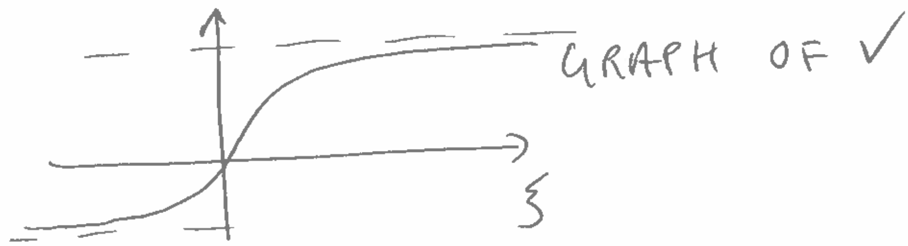
TRAVELLING WAVES (TWs)

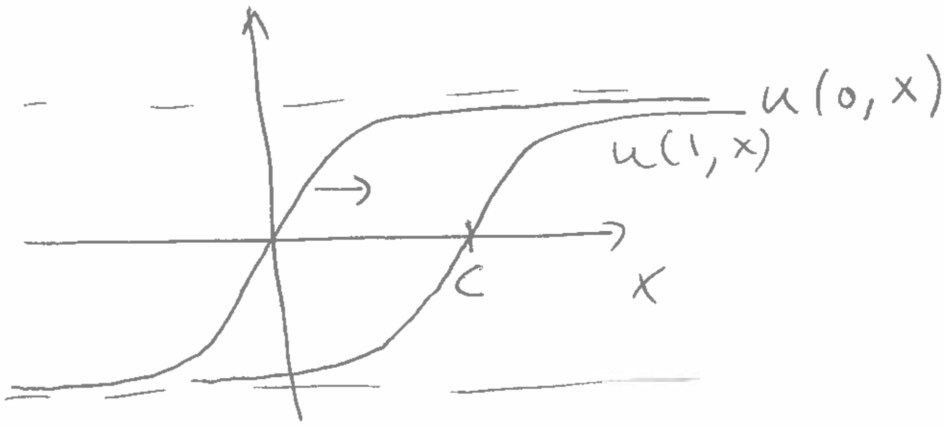
TWs ARE SOLUTIONS TO PDEs $u(t,x)$ THAT MOVE WITH CONST. SPEED c WHILE MAINTAINING THEIR SHAPE.

IN OTHER WORDS:

$$u(t,x) = v(x-ct)$$

EXAMPLE: TRAVELLING FRONT!





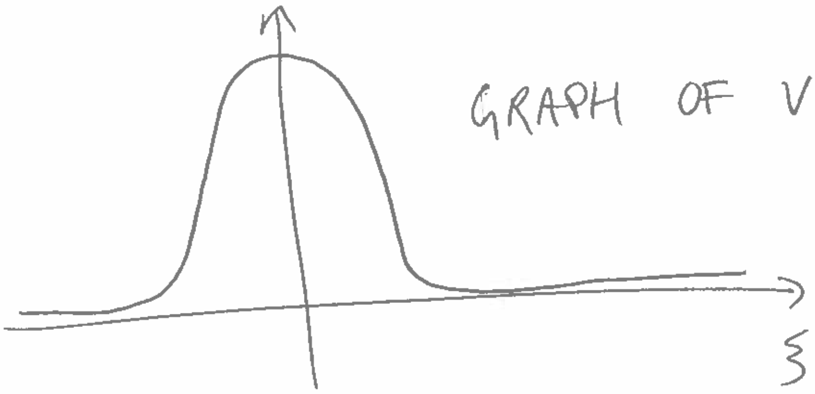
SIMPLEST EXAMPLE

$$u_t = \varepsilon u_{xx} + f(u), \quad x \in \mathbb{R}$$

APPLICATIONS: 1/ WAVES, 2/ IMPULSES IN NERVE FIBERS, 3/ FLAME FRONTS IN COMBUSTION

TWs CAN BE OF DIFFERENT TYPES

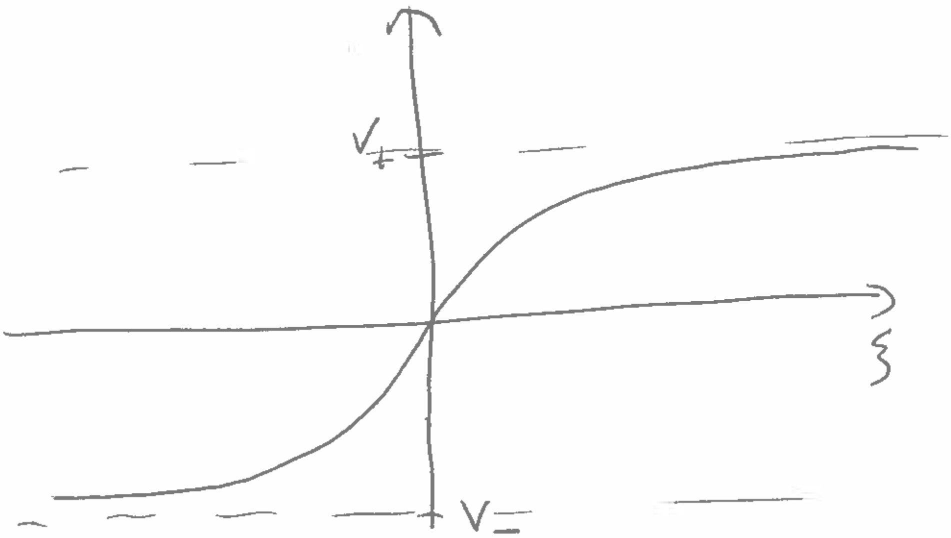
PULSE



GRAPH OF $V(\xi)$

$$\lim_{\xi \to \pm\infty} V(\xi) = 0$$

FRONTS



$$\lim_{\xi \to \pm\infty} V(\xi) = V_\pm$$

EXAMPLE

$$u_t = u_{xx} - \lambda u_x - u(1-u)$$

ANSATZ

$u(t,x) = v(x-ct)$, c UNKNOWN

$$u_t = v'(x-ct)\,(-c)$$
$$= -c\,v'(x-ct) = -c\,v'(\xi)$$
$$u_{xx} = v''(x-ct) = v''(\xi)$$
$$u_x = v'(x-ct) = v'(\xi)$$

WITH $\xi = x - ct$.

INSERTING GIVES

$$-cv' = v'' - \lambda v' - v(1-v)$$

$$v'' = +(\lambda - c)v' + v(1-v)$$

PUT $w = v'$. THEN

$$v' = w$$
$$w' = (\lambda - c)w + v(1-v).$$

EQ. AT $(v, w) = 0$.

LINEARIZATION:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & \lambda - c \end{bmatrix}$$

EIGENVALUES:

$$\mu_+ = \frac{1}{2}\left\{ \lambda - c + \sqrt{(\lambda-c)^2+4} \right\}$$

$$> 0$$

$$\mu_- = \frac{1}{2}\left\{ \lambda - c - \sqrt{(\lambda-c)^2+4} \right\}$$

$$< 0$$

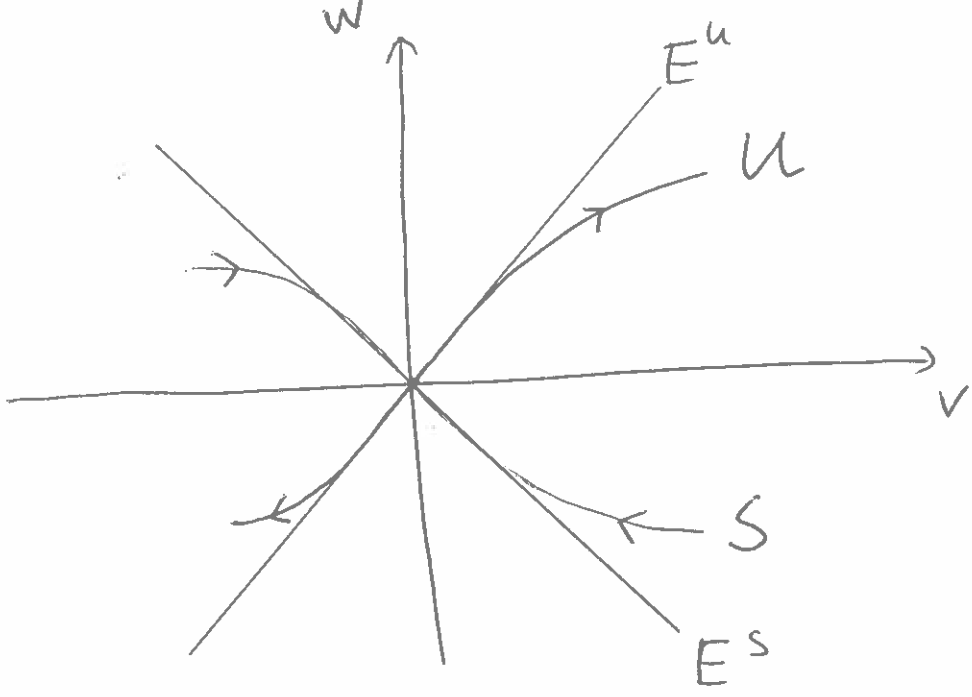
EIGENVECTORS

$$\underline{v}_+ = \begin{bmatrix} 1 \\ \mu_+ \end{bmatrix}$$

$$\underline{v}_- = \begin{bmatrix} 1 \\ \mu_- \end{bmatrix}$$

UNSTABLE SPACE $E^u = \text{SPAN } \underline{v}_+$

STABLE SPACE $E^s = \text{SPAN } \underline{v}_-$



$\mathcal{U}$: $w = h_u(v)$, $h_u(0) = 0$, $h_u'(0) = \mu_+$

$\mathcal{S}$: $w = h_s(v)$, $h_s(0) = 0$, $h_s'(0) = \mu_-$

FOR $c = \lambda$: THE SET DEFINED BY

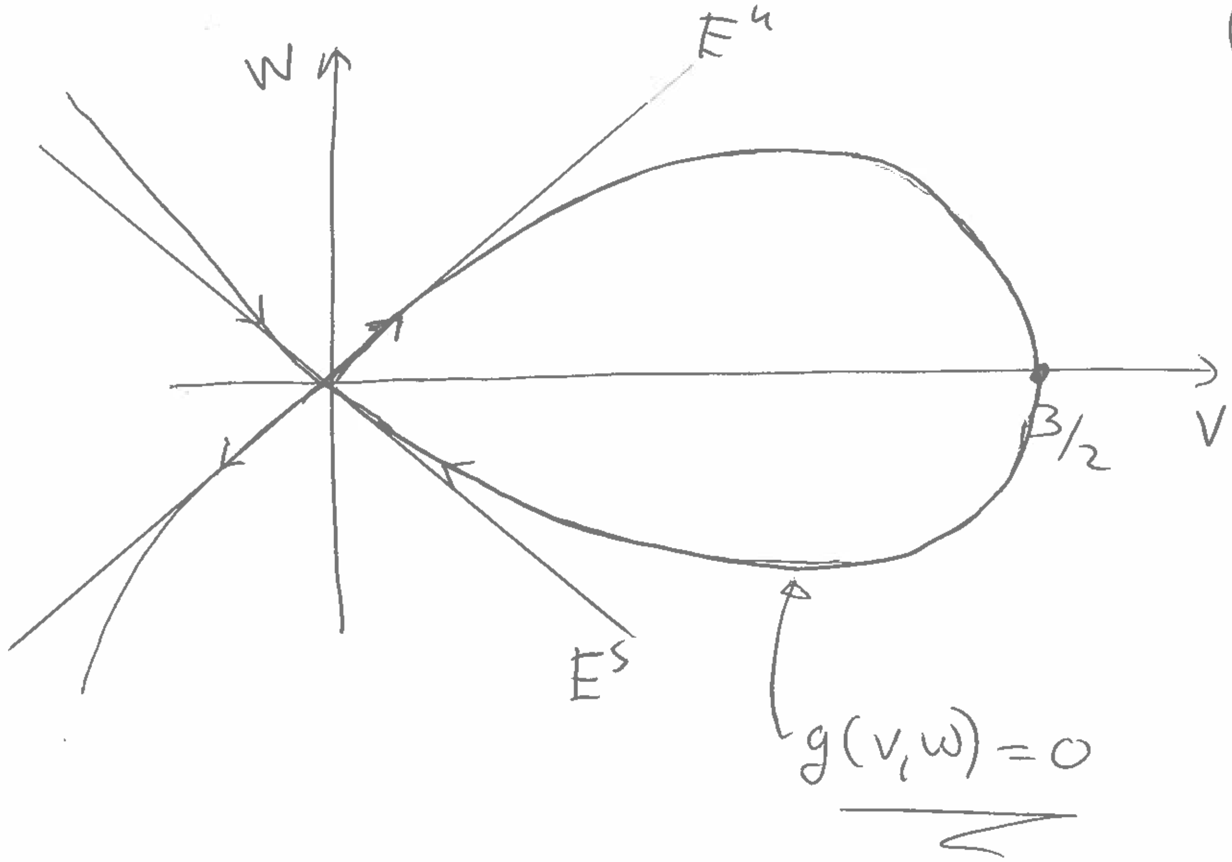
$$0 = \underbrace{\frac{1}{2} w^2 - \frac{1}{2} v^2 + \frac{1}{3} v^3}_{= g(v,w)}$$

IS INVARIANT!

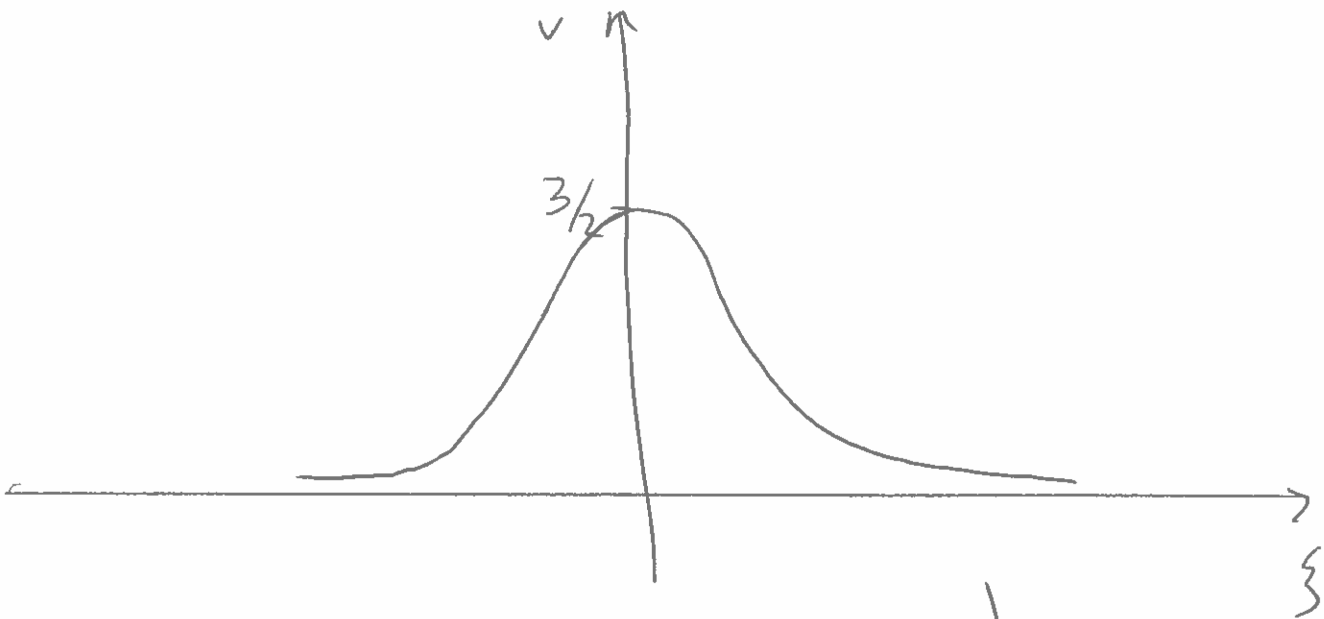
WHY?

$$\frac{d}{dt} g(v,w) = \underbrace{\begin{bmatrix} -v + v^2 & w \end{bmatrix}}_{Dg} \underbrace{\begin{bmatrix} w \\ v(1-v) \end{bmatrix}}_{\binom{\dot v}{\dot w}}$$

$= 0$!



FOR $c = \lambda$:



TRAVELLING FRONT!