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# Week 6

## On today

In week 1 we gave a definition of stability for $1D$ systems. Today we present a general definition of stability of an equilibrium point $x = 0$ of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$. We will discuss sufficient and necessary conditions for stability based on linearization $\dot{x} = Ax$, $A = Df(0)$. The definition of stability is formulated in terms of the flow and it therefore also applies directly to the discrete case where time is discrete: $t \in \mathbb{Z}$. We will use this in exercise 4 below where we work on stability of fix points of maps. We illustrate the role of maps on a simple continuous dynamical system in exercise 5.

## NB!

I aim to give a short lecture (of length $\approx 1$hr) so you will have time to reach exercise $5$!

## Read

Section 2.9 of [Per00] and for exercise 5 below look through section 3.4 up until end of p. 214. Definition 1 on p. 202 defines stability of periodic orbits. But it is perhaps simpler to (equivalently) define stability through the fix point of the associated Poincaré map. We will address this in exercise 4 (which you might want to read before the lecture) and in exercise 5 below.

## Key words

Stability.

## Exercises

**Exercise 1:** Problem 2 of problem set 9 on p. 134 of [Per00]. NB! For (a) you do not even need to find the equilibria. Can you see why? :) In (b) classify each equilibria according to the classification: stable (unstable) node, saddle, and stable (unstable) focus. Also for (c) (if you have the steam) indicate $E^s$ and $E^u$ for each equilibrium in the plane. Try to sketch the phase portrait (possibly using Maple for assistance).

**Exercise 2:** Problem 3 of problem set 9 on p. 135 of [Per00].

**Exercise 3:** Consider $\dot{x} = f(x)$ and let $f(0) = 0$. Suppose that $W^s(0) = \mathbb{R}^n$. Is it true that $x = 0$ is asymptotically stable? If not can you sketch a planar counter-example?

**Exercise 4:** First some **background**: Consider a diffeomorphism

$$f : P \to P.$$

Then: **Definition**: A discrete dynamical system is defined by the flow:

$$\phi_n(x) = f^n(x), \quad x \in P, n \in \mathbb{Z},$$

obtained from iterates of $f$. This map satisfies the usually flow-properties: $\phi_{n+m} = \phi_n \circ \phi_m$. **WHY?**

**Definition**: A *fix-point* $x_*$ "fixes" $f$: $f(x_*) = x_*$. A fix-point $x_*$ is hyperbolic as long as no eigenvalues of $A = Df(x_*)$ lie on the unit circle in the complex plane. That is:

$$\text{all eigenvalues } \lambda \text{ of} \quad A = Df(x_*) \quad \text{satisfy:} \quad |\lambda| \neq 1.$$

The stability of a fix-point of a discrete system is completely analogous to the continuous version.

**Example**: $f(x_1, x_2) = (x_2, x_1/4 + x_2 - x_2^3)$ is a diffeomorphism on $\mathbb{R}^2$ with a single fix-point: $(1/2, 1/2)$. **WHY?**

Now to the exercise: $x \mapsto Ax$ with $A$ invertible defines a linear discrete dynamical system through $\phi_n(x) = A^n x$. Suppose first that $A$ is semi-simple. Then:

- Show that $x = 0$ is stable if and only if $|\lambda| \leq 1$.
- Show that $x = 0$ is asymptotically stable if $|\lambda| < 1$.

**-question (exercise 5 is more important!): Can you describe stability for general $A$'s, not necessarily semi-simple?

Further **background**: **Theorem**: Suppose $f(0) = 0$. Then $x = 0$ is

- asymptotically stable if all eigenvalues of $Df(0)$ satisfy $|\lambda| < 1$.
- unstable if there exists an eigenvalue of $Df(0)$ satisfying $|\lambda| > 1$.

**Example**. The fix-point $(1/2, 1/2)$ of $f$ above is asymptotically stable since the eigenvalues are

$$\frac{1}{8} \pm \frac{1}{8}\sqrt{17},$$

which are inside the unit circle.

**Exercise 5:** Consider the system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - 2\beta y + \gamma \cos(\theta) + \epsilon f(x, y, \theta), \\ \dot{\theta} &= \omega, \end{aligned} \tag{1}$$

with $\beta$, $\gamma$, $\omega$ positive constants and $f$ some smooth function periodic in its third argument. Such systems appear frequently in structural mechanics. In particular, the system with $\epsilon = 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - 2\beta y + \gamma \cos(\theta), \\ \dot{\theta} &= \omega, \end{aligned} \tag{2}$$

models a rigid body, positioned at $x$ with velocity $y$, under the influence of a linear restoring force (spring), damping (strength measured by $\beta$), and external periodic forcing (strength measured by $\gamma$, frequency measured by $\omega$). We shall in this problem primarily focus on (2). But eventually, in (f*) below, we will apply results on (2) to establish the existence of periodic orbits of (1) for small values of $\epsilon$.

**NB!** The right hand side of (2) is periodic in $\theta$ so we can therefore treat $\theta \in [0, 2\pi)$ as an angle, identifying $0$ with any multiple of $2\pi$.

**NB!** The analysis and the calculations can easily be repeated for any $\beta > 0$ and $\gamma > 0$ but for simplicity you may focus on

$$\beta = \gamma = \omega = 1.$$

Let $\phi_t(x, y, \theta)$ be the flow of (2).

(a) Solve the linear system and find a $\Psi_t(x, y, \theta)$ so that

$$\phi_t(x, y, \theta) = (\Psi_t(x, y, \theta), \omega t + \theta).$$

Do not worry about solving for integration constants in terms of initial conditions. Just write $\Psi_t$ as

$$\Psi_t(x_0, y_0, \theta_0) = e^{At} \begin{pmatrix} c_1(x_0, y_0, \theta_0) \\ c_2(x_0, y_0, \theta_0) \end{pmatrix} + \begin{pmatrix} x_{par}(t) \\ y_{par}(t) \end{pmatrix},$$

determining $e^{At}$, $x_{par}$ and $y_{par}$.

**Hint**: To solve the system consider first the homogeneous system:

$$\dot{x} = y,$$
$$\dot{y} = -x - 2\beta y,$$

and apply Theorem 1 p.33. For the particular solution it may be useful to write the system as a second order system:

$$\ddot{x} + x + 2\beta\dot{x} = \gamma\cos(\omega t + \theta_0).$$

(b) Use (a) to find a periodic orbit. **Hint**: Recall that $\theta$ is an angle.

(c) Show that this orbit attracts every orbit of the system as $t \to \infty$. **Hint**: What happends to the homogeneous part of the solution in (a)?

In the following we will approach the study of periodic orbits of (2) from another perspective that generalizes to situations where $\phi_t$ is not explicitly known.

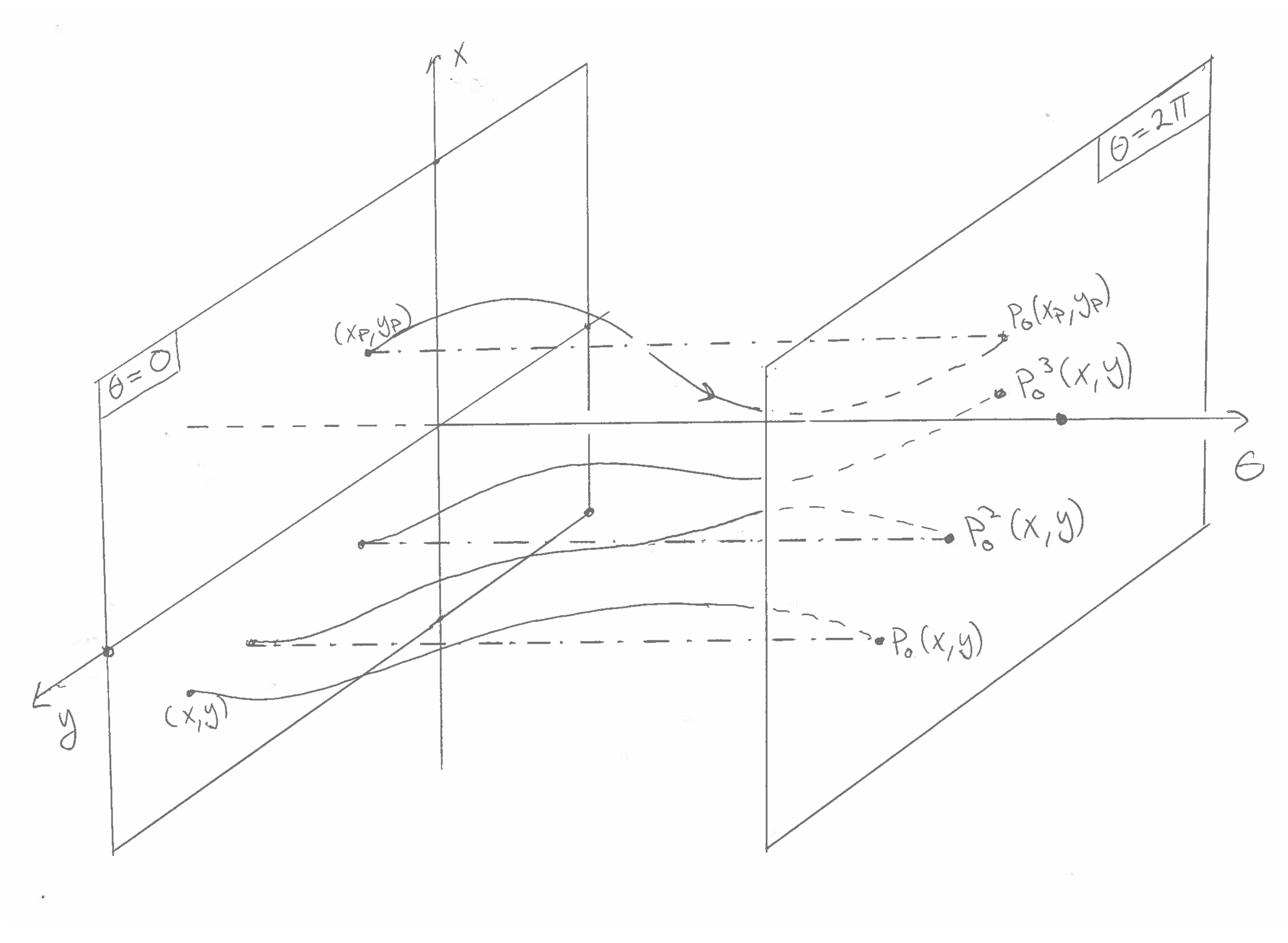


Figure 1: Illustration of the map $P_0$. Note that the planes $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ coincides upon identification of $\theta$ as an angle.

Let

$$P_0 : (x, y) \mapsto \Psi_{2\pi/\omega}(x, y, 0). \tag{3}$$

The mapping $P_0$ is an example of a Poincare map (or stroboscopic mapping). It maps initial conditions $(x, y, 0)$ within the $\{\theta = 0\}$-plane to final conditions $(P_0(x, y), 2\pi)$ (since time in (3) is $2\pi/\omega$)) within the $\{\theta = 2\pi\}$-plane. But since $\theta$ is an angle, and we identify $0$ with multiples of $2\pi$, $P_0$ effectively defines a mapping from $\{\theta = 0\}$ to itself. See Fig. 1. This replaces the continuous-time dynamical system with a discrete one for which the periodic orbit becomes a fix-point:

(d) Denote the intersection of the periodic orbit from (b) with $\theta = 0$ by $(x, y, \theta) = (x_p, y_p, 0)$. Find $(x_p, y_p)$ and show that $(x_p, y_p)$ is a fixed point of $P_0$ so that $P_0(x_p, y_p) = (x_p, y_p)$. **Hint**: Just put $\theta = 0$ in (a).

(e) Show that $P_0^n(x, y) \to (x_p, y_p)$ for every $(x, y) \in \mathbb{R}^2$ as $n \to \infty$. **Hint**: You can use exercise $4$.

(f*) Use (d) and (e), the smooth dependence of flow map on parameters (see [Per00, Theorem 2, p. 84]), and the implicit function theorem, to show that that (1) has a periodic orbit for $\epsilon$ sufficiently small. **Additional hint**: Define a Poincare mapping $P_\epsilon(x, y)$ of (1). Then apply the implicit function theorem to

$$F(x, y, \epsilon) = P_\epsilon(x, y) - (x, y),$$

using that $F(x_p, y_p, 0) = 0$ and properties of the Jacobian $DP_0(x_p, y_p)$.

If this exercise has drawn your interest, it is worth mentioning that one of the projects in $DS1.5$ : January course 01257 is about (1) and bifurcations of periodic orbits!

## Next week

Next week (course week 7) we will do our mid-term evaluation, summarize the first six course weeks and have our 1hr mid-term test. The test will be without aids; the questions will therefore only involve minor calculations. We will start at 3pm in Building 303, Room 225. To prepare for the test: Make sure that you are on top of all the weekly exercises.

# References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.