Dynamical Systems 1
Course 01617
E16

Poul G. Hjorth and Kristian Uldall Kristiansen
Institut for Matematik og Computer Science
Technical University of Denmark

Assignment 2

Note: Please include references when applying results of [Per00].

Problem 1: Consider the 2-dimensional system of non-linear ODEs

$$\begin{aligned} \dot{x} &= \alpha x(1-x) - xy, \\ \dot{y} &= y(x-y) - \beta y, \end{aligned}$$

with

$$x \geq 0, \quad y \geq 0. \tag{1}$$

This is a simplified instance of a predator-prey population model, where x is the scaled population of prey and y the scaled population of the predator. The parameter α represents the growth rate of the prey and β the starvation rate of the predator population.

In the following, set

$$\alpha = 1 \quad \text{and} \quad \beta = 1/2.$$

Then:

(a) Compute all equilibrium points within (1) and determine their stability. For all equilibrium points in (a) of saddle type, compute the invariant subspaces E^s and E^u of their linearization.

(b) Draw a sketch in Cartesian coordinates showing:

(i) The location of all equilibrium points;

(ii) the local dynamics around the equilibrium points as described by the linearization;

(iii) and finally highlighting for the saddles the associated linear subspaces E^s and E^u.

(c) Why does $x(0) > 0$ and $y(0) > 0$ imply that $x(t) > 0$ and $y(t) > 0$ for all $t \geq 0$?

(d) For initial conditions as in (c), why does the additional condition $x(0) < 1$ imply that $x(t) < 1$ for all $t \geq 0$?

(e) Do initial conditions satisfying (c) and (d) exist such that one or both species go extinct as $t \to \infty$? Explain.

(f) From your results above, sketch the phase portrait on the unit square $[0,1] \times [0,1]$, highlighting, in particular, the stable and unstable manifolds for each of the saddle equilibria. **Hint**: You may use software packages such as Maple's phaseportrait command for assistance.

Problem 2: Consider the following system:

$$\dot{x} = \epsilon x - x^3 + xy, \tag{2}$$

$$\dot{y} = -y + y^2 - x^2, \tag{3}$$

with $0 < \epsilon \ll 1$.

(a) Show that $(x,y) = (0,0)$ is hyperbolic. Apply the Hartman-Grobman theorem to sketch the *local* phaseportrait.

We will in the following study (2) for every (x, y, ϵ) within

$$U_\delta \equiv \{(x,y,\epsilon) | \epsilon > 0, \ \sqrt{x^2+y^2+\epsilon^2} < \delta\},$$

with $\delta > 0$.

(b) Use center manifold theory to show that there exists a $\delta > 0$ so that

$$y = m(x, \epsilon), \quad \text{for} \quad (x, y, \epsilon) \in U_\delta,$$

with m smooth and satisfying $m(0,0) = 0$, $Dm(0,0) = 0$, is an attracting center manifold. **Hint**: Augment ϵ as a dynamic variable $\dot{\epsilon} = 0$. Consider the equilibrium $(x, y, \epsilon) = (0,0,0)$ of the extended system and compute E^c.

It can be shown that $m(x, \epsilon)$ actually takes the following form:

$$m(x, \epsilon) = x^2(c + \epsilon m_1(\epsilon) + x^2 m_2(x^2, \epsilon)), \tag{4}$$

for some constant c and smooth functions m_1 and m_2.

(c) Find c.

(d) Reduce to the center manifold and obtain an ODE for x only. **You may ignore m_1 and m_2 in (4) when you reduce to the center manifold**. Analyse the resulting system and illustrate your findings by sketching the phase portrait in the (x, y)-plane for $(x, y, \epsilon) \in U_\delta$.

(e) How does your findings in (d) compare with (a)? Discuss the differences.

References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.