Dynamical Systems 1 Kristian Uldall Kristiansen and Frank Schilder
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Week 1

On today

Welcome to the course. We start out with some practical information, e.g., how the course is evaluated, what teaching material will be used and how the teaching will be organised. Then we move on to the subject: Dynamical Systems Theory. We will hear about its history and background, and we will see some motivating examples. We will then give some basic definitions. For example, a dynamical system is the collection of:

(a) a *phase space*: $P = \{x\}$; and

(b) a *time* $t \in \mathbb{R}$ (continuous dynamical system) or $t \in \mathbb{Z}$ (discrete dynamical system); and

(c) a *flow* ϕ_t on P. That is a family (paramatrized by time t) of *diffeomorphisms*

$$\phi_t : P \to P$$

satisfying:

$$\phi_0(x) = x, \quad \phi_{t+s}(x) = \phi_t \circ \phi_s(x),$$

for all $x \in P$ and t, s.

In this course our dynamical systems will typically be generated by ODEs

$$\dot{x}(t) = f(x(t)). \tag{1}$$

Then we define the flow ϕ_t associated with (1) by letting

$$\phi_t(x_0) \quad \text{denote the unique solution } x(t) \text{ of (1) at time } t \text{ with initial condition } x(0) = x_0. \tag{2}$$

More on this later in the course. Today we will primarily focus on $x(t) \in \mathbb{R}$.

Read

Lecture notes on 1D systems.

Key words

Initial value problem, flow, 1D systems, phase portrait.

Exercises

The exercises are (more or less) listed in order of importance. But that does not mean that you should go for the first one first. Instead, I suggest that you have a look through all the exercises before you start. You are welcome to come back to these exercises later. I highly recommend you to solve all the exercises.

Exercise 1 Solve problems 2.2.1, 2.2.3, 2.2.8, and 2.2.9 in [Str94, pp. 36-37].

Exercise 2 Given

$$\dot{x}(t) = ax(t), \quad x(0) = x_0,$$

with $a \in \mathbb{R}$.

(a) Find ϕ_t as in (2).

(b) What is $\phi_1(x)$?

(c) What is $\phi_t(1)$?

(d) Show that the linear initial value problem defines a dynamical system. Hint: Verify each of the ingredients above with ϕ_t as in (a).

Exercise 3 Assume that the $1D$ initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{3}$$

with f continuous, has a unique, continuously differentiable solution $x(t)$ for every x_0 defined within some open interval $t \in (a, b)$. Show that $x(t)$ is either

- constant,
- strictly increasing, or
- strictly decreasing.

Exercise 4* Consider 2.2.8 in [Str94] again. Replace the right hand side $f(x)$ by $f(x) + \mu$ and describe what happends when you vary μ near $\mu = 0$. Sketch qualitatively different phase portraits.

Exercise 5* Consider again (3) and suppose that $x(t)$ is defined for all $t \in \mathbb{R}$. Show that one of the following three statements is true:

- $x(t) \to \infty$ as $t \to \infty$
- $x(t) \to -\infty$ as $t \to \infty$
- $x(t) \to x_0$ as $t \to \infty$ where $f(x_0) = 0$.

Next week

Next week we will revisit linear systems $\dot{x} = Ax$ from a dynamical systems point of view.

Office time

For discussions/questions/etc. contact me on krkri@dtu.dk. My office is in Building 303B office 155.

References

[Str94] Strogatz, S. H., Nonlinear Dynamics and Chaos - with Applications to Physics, Biology, Chemistry, and Engineering, Studies in Nonlinearity, 1994.