Linear Algebra Brushup

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2 September 2016

Vector Spaces

A vector space is a space V where you can take *linear combinations* without leaving the space, i.e.,

$$a\mathbf{u} + b\mathbf{v} \quad \in V,$$

for any vectors $\mathbf{u}$, $\mathbf{v} \in V$ and scalars a, $b \in \mathbb{R}$.

Examples

- $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \ldots$.
- The set of *solutions* to a homogeneous linear equation:

$$A\mathbf{x} = 0.$$

- The set of real (or complex) polynomials of degree n.

Vector space axioms: see textbook.

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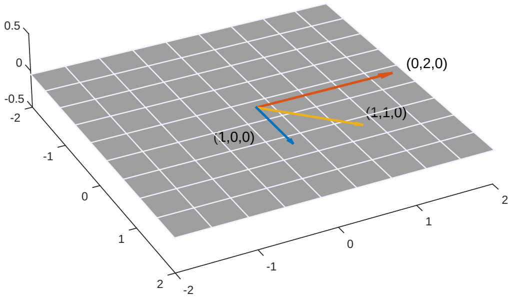
Span and spanning set

The *span* of a set of vectors $T \subseteq V$:

$$\operatorname{span} T = \{a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \mid a_i \in \mathbb{R},\ \mathbf{v}_i \in T\}.$$

If $V = \operatorname{span} T$, then T is called a *spanning set* for V.



$$\text{e.g.} \qquad T = \{(1,0,0), \quad (0,2,0), \quad (1,1,0)\}$$

is a spanning set for the *xy*-plane $V = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

Linearly independent set

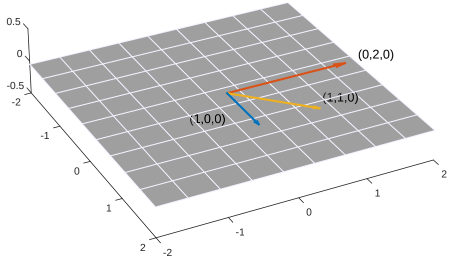
$T \subseteq V$ is called *linearly dependent* if there exists a non trivial linear combination of vectors in T that gives zero, i.e. :

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = 0,$$

for some scalars a_i and vectors $v_i \in T$.

e.g. $T = \{(1,0,0), \quad (0,2,0), \quad (1,1,0)\}$ is linearly dependent:

$$(1,0,0) + (1/2)(0,2,0) - (1,1,0) = (0,0,0).$$



Otherwise we call the set T *linearly independent*.

Basis and dimension

A *basis* of a vector space V is a linearly independent spanning set.

e.g. standard basis for $\mathbb{R}^2$

$$e_1 = (1, 0), \quad e_2 = (0, 1).$$

Another basis:

$$\tilde{e}_1 = (1, 1), \quad \tilde{e}_2 = (1, -1).$$

Check for linear independence:

$$\det[\tilde{e}_1; \tilde{e}_2] = \det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0.$$

The cardinality of any basis for V is the same, called the *dimension*.

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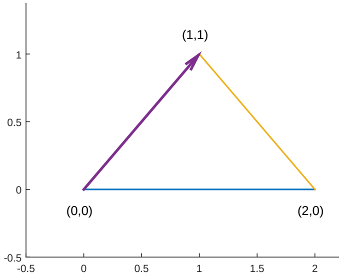
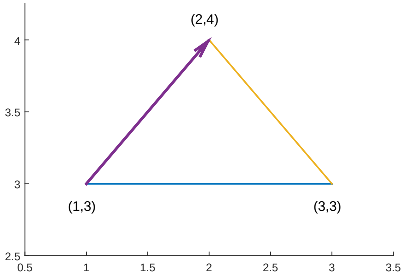
Digression: Affine Spaces

An *affine space* is like a vector space if you forget about the origin.





Cannot add points consistently:

$$(0,0) + \color{red}{(1,1) = (1,1)} \qquad \text{but} \qquad (1,3) + \color{red}{(2,4) = (3,7)}$$

Can *subtract* points to get a *vector* between them:

$$(1,1) - (0,0) = \color{red}{(1,1)} \qquad \text{and} \qquad (2,4) - (1,3) = \color{red}{(1,1)}$$

Affine Combinations

Arbitrary linear combinations make no sense in affine space. Instead consider *affine combinations*

$$\sum_{j=1}^{n} a_j \mathbf{v}_j, \qquad \text{where} \quad \sum_{j=1}^{n} a_j = 1.$$

If we translate the $\mathbf{v}_j$ by T then the affine combination is also translated by T:

$$\begin{aligned}
\sum_{j=1}^{n} a_j(\mathbf{v}_j + T) &= \sum_{j=1}^{n} a_j \mathbf{v}_j + \left(\sum_{j=1}^{n} a_j\right) T \\
&= \left(\sum_{j=1}^{n} a_j \mathbf{v}_j\right) + T.
\end{aligned}$$

Linear Maps between Vector Spaces

Let U and V be vector spaces. A map $L : U \to V$ is called *linear* if it preserves linear combinations. Equivalently, we require:

- $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in U$.
- $L(\alpha \cdot \mathbf{u}) = a \cdot L(\mathbf{u})$ for all $\mathbf{u} \in U$ and $a \in \mathbb{R}$.

Examples of linear maps are

- Rotation, scaling, and shearing of geometric vectors.
- Projection from vectors in space to a vectors in a plane.
- Differentiation $C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R}) : f \mapsto f'$.
- Differentiation $C^n(\mathbb{R}, \mathbb{R}) \to C^{n-1}(\mathbb{R}, \mathbb{R}) : f \mapsto f'$.
- Integration $C^0(\mathbb{R}, \mathbb{R}) \to C^0(\mathbb{R}, \mathbb{R}) : f \mapsto (x \mapsto \int_0^x f(t)\,\mathrm{d}t)$

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Matrix of a linear map of finite dimensional spaces

Let $L : U \to V$ be a linear map, and let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and $\mathbf{f}_1, \ldots, \mathbf{f}_m$ be bases for U and V respectively.
For each $\mathbf{e}_j$ he vector $L(\mathbf{e}_j)$ can be expressed uniquely in terms of $\mathbf{f}_1, \ldots, \mathbf{f}_m$:

$$L(\mathbf{e}_j) = \sum_{i=1}^{m} a_{i\,j}\mathbf{f}_i.$$

The matrix of the linear map L with respect to these bases is

$$\mathbf{A} = \begin{pmatrix} a_{1\,1} & \ldots & a_{1\,n} \\ \vdots & \ddots & \vdots \\ a_{m\,1} & \ldots & a_{m\,n} \end{pmatrix}.$$

- Addition and scalar multiplication of linear maps
  $\longleftrightarrow$
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- Composition of linear maps $\longleftrightarrow$ matrix multiplication.

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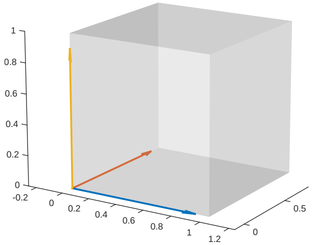
Determinants

An $n \times n$ matrix A corresponds to a linear map from $\mathbb{R}^n \to \mathbb{R}^n$.

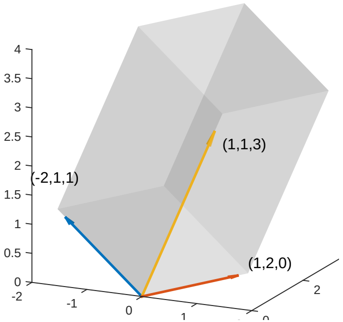
The *determinant* of A tells you how volumes are distorted.

$$\text{e.g.} \qquad A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \qquad \det A = -16.$$



Volume $= 1$

$\mapsto$



Volume $= |\det A| = 16$

Computing determinants

The determinant can be defined recursively in terms of $(n-1) \times (n-1)$ sub-matrices.

e.g. cofactor expansion along the first row: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1);$$

or along the second column:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

$$= -a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1);$$

Eigenvalues and eigenvectors

Le $L : V \to V$ be a linear map from a vector space to itself. An *eigenvector* $\mathbf{v}$ with *eigenvalue* λ is non zero vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that

$$L(\mathbf{v}) = \lambda \cdot \mathbf{v}$$

If V has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ consisting of eigenvectors of L then the corresponding matrix is

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

where the λ_i'es are the eigenvalues. We say that L is *diagonalised*.

A scalar λ is an eigenvalue if and only if

$$\det(\mathbf{A} - \lambda I) = 0,$$

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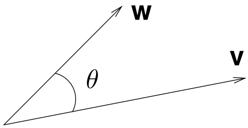

$$\langle \mathbf{v}, \mathbf{w} \rangle = |\mathbf{v}||\mathbf{w}| \cos\theta$$

The inner product between two vectors in the plane or space.

An Euclidian vector space is a real vector space V equipped with an inner product. That is, a positive definite, symmetric, bilinear pairing $\langle , \rangle : V \times V \to \mathbb{R}$, i.e.,

- $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$.
- $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \iff \mathbf{v} = \mathbf{0}$.
- $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.
- $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$.
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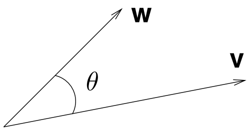

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Orthonormal basis

Orthonormal set of vectors:

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

It is easy to expand in an orthonormal basis:

$$\mathbf{v} = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{e}_i \rangle \, \mathbf{e}_i .$$

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Let V be an Euclidean vector space.
An *isometry* is a linear map $L : V \to V$ such that $\langle L(\mathbf{u}), L(\mathbf{v}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

If we equip V with an orthonormal basis then a linear map is an isometry if and only if its matrix is orthogonal: $AA^t = I$.

Eigenvalues and eigenvectors

A *symmetric* map is a linear map $L : V \to V$ such that $\langle L(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, L(\mathbf{v}) \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.

If we equip V with an orthonormal basis then a linear map is symmetric if and only if its matrix is symmetric.

If $L : V \to V$ is a symmetric linear map then V has an orthonormal basis consisting of eigenvectors.

Equivalently:
A symmetric $n \times n$ matrix $\mathbf{A}$ can be factorized as $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^T$, where $\mathbf{U}$ is orthogonal and $\mathbf{\Lambda}$ is diagonal.
The elements of $\mathbf{\Lambda}$ are the eigenvalues and the columns of $\mathbf{U}$ are the (coefficients of) the eigenvectors.

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Singular Value Decomposition (motivation)

e.g.: Iterated closest point algorithm

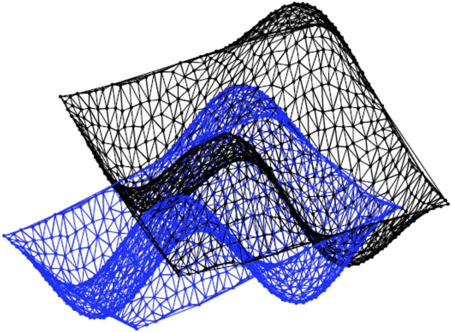

Fig. 15.3 Two similar meshes, one rotated and translated w.r.t. the other

1. For each vertex $p_i \in M_1$ find closest vertex $q_i \in M_2$.
2. Find the rotation R and translation T that brings all p_i as close as possible to q_i:

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One can find T by center of mass considerations. Thus we end up with a matrix problem of the form

$$\min_R ||PR - Q||^2,$$

for some known matrices P and Q.

This can be solved by a *singular value decomposition*.

Singular value decomposition

Any $m \times n$ matrix $\mathbf{A}$ can be factorized as

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T,$$

$\mathbf{U}$ orthogonal $m \times m$, $\mathbf{V}$ orthogonal $n \times n$, and Σ is a *diagonal* is $m \times n$ matrix with non-negative diagonal elements:

$$\sigma_1 \geq \ldots, \geq \sigma_k \geq 0,$$

where $k = min(m, n)$.

The scalars σ_i are called the *singular values* of $\mathbf{A}$.

Note, writing $D = \text{diag}(\sigma_1, \cdots, \sigma_k)$, then

$$\mathbf{\Sigma} = [D, \ \mathbf{0}], \quad \text{or} \quad \mathbf{\Sigma} = \begin{bmatrix} D \\ \mathbf{0} \end{bmatrix}.$$

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More on SVD

If $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, then

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^T\mathbf{U}^T, \qquad \mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{\Sigma}^T\mathbf{\Sigma}\mathbf{V}^T.$$

So the columns of $\mathbf{U}$ and $\mathbf{V}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ respectively. The square of the singular values $\sigma_1^2, \ldots, \sigma_k^2$ are the (non-zero) eigenvalues of both $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$.
This is one way to find the SVD.

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SVD Example

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Applications of SVD

The *Moore-Penrose pseudo inverse* of a matrix $A = U\Sigma V^t$ is

$$A^+ = V\Sigma^+ U^t,$$

where, if the diagonal elements of Σ are σ_i, then the diagonal elements of Σ^+ are $1/\sigma_i$.

If A is invertible then $A^+ = A^{-1}$.

Consider the problem $Ax = b$, and set $\tilde{x} = A^+ b$. Then

- If the system $Ax = b$ is *overdetermined*, $\tilde{x}$ is the *least square solution*, i.e. the solution to

$$\min_x ||Ax - b||^2.$$

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