Dynamical Systems 1
Course 01617
E16

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"So many books, so little time"
- Frank Zappa

Week 5

On today

This week we will study a famous result, the Hartman-Grobman theorem. It is a theorem that reassures us that near hyperbolic stationary points (for definition af those, see Perko p. 102), the linearized system will show the same qualitative phase flow as the nonlinear system; in other words as long as we are very close to the stationary point, the linearized and nonlinear system have very much the same phase flow, in a sense to be specified by the theorem. The name for the similarity is *Topological Equivalent*. The theorem is *constructive* in the sense that it actually construct a sequence of maps that converge to the desired homeomorphism.

Read

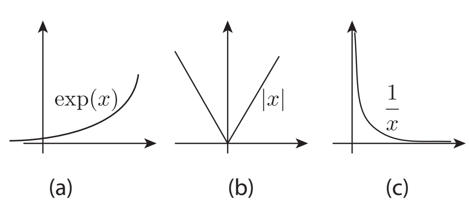
Perko, section 2.8: The Hartman-Grobman theorem. But beware, it is a lengthy chunk of medium strength theory, and even Perko skips some technicalities. Try to get the grand picture. The lecture, and imbedded example, will follow the book quite closely and will thus also skip technicalities.

Keywords

Topological equivalence, homeomorphism, hyperbolic systems, stable and unstable manifolds.

Exercises

The first exercises are easy review questions; do not idle endlessly with them, neglecting the more tricky later exercises which describe the inner workings of the Hartman-Grobman construction. Only embark on the final exercise (Perko 8.1) if you feel you understand everything else that has been going on. Do not despair if you have no time left over to complete the final exercise.



Exercise 1 Which of the above pictures show graphs of functions that are homeomorphisms $\mathbb{R} \to \mathbb{R}$?

Exercise 2 Solve (by paper and pencil) the initial-value problem (consider y_0 a constant):

$$\begin{aligned} \dot{z} - z &= y_0 \exp(-2t) \\ z(0) &= z_0 \end{aligned}$$

Exercise 3 Explain why

$$\begin{aligned} \dot{y} &= -y + z + yz \\ \dot{z} &= z + yz^3 \end{aligned}$$

is topologically equivalent to

$$\begin{aligned} \dot{y} &= -y + z \\ \dot{z} &= z \end{aligned}$$

Exercise 4 Be a serious student. Verify that (p. 125 in Perko)

$$\Psi(y,z) = z + ke^{-1}(\sum_{n=1}^{\infty} e^{-3n})y^2 = z + \frac{y^2}{3}$$

Exercise 5 Check that (from p. 126 in Perko) $H_0 = L^{-1}H_0T$ and that consequently

$$\int_{-t}^{0} L^{-s}H_0T^s \mathrm{d}s = \int_{1-t}^{1} L^{-s}H_0T^s \mathrm{d}s$$

Exercise 6 Show that indeed

$$H_0^{-1}(y,z) = \begin{bmatrix} y \\ z + \frac{1}{3}y^2 \end{bmatrix} \qquad \text{is the inverse of} \qquad H_0(y,z) = \begin{bmatrix} y \\ z - \frac{1}{3}y^2 \end{bmatrix}$$

Exercise 7 Perko 8.1 : Construct H for the map

$$\begin{aligned} \dot{y}_1 &= -y_1 \\ \dot{y}_2 &= -y_2 + z^2 \\ \dot{z} &= z \end{aligned}$$

Next week

Next week we will study stability, Liapunov functions (Perko section 2.9).