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1 Algebraic Multiscale Solver

Let the fine-scale elliptic system of equations be define by:

$$T_f p_f = F_f \tag{1}$$

where T_f, p_f and F_c represent respectively, the transmissibility matrix, the pressure solution and the source and sink terms on the fine-scale.

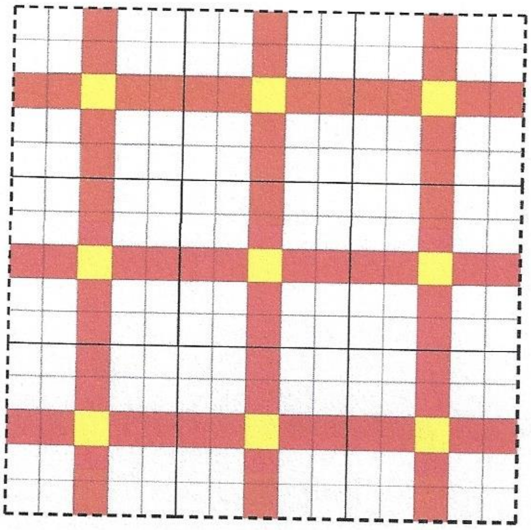

Figure 1: Wirebasket hierarchical segregation of fine-scale volumes in a two-dimension computational DOMAIN: Nodes (yellow) , Edges (red), Internals (white)

REFER FIRST TO DUAL COARSE MESH →

We define a hierarchical wirebasket segregation, the grouping of fine-scale volumes into different categories: (See Figure 1) Nodes, Edges, Surfaces and Internals in 3 d, or Nodes, Edges, and Internals in 2 d.

Let G be a permutation matrix that sorts a vector following the wirebasket CONFIGURATION

$$\widetilde{X}_f = \begin{bmatrix} x_i \\ x_s \\ x_e \\ x_n \end{bmatrix} = GX_f \tag{2}$$

where the subscript n, e, s and i stand for all groups of the wirebasket segregation: nodes, edges, surfaces and internals.

Analogously, the wirebasket matrix is the transmissibility matrix permuted to group the wirebasket categories together. Thus, we define it as:

$$\widetilde{A} = \begin{bmatrix} \widetilde{A}_{ii} & \widetilde{A}_{is} & \widetilde{A}_{ie} & \widetilde{A}_{in} \\ \widetilde{A}_{si} & \widetilde{A}_{ss} & \widetilde{A}_{se} & \widetilde{A}_{sn} \\ \widetilde{A}_{ei} & \widetilde{A}_{es} & \widetilde{A}_{ee} & \widetilde{A}_{en} \\ \widetilde{A}_{ni} & \widetilde{A}_{ns} & \widetilde{A}_{ne} & \widetilde{A}_{nn} \end{bmatrix} = GT_f G^T \tag{3}$$

VOLUME CATEGORY

where the block matrix $\widetilde{A}_{xy}$ stands for the influence of x upon y, and where x and y are one of the groups of the wirebasket hierarchy.

Hence, the linear system of equations reordered is given by:

$$\widetilde{A}\widetilde{p}_f = \widetilde{F}_f \tag{4}$$

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The AMS devises a upper triangular matrix by neglecting the influence of block matrices bellow the main diagonal. This way, the hierarchy of the wirebasket matrix is achieved, as the volumes that belong to a group only have influence in other volumes of the same or higher dimension categories.

$$M = \begin{bmatrix} \widetilde{A}_{ii} & \widetilde{A}_{is} & \widetilde{A}_{ie} & \widetilde{A}_{in} \\ 0 & M_{ss} & \widetilde{A}_{se} & \widetilde{A}_{sn} \\ 0 & 0 & M_{ee} & \widetilde{A}_{en} \\ 0 & 0 & 0 & M_{nn} \end{bmatrix} \tag{5}$$

Nonetheless, in order to ensure mass conservation the influence of those terms are redistributed SUCH as:

$$M_{ss} = \widetilde{A}_{ss} + diag\left(\sum_k^s \widetilde{A}_{ki}\right) \tag{6}$$

$$M_{ee} = \widetilde{A}_{ee} + diag\left(\sum_k^e (\widetilde{A}_{ki} + \widetilde{A}_{ks}\right) \tag{7}$$

The modified system of equations is defined as:

$$Mp = \widetilde{F} \tag{8}$$

where p is an approximation of $\widetilde{p}_f$ and $\widetilde{F}$ is the source and sink term wirebasket sorted.

Therefore, M^{-1} can be found by a simple backward substitution due to the upper triangular nature of M.

$$M^{-1} = BM_{nn}^{-1} + C \tag{9}$$

DISCUTIR MELHOR ESSA SEPARAÇÃO JÁ AQUI !!

where:

$$B = \begin{bmatrix} 0 & 0 & 0 & A_{ii}^{-1}[\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} - \widetilde{A}_{in} - \widetilde{A}_{is}M_{ss}^{-1}(\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} - \widetilde{A}_{sn})] \\ 0 & 0 & 0 & M_{ss}^{-1}(\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} - \widetilde{A}_{sn}) \\ 0 & 0 & 0 & -M_{ee}^{-1}\widetilde{A}_{en} \\ 0 & 0 & 0 & I_{nn} \end{bmatrix} \tag{10}$$

and ~~where~~:

$$C = \begin{bmatrix} \widetilde{A}_{ii}^{-1} & -\widetilde{A}_{ii}^{-1}\widetilde{A}_{is}M_{ss}^{-1} & \widetilde{A}_{ii}^{-1}(\widetilde{A}_{is}^{-1}M_{ss}^{-1}\widetilde{A}_{se}M_{ee}^{-1} - \widetilde{A}_{ie}M_{ee}^{-1}) & 0 \\ 0 & M_{ss}^{-1} & -M_{ss}^{-1}\widetilde{A}_{se}M_{ee}^{-1} & 0 \\ 0 & 0 & M_{ee}^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

By premultiplying ~~the~~ Equation (8) by (9), we find:

$$\widetilde{p}_f \simeq p = (BM_{nn}^{-1} + C)\widetilde{F} = BM_{nn}^{-1}\widetilde{F}_c + C\widetilde{F} \tag{12}$$

Thus, we define pressure on the coarse scale as:

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$$p_c = M_{nn}^{-1}\widetilde{F}_c \tag{13}$$

Hence, we use this definition to rewrite Equation (12) as:

$$p = Bp_c + C\widetilde{F}_c \tag{14}$$

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As B projects coarse-scale pressure solution on the high-resolution grid, it can be also referred as the Prolongation Operator. On the other hand, C is known as the Correction Fuction as it improves the multiscale solution by capturing information, such as complex well behaviors, not take in account by the basis functions.

REESCREVER

At this point, we still lack a proper definition M_{nn}. To do so, we need to define a restriction operator. We can do so inserting equation 14 into 8.

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$$\chi(i,j) = \begin{cases} 1 & \text{se } \Omega_j^f \subset \Omega_i^c \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

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By inserting Equation (14) into ~~(4), and~~ premultiplying by the restriction operator we have:

$$\chi\widetilde{A}(Bp_c + C\widetilde{F}_c) = \chi\widetilde{F}_f \tag{16}$$

$$\chi\widetilde{A}Bp_c = \chi\widetilde{F}_f - \widetilde{A}C\widetilde{F}_c \tag{17}$$

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Thus:

$$M_{nn} = \chi\widetilde{A}B \quad \text{and} \quad q_n = \chi\widetilde{F}_f - \widetilde{A}C\widetilde{F}_c \tag{18}$$

Therefore, the multiscale solution can be obtained by substitute ING Equation (18) into (12) resulting in:

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DÁ PARA MELHORAR A ESCRITA, TEM PARTES CONFUSAS, SEM CONEXÃO CLARA

MAS CUIDADO COM NOTAÇÃO

ALGEBRAIC MULTI-SCHEME

Thus:

$$p_f \simeq= p_{ms} = B(\chi\widetilde{A}B)^{-1}q_n + C\widetilde{F}_f \tag{19}$$

1.1 AMS in structured grids using Two Point Flux Approximation

Two fine-scale elements a and b are connected, if the entries a,b and b,a in the transmissibility matrix T_f are non zero. In a standard two-point flux approximation (TPFA) , connections can be asserted if volumes share an edge in 2d, or a face in 3d. In the AMS context, this creates an natural uncoupling, as several entries in the wirebasket matrix are zero. Therefore, we can rewrite Equation (3) for a TPFA as:

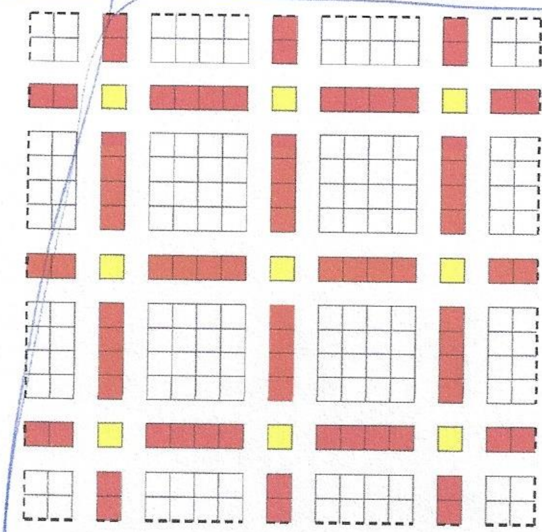

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Figure 2: Wirebasket hierarchical segregation of fine-scale volumes in a two-dimension computational : Nodes (yellow) , Edges (red), Internals (white)

$$\widetilde{A}_{\mathrm{TPFA}} = \begin{bmatrix} \widetilde{A}_{ii} & \widetilde{A}_{is} & 0 & 0 \\ \widetilde{A}_{si} & \widetilde{A}_{ss} & \widetilde{A}_{se} & 0 \\ 0 & \widetilde{A}_{es} & \widetilde{A}_{ee} & \widetilde{A}_{en} \\ 0 & 0 & \widetilde{A}_{ne} & \widetilde{A}_{nn} \end{bmatrix} \tag{20}$$

This leads to B and C being defined as:

$$\widetilde{B} = \begin{bmatrix} 0 & 0 & 0 & -A_{ii}^{-1}A_{if}M_{ss}^{-1}\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} \\ 0 & 0 & 0 & M_{ss}^{-1}\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} \\ 0 & 0 & 0 & -M_{ee}^{-1}\widetilde{A}_{en} \\ 0 & 0 & 0 & I_{nn} \end{bmatrix} \tag{21}$$

and where:

$$\widetilde{C} = \begin{bmatrix} \widetilde{A}_{ii}^{-1} & -\widetilde{A}_{ii}^{-1}\widetilde{A}_{is}M_{ss}^{-1} & \widetilde{A}_{ii}^{-1}\widetilde{A}_{is}^{-1}M_{ss}^{-1}\widetilde{A}_{se}M_{ee}^{-1} & 0 \\ 0 & M_{ss}^{-1} & -M_{ss}^{-1}\widetilde{A}_{se}M_{ee}^{-1} & 0 \\ 0 & 0 & M_{ee}^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

Figure

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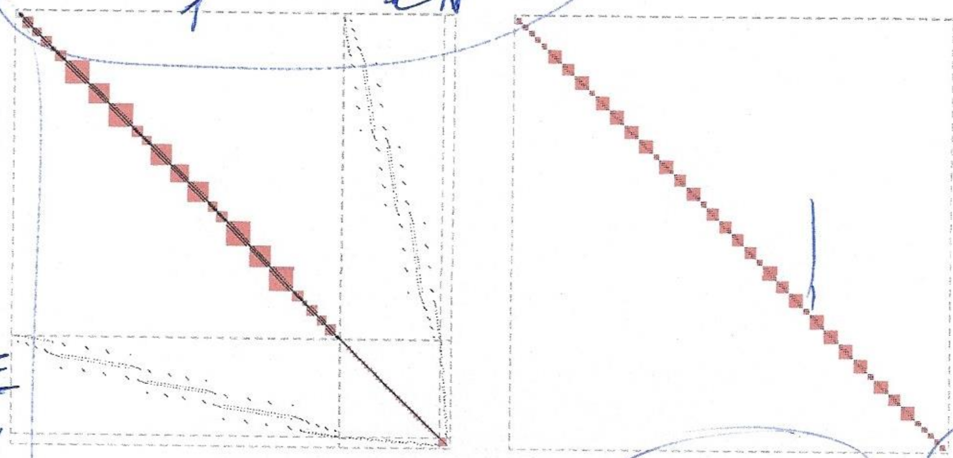

(a) $\widetilde{A}_{\text{TPFA}}$ (b) $\widetilde{A}_{ee}$ / M_{ee}

e A_{SS}/M_{SS}!

Figure 3: Sparsity pattern representation of a TPFA wirebasket matrix for a 2d structured mesh: The natural uncoupling of the duals grid 2 is reflected on the wirebasket matrix creating several block matrices (red).

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The sparsity representation of the TPFA wirebasket matrix illustrated in Figure 3 reproduces this natural uncoupling as $\widetilde{A}_{\text{TPFA}}$, $\widetilde{A}_{ee}/M_{ee}$ and $\widetilde{A}_{ss}/M_{ss}$ are comprised of several independent block matrices. Computational wise, finding M_{ee}^{-1}, M_{ss}^{-1}, and $\widetilde{A}_{ii}^{-1}$ becomes a compilation of easy and cheap parallelizable problems. Like the original MsFV, the AMS solves hierarchy ONLY the uncoupled basis functions on the edges by employing normalized Dirichlet boundary conditions on the nodes. The basis functions on edges become boundary conditions to compute the value of the surfaces basis functions. In similar way they are later used to compute the value of the internal basis functions. Partition of unity on the edges and surfaces is achieved because each fine-scale volume belong exclusively to a single edge or surface, and this group is isolated from the other groups by the definition.

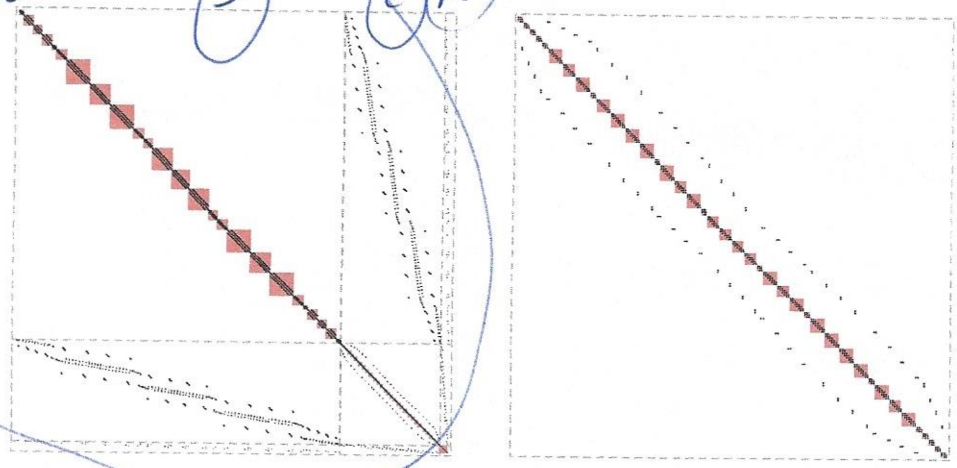
1.2 AMS in structured grids using a Multi Point Flux Approximation

Meanwhile, schemes that rely in a multi point flux approximation will give birth to full wirebasket matrices, such as defined in Equation 1. In these schemes, a connection may be established if two fine-scale elements share a common vertex. Figure 4 shows this phenomena, as it illustrates the sparsity pattern of the wirebasket MPFA matrix for standard 2d structured grid of the classical AMS. It is easy to notice that natural uncoupling of the matrices M_{ee} and M_{ss} in 3d is broken.

? (3)

ALTERNA MUITO 2D COM 3D FICANDO AS VEZES CONFUSO!

? PARECE A MESMA FIGURA QUE 3(a)?



(a) $\widetilde{A}_{\mathrm{MPFA}}$ (b) $\widetilde{A}_{ee}$ / M_{ee}

Figure 4: Sparsity pattern representation of a MPFA wirebasket matrix for a 2d structured mesh: The natural uncoupling of the duals grid is reflected on the wirebasket matrix creating several block matrices (red).

The problem worsens as we bring it to the context of unstructured grids in general. As illustrated in Figure 5, it is not possible to separate the edges. Thus, an edges may share a common face or a fine-scale volume. Once again, this is reflected on the sparsity pattern of the wirebasket matrix in Figure 6. It worth noting that even in simple case several edges were clustered together in a bigger block matrix. In turn, this has two major consequences on the AMS solution. Firstly, this clustering of the edges leads to the leakage of the basis functions out the support region of each coarse volume. Secondly, the loss of partition of unity as the shared fine-scale volumes are submitted to different boundary conditions in each edge it belongs. This turns the use of the standard AMS on unstructured grids as it is unfeasible.

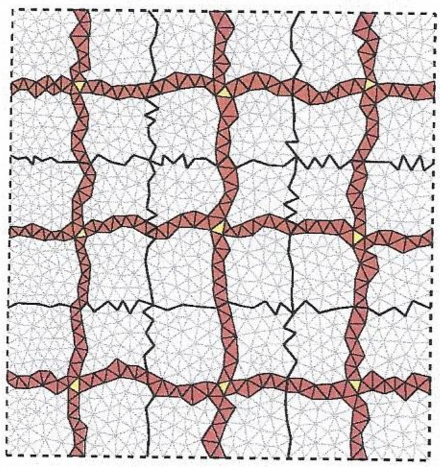


Figure 5: Wirebasket hierarchical segregation of unstructured fine-scale volumes in a two-dimension computational : Nodes (yellow) , Edges (red), Internals (white)

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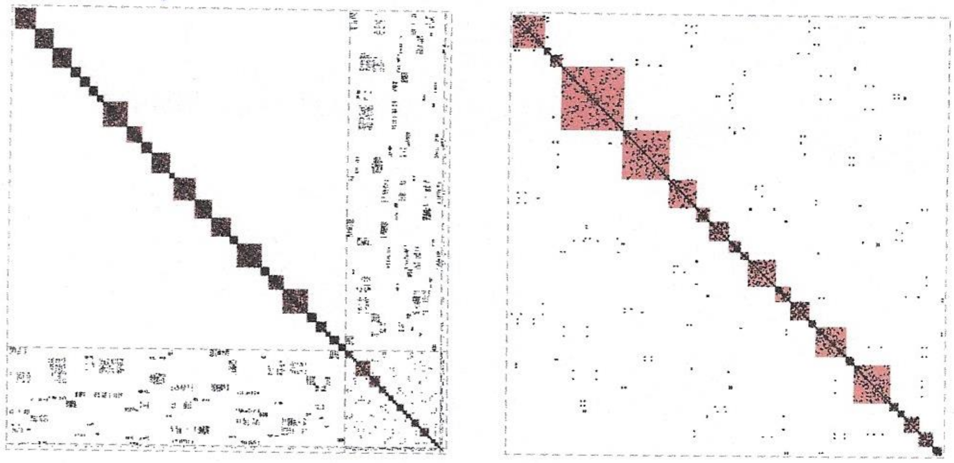

(a) $\widetilde{A}_{\text{MPFA}}$ (b) $\widetilde{A}_{ee}$ / M_{ee}

Figure 6: Sparsity pattern representation of a MPFA wirebasket matrix for a 2d unstructured fine and coarse scale mesh: The natural uncoupling of the duals grid is reflected on the wirebasket matrix creating several block matrices (red).

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1.3 AMS for unstructured grids

The main objective of this work is to enable the use of the AMS in truly unstructured grids. To do so, we devised two algorithms that prevent ~~the~~ leakage of the basis function whilst maintaining the partition of unity. For both of the algorithm s it is possible to recover ~~the uncoupled~~ nature that allow the AMS to be speeded up.

1.3.1 AMS with restricted basis function

SECOND and THIRD

The idea behind this fix is to ~~analyse the third~~ and ~~forth~~ row of the B matrix in Equation (10).

$$B_{sn} = M_{ss}^{-1}(\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} - \widetilde{A}_{sn}) \tag{23}$$

$$B_{en} = -M_{ee}^{-1}\widetilde{A}_{en} \tag{24}$$

Physically speaking, solving B_{en} is equivalent to solve:

$$M_{ee}B_{en} = -\widetilde{A}_{en} \tag{25}$$

In turn, this is equivalent of solving the M_{ee} transmissibility matrix of the edges restricted for all the n coarse volumes. As we know, the support region of a coarse volume IS the subset of a domain which contains non-zero elements only. In the context of multiscale problems, the support region of a basis function of the coarse volume i is comprised by ~~the~~ all dual coarse volumes that share the node i, excluding the boundaries of this region. Hence, instead of solving the Equation (25), we reformulate the problem as:

WHAT COARSE VOLUMES

COARSE VOLUME

$$M_{e^i}B_{e^ii} = -\widetilde{A}_{e^ii} \tag{26}$$

where i is one of the n, and e^i stands for the edges included in the support region of i.

In other words, for each i in n, we solve the problem restricted to the support region of the coarse volume i. In this way, we are explicitly imposing that the edges in this regions have no influence outside the of their respective support regions. Therefore preventing the leakage of the basis functions.

This problem can be similarly reframed for the surfaces. Thus, instead of solving Equation 23, we solve:

$$M_{ss}B_{sn} = H_{sn} \tag{27}$$

where $H_{sn} = (\widetilde{A}_{se}M_{ee}^{-1}\widetilde{A}_{en} - \widetilde{A}_{sn})$

Equation 27 is solved restricted to the support region of each coarse volume n.

$$M_{s^i}B_{s^ii} = H_{s^ii} \tag{28}$$

where i is one of the n coarse volumes, and s^i stands for the surfaces included in the support region of i

This easy fix eliminates the so undesirable leakage of the basis functions. However, as we still have fine-scale volumes connecting different blocks of edges and surfaces, we cannot assure partition of unity. In order to do so, we use the solution proposed by Moyner to explicitly normalize the B_{sn} and B_{en}: ref

$$B_{in}^{cor} = \frac{B_{in}}{\sum_{k=1}^{n} B_{ik}} \tag{29}$$

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i → INTERNAL

where i is either s or e and where B_{in}^{cor} stands for the corrected basis functions .

1.3.2 AMS with restricted edges

Another possible approach is to tackle the problem of finding $M_{ee}^{-}1$ and $M_{ss}^{-}1$ directly . By definition, we know that:

$$M_{ee}M_{ee}^{-1} = I_{ee} \tag{30}$$

Once again, we can look at this problem as a collection of e different problems:

$$M_{ee}M_{ei}^{-1} = \delta_{ei} \tag{31}$$

where M_{ei}^{-1} is the i^{th} column of M_{ee}^{-1} and where:

$$\delta_{ei} = \begin{cases} 1 & \text{if } i = e \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

which is the i^{th} of column of the identity matrix I_{ee}.

This equations can understood as solving the flux problem on the edges for each one of the e edges. In each a unitary source and sink term is added on the edge i. Mathematically speaking, if we solve this equations as it is, the coupling of the edges as showed on Figure 6b will spread the influence to strip of edges that were not meant to have it causing the leaking of the basis functions. This can be avoided by restricting this problem to the support region of each fine-scale volume of the edge type. We can define this support region of each as the collection of edges strips in which this volume is contained. Thus, the problem above can be rewritten as:

$$M_{e^ie^i}M_{e^si}^{-1} = \delta_{e^ii} \tag{33}$$

where $M_{e^ie^i}$ is transmissibility matrix of the edges M_{ee}, and where δ_{e^ii} the source and sink term excluding the influence outside the support region of the i^{th} edge volume.

Similarly, we define the problem for the surfaces as

$$M_{s^isi}M_{s^si}^{-1} = \delta_{s^ii} \tag{34}$$

where $M_{s^is^i}$ is transmissibility matrix of the surfaces M_{ss}, and where δ_{s^ii} the source and sink term excluding the influence outside the support region of the i^{th} surface volume.

This modifications also prevent the leaking of the basis functions outside the support regions of each coarse volume. Partition of unity is likewise lost, therefore we need to solve Equations 23 and 24 and normalize as described in Equation 29. In comparison, this correction seems to maintain a better coupling of the edges. This is noticeable because, prior to the normalization, the partition of unity remains closer to 1 than the previous fix. As a consequence, the multiscale solution seems to have a lower impact of the spurious oscillations,

1.4 AMS Prolongation Operator and Correction Functions

With a B_{en}^{cor} properly defined, we can use the result to compute and B_{sn}^{cor} and consequently B_{in}^{cor}. Therfore, we define the AMS corrected prolongation operator and correction function as:

$$\widetilde{B}^{cor} = \begin{bmatrix} 0 & 0 & 0 & -A_{ii}^{-1}(\widetilde{A}_{se}B_{en}^{cor} + \widetilde{A}_{in} + \widetilde{A}_{is}B_{sn}^{cor}) \\ 0 & 0 & 0 & B_{sn}^{cor} \\ 0 & 0 & 0 & B_{en}^{cor} \\ 0 & 0 & 0 & I_{nn} \end{bmatrix} \tag{35}$$

The algebraic manipulation done to write B in function of the B_{en}^{cor} and B_{sn}^{cor} cannot be done for C.

Knowing that M_{ee} and M_{ss} are approximations, and they do not guarantee that each basis function is restricted to its support regions and partition of union, we can find a better approximation for these matrices. By looking at the definition of C in Equation 11, we only need to find a most suitable approximation for M_{ee}^{-1} and M_{ss}^{-1} that we are going to call $\mathscr{M}_{ee}^{-1}$ and $\mathscr{M}_{ss}^{-1}$.

$$B_{sn}^{cor} = -\mathscr{M}_{ss}^{-1}(\widetilde{A}_{se}B_{en}^{cor} + \widetilde{A}_{sn}) \tag{36}$$

$$B_{en}^{cor} = -\mathscr{M}_{ee}^{-1}\widetilde{A}_{en} \tag{37}$$

BE

(33)

(34)

By isolating $\mathscr{M}_{ee}^{-1}$ and $\mathscr{M}_{ss}^{-1}$, we have:

$$\mathscr{M}_{ss}^{-1} = B_{sn}^{cor} / (\widetilde{A}_{se} B_{en}^{cor} + \widetilde{A}_{sn}) \tag{38}$$

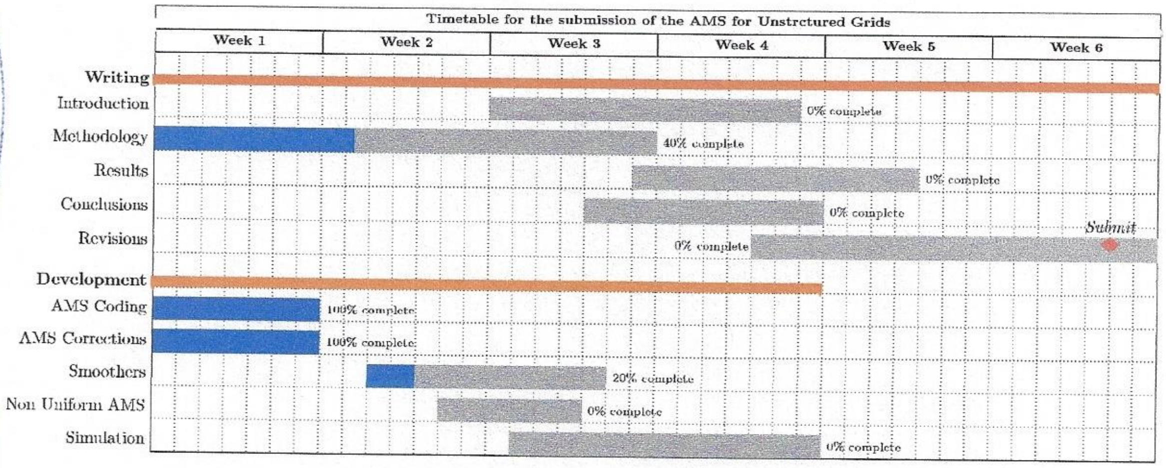
$$\mathscr{M}_{ee}^{-1} = -B_{en}^{cor} / \widetilde{A}_{en} \tag{39}$$

and where:

$$\widetilde{C}^{cor} = \begin{bmatrix} \widetilde{A}_{ii}^{-1} & -\widetilde{A}_{ii}^{-1}\widetilde{A}_{is}\mathscr{M}_{ss}^{-1} & \widetilde{A}_{ii}^{-1}(\widetilde{A}_{is}^{-1}\mathscr{M}_{ss}^{-1}\widetilde{A}_{se}\mathscr{M}_{ee}^{-1} - \widetilde{A}_{ie}\mathscr{M}_{ee}^{-1}) & 0 \\ 0 & \mathscr{M}_{ss}^{-1} & -\mathscr{M}ss^{-1}\widetilde{A}_{se}\mathscr{M}_{ee}^{-1} & 0 \\ 0 & 0 & \mathscr{M}_{ee}^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

1.5 Timetable for the release of the AMS paper

The following timetable is an estimative of the several stages and the time they will consume before releasing the corrected AMS paper.



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