Dynamical Systems 1
Course 01617
E15

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Solution of Problem 4 in problem set 5 on p. 26 in [Per00]

[No need to include the problem text but you may need to write the problem with your own words to make sure that it is clear where and how your answer relates to the questions posed]
We consider $\dot{x} = Ax$ with

(a) (Semi-simple) $A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ with $\lambda \neq 0$.

(b) (Nilpotent) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(c) (Trivial) $A = 0$

all satisfying

$$\det A = 0.$$

We will determine the solution and the corresponding phase portraits for each of the systems in the following.

Case (a)

We can write $\dot{x} = Ax$ as

$$\dot{x}_1 = \lambda x_1,$$
$$\dot{x}_2 = 0.$$

Hence $x_1(t) = e^{\lambda t} x_{10}$ and $x_2(t) = x_{20}$ so that

$$x(t) = e^{At} x_0 = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & 1 \end{pmatrix} x_0.$$

The x_2-axis is a line of equilibria. There is only dynamics in the x_1-direction. For $\lambda > 0$ every point contracts horizontally to an equilibrium on the x_2-axis as $t \to -\infty$. This gives the phaseportraits shown below. The phaseportrait for $\lambda < 0$ is similar: The arrows are just reversed since every point contracts horizontally to an equilibrium on the x_2-axis as $t \to \infty$.

Case (b)

We can write $\dot{x} = Ax$ as

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = 0.$$

Hence $x_2(t) = x_{20}$ and therefore $x_1(t) = x_{10} + t x_{20}$ so that

$$x(t) = e^{At} x_0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0.$$

Alternatively: A is nilpotent since $A^2 = 0$. Therefore $A^k = 0$ for all $k \geq 2$ and hence by the definition of e^{At}; see [Per00, Definition 2, p.12], we have

$$x(t) = e^{At} x_0 = (I + At + 0) x_0 = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0.$$

Here the x_1-axis is a line of equilibria. Since $x_1(t) = x_{10} + t x_2$ with x_2 constant, it follows that the dynamics for $x_2 \neq 0$ is purely in the x_1-direction. It moves to the right for $x_2 > 0$ and to the left for $x_2 < 0$. The speed increases with increasing values of $|x_2|$. This gives the phaseportraits shown below.

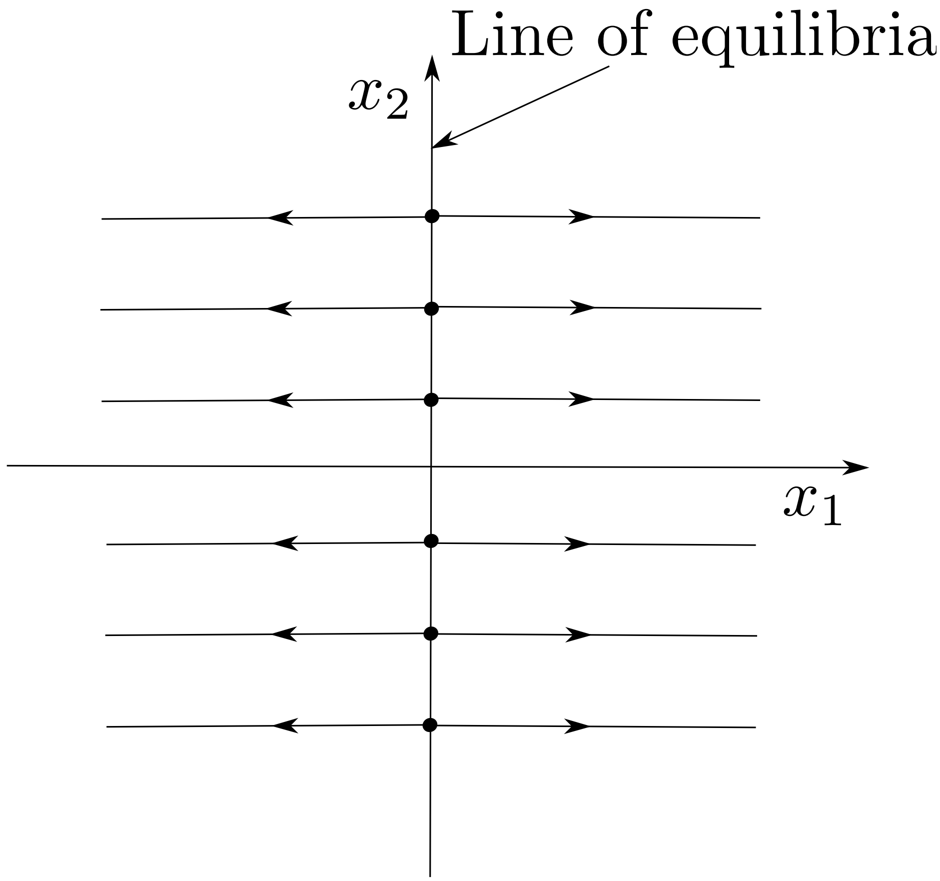

Figure 1: Phase portrait for case (a) with $\lambda > 0$. The x_2-axis is a line of equilibria. The case with $\lambda < 0$ is obtained by reversing the arrows.

Case (c)

In this case

$$x(t) = x_0,$$

for all $x_0 \in \mathbb{R}$ and all t. The whole plane is a set of equilibria. There is no dynamics and the phaseportrait is trivial. Each orbit is a single point.

References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.

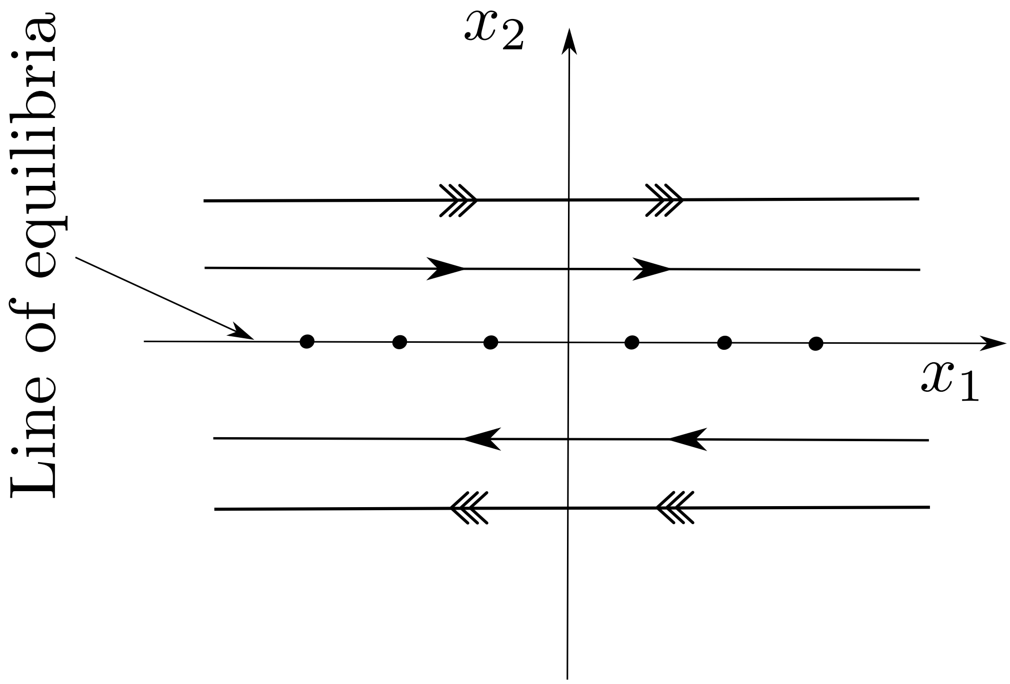

Figure 2: Phase portrait for case (b). The x_1-axis is a line of equilibria.

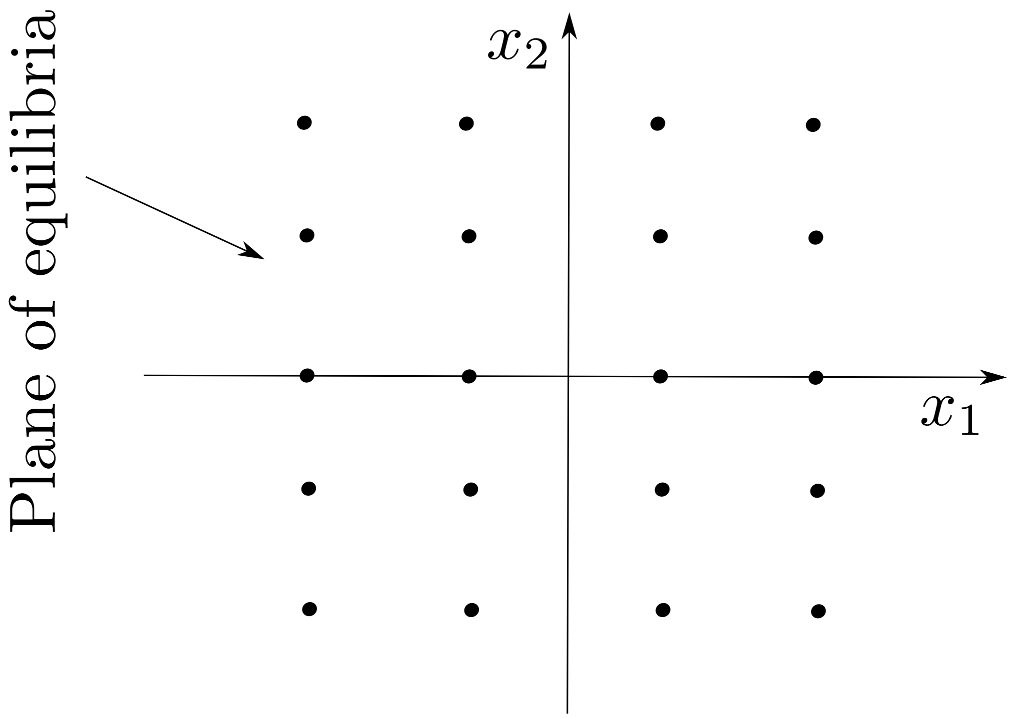

Figure 3: The trivial phase portrait for case (c). The whole plane is a set of equilibria.