WEEK 2: LINEAR SYSTEMS $\dot{x} = Ax$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $x(0) = x_0$

- 2D EXAMPLE: PHASE PORTRAIT / ORBITS
- SOLUTION THEORY: MATRIX EXPONENTIAL
- INVARIANT SUBSPACES E^u, E^s, E^c w. PROPERTIES

EXAMPLE $\dot{x} = ax \in \mathbb{R}$ $\exists!$ $x(t) = e^{at}x_0$ WITH $x(0) = x_0$

WHAT IF $a \in \mathbb{R}^{n\times n}$? LET $\varphi_t(x_0) = e^{at}x_0$

$x_0 \to \varphi_t(x_0)$

EXAMPLE

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x, \quad x(0) = x_0 \in \mathbb{R}^2 \qquad (2)$$

ALTERNATIVELY $\dot{x}_1 = -x_1$, $\dot{x}_2 = 2x_2$ DECOUPLED

SOLUTION $x_1 = e^{-t}x_{10}$, $x_2 = e^{2t}x_{20}$ (*)

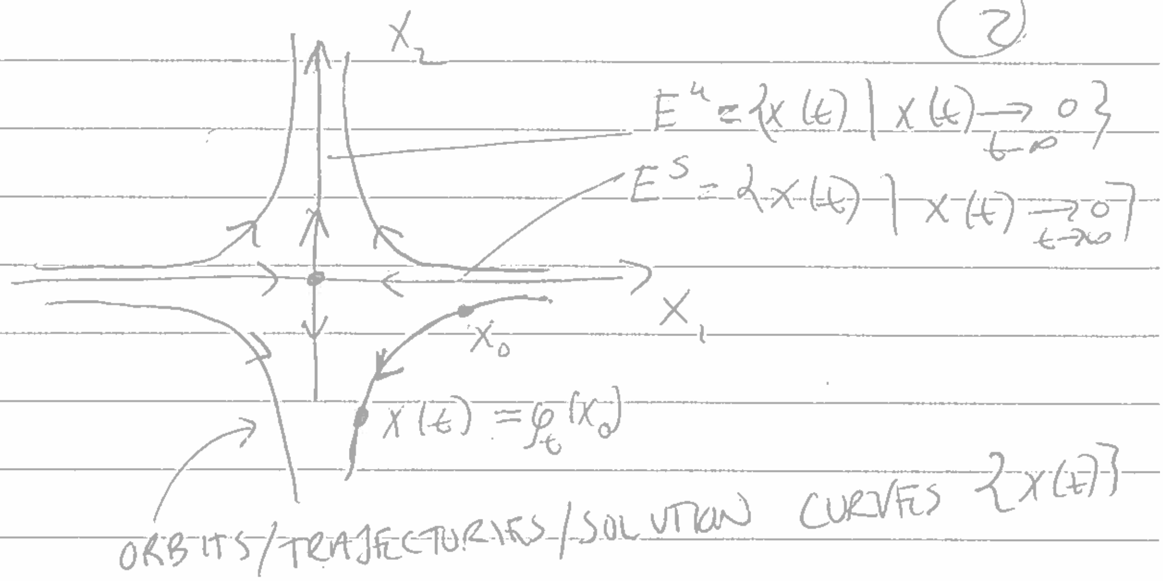
OR IN VECTOR-FORM

$$x(t) = \varphi_t(x_0) \equiv \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} x_0$$

NOTICE: $x_2 * x_1^2 = e^{2t}x_{20}e^{-2t}x_{10}^2 = x_{20}x_{10}^2$ //

ALGEBRAIC CURVE IN (x_1, x_2)-PLANE

(*) DEFINES MOTION ALONG THESE CURVES



PHASE PORTRAIT = COLLECTION OF ALL ORBITS

"SKETCH" = COLLECTION OF ALL QUALITATIVELY DISTINCT ORBITS

~~[illegible]~~

(2) DEFINES A DS WITH FLOW

$$\varphi_t(x) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} x \quad \Leftarrow \text{TODAY: GENERALISING THIS}$$

EXERCISE: CHECK

EXERCISE TODAY: SKETCH ALL DIFFERENT 2D PHASE PORTRAIT.

SOLUTION THEORY

$\dot{x} = Ax$, $x(0) = x_0$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$.

HOW TO SOLVE THIS?

EXAMPLE: SEMI-SIMPLE CASE

③

$$\dot{x}_1 = -x_1 - 3x_2$$
$$\dot{x}_2 = 2x_2$$
$$, \quad A = \begin{bmatrix} -1 & -3 \\ 0 & 2 \end{bmatrix}$$

EIGENVALUES λ: $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} =$

$= (-1-\lambda)(2-\lambda) = 0$

$\lambda = -1, 2$ //

EIGENVECTORS v:

$\lambda = -1$: $(A - \lambda I)v = (A + I)v = \begin{bmatrix} 0 & -3 \\ 0 & 3 \end{bmatrix} v = 0$

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $E_{-1} = \ker(A+I) = \mathrm{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 2$: $(A - \lambda I)v = (A - 2I)v = \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} v = 0$

$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. $E_2 = \ker(A - 2I) = \mathrm{SPAN}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

LET $P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

THEN $A = P \Lambda P^{-1}$ AND $y = P^{-1} x$

GIVES

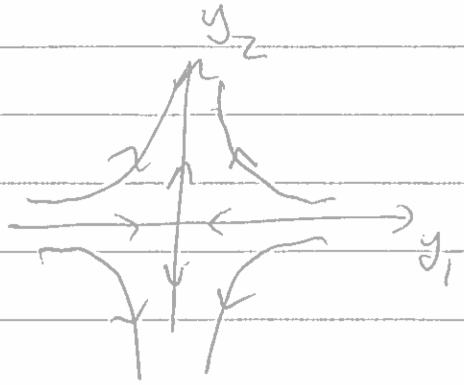
$\dot{y} = P^{-1}\dot{x} = P^{-1} A P y = \Lambda y$

SEMI-SIMPLE

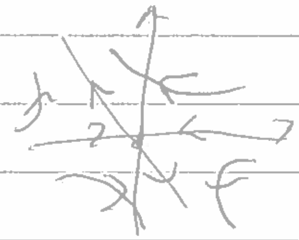
$y(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} y_0$

OR

$x(t) = P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1} x_0$ //



$X = Py$



EXAMPLE ← NOT SEMISIMPLE

$$\dot{X} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} X, \quad X(0) = X_0$$

EIGENVALUES λ: $\lambda = a$, $am = 2$

EIGENVECTORS v:

$$(A - \lambda I)v = (A - aI)v = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} v = 0$$

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad gm = 1$$

$$E_a = \text{SPAN} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

HOW?

DEFINITION: MATRIX EXP
$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

THM: e^{At} EXISTS

PROOF: $\cancel{\|e^{At}x\| \le e^{\|At\|}|x|}$

$$\|e^{At}x\| \le \sum_{k=0}^{\infty} \left|\frac{(At)^k x}{k!}\right| \le \sum_{k=0}^{\infty} \frac{\|(At)\|^k}{k!} |x| = e^{\|At\|}|x|.$$

NOTE: $e^{A+B} \neq e^A e^B$ IN GENERAL

HOWEVER IF $AB = BA$: A AND B COMMUTE
THEN $e^{A+B} = e^A e^B$. (II)

~~THM (p. 12) $\exists!$ SOLUTION OF (1)~~

THM (p. 17) $x(t) = e^{At} x_0$ IS THE UNIQUE SOLUTION OF (1)

PROOF:
EXISTENCE:

$$(e^{At})' = \lim_{h \to 0} \frac{e^{A(t+h)} - e^{At}}{h}$$

$$\overset{\text{USING (II)}}{=} \lim_{h \to 0} \frac{(e^{Ah} - I)}{h} e^{At}$$

$$\overset{\text{USING DEF}}{=} A e^{At}$$

UNIQUENESS: EXERCISE.

COR: IF $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ THEN $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$

SEMISIMPLE/ DIAGONALIZABLE

PROP: IF $A = P \Lambda P^{-1}$ THEN

$$e^{At} = P e^{\Lambda t} P^{-1}.$$

PROOF: $A^k = P \Lambda^k P^{-1}$

COMPUTE e^{At}, $A \in \mathbb{R}^{n \times n}$.

DEFINITION OF GEN. EIGENVECTORS

LET λ BE EIGENVALUES WITH $am = m$.

THEN $\hat{E}_\lambda = \ker (A - \lambda I)^m$

IS THE SPACE OF GEN. EIGENVECTORS.

RECALL: $E_\lambda = \ker (A - \lambda I)$ AND

$$1 \leq \dim E_\lambda \leq m$$

THM: $\dim \hat{E}_\lambda = m$, $E_\lambda \subset \hat{E}_\lambda$, $A\hat{E}_\lambda \subset \hat{E}_\lambda$

~~(p. 33)~~

NILPOTENT: N IS NILPOTENT IF $\exists k \in \mathbb{N}$:

$$N^k = 0 \quad (N^{k-1} \neq 0)$$

THM 1 (p. 33) SLIDE

NOW

$$e^{At} = e^{(S+N)t} \overset{\text{USING } \square}{=} e^{St} e^{Nt}$$

$$= \underset{\text{SEMISIMPLE}}{e^{St}} \left(I + N + \text{———} + \frac{N^{k-1} t^{k-1}}{(k-1)!} \right)$$

THM 2 (p. 36) SLIDE

COMPUTE e^{At}, $A \in \mathbb{R}^{n \times n}$ (6)

DEFINITION OF GEN. EIGENVECTORS

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IS THE SPACE OF GEN. EIGENVECTORS.

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THM 2 (p. 36) SLIDE

EXAMPLE

$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. COMPUTE e^{At}

$\lambda = a$, $am = 2$

$E_a = \cancel{\text{SPAN}}\ \ker(A - aI) = \text{SPAN}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\dim E_a = 1 < 2$: NOT ~~SEMISIMPLE~~ SEMISIMPLE.

$\hat{E}_a = \cancel{\text{SPAN}}\ \ker(A - aI)^2$

$= \ker\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}^2 = \ker\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \text{SPAN}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$. LET

$A = P\Lambda P^{-1} + N = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$

COMMUTE

$e^{At} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} e^{\begin{bmatrix} 0 & bt \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} \; \end{bmatrix}\left(I + \begin{bmatrix} 0 & bt \\ 0 & 0 \end{bmatrix}\right)$

$\blacksquare$

STABILITY THEORY

DEFINITION

STABLE SPACE:

E^s = DIRECT SUM OF ALL $\hat{E}_\lambda$ WITH $\mathrm{Re}\,\lambda < 0$,

UNSTABLE SPACE

E^u = DIRECT SUM OF ALL $\hat{E}_\lambda$ WITH $\mathrm{Re}\,\lambda > 0$.

CENTER SPACE

E^c = DIRECT SUM OF ALL $\hat{E}_\lambda$ WITH $\mathrm{Re}\,\lambda = 0$.

WHAT IF $\lambda = \alpha \pm i\beta$?
READ ~~SECTION~~ SECTION 1.6

~~HYPERBOLIC~~ C: $\dot{x} = Ax$ IS HYPERBOLIC IF $E^c = \varnothing$.

EXAMPLE

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} x, \quad x \in \mathbb{R}^4$$

FIND E^s, E^u.

~~$\lambda = -1, -1, 2, 6$~~

$\lambda = -1$, am $= 2$

$\lambda = 2$, am $= 1$

$\lambda = 4$, am $= 1$

$$E_s = \hat{E}_{-1} = \cancel{\text{span}}\ \ker (A - \lambda I)^{am} = \ker (A + I)^2$$

$$= \cancel{\text{span}}\ \ker \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{SPAN}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$E_2 = \hat{E}_2 = \text{SPAN}\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \hat{E}_4 = E_4 = \text{SPAN}\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$ 9

$E_u = E_2 \oplus E_4 = \text{SPAN}\left\{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$

THM 1 (P. 55)

$$\left.\begin{array}{l} - \ \mathbb{R}^n = E^s \oplus E^u \oplus E^c \\ - \ AE^{s,u,c} \subset E^{s,u,c} \ \text{RESP} \end{array}\right\} \Rightarrow A = \begin{bmatrix} S & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & C \end{bmatrix}$$

- $e^{At}E^{s,u,c} \subset E^{s,u,c}$ RESP.

EXP
- LET $x \in E^s$ THEN $\exists\, m, M > 0$ $|e^{At}x| \le Me^{-mt}|x| \xrightarrow[t\to\infty]{} 0$
- LET $x \in E^u$ THEN $\exists\, m, M > 0$ $|e^{At}x| \le Me^{mt}|x|$ $\forall t$ $\xrightarrow[t\to-\infty]{} 0$

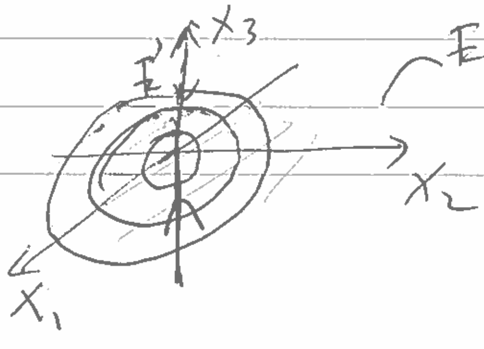
POL
- LET $x \in E^c$ THEN $\exists\, M$ AND POS POL $Q(t)$:

$$|e^{At}x| \le M|Q(t)|$$

EX

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x, \quad \lambda = \pm i, \quad v = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$$

$$E^s = \text{SPAN}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}, \quad E^c = \text{SPAN}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$



E^c — SPAN OF REAL, IMAG PART OF $\begin{pmatrix} i \\ 1 \end{pmatrix}$

SEE SECTION 1.6