PROOF OF STABLE MANIFOLD THEOREM

$x = 0$: HYPERBOLIC EQ OF $\dot{x} = f(x)$.

WRITE $\dot{x} = Ax + \hat{g}(x)$, $A = Df(0)$, $\hat{g}$ NONLINEAR.

THEN

$$x = \big[\underbrace{v_1 \ \text{---} \ v_k}_{\text{BASIS OF } E^s} \quad \overbrace{v_{k+1} \ \text{---} \ v_n}^{\text{BASIS OF } E^u}\big] \begin{bmatrix} y_s \\ y_u \end{bmatrix}$$

GIVES

$$\dot{y} = \begin{bmatrix} \dot{y}_s \\ \dot{y}_u \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} y_s \\ y_u \end{bmatrix} + g(y)$$

PUT $B = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$.

EIGENVALUES OF P: $\lambda_1, \text{---}, \lambda_k$ SATISFYING $Re(\lambda_i) \le -\alpha < 0$, $i = 1, \text{---}, k$

EIGENVALUES OF Q: $\lambda_{k+1}, \text{---}, \lambda_n$ SATISFYING $Re(\lambda_i) \ge \alpha > 0$, $i = k+1, \text{---}, n$.

RECALL: $\|e^{Pt} y_s\| \le K e^{-\alpha t} |y_s|$

$$\|e^{-Qt} y_u\| \le K e^{-\alpha t} |y_u| \qquad (*)$$

LET $U(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix}$, $V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$

SUCH THAT $e^{Bt} = U(t) + V(t)$.

LEMMA 1 $\dot{y} = By + \gamma(t)$.

SUPPOSE $\gamma(t)$ CONTINUOUS AND BOUNDED $|\gamma(t)| \leq \delta$ FOR $t \geq 0$.

THEN $\forall\, y_{s0} \in \mathbb{R}^k$ $\exists!$ $y(t; y_{s0})$ SOLUTION WHICH IS BOUNDED FOR ALL $t \geq 0$:

$$y(t; y_{s0}) = U(t) \begin{bmatrix} y_{s0} \\ 0 \end{bmatrix} + \int_0^t U(t-s)\gamma(s)\,ds$$
$$- \int_t^\infty V(t-s)\gamma(s)\,ds$$

PROOF: LET $y(t) = e^{Bt}\xi(t)$. THEN

$\dot{\xi}(t) = e^{-Bt}\gamma(t)$. HENCE

$\xi(t) = \xi(0) + \int_0^t e^{-Bs}\gamma(s)\,ds$ SO THAT

$$y(t) = e^{Bt}y(0) + \int_0^t e^{B(t-s)}\gamma(s)\,ds.$$

THUS

$$y_s(t) = e^{Pt}y_{s0} + \int_0^t e^{B(t-s)}\gamma_a(s)\,ds$$

WHICH IS BOUNDED:

$$|y_s(t)| \leq Ke^{-\alpha t}|y_{s0}| + \int_0^t Ke^{-a(t-s)}\delta\,ds$$
$$\leq Ke^{-at}|y_{s0}| + \frac{K}{\alpha}\delta < \infty$$

USING (*)

NEXT $y_u(t) = e^{Qt} y_{u0} + \int_0^t e^{Q(t-s)} \gamma_u(s) ds$

$$\underset{\triangle}{=} e^{Qt} \left[y_{u0} + \int_0^t e^{-Qs} \gamma_u(s) ds \right]$$

$e^{Qt} \longrightarrow \infty$ FOR $t \to \infty$. THEREFORE

$y_u(t)$ IS ONLY BOUNDED IF

$$\left[y_{u0} + \int_0^\infty e^{-Qs} \gamma_u(s) ds \right] = 0$$

$$y_{u0} = - \int_0^\infty e^{-Qs} \gamma_u(s) ds$$

INTEGRAL EXISTS CF $*$

INSERTING INTO $\triangle$ GIVES

$$y_u(t) = e^{Qt} \left[- \int_0^\infty + \int_0^t e^{-Qs} \gamma_u(s) ds \right]$$

$$= e^{Qt} \left[- \int_t^\infty e^{-Qs} \gamma_u(s) ds \right]$$

$$= - \int_t^\infty e^{Q(t-s)} \gamma_u(s) ds.$$ THEN

$$|y_u(t)| \leq K \int_t^\infty e^{\alpha(t-s)} \delta \leq \frac{K}{\alpha} \delta$$ // ⊠

FIXED POINT ARGUMENT

DEFINE $T: C^0(\mathbb{R}_+) \longrightarrow C^0(\mathbb{R}_+)$ BY

$$(Ty)(t) = U(t)\begin{bmatrix} y_{s0} \\ 0 \end{bmatrix} + \int_0^t U(t-s)g(y(s))ds$$
$$- \int_t^\infty V(t-s)\, g(y(s))ds$$

$$\forall\, y_{s0} \in \mathbb{R}^k.$$

NOTE: $(Ty)(t) \in C^0(\mathbb{R}_+)$ CF. PREVIOUS LEMMA 1

FURTHERMORE, SETTING $\gamma(s) = g(y(s))$, WE REALISE THAT A FIXED POINT $y(t; y_{s0})$ OF T: $Ty(\cdot; y_{s0}) = y(\cdot, y_{s0})$ SOLVES OUR ODE!

$y_u(0; y_{s0})$ WILL BE OUR STABLE MANIFOLD FOR $y_{s0} \subset B_\delta(0) \subset \mathbb{R}^k$.

LEMMA 2 $\forall \varepsilon > 0\ \exists \delta$:

$$|g(y)| \leq \varepsilon |y|$$
$$|g(x) - g(y)| \leq \varepsilon |x - y|$$

$\forall\, x, y \in B_\delta(0)$.

PROOF: BY TAYLOR'S THM:

$g(x) = \int_0^1 Dg(sx)\,x\,ds$. RECALL $Dg(0) = 0$

HENCE $\forall \varepsilon \; \exists \delta : \; \| Dg(x) \| \leq \varepsilon \;\; \forall x \in B_\delta(0)$

$$|g(x)| \leq \sup_{B_\delta(0)} \| Dg(x) \| \, |x| \leq \varepsilon |x|$$

SIMILARLY

$$g(x) - g(y) = \int_0^1 Dg(y + s(x-y))(x-y)\,ds$$

$$\Rightarrow |g(x) - g(y)| \leq \varepsilon |x-y| \qquad \boxtimes$$

LET $W = \{ y \in C^0(\mathbb{R}_+) \mid |y(t)| \leq \delta \}$

W IS CLOSED.

LEMMA 3 $T : W \longrightarrow W$ IF

$$|y_{s0}| < \delta/2K, \quad \varepsilon \leq \alpha/4K \qquad \square$$

PROOF:

$$|(Ty)(t)| \leq K e^{-t\alpha} |y_{s0}| + K\varepsilon \left[\int_0^t e^{-(t-s)} |y(s)|\,ds + \int_t^\infty e^{(t-s)\alpha} |y(s)|\,ds \right]$$

USING LEMMA 2

THEREFORE FOR $y(t) \in W$:

$$|(Ty)(t)| \leq K e^{-t\alpha} |y_{s0}| + K\varepsilon \frac{2\delta}{\alpha}$$

$$\leq \delta \quad \text{USING } \square$$

LEMMA 4: SUPPOSE (I). THEN

$T: W \longrightarrow W$ IS A CONTRACTION.

PROOF: ESTIMATION GIVES:

$$\|T(x)-T(y)\| \le K\varepsilon \|x-y\| \left(\int_0^t e^{-(t-s)\alpha}\, ds + \int_t^\infty e^{(t-s)\alpha}\, ds \right)$$

$$\le 2\frac{K\varepsilon}{\alpha}\|x-y\| \le \frac{1}{2}\|x-y\|$$

$$< \|x-y\|. \quad \blacksquare$$

BY BANACH'S FIXED POINT THEOREM

$\exists!\ y(t; y_{s0})$ FIXED POINT OF T IN W.

$y(0; y_{s0})$ IS OUR S. IT IS A GRAPH

$$y(0; y_{s0}) = \begin{bmatrix} y_{s0} \\ y_u(0; y_{s0}) \end{bmatrix} \quad //$$

IT IS EASY TO SHOW THAT $y(0; y_{s0})$ IS INVARIANT. ALSO USING GRONWALL'S INEQUALITY IT CAN BE SHOWN THAT

$$|y(t; y_{s0})| \le 2K e^{-\alpha/2\, t} |y_{s0}| \quad //$$