# Assignment 5

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## 1 Question 1: State space formulation

In this problem, the trajectory of a satellite is observed, taking 50 measurements of the radius and the angle. The measurement are poisoned with withe noise, and it can be described as follow:

$$r_t^m = r_t^p + \epsilon_{rt}$$
$$\theta_t^m = \theta_t^p + \epsilon_{\theta t}$$

where $\epsilon_r t \sim N(0, 2000^2)$, $\epsilon_{\theta t} \sim N(0, 0.03^2)$ and $\epsilon_r t$, $\epsilon_{\theta t}$ are independent.

The model of the trajectory, can be described with the following stochastic, equations:

$$r_t^p = r_{t-1}^p + \varepsilon_{rt}^p$$
$$\theta_t^p = \theta_{t-1}^p + v_{t-1}^p + \varepsilon_{\theta t}^p$$
$$v_t^p = v_{t-1}^p + \varepsilon_{v_\theta t}^p$$

where $r_t^p$ is the real radius, $\theta_t^p$ is the angle, and $v_{\theta t}^p$ is the angular velocity. The model describe a circular orbit with some perturbations, which are modeled by the following distributions:

$$\varepsilon_{rt}^p \sim N(0, 500^2), \quad \varepsilon_{\theta t}^p \sim N(0, 0.005^2), \quad \varepsilon_{v_\theta t}^p \sim N(0, 0.005^2)$$

With the given information we can construct the stochastic space model as

$$X_t = AX_{t-1} + e_1$$

$$Y_t = CX_t + e_2$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$X_t = \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} = \begin{pmatrix} r_t^p \\ \theta_t^p \\ v_{\theta t}^p \end{pmatrix}, \quad Y_t = \begin{pmatrix} r_t^m \\ \theta_t^m \end{pmatrix} = \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix}$$

$$e_1 \sim N(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}500^2 & 0 & 0\\ 0 & 0.005^2 & 0\\ 0 & 0 & 0.005^2\end{bmatrix}), \quad e_2 \sim N(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2000^2 & 0\\ 0 & 0.03^2\end{bmatrix}))$$

The third state, the satellite velocity, is not directly measured. However, thanks to the correlation between the velocity and the angle this variable is possible to estimate. Precisely, the Kalman filter implemented in the next section address, among others, this problem.

## 2 Question 2: Kalman filter implementation

Below it can found the implementation of the kalaman filter. It is based in the formulas provides in the slides from lecture 11.

```
############## Kalaman filter initalization
X_estimated <- matrix(0,nrow = 3, ncol = 60)
stored_K <- matrix(0,nrow = 3, ncol = 60)
stored_klist <- list()
stored_SigmaXX <- list()

Xhat <- matrix(c(38000,0,0.0269356),nrow=3)
SigmaXXhat<- matrix(c(1000,0,0,0,0,0,0,0,0),nrow = 3)
SigmaYYhat <- matrix(c(1,0,0,1),nrow = 2)
SigmaXYhat <- SigmaXXhat%*%t(C)

for(n in 1:50){
  #estimation
  #K = SigmaXYhat%*%solve(SigmaYYhat)
  K = SigmaXXhat%*%t(C)%*%solve(SigmaYYhat)
  Xestim = Xhat + K%*%(Y[,n]-C%*%Xhat)
  SigmaXX <- SigmaXXhat - K%*%SigmaYYhat%*%t(K)

  #Storing data
  X_estimated[,n] <-Xestim
  stored_SigmaXX[[n]] <- SigmaXX

  #prediction
  Xhat <- A%*%Xestim
  SigmaXXhat <- A%*%SigmaXX%*%t(A) + Sigma1
  SigmaYYhat <- C%*%SigmaXXhat%*%t(C) + Sigma2
  SigmaXYhat <- SigmaXXhat%*%t(C)
}
```

An important issue to point is the initialization of the filter. The chosen starting point is:

$$X_t = \begin{pmatrix} X_{1,t=0} \\ X_{2,t=0} \\ X_{3,t=0} \end{pmatrix} = \begin{pmatrix} 38000 \\ 0 \\ 0.026 \end{pmatrix}$$

The starting radius and angle are extracted from the first measurement. On the other hand, the angular velocity is calculated for the given radius. It is the necessary velocity for the satellite to be on that specific orbit.

## 3 Question 3: Trajectory reconstruction

The Kalman filter is used for reconstruct every of the states. In the following graphs are presented the estimation plus the 95% confidence interval. Worth to point out, how the confidence interval do not cover all the sampled data. This is because Kalman filter give the best estimation possible with the available and model, minimizing the uncertainty. Also, notice how for radius and angle are perfect from the very beginning, since our starting point was a good point.

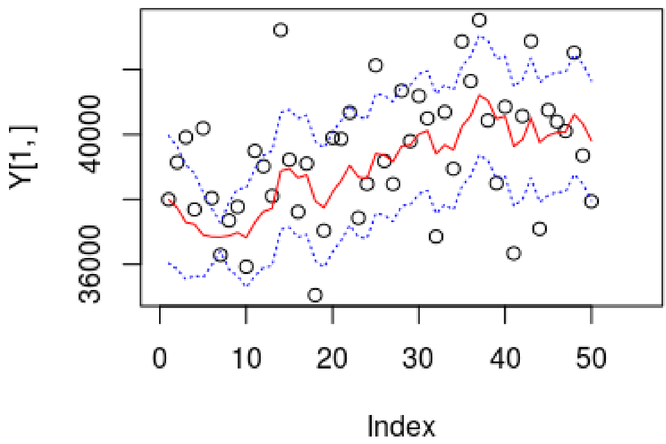


Figure 1: Radius reconstruction

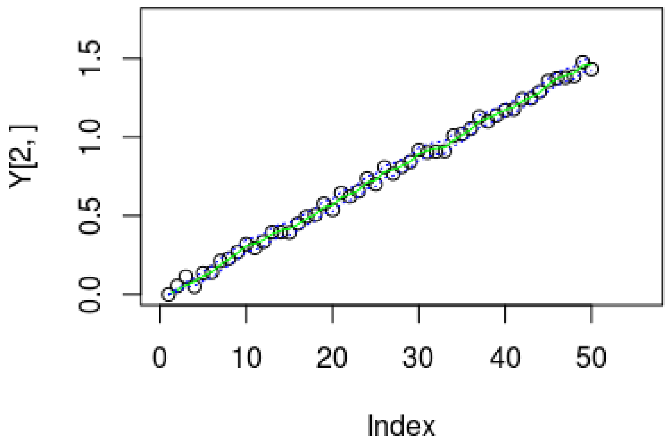


Figure 2: Angle reconstruction

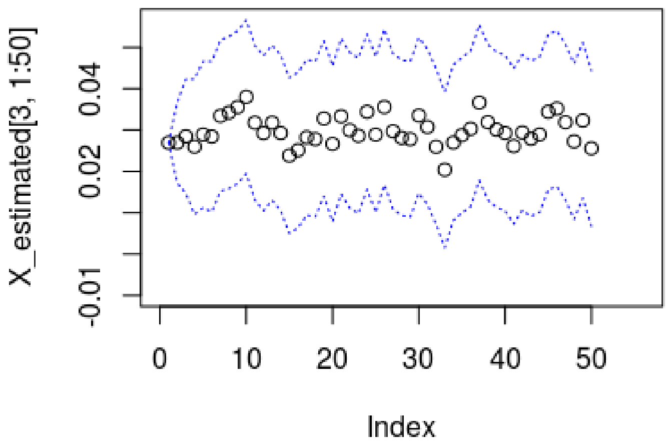


Figure 3: Angular velocity estimation

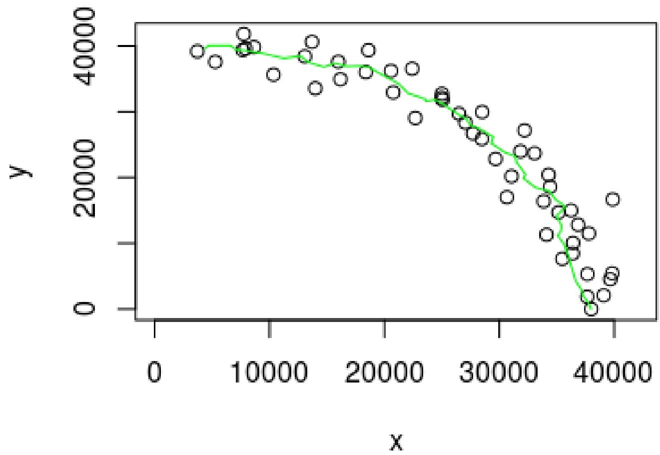


Figure 4: Estimated trajectory

# 4 Question 4: Predictions

The code written for predicting is the following:

```
num_pred = 6  #ahead steps predictions
Xhat = matrix(X_estimated[,50], ncol = 1)
SigmaXXhat <- stored_SigmaXX[[50]]
SigmaXX_pred[[1]] <- SigmaXXhat
for(n in 2:(num_pred+1)){
  Xhat <- A%*%Xhat
  SigmaXXhat <- A%*%SigmaXXhat%*%t(A) + Sigma1
}
```

The prediction is based in a recursive 1-step predictions. This is something that is already done during Kalmana filtering. The difference is that for more than one step prediction, the predicted state depends on the previous prediction, rather than the state estimation, for which is needed new measurements. Basically, we are doing predictions based completely in the developed model. For the variance prediction, normal variance propagation is used. In all the following graphs the prediction is plotted plus the 95 % confidence interval. From the graphs we can see how the uncertainty grow, the further we want to predict.

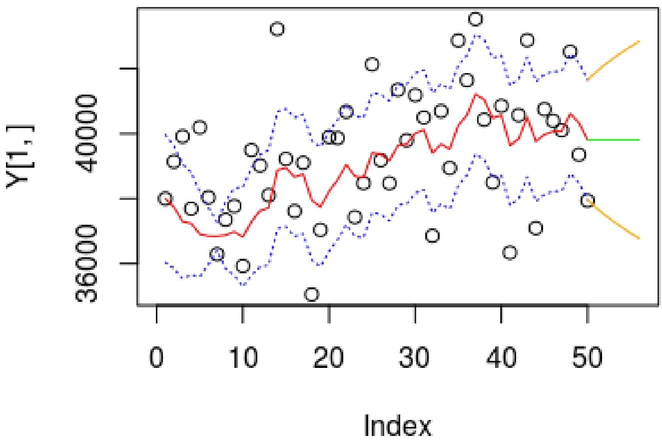


Figure 5: Radius prediction

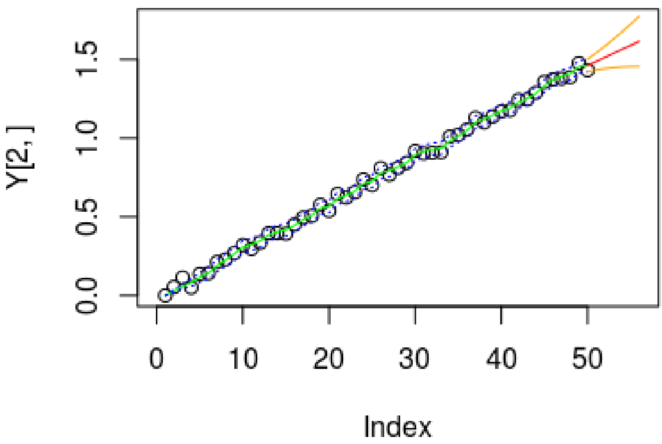


Figure 6: Angle prediction

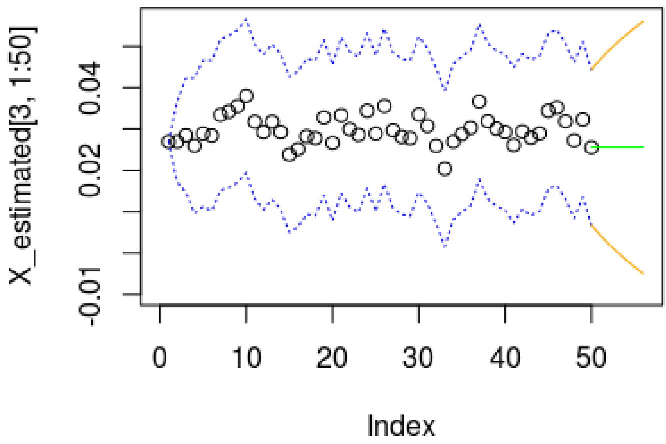


Figure 7: Angular velocity prediction

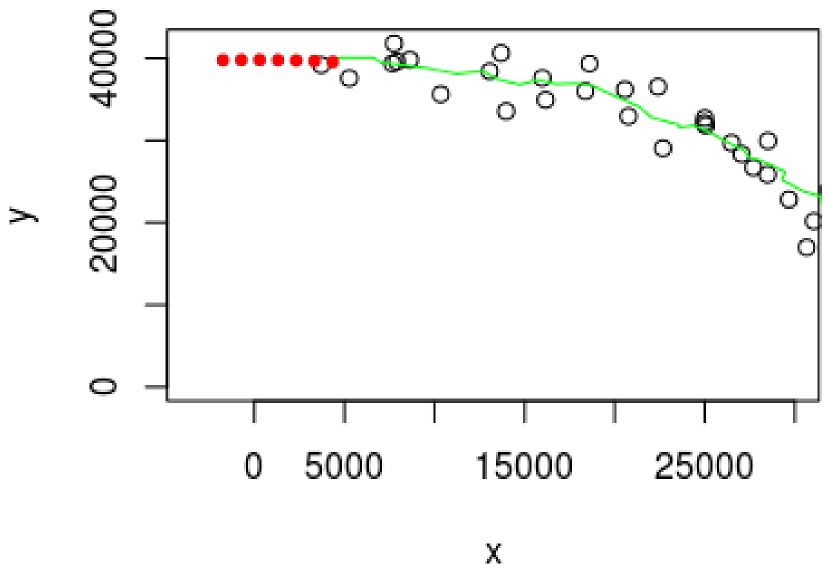


Figure 8: Multi-step prediction

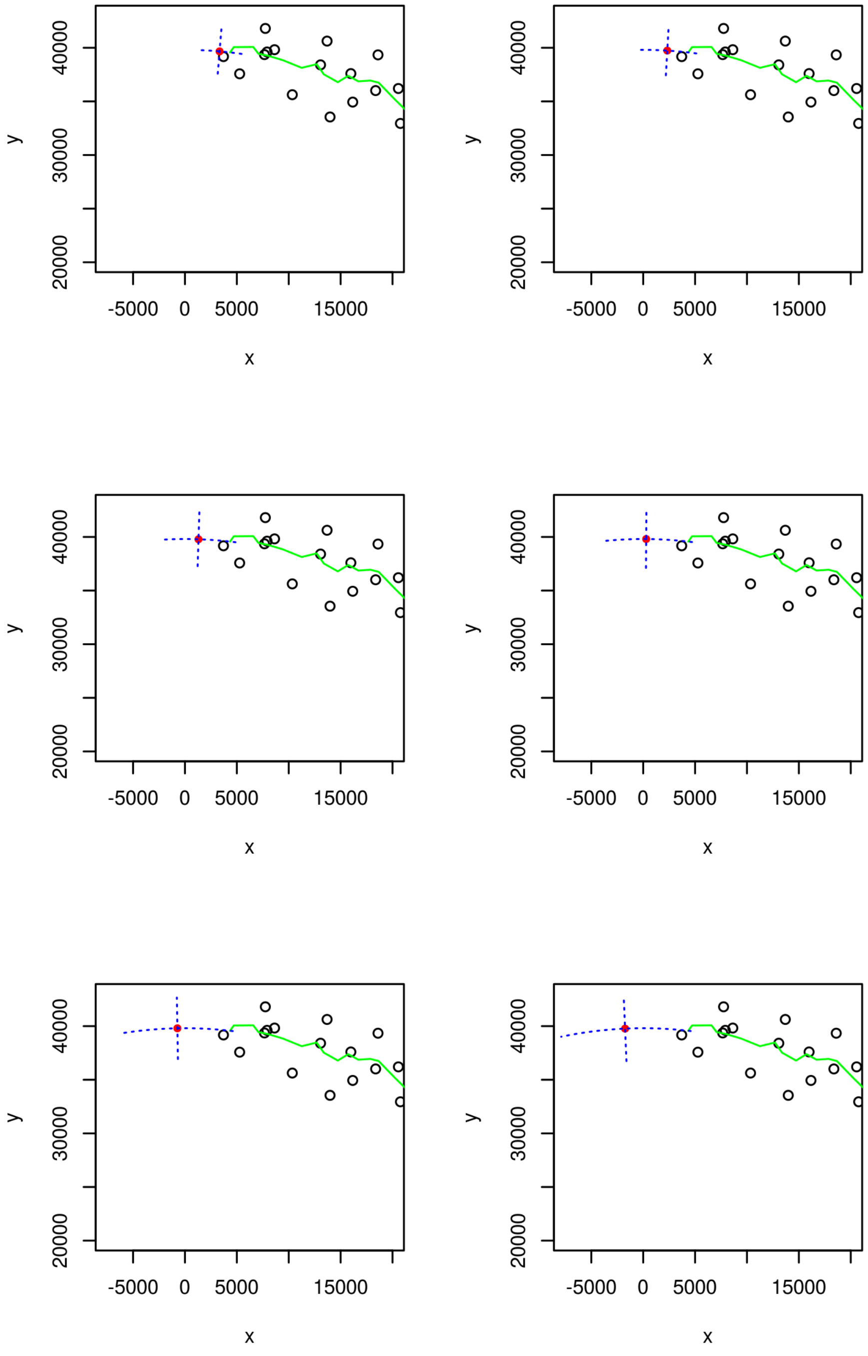


Figure 9: Step predictions