EXERCISES WEEK 7 (1)

PROBLEM 1

$$\dot{x} = \begin{bmatrix} -1 & a \\ ab & -1 \end{bmatrix}, \quad a>0,\ b\in\mathbb{R}$$

$$|A-\lambda I| = (-1-\lambda)^2 - a^2 b$$

$\lambda$: COMPLEX IF $b=-w^2<0$,
$\lambda=-1$ IF $b=0$,
$\lambda$ REAL IF $b=w^2>0$

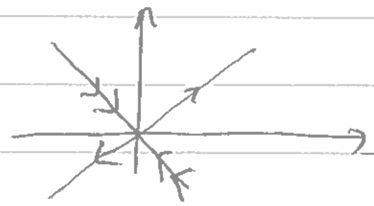
$\boxed{b=-w^2}$

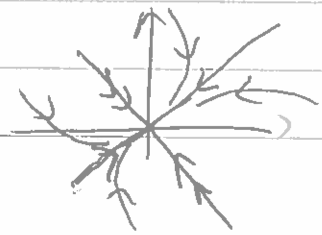
$\lambda = -1 \pm i a w$, $\underline{v} = \begin{bmatrix} 1 \\ \pm i w \end{bmatrix}$

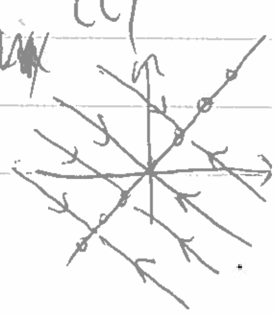

$\boxed{b=w^2}$

$\lambda = -1 \pm a w$, $v = \begin{bmatrix} 1 \\ \pm w \end{bmatrix}$

IF i/ $aw>1$ : SADDLE
ii/ $aw=1$ : SADDLE-NODE
iii/ $0<aw<1$ : NODE

i/ 

ii/ 

iii/ 

$b = 0, \quad \lambda = -1, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ : ST. NODE ②

$\dot{x}_1 = -x_1 + a x_2,$

$\dot{x}_2 = -x_2, \quad x_2 = e^{-t} x_{20}, \quad \left( e^{-t} = \frac{x_2}{x_{20}} \right)$

$\dot{x}_1 = -x_1 + a e^{-t} x_{20}$ $\quad +t = \ln^{-1} \frac{x_2}{x_{20}}$

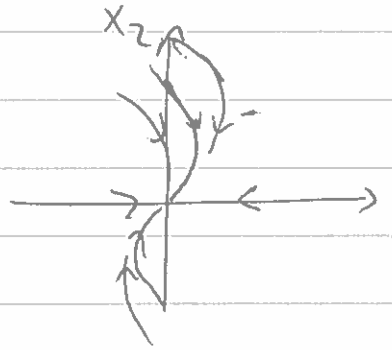
$e^t x_1 + e^t \dot{x}_1 = a x_{20}$

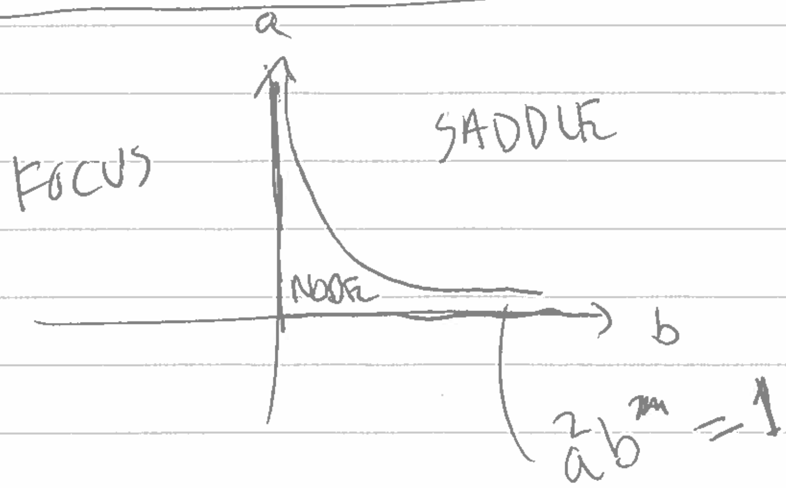
$\overset{\shortparallel}{(x_1 e^t)^{\cdot}} = a x_{20}$

$x_1 e^t - x_{10} = a x_{20} t$

$x_1 = e^{-t} \left[ x_{10} + a x_{20} t \right]$

$= \frac{x_2}{x_{10}} \left[ x_{10} + a x_{20} \ln^{-1} \frac{x_2}{x_{20}} \right]$



PROBLEM 2a

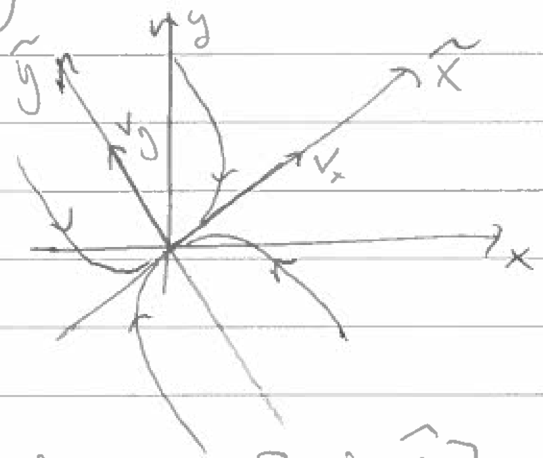
FIG a/ SADDLE, FIG b/ CENTER, FIG c/ FOCUS

FIGS d–f: ALL NODES.

2b CASE d, e, f CAN BE WRITTEN AS

$$\dot{\tilde{x}} = -\tilde{x}$$
$$\dot{\tilde{y}} = -a\tilde{y} \qquad a>0$$

WITH $\tilde{x}, \tilde{y}$ EIGENKOORDINATES




$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} v_x & v_y \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

IN d/ $a>1$ IS LARGER THAN a IN e/.

IN f/ $a<1$

2c FROM FIGS $v_x \hat{=} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_y \hat{=} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -a \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -a \\ -1 & -2a \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4-a & -2-2a \\ -2-2a & -1-4a \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

PROBLEM 3

FIND $A, B$ : $e^{A+B} \neq e^{A} e^{B}$

TAKE $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$$e^{A+B} = \begin{bmatrix} e & e-e^2 \\ 0 & e^2 \end{bmatrix} \quad \text{USING}$$

$$A+B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e^{A+B} = \begin{bmatrix} & \\ & \end{bmatrix}^{-1} \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix}.$$

PROBLEM 4

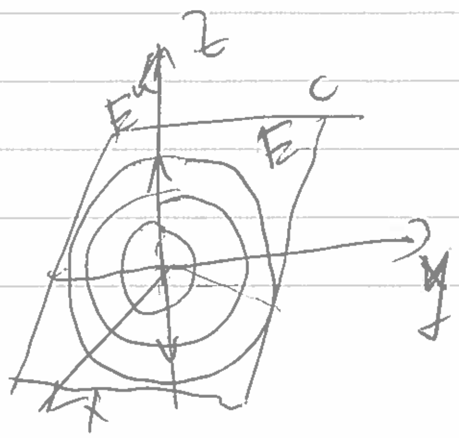
$$A = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{bmatrix}$$

$\lambda = 6, \pm 2i$

$\lambda = 6$ : $v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = \pm 2i$ : $v_1 = \begin{bmatrix} -3 \pm i \\ -1 \mp 3i \\ 1 \end{bmatrix}$, $\mathrm{Re}\, v = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$

$\mathrm{Im}\, v = \pm \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$



PROBLEM 5

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $x_1 = t x_{20} + x_{10}$, $x_2 = x_{20}$

$|x_1| \xrightarrow[t \to \infty]{} \infty$ FOR $x_{20} \neq 0$.

PROBLEM 6

$\dot{x} = A x$, $\dot{x} = K^{-1} x'$ IF $\dot{()} = \frac{d}{dt}$

$()' = \frac{d}{d\tau}$, $\frac{d\tau}{dt} = K^{-1}$.

THUS

$x' = K A x$

PROBLEM 7

$\varphi_t = e^{At}$.

i/ $\varphi_0 = e^{A0} = I$

ii/ $\varphi_t(\varphi_s) = e^{At}(e^{As}) = e^{A(t+s)}$

SINCE $[At, As] = 0$

iii/ $\varphi_{-t}(\varphi_t) = I$ BY ii,