1)

k	x_k	y_k			
0	-3	-16			
1	-1	0	8		
2	0	-16	-16	-8	
3	1	32	48	32	10

$$L_n(x) = -16 + 8(x+3) - 8(x+3)(x+1) + 10(x+3)(x+1)x$$

b)

0	-3	...				
1	-1	...				
2	0	-16	-16	-8		
3	1	32	48	32	10	
4	3	560	264	72	10	0

$$L_n(x) = -16 + 8(x+3) - 8(x+3)(x+1) + 10(x+3)(x+1)\cancel{x}$$

c)

0	-3	-16	...		
1	-1	0	...		
2	0	-16	-16	-8	
3	1	-8	8	12	5
4					

$$L_n(x) = -16 + 8(x+3) - 8(x+3)(x+1) + 5(x+3)(x+1)x$$

(2) x_0 $f(x_0)$

x_1 $f(x_1)$ $f[x_0,x_1]$

x_2 $f(x_2)$ $f[x_1,x_2]$ $f[x_0,x_1,x_2]$

$\vdots \qquad \ddots$

x_n $f[x_{n-1},x_n]$ $\cdots$ $f[x_0,\dots,x_n]$

Ile (1na iterację) dzieleń? $f[x_0,\dots,x_n] = \dfrac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}$

Dla $[x_0 \dots x_n]$ punktów ~~x~~ $\Rightarrow D(n) = 2D(n-1)+1$

(można indukcyjnie) czyli $D(n) = 2^n - 1$, $D(0) = 0 = 2^0 - 1 = 0$, $D(1) = 1$

Odejmowań (2na iterację) $2(2^n-1) = 2^{n+1}-2$

$2^n - 1$ dzieleń, $2^{n+1}-2$ odejmowań dla $f[x_0,\dots,x_n]$

Algorytm

```
x[] = [x0, x1, ..., xn]
f[] = [y0, y1, ..., yn]
for (i = 1; i <= n, i++)
    for (j = n; j >= i; j--)
        f[j] = (f[j] - f[j-1]) / (x[j] - x[j-1])
```

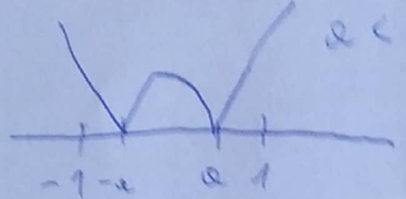
ANL - 7 $\qquad$ 2/

(3) a) $|(x-a)(x+a)|$

$|x^2-a^2|$

$a>1$:

$\max_{x\in[-1,1]} |\ | = a^2 > 1$ dla $x=0$



$a = 1$

$\max_{x\in[-1,1]} |x^2-1| = 1$

$a<1$

$\max_{x\in[-1,1]} |(x-a)(x+a)| = \underbrace{\max\{a^2, 1-a^2\}}_{\text{Dobieramy tak by zminimalizować}}$, ale $a^2>0$

$a^2 = 1-a^2$

$a^2 = \frac{1}{2}$

$a = \pm\frac{\sqrt{2}}{2}$, $a>0$, $a = \frac{\sqrt{2}}{2}$

$x = \cos\frac{2k+1}{2n+2}\pi$

$n=1$: $x_0 = \cos\frac{1}{4}\pi = \frac{\sqrt{2}}{2}$

$x_1 = \cos\frac{3}{4}\pi = -\frac{\sqrt{2}}{2}$

b) $|(x-b)x(x+b)| = |x^3-b^2x|$

$f'(x) = 3x^2-b^2$

$f'(0)$: $3x^2-b^2=0 \Rightarrow x = \pm\frac{b}{\sqrt{3}}$

$f'(-1) =$

$f(-1) = f(1) = |1-b^2|$ $\qquad$ $f\left(\frac{b}{\sqrt{3}}\right) = f\left(\frac{-b}{\sqrt{3}}\right) = \frac{b^3}{(\sqrt{3})^3} - \frac{b^3}{\sqrt{3}} = \frac{2b^3}{3\sqrt{3}}$

$\max_{[-1,1]}|f(x)| = \underbrace{\max\left\{1-b^2, \frac{2b^3}{3\sqrt{3}}\right\}}_{\text{Minimalizujemy}}$

$1-b^2 = \frac{2b^3}{3\sqrt{3}}$ $\qquad$ $1-\frac{3}{4} = \frac{2\cdot\frac{3\sqrt{3}}{8}}{3\sqrt{3}} \Rightarrow \frac{1}{4} = \frac{1}{4} = 0 = 0$

$b = \frac{\pm\sqrt{3}}{2}$

$x = \cos\frac{2k+1}{2n+2}\pi$

$n=2$ $\quad x_0 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$x_1 = \cos\frac{3\pi}{6} = 0$

$x_2 = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

⑥ z wykładu:

$$\max_{-1\le x\le 1} |f(x) - L_n(x)| \le \max_{-1\le x\le 1} \frac{|f^{(n+1)}(x)|}{(n+1)!} \cdot \max_{-1\le x\le 1} |(x-x_0) \dots (x-x_n)|$$

wikipedia → Polynomial interpolation → equally spaced intervals:

$$\left| \prod_{i=0}^{n} (x-x_i) \right| = \prod_{i=0}^{n} |x-x_i| \le \frac{n!}{4} \cdot h^{n+1}, \quad h = \frac{b-a}{n} = ?$$

$$\le \frac{n!}{4} \cdot \left(\frac{2}{n}\right)^{n+1}$$

$$f(x) = e^{\frac{x}{3}}$$

$$f'(x) = \frac{1}{3} e^{\frac{x}{3}}, \quad f''(x) = \frac{1}{9} e^{\frac{x}{3}}$$

$$f^{(n)}(x) = \left(\frac{1}{3}\right)^n e^{x/3}, \quad f^{(n+1)} = \left(\frac{1}{3}\right)^{n+1} e^{x/3}$$

rosnące max: $x=1$

$$\max_{-1\le x\le 1} |f(x) - L_n(x)| \le \max_{-1\le x\le 1} \frac{\left|\left(\frac{1}{3}\right)^{n+1} \cdot e^{x/3}\right|}{(n+1)!} \cdot \frac{n!}{4} \cdot \left(\frac{2}{n}\right)^{n+1}$$

$$\le \frac{\left(\frac{1}{3}\right)^{n+1} e^{1/3}}{(n+1)} \cdot \frac{1}{4} \cdot \left(\frac{2}{n}\right)^{n+1} \le 10^{-16}$$

$$n = 11 \rightarrow \quad = \frac{\left(\frac{1}{3n}\right)^{n+1} \cdot e^{1/3} \cdot \left(\frac{2}{n}\right)^{n+1}}{4(n+1)} \le 10^{-16}$$

Wielomiany Czebyszewa ?:

$$\max_{x\in[-1,1]} |p_{n+1}(x)| = \frac{1}{2^n},$$

$$\max_{-1\le x\le 1} |f(x) - L_n(x)| \le \frac{\left(\frac{1}{3}\right)^{n+1} e^{1/3}}{(n+1)!\, 2^n} \le 10^{-16}$$

$$\Rightarrow \boxed{n = 11}$$

ANL-7

3/

7. $\overset{[x_0, x_1, \dots, x_n]}{\text{Interp\_Newton}}(x, \overset{[f_0, f_1, \dots, f_n]}{f})$: $b_0, b_1, \dots, b_{30}$ ← co najwyżej

$$L_n(x) = b_0 + b_1(x-x_0) + \dots + b_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$\forall i:\ L_n(x_i) = f_i \quad (i=0,\dots,n)$$

Fakt z wykładu: $L_{n+1}(x) = L_n(x) + b_{n+1}\,p_{n+1}(x)$

(dodanie obserwacji ⇒ liniowe)

W szczególności:

$$L_{31}(x_{31}) = L_{30}(x_{31}) + b_{31}\,p_{31}(x_{31}) \qquad (1)$$

Z jednoznaczności: $f(x_{31}) = L_{31}(x_{31}) \qquad (2)$

z (1) i (2):

$$f(x_{31}) = L_{30}(x_{31}) + b_{31}\,p_{31}(x_{31})$$

$$\boxed{b_{31} = \frac{f(x_{31}) - L_{30}(x_{31})}{p_{31}(x_{31})}} - O(n)$$

liniowy czas

$\{b_0, b_1, \dots, b_n\}$ - Wyliczamy z Interp_Newton

$$L_{30}(x_{31}) = b_0 + b_1(x_{31}-x_0) + b_2(x-x_0)(x-x_1) + \dots + b_{30}(x-x_0)\cdots(x-x_{29})$$

$$\overset{?}{=}\ b_0 + (x-x_0)[b_1 + (x-x_1)(b_2 + \dots (b_{29} + (x-x_{28})b_{30}))))$$

$L_{30}(x_{31})$

^ ~~Horner~~ Hornera $x := (x - x_i)$ $\quad O(n)$

$$p_{31}(x_{31}) = (x_{31}-x_0)(x_{31}-x_1)\cdots(x_{31}-x_{30}) \qquad O(n)$$