Assignment 2

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Question 1

Let the process Xt be given by

$$X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} = \epsilon_t$$

where ϵ_t is a white noise process. Investigate analytically for which values of ϕ_2 the process is stationary when $\phi_1 = 0.75$. In addition it should be investigated for which values of ϕ_2 the autocorrelation function shows damping harmonic oscillations. Still for $\phi_1 = 0.75$.

To study the dynamics of the system, we have to solve the difference equation. For this concern, we can rewrite the equation as:

$$X_{t+2} + \phi_1 X_{t+1} + \phi_2 X_t = \epsilon_t$$

If we guess one of the solutions as λ^t then we have:

$$\lambda^t(\lambda^2 + \phi_1 \lambda + \phi_2) = \epsilon_t$$

and then the characteristic polynomial will give as the rest of the roots:

$$(\lambda^2 + \phi_1 \lambda) + \phi_2) = 0$$

$$\lambda = \frac{-\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_2}}{2}$$

Then we have 3 possibles scenarios in base of the last equation:

- **Distinct real roots** $\phi_1^2 > 4\phi_2$ where the general solution will look like $A\lambda_1^t + B\lambda_2^t$
- **Repeated real roots** $\phi_1^2 = 4\phi_2$ where the general solution will look like $(A + Bt)\lambda^t$
- **Complex Roots** $\phi_1^2 < 4\phi_2$ where the general solution will look like $Ar^t cos(\theta t + \omega)$

In any of the cases we can clearly see how $|\lambda| > 1$ will result in exponential growth, while $|\lambda| < 1$ will decay and converge at a stationary state. Hence we have to find the values of ϕ_2 given $\phi_1 = 0.75$ which fulfill:

$$|\lambda| < 1$$

The solution is $-\frac{7}{4} < \phi_2 \leq \frac{9}{64}$

Damping of correlation function As exaplained in the therem 5.10 of the books, the autocovariance function, of the AR(2) process, satisfy the following difference equation:

$$\gamma(k) + \phi_1 \gamma(k-1) + \phi_2 \gamma(k-2) = 0$$

with initial condistions:

$$\gamma(0) + \phi_1 \gamma(1) + \phi_2 \gamma(2) = \sigma_\epsilon$$

Our functions to solve is exactly the same that we did in the first part of the exercise. We can then assume that for $\phi_1^2 < 4\phi_2$ we get complex roots, hence damped behaviour in the function.

For $\phi_2 > \frac{9}{64}$ the autocorrelation function is damped.

Question 2

$$(1 - 0.8B)(1 - 0.2B^6)(1 - B)Y_t = \epsilon_t$$

The equation can be expanded as:

$$(-0.16B^8 + 0.36B^7 - 0.2B^6 + 0.8B^2 - 1.8B + 1)Y_t = \epsilon_t$$

and then removing the B operator:

$$Y_t = 0.16Y_{t-8} - 0.36Y_{t-7} + 0.2Y_{t-6} - 0.8Y_{t-2} + 1.8Y_{t-1} + \epsilon_t$$

Then the estimation of Y will be:

$$\hat{Y}_{11} = 0.16Y_3 - 0.36Y_4 + 0.2Y_5 - 0.8Y_9 + 1.8Y_{10} = -5.44$$

The variance of the estimation is the variance of the error:

$$Var[Y_{11} - \hat{Y}_{11}] = Var[\epsilon_{11}] = \sigma_\epsilon = 0.31$$

The 95% confidence interval is given by:

$$-5.44 \pm 1.96\sqrt{0.31} = [-6.53, -4.34]$$

For prediction at time t=12 we use the estimation at time t=11 as we do not have measurment.

$$\hat{Y}_{12} = 1.8\hat{Y}_{11} + 0.16Y_4 - 0.36Y_5 + 0.2Y_6 - 0.8Y_{10} = -6.43$$

Now the variance:

$$\begin{aligned} Var[Y_{11} - \hat{Y}_{11}] &= Var[1.8\hat{Y} + \epsilon_{12}] \\ &= (1.8^2 + 1)\sigma_\epsilon \\ &= 1.31 \end{aligned}$$

The 95% confidence interval is given by:

$$-6.43 \pm 1.96\sqrt{1.31} = [-8.68, -4.18]$$

Question 3

$$Y_t = \frac{22}{7} + \sum_{i=0}^{t} \epsilon_i$$

3.1 Mean, variance and covariance functions

$$E[Y_t] = E[\frac{22}{7}] + E[\epsilon_0] + E[\epsilon_1] + E[\epsilon_2] + .. = \frac{22}{7} + 0 + 0... = \frac{22}{7}$$

$$Var[Y_t] = Var[\frac{22}{7}] + Var[\epsilon_0] + Var[\epsilon_1] + Var[\epsilon_2] + ... = 0 + \sigma_\epsilon + \sigma_\epsilon + \sigma_\epsilon ... = t\sigma_\epsilon$$

Since each sequence of white noise is uncorrelated:

$$\begin{aligned}\gamma(Y_t, Y_{t-1}) &= Cov[Y_t, Y_{t-1}] \\ &= E[\frac{22}{7} + \sum_{i=0}^{t} \epsilon_i, \frac{22}{7} + \sum_{i=0}^{t-1} \epsilon_i] \\ &= (t-1)\sigma_\epsilon\end{aligned}$$

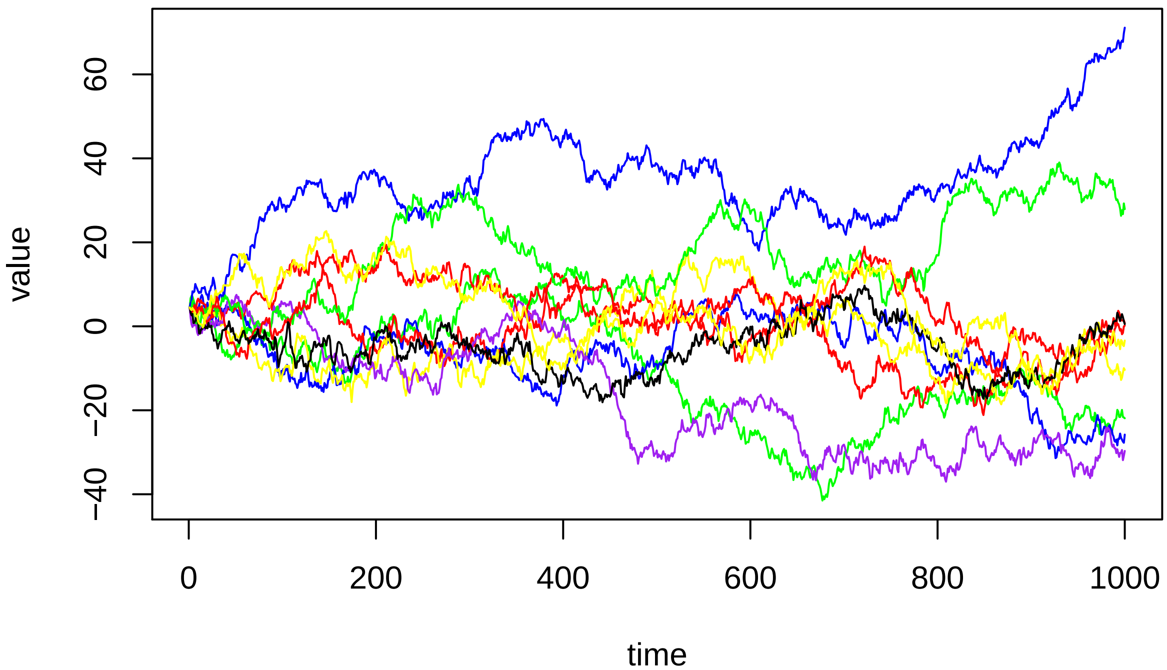
3.2 Stationarity We can state that the process is not stationary since the variance is not constant on time.

3.3 and 3.4 Simulated data

For simulating concerns we can rewrite the random walking functions as:

$$Y_t = \frac{22}{7} + Y_{t-1} + \epsilon_t$$

With a loop we can easily generate new values of Y.



As we can see the variance of the process grow linearly. However, the mean remains constant.