LECTURE NOTES:

1D SYSTEMS

IN THIS NOTE WE CONSIDER ODES OF THE FORM

$$\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}$$

WITH $f \in C^1$ - FUNCTION.

WE SHALL OFTEN SUPPRESS "(t)" AND JUST WRITE

$$\dot{x} = f(x), \quad x \in \mathbb{R}.$$

WE ALSO CONSIDER THE ASSOCIATED INITIAL VALUE PROBLEM (IVP)

$$\dot{x} = f(x), \quad x(0) = x_0 \in \mathbb{R} \qquad (1)$$

WITH x_0 GIVEN.

IN A ODE - COURSE YOU HAVE BEEN TAUGHT TO SOLVE (1) BY SEPARATION OF VARIABLES:

$$dx = f(x)\,dt, \quad f(x) \neq 0 \implies$$

$$\int_{x_0}^{x} \frac{dx}{f(x)} = \int_0^t dt = t \implies$$

WRITE AS $\psi(x, x_0) = t$ ②
FOR SOME $\psi \longrightarrow$ INVERT ψ
TO OBTAIN $x = \phi(t, x_0)$.

REMARK: LET x_* BE SO THAT $f(x_*) = 0$. THEN x_* IS SAID TO BE AN EQUILIBRIUM.

NOTE: IF $x_0 = x_*$, THEN $x(t) = x_*$ IS CONSTANT!

EXAMPLE

CONSIDER $\dot{x} = \sin(x)$, $x(0) = x_0$.

THEN FOR $x \neq n\pi$, $n \in \mathbb{Z}$ WE HAVE

$$\int_{x_0}^{x} \frac{du}{\sin u} = \int_0^t ds = t$$

EVALUATING THE LEFT HAND SIDES GIVES

$$\ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right| = t$$

... !! TIDEOUS

NOT ONLY IS THE APPROACH TIME-CONSUMING IT IS ALSO INCAPABLE (TO A LARGE EXTEND) TO DEAL WITH QUESTIONS LIKE

Q: WHAT IS $\lim_{t \to \infty} x(t)$ IF $x(0) = \pi/4$?

(THESE ISSUES BECOME EVEN MORE PROFOUND WHEN WE CONSIDER HIGHER DIMENSIONS WHERE MOST FREQUENTLY NO EXPLICIT SOLUTION-FORM EXISTS!)

REMARK

THE SYSTEM $\dot{x} = f(x, t)$ IS A NON-AUTONOMOUS SYSTEM. IT DEPENDS EXPLICITLY UPON TIME.

IT IS NOT 1D! IT IS 2D! LET $x_1 = x$, $x_2 = t$, THEN

$$\left.\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= 1 \end{aligned}\right\}$$ THIS IS 2D AUTONOMOUS SYSTEM

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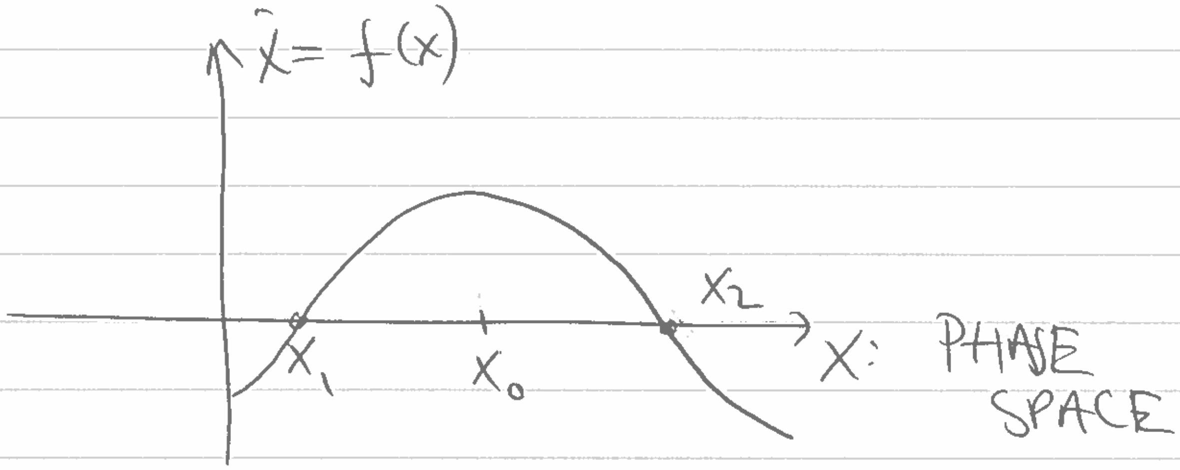
PHASE - SPACE GEOMETRY

HOW CAN WE ANSWER QUESTIONS LIKE Q ABOVE?

CONSIDER $\dot{x} = f(x)$, $x(0) = x_0$ AGAIN. SUPPOSE THE GRAPH OF $f(x)$ IS LIKE IN THE FIGURE BELOW.



THEN INTERSECTIONS WITH THE X-AXIS CORRESPOND TO EQUILIBRIA: TWO x_1 AND x_2 ARE SHOWN ABOVE.

NOW THINK OF $x(t)$ = POSITION OF A PARTICLE AT TIME t. THEN $\dot{x}(t)$ = VELOCITY OF PARTICLE.

LET x_0 BE AS IN THE FIGURE, THEN $\dot{x} > 0$ AND THE PARTICLE MOVES TO THE RIGHT. CLEARLY (?)

$$x(t) \xrightarrow[t \to \infty]{} x_2$$

$$x(t) \xrightarrow[t \to -\infty]{} x_1$$

DEFINITION: AN ORBIT OF $\dot{x}=f(x)$
IS A SET

$$\mathcal{O}_{x_0} = \{ x(t) \mid x(t) \text{ SOLVES } (1) \}$$

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EXERCISE:
HOW MANY DIFFERENT ORBITS ARE THERE IN THE FIGURE ON P.4?

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STABILITY FOR 1D SYSTEMS

LET x_* : $f(x_*)=0$. (x_* IS AN EQUILIBRIUM).

DEFINITION:

a/ x_* IS STABLE IF THERE EXISTS $\varepsilon_0>0$: $\forall x_0 \in [x_*-\varepsilon, x_*+\varepsilon]$, $\varepsilon \le \varepsilon_0$
THEN $x(t)$ OF (1) SATISFIES
$x(t) \in [x_*-\varepsilon, x_*+\varepsilon]$

b/ x_* IS ASYMPTOTICALLY STABLE IF x_* IS STABLE AND
$x(t) \xrightarrow[t\to\infty]{} x_*$ FOR $\forall x_0 \in [x_*-\varepsilon, x_*+\varepsilon]$
$\varepsilon \le \varepsilon_0$

c/ UNSTABLE IF NOT STABLE.

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REMARK

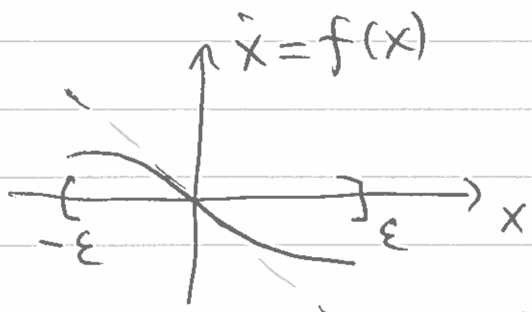
CARE WHEN GENERALISING TO $\mathbb{R}^n$ — SEE WEEK 6.

THM: LET $f(x_*)=0$. THEN

a/ IF $f'(x_*)<0$ THEN x_* IS ASYMPTOTICALLY STABLE

b/ IF $f'(x_*)>0$ THEN x_* IS UNSTABLE $\square$

PROOF: SUPPOSE W.L.O.G. $x_*=0$. CONSIDER FIRST a/. THEN THERE EXISTS AN $\varepsilon>0$ SUCH THAT THE GRAPH OF f LOOKS LIKE THE FIGURE BELOW IN $x\in[-\varepsilon,\varepsilon]$:



IN PARTICULAR $\text{sign}(f(x)) = -\text{sign}(x)$ $\forall x\in[-\varepsilon,\varepsilon]\setminus\{0\}$. $\Longrightarrow$ $x=0$ STABLE

x_* IS ASYMPTOTICALLY STABLE SINCE: $x(t)$ FOR $x_0>0$ $(x_0<0)$ IS

MONOTONICALLY DECREASING (INCREASING) AND BOUNDED FROM BELOW (ABOVE) BY $X=0$ $\Longrightarrow$ LIMIT $\lim_{t\to\infty} x(t) = 0$ EXISTS.

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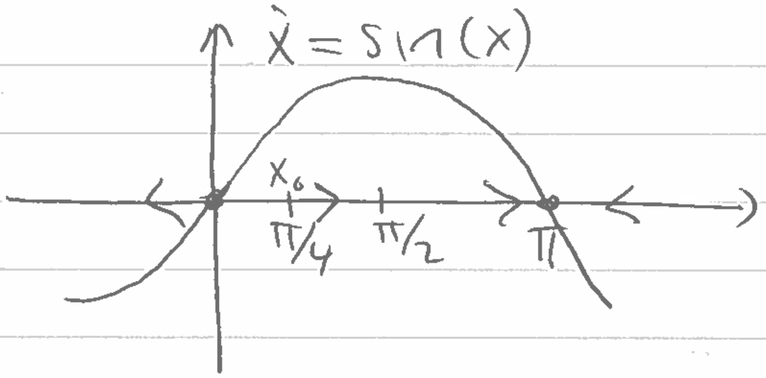
EXAMPLE

$\dot{x} = \sin(x)$ AGAIN. LET $x(0) = \pi/4$.
WE WILL NOW ANSWER QUESTION Q.



FROM THE GRAPH WE DIRECTLY DEDUCE $x(t) \xrightarrow[t\to\infty]{} \pi$ FOR $x(0) = \pi/4$.

NOTE: $X = \pi$ IS ASYMPTOTICALLY STABLE WHEREAS $X=0$ IS UNSTABLE.

REMARK: $X=\pi$ IS SAID TO BE A SINK WHEREAS $X=0$ IS CALLED A SOURCE

REMARK: DS IS A QUALITATIVE THEORY. IT DOES NOT ADDRESS QUESTIONS LIKE: (8)

WHAT IS TIME WHEN $x(t) = \pi/2$ GIVEN THAT $x(0) = \pi/4$?