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Assignment 1

Note: Please include references when applying results of [Per00].

Problem 1: You have a friend. He is an experimentalist, but not strong in mathematics. He has made an experiment measuring a non-constant quantity x as a function of time. He says that $x(t)$ is periodic:

Definition: An orbit $O = \{x(t)|t \in \mathbb{R}\}$ is periodic if there exists $T > 0$: $x(t+T) = x(t) \in O$ for all t.

Your friend would like to have you, as a student studying mathematics, build a $1D$ dynamical model $\dot{x} = f(x)$ with f smooth for $x = x(t)$. You tell him: "I am afraid that is not possible. I'll tell you why..." Write down a proof why the observed $x(t)$ cannot satisfy a $1D$ model $\dot{x} = f(x)$.

Problem 2: Consider a mass connected to a spring. Then Newton's law gives $m\ddot{x}(t) = -kx(t)$, $x(t)$ measuring the deformation of the spring. Show that every non-constant orbit is periodic. Why is this not in contradiction with your conclusions in problem 1?

Problem 3: Consider the following equation

$$\dot{x}(t) = ax(t)(1 - K(t)^{-1}x(t)), \tag{1}$$

with $K(t)$ a periodic function: There exists a $T > 0$ such that $K(t+T) = K(t)$ for all $t \in \mathbb{R}$. Here x represents a population size. The system with $K(t) \equiv K_0 > 0$ *constant*:

$$\dot{x}(t) = ax(t)(1 - K_0^{-1}x(t)), \tag{2}$$

is called the logistic model. In this situation the parameter $a > 0$ is called the growth rate while $K_0 > 0$ is called the carrying capacity. The system (1) reflects the fact that environmental conditions (such as different seasons) influence the carrying capacity.

Now

(a) Sketch the phase portrait of (2), list all the different orbits of the system, and finally: describe every possible limit of $x(t)$ as $t \to \infty$.

(b) Sketch the graph of $x(t)$ for different initial conditions.

(c) For system (1) with $K(t+T) = K(t)$ it can be shown that there exists a periodic solution of (1). You do not need to show this but explain why this does not contradict your findings in Problem 1.

Problem 4: Consider the linear system of ODEs $\dot{x}(t) = -x(t), \dot{y}(t) = -ay(t)$, $a \in (0, \infty)$.

(a) Prove that for $a > 1$ all orbits, corresponding to initial conditions satisfying $x(0) \neq 0$, approach the origin tangentially to the x axis. Hint: One way to do this is to write the orbits as graphs. To do this solve the linear system and eliminate time.

(b) How do solutions curves with $x(0) = 0$ approach the origin?

(c) What happens for $a = 1$ and $0 < a < 1$? Sketch the phase portraits in the three separate cases $0 < a < 1$, $a = 1$, $a > 1$.

References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.