined ordering utilized is the following:

- $f_1(s) = -|s.B|$: we prefer states that opened fewer bins.
- $f_2(s) = \text{avgvol}(s)$: we prefer states that have packed more average volume between bins.
- $f_3(s) = \text{avgcageratio}(s)$: we prefer states that have better average cage ratio (eq. (3.1)) between bins.

4.3. Support Planes

Support Planes (SP) is a heuristic detailed in the following section based on an underlying 2DBPP heuristic which is used to evaluate feasible insertions starting from a given state. Since the insertions must be feasible SP mantains an internal structure to facilitate the check for feasibility. The idea at the base of SP is to build a solution to the 3DBPP by filling 2D planes called *support planes*.

Each support plane can be characterized by the triple $S_z = (z, I_{support}, I_{upper})$ where

– z: the height of the plane

– $I_{support}$: the set of the items that can offer support to items placed on the plane

– I_{upper}: the set of items that will be obstacles to potential new items placed on the plane

Let *coords* be the set of possible coordinate changes which allow for the problem to evaluate insertions starting from different corners of the bin.

Given a function $IsFeasible(i, bin, I_{support}, I_{upper}, aabb)$ which evaluates if a packing of item i in bin bin is feasible, and the function $ComparePacking(p, p')$ which defines a ranking over insertions in the same plane, the SP algorithm can be written as algorithm 7.

Algorithm 7: SP Best Insertion

input : s_b, I
output: *placement*

```
placement ← ∅
forall S_z ∈ planes do
    I_p ← I \ {i ∈ I : z + i.h > H_b}
    forall change ∈ coords do
        I'_upper ← CoordinateChange(change, I_upper)
        I'_p ← CoordinateChange(change, I_p)
        P' ← SPPackPlane(W_b, D_b, I'_upper, I'_p) (Algorithm 8)
        P ← CoordinateChange(change, P')
        P ← {i ∈ P : IsFeasible(i, bin, I_support, I_upper, aabb)}
        if ComparePacking(placement, P) then
            placement ← P
        end
    end
    if placement ≠ ∅ then
        return placement
    end
end
return placement
```

To evaluate a packing on a plane a heuristic to solve the 2DBPP is used with the introduction of fixed insertions which represent items on other planes that will be obstacles in the current one.

Given the dimensions of the 2D bin (W_b, D_b), the set of obstacles I_o and the set of items to pack I_p a new insertion can be computed following algorithm 8

Algorithm 8: SP Pack Plane

input : W_b, D_b, I_o, I_p
output: P
$P \leftarrow \emptyset$
$2dPacking \leftarrow \emptyset$
foreach $i \in I_o$ **do**
    `//Initialize the 2D bin packing instance with each obstable already placed`
    $2DPlaceRect(2dPacking, i)$
end
repeat
    `//Pack untill full`
    $p \leftarrow 2DPackRect(2dPacking, W_b, D_b, i)$
    $P \leftarrow P \cup \{p\}$
until $p \neq \emptyset$
return P

Commit Extension We now describe an extension to *Commit* (algo. 1) to update the structures needed by SP.

When a plane is filled, new insertions become less likely to be feasible. To avoid evaluating planes where no insertion is possible a mechanism to prune dead planes can be introduced.

Since best insertions for a bin are always evaluated by considering lower planes first, if all the insertions in *Expand* (algo. 5) happened over a z_{min} then we can safely remove the opened planes with $z < z_{min}$ for that bin. Let us introduce a z_{min} variable carried over in q_b for each bin, which is updated during the *Expand* phase with the minimum z of all the insertions on bin b. Once the best states are computed and *Commit* is called we can then use its value to prune planes in each q_b. Other operations are also necessary in the *Commit* algorithm to allow SP to update its data structures accordingly to the insertion.

Given a state s and an insertion p where each packed item $i \in p.I$ in bin b has z_i within tolerance of z and the minimum height for the considered bin $q_b.z_{min}$. The algorithm which updates the structures for a given bin b is represented by algorithm 9. This new algorithm can be used as the last step of the *Commit* algorithm for each $b \in s'.B$.

Algorithm 9: SP Apply and Filter

```
input  : s, p, z, z_min, β_s
output: s
q_b ← (q_i ∈ s.Q : i = p.b)
//Filter bad planes
q_b.Z ← q_b.Z \ {(z', I_support, I_upper) ∈ q_b.Z : z' < z_min}
//Apply insertion
forall i ∈ p.I do
    q_b.T ← InsertAABB(i, q_b.T) //If balanced O(log(n))
    generate ← true
    forall (z', I_support, I_upper) ∈ q_b.Z do
        //Based on the distance from the top of the item
        dz ← z' − (z_i + h_i)
        if 0 ≤ dz ≤ β_s then
            generate ← false
            I_support ← I_support ∪ i
        end
        else if dz < 0 then
            I_upper ← I_upper ∪ i
        end
    end
    if generate then
        q_b.Z ← q_b.Z ∪ (z_i + h_i, {i}, ∅)
    end
end
return s
```

4.3.1. Scoring Insertions

5 | Computational experiments

5.1. Model validation

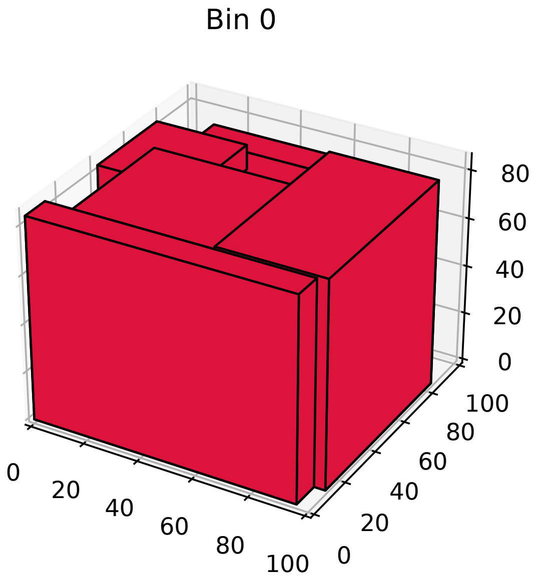
Table 5.1: Comparison with MILP model on limited set of boxes

N	Group By Hash		Single Placement		MILP Model	
	Max Z	TT(s)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.00	85	0.00	85	0.01
2	85	0.00	85	0.00	85	0.09
3	85	0.00	85	0.00	85	0.24
4	85	0.01	85	0.01	85	0.64
5	85	0.02	85	0.02	85	16.48
6	158	0.06	158	0.05	158*	594.32
7	158	0.07	158	0.08	158*	3,178.00
8	160	0.10	160	0.08	-	-
9	169	0.09	161	0.10	-	-
10	218	0.12	218	0.13	-	-
11	240	0.12	240	0.12	-	-
12	310	0.13	316	0.16	-	-
13	310	0.15	333	0.18	-	-
14	310	0.20	333	0.22	-	-
15	406	0.21	397	0.27	-	-
16	435	0.23	452	0.36	-	-
17	429	0.27	515	0.41	-	-
18	432	0.32	522	0.47	-	-
19	458	0.35	522	0.55	-	-
20	539	0.37	564	0.62	-	-

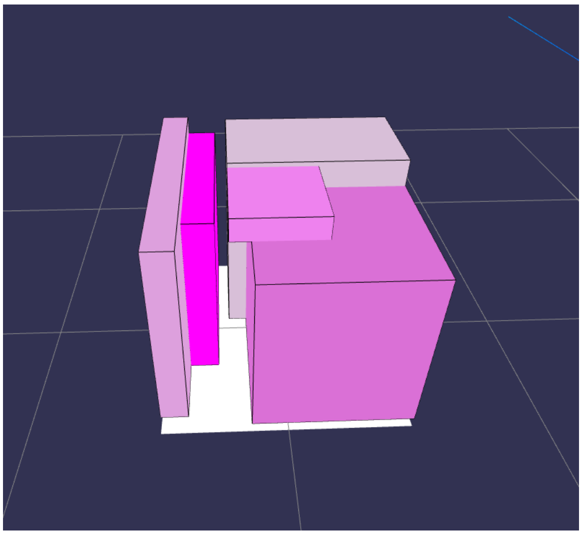
* Some boxes had lower support than expected due to discretization errors within the $0.65 \leq \beta_s \leq 0.7$ range.

5.2. Literature results

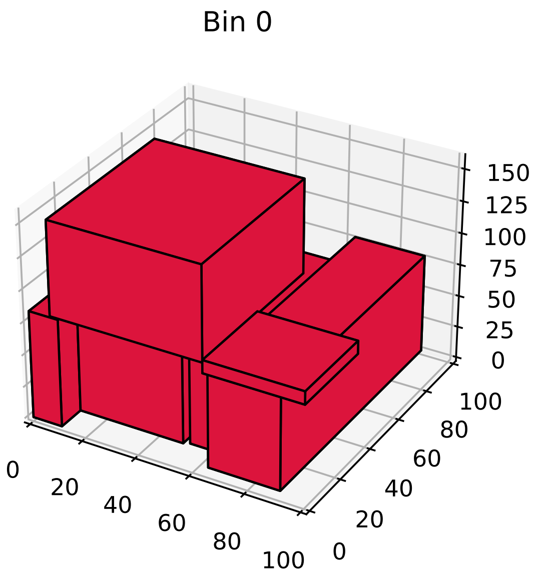
5.3. Use-case results



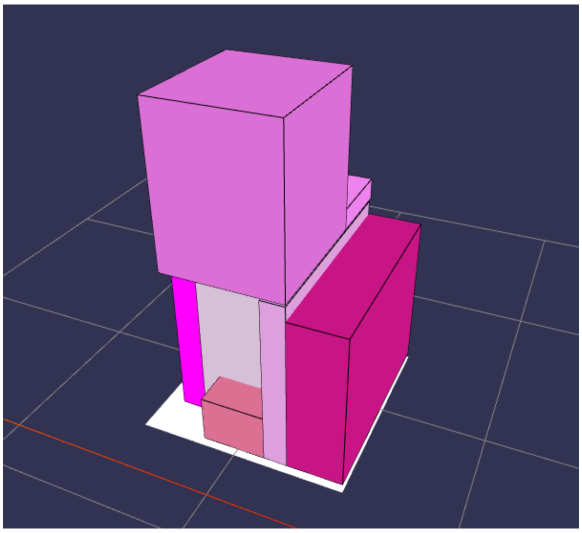
(a) MILP, Instance 5



(b) Heuristic k=200, Instance 5



(c) MILP, Instance 7

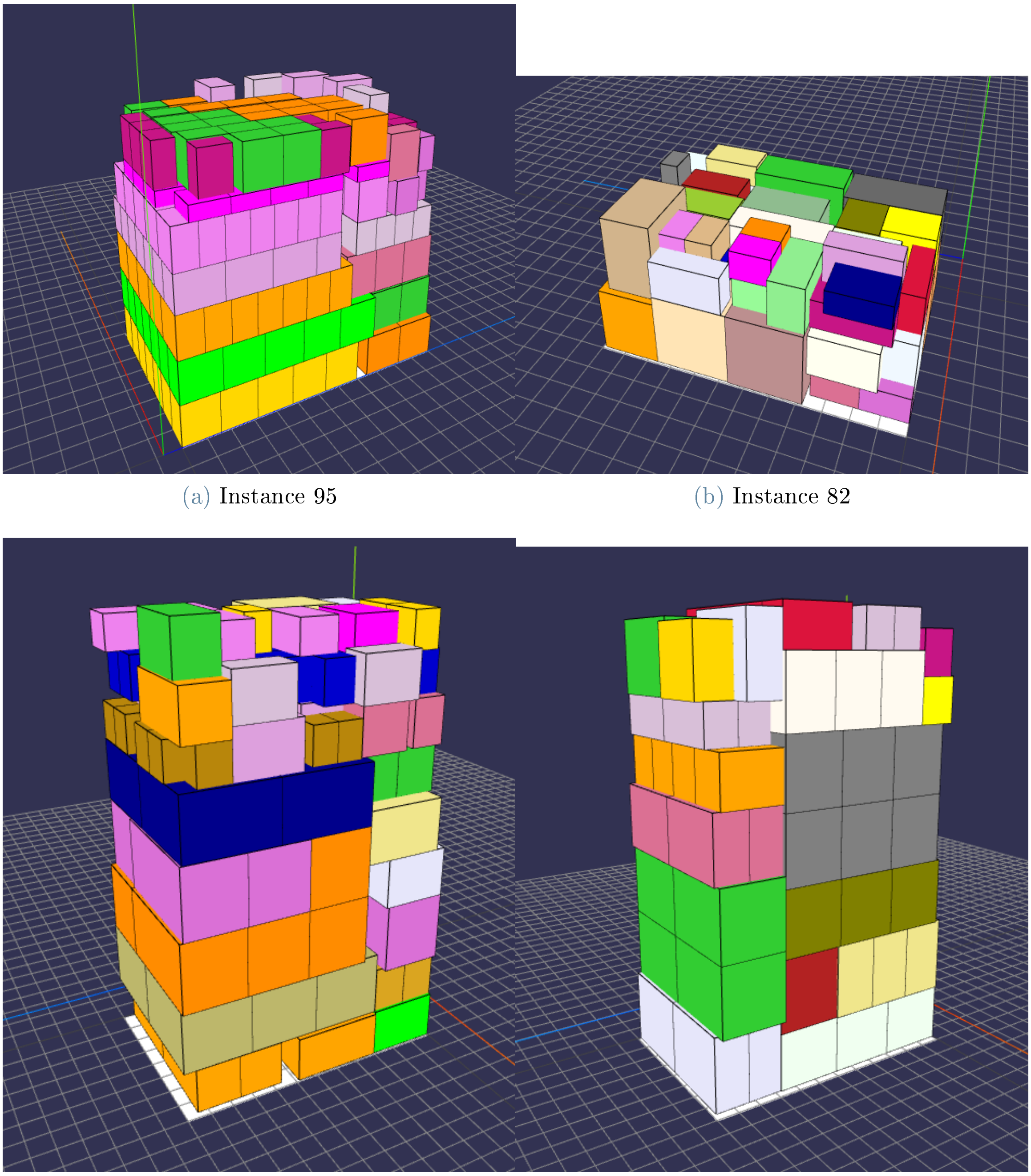


(d) Heuristic k=200, Instance 7

Figure 5.1: Graphical comparison between solutions from the heuristic and from the MILP model

Table 5.2: Summary of use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
Global	k=1	423.87	1.37	65.87	65.18	1.31	**70.70**
	k=5	1,597.54	1.34	69.19	185.22	1.29	**73.08**
	k=10	2,627.52	1.32	70.35	344.90	1.27	**73.56**
	k=20	5,373.79	1.34	70.78	620.95	1.27	**74.57**
	k=50	14,203.10	1.31	72.11	1,279.96	1.29	**74.61**
	k=100	26,934.21	1.31	73.23	2,340.37	1.26	**75.36**
	k=200	48,944.90	1.30	73.89	4,465.78	1.25	**76.39**
Class 0-19	k=1	187.25	1.15	64.10	54.95	1.05	**70.69**
	k=5	489.40	1.05	70.38	111.75	1.00	**75.36**
	k=10	861.30	1.05	71.94	182.20	1.00	**75.77**
	k=20	1,588.15	1.05	72.04	308.45	1.00	**76.60**
	k=50	3,896.40	1.05	73.07	690.80	1.00	**76.95**
	k=100	7,789.90	1.00	75.45	1,204.35	1.00	**78.46**
	k=200	15,817.20	1.05	74.99	2,192.75	1.00	**78.27**
Class 20-39 N = [50, 70]	k=1	50.90	1.00	68.21	17.80	1.00	**73.66**
	k=5	138.40	1.00	71.92	39.20	1.00	**74.78**
	k=10	253.10	1.00	73.15	74.95	1.00	**75.28**
	k=20	483.85	1.00	73.86	124.30	1.00	**76.46**
	k=50	1,193.55	1.00	74.77	288.50	1.00	**77.02**
	k=100	2,358.50	1.00	75.08	535.30	1.00	**77.11**
	k=200	4,769.85	1.00	76.69	1,033.00	1.00	**78.64**
Class 40-59 N = [70, 120]	k=1	292.35	1.30	65.62	60.55	1.25	**71.34**
	k=5	1,025.65	1.30	67.97	172.35	1.30	**72.53**
	k=10	1,910.60	1.30	68.46	304.25	1.25	**72.04**
	k=20	3,666.40	1.30	68.68	571.90	1.25	**74.01**
	k=50	7,649.95	1.25	71.32	1,152.40	1.25	**75.25**
	k=100	15,848.15	1.25	72.90	1,956.55	1.20	**75.67**
	k=200	32,420.40	1.25	73.29	3,472.50	1.20	**76.10**
Class 60-79 N = [120, 200]	k=1	1,371.00	2.20	64.68	158.00	2.05	**69.11**
	k=5	5,751.95	2.15	66.66	531.80	1.95	**71.31**
	k=10	9,040.85	2.05	68.56	1,033.15	1.90	**72.69**
	k=20	19,116.60	2.15	67.81	1,881.70	1.90	**73.84**
	k=50	52,937.40	2.05	69.94	3,744.70	2.00	**71.25**
	k=100	98,271.55	2.10	70.04	7,010.65	1.90	**73.80**
	k=200	170,191.55	2.00	71.15	13,544.15	1.90	**75.01**
Class 80-99	k=1	217.85	1.20	66.74	34.60	1.20	**68.68**
	k=5	582.30	1.20	69.03	71.00	1.20	**71.41**
	k=10	1,071.75	1.20	69.65	129.95	1.20	**72.00**
	k=20	2,013.95	1.20	71.52	218.40	1.20	**71.97**
	k=50	5,338.20	1.20	71.44	523.40	1.20	**72.57**
	k=100	10,402.95	1.20	**72.68**	995.00	1.20	71.74
	k=200	21,525.50	1.20	73.30	2,086.50	1.15	**73.95**



(a) Instance 95

(b) Instance 82

(c) Instance 56

(d) Instance 66, Bin 1

Figure 5.2: Some solutions of the use-case tests with the "Group by Hash" behaviour and $k = 200$

6 | Conclusions and future developments

A final chapter containing the main conclusions of your research/study and possible future developments of your work have to be inserted in this chapter.

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A | Appendix A

Table A.1: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
1	k=1	403	1	69.54	218	1	69.82
	k=5	384	1	70.9	157	1	74.64
	k=10	502	1	71.47	151	1	74.64
	k=20	786	1	71.47	187	1	73.33
	k=50	1732	1	71.47	357	1	74.26
	k=100	3524	1	71.47	613	1	76.54
	k=200	6892	1	74.71	1020	1	74.64
2	k=1	266	2	48.2	67	1	77.19
	k=5	835	1	78.58	196	1	84.69
	k=10	1537	1	78.58	311	1	86.65
	k=20	2045	1	83.22	607	1	87.59
	k=50	5233	1	83.22	1706	1	87.84
	k=100	11422	1	83.22	3226	1	86.94
	k=200	22911	1	83.22	3860	1	85.87
3	k=1	169	1	73.36	62	1	65.48
	k=5	335	1	73.36	104	1	73.31
	k=10	532	1	73.36	141	1	72.73
	k=20	1003	1	73.36	245	1	74.86
	k=50	2621	1	73.46	457	1	74.51
	k=100	5209	1	73.46	897	1	75.02
	k=200	10781	1	74.2	1676	1	78.9
4	k=1	384	1	53.7	57	1	79.2
	k=5	1048	1	59.27	153	1	79.91
	k=10	1934	1	59.27	203	1	76.37
	k=20	3754	1	59.27	313	1	79.44
	k=50	9266	1	65.04	754	1	82.18
	k=100	18445	1	72.44	1467	1	82.18
	k=200	36636	1	72.44	2956	1	82.18
5	k=1	52	1	67.48	25	1	74.44
	k=5	192	1	73.22	75	1	76.16
	k=10	324	1	73.22	104	1	69.76
	k=20	641	1	73.22	144	1	69.18
	k=50	1613	1	73.22	255	1	68.65
	k=100	3466	1	73.22	518	1	68.65
	k=200	7149	1	73.22	1050	1	68.74

Table A.2: Use-case tests

Instance		**Single Placement**			**Order by Hash**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
6	k=1	357	2	35.53	76	1	78.78
	k=5	939	1	65.88	196	1	79.74
	k=10	1638	1	74.23	419	1	82.46
	k=20	3257	1	73.74	624	1	80.73
	k=50	8533	1	73.74	1295	1	80.52
	k=100	16594	1	75.36	2019	1	78.53
	k=200	33658	1	75.36	3925	1	77.35
7	k=1	308	1	62.38	32	1	68.32
	k=5	774	1	62.38	98	1	71.36
	k=10	1052	1	69.06	148	1	81.03
	k=20	2003	1	69.06	299	1	82.35
	k=50	4828	1	71.32	697	1	79.09
	k=100	10009	1	71.32	1138	1	82.12
	k=200	19931	1	71.32	2289	1	82.12
8	k=1	50	1	74.2	36	1	79.27
	k=5	142	1	74.2	46	1	78.78
	k=10	240	1	76.51	66	1	83.85
	k=20	472	1	80.77	126	1	83.85
	k=50	1196	1	82.12	317	1	83.85
	k=100	2410	1	82.12	617	1	83.85
	k=200	4844	1	82.12	1212	1	83.85
9	k=1	188	1	67.28	41	1	69.6
	k=5	580	1	74.36	135	1	73.26
	k=10	989	1	74.36	319	1	81.8
	k=20	1795	1	75.21	364	1	77.87
	k=50	4573	1	78.8	1377	1	80.34
	k=100	8641	1	78.8	1557	1	76.19
	k=200	18028	1	78.8	3058	1	76.07
10	k=1	37	1	75.91	24	1	72.18
	k=5	229	1	76.34	65	1	74.73
	k=10	321	1	76.34	102	1	74.73
	k=20	645	1	76.34	186	1	74.73
	k=50	1641	1	76.34	413	1	80.24
	k=100	3177	1	76.34	685	1	79.76
	k=200	6562	1	76.34	1376	1	79.76

Table A.3: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
11	k=1	83	1	66.82	26	1	69.88
	k=5	227	1	66.82	88	1	73.04
	k=10	437	1	73.04	94	1	73.04
	k=20	901	1	73.04	182	1	73.73
	k=50	2163	1	74.9	374	1	72.55
	k=100	4271	1	75.09	656	1	70.51
	k=200	8965	1	75.09	1338	1	73.61
12	k=1	290	2	54.85	41	2	38.76
	k=5	905	2	49.36	149	1	79.67
	k=10	1746	2	49.36	153	1	77.74
	k=20	3048	2	40.42	304	1	76.43
	k=50	6768	2	40.91	526	1	77.53
	k=100	13867	1	73.25	1054	1	79.32
	k=200	29292	2	42.83	2136	1	79.32
13	k=1	161	1	53.61	23	1	68.76
	k=5	333	1	69.77	62	1	72.32
	k=10	585	1	69.77	120	1	73.12
	k=20	1140	1	70.64	156	1	64.05
	k=50	2821	1	70.64	420	1	65.88
	k=100	5610	1	70.64	833	1	73.76
	k=200	11427	1	75.46	1119	1	72.44
14	k=1	209	1	66.77	30	1	70.43
	k=5	512	1	66.77	71	1	69.51
	k=10	959	1	71.72	229	1	80.74
	k=20	1773	1	71.72	442	1	73.78
	k=50	4093	1	72	993	1	78.03
	k=100	8228	1	72	1915	1	77.62
	k=200	16602	1	75.58	2885	1	74.15
15	k=1	216	1	79.47	74	1	78.65
	k=5	710	1	79.47	161	1	66.78
	k=10	1415	1	80.62	245	1	70.17
	k=20	2661	1	80.62	365	1	77.37
	k=50	6673	1	80.62	865	1	80.52
	k=100	12879	1	80.62	1530	1	85.66
	k=200	25418	1	80.62	3552	1	85.66

Table A.4: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
16	k=1	114	1	68.8	125	1	80.6
	k=5	471	1	71.42	139	1	76.27
	k=10	808	1	71.42	265	1	76.27
	k=20	1529	1	72.13	473	1	78.77
	k=50	3901	1	72.92	1081	1	78.77
	k=100	7624	1	76.33	1654	1	79.27
	k=200	15602	1	76.33	3217	1	78.91
17	k=1	98	1	71.2	27	1	73.37
	k=5	263	1	77.41	61	1	71.09
	k=10	535	1	77.41	148	1	72.96
	k=20	1014	1	77.41	276	1	76
	k=50	2540	1	78.54	616	1	75.31
	k=100	4890	1	78.54	989	1	79.99
	k=200	10395	1	78.54	1790	1	79.92
18	k=1	108	1	60.55	36	1	69.39
	k=5	244	1	75.18	59	1	80.2
	k=10	434	1	75.18	127	1	78
	k=20	801	1	75.18	204	1	78
	k=50	1957	1	77.61	376	1	78.85
	k=100	3807	1	77.61	796	1	78.85
	k=200	7824	1	80	1575	1	77.54
19	k=1	113	1	67.04	52	1	66.58
	k=5	330	1	77.57	133	1	77.82
	k=10	623	1	77.57	172	1	59.29
	k=20	1289	1	77.57	435	1	77.44
	k=50	2977	1	77.57	589	1	74.24
	k=100	6064	1	77.57	1235	1	83.16
	k=200	12258	1	78.13	2461	1	83.16
20	k=1	139	1	65.36	27	1	63
	k=5	335	1	65.36	87	1	73.91
	k=10	615	1	66.36	127	1	70.1
	k=20	1206	1	66.36	237	1	72.51
	k=50	2799	1	66.92	348	1	65.81
	k=100	5661	1	69.53	688	1	71.21
	k=200	11169	1	75.42	1360	1	71.21

Table A.5: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
21	k=1	33	1	67.44	18	1	77.23
	k=5	97	1	75.35	38	1	71.08
	k=10	166	1	75.35	60	1	72.69
	k=20	298	1	76.28	108	1	71.65
	k=50	741	1	76.28	173	1	69.42
	k=100	1399	1	76.28	349	1	69.42
	k=200	2870	1	76.28	692	1	71.4
22	k=1	29	1	72.11	9	1	72.94
	k=5	69	1	73.29	26	1	74.94
	k=10	141	1	74.21	50	1	74.14
	k=20	277	1	74.94	80	1	74.65
	k=50	706	1	74.94	184	1	75.91
	k=100	1385	1	78.81	354	1	75.91
	k=200	2802	1	78.81	716	1	75.91
23	k=1	34	1	63.84	13	1	81.33
	k=5	94	1	73.15	31	1	78.01
	k=10	173	1	75.99	57	1	77.42
	k=20	329	1	75.99	84	1	81.09
	k=50	808	1	76.28	211	1	83.06
	k=100	1594	1	76.28	381	1	83.06
	k=200	3164	1	79.14	743	1	84.51
24	k=1	26	1	73.86	8	1	73.12
	k=5	65	1	75.64	23	1	73.12
	k=10	110	1	79.19	40	1	78.76
	k=20	207	1	79.45	77	1	77
	k=50	511	1	81.14	173	1	70.35
	k=100	1076	1	81.14	349	1	70.35
	k=200	2148	1	81.14	682	1	79.63
25	k=1	82	1	70.7	37	1	71.53
	k=5	229	1	70.7	99	1	69.62
	k=10	431	1	70.7	181	1	69.62
	k=20	816	1	70.7	283	1	78.27
	k=50	2005	1	70.7	462	1	75.37
	k=100	4011	1	71.74	933	1	75.37
	k=200	8134	1	72.09	1759	1	75.37

Table A.6: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
26	k=1	35	1	71.69	16	1	73.01
	k=5	102	1	75.92	26	1	77.99
	k=10	194	1	75.92	49	1	76.17
	k=20	400	1	75.92	68	1	72.31
	k=50	1008	1	75.92	161	1	73.89
	k=100	2126	1	76.05	327	1	73.89
	k=200	4295	1	76.05	644	1	73.89
27	k=1	63	1	66.47	22	1	76.56
	k=5	164	1	70.9	38	1	77.73
	k=10	276	1	70.9	59	1	72.99
	k=20	522	1	70.9	105	1	77.81
	k=50	1291	1	71.76	249	1	77.81
	k=100	2563	1	71.76	483	1	77.81
	k=200	5463	1	77.81	947	1	77.81
28	k=1	55	1	68.54	17	1	77.78
	k=5	136	1	68.54	36	1	78.32
	k=10	236	1	70.2	59	1	79.3
	k=20	429	1	73.5	103	1	83.47
	k=50	1096	1	73.5	272	1	83.95
	k=100	2151	1	73.5	451	1	83.95
	k=200	4486	1	74.86	931	1	86.45
29	k=1	48	1	73.14	17	1	73.85
	k=5	144	1	75.49	41	1	74.03
	k=10	244	1	79.77	68	1	76.69
	k=20	462	1	79.77	132	1	77.94
	k=50	1104	1	81.33	459	1	80.83
	k=100	2230	1	81.33	706	1	84.8
	k=200	4511	1	84.64	1425	1	84.8
30	k=1	25	1	72.31	8	1	75.76
	k=5	111	1	72.31	18	1	75.63
	k=10	199	1	76.44	34	1	75.63
	k=20	349	1	76.44	71	1	75.76
	k=50	946	1	76.44	163	1	80.23
	k=100	1865	1	76.57	317	1	80.23
	k=200	3472	1	79.56	645	1	82.23

Table A.7: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
31	k=1	37	1	59.65	14	1	74.05
	k=5	106	1	71.6	38	1	72.43
	k=10	191	1	72.36	61	1	72.43
	k=20	365	1	72.36	103	1	73.48
	k=50	918	1	75.08	247	1	80.34
	k=100	1745	1	75.08	406	1	77.23
	k=200	3442	1	77.46	798	1	78.02
32	k=1	61	1	65.02	24	1	69.44
	k=5	148	1	72.09	30	1	78.06
	k=10	253	1	73.71	45	1	72.09
	k=20	515	1	73.71	83	1	72.09
	k=50	1229	1	73.71	214	1	72.09
	k=100	2481	1	73.71	425	1	72.09
	k=200	5037	1	73.71	850	1	72.37
33	k=1	106	1	71.14	40	1	79.25
	k=5	239	1	75.74	64	1	78.21
	k=10	462	1	75.74	120	1	78.21
	k=20	854	1	78.73	192	1	78.21
	k=50	2130	1	78.73	412	1	83.68
	k=100	4239	1	78.73	837	1	83.68
	k=200	8519	1	80.04	1685	1	83.68
34	k=1	36	1	61.8	12	1	62.54
	k=5	95	1	64.99	50	1	69.19
	k=10	170	1	64.99	78	1	73.28
	k=20	321	1	67.87	150	1	73.28
	k=50	735	1	73.08	250	1	75.11
	k=100	1390	1	73.08	504	1	75.11
	k=200	2791	1	79.15	868	1	72.67
35	k=1	36	1	69.35	13	1	69.86
	k=5	106	1	69.35	37	1	71.2
	k=10	253	1	72.43	81	1	71.2
	k=20	480	1	74.53	85	1	73.06
	k=50	1277	1	74.58	188	1	71.65
	k=100	2378	1	74.58	373	1	71.65
	k=200	4929	1	74.58	755	1	71.65

Table A.8: Use-case tests

Instance		**Single Placement**			**Order by Hash**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
36	k=1	121	1	66.81	28	1	75
	k=5	343	1	66.81	53	1	73.69
	k=10	618	1	66.81	180	1	78.01
	k=20	1136	1	67.11	291	1	82.68
	k=50	2664	1	71.94	711	1	76.98
	k=100	5253	1	73.12	1223	1	80.92
	k=200	10658	1	75.67	2579	1	81.79
37	k=1	44	1	73.94	13	1	68.3
	k=5	106	1	73.94	36	1	71.96
	k=10	184	1	73.94	92	1	74.65
	k=20	384	1	73.94	124	1	76.03
	k=50	909	1	73.94	270	1	76.34
	k=100	1883	1	73.94	461	1	68.16
	k=200	3836	1	73.94	907	1	79.57
38	k=1	39	1	70.69	17	1	74.76
	k=5	132	1	70.69	43	1	77.39
	k=10	259	1	70.69	83	1	77.39
	k=20	522	1	71.28	160	1	77.39
	k=50	1289	1	72.24	396	1	77.39
	k=100	2606	1	72.24	950	1	78.95
	k=200	5212	1	72.24	1208	1	79.17
39	k=1	24	1	62.25	7	1	76.42
	k=5	71	1	69.91	14	1	79.74
	k=10	121	1	71.76	23	1	79.74
	k=20	230	1	71.76	47	1	80.77
	k=50	559	1	71.76	110	1	80.77
	k=100	1065	1	71.76	216	1	80.77
	k=200	2097	1	74.15	444	1	80.77
40	k=1	84	1	63.4	23	1	70.51
	k=5	211	1	71.97	43	1	73.19
	k=10	381	1	71.97	79	1	75.09
	k=20	781	1	71.97	140	1	72.27
	k=50	1945	1	71.97	465	1	75.22
	k=100	3730	1	71.97	661	1	78.91
	k=200	7531	1	72.51	1382	1	81.13

Table A.9: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
41	k=1	326	1	68.36	38	1	70.62
	k=5	697	1	74.25	104	2	52.93
	k=10	1062	1	75.63	122	1	74.21
	k=20	1948	1	77.68	262	1	78.73
	k=50	4790	1	77.68	668	1	74.45
	k=100	10170	1	79.15	1062	1	77.1
	k=200	20996	1	79.15	2013	1	78.91
42	k=1	149	1	70.28	20	1	67.43
	k=5	479	1	75.83	108	1	68.59
	k=10	849	1	75.83	214	1	73.76
	k=20	1623	1	75.83	374	1	75.83
	k=50	4004	1	75.83	907	1	81.2
	k=100	8093	1	75.83	1923	1	77.36
	k=200	16798	1	75.83	2991	1	77.3
43	k=1	174	2	64.74	96	2	65.13
	k=5	1628	2	71.89	370	2	64.66
	k=10	3028	2	71.89	515	2	58.88
	k=20	6518	2	70.91	1089	2	67.08
	k=50	9427	2	68.02	2335	2	80.84
	k=100	21206	2	71.12	3015	2	76.47
	k=200	67001	2	67.51	5429	2	77.53
44	k=1	367	1	61.69	61	1	71.18
	k=5	868	1	69.72	205	1	83
	k=10	1506	1	69.72	310	1	78.48
	k=20	3062	1	69.72	760	1	76.04
	k=50	7259	1	69.72	1567	1	80.62
	k=100	14367	1	69.72	3204	1	81.5
	k=200	29580	1	69.72	5594	1	80.33
45	k=1	604	2	46.66	65	1	75.62
	k=5	1502	2	44.05	359	1	83.12
	k=10	2890	2	43.78	529	1	83.12
	k=20	4877	2	38.4	810	1	78.72
	k=50	9384	1	76.94	2070	1	83.31
	k=100	22382	1	76.94	4241	1	86.62
	k=200	36854	1	79.15	6044	1	85.35

Table A.10: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
46	k=1	259	2	77.06	78	2	71.7
	k=5	2706	2	72.32	101	2	66.68
	k=10	4194	2	70.3	280	2	72.68
	k=20	8229	2	70.3	456	2	77.02
	k=50	14847	2	76.33	1201	2	73.15
	k=100	26144	2	76.15	1841	2	74.16
	k=200	45738	2	75.98	4028	2	75.2
47	k=1	395	1	70.04	73	1	72.15
	k=5	1082	1	70.04	188	1	75.28
	k=10	2024	1	70.04	439	1	72.72
	k=20	3741	1	70.04	737	1	78.82
	k=50	8428	1	70.04	1798	1	78.82
	k=100	17165	1	72.83	3143	1	78.82
	k=200	34689	1	72.83	6153	1	78.82
48	k=1	208	2	63.3	75	2	62.24
	k=5	1616	2	69.72	179	2	61.82
	k=10	3395	2	63.35	410	2	54.88
	k=20	6845	2	62.39	749	2	66.88
	k=50	7946	2	65.44	913	2	63.9
	k=100	20583	2	65.15	2056	2	64.02
	k=200	43066	2	70.5	4014	2	66.03
49	k=1	152	1	64.08	36	1	79
	k=5	617	1	69.12	79	1	76.7
	k=10	1140	1	72.53	128	1	76.11
	k=20	2039	1	72.53	252	1	79.88
	k=50	4622	1	74.03	603	1	79.88
	k=100	8767	1	74.03	1170	1	81.7
	k=200	16856	1	74.03	2384	1	81.7
50	k=1	421	1	63.37	63	1	70.85
	k=5	899	1	67.26	279	1	74.19
	k=10	1950	1	70.81	283	1	74.19
	k=20	3077	1	70.81	670	1	67.69
	k=50	6702	1	70.81	1476	1	74.15
	k=100	14720	1	72.07	1272	1	67.3
	k=200	31094	1	72.29	2618	1	67.3

Table A.11: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
51	k=1	199	1	70.66	62	1	71.17
	k=5	1189	1	71.25	157	1	70.5
	k=10	2375	1	71.25	297	1	69.81
	k=20	4786	1	71.25	1206	1	71.6
	k=50	11267	1	71.25	879	1	64.67
	k=100	20929	1	71.25	1761	1	64.67
	k=200	41807	1	73.87	3514	1	64.67
52	k=1	54	1	71.22	22	1	77.21
	k=5	170	1	74.69	51	1	80.62
	k=10	332	1	74.69	85	1	76.17
	k=20	589	1	76.01	169	1	78.28
	k=50	1589	1	76.01	424	1	80.14
	k=100	3017	1	77.1	517	1	81.3
	k=200	6764	1	77.1	1030	1	81.3
53	k=1	434	1	63.73	54	1	70.09
	k=5	1510	1	63.73	217	1	75.61
	k=10	2843	1	63.73	365	1	69.2
	k=20	5270	1	63.73	544	1	75.31
	k=50	12437	1	68.54	1198	1	77.54
	k=100	22733	1	69.71	1561	1	72.15
	k=200	46766	1	70.13	3089	1	71.75
54	k=1	889	2	71.9	75	2	71.76
	k=5	1611	2	74.52	186	2	77.16
	k=10	3338	2	74.52	452	2	70.11
	k=20	7081	2	72.86	771	2	77.55
	k=50	17401	2	72.55	1609	2	77.67
	k=100	36080	2	77.76	2773	2	71.73
	k=200	69866	2	71.95	5305	2	75.45
55	k=1	136	1	58.5	46	1	69.04
	k=5	455	1	70.35	89	1	71.56
	k=10	786	1	74.32	168	1	71.98
	k=20	1501	1	78.76	303	1	72.31
	k=50	3475	1	78.76	748	1	72.31
	k=100	7132	1	78.76	1376	1	72.53
	k=200	14863	1	79.02	1909	1	79.6

Table A.12: Use-case tests

Instance		**Single Placement**			**Order by Hash**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
56	k=1	319	1	63.92	141	1	79.42
	k=5	828	1	71.46	236	1	77.95
	k=10	1476	1	74.87	420	1	80.12
	k=20	2840	1	74.87	550	1	76.63
	k=50	7310	1	74.87	1478	1	81.85
	k=100	15072	1	74.87	2233	1	75.5
	k=200	30574	1	78.73	3703	1	73.29
57	k=1	249	2	64.23	53	2	55.11
	k=5	743	2	47.8	137	2	58.35
	k=10	1343	2	45.86	295	2	48
	k=20	2089	2	42.54	398	2	48.64
	k=50	5008	2	44.26	1116	2	54.39
	k=100	13920	2	53.72	2074	1	82.3
	k=200	26749	2	51.17	2753	1	82.01
58	k=1	106	1	65	34	1	77.03
	k=5	809	1	65	135	1	76.3
	k=10	1525	1	67.19	195	1	76.3
	k=20	3043	1	67.99	328	1	74.82
	k=50	7037	1	67.99	539	1	76.89
	k=100	14984	1	71.01	1032	1	76.84
	k=200	29794	1	71.48	2064	1	77.89
59	k=1	64	1	69.12	54	1	78.11
	k=5	278	1	69.66	131	1	78.2
	k=10	658	1	69.66	250	1	77.18
	k=20	1229	1	75.21	440	1	76.43
	k=50	3024	1	75.21	643	1	77.6
	k=100	5983	1	75.21	1248	1	77.6
	k=200	11839	1	75.21	2491	1	77.6
60	k=1	342	1	64.54	65	1	71.84
	k=5	826	1	66.78	136	1	77.47
	k=10	1498	1	69.14	328	1	82.87
	k=20	2941	1	71.75	570	1	81.86
	k=50	7042	1	72.18	876	1	71.62
	k=100	13516	1	75.64	1629	1	73.71
	k=200	26714	1	80.19	2324	1	69.89

Table A.13: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
61	k=1	2470	3	61.14	194	2	75.03
	k=5	7566	3	61.46	458	2	79.58
	k=10	7705	2	71.35	882	2	74.15
	k=20	30277	3	57.73	2574	2	78.49
	k=50	41450	2	71.61	4022	2	79.87
	k=100	69808	3	58.95	6114	2	79.77
	k=200	197317	2	71.15	13254	2	77.35
62	k=1	2572	2	66.32	179	2	73.37
	k=5	10605	2	69.85	507	2	73.55
	k=10	17406	2	73.78	1380	2	73.49
	k=20	45291	2	66.46	2006	2	72.71
	k=50	44703	2	72.57	6606	2	77.29
	k=100	147556	2	69.57	7603	2	76.6
	k=200	103674	2	73.59	20459	2	76.31
63	k=1	1665	2	64.66	269	2	75.39
	k=5	11233	2	63.95	956	2	77.32
	k=10	13900	2	68.69	1656	2	77.37
	k=20	28438	2	72.53	2944	2	75.38
	k=50	99439	2	72.61	9805	2	75.62
	k=100	71230	2	72.6	15467	2	77.67
	k=200	137283	2	73.18	12330	2	78.89
64	k=1	2452	2	66.08	156	2	76.31
	k=5	5607	2	71.17	392	2	75.41
	k=10	4290	2	67.29	713	2	72.47
	k=20	7240	2	65.83	1005	2	76.43
	k=50	60548	2	69.7	2428	2	77.74
	k=100	70161	2	73.64	6411	2	79.81
	k=200	144912	2	73.64	10006	2	80.02
65	k=1	1391	2	61.23	126	2	62.06
	k=5	8206	2	61.29	482	2	62.92
	k=10	17117	2	64.36	1116	2	61.67
	k=20	15367	2	69.79	2358	2	65.05
	k=50	78715	2	63.81	3023	2	60.12
	k=100	146277	2	72.45	7925	2	61.3
	k=200	406884	2	63.72	15111	2	64.17

Table A.14: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
66	k=1	1904	3	58.8	175	2	81.55
	k=5	10568	3	67.91	485	2	79.54
	k=10	5507	3	65.99	1231	2	77.61
	k=20	11187	3	65.99	1804	2	78.44
	k=50	119767	3	56.88	3802	2	81.5
	k=100	256247	3	68.01	6740	2	80.58
	k=200	222528	2	77.34	13746	2	82.84
67	k=1	925	2	70.81	122	2	68.56
	k=5	5226	2	69.78	439	2	66.14
	k=10	10700	2	65.29	901	2	66.67
	k=20	10307	2	70.82	1418	2	75.99
	k=50	48933	2	69.91	2782	2	68.27
	k=100	72109	2	68.5	6219	2	70.04
	k=200	80496	2	72.21	13593	2	75.07
68	k=1	662	2	67.89	266	2	74.12
	k=5	2206	2	70.69	485	2	71.13
	k=10	3601	2	70.59	933	2	72.56
	k=20	6176	2	69.41	3152	2	75.02
	k=50	21427	2	72.94	3732	2	75.85
	k=100	46567	2	74.75	5835	2	73.27
	k=200	136385	2	72.67	13022	2	72.26
69	k=1	533	2	67.01	164	2	70.51
	k=5	2336	2	66.89	645	2	56.89
	k=10	3697	2	69.5	906	2	57.8
	k=20	34601	2	68.7	1629	2	63.27
	k=50	25976	2	68.5	2795	2	55.69
	k=100	45713	2	68.21	7779	2	57.47
	k=200	95060	2	69.61	11836	2	67.42
70	k=1	706	2	66.92	114	2	48.99
	k=5	3006	2	66.92	228	2	50.21
	k=10	9862	2	67.94	923	2	49.66
	k=20	18979	2	67.95	1259	2	47.25
	k=50	47210	2	67.95	3296	2	45.48
	k=100	93287	2	67.94	5671	2	50.66
	k=200	175227	2	69.96	10166	2	46.7

Table A.15: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
71	k=1	1937	2	68.46	275	2	78.92
	k=5	6324	2	67.23	923	2	79.06
	k=10	12072	2	60.76	1845	2	77.33
	k=20	43571	2	69.7	2488	2	77.21
	k=50	107498	2	69.7	4843	2	79.95
	k=100	196533	2	74.01	9836	2	80.56
	k=200	165421	2	70.67	30888	2	77.79
72	k=1	3135	2	69.36	180	2	75.19
	k=5	12714	2	70.89	841	2	78.12
	k=10	13866	2	73.13	1489	2	75.76
	k=20	17588	2	74.99	2531	2	80.41
	k=50	63019	2	72.49	4930	2	82.07
	k=100	99912	2	73.73	8868	2	76.87
	k=200	202486	2	73.73	15071	2	80.41
73	k=1	543	2	62.87	106	2	59.24
	k=5	1435	2	61.18	434	2	49.78
	k=10	2961	2	61.18	912	1	81.44
	k=20	5741	2	61.18	1491	1	78.06
	k=50	14454	2	59.95	2519	2	61
	k=100	48160	2	53.75	3573	1	81.19
	k=200	55897	2	56.57	16018	1	81.52
74	k=1	1141	3	59.03	110	3	51.47
	k=5	2047	2	74.12	573	2	78.35
	k=10	11023	2	75.26	647	2	81.52
	k=20	21326	2	78.24	1739	2	78.18
	k=50	22942	2	73.59	3399	3	52.15
	k=100	99298	2	76.12	4867	2	80.29
	k=200	187444	2	76.64	5707	2	78.08
75	k=1	1626	3	67.53	146	3	56.95
	k=5	7600	3	63.87	548	2	79.31
	k=10	14729	3	63.87	1095	2	82.81
	k=20	22788	3	68.43	1881	2	79.28
	k=50	124694	3	76.24	3366	2	78.77
	k=100	205410	3	66.02	7725	2	80.9
	k=200	537763	3	65.18	24478	2	84.28

Table A.16: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
76	k=1	1376	2	63.96	177	2	74.41
	k=5	8926	2	74.31	519	2	72.4
	k=10	17086	2	74.31	751	2	78.52
	k=20	33521	2	74.31	1159	2	78.81
	k=50	60053	2	73.48	3003	2	84.23
	k=100	135048	2	73.09	7363	2	78.8
	k=200	231300	2	71.65	16926	2	80.54
77	k=1	1023	2	49.84	120	1	71.99
	k=5	4338	2	39.27	651	1	82.65
	k=10	5616	1	71.67	1016	1	76.75
	k=20	10976	2	40.22	2028	1	80.46
	k=50	29232	1	72.87	3568	1	75.93
	k=100	41278	1	77.2	8318	1	73.59
	k=200	88763	1	77.2	5868	1	80.87
78	k=1	265	2	69.04	49	2	63.24
	k=5	1194	2	71.41	262	2	67.07
	k=10	2336	2	65.69	571	2	68.79
	k=20	4516	2	69.82	902	2	68.92
	k=50	12694	2	69.79	2142	2	68.06
	k=100	37343	2	70.11	3265	2	71.6
	k=200	68154	2	69.01	5888	2	68.1
79	k=1	619	2	65.18	142	2	68.24
	k=5	2508	2	68	521	2	73.19
	k=10	4839	2	68	912	2	73.2
	k=20	9629	2	68	1748	2	73.2
	k=50	24561	2	68	2860	2	69.99
	k=100	59050	2	70.69	6388	2	68.51
	k=200	119607	2	68.91	8898	2	70.83
80	k=1	475	2	67.45	90	2	76.64
	k=5	1394	2	73	287	2	73.64
	k=10	2504	2	72.45	784	2	74.19
	k=20	4813	2	76.09	1518	2	74.19
	k=50	11433	2	76.26	1973	2	75.36
	k=100	24444	2	71.36	4246	2	76.51
	k=200	47230	2	76.39	7618	2	76.66

Table A.17: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
81	k=1	8	1	71.74	9	1	73.14
	k=5	32	1	71.74	17	1	71.74
	k=10	49	1	73.14	27	1	71.74
	k=20	94	1	73.14	49	1	71.74
	k=50	226	1	73.14	99	1	71.74
	k=100	451	1	74.59	185	1	71.74
	k=200	916	1	74.59	345	1	74.59
82	k=1	30	1	62.99	67	1	73.56
	k=5	128	1	75.39	124	1	77.7
	k=10	242	1	75.39	244	1	77.7
	k=20	460	1	75.89	577	1	75.39
	k=50	1150	1	75.89	1658	1	76.65
	k=100	2312	1	75.89	3109	1	76.14
	k=200	4703	1	75.89	4769	1	76.65
83	k=1	6	1	63.86	4	1	59.07
	k=5	16	1	63.86	11	1	65.64
	k=10	31	1	63.86	19	1	65.64
	k=20	52	1	64.13	39	1	65.64
	k=50	125	1	65.64	87	1	65.64
	k=100	245	1	65.64	173	1	65.64
	k=200	489	1	65.64	347	1	65.64
84	k=1	3	1	66.1	2	1	73.29
	k=5	9	1	73.29	5	1	74.06
	k=10	17	1	73.29	8	1	74.06
	k=20	32	1	73.29	11	1	74.06
	k=50	75	1	73.29	10	1	74.06
	k=100	150	1	73.29	10	1	74.06
	k=200	307	1	73.29	10	1	74.06
85	k=1	24	1	69.56	21	1	66.42
	k=5	75	1	69.56	33	1	79.68
	k=10	140	1	69.56	54	1	79.68
	k=20	267	1	69.56	94	1	80.26
	k=50	670	1	69.56	208	1	80.26
	k=100	1311	1	69.56	378	1	80.26
	k=200	2645	1	69.56	798	1	80.26

Table A.18: Use-case tests

Instance		**Single Placement**			**Order by Hash**		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
86	k=1	34	1	82.3	17	1	75.61
	k=5	80	1	82.3	20	1	82.3
	k=10	133	1	82.3	28	1	82.3
	k=20	259	1	82.3	53	1	82.3
	k=50	604	1	82.3	104	1	82.3
	k=100	1174	1	82.3	180	1	82.3
	k=200	2400	1	82.3	359	1	82.3
87	k=1	250	1	68.79	111	1	71.46
	k=5	380	1	70.77	189	1	76.67
	k=10	617	1	70.77	138	1	70.1
	k=20	1072	1	72.52	274	1	75.49
	k=50	2504	1	76.27	771	1	78.3
	k=100	4903	1	76.27	1531	1	78.3
	k=200	9892	1	77.89	2544	1	77.07
88	k=1	109	1	63.76	53	1	77.67
	k=5	244	1	67.29	98	1	77.67
	k=10	375	1	69.53	145	1	76.85
	k=20	690	1	73.38	310	1	73.38
	k=50	1581	1	75.66	752	1	76.45
	k=100	3019	1	77.26	1447	1	76.45
	k=200	6146	1	77.26	2800	1	76.45
89	k=1	32	1	67.75	22	1	75.37
	k=5	210	1	75.72	24	1	66.46
	k=10	376	1	75.72	41	1	66.46
	k=20	601	1	75.72	76	1	75.37
	k=50	1500	1	75.72	176	1	75.37
	k=100	3224	1	75.72	321	1	75.37
	k=200	5858	1	76.95	619	1	76.59
90	k=1	12	1	80.24	10	1	80.24
	k=5	41	1	80.24	18	1	80.24
	k=10	76	1	80.24	20	1	80.24
	k=20	148	1	80.24	23	1	80.24
	k=50	372	1	80.24	20	1	80.24
	k=100	738	1	80.24	20	1	80.24
	k=200	1517	1	80.24	20	1	80.24

Table A.19: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
91	k=1	27	2	61.65	23	2	62.87
	k=5	167	2	60.72	51	2	67.36
	k=10	285	2	65.81	99	2	69.17
	k=20	596	2	67.21	189	2	70.48
	k=50	1239	2	64.03	422	2	73.88
	k=100	2370	2	71.91	631	2	64.22
	k=200	4631	2	71.91	1244	2	63.94
92	k=1	41	2	68.98	17	2	64.7
	k=5	102	2	68.5	46	2	60.91
	k=10	180	2	68.5	87	2	63.43
	k=20	343	2	68.5	200	2	64.45
	k=50	1081	2	68.5	481	2	64.45
	k=100	2487	2	75.09	848	2	62.44
	k=200	4615	2	74.62	1428	2	61.01
93	k=1	11	2	53.18	6	2	49.39
	k=5	29	2	60.58	13	2	63.43
	k=10	62	2	60.75	23	2	63.43
	k=20	114	2	60.97	44	2	62.4
	k=50	251	2	53.59	102	2	62.4
	k=100	480	2	53.59	207	1	70.47
	k=200	1141	2	53.59	408	1	70.85
94	k=1	7	2	60.41	6	2	61.33
	k=5	18	2	60.41	10	2	62.49
	k=10	32	2	60.41	17	2	64.19
	k=20	65	2	72.49	30	2	64.19
	k=50	166	2	72.49	72	2	64.19
	k=100	332	2	72.49	146	2	74.4
	k=200	675	2	75.2	247	2	71.17
95	k=1	2596	1	71.15	217	1	80.4
	k=5	5493	1	71.15	345	1	81.11
	k=10	10208	1	71.15	859	1	78.75
	k=20	19066	1	71.15	1031	1	76.3
	k=50	53469	1	71.15	2902	1	82.38
	k=100	101264	1	71.15	5778	1	82.38
	k=200	198827	1	71.15	11145	1	82.38

Table A.20: Use-case tests

Instance		Single Placement			Order by Hash		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
96	k=1	447	1	60.54	29	1	64.78
	k=5	1417	1	60.54	108	1	67.96
	k=10	2778	1	60.54	191	1	68.33
	k=20	5405	1	66.23	685	1	71.57
	k=50	13724	1	66.23	1133	1	68.28
	k=100	26596	1	67	2122	1	68.28
	k=200	55258	1	67	3725	1	69.27
97	k=1	625	1	56.36	40	1	70.47
	k=5	2977	1	61.41	228	1	69.38
	k=10	5436	1	61.41	459	1	68.66
	k=20	10287	1	62.02	458	1	66.34
	k=50	26357	1	63.42	988	1	58.96
	k=100	53737	1	65.76	1873	2	36.16
	k=200	123811	1	72.94	9152	1	74.29
98	k=1	78	1	79.05	23	1	64.42
	k=5	172	1	79.05	51	1	64.42
	k=10	288	1	81.04	94	1	77.16
	k=20	530	1	81.04	137	1	67.57
	k=50	1193	1	81.04	319	1	73.63
	k=100	2307	1	81.04	621	1	73.75
	k=200	4741	1	81.04	1182	1	77.16
99	k=1	8	1	71.74	9	1	73.14
	k=5	26	1	71.74	17	1	71.74
	k=10	55	1	73.14	27	1	71.74
	k=20	97	1	73.14	53	1	71.74
	k=50	234	1	73.14	93	1	71.74
	k=100	464	1	74.59	186	1	71.74
	k=200	921	1	74.59	338	1	74.59
100	k=1	9	1	54.55	6	1	56.35
	k=5	30	1	56.35	12	1	67.25
	k=10	55	1	56.35	19	1	70.44
	k=20	101	1	67.45	35	1	70.44
	k=50	243	1	67.45	71	1	70.44
	k=100	495	1	70.22	134	1	70.44
	k=200	1017	1	70.44	250	1	70.44

B | Appendix B

It may be necessary to include another appendix to better organize the presentation of supplementary material.

List of Figures

List of Tables

List of Symbols

Variable	**Description**	**SI unit**
$\boldsymbol{u}$	solid displacement	m
$\boldsymbol{u}_f$	fluid displacement	m

Acknowledgements

Here you might want to acknowledge someone.