Dynamical Systems 1 Kristian Uldall Kristiansen and Poul G. Hjorth
Course 01617 Department of Applied Mathematics and Computer Science
E16 Technical University of Denmark

"All the world's a stage. And all the men and women merely players - "
- Shakespeare [*As You Like It, II.5*]

Week 8

On today

We have now set the stage for one of the, well, central, topics of this course: Center Manifolds. Near a non-hyperbolic stationary point there is in general a manifold, of smaller dimension than the full phase space, and the flow on this manifold near the stationary point contains the essential dynamics of the full system near the stationary point. The determination of this manifold and the flow on it is known as *center-manifold reduction*.

Read

Perko, section 2.12: The Center-Manifold Theorem theorem. Review from previous courses: Taylor series, Power series.

Keywords

Center-Manifold, reduced dynamics.

Exercises

Do not despair if you have no time left over to complete the final exercise.

Exercise 1: Compute (to order x^5) a reduction to center manifold for the system

$$\begin{aligned} \dot{x} &= xy - x^3 + xy^2 \\ \dot{y} &= -y + x^2 + x^2 y \end{aligned}$$

Exercise 2: Compute a reduction (to 2. order) to center manifold for the system.

$$\begin{aligned} \dot{x} &= 10(y-x) + x^2 \\ \dot{y} &= -y + x - (1/11x + 10/11y)^2 \end{aligned}$$

Hint: Recall that the CenterManifold Theorem assumes a diagonal linear part. Begin by changing coordinates so this becomes the case.

Next page!

Exercise 3: Consider the following system

$$\begin{aligned}\dot{x} &= \mu - x^2 + \mu x y, \\ \dot{y} &= -y + x^2,\end{aligned} \tag{1}$$

with μ a real-valued parameter. We will study this system near $(x, y) = 0$ for μ small. We therefore consider the the following extended system

$$\begin{aligned}\dot{x} &= \mu - x^2 + \mu x y, \\ \dot{y} &= -y + x^2, \\ \dot{\mu} &= 0.\end{aligned} \tag{2}$$

Do the following:

(a) Show that $(x, y, \mu) = 0$ is a non-hyperbolic equilibrium and

$$E^c = \ker A^2 = \{(x, y, \mu) | y = 0\}, \quad A = Df(0),$$

f being the right hand side of (2).

(b) Argue that there exists a center manifold of the following form:

$$y = h_c(x, \mu) = ax^2 + bx\mu + c\mu^2 + \cdots, \tag{3}$$

with $\cdots$ denoting terms of order cubic or higher.

(c) Determine the constants a, b and c in (3) by direct insertion of (3) into (1).

(d) Reduce to (3) ignoring terms of cubic and higher order and describe the reduced system for different values of μ.

(e) Sketch the phase portrait of the system (1) near $(x, y) = 0$ for various values of small μ, in particular for (i) $\mu < 0$, (ii) $\mu = 0$ and (iii) $\mu > 0$.

Next week

Next week we will study local bifurcations (Perko section 4.2). This is in the direction where Problem 3(e) is taking us.