stepsize $\Delta t = 0.1$. The solutions have the shape expected from Section 2.3. ■

Computers are indispensable for studying dynamical systems. We will use them liberally throughout this book, and you should do likewise.

EXERCISES FOR CHAPTER 2

2.1 A Geometric Way of Thinking

In the next three exercises, interpret $\dot{x} = \sin x$ as a flow on the line.

2.1.1 Find all the fixed points of the flow.

2.1.2 At which points x does the flow have greatest velocity to the right?

2.1.3

a) Find the flow's acceleration $\ddot{x}$ as a function of x.
b) Find the points where the flow has maximum positive acceleration.

2.1.4 (Exact solution of $\dot{x} = \sin x$) As shown in the text, $\dot{x} = \sin x$ has the solution $t = \ln\left|(\csc x_0 + \cot x_0)/(\csc x + \cot x)\right|$, where $x_0 = x(0)$ is the initial value of x.

a) Given the specific initial condition $x_0 = \pi/4$, show that the solution above can be inverted to obtain

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).$$

Conclude that $x(t) \to \pi$ as $t \to \infty$, as claimed in Section 2.1. (You need to be good with trigonometric identities to solve this problem.)

b) Try to find the analytical solution for $x(t)$, given an *arbitrary* initial condition x_0.

2.1.5 (A mechanical analog)

a) Find a mechanical system that is approximately governed by $\dot{x} = \sin x$.
b) Using your physical intuition, explain why it now becomes obvious that $x^* = 0$ is an unstable fixed point and $x^* = \pi$ is stable.

2.2 Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1 $\dot{x} = 4x^2 - 16$

2.2.2 $\dot{x} = 1 - x^{14}$

2.2.3 $\dot{x} = x - x^3$

2.2.4 $\dot{x} = e^{-x}\sin x$

2.2.5 $\dot{x} = 1 + \frac{1}{2}\cos x$

2.2.6 $\dot{x} = 1 - 2\cos x$

2.2.7 $\dot{x} = e^x - \cos x$ (Hint: Sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still find the qualitative behavior.)

2.2.8 (Working backwards, from flows to equations) Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers—and wrong ones too.)

Figure 1

2.2.9 (Backwards again, now from solutions to equations) Find an equation $\dot{x} = f(x)$ whose solutions $x(t)$ are consistent with those shown in Figure 2.

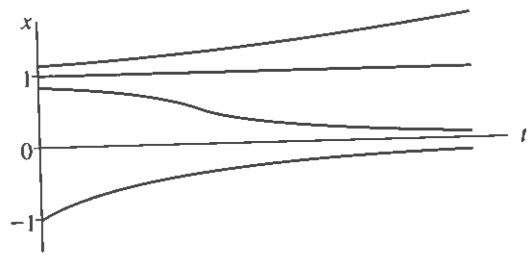

Figure 2

2.2.10 (Fixed points) For each of (a)–(e), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

a) Every real number is a fixed point.
b) Every integer is a fixed point, and there are no others.
c) There are precisely three fixed points, and all of them are stable.
d) There are no fixed points.
e) There are precisely 100 fixed points.

2.2.11 (Analytical solution for charging capacitor) Obtain the analytical solution of the initial value problem $\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$, with $Q(0) = 0$, which arose in Example 2.2.2.

2.2.12 (A nonlinear resistor) Suppose the resistor in Example 2.2.2 is replaced by a nonlinear resistor. In other words, this resistor does not have a linear