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Abstract

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Abstract in lingua italiana

Qui va l'Abstract in lingua italiana della tesi seguito dalla lista di parole chiave.

Parole chiave: qui, vanno, le parole chiave, della tesi

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3.1. 3D Bin Packing Problem

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Conceptual model

A conceptual model of the problem we are trying to solve would be:

minimize	unused volume in used bins
subject to	all items assigned to one and only one bin
	all items within the bin dimensions
	no overlaps between items in the same bin
	all items with support

We can now provide the formal definition of the 3DBPP by formulating a mixed integer linear programming problem model.

Formal model

We'll now introduce a MILP model for the standard 3DBPP problem definition and then we'll expand it to address the stability constraint afterwards.

We start by defining the known sets and parameters of the problem.

Sets

$$I = \{1, \dots, n\}: \quad \text{set of items}$$
$$B = \{1, \dots, m\}: \quad \text{set of bins}$$

Parameters

$$\begin{aligned} W \times D \times H &\quad \text{width} \times \text{depth} \times \text{height of a bin} \\ V &\quad \text{bin volume} \\ w_i \times d_i \times h_i &\quad \text{width} \times \text{depth} \times \text{height of item } i \qquad \forall i \in I \end{aligned} \tag{3.1}$$

Variables We can now introduce the following sets of integer variables

$$(x_i, y_i, z_i) \quad \text{bottom front left corner of an item} \qquad \forall i \in I \tag{3.2}$$

$$v_b \quad \begin{cases} 1, \text{if bin } b \text{ is used} \\ 0, \text{otherwise} \end{cases} \qquad \forall i \in I, \forall b \in B$$

$$u_{ib} \quad \begin{cases} 1, \text{if item } i \text{ is placed in bin } b \\ 0, \text{otherwise} \end{cases} \qquad \forall i \in I, \forall b \in B$$

$$x_{ij}^p \quad \begin{cases} 1, \text{if } x_i + w_i \leq x_j \\ 0, \text{otherwise} \end{cases} \qquad \forall i, j \in I$$

$$y_{ij}^p \quad \begin{cases} 1, \text{if } y_i + d_i \leq y_j \\ 0, \text{otherwise} \end{cases} \qquad \forall i, j \in I$$

$$z_{ij}^p \quad \begin{cases} 1, \text{if } z_i + h_i \leq z_j \\ 0, \text{otherwise} \end{cases} \qquad \forall i, j \in I$$

$$z_b^{\max} \quad \text{maximum height of bin } b \qquad \forall b \in B$$

Given a coordinate system, each item i can be represented univocally in 3D space by eqs. (3.1) and (3.2) as seen in figure 3.1

Othogonal rotations

Let us extend the definition of the bin packing problem without rotations with a new formulation which allows 90 degrees rotations of each item

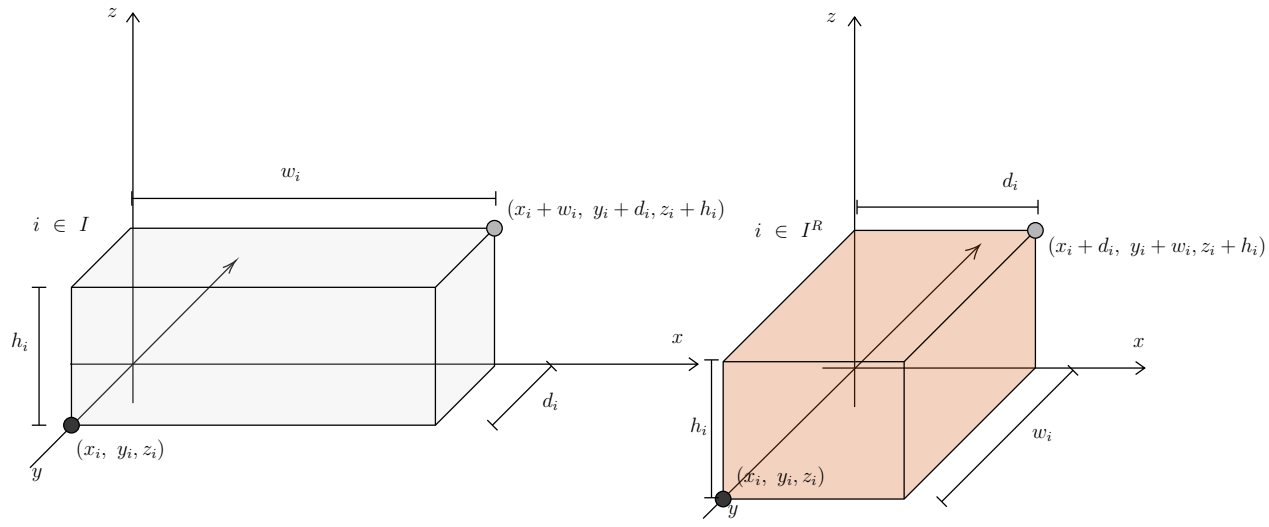

Figure 3.1: Coordinate system representation for a generic item i and it's rotated clone $i \in I^R$

Static stability constraints

We now extend the model introduced in section 3.3 to introduce constraints addressing static stability.

Additional Parameters

α_s : support area threshold

β_s : support height tolerance

4 | Solution algorithms

In this chapter, we describe a solution to the 3D bin packing problem with static stability. A solution candidate to the problem can be found by conducting a search over the tree of possible packings or states. In section 4.1 we describe what a state or packing is and its representation. Since an exhaustive search isn't feasible, a heuristic search is conducted by combining a beam search algorithm described in section 4.2 and a constructive heuristic described in section 4.3. The proposed algorithm takes in input an initial feasible state (as defined in section 4.1.2) usually represented by the empty state (4.3) and outputs the best scoring state based on an ordering function defined in section 4.2.1.

4.1. State

States or packings are partial solutions to the 3DBPP. Given the formal definition of the problem (3.3) a few new definitions are introduced to facilitate the algorithm's definition.

Definition 4.1 (Unpacked item). *An item $i \in I$ is unpacked* ***iff***

$$\sum_{b \in B} u_{ib} = 0$$

It is also assumed that variables identifying an item's position, and rotation are independent between states (changes to their values in state s won't affect state s').

A state s can then be defined as follows

- U: the set of unpacked items
- B: the set of bins
- $Q = (q_1, q_2, \ldots, q_b)$: the set of supporting structures for each bin $b \in B$
- p: the insertion pending on this state (described by def. 4.4)

Observation 4.1. *Given two states s and s' we can have that $|s.B| \neq |s'.B|$ since the number of bins is also a variable in the proposed heuristic*

We can also trivially define a function that determines if a state is a final state

Definition 4.2. *A state s is final if there are no more items to pack*

$$IsFinal(s) = \begin{cases} 1, & s.U = \emptyset \\ 0, & otherwise \end{cases} \tag{4.1}$$

Each bin b has additional data that is contained in a structure $q_b \in s.Q$ used to facilitate the execution of the algorithm.

Let us introduce the concept of packed items inside a bin:

Definition 4.3 (Packed item). *Given a state s and a bin $b \in s.B$, we say that item*

$$i \in I \text{ is packed in } b \textbf{ iff } \quad u_{ib} = 1$$

Given a bin $b \in s.B$ we can then define structure q_b as follows

- J: the set of items that are packed inside b
- Z: the set of planes inside b (section 4.3)
- T: the AABB Tree (section 4.1.1) representing the items inside b

Notice that two separate sets containing the items packed in b are present inside q_b but adding and accessing items in $q_b.J$ has a time complexity of $O(1)$ given an underlying implementation as HashSet while maintaining $q_b.T$ usually has a time complexity of $O(log(|q_b.J|))$.

The reason to include an AABB Tree inside this structure is further explained in sections 4.1.2 and 4.3.1

4.1.1. AABB Tree

In order to determine the feasibility of a given state, a way of checking for overlaps with items already placed is needed. Since our formulation of the problem only allows for 90 deg rotations over the z-axis. Every item in a solution, by the problem formulation (3.1), is contained inside a bounding box and this box is axis-aligned. An adequate structure to compute overlaps is then an Axis-Aligned Bounding Box Tree (AABB Tree) [1].

AABB Trees are bounding volume hierarchies typically used for fast collision detection and they usually offer a few operations:

- $AABBInsert(i)$: which allows inserting an axis-aligned box i in the tree
- $AABBOverlaps(i)$: which allows determining if an axis-aligned box i overlaps an element in the tree
- $AABBClosest(i, d)$: which, given an axis-aligned box i and a direction $d \in \{XP, XN, YP, YN, ZP, ZN\}$ along an axis, returns the closest element inside the tree following that direction starting from the box i

If the tree is properly balanced each operation on average has a time complexity of $O(log(n))$ where n is the number of elements in the tree. Maintaining an AABB Tree in the state allows us to do checks for feasibility during the construction of a solution (as detailed in 4.3.1) and feasibility checks on the final states to allow for error detection.

4.1.2. Feasibility

A state s is said to be feasible if the currently packed items for every bin $b \in s.B$ respects the constraints defined in the problem formulation (3.3)

Since the proposed heuristic is constructive it is more convenient to define the concept of feasibility relative to a change in the state.

Insertions Given a state s and $b \in s.B$, an insertion of items is a set of items that are placed in b and have their z_i within a tolerance of a certain z.

Definition 4.4 (Insertion). *Given a state s and a tolerance β_s we define an insertion or placement p a tuple (b, I) where b is a bin and I is a set of items that are going to be packed in b such that, $I \subseteq s.U \wedge \exists z(z \in \mathbb{Z} \wedge \forall i(i \in I \wedge |z_i - z| \leq \beta_s))$*

Observation 4.2. *Given s and $p = (b, \emptyset)$ where $b \notin s.B$, p is an insertion which will open bin b in s.*

Definition 4.5 (Next). *Let p be an insertion over a state s we can then define $s' = Next(s, p)$ as the "copy" of state s with $s'.p = p$. And p is then pending on s'.*

In this way, we can evaluate the changes to the score of a state based on its pending insertion without having to update all the structures for every evaluated state. This property will become apparent in section 4.2.

We can then define an algorithm that applies insertions to a given state s with pending insertions with the help of a function $OpenBin(b)$ which initializes a new structure q_b with every element at its empty value. The proposed algorithm is shown in 1.

Algorithm 1: Commit

input : s

output: s'

$(b, I) \leftarrow s.p$

$s' \leftarrow Clone(s)$ `//Memory clone of` s

if $b \in s'.B$ **then**

  $q_b \leftarrow (q_i \in s'.Q : i = b)$

  $q_b.J \leftarrow q_b.J \cup I$

  $s'.U \leftarrow s'.U \setminus I$

end

else Open a new bin

  $s'.B \leftarrow s'.B \cup b$

  $s'.Q \leftarrow s'.Q \cup OpenBin(b)$

end

$s'.p \leftarrow$ none

return s'

Insertion feasibility Describe insertion feasibility givine the sets defined

Algorithm 2: Is Insertion Feasible

input : $b, I, z, I_{support}, I_{upper}$

output: $isFeasible$

return *true*

State feasibility Describe how to compute the sets efficiently to use the insertion feasibility logic

Algorithm 3: Is State Feasible

input : s

output: $isFeasible$

return *true*

Proposition 4.1. *A state s' derived by committing a feasible insertion p to a feasible state s is feasible.*

Observation 4.3. *We can always define the empty state s_e where*

$$\begin{cases} s_e.U = I \\ s_e.Q = \emptyset \\ s_e.B = \emptyset \end{cases}$$

and it is always feasible

4.2. Beam Search

Beam Search (BS) is a heuristic tree search algorithm designed for systems with limited memory where expanding every possible node is unfeasible. The idea behind BS is to conduct an iterative truncated breadth-first search where, at each iteration, only a limited number of k nodes is expanded. After the expansion, every new node needs to be evaluated and sorted in order to prune the number of nodes down to the k best ones. The algorithm keeps exploring until no further node can be expanded.

To perform BS one must define the node structure, an expansion function to generate new nodes from existing ones, a ranking between nodes, and a function to determine if a node is final.

A node in the tree can be represented as the state in section 4.1 and eq. (4.1) can be used to determine if a state is final. We also know that a new state s' derived by s by applying a feasible insertion p can be computed as in definition 4.5. This state expansion procedure, with the exception of empty insertions, will generate new states in our tree which will add a positive number of bins or packed items to the solution so, eventually, it will generate a final state.

If the starting state for the search is feasible every new state generated will be feasible and if a final state is found it will be feasible (proposition 4.1). We also note that starting from state s the time complexity to compute feasible insertions can be lower than the complexity required to update the structures that will be used for further expansions (AABB Tree insertion and balancing, memory cloning, etc.) so we modified the standard BS algorithm to separate the expansion phase from the commit phase.

Given S^0 the set of initial states and k the number of best states to expand at each iteration, the described procedure is represented by algorithm 4. As observed in observation 4.3 it's possible to start the search from $S^0 = \{s_e\}$.

Algorithm 4: Beam search

input : S^0, k
output: s_{best}
$S^t \leftarrow S^0$
$S_{final} \leftarrow \emptyset$
repeat
  $S^{t+1} \leftarrow Expand(S^t)$ (algo. 5)
  $S_{final} \leftarrow S_{final} \cup \{s \in S^{t+1} : IsFinal(s)\}$ (def. 4.2)
  $S^{t+1} \leftarrow S^{t+1} \setminus S_{final}$
  $S^{t+1} \leftarrow Sort(S^{t+1})$ (sec. 4.2.1)
  $S^t \leftarrow \emptyset$
  $i \leftarrow 0$
  forall $s \in S^{t+1}$ **do**
    $S^t \leftarrow S^t \cup Commit(s)$ (algo. 1)
    $i \leftarrow i + 1$
    if $i > k$ **then**
      break
    end
  end
until $S^t \neq \emptyset$
$S_{final} \leftarrow Sort(S_{final})$
return *first element of* S_{final}

State Expansion An expansion of a state s can be seen as a new set of states S_{new} derived by a set of feasible insertions. In order to determine these insertions, an underlying heuristic is used (described in section 4.3).

The main idea in this phase of the algorithm is to find feasible insertions in all the bins for items that still need to be packed and that are of the same height. With this approach, the solutions given by the algorithm will start by trying to fill lower layers with items of the same height if possible and they'll become more heterogeneous in upper layers where the classes of height will start to mix up. The underlying heuristic will also use a scoring mechanism to select the best insertions for a given class of heights in order to avoid having too many states to sort.

Given a set of items I and a tolerance β_s we can introduce an algorithm to group them by height and produce a set G of tuples (h, I') where h is the height of the group and I' is the set of items grouped as in algo. 6.

Once items are grouped by height the best insertion for each class of items can be computed for each open bin. If no insertion is possible in any bin, then the only viable insertion is the bin opening insertion (observation 4.2). The described procedure is detailed in algo. 5.

Algorithm 5: Expand

input : S
output: S_{new}
forall $s \in S$ **do**
  $S_{new} \leftarrow \emptyset$
  $G \leftarrow GroupByHeight(s.U)$ (algo. 6)
  $placed \leftarrow false$
  forall $(h, I) \in G$ **do**
    forall $q_b \in s.Q$ **do**
      $P \leftarrow SPBestInsertion(q_b, I)$ (algo. 7)
      if $P \neq \emptyset$ **then**
        $placed \leftarrow true$
        forall $p \in P$ **do**
          $S_{new} \leftarrow S_{new} \cup Next(s, p)$ (def. 4.5)
        end
      end
    end
  end
  if $placed = false$ **then**
    `Open a new bin with index` $|s.B|$ `(oss. 4.2)`
    $S_{new} \leftarrow S_{new} \cup Next(s, (|s.B|, \emptyset))$
  end
end
return S_{new}

Algorithm 6: Group By Height

input : I, β_s
output: G
$G \leftarrow \emptyset$
forall $i \in I$ **do**
  $generate \leftarrow$ true
  forall $(h, I') \in G$ **do**
    if $|h_i - h| \leq \beta_s$ **then**
      $generate \leftarrow$ false
      $I' \leftarrow I' \cup i$
      break
    end
  end
  if $generate = true$ **then**
    $G \leftarrow G \cup (h_i, \{i\})$
  end
end
return G

4.2.1. Scoring States

In order to sort states, a scoring function needs to be defined over them. Since the scoring of the states is what will influence the final solution the most, parameters that are directly related to minimizing the objective function are selected.

In the proposed solution to handle multiple objective functions, lexicographic ordering is used.

Definition 4.6. *Let $f_1(s), f_2(s), f_i(s), \ldots, f_n(s)$ be objective functions ordered by precedence based on index i, then*

$$s < s' \textit{ iff } \exists j \in \mathbb{Z} : \begin{cases} f_j(s) < f_j(s') \\ f_k(s) = f_k(s'), \quad \forall k \in \mathbb{Z} : 0 \leq k < j \end{cases}$$

Scoring metrics for each state s that we want to evaluate can then be computed in the *Next* algorithm by considering the contents of the pending insertions and updating each parameter differentially.

The defined ordering utilized is the following:

- $f_1(s) = -|s.B|$: we prefer states that opened fewer bins.
- $f_2(s) = \text{avgvol}(s)$: we prefer states that have packed more average volume between bins.
- $f_3(s) = \text{avgcageratio}(s)$: we prefer states that have better average cage ratio (def. **??**) between bins.

4.3. Support Planes

Support Planes (SP) is a heuristic detailed in the following section based on an underlying 2DBPP heuristic which is used to evaluate feasible insertions starting from a given state. Since the insertions must be feasible SP mantains an internal structure to facilitate the check for feasibility. The idea at the base of SP is to build a solution to the 3DBPP by filling 2D planes called *support planes*.

Each support plane can be characterized by the triple $S_z = (z, I_{support}, I_{upper})$ where

- z: the height of the plane
- $I_{support}$: the set of the items that can offer support to items placed on the plane
- I_{upper}: the set of items that will be obstacles to potential new items placed on the plane

Let *coords* be the set of possible coordinate changes which allow for the problem to evaluate placements starting from different corners of the bin.

Given a function $IsFeasible(i, bin, I_{support}, I_{upper}, aabb)$ which evaluates if a packing of item i in bin bin is feasible, and the function $ComparePacking(p, p')$ which defines a ranking over placements in the same plane, the SP algorithm can be written as algorithm 7.

Algorithm 7: SP Best Insertion

input : s_b, I
output: *placement*
$placement \leftarrow \emptyset$
forall $S_z \in planes$ **do**
  $I_p \leftarrow I \setminus \{i \in I : z + i.h > H_b\}$
  forall $change \in coords$ **do**
    $I'_{upper} \leftarrow CoordinateChange(change, I_{upper})$
    $I'_p \leftarrow CoordinateChange(change, I_p)$
    $P' \leftarrow SPPackPlane(W_b, D_b, I'_{upper}, I'_p)$ (Algorithm 8)
    $P \leftarrow CoordinateChange(change, P')$
    $P \leftarrow \{i \in P : IsFeasible(i, bin, I_{support}, I_{upper}, aabb)\}$
    if $ComparePacking(placement, P)$ **then**
      $placement \leftarrow P$
    end
  end
  if $placement \neq \emptyset$ **then**
    return *placement*
  end
end
return *placement*

To evaluate a packing on a plane a heuristic to solve the 2DBPP is used with the introduction of fixed placements which represent items on other planes that will be obstacles in the current one.

Given the dimensions of the 2D bin (W_b, D_b), the set of obstacles I_o and the set of items to pack I_p a new placement can be computed following algorithm 8

Algorithm 8: SP Pack Plane

input : W_b, D_b, I_o, I_p

output: P

$P \leftarrow \emptyset$

$2dPacking \leftarrow \emptyset$

foreach $i \in I_o$ **do**

    `//Initialize the 2D bin packing instance with each obstable already placed`

    $2DPlaceRect(2dPacking, i)$

end

repeat

    `//Pack untill full`

    $p \leftarrow 2DPackRect(2dPacking, W_b, D_b, i)$

    $P \leftarrow P \cup \{p\}$

until $p \neq \emptyset$

return P

Commit Extension We now describe an extension to *Commit* (algo. 1) to update the structures needed by SP.

When a plane is filled, new insertions become less likely to be feasible. To avoid evaluating planes where no insertion is possible a mechanism to prune dead planes can be introduced.

Since best insertions for a bin are always evaluated by considering lower planes first, if all the insertions in *Expand* (algo. 5) happened over a z_{min} then we can safely remove the opened planes with $z < z_{min}$ for that bin. Let us introduce a z_{min} variable carried over in q_b for each bin, which is updated during the *Expand* phase with the minimum z of all the insertions on bin b. Once the best states are computed and *Commit* is called we can then use its value to prune planes in each q_b. Other operations are also necessary in the *Commit* algorithm to allow SP to update its data structures accordingly to the insertion.

Given a state s and an insertion p where each packed item $i \in p.I$ in bin b has z_i within tolerance of z and the minimum height for the considered bin $q_b.z_{min}$. The algorithm which updates the structures for a given bin b is represented by algorithm 9. This new algorithm can be used as the last step of the *Commit* algorithm for each $b \in s'.B$.

Algorithm 9: SP Apply and Filter

input : $s, p, z, z_{min}, \beta_s$

output: s

$q_b \leftarrow (q_i \in s.Q : i = p.b)$

`//Filter bad planes`

$q_b.Z \leftarrow q_b.Z \setminus \{(z', I_{support}, I_{upper}) \in q_b.Z : z' < z_{min}\}$

`//Apply insertion`

forall $i \in p.I$ **do**

  $q_b.T \leftarrow InsertAABB(i, q_b.T)$ `//If balanced` $O(log(n))$

  $generate \leftarrow true$

  forall $(z', I_{support}, I_{upper}) \in q_b.Z$ **do**

    `//Based on the distance from the top of the item`

    $dz \leftarrow z' - (z_i + h_i)$

    if $0 \leq dz \leq \beta_s$ **then**

      $generate \leftarrow false$

      $I_{support} \leftarrow I_{support} \cup i$

    end

    else if $dz < 0$ **then**

      $I_{upper} \leftarrow I_{upper} \cup i$

    end

  end

  if $generate$ **then**

    $q_b.Z \leftarrow q_b.Z \cup (z_i + h_i, \{i\}, \emptyset)$

  end

end

return s

4.3.1. Scoring Insertions

5 | Computational experiments

Table 5.1: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
1	k=1	320	1	73.29	317	1	66.1
	k=5	80	1	74.06	193	1	73.29
	k=10	110	1	74.06	183	1	73.29
	k=20	101	1	74.06	229	1	73.29
	k=50	100	1	74.06	417	1	73.29
	k=100	83	1	74.06	617	1	73.29
	k=200	65	1	74.06	1165	1	73.29
2	k=1	102	1	66.42	88	1	69.56
	k=5	154	1	79.68	220	1	69.56
	k=10	222	1	79.68	373	1	69.56
	k=20	349	1	80.26	691	1	69.56
	k=50	844	1	80.26	1709	1	69.56
	k=100	1407	1	80.26	3285	1	69.56
	k=200	3193	1	80.26	6738	1	69.56
3	k=1	44	1	75.61	89	1	82.3
	k=5	64	1	82.3	271	1	82.3
	k=10	92	1	82.3	391	1	82.3
	k=20	146	1	82.3	688	1	82.3
	k=50	348	1	82.3	1629	1	82.3
	k=100	482	1	82.3	3109	1	82.3
	k=200	1017	1	82.3	6214	1	82.3
4	k=1	253	1	71.46	536	1	68.79
	k=5	519	1	76.67	1148	1	70.77
	k=10	467	1	70.1	1995	1	70.77
	k=20	662	1	70.1	3172	1	72.52
	k=50	1970	1	77.07	8302	1	75.49
	k=100	3490	1	77.07	14977	1	76.27
	k=200	6444	1	69.44	30086	1	77.89
5	k=1	166	1	77.67	253	1	63.76
	k=5	224	1	79.79	619	1	67.29
	k=10	381	1	79.79	1118	1	69.53
	k=20	574	1	76.85	2056	1	73.38
	k=50	2034	1	76.45	4601	1	75.66
	k=100	3587	1	76.45	9288	1	77.26
	k=200	7394	1	76.45	18662	1	77.26
6	k=1	59	1	75.37	82	1	67.75
	k=5	69	1	66.46	429	1	75.72
	k=10	107	1	66.46	980	1	75.72
	k=20	176	1	75.37	1824	1	75.72
	k=50	401	1	75.37	4033	1	75.72
	k=100	759	1	75.37	9494	1	75.72
	k=200	1479	1	76.59	17487	1	76.95

Table 5.2: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
7	k=1	25	1	80.24	29	1	80.24
	k=5	37	1	80.24	108	1	80.24
	k=10	45	1	80.24	193	1	80.24
	k=20	47	1	80.24	389	1	80.24
	k=50	45	1	80.24	834	1	80.24
	k=100	52	1	80.24	1659	1	80.24
	k=200	45	1	80.24	3263	1	80.24
8	k=1	41	2	65.22	74	2	61.66
	k=5	146	2	67.89	278	2	60.13
	k=10	293	2	64.14	621	2	60.13
	k=20	481	2	64.14	793	2	60.13
	k=50	801	2	60.98	2139	2	63.68
	k=100	1335	2	59.87	4022	2	56.46
	k=200	2462	2	60.18	8479	2	56.46
9	k=1	32	2	64.21	57	2	61.02
	k=5	84	2	65.7	183	2	60.55
	k=10	154	2	65.7	361	2	60.55
	k=20	283	2	65.7	705	2	60.55
	k=50	763	2	64.5	2064	2	61.5
	k=100	1489	2	64.5	3281	2	60.21
	k=200	2499	2	63.38	9709	2	57.02
10	k=1	56	2	62.87	82	2	61.65
	k=5	136	2	67.21	433	2	60.72
	k=10	229	2	74.79	895	2	60.72
	k=20	454	2	74.79	1733	2	60.72
	k=50	927	2	73.88	3173	2	64.03
	k=100	1461	2	64.48	5933	2	69.92
	k=200	2823	2	64.17	11555	2	70.28
11	k=1	57	2	71.38	27	2	61.85
	k=5	120	2	67.19	97	2	71.18
	k=10	80	2	67.02	154	2	72.41
	k=20	128	2	68.11	288	2	73.33
	k=50	274	2	68.11	621	2	71.19
	k=100	523	2	71.87	1285	2	71.45
	k=200	1086	2	72.86	2521	2	71.45
12	k=1	25	2	69.62	54	2	62.51
	k=5	44	2	66.29	147	2	72.65
	k=10	100	2	73	221	2	74.31
	k=20	179	2	73	397	2	73.87
	k=50	376	2	79.91	1036	2	72.72
	k=100	692	2	73.26	2137	2	72.9
	k=200	1532	2	76.49	4148	2	74.19

Table 5.3: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
13	k=1	21	2	69.62	32	2	70.03
	k=5	43	2	66.29	100	2	75.57
	k=10	101	2	72.97	184	2	74.51
	k=20	180	2	72.97	373	2	73.76
	k=50	380	2	79.04	828	2	73.76
	k=100	770	2	79.88	2026	2	75.09
	k=200	1560	2	79.88	4659	2	73.7
14	k=1	38	2	64.7	106	2	68.98
	k=5	149	2	69.75	257	2	60.37
	k=10	227	2	65.15	477	2	65.31
	k=20	462	2	64.45	1010	2	60.37
	k=50	1179	2	64.45	2273	2	68.75
	k=100	2024	2	61.66	5963	2	66.68
	k=200	3939	2	59.11	12930	2	75.09
15	k=1	54	2	49.39	30	2	53.18
	k=5	117	2	63.43	74	2	60.23
	k=10	75	2	63.43	153	2	60.58
	k=20	98	2	62.4	265	2	60.75
	k=50	226	2	62.4	635	2	60.75
	k=100	528	1	70.47	1164	2	60.97
	k=200	979	1	70.85	2656	2	55.89
16	k=1	10	2	55.38	10	1	58.79
	k=5	21	2	59.28	36	1	66.99
	k=10	38	1	58.94	68	1	66.99
	k=20	72	1	58.94	120	1	66.99
	k=50	154	1	58.94	278	1	66.99
	k=100	303	1	58.94	563	1	67.19
	k=200	587	1	58.94	1172	1	67.78
17	k=1	20	2	49.39	31	2	53.18
	k=5	39	2	63.43	80	2	60.23
	k=10	66	2	63.43	173	2	60.58
	k=20	106	2	62.4	258	2	60.75
	k=50	221	2	62.4	691	2	60.75
	k=100	471	1	70.47	1130	2	60.97
	k=200	948	1	70.85	2596	2	55.89
18	k=1	13	2	61.33	20	2	60.41
	k=5	23	2	62.49	44	2	60.41
	k=10	40	2	64.19	79	2	60.41
	k=20	73	2	64.19	155	2	72.49
	k=50	155	2	64.19	404	2	72.49
	k=100	310	2	74.4	813	2	72.49
	k=200	510	2	71.17	1575	2	75.2

Table 5.4: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
19	k=1	11	1	78.52	81	2	49.14
	k=5	27	1	79.08	163	1	69.63
	k=10	43	1	73.81	343	1	69.63
	k=20	76	1	80.23	715	1	70.97
	k=50	192	1	80.23	1813	1	76.35
	k=100	362	1	80.23	3687	1	76.35
	k=200	717	1	80.23	7508	1	77.97
20	k=1	28	1	86	250	1	67.37
	k=5	96	1	83.63	1073	1	73.49
	k=10	139	1	89.17	2126	1	73.49
	k=20	235	1	90.5	4178	1	73.49
	k=50	546	1	90.5	11453	1	73.49
	k=100	1065	1	90.5	18290	1	78.24
	k=200	2056	1	90.5	34020	1	78.74
21	k=1	150	1	70.43	506	1	66.77
	k=5	183	1	69.51	1700	1	66.77
	k=10	773	1	76.27	2993	1	71.72
	k=20	1182	1	76.98	5696	1	71.72
	k=50	2672	1	73.78	13024	1	71.72
	k=100	4535	1	76.19	25470	1	71.72
	k=200	7117	1	72.63	51201	1	75.58
22	k=1	179	1	78.65	560	1	79.47
	k=5	403	1	70.36	2329	1	79.47
	k=10	619	1	76.27	4774	1	80.62
	k=20	973	1	77.37	9270	1	80.62
	k=50	2319	1	82.06	21577	1	80.62
	k=100	3652	1	85.66	42499	1	80.62
	k=200	8674	1	80.52	84788	1	80.62
23	k=1	310	1	80.6	264	1	68.8
	k=5	333	1	76.13	1426	1	71.42
	k=10	609	1	76.13	2473	1	71.42
	k=20	1140	1	82.92	5071	1	71.42
	k=50	2746	1	78.77	13208	1	72.92
	k=100	4120	1	77.15	25710	1	72.92
	k=200	7811	1	84.29	53206	1	76.33
24	k=1	136	1	73.37	236	1	71.2
	k=5	150	1	71.09	747	1	77.41
	k=10	370	1	72.96	1640	1	77.41
	k=20	673	1	72.56	3109	1	77.41
	k=50	1479	1	73.55	7597	1	78.54
	k=100	2701	1	85.37	15296	1	78.54
	k=200	4246	1	79.92	32201	1	78.54

Table 5.5: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
25	k=1	87	1	69.39	263	1	60.55
	k=5	155	1	80.2	601	1	75.18
	k=10	315	1	78	1139	1	75.18
	k=20	521	1	78.65	2336	1	75.18
	k=50	990	1	76.84	5600	1	75.18
	k=100	1892	1	78.85	11709	1	77.61
	k=200	3891	1	77.54	23801	1	77.61
26	k=1	138	1	66.58	265	1	67.04
	k=5	315	1	77.82	983	1	77.57
	k=10	417	1	59.33	2101	1	77.57
	k=20	1062	1	77.44	3995	1	77.57
	k=50	1726	1	72.79	9543	1	77.57
	k=100	2973	1	83.16	19433	1	77.57
	k=200	5981	1	82.8	39231	1	77.82
27	k=1	68	1	63	332	1	65.36
	k=5	205	1	75.2	954	1	65.36
	k=10	324	1	70.82	1923	1	66.36
	k=20	543	1	70.82	3611	1	66.36
	k=50	853	1	64.74	8819	1	66.92
	k=100	1738	1	71.21	17305	1	66.92
	k=200	3244	1	71.21	36103	1	73.44
28	k=1	29	1	69.82	232	1	69.54
	k=5	95	1	70.61	608	1	70.9
	k=10	165	1	70.61	1135	1	71.47
	k=20	284	1	73.33	2224	1	71.47
	k=50	633	1	74.14	5250	1	71.47
	k=100	1246	1	76.54	10356	1	71.47
	k=200	2289	1	74.64	22032	1	74.71
29	k=1	164	1	77.19	706	2	48.2
	k=5	441	1	84.69	2445	1	78.58
	k=10	885	1	84.69	4503	1	78.58
	k=20	1725	1	84.69	7000	1	78.7
	k=50	3360	1	87.49	14575	1	83.22
	k=100	7270	1	86.65	29428	1	83.22
	k=200	8574	1	80.85	62878	1	83.22
30	k=1	158	1	65.48	414	1	73.36
	k=5	239	1	73.31	779	1	73.36
	k=10	377	1	72.78	1300	1	73.36
	k=20	603	1	74.86	2554	1	73.36
	k=50	1161	1	74.51	5771	1	73.46
	k=100	2376	1	78.17	12841	1	73.46
	k=200	3914	1	74.51	31508	1	73.46

Table 5.6: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
31	k=1	159	1	79.2	911	1	53.7
	k=5	388	1	78.44	3291	1	59.27
	k=10	453	1	79.14	6425	1	59.27
	k=20	770	1	79.44	12605	1	59.27
	k=50	1935	1	82.18	30437	1	65.04
	k=100	3631	1	82.18	62640	1	70.81
	k=200	7544	1	82.18	125741	1	70.81
32	k=1	71	1	74.44	134	1	67.48
	k=5	186	1	76.04	518	1	73.22
	k=10	315	1	77.11	918	1	73.22
	k=20	355	1	66.61	1864	1	73.22
	k=50	603	1	68.08	4444	1	73.22
	k=100	1362	1	68.65	9625	1	73.22
	k=200	2502	1	68.74	20555	1	73.22
33	k=1	181	1	78.78	899	2	35.53
	k=5	489	1	80.63	2859	1	65.88
	k=10	1039	1	80.42	5214	1	74.23
	k=20	1635	1	79.85	10523	1	73.74
	k=50	3207	1	79.9	27746	1	73.74
	k=100	6681	1	80.16	55735	1	75.36
	k=200	8630	1	78.08	113254	1	75.36
34	k=1	115	1	68.32	708	1	62.38
	k=5	272	1	71.36	2227	1	62.38
	k=10	396	1	78.23	3636	1	64.08
	k=20	756	1	82.35	6618	1	66.35
	k=50	1792	1	79.45	15946	1	71.32
	k=100	2764	1	81.47	32405	1	71.32
	k=200	5568	1	82.12	66746	1	71.32
35	k=1	79	1	79.27	139	1	74.2
	k=5	110	1	78.78	403	1	74.2
	k=10	190	1	83.85	747	1	76.51
	k=20	333	1	83.85	1311	1	76.51
	k=50	820	1	83.85	3200	1	82.12
	k=100	1615	1	83.85	7230	1	82.12
	k=200	3047	1	83.85	15202	1	82.12
36	k=1	100	1	69.6	425	1	67.28
	k=5	284	1	74.25	1712	1	74.36
	k=10	758	1	77.93	3051	1	74.36
	k=20	1046	1	76.66	5860	1	75.21
	k=50	3350	1	80.34	15087	1	78.8
	k=100	3890	1	76.19	30953	1	78.8
	k=200	7178	1	76.07	61778	1	78.8

Table 5.7: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
37	k=1	149	1	72.18	90	1	75.91
	k=5	188	1	72.96	566	1	76.34
	k=10	251	1	72.96	903	1	76.34
	k=20	534	1	74.73	2020	1	76.34
	k=50	1099	1	75.15	5013	1	76.34
	k=100	2101	1	79.76	9900	1	76.34
	k=200	3794	1	79.76	20435	1	76.34
38	k=1	83	1	69.88	222	1	66.82
	k=5	213	1	73.04	658	1	66.82
	k=10	221	1	73.04	1307	1	73.04
	k=20	424	1	73.73	2715	1	73.04
	k=50	945	1	72.55	6653	1	73.04
	k=100	1693	1	69.88	13810	1	75.09
	k=200	3245	1	73.61	27200	1	75.09
39	k=1	110	2	38.76	862	2	54.85
	k=5	350	1	79.19	2901	2	49.23
	k=10	397	1	77.74	5583	2	49.23
	k=20	709	1	76.43	11148	2	44.91
	k=50	1347	1	77.53	21052	2	46.95
	k=100	2597	1	79.32	42939	2	46.95
	k=200	5231	1	79.32	106093	1	72.74
40	k=1	56	1	68.76	376	1	53.61
	k=5	167	1	74.4	1035	1	69.77
	k=10	280	1	73.12	1828	1	69.77
	k=20	368	1	64.05	3553	1	70.64
	k=50	1046	1	65.88	9077	1	70.64
	k=100	2003	1	74.6	17886	1	70.64
	k=200	2743	1	72.44	36764	1	75.46
41	k=1	451	1	80.4	5696	1	71.15
	k=5	898	1	81.11	16383	1	71.15
	k=10	2315	1	79.38	30746	1	71.15
	k=20	2907	1	76.3	60288	1	71.15
	k=50	8229	1	82.38	161724	1	71.15
	k=100	16738	1	82.38	325335	1	71.15
	k=200	32727	1	82.38	643344	1	71.15
42	k=1	109	1	64.78	1310	1	60.54
	k=5	290	1	67.96	4503	1	60.54
	k=10	446	1	67.96	8606	1	60.54
	k=20	1671	1	72.36	16856	1	66.23
	k=50	2745	1	68.83	45823	1	66.23
	k=100	5358	1	68.23	88785	1	67
	k=200	9904	1	68.83	184172	1	67

Table 5.8: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	***B***	***CR***	***TT (ms)***	***B***	***CR***
43	k=1	106	1	70.47	1913	1	56.36
	k=5	580	1	68.82	10128	1	61.01
	k=10	1280	1	68.66	19809	1	61.01
	k=20	1187	1	65.88	40672	1	62.02
	k=50	2415	1	58.17	96108	1	63.42
	k=100	4518	2	36.16	196877	1	65.11
	k=200	12388	1	70.47	419267	1	65.11
44	k=1	44	1	65.86	120	1	65.35
	k=5	106	1	63.52	302	1	67.29
	k=10	172	1	61.79	530	1	67.29
	k=20	316	1	66.63	923	1	67.29
	k=50	616	1	62.7	2153	1	68.78
	k=100	1272	1	66.11	4091	1	68.78
	k=200	2669	1	71.23	8424	1	68.78
45	k=1	110	1	64.42	182	1	79.05
	k=5	108	1	64.42	430	1	79.05
	k=10	244	1	77.16	870	1	79.05
	k=20	354	1	67.57	1441	1	79.05
	k=50	781	1	69.71	3601	1	81.04
	k=100	1550	1	73.63	7255	1	81.04
	k=200	2935	1	77.16	13512	1	81.04
46	k=1	17	1	73.14	22	1	71.74
	k=5	41	1	71.74	69	1	71.74
	k=10	69	1	71.74	133	1	71.74
	k=20	124	1	71.74	234	1	73.14
	k=50	219	1	71.74	568	1	73.14
	k=100	442	1	71.74	1127	1	73.14
	k=200	835	1	74.59	2351	1	74.59
47	k=1	13	1	56.35	25	1	54.55
	k=5	27	1	67.25	80	1	56.35
	k=10	47	1	70.44	139	1	56.35
	k=20	86	1	70.44	248	1	67.45
	k=50	166	1	70.44	605	1	67.45
	k=100	304	1	70.44	1170	1	67.45
	k=200	562	1	70.44	2409	1	70.44
48	k=1	21	1	73.14	21	1	71.74
	k=5	46	1	71.74	68	1	71.74
	k=10	69	1	71.74	119	1	71.74
	k=20	120	1	71.74	226	1	73.14
	k=50	228	1	71.74	553	1	73.14
	k=100	451	1	71.74	1070	1	73.14
	k=200	869	1	74.59	2240	1	74.59

Table 5.9: Use case instances with different configurations

Instance		Order by Hash			Single Placement		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
49	k=1	162	1	73.56	100	1	62.99
	k=5	309	1	77.7	362	1	75.39
	k=10	602	1	77.7	677	1	75.39
	k=20	1243	1	77.7	1304	1	75.89
	k=50	3151	1	74.65	3390	1	75.89
	k=100	6994	1	76.78	6913	1	75.89
	k=200	15553	1	76.91	14112	1	75.89
50	k=1	9	1	59.07	17	1	63.86
	k=5	34	1	65.64	46	1	63.86
	k=10	48	1	60.83	79	1	63.86
	k=20	89	1	60.83	147	1	64.13
	k=50	227	1	65.64	340	1	64.26
	k=100	426	1	65.64	666	1	65.64
	k=200	910	1	65.64	1320	1	65.64
51	k=1	92	1	68.28	2296	1	56.34
	k=5	302	1	62.41	7617	1	63.05
	k=10	527	1	67.92	14770	1	63.05
	k=20	1578	1	75.57	32100	1	66.18
	k=50	3877	1	73.11	84738	1	66.68
	k=100	4986	1	68.5	170373	1	66.68
	k=200	9969	1	68.5	345136	1	66.76

Table 5.10: Summary of the use case test results

Instance		Order by Hash			Single Placement		
		TT (ms)	*B*	*CR*	*TT (ms)*	*B*	*CR*
Average	k=1	98.65	1.24	69.23	441.06	1.27	64.39
	k=5	198.02	1.22	72.38	1,464.37	1.22	69.02
	k=10	352.59	1.20	72.84	2,767.82	1.22	69.72
	k=20	592.02	1.20	73.43	5,442.37	1.22	70.47
	k=50	1,346.53	1.20	73.49	13,571.67	1.22	71.62
	k=100	2,452.39	1.18	74.37	27,246.18	1.22	72.03
	k=200	4,625.08	1.16	74.78	55,963.67	1.20	73.15

6 | Conclusions and future developments

A final chapter containing the main conclusions of your research/study and possible future developments of your work have to be inserted in this chapter.

Bibliography

[1] G. v. d. Bergen. Efficient collision detection of complex deformable models using aabb trees. *Journal of graphics tools*, 2(4):1–13, 1997.

A | Appendix A

If you need to include an appendix to support the research in your thesis, you can place it at the end of the manuscript. An appendix contains supplementary material (figures, tables, data, codes, mathematical proofs, surveys, ...) which supplement the main results contained in the previous chapters.

B | Appendix B

It may be necessary to include another appendix to better organize the presentation of supplementary material.

List of Figures

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List of Symbols

Variable	Description	SI unit
$\boldsymbol{u}$	solid displacement	m
$\boldsymbol{u}_f$	fluid displacement	m

Acknowledgements

Here you might want to acknowledge someone.