Three-Dimensional Bin Packing with Vertical Support

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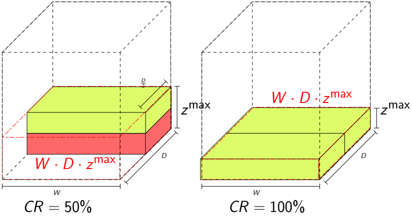
Case study

- Large warehouses
- Mixed-case palletization
- No control over items' shape (strongly heterogeneous)
- Pallets wrapped during loading procedure

Example of pit pallettization



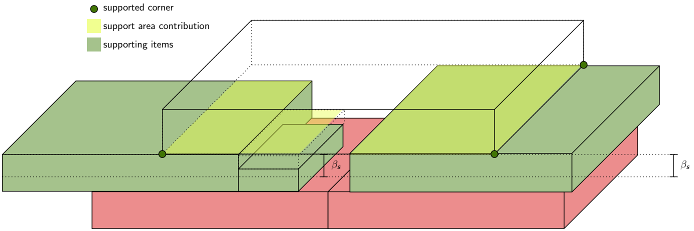
Cage ratio of two different bin configurations

Vertical Support

Definition

An item has vertical support if one of the following conditions holds:

- **Condition 1**: at least a percentage α_s of its base area is resting on other items
- **Condition 2**: at least 3 of its vertices are resting over other items and **Condition 1** holds with a lower percentage

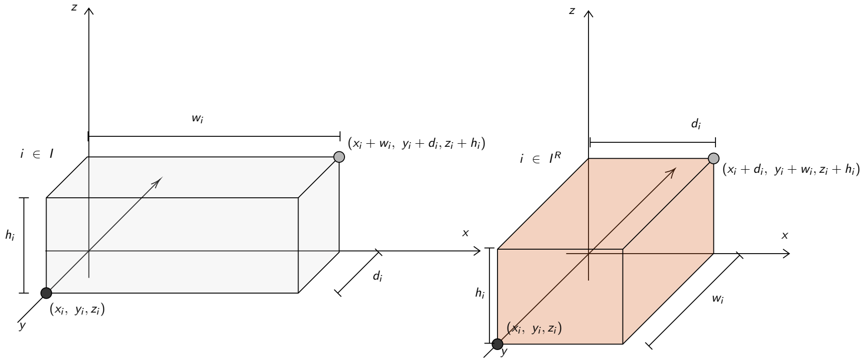


Literature Review

Problem: pack a set of cuboids into the minimum amount of bins, maximizing the cage ratio of used bins, without overlaps, and with every item having vertical support

- The problem is NP-Hard
- Exact methods may lead to optimal solutions only in fairly small instances
- Existing 3D-BPP heuristics don't consider vertical support
- Solutions for container loading and pallet loading problems are layer based

MILP Model - Coordinate System



Coordinate system representation for a generic item i and its rotated clone $i \in I^R$

Conceptual Formulation

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

MILP Model - Objective Function

minimize number of used bins

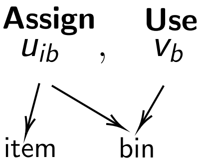
then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support



$$
\begin{array}{lll}
\min & \sum_{b \in B} (H v_b + z_b^{max}) & \\
\text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
& u_{ib} \le v_b & \forall i \in I, \forall b \in B \\
& v_b \ge v_c & \forall (b,c) \in B : b < c \\
& x_i + w_i \le W & \forall i \in I \\
& y_i + d_i \le D & \forall i \in I \\
& z_i + h_i \le H & \forall i \in I \\
& (x_i + w_i) - x_j \le W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \le W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \le D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \le D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \le H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \le H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \ge u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \ge (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}
$$

MILP Model - Geometric Constraints 1

minimize number of used bins

then, maximize average cage ratio of the used bins

subject to all items are assigned to one and only one bin

all items are inside the bin's bounds

no overlaps between items in the same bin

all items have vertical support

$$
\begin{array}{lll}
\min & \sum_{b \in B} (H v_b + z_b^{max}) & \\
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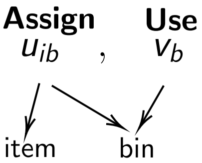
& z_i + h_i \leq H & \forall i \in I \\
& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}
$$



MILP Model - Geometric Constraints 2

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

$$
\begin{array}{lll}
\min & \sum_{b \in B} (Hv_b + z_b^{max}) & \\
\text{s.t.} & \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 & \forall (i,j) \in I^{OR} \\
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& (x_i + w_i) - x_j \leq W(1 - x_{ij}^p) & \forall i,j \in I \\
& x_j - (x_i + w_i) + 1 \leq W x_{ij}^p & \forall i,j \in I \\
& (y_i + d_i) - y_j \leq D(1 - y_{ij}^p) & \forall i,j \in I \\
& y_j - (y_i + d_i) + 1 \leq D y_{ij}^p & \forall i,j \in I \\
& (z_i + h_i) - z_j \leq H(1 - z_{ij}^p) & \forall i,j \in I \\
& z_j - (z_i + h_i) + 1 \leq H z_{ij}^p & \forall i,j \in I \\
& x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 & \forall i,j \in I, \forall b \in B \\
& z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) & \forall i \in I, \forall b \in B
\end{array}
$$

MILP Model - Geometric Constraints 3

minimize number of used bins
then, maximize average cage ratio of the used bins
subject to all items are assigned to one and only one bin
all items are inside the bin's bounds
no overlaps between items in the same bin
all items have vertical support

$$\min \quad \sum_{b \in B} (Hv_b + z_b^{max})$$

$$\text{s.t.} \quad \sum_{b \in B} u_{ib} + \sum_{b \in B} u_{jb} = 1 \qquad \forall (i,j) \in I^{OR}$$

$$u_{ib} \leq v_b \qquad \forall i \in I, \forall b \in B$$

$$v_b \geq v_c \qquad \forall (b,c) \in B : b < c$$

$$x_i + w_i \leq W \qquad \forall i \in I$$

$$y_i + d_i \leq D \qquad \forall i \in I$$

$$z_i + h_i \leq H \qquad \forall i \in I$$

$$(x_i + w_i) - x_j \leq W(1 - x_{ij}^p) \qquad \forall i,j \in I$$

$$x_j - (x_i + w_i) + 1 \leq W x_{ij}^p \qquad \forall i,j \in I$$

$$(y_i + d_i) - y_j \leq D(1 - y_{ij}^p) \qquad \forall i,j \in I$$

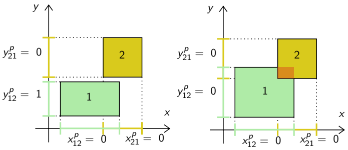
$$y_j - (y_i + d_i) + 1 \leq D y_{ij}^p \qquad \forall i,j \in I$$

$$(z_i + h_i) - z_j \leq H(1 - z_{ij}^p) \qquad \forall i,j \in I$$

$$z_j - (z_i + h_i) + 1 \leq H z_{ij}^p \qquad \forall i,j \in I$$

$$x_{ij}^p + x_{ji}^p + y_{ij}^p + y_{ji}^p + z_{ij}^p + z_{ji}^p \geq u_{ib} + u_{jb} - 1 \qquad \forall i,j \in I, \forall b \in B$$

$$z_b^{max} \geq (z_i + h_i) - H(1 - u_{ib}) \qquad \forall i \in I, \forall b \in B$$

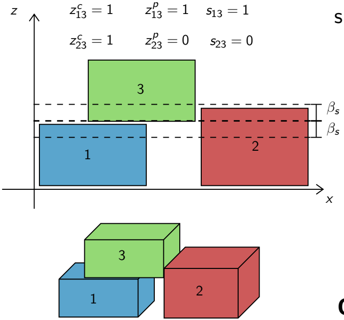


Precedences variables (2D case)

MILP Model - Vertical Support

minimize	number of used bins
then, maximize	average cage ratio of the used bins
subject to	all items are assigned to one and only one bin
	all items are inside the bin's bounds
	no overlaps between items in the same bin
	all items have vertical support

MILP Model - Closeness



Closeness variables example

$$
\begin{aligned}
\text{s.t.} \quad & z_j - (z_i + h_i) \leq \beta_s + H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& z_j - (z_i + h_i) \geq -\beta_s - H(1 - z^c_{ij}) && \forall (i,j) \in I : i \neq j \\
& s_{ij} \leq z^p_{ij} && \forall (i,j) \in I \\
& s_{ij} \leq z^c_{ij} && \forall (i,j) \in I \\
& s_{ij} \geq z^p_{ij} + z^c_{ij} - 2 && \forall (i,j) \in I : i \neq j \\
& \sum_{j \in I} s_{ij} \leq \sum_{b \in B} u_{ib} && \forall i \in I
\end{aligned}
$$

Can Support $=$ **Close** $\wedge$ **Precedes**

$$s_{ij} = z^c_{ij} \wedge z^p_{ij}$$

MILP Model - Discretized Vertical Support

Pre-Computed Parameter

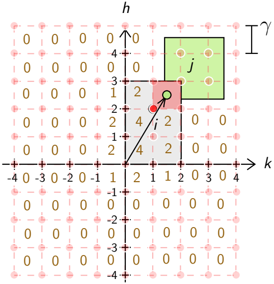
$O(i,j,k,h)$

$O(i,j,1,2) = 1$

Variables

$s_{ijb}^{12} = 1$

$(x_j - x_i, y_j - y_i)$



Space discretization

$$
\begin{aligned}
\text{s.t.} \quad & z_i \le H(1 - g_i) && \forall i \in I \\
& \sum_{(k,h)\in\Delta, b\in B: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le s_{ij} && \forall (i,j) \in I \\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le u_{ib} && \forall (i,j,b) \in I^B \\
& \sum_{(k,h)\in\Delta: O(i,j,k,h)\neq 0} s_{ijb}^{kh} \le u_{jb} && \forall (i,j,b) \in I^B \\
& x_j - x_i \ge \gamma k - 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& x_j - x_i \le \gamma (k+1) + 2W(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& y_j - y_i \ge \gamma h - 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& y_j - y_i \le \gamma (h+1) + 2D(1 - s_{ijb}^{kh}) && \forall (k,h) \in \Delta, \forall (i,j,b) \in I^B : O(i,j,k,h) \neq 0 \\
& \sum_{(k,h)\in\Delta, b\in B, j\in I: i\neq j \wedge O(i,j,k,h)\neq 0} O(i,j,k,h) s_{jib}^{kh} \ge \alpha_s w_i d_i - w_i d_i g_i && \forall i \in I
\end{aligned}
$$

MILP Model - Results

	MILP Model		
n	**Max Z**	**TT(s)**	**Gap(%)**
1	85	0.01	0.00
2	85	0.07	0.00
3	85	0.13	0.00
4	85	0.20	0.00
5	85	2.02	0.00
6	158	90.58	0.00
7	158	1,369.24	0.00
8	161*	3,600.00	1.86
9	-	-	-

* Some boxes had lower support than expected due to discretization errors.

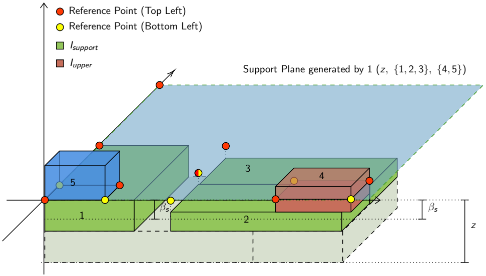
Overview

Composed of:

- Constructive heuristic (Support Planes)
- Beam-Search

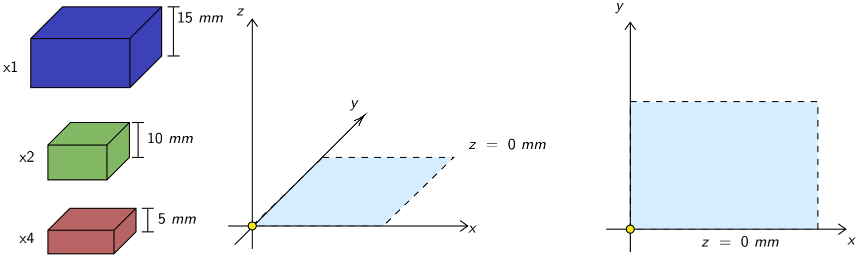
Support Planes

- Operates on a single bin
- Exploits a modified 2D-BPP heuristic
- No explicit layers
- Guarantees vertical support

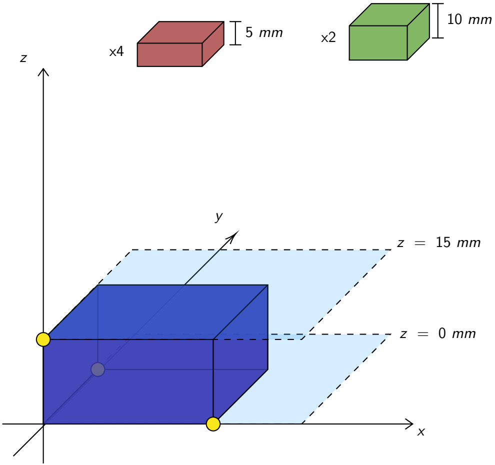


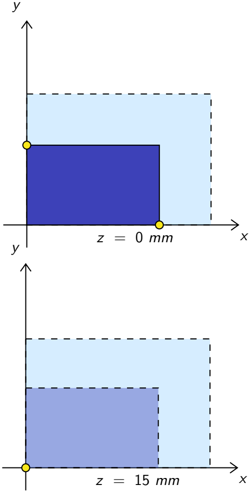
An example of a support plane generated by item 1

Support Planes

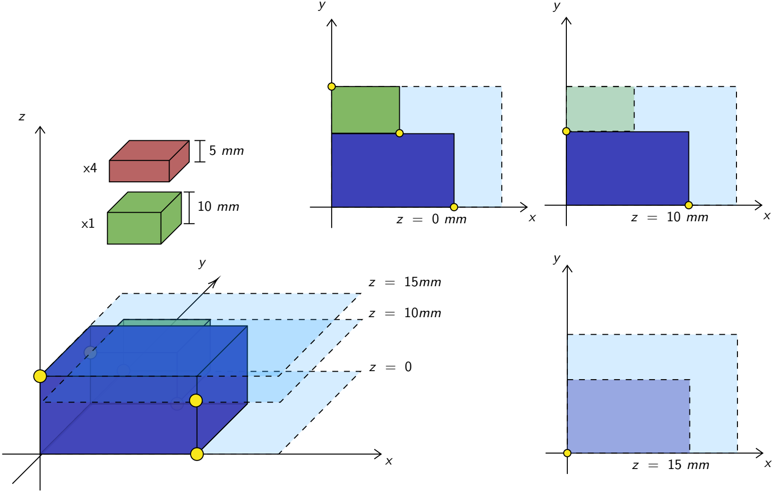


Support Planes

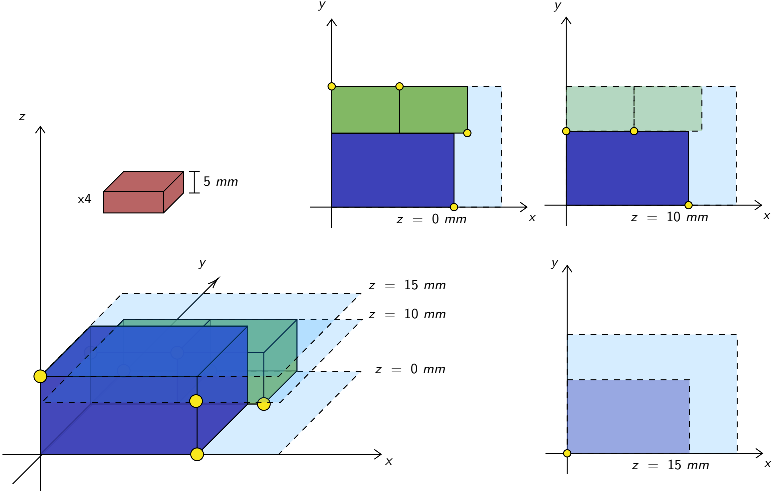




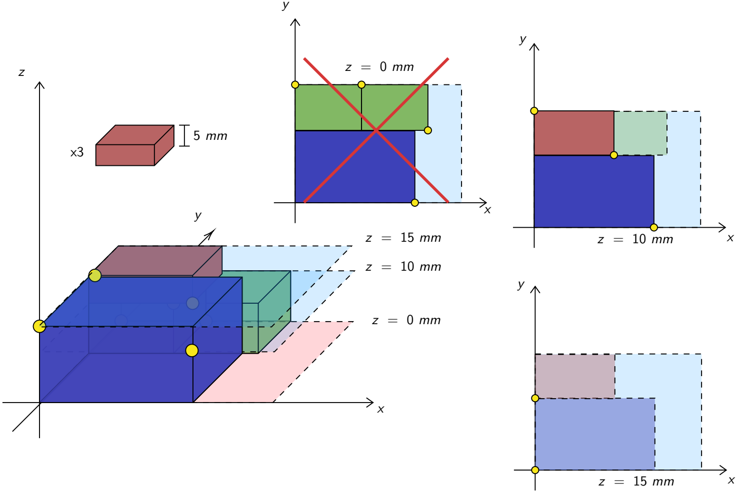
Support Planes



Support Planes



Support Planes



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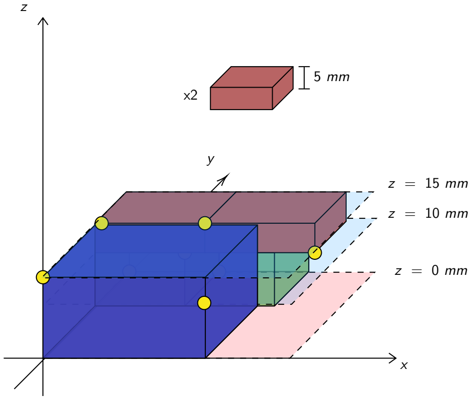

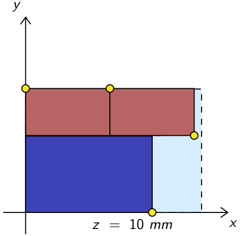

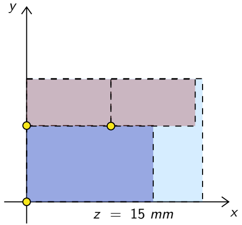

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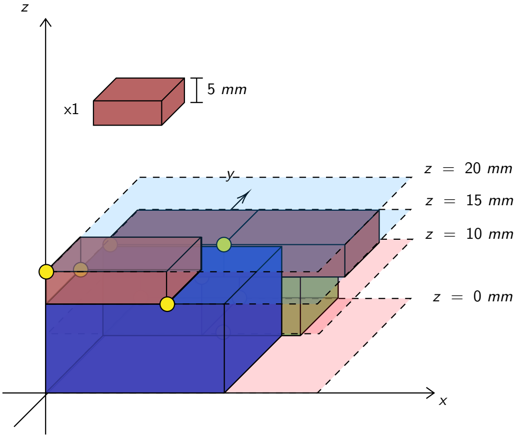
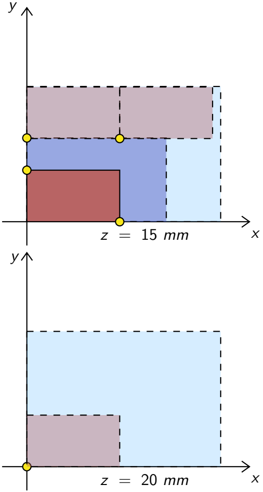
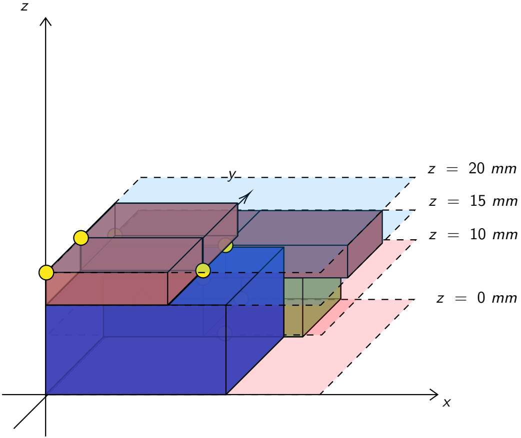
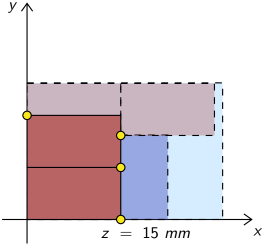
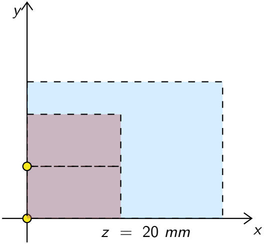




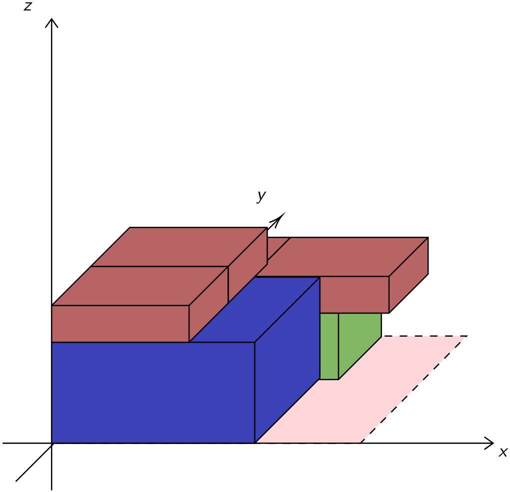
Support Planes







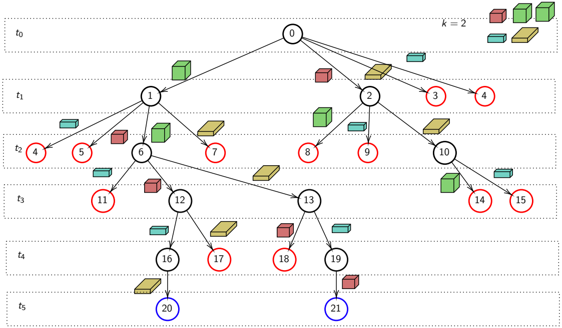
Support Planes



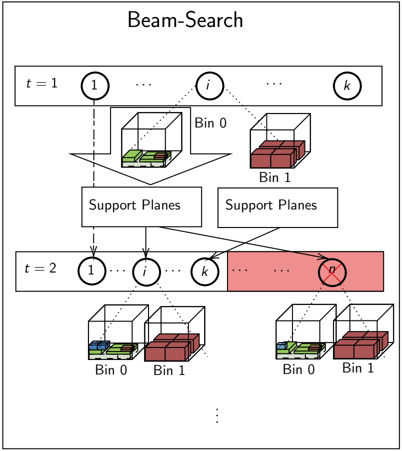
Beam Search

- Exploits support planes
- Branches on the type of item to place next

Beam-Search: truncated breadth-first tree search, limited to k best nodes

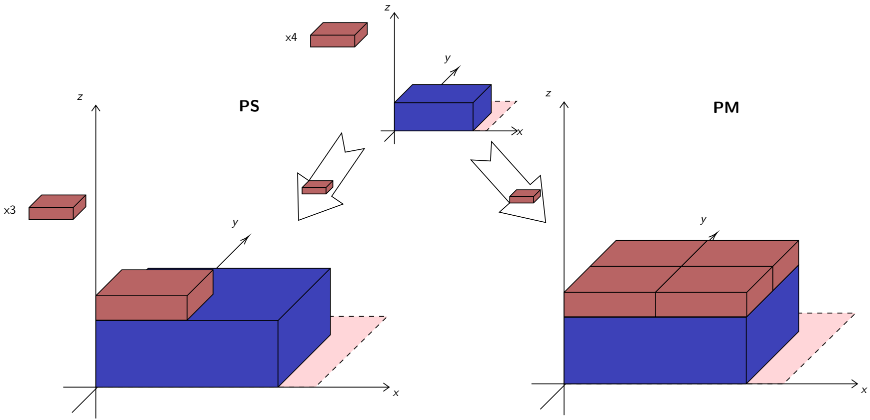


Beam Search



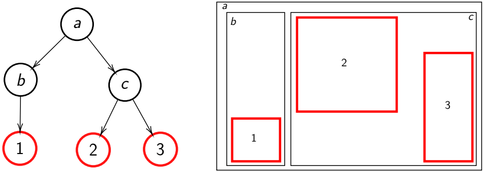
Conceptual representation of the proposed heuristic

Beam Search



Proposed placement modes

- Prune duplicate nodes
- Fast overlap checks
- Lazy node updates



AABB Tree, self-balancing tree of axis-aligned bounding boxes

Computational Experiments

Comparison with MILP model on limited set of boxes

	MILP Model			PM		PS	
n	Max Z	TT(s)	Gap(%)	Max Z	TT(s)	Max Z	TT(s)
1	85	0.01	0.00	85	0.00	85	0.00
2	85	0.07	0.00	85	0.00	85	0.00
3	85	0.13	0.00	85	0.00	85	0.00
4	85	0.20	0.00	85	0.01	85	0.01
5	85	2.02	0.00	85	0.02	85	0.02
6	158	90.58	0.00	158	0.06	158	0.05
7	158	1,369.24	0.00	158	0.07	158	0.08
8	161*	3,600.00	1.86	160	0.10	160	0.08
9	-	-	-	169	0.09	161	0.10
10	-	-	-	218	0.12	218	0.13
11	-	-	-	240	0.12	240	0.12
12	-	-	-	310	0.13	316	0.16
13	-	-	-	310	0.15	333	0.18
14	-	-	-	310	0.20	333	0.22
15	-	-	-	406	0.21	397	0.27
16	-	-	-	435	0.23	452	0.36
17	-	-	-	429	0.27	515	0.41
18	-	-	-	432	0.32	522	0.47
19	-	-	-	458	0.35	522	0.55
20	-	-	-	539	0.37	564	0.62

* Some boxes had lower support than expected due to discretization errors.

Average execution time of literature results with bin gap

Heuristic		Execution Time (s)				Bin Gap (%)
		$n = 50$	$n = 100$	$n = 150$	$n = 200$	
PM	$k = 1$	0.05	0.11	0.28	0.55	4.57
	$k = 5$	0.08	0.39	1.02	2.16	4.32
	$k = 10$	0.15	0.74	1.98	4.12	4.29
	$k = 20$	0.29	1.45	3.89	8.07	4.05
	$k = 50$	0.72	3.63	9.72	20.47	3.95
PS	$k = 1$	0.04	0.18	0.51	1.08	4.35
	$k = 5$	0.12	0.74	2.19	4.79	4.01
	$k = 10$	0.23	1.43	4.19	9.39	3.94
	$k = 20$	0.47	2.81	8.48	18.93	3.74
	$k = 50$	1.15	6.74	21.03	45.78	3.52
BRKGA-VD		17.13	80.63	190.50	369.75	0.00

Computational Experiments

Class	n	PM $k = 50$	PS $k = 50$	TS3	GLS	GASP	EHGH2	GVN	BRKGA	BRKGA-VD
1	50	14.10	*14*	**13.4**	**13.4**	**13.4**	13.8	**13.4**	**13.4**	**13.4**
	100	28.3	*27.7*	**26.6**	**26.6**	26.9	27.6	**26.6**	**26.6**	**26.6**
	150	*38.1*	*38.1*	36.7	37	37	39.8	36.4	36.4	**36.3**
	200	52.9	*52.6*	51.2	51.2	51.6	**50.6**	50.9	50.8	50.8
2	50	*14.7*	*14.7*	**13.8**	-	-	-	**13.8**	**13.8**	**13.8**
	100	26.6	26.6	25.7	-	-	-	25.7	25.6	**25.5**
	150	38.3	38.7	37.2	-	-	-	36.9	**36.6**	**36.6**
	200	51.1	51.6	50.1	-	-	-	**49.4**	**49.4**	**49.4**
3	50	*13.7*	13.8	**13.3**	-	-	-	**13.3**	**13.3**	**13.3**
	100	27.7	27.3	26	-	-	-	26	**25.9**	**25.9**
	150	39.4	39	37.7	-	-	-	37.6	**37.5**	**37.5**
	200	51.6	*51.3*	50.5	-	-	-	50	**49.8**	**49.8**
4	50	*29.7*	*29.7*	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**	**29.4**
	100	*59.2*	*59.2*	59	59	59	59.5	59	59	**58.9**
	150	87.7	*87.6*	**86.8**	**86.8**	**86.8**	90.4	**86.8**	**86.8**	**86.8**
	200	*119.5*	*119.5*	**118.8**	119	**118.8**	119	**118.8**	**118.8**	**118.8**
5	50	*8.6*	*8.6*	8.4	8.3	8.4	**7.9**	8.3	8.3	8.3
	100	16.1	15.6	15	15.1	15.1	**14.6**	15	15	15
	150	21.8	*21.3*	20.4	20.2	20.6	21.5	20.4	20.1	**19.9**
	200	29.1	28.3	27.6	27.2	27.7	29.6	**27.1**	**27.1**	**27.1**
6	50	10.3	10.3	9.9	9.8	9.9	11.8	9.8	**9.7**	**9.7**
	100	19.7	19.7	19.1	19.1	19.1	19.2	19	**18.9**	**18.9**
	150	30.2	*30.1*	29.4	29.4	29.5	29.8	29.2	**29**	**29**
	200	39	*38.4*	37.7	37.7	38	38.7	37.4	**37.3**	**37.3**
7	50	7.7	7.8	7.5	**7.4**	7.5	**7.4**	**7.4**	**7.4**	**7.4**
	100	13.3	*13.1*	12.5	12.3	12.7	13.5	12.5	**12.2**	**12.2**
	150	17.1	*16.8*	16.1	15.8	16.6	18.2	16	15.3	**15.2**
	200	24.8	24.7	23.9	23.5	24.2	24.1	23.5	**23.4**	**23.4**
8	50	9.9	9.7	9.3	**9.2**	9.3	9.4	**9.2**	**9.2**	**9.2**
	100	*19.6*	19.9	18.9	18.9	19	18.9	18.9	18.9	**18.8**
	150	25.7	25.6	24.1	23.9	24.8	26	24.1	**23.6**	**23.6**
	200	31.6	*31.1*	30.3	29.9	31.1	35.8	29.8	**29.3**	**29.3**

Computational Experiments

Summary of case study tests

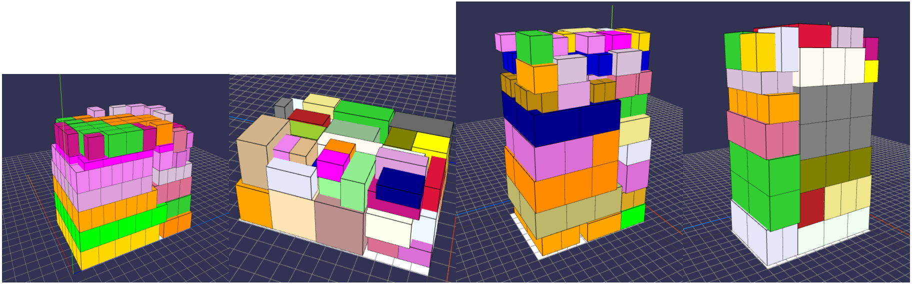
	PS			**PM**		
k	**TT (ms)**	**B**	**CR (%)**	**TT (ms)**	**B**	**CR (%)**
1	423.87	1.37	65.87	65.18	1.31	**70.70**
5	1,597.54	1.34	69.19	185.22	1.29	**73.08**
10	2,627.52	1.32	70.35	344.90	1.27	**73.56**
20	5,373.79	1.34	70.78	620.95	1.27	**74.57**
50	14,203.10	1.31	72.11	1,279.96	1.29	**74.61**
100	26,934.21	1.31	73.23	2,340.37	1.26	**75.36**
200	48,944.90	1.30	73.89	4,465.78	1.25	**76.39**

Computational Experiments

Case study experiments trade off between average execution times and average cage ratio

k	PS		PM	
	CR* – CR (%)	**TT – TT* (ms)**	**CR* – CR (%)**	**TT – TT* (ms)**
1	10.56	358.69	5.73	0.00
5	7.24	1,532.36	3.35	120.04
10	6.08	2,562.34	2.87	279.72
20	5.65	5,308.61	**1.85**	**555.77**
50	4.32	14,137.92	**1.82**	**1,214.78**
100	3.20	26,869.03	1.07	2,275.19
200	2.54	48,879.72	0.04	4,400.60

Results



(a) Instance 95 (b) Instance 82 (c) Instance 56 (d) Instance 66, Bin 1

Solutions of case study tests with the "PM" placement and $k = 200$

Future Developments

- More practical constraints
- Solution optimization heuristics
- Improvements to the underlying 2D-BPP heuristic

Thank you for your attention!
Questions?