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Week 2

On today

Today we will talk about linear systems

$$\dot{x} = Ax. \tag{1}$$

Linear systems appear naturally in the study of real-life nonlinear problems through the process of *linearization* (see Definition 1 on p. 102). We will solve linear systems: first in the simple case where A can be diagonalized (semisimple case), and then introduce matrix exponentials to obtain a compact and general solution representation. We will discuss ways to compute the matrix exponential, also in the non-semisimple case, and introduce the following spaces: the stable space E^s, the unstable space E^u and the center space E^c. We will discuss the role of these spaces in the geometry of the phase space.

Read

Sections 1.4-1.7 and 1.9 in [Per00].

Key words

Linear systems, matrix exponential, semisimple and nilpotent, invariant subspaces.

Exercises

Exercise 1 Consider the $2D$ linear system

$$\dot{x} = \begin{pmatrix} -1 & a \\ ab & -1 \end{pmatrix},$$

with parameters $a > 0$ and $b \in \mathbb{R}$. Read section 1.5 pp. 20-24 and note the classification by saddles, nodes, foci and centers. Do the following (in any order):

(a) Solve the system for all a and b.

(b) Divide the (a, b)-plane into regions where $x = 0$ is (i) a saddle, (ii) a node, and (iii) a focus.

(c) Sketch all of the distinct phase portrait, highlighting the invariant spaces E^s, E^u and E^c. Use Maple's phaseportrait command! A template available on Campusnet.

Exercise 2 Consider the phase portraits of the $2D$ linear systems in the figure below.

(a) Classify each of the phase portraits according to the classification in section 1.5.

(b) Explain the difference between (d), (e) and (f).

(c) Try to write down simple equations that qualitatively match figures (d), (e) and (f).

Exercise 3 Do the following:

(a) Find two matrices A and B so that

$$e^{A+B} \neq e^A e^B.$$

Hint: Consider 2×2 matrices and A diagonal and B nilpotent.

(b) Find a quadratic matrix $A \in \mathbb{R}^{2\times 2}$ where both eigenvalues satisfy $\mathrm{Re}\,(\lambda) \leq 0$ yet there exists a solution of $\dot{x} = Ax$ satisfying $|x(t)| \to \infty$ for $t \to \infty$.

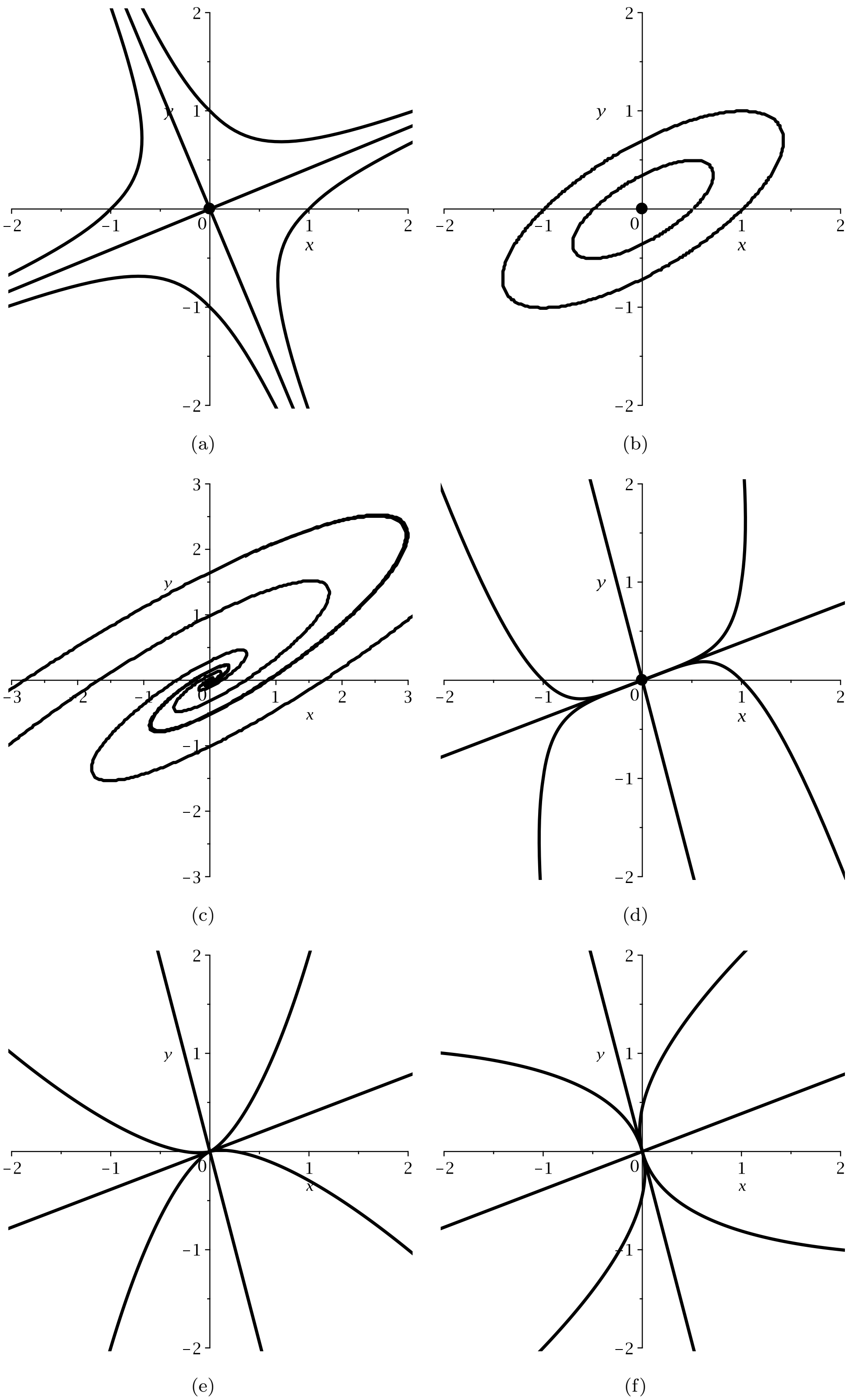

Figure 1: Phase portraits of $2D$ linear systems.

(c) Determine all $A \in \mathbb{R}^{2\times 2}$ so that

(i) A is semisimple.

(ii) A is nilpotent.

Exercise 4 Problem 3 in problem set 9 on p. 59.

Exercise 5 Problem 5 in problem set 1 on p. 6.

Exercise 6 Let $t \in \mathbb{R}$ and set $\phi_t(x) = e^{At}x$. Show that the mapping $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ defines a flow satisfies

(i) $\phi_0(x) = x$ for all x;

(ii) $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$ for all t and x;

(iii) $\phi_{-t}(\phi_t(x)) = x$ for all t and x.

Exercise 7* Suppose that E is a subspace of $\mathbb{R}^n$ and that the matrix $A \in \mathbb{R}^{n\times n}$ leaves E invariant:

$$AE = \{Av | v \in E\} \subset E.$$

Then show that if $x(t)$ is the solution of the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0,$$

with $x_0 \in E$, then $x(t) \in E$ for all $t \in \mathbb{R}$.

Exercise 8** Problem 5 in problem set 9 on p. 59.

Next week

When does $\dot{x} = f(x)$ generate a flow? Next week we will address this issue by considering well-posedness of initial value problems.

References

[Per00] Perko, L., Differential Equations and Dynamical Systems, Texts in Applied Mathematics 7, Springer-Verlag, New York, 2000.