① a) $\sum_{k=0}^{n} \binom{n}{k} p^{k}(1-p)^{n-k} \overset{?}{=} 1$

Wzór dwumianowy Newtona:

$[p+(1-p)]^{k} = 1^{k}$

b) $\sum_{k=0}^{n} \binom{n}{k} k\, p^{k}(1-p)^{n-k} = np$

$\| \; k=0$

$\sum_{k=1}^{n} \binom{n}{k} k\, p^{k}(1-p)^{n-k} = *$

$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} = \frac{(n-1)!}{(k-1)!\,(n-k)!}$

$\binom{n-1}{k-1}\frac{n}{k} = \binom{n}{k} \quad , k \neq 0$

$* = \sum_{k=1}^{n} k \cdot \frac{n}{k} \binom{n-1}{k-1} p^{k}(1-p)^{n-k}$

$= n \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k}(1-p)^{n-k}$

$= n \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k+1}(1-p)^{n-k-1} =$

$= np \underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} p^{k}(1-p)^{\overset{n-k-1}{n-1-k}}}_{(p+1-p)^{n-1}} = np$

② a) $\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = 1$

$\frac{1}{e^{\lambda}}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$ Szereg Maclaurina dla $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\frac{1}{e^{\lambda}}\cdot e^{\lambda} = 1$

b) $\sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \lambda$

$$\frac{1}{e^{-\lambda}}\sum_{k=0}^{\infty}\frac{k\lambda^k}{k!} = \frac{1}{e^{-\lambda}}\sum_{k=0}^{\infty}\frac{\lambda^k}{(k-1)!} = \frac{1}{e^{-\lambda}}\sum_{k=1}^{\infty}\frac{\lambda\lambda^k}{k!}$$

$$= \frac{\lambda}{e^{-\lambda}}\cdot\underbrace{\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}}_{e^{-\lambda}\text{ Maclaurin}} = \frac{\lambda}{e^{-\lambda}}\cdot e^{-\lambda} = \lambda$$

RPIS 2/4

③ $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt \, , p>0$

T. $\Gamma(n) = (n-1)! \, , n \in \mathbb{N}$

$\Gamma(1) = \int_0^\infty t^0 e^{-t} = \int_0^\infty e^{-t} = -e^{-t}\Big|_0^\infty = 1$

Wystarczy pokazać że $\Gamma(n) = n\Gamma(n-1)$

Całkujemy przez części $\left[\int f g' = fg - \int f' g\right]$

$$\int_0^\infty t^{p-1}(-e^{-t})' dt = \left[t^{p-1} \cdot -e^{-t}\right]_{t=0}^{\infty} - \int_0^\infty (p-1) t^{p-2} \cdot (-e^{-t}) dt$$

$$\underbrace{\left[\frac{t^{p-1}}{e^t}\right]_{t=0}^{t=\infty}}_{0} + \int_0^\infty (p-1) t^{p-2} e^{-t} dt =$$

$$(p-1) \int_0^\infty t^{p-2} e^{-t} dt = (p-1)\Gamma(p-1) \; ■$$

④ $f(x)=\lambda \exp(-\lambda x), \lambda>0$

a) $\int_0^\infty f(x)dx$

$$\int_0^\infty \lambda e^{-\lambda x}dx \left[\begin{array}{l} -\lambda x = t \\ -\lambda dx = dt \end{array} \right.$$

$$-\int_0^\infty e^t dt = -(e^t)\Big|_{t=0}^{\infty}$$

$$\lim_{T\to\infty}\left(-\frac{1}{\underset{\searrow \infty}{e^{\lambda T}}} - (-e^0)\right) = 1$$

b) $\int_0^\infty x f(x)dx = \int_0^\infty x\lambda e^{-\lambda x}dx$

„przez części"

$$\int_0^\infty x\cancel{\lambda}(-e^{-\lambda x})' = \underbrace{\left[x(-e^{-\lambda x})\right]_0^\infty}_{0} - \int_0^\infty -e^{-\lambda x}dx =$$

$$\left[-\frac{e^{-\lambda x}}{-\lambda}\right]_{x=0}^{x=\infty} = \frac{e^0}{\lambda} = \frac{1}{\lambda}$$

⑤ $$D_n = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{bmatrix} \to \begin{bmatrix} n & 0 & 0 & 0 \cdots 0 \\ 1 & 1 & & \\ 1 & & 1 & \\ \vdots & & & \ddots \\ 1 & & & & 1 \end{bmatrix}$$

Dodajemy $w_1 := \sum_{k=2}^{n} w_n$, gdzie w_i - i-ty wiersz macierzy

Otrzymujemy macierz trójkątną której wyznacznik łatwo policzyć

$D_n = n \cdot 1 \cdot 1 \cdot 1 \ldots \cdot 1 = n$

RPIS 4/4

⑥ $I = \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{x^2+y^2}{2}\right\} dy\, dx$$

$x = r\cos\varphi$, $y = r\sin\varphi$ – współrzędne biegunowe

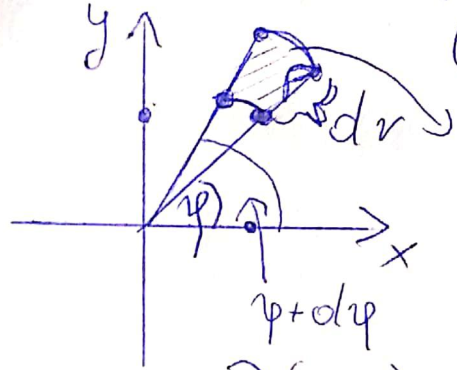
$\sin\varphi = \frac{y}{r}$

$\cos\varphi = \frac{x}{r}$

T: $I^2 = 2\pi$

$x^2+y^2 = r^2$ – koło

$(x,y) \to (r,\varphi)$



$\int dr\, d\varphi$

Jakobian przekształcenie liniowego

$x = r\cos\varphi$

$y = r\sin\varphi$

$$|J| = \frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$= r\cos^2\varphi + r\sin^2\varphi = r(\cos^2\varphi + \sin^2\varphi) = \underline{r}$$

Czyli liczymy $\int_0^{2\pi}\int_0^{\infty} \exp\left\{-\frac{r^2}{2}\right\} r\, dr\, d\varphi$

~~x~~ $:(-\infty,+\infty) \to r : (0,+\infty)$, $y : (-\infty,+\infty) \to \varphi : (0, 2\pi)$

$u = -\frac{r^2}{2}$

$du = -r\, dr$

$$\int_0^{2\pi}\int_0^{\infty} -e^u\, du\, d\varphi = \int_0^{2\pi}\int_0^{-\infty} e^u\, du\, d\varphi = \int_0^{2\pi} 1\, d\varphi = \underline{2\pi}$$

Czyli $I^2 = 2\pi$ ▨

⑦ $\bar{S}$ - średnia $s_1, \dots, s_n$

a)

$$\sum_{k=1}^{n}(x_k-\bar{x})^2=\sum_{k=1}^{n}x_k^2-n\bar{x}^2$$

$$\sum_{k=1}^{n}(x_k^2-2x_k\bar{x}+\bar{x}^2)-\sum_{k=1}^{n}x_k^2+n\bar{x}^2=$$

$$\sum_{k=1}^{n}-2x_k\bar{x}+2(n\bar{x}^2)=-2\bar{x}\underbrace{\left(\sum_{k=1}^{n}x_k-n\bar{x}\right)}_{0}=0$$

b)

$$\sum_{k=1}^{n}(x_k-\bar{x})(y_k-\bar{y})=\sum_{k=1}^{n}x_ky_k-n\bar{x}\bar{y}$$

$$\cancel{\sum_{k=1}^{n}x_ky_k}+\underbrace{\sum_{k=1}^{n}\bar{x}_k\bar{y}_k}_{n\cdot}-\sum_{k=1}^{n}x_k\bar{y}-\sum_{k=1}^{n}\bar{x}y_k-\cancel{\sum_{k=1}^{n}x_ky_k}+n\bar{x}\bar{y}$$

$$2n\bar{x}\bar{y}-\bar{y}\sum_{k=1}^{n}x_k-\bar{x}\sum_{k=1}^{n}y_k=$$

$$2n\bar{x}\bar{y}-\left(\frac{\sum_{k=1}^{n}y_k}{n}\cdot\sum_{k=1}^{n}x_k\right)-\left(\frac{\sum_{k=1}^{n}x_k}{n}y_k\right)=0$$ ▨

⑧ $\vec{u}, X \in R^n, \Sigma \in R^{n \times n}$ RPIS 3/4

$S = (X - \vec{u})^T \Sigma^{-1} (X - \vec{u})$, $Y = AX$

$T: S = (Y - A\vec{u})^T (A \Sigma A^T)^{-1} (Y - A\vec{u})$

$(AB)C = A(BC)$

przydatne: $Id A = A Id = A$ (1), $A(B + C) = AB + AC$ (2)

$M = (M^T)^T$, $(M + N)^T = M^T + N^T$, $(MN)^T = N^T M^T$ (3)

$M = (M^{-1})^{-1}$, $(M^T)^{-1} = (M^{-1})^T$, $(MN)^{-1} = N^{-1} M^{-1}$ (4)

Najpierw $Y = AX$

$$S = (AX - A\vec{u})^T (A \Sigma A^T)^{-1} (AX - A\vec{u})$$

$$[A(X - \vec{u})]^T (A \Sigma A^T)^{-1} A (X - \vec{u}) =$$

$$(X - \vec{u})^T A^T \underbrace{[A(\Sigma A^T)]^{-1}} A (X - \vec{u}) \cancel{A} = (*)$$

$$[A(\Sigma A^T)]^{-1} = (\Sigma A^T)^{-1} A^{-1} = (A^T)^{-1} \Sigma A^{-1}$$

$$(*)\ (X - \vec{u})^T \underbrace{A^T (A^T)^{-1}} \Sigma^{-1} \underbrace{A^{-1} A} (X - \vec{u}) =$$

$$(X - \vec{u})^T \Sigma^{-1} (X - \vec{u})$$