WEEK 4: LINEARIZATION, HYPERBOLICITY, STABLE AND UNSTABLE MANIFOLDS

LOCAL THEORY: WEEK 4–10

$\dot{x} = f(x) \quad (1)$, $f(x_*) = x_*$ EQUILIBRIUM, $x \in B_\varepsilon(x_*)$
$= \{x \mid |x - x_*| < \varepsilon\}$

NOTE: $\varphi_t(x_*) = x_*$: FIXED POINT OF φ_t

DEFINITION: LINEARIZATION OF (1) AROUND EQUILIBRIUM $x = x_*$ IS THE LINEAR SYSTEM

$$(2) \quad \dot{x} = Ax, \quad A = Df(x_*).$$

SUPPOSE $x_* = 0$. IF NOT REPLACE x BY $x - x_*$. THEN

$$f(x) = f(0) + Df(0)x + \mathcal{O}(x^2)$$
$$= 0 + Ax + \mathcal{O}(x^2)$$

HENCE (2) FIRST ORDER/LEADING ORDER APPROX

CENTRAL QUESTION: HOW "CLOSE" IS (2) TO (1)?

DEFINITION: LET $\lambda_1, \ldots, \lambda_n$ BE THE EIGENVALUES (WITH MULTIPLICITY) OF $A = Df(x_*)$.

THEN:

x_* IS HYPERBOLIC IF

$$\mathrm{Re}\,\lambda_i \neq 0 \quad \forall i$$

DEFINITION: IF x_* IS HYPERBOLIC

THEN:

1/ x_* IS A SINK IF $\mathrm{Re}\,\lambda_i < 0 \ \forall i$

2/ x_* IS A SOURCE IF $\mathrm{Re}\,\lambda_i > 0 \ \forall i$

3/ x_* IS A SADDLE IF $\exists i, j$:

$$(\mathrm{Re}\,\lambda_i)(\mathrm{Re}\,\lambda_j) < 0.$$

EXAMPLE: $\dot{x}_1 = x_1^2 - x_2^2 - 1,$

$\dot{x}_2 = 2x_2.$

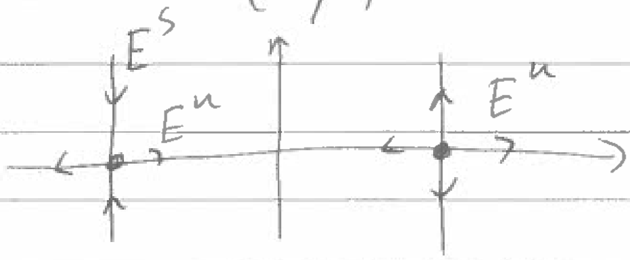
EQUILIBRIUM: $x_2 = 0$, $x_1^2 = 0 \iff x_1 = \pm 1$

$$Df(x) = \begin{bmatrix} 2x_1 & -2x_2 \\ 0 & 2 \end{bmatrix}. \quad \text{THEREFORE}$$

$$Df(\pm 1, 0) = \begin{bmatrix} \pm 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

THEN $(1,0)$: SOURCE $\lambda = 2$ an (2)

$(-1,0)$: SINK $\lambda = \pm 2$.



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NONLINEAR VERSIONS OF $E^{s,u}$?
MANIFOLDS S, U OF THE SAME DIMENSION,
TANGENT TO $E^{s,u}$.

MANIFOLD $M \subset \mathbb{R}^n$ IS A K-DIM SMOOTH MANIFOLD IF $\forall x \in M$ $\exists\, U \subset \mathbb{R}^n$ OPEN SUCH THAT: $\exists\, h: B \subset \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$ SMOOTH EXPRESSING $n-k$ COORD IN TERMS OF REMAING K COORD: $M \cap U = \text{GRAPH}(h)$

EXAMPLES: 1/ $S^1 = \{x^2+y^2=1\}$ IS A 1D SMOOTH MANIFOLD! EACH POINT HAS A NEIGHBORHOOD U: $S \cap U$ IS DESCRIBED BY THE GRAPHS $x = \pm\sqrt{1-y^2}$, $y \in (-1,1)$,
$y = \pm\sqrt{1-x^2}$, $x \in (-1,1)$.

2/ ALL GRAPHS ARE MANIFOLDS
$y = h(x)$, $x \in U$ OPEN
$x = g(y)$, $y \in U$ OPEN

3/ $M = \{xy = 0\} \subset \mathbb{R}^2$ NOT MANIFOLD 

4/ SUPPOSE $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ SMOOTH.

IF $Df = [\partial_x f \;\; \partial_y f] \neq 0$ THEN

$M = \{f(x,y) = 0\}$ 1D SMOOTH MANIFOLD

PROOF: $x_0 \in M$: $f(x_0) = 0$, $\partial_y f(x_0) \neq 0$ (WLOG)
$\Longrightarrow$ IFT $\exists a$: $h: I(x_0) \equiv [-a, a] \rightarrow \mathbb{R}$ SMOOTH SATISFYING $y_0 = h(x_0)$, $f(x, h(x)) = 0$ $\forall x \in I(x_0)$
$\Longrightarrow$ $M \cap B_a(x_0) = \text{GRAPH}(h) \cap B_a(x_0)$

DEFINITION: STABLE SET $W^s(0) = \{x \mid \varphi_t(x) \xrightarrow[t\to\infty]{} 0\}$.
UNSTABLE SET $W^u(0) = \{x \mid \varphi_t(x) \xrightarrow[t\to-\infty]{} 0\}$

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THM (p. 107)

SUPPOSE: $x=0$ HYPERBOLIC EQ.
$\exists k$ EIGENVALUES $\mathrm{Re}\,\lambda < 0$
$\exists (n-k)$ — " — $\mathrm{Re}\,\lambda > 0$

THEN $\exists$ K-DIM (n-k-DIM) SMOOTH MANIFOLD S (U):

- S (U) IS TANGENT TO E^s (E^u) AT $\underline{x=0}$
- S (U) IS FORWARD (BACKWARD) INVARIANT:

$$\varphi_t(S) \subset S \quad \forall t \geq 0$$

$$\left(\varphi_t(U) \subset U \quad \forall t \leq 0 \right)$$

- $\exists N \subset \mathbb{R}^n$, $0 \in N$:

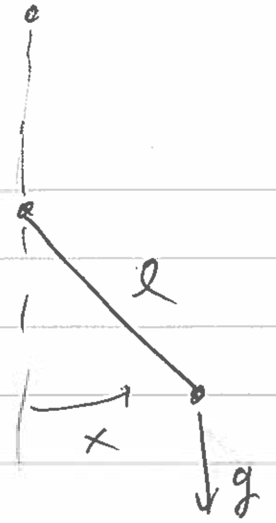
$$S = \{x \in W^s(0) \mid \varphi_t(x) \in N \quad \forall t \geq 0\}$$

$$\left(U = \{x \in W^u(0) \mid \varphi_t(x) \in N \quad \forall t \leq 0\} \right)$$

NOTE: OFTEN $S = W^s_{loc}(0)$
$U = W^u_{loc}(0)$

EX. p. 2

EX



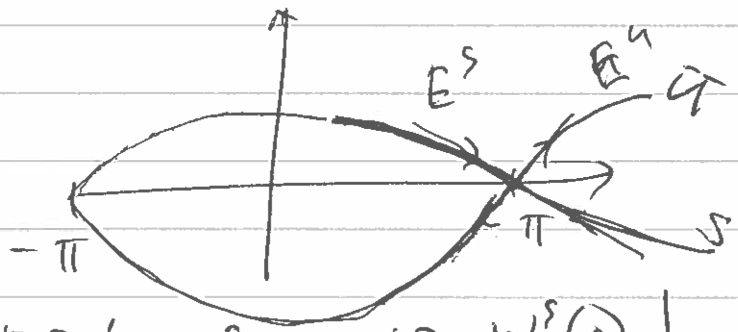
$$\dot{x} = y$$
$$\dot{y} = -\frac{g}{l}\sin x$$

EQ: $(x=0, y=0)$, $x=\pi, y=0$

$$A = Df(\pi,0) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix}$$

$\lambda = \pm\sqrt{\frac{g}{l}}$: SADDLE

$v = \begin{bmatrix} 1 \\ \pm\sqrt{\frac{g}{l}} \end{bmatrix}$



DIFFERENCE BETWEEN S AND $W^s(0)$!

EX LET $\tilde{S}, \tilde{U}$ BE ST-, UST-MANIFOLDS OF $\dot{\tilde{x}} = f(\tilde{x})$, $\hat{x} = 0$

CONSIDER $\dot{x} = -f(x)$ $(t \mapsto -t)$.

THEN $U = \tilde{S}$,
$S = \tilde{U}$.

EX

COR (P. 115)

LET $Re(\lambda_1), \ldots, Re(\lambda_k) < -\alpha$.
THEN $\forall \varepsilon\ \exists \delta : \forall x_0 \in B_\delta(0) \cap S$

$$|\varphi_t(x_0)| \le \varepsilon e^{-\alpha t}.$$

GLOBAL MANIFOLDS

S, U ONLY LOCALLY DEFINED.

~~DEFINITION: GLOBAL ST. MAN~~

THEOREM

$$W^s(0) = \{ \varphi_t(x) \mid \underset{t \le 0}{x \in S} \}$$

~~PROOF~~

$$W^u(0) = \{ \varphi_t(x) \mid x \in U,\ t \ge 0 \}$$

PROOF:

LET $x_0 \in W^s(0) \Rightarrow \exists T : \varphi_t(x_0) \subset N \quad \forall t \ge T.$

$\Rightarrow \varphi_t(x_0) \in S \quad \forall t \ge T.$

$\Rightarrow W^s(0) \subset RHS$

$RHS \subset W^s(0)$ TRIVIAL.

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φ_t SMOOTH $\Rightarrow$ $\varphi_t(S)$ SMOOTH k-DIM MANIFOLD.

BUT WHAT ABOUT $t \to -\infty$?

HOW CRAZY/WILD CAN W^s BE?

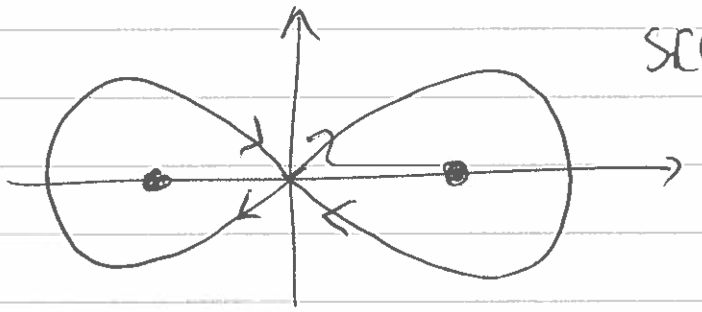
EX DUFFING EQN (BJÆLKE UDBØJNING)

$$\dot{x} = y$$

$$\dot{y} = x - x^3 + \varepsilon(\mu\cos(\theta) - 2.5y)$$

$$\dot{\theta} = 1$$

FOR $\varepsilon = 0$:



SELFINTERSECTION!

FOR $\varepsilon \ll 1$: MATLAB (3D)

LET $u(t) = \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix}$, $V(t) = \begin{bmatrix} 0 & 0 \\ 0 & e^{Qt} \end{bmatrix}$

$\dot{u} = Bu$, $\dot{V} = BV$, AND $e^{Bt} = u + V$.

SHOW: $\|e^{Pt} y_s\| \le Ke^{-at}|y_s|$, $\|e^{-Qt} y_u\| \le Ke^{-at}|y_u|$

LEMMA: $\dot{y} = By + \gamma(t)$. SUPPOSE $\gamma(t)$ CONT, BOUNDED FOR $t \ge 0$.
THEN $\forall y_{s0} \in \mathbb{R}^k \; \exists! \; y(t; y_{s0})$ SOLUTION WHICH IS BOUNDED FOR ALL $t \ge 0$:

$$y(t; y_{s0}) = u(t) \begin{bmatrix} y_{s0} \\ 0 \end{bmatrix} + \int_0^t u(t-s)\gamma(s)ds$$
$$- \int_t^\infty V(t-s)\gamma(s)ds$$

PROOF: USE $y(t) = e^{Bt}\xi(t)$ TO SHOW THAT

$$y(t) = e^{Bt} y_0 + \int_0^t e^{B(t-s)}\gamma(s)ds.$$

$$y_s(t) = e^{Pt} y_{s0} + \int_0^t e^{P(t-s)}\gamma_s(s)ds$$

SHOW y_s IS BOUNDED.

$$y_u(t) = e^{Qt} y_{u0} + \int_0^t e^{Q(t-s)}\gamma_u(s)ds$$
$$= e^{Qt}\left[y_{u0} + \int_0^t e^{Q(-s)}\gamma_u(s)ds\right]$$

$\|e^{Qt}\| \to \infty$ AS $t \to \infty$. FOR BOUNDEDNESS

A NECESSARY COND IS THEREFORE:

$$y_{u0} + \int_0^\infty e^{Q(-s)}\gamma_u(s)ds = 0.$$

SHOW THAT ~~IT~~ IS SUFFICIENT!

HINT: $y_u(t) = e^{Qt}\left[-\int_0^\infty e^{-Qs}\gamma_u(s)ds + \int_0^t e^{-Qs}\gamma_u(s)ds\right]$

$$= -\int_0^\infty e^{Q(t-s)}\gamma_u(s)ds.$$

FIXED POINT ARGUMENT

LET

$T: C^0(\mathbb{R}_+;\mathbb{R}^n) \longrightarrow C^0(\mathbb{R}_+;\mathbb{R}^n)$

BE

$$(Ty)(t) = e^{Bt}\begin{bmatrix} y_{s0} \\ 0 \end{bmatrix} + \int_0^t U(t-s)\, g(y(s))ds - \int_t^\infty V(t-s)\, g(y(s))ds$$

LET $W = \{y \in C^0(\mathbb{R}_+) \mid \|y\| < \delta\}$ CLOSED

SINCE g IS QUADRATIC: $\forall \varepsilon\ \exists \delta$:

$|g(y(s))| \leq \varepsilon |y(s)| \quad \forall y(s) \in W$

SHOW THIS.

TAKE $|y_{s0}| \leq \delta/2K$, $\varepsilon \leq \alpha/4K$ AND

SHOW THAT $T: W \longrightarrow W$.

NEXT: SHOW THAT T IS A CONTRACTION

USING $|g(x) - g(y)| \leq \varepsilon |x-y| \quad \forall x,y \in B_\delta(0)$

$\Rightarrow$ UNIQUE FIX POINT $y(t; y_{s0})$ OF T. ∎