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> "In the midst of this perplexity, I received from Oxford the manuscript you have examined. I lingered, naturally, on the sentence: *I leave to the various futures (not to all) my garden of forking paths*. Almost instantly, I understood: 'the garden of forking paths' was the chaotic novel; the phrase 'the various futures (not to all)' suggested to me the forking in time, not in space."
>
> - J. L. Borges [*The Garden of Forking Paths*]

Week 9

Bifurcations-I - Perko 4.2

It often happens that the equations describing a system contain one or more parameters. Different values of these parameters may result in different system types, in particular different sets of equilibrium points. A change from one system type to another is called a bifurcation. The study of these phenomenae is known as bifurcation analysis.

Read

Perko, section 4.2. Review Implicit Function Theorem.

Keywords

Saddle-node bifurcation, transcritical bifurcation, pitchfork bifurcation, normal forms.

Exercises

You should have time to complete these exercises before the werewolves begin to howl.

Exercise 1: (Perko, Problem 1) Consider the one-dimensional system

$$\dot{x} = -x^4 + 5\mu x^2 - 4\mu^2$$

Determine the critical points and the bifurcation value for this ODE. Draw phase portraits for various values of μ and construct the bifurcation diagram.

Exercise 2: Classify the 1D systems below in terms of which bifurcations they undergo when μ is varied. For each, specify the value of μ for which the bifurcation takes place, and sketch the bifurcation diagram:

$$(a)\ \dot{x} = \mu - 3x^2 \qquad (b)\ \dot{x} = \mu x - \frac{x}{1+x} \qquad (c)\ \dot{x} = 5 - \mu\exp(-x^2)$$
$$(d)\ \dot{x} = \mu x - \frac{x}{1-x^2} \qquad (e)\ \dot{x} = x + \tanh\mu x \qquad (f)\ \dot{x} = \mu x + \frac{x^2}{1+x^2}$$

Exercise 3: This exercise (from Strogatz, Chapter 3) looks at a model of insect populations. Sometimes conditions change and a local insect population undergoes a bifurcation to a new stable equilibrium. Being able to predict this is vital for agriculture. The equation is a modified version of the *logistic equation*. The variable $x(t) \geq 0$ gives a dimensionless measure of the insect population size.

$$\dot{x} = \mu x\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2} \tag{1}$$

The parameter μ is the (dimensionless) population growth rate (which can, for instance, increase with climate change, increased use of fertilizers or by a natural increase in conditions beneficial for the insects) and k is the so-called carrying capacity (a measure of how many insects the local enviroment can maximally sustain).

(a) Show that the value $x = 0$ is always an unstable stationary point for (1).

(b) Show that, for a large value of k, there can, in addition to $x = 0$ be three stationary points: A stable stationary point for small values of x, an unstable stationary point for larger values of x, and finally a stable stationary point near the value $x = k$. This high-x point represents a situation with a catastrophically high insect population, known as an *outbreak*. Argue that, when μ is increased, the middle unstable stationary may merge with the low-x stationary and the pair vanish in a saddle-node bifurcation, causing the outbreak point to be the only stationary point.(Hint: use a graph method to visualize the position of the three stationary points).

(c) The condition for the saddle-node bifurcation in (b) occuring at a given x-value is that

$$\mu(1 - \frac{x}{k}) = \frac{x}{1+x^2} \tag{2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu(1 - \frac{x}{k}) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{1+x^2} \tag{3}$$

Use these equations to isolate $\mu = \mu(x)$ and $k = k(x)$ for a given x. Use MAPLE to plot in a (k, μ) diagram values of $\mu(x)$ and $k(x)$. Follow a line with fixed k and increasing μ. Where is the transition to an insect outbrak occur ?

Next week

Next week we will continue with bifurcations. Read Perko sections 3.3 and 4.4.