Dynamical Systems 1 Kristian Uldall Kristiansen and Poul Hjorth
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Week 4

On today

Today we will start to talk about local dynamical systems theory. This theory focusses on the phase portrait of $\dot{x} = f(x)$ in a (small) neighborhood of an equilibrium $f(x_*) = 0$. We will talk about local stuff up until course 11. In this week we will introduce linearization, hyperbolicity and stable and unstable manifolds. Finally we will present and discuss the implications of The (Local) Stable Manifold Theorem (pp. 107-108).
You will encounter manifolds several times in this course. There is a definition on p. 107. But for our purposes the following simpler definition of sub-manifolds of $\mathbb{R}^n$ will suffice:

- A set $M \subset \mathbb{R}^n$ is an r-dimensional C^k-(sub)manifold of $\mathbb{R}^n$ if for every $x \in M$ there is an open set $U \subset \mathbb{R}^n$ containing x so that $M \cap U$ is a graph of C^k-function

$$h : B \subset \mathbb{R}^r \to \mathbb{R}^{n-r},$$

expressing $(n-r)$-coordinates in terms of the remaining r-coordinates.

Examples:

- $S^1 = \{(x,y)|x^2+y^2=1\} \subset \mathbb{R}^2$ is an example of a smooth one-dimensional manifold. Indeed all points on S^1 have neighborhoods described by either of the following graphs:

$$x = \pm\sqrt{1-y^2}, \quad y \in (-1,1),$$
$$y = \pm\sqrt{1-x^2}, \quad x \in (-1,1).$$

- The set $M = \{xy = 0\} \subset \mathbb{R}^2$, consisting of the union of the x and the y-axis, is not a manifold because there is no graph represented M in a neighborhood of $(x,y) = 0$. If we remove $(x,y) = 0$ from M and consider $N = M\backslash\{0\}$ then we obtain a smooth one-dimensional manifold.

- Consider $f : \mathbb{R}^n \to \mathbb{R}$ smooth satisfying $Df(x) \neq 0$. Then by the implicit function theorem, the level sets $f^{-1}(c) = \{x|f(x) = c\}$, are $(n-1)$-dimensional smooth manifolds. Indeed, consider $x_0 \in f^{-1}(c)$. Then by assumption:

$$f(x_0) = c, \quad \partial_{x_i} f(x_0) \neq 0,$$

for some i. Take $i = 1$ (without loss of generality). Then the implicit function provides the existence of a function $h : B \subset \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$x_{10} = h(x_{20}, \ldots, x_{n0}),$$

and

$$f(h(x_2, \ldots, x_n), x_2, \ldots, x_n) = c.$$

(If you do not know about the implicit function theorem, or have forgotten then we will introduce it formally later in the course).

Theorem 5.3 will be illustrated through several examples.

NB!

I will start the lecture by going through some basic geometry and expand a bit on the "easy" way to approach invariance that I presented at the end of last week's lecture. I have uploaded a document **geometryAndInvariance.pdf** on file-share's "lecture notes".

Notation

[Per00] uses the notation S and U for local stable and unstable manifolds. Many authors use W^s_{loc} and W^u_{loc} instead.

Read

Sections 2.6-2.7 in [Per00].

Key words

Stable and unstable manifolds, local stable manifold theorem.

Exercises

Exericse 1: Find W^s and W^u for

$$\begin{aligned}\dot{x}_1 &= -x_1,\\ \dot{x}_2 &= x_2 + x_1^2.\end{aligned}$$

Show that S and U are tangent to E^s and E^u respectively. Use Maple and the template from week 2 to sketch the phase portrait. Discuss your findings. Compare with the phase portrait of the linearized system:

$$\begin{aligned}\dot{x}_1 &= -x_1,\\ \dot{x}_2 &= x_2.\end{aligned}$$

Exericse 2: Problem 1(a) in problem set 6 on p. 104. Show that the lines $x_2 = \pm x_1$ are invariant. Draw a sketch showing E^s and E^u together with their local nonlinear versions S and U. *Use the Maple template from week 2 to sketch the phase portrait.

Exercise 3: Consider the nonlinear example:

$$\begin{aligned}\dot{x}_1 &= -x_1 - 4x_3 - 4x_2 - (x_1+x_3)^2,\\ \dot{x}_2 &= -3x_2 - 2x_3,\\ \dot{x}_3 &= 4x_2 + 3x_3 + (x_1+x_3)^2.\end{aligned} \tag{1}$$

(a) Show that

$$A = Df(0) = \begin{pmatrix} -1 & -4 & -4 \\ 0 & -3 & -2 \\ 0 & 4 & 3 \end{pmatrix},$$

and that

$$\begin{aligned}E^s &= \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}\right\},\\ E^u &= \text{span}\left\{\begin{pmatrix}-2\\-1\\2\end{pmatrix}\right\}.\end{aligned}$$

(b) Introduce new coordinates y based on the eigenvectors in (a) so that

$$\begin{aligned}\dot{y}_1 &= -y_1,\\ \dot{y}_2 &= -y_2 + y_1^2,\\ \dot{y}_3 &= y_3 + y_1^2.\end{aligned} \tag{2}$$

Hint: You may find the following useful. If

$$V = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix},$$

then

$$V^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(c) What is E^s and E^u in the y-coordinates for the linearization of (2)?

(d) Write the unknown manifolds S and U as graphs and explicit the properties of the graphs relative to E^s and E^u in the y-coordinates used.

Exercise 4*: Consider (2).

(a) Show that S takes the following form:

$$y_3 = h_s(y_1, y_2) = ay_1^2 + by_1y_2 + cy_2^2 + \cdots, \tag{3}$$

with $\cdots$ representation terms that are at least cubic in (y_1, y_2).

(b) Insert (3) into (2) and collect powers of y_1, y_2 (ignoring cubic terms or higher) to determine a, b and c.

(c) Compare your result in (c) with the calculations on p. 106 in [Per00]. Discuss.

Exercise 5*: Read the lecture notes "Proof of stable manifold theorem" available on Campusnet. Can you redo the proof?

Next week

When is a nonlinear system locally near an equilibrium "similar" to the linearization? And how do we define "similar"? These questions will be addressed next week.

References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.