(1) $\int_a^b g(x)\,dx \qquad a<b,\ g\in C[a,b]$

$x = a \Rightarrow -2$

$a \quad b \longrightarrow -2 \quad 2$

$x = b \Rightarrow 2$

$$\begin{cases} y = -2 + 4\,\dfrac{x-a}{b-a} \\[2mm] \dfrac{dy}{dx} = \dfrac{4}{b-a} \end{cases}$$

$\Downarrow$

$$\begin{cases} x = \dfrac{(y+2)(b-a)}{4} + a \\[2mm] dx = \left(\dfrac{b-a}{4}\right) dy \end{cases}$$

$$\int_a^b g(x)\,dx = \int_{-2}^{2} g\left(\frac{(y+2)(b-a)}{4} + a\right)\left(\frac{b-a}{4}\right) dy$$

$$= \left(\frac{b-a}{4}\right)\int_{-2}^{2} g\left(\frac{(y+2)(b-a)}{4} + a\right) dy$$

(2) $Q_n(f) = \sum_{k=0}^{n} A_k f(x_k)$

$Q_n(f)$ ma rząd $\geq n+1$ <=> Q_n jest kwadraturą interpolacyjną

($\Rightarrow$) Załóżmy że rząd $\geq n+1$.

Czyli z def rzędu każdy wielomian $w_n \in \Pi_n$ spełnia:

$Q_n(w_n) = \int_a^b w_n(x)\,dx$

Niech $w_n = \lambda_i$, gdzie $\quad w_n(x) = \sum_{k=0}^{n} w(x_k)\lambda_k(x)$, λ_i st. co najwyżej n

$\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j} \Rightarrow \lambda_i(x_k) = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$

$\int_a^b (\lambda_i(x)) \overset{\text{z treści zad.}}{=} Q_n(\lambda_i(x)) = \sum_{k=0}^{n} A_k \lambda_i(x_k) = A_i$

Czyli $Q_n(f) = \sum_{k=0}^{n} A_k f(x_k)$, gdzie $A_k = \int_a^b \lambda_i(x)\,dx$

Czyli Q_n to kwadratura interpolacyjna

($\Leftarrow$) Załóżmy, że Q_n jest kw. interpolacyjną

$Q(f) = Q(L_n(f))$

Weźmy dowolny wielomian $w \in \Pi_n$ $\quad$ ($w = L_n(w)$) z jednoznaczności

~~$Q_n(w) = Q_n(L_n(w)) \overset{\text{z treści}}{=} \sum_{k=0}^{n} \int_a^b \lambda_k(x)\, w(x_k) \overset{\text{„wciągamy" całkę}}{=}$~~

~~$= \int_a^b \sum_{k=0}^{n} w(x_k) \lambda_k(x) = \int_a^b L_n(w$~~

$Q_n(w) \overset{\text{z treści def. } Q_n}{=} Q_n(L_n(w)) = \sum_{k=0}^{n} \int_a^b \lambda_k(x)\,dx\; w(x_k) \overset{\text{„wciągamy" całkę}}{=}$

$= \int_a^b \Big[\underbrace{\sum_{k=0}^{n} w(x_k)\lambda_k(x)}_{L_n(w)}\Big]dx = \int_a^b L_n(w) = \int_a^b w$

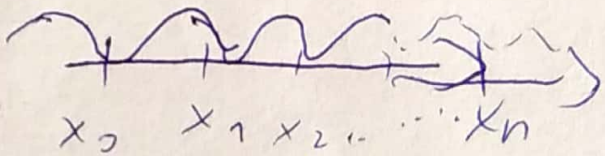
Czyli jest rzędu co najmniej $n+1$

(8) $Q_n(f) = \sum_{k=0}^{n} A_k f(x_k)$, $r \leq 2n+2$

Pokażę, że $\exists w \in \Pi_{2n+2} / \Pi_{2n+1}$ t. że

$Q_n(w) \neq \int_a^b f(x)$ $\in \Pi_{2n+2}$ $\rightarrow$

Weźmy $f(x) = \prod_{k=0}^{n} (x-x_k)^2$, $f(x) \geq 0$



Więc skoro $f(x)=0 \iff x = x_k$, to poza tymi $n+1$ punktami

$f(x)>0$ czyli $\int_a^b f(x)dx > 0$

Ale $Q_n(f) = \sum_{k=0}^{n} A_k f(x_k) = 0$

⑤ $L_n(x) = \sum_{k=0}^{n} f(x_k) \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{x-x_j}{x_k-x_j}$

$x_k = a + \frac{b-a}{n} k$ – węzły równoodległe

1) • $x_k - x_j = \left(a + \frac{b-a}{n}k\right) - \left(a + \frac{b-a}{n}j\right) = \overbrace{\frac{b-a}{n}}^{=h}(k-j) = h(k-j)$

2) • $x - x_j = x - \left(a + \frac{b-a}{n}j\right) = x - a - hj = x - (a + hj)$

$x = a + th$ czyli $x \in [x_0, x_n] \Rightarrow t \in [0, n]$

2) • $x - x_j = a + th - (a + hj) = h(t-j)$

$L_n(t) = \sum_{k=0}^{n} f(x_k) \underbrace{\prod_{j=0}^{n} \frac{t-j}{k-j}}$ $\quad j = 0, 1, 2, \dots, k-1 \rightarrow k-j = k, k-1, k-2, \dots, 1$ | $j = k+1, k+2, \dots, n$ $\quad (k-j) = 1, 2, \dots, n-k$

Czyli mamy $k!$ i $(n-k)!$ oraz -1 ; $n-k$ razy

$L_n(t) = \sum_{k=0}^{n} f(t_k) \frac{(-1)^{n-k}}{k!(n-k)!} \prod_{\substack{j=0 \\ j\neq k}}^{n} (t-j) \rightarrow \lambda_k(t)$

$t_k = \frac{x_k - x_0}{h}$

⑥ $N_n(f) = \sum_{k=0}^{n} A_k f(a + k h_n) \quad h_n = \frac{b-a}{n}$

T: $A_k = A_{n-k}$

$$\int_a^b L_n(x)\,dx = \sum_{k=0}^{n} f(x_k) \underbrace{\int_a^b \lambda_k \,dx}_{A_k}$$

$$x = a + th \Rightarrow t = \frac{x-a}{h} \Rightarrow \frac{dt}{dx} = \frac{1}{h} \Rightarrow dx = h\,dt$$

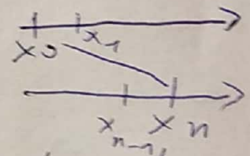
granice całkowania $x = a \Rightarrow t = 0 \qquad h = \frac{b-a}{n}$

$x = b \Rightarrow t = \frac{b-a}{h} = n$

$$(*) A_k = \int_a^b \lambda_k\,dx = \int_0^n \lambda_k h\,dt = \frac{(-1)^{n-k}}{k!\,(n-k)!} h \int_0^n \prod_{\substack{j=0\\ j\neq k}}^{n} (t-j)\,dt =$$

$s = n - t \Rightarrow t = n - s$ (symetria)

$dt = -ds$



$$A_k = -\Big(\int_n^0 \underbrace{\prod_{\substack{j=0\\ j\neq k}}^{n} (n-s-j)}_{(-1)\cdot} ds\Big) \cdot \frac{(-1)^{n-k}\cdot h}{k!\,(n-k)!}$$ // odwracamy granice i wyciągamy minus

$$= \int_0^n \Big[\prod_{\substack{j=0\\ j\neq k}}^{n} \big(s-(n-j)\big) ds\Big] \cdot \frac{(-1)^{2n-k}\cdot h}{k!\,(n-k)!}$$

$$= \int_0^n \Big[\prod_{\substack{l=0\\ l\neq n-k}}^{n} (s-l)\,ds\Big] \cdot \frac{(-1)^{2n-k}\cdot h}{k!\,(n-k)!}$$

$= ?$

Jak podstawimy $k := n-k$ do $*$ to otrzymamy

$$A_{n-k} = \int_0^n \prod_{\substack{j=0\\ j\neq n-k}}^{n} (t-j)\,dt \cdot \cancel{\frac{(-1)^{2n-k}}{k!\,(n-k)}} \cdot \frac{(-1)^k \cdot h}{k!\,(n-k)!}$$

Ale $$\frac{(-1)^{2n-k}}{(-1)^k} = (-1)^{2n-2} = (-1)^{2(n-1)} = 1$$

czyli $A_{n-k} = A_n$

(7) $\frac{A_k}{b-a}$ jest wymierne

$$A_k = \frac{(-1)^{n-k}}{k!(n-k)!} h \cdot \int_0^n \prod_{j=0}^{n} (t-j)\,dt$$

$$\frac{A_k}{b-a} = \underbrace{\frac{(-1)^{n-k}}{k!(n-k)!}}_{\in \mathbb{Q}} \cdot \underbrace{\frac{\cancel{b-a}}{n} \cdot \frac{1}{\cancel{b-a}}}_{\in \mathbb{Q}} \cdot \underbrace{\int_0^n \prod_{\substack{j=0 \\ j\neq k}}^{n} (t-j)\,dt}_{(*)}$$

$$(*) \int_0^n (t-0)(t-1)\cdots(t-n)\,dt$$

$$= \int_0^n t(t-1)(t-2)\ldots(t-n)\,dt$$

$$= \int_0^n (\alpha_n t^{n} + \alpha_{n-1} t^{n-1} + \alpha_{n-2} t^{n-2} + \ldots + \alpha_1 t + \alpha_0)\,dt$$

$$= \int_0^n \alpha_n t^n\,dt + \int_0^n \alpha_{n-1} t^{n-1}\,dt + \int_0^n \alpha_{n-2} t^{n-2}\,dt \ldots + \int_0^n \alpha_1 t\,dt + \int_0^n \alpha_0\,dt =$$

$$= \underbrace{\alpha_n \frac{n^{n+1}}{n+1}}_{\in \mathbb{Q}} + \underbrace{\alpha_{n-1} \frac{n^n}{n}}_{\in \mathbb{Q}} + \ldots + \underbrace{\alpha_0 n}_{\in \mathbb{Q}}$$

bo $\alpha_i \in \mathbb{Z}$, $n \in \mathbb{N}$

Więc $\frac{A_k}{b-a} \in \mathbb{Q}$