(1)

k	0	1	2
x_k	$a=0$	2	$4=b$
y_k	-8	8	-8

a)

$$s(x)=\begin{cases} s_1(x)=Ax^3+Bx^2+Cx+D & : x\in[0,2] \\ s_2(x)=Ex^3+Fx^2+Gx+H & : x\in[2,4] \end{cases}$$

$s_1(0)=D=-8$

$s_1(2)=8 \Rightarrow \begin{cases} 8A+4B+2C+D=8 & (1) \\ 8E+4F+2G+H=8 & (2) \end{cases}$

$s_2(4)=-8 \Rightarrow 64E+16F+4G+H=-8 \ (3)$

$$s'(x)=\begin{cases} s_1'(x)=3Ax^2+2Bx+C, & x\in[0,2] \\ s_2'(x)=3Ex^2+2Fx+G, & x\in[2,4] \end{cases}$$

$s_1'(2)=s_2'(2) \Rightarrow 12A+4B+C=12E+4F+G \ (4)$

$$s''(x)=\begin{cases} s_1''(x)=6Ax+2B, & x\in[0,2] \\ s_2''(x)=6Ex+2F, & x\in[0,2] \end{cases}$$

$s_1''(2)=s_2''(2) \Rightarrow 12A+2B=12E+2F \ (5)$

naturalna $\Rightarrow s_1''(0)=s_2''(4)=0$ $\begin{cases} 2B=0 \Rightarrow B=0 \\ 24E+2F=0 \quad F=-12E \ (6) \\ \cancel{24E+2F=2B} \end{cases}$

Podstawiamy $\begin{cases} B=0 \\ F=-12E \end{cases}$

$$\begin{cases} 8A+\cancel{4B}+2C=16 \ (1) \\ 8E-48E+2G+H=8 \ (2) \\ 64E-192E+4G+H=-8 \ (3) \\ 12A+C=12E-48E+G \ (4) \\ 12A=12E-24E \Rightarrow A=-E \end{cases} \Rightarrow$$

(4) $12A+C=-36E+G$

$C=-24E+G$

(1) $2C=16+8E$

$C=8+4E$

$G=8+28E$

(2) $-40E+2(8+28E)+H=8$

$16E+H=-8 \Rightarrow H=-8-16E$

(3) $-128E+4(8+28E)+(-8)-16E=-8 \Rightarrow E=1 \Rightarrow A=-1$

$G=36 \quad H=-24 \quad C=12$

$$s(x)=\begin{cases} -x^3+12x-8 & x\in[0,2] \\ x^3-12x^2+36x-24 & x\in[2,4] \end{cases}$$

b) x_k $\quad -1 \quad -1/2 \quad 1/2 \quad 1$ $\qquad h_k = x_k - x_{k-1}$

y_k $\quad 4 \quad 2 \quad 6 \quad -24$

$h_1 \quad h_2 \quad h_3$ $\qquad h_1 = \frac{1}{2},\ h_2 = 1,\ h_3 = \frac{1}{2}$

$\lambda_1 \quad \lambda_2$ $\qquad \lambda_k = \frac{h_k}{h_k + h_{k+1}}$

$0 = M_0 \quad M_1 \quad M_2 \quad M_3 = 0$

$\lambda_1 = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3} \qquad \lambda_2 = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

$$s_k(x) = h_k^{-1}\left[\frac{1}{6}M_{k-1}(x_k - x)^3 + \frac{1}{6}M_k(x - x_{k-1})^3 + \left(y_{k-1} - \frac{1}{6}M_{k-1}h_k^2\right)(x_k - x) + \left(y_k - \frac{1}{6}M_k h_k^2\right)(x - x_{k-1})\right]$$

Ilorazy:

-1	4		
$-\frac{1}{2}$	2	-4	
$\frac{1}{2}$	6	-8	$-\frac{8}{3}$ $f[x_0,x_1,x_2]$
1	-24	-36	$-\frac{56}{3}$ $f[x_1,x_2,x_3]$

Momenty:

$$\lambda_k M_{k-1} + 2M_k + (1-\lambda_k)M_{k+1} = 6f[x_{k-1},x_k,x_{k+1}]$$

$$\begin{cases} 0 + 2M_1 + (1-\lambda_1)M_2 + 0 = -16 \\ 0 + \lambda_2 M_1 + 2M_2 + 0 = -112 \end{cases}$$

$$\begin{cases} 2M_1 + \frac{2}{3}M_2 = -16 \Rightarrow 6M_1 + 2M_2 = -48 \\ \frac{2}{3}M_1 + 2M_2 = -112 \end{cases}$$

$\frac{16}{3}M_1 = 64 \Rightarrow M_1 = 12,\ M_2 = -60$

$k=1 \quad x \in [-1, -\frac{1}{2}]$ $\quad (x_{k-1}, x_k)$

$M_{k-1} = M_0 = 0$

$$s_1(x) = 2\left[\frac{1}{6}\cdot 12\cdot(x+1)^3 + \frac{1}{6}\cdot 4\left(-\frac{1}{2} - x\right) + \left(2 - \frac{1}{6}\cdot 12\cdot\frac{1}{4}\right)(x+1)\right]$$

$$4(x+1)^3 + (-4-8x) + 3(x+1)$$

$$= 4x^3 + 12x^2 + 12x + 4 - 4 - 8x + 3x + 3 = 4x^3 + 12x^2 + 7x + 3$$

$k=2$

$$s_2(x) = \left[\frac{1}{6}\cdot 12\left(\frac{1}{2}-x\right)^3 + \frac{1}{6}\cdot(-60)\left(x+\frac{1}{2}\right)^3 + \left(2 - \frac{1}{6}12\right)\left(\frac{1}{2}-x\right) + \left(-6 - \frac{1}{6}(-60)\right)\left(x + 1/2\right)\right]$$

$$= -12x^3 - 12x^2 - 5x + 1$$

$$s_3(x) = 2\left[\frac{1}{6}\cdot(-60)(1-x)^3 + \left(-6 - \frac{1}{6}\cdot(-60)\cdot\frac{1}{4}\right)(1-x) + (-24)\left(x - \frac{1}{2}\right)\right]$$

$$= 20x^3 - 60x^2 + 19x - 3$$

$$S(x) = \begin{cases} 4x^3 + 12x^2 + 7x + 3 & ;\ x \in [-1, \frac{1}{2}) \\ -12x^3 - 12x^2 - 5x + 1 & ;\ x \in [-\frac{1}{2}, \frac{1}{2}] \\ 20x^3 - 60x^2 + 19x - 3 & ,\ x \in [\frac{1}{2}, 1] \end{cases}$$

② 1° $f(x_k) = y_k$

2° ciągłość f $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

w $-1 = x$: $(-1)^3 + 6(-1)^2 + 18(-1) + 13 = -5(-1)^3 - 12(-1)^2 + 7$

$-1 + 6 - 18 + 13 = 5 - 12 + 7$

$0 = 0$

✓

w $0 = x$: $7 = 7$ ✓

w $1 = x$: $5 - 12 + 7 = -1 + 6 - 18 + 13$ ✓

3° ciągłość $f'(x)$

$f_1'(x) = 3x^2 + 12x + 18$

$f_2'(x) = -15x^2 - 24x$

$f_3'(x) = 15x^2 - 24x$

$f_4'(x) = -3x^2 + 12x - 18$

w -1: $3 - 12 + 18 = -15 + 24$ ✓

w 0: ✓

w 1: $15 - 24 = -3 + 12 - 18$ ✓

4° ciągłość $f''(x)$

$f_1''(x) = 6x + 12$

$f_2''(x) = -30x - 24$

$f_3''(x) = 30x - 24$

$f_4''(x) = -6x + 12$

w -1: $-6 + 12 = 30 - 24$ ✓

w 0: ✓

w 1: $30 - 24 = -6 + 12$ ✓

5° $f_{1,2,3,4}(x) \in \Pi_3$ ✓

6° $f''(a) = f''(b) = 0$

$f''(-2) = 0$ $f''(2) = 0$ ✓

Zatem funkcja jest NIFS3

③ $f(x)=\begin{cases} s_1(x)=\cancel{+}2020x & x\in[-3,-1) \\ s_2(x)=ax^3+bx^2+cx+d & x\in[-1,1) \\ s_3(x)=-2020x & x\in[1,2] \end{cases}$

1° ciągłość

$\cancel{f}_1(-1)=-2020$ $\qquad$ $s_3(1)=-2020$

$\qquad\qquad\qquad\qquad \parallel$ $\qquad\qquad\qquad\qquad\qquad \parallel$

$s_2(-1)=-a+b-c+d$ $\qquad$ $s_2(1)=a+b+c+d$

2° ciągłość f'

$s_1'(-1)=2020$ $\qquad$ $s_3'(1)=-2020$

$3ax^2+2bx+c$ $\qquad\qquad\qquad\qquad \parallel$

$s_2'(-1)=3a-2b+c$ $\qquad$ $s_2'(1)=3a+2b+c$

3° $6ax+2b=f$ ciągłość f''

$\cancel{f}s_1''(-1)=0=f''(a)$ $\qquad$ $s_3''(1)=0=f''(b)$

$s_2''(-1)=-6a+2b$ $\qquad$ $s_2''(1)=6a+2b$

z 2° $\left.\begin{matrix} 3a-2b+c=2020 \\ 3a+2b+c=-2020 \end{matrix}\right\} 4b=-4040$

/ sprzeczność

z 3° $\begin{cases} -6a+2b=0 \\ 6a+2b=0 \end{cases} \Rightarrow b=0$ więc nie istnieją takie a,b,c,d że $f(x)$ to NFS3

(4) Chcemy policzyć:

$$d_k = \lambda_k M_{k-1} + 2M_k + (1-\lambda_k)M_{k+1}, \quad (k = 1, 2, \dots, n-1)$$

$$d_k = 6f[x_{k-1}, x_k, x_{k+1}], \quad \lambda_k = \frac{h_k}{h_k + h_{k+1}}, \quad h_k = x_k - x_{k-1}$$

Algorytm I

$q_0 := 0$
$u_0 := 0$
$p_k := \lambda_k q_{k-1} + 2$
$q_k := (\lambda_k - 1)/p_k$
$u_k := (d_k - \lambda_k u_{k-1})/p_k$
$\Big\} \; k = 1, 2, \dots, n-1$

Algorytm II $(M_n = M_0 = 0)$

$M_{n-1} := u_{n-1}$
$M_k := u_k + q_k M_{k+1}$
$(k = n-2, \dots, 1)$

$$2M_1 + (1-\lambda_1)M_2 = d_1 \quad (1)$$
$$\lambda_2 M_1 + 2M_2 + (1-\lambda_2)M_3 = d_2 \quad (2)$$
$$\lambda_3 M_2 + 2M_3 + (1-\lambda_3)M_3 = d_3 \quad (3)$$
$$\dots$$

$\frac{d_1}{2} = \frac{d_1 - \lambda_1 u_0}{\lambda_1 q_0 + 2}$, $\lambda_1 = 0$

z (1) $M_1 = \frac{d_1}{2} + \left(\frac{\lambda_1 - 1}{2}\right) M_2 = u_1 + q_1 M_2$, $\quad \frac{\lambda_1 - 1}{2}$

wstawiamy do (2) $\lambda_2 u_1 + \lambda_2 q_1 M_2 + 2M_2 + (1-\lambda_2)M_3 = d_2$

$\lambda_2 q_1 M_2 + 2M_2 = d_2 - \lambda_2 u_1 + (\lambda_2 - 1)M_3$ $\quad \frac{d_2 - \lambda_2 u_1}{\lambda_2 q_1 + 2}$ //

$$M_2 = \left(\frac{d_2 - \lambda_2 u_1}{\lambda_2 q_1 + 2}\right) + \left(\frac{\lambda_2 - 1}{\lambda_2 q_1 + 2}\right) M_3 = u_2 + q_2 M_3 \quad \left(\frac{\lambda_2 - 1}{\lambda_2 q_1 + 2}\right)$$

$M_{n-2} = u_{n-2} + q_{n-2} M_{n-1}$ - przedostatnie

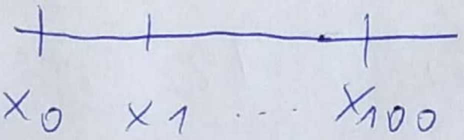
do (n-1) $M_n = 0$: $\lambda_{n-1} M_{n-2} + 2M_{n-1} = d_{n-1}$ - ostatnie

$\lambda_{n-1} u_{n-2} + \lambda_{n-1} q_{n-2} M_{n-1} + 2M_{n-1} = d_{n-1}$

$$M_{n-1} = \frac{d_{n-1} - \lambda_{n-1} u_{n-2}}{\lambda_{n-1} q_{n-2} + 2} = \frac{u_{n-1}}{p_{n-1}}$$

liniowo liczymy M_k

(5) $x := [x_0, \dots, x_n]$, $y := [y_0, \dots, y_n]$



$s_n(x_k) = y_k$

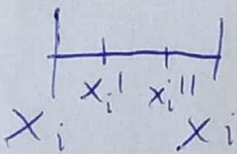
$z := [z_0, z_1, \dots, z_m]$ ← NSpline3(x, y, z)

$Z := [s_n(z_0), s_n(z_1), \dots, s_n(z_m)]$ $m < 2n$

$m < \overset{2\cdot 100}{200}$ → NSPline ~~przybliża~~ liczymy w 200 punktach

Do każdego podprzedziału $[x_i, x_{i+1}]$ $i = 0, \dots, 98$

wstaw liczymy po 2 punkty równolegle ~~X~~ x_i', x_i''



Dla każdego z podprzedziałów mamy wtedy po 4 punkty więc możemy jednoznacznie wyznaczyć wielomian $\in \Pi_3$

Wyznaczamy A, B, C, D dla $\overset{s(x)}{Ax^3 + Bx^2 + Cx + D}$

Następnie $s'(x) = 3Ax^2 + Bx + C = 0 \to \Delta \; x_1, x_2$

i znajdujemy miejsca zerowe, gdzie pochodna jest zerowa (możliwe ekstrema)

Dla tych miejsc zerowych (jeśli $Mz \in [x_i, x_{i+1}]$) i ustalonego ~~dostatecznie~~ małego ε sprawdzamy czy ~~$s(x+$~~ ~~$s(x+\varepsilon)$~~

$sgn(s(x+\varepsilon)) \cdot sgn(s(x-\varepsilon)) < 0$

Jeśli tak to wyznaczyliśmy ekstremum o wartości $s(x)$, $x \in [x_i, x_{i+1}]$