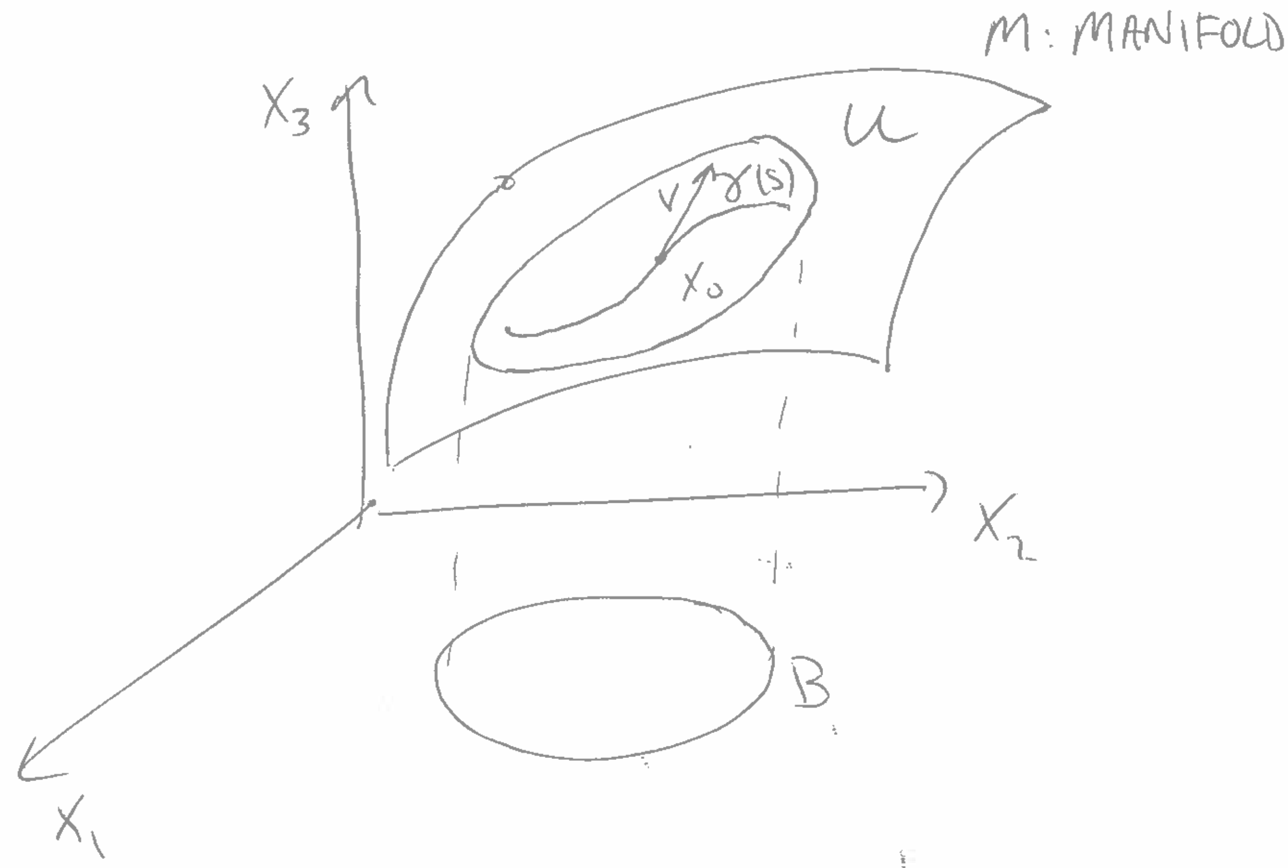
GEOMETRY & INVARIANCE



$$M \cap U = \text{GRAPH}(h): \quad x_3 = h(x_1, x_2), \quad (x_1, x_2) \in B.$$

LET $g(x) = x_3 - h(x_1, x_2)$:

$$M \cap U: \quad g(x) = 0, \quad (x_1, x_2) \in B$$

<u>TANGENT VECTORS</u>: LET $\gamma(s)$ BE A CURVE IN U, $s \in (-\varepsilon, \varepsilon), \varepsilon > 0$, γ SMOOTH, $\gamma(0) = x_0$.

2

THEN $g(\gamma(s)) = 0 \Rightarrow$

$$LHS = \frac{d}{ds} g(\gamma(s))\Big|_{s=0} = Dg(\gamma(0))\gamma'(0),$$

$$RHS = 0,$$

BY DIFFERENTIATION.

THEREFORE :

SET OF ALL TANGENT VECTORS TO M AT $X_0 = T_{X_0}M$

$$= \{ v \in \mathbb{R}^n \mid Dg(x_0)v = 0 \}$$

REMARK: Dg JACOBIAN MATRIX.

FROM EQ. (1)

$$Dg(x) = \left[-\frac{\partial h}{\partial x_1} \quad -\frac{\partial h}{\partial x_2} \quad 1 \right]$$ — ROW VECTO

LET φ_t BE A FLOW.

CONSIDER $\mathcal{U} = \{x \mid g(x) = 0\}$

THEN: DEFINITION: $\mathcal{U}$ IS INVARIANT IF $\varphi_t(\mathcal{U}) \subset \mathcal{U} \quad \forall t \in \mathbb{R}$

OR SIMPLY

$$\varphi_t(x) \in \mathcal{U}, \quad \forall x \in \mathcal{U}, \forall t \in \mathbb{R}$$

ALSO: DEFINITION: $\mathcal{U}$ IS LOCALLY INVARIANT IF $\forall x \in \mathcal{U} \; \exists \varepsilon:$

$$\varphi_t(x) \in \mathcal{U} \quad \forall t \in [-\varepsilon, \varepsilon].$$

THEOREM $\mathcal{U}$ IS LOCALLY INVARIANT

$\Longleftrightarrow$ $\mathcal{U}$ IS A UNION OF SOLUTION CURVES $\Longleftrightarrow$

$$\forall x_0 \in \mathcal{U}, \quad \frac{d}{dt} \varphi_t(x_0)\Big|_{t=0} = f(x_0) \in T_{x_0}\mathcal{U}$$

$$\Longleftrightarrow \quad \forall x_0 \in \mathcal{U}, \; \frac{d}{dt} g(\varphi_t(x_0))\Big|_{t=0} = Dg(x_0) f(x_0) = 0.$$

EX
$$\dot{x}_1 = -x_1$$
$$\dot{x}_2 = -x_2 + x_1^2$$
$$\dot{x}_3 = x_3 + x_1^2$$

SHOW THAT $S = \{x \in \mathbb{R}^3 \mid x_3 = -x_1^2/3\}$

IS INVARIANT. HERE

$g(x) = x_3 + x_1^2/3$. HENCE

$$Dg = [2x_1/3 \quad 0 \quad 1].$$

THEREFORE LET $x(t) = \varphi_t(x_0)$, $x_0 \in S$.

$$\frac{d}{dt} g(x(t))\Big|_{t=0} = Dg(x_0) f(x_0)$$
$$= [2x_1/3 \quad 0 \quad 1] \begin{bmatrix} -x_1 \\ -x_2 + x_1^2 \\ x_3 + x_1^2 \end{bmatrix}$$
$$= -2x_1^2/3 + x_3 + x_1^2$$
$$= x_3 + x_1^2/3$$
$$= 0 \quad \text{SINCE } x_3 = -\frac{x_1^2}{3}.$$

S HAS NO BOUNDARIES!

EX

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2 + x_1^2$$

SHOW THAT $x_1 = 0$ IS INVARIANT.

$$\dot{x}_1 \Big|_{x_1 = 0} = 0 \;!$$

IS $x_2 = 0$ INVARIANT?

NO!