LAB 2: RC CIRCUIT ANALYSIS

Circuit Theory and Electronics Fundamentals

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April 6, 2021

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1 Introduction

In this laboratory assignment, a circuit containing both an independent (first constant and then sinusoidal) and a current controlled voltage sources (V_s and V_d, respectively), a voltage controlled current source (I_b), connected to multiple resistors (from R_1 to R_7) and a capacitor (C) is going to be studied. The described circuit can be observed in detail in Figure 1.

In Section 2, a theoretical analysis of the circuit is presented, studying the static, time and frequency responses, using the Octave tool. After that, in Section 3, the circuit is analysed via simulation, using the software Ngspice. The obtained results are then compared, explaining the reasons behind the differences and similarities found. Finally, one can find the conclusions of this study outlined in Section 4.

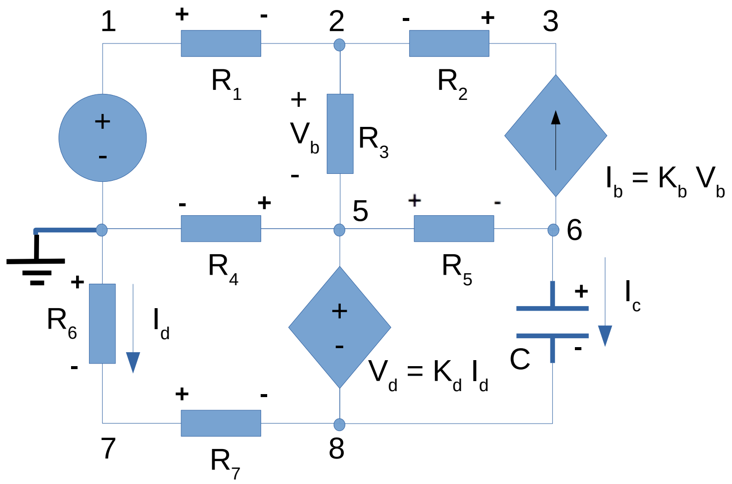

Figure 1: Circuit diagram.

2 Theoretical Analysis

In this section, the previously shown circuit is theoretically analised. It is important to remember that its behaviour depends on the consideration of two different time intervals, as $v_s(t)$ is defined as a piecewise function:

$$v_s(t) = \begin{cases} V_s & ,t < 0 \\ sin(2\pi f t) & ,t \geq 0 \end{cases}$$

2.1 Nodal Analysis for $t < 0$

For $t < 0$, $v_s(t)$ assumes a constant value V_s for excitation. Besides that, the system has had enough time for it to overcome any transitional period, which means all the electric quantities in the circuit have had the time to stabilize, and so the current that passes through the capacitor $i_c(t)$ is null, because $i_c(t) = C\frac{dv_c(t)}{dt}$ and $\frac{dv_c(t)}{dt} = 0$ (since $v_c(t)$ is no longer varying). Therefore, we can consider an open circuit as a replacement for the capacitor branch.

In order to determine the voltages in all nodes and, hence, the currents in all branches, nodal analysis is applied to the circuit, as suggested by the following system:

$$\begin{cases} V_1 = V_s \\ (V_2 - V_1)G_1 + (V_2 - V_3)G_2 + (V_2 - V_5)G_3 = 0 \\ (V_3 - V_2)G_2 - K_b(V_2 - V_5) = 0 \\ (V_5 - V_2)G_3 + (V_5 - V_6)G_5 + V_5G_4 + (V_8 - V_7)G_7 = 0 \\ K_b(V_2 - V_5) + (V_6 - V_5)G_5 = 0 \\ (V_7 - V_8)G_7 + V_7G_6 = 0 \\ V_8 - V_5 - K_dV_7G_6 = 0 \end{cases}$$

Note that some substitutions were made, such as:

$$I_b = K_b(V_2 - V_5) \tag{1}$$

$$V_d = -K_dV_7G_6 \tag{2}$$

By solving this system using Octave, the voltages that follow are obtained for every node. On the other hand, defining branch currents accordingly to the conventional electric current flow (from the plus to the minus sign defined in the figure), one obtains the subsequent values for each branch (using Ohm's law). For the sake of direct comparison, the results that follow the simulation analysis are already shown in this segment as well.

Variable name	Value [A or V]
I_b	-2.540344e-04
I_{R_1}	2.428274e-04
I_{R_2}	-2.540344e-04
I_{R_3}	-1.120696e-05
I_{R_4}	1.176699e-03
I_{R_5}	-2.540344e-04
I_{R_6}	9.338720e-04
I_{R_7}	9.338720e-04
V_1	5.046144e+00
V_2	4.802884e+00
V_3	4.276252e+00
V_5	4.837826e+00
V_6	5.626501e+00
V_7	-1.934734e+00
V_8	-2.909165e+00
I_s	-2.428274e-04
I_d	-9.338720e-04

Variable name	Value [A or V]
@gb[i]	-2.54035e-04
@r1[i]	2.428275e-04
@r2[i]	-2.54035e-04
@r3[i]	-1.12071e-05
@r4[i]	1.176699e-03
@r5[i]	-2.54035e-04
@r6[i]	9.338720e-04
@r7[i]	9.338720e-04
v(1)	5.046144e+00
v(2)	4.802883e+00
v(3)	4.276251e+00
v(4)	0.000000e+00
v(5)	4.837826e+00
v(6)	5.626502e+00
v(7)	-1.93473e+00
v(8)	-2.90916e+00

Table 1: On the left, nodal analysis and current values for $t < 0$ by Octave. On the right, operating point analysis for $t < 0$ by Ngspice: a variable preceded by @ is of type *current* and expressed in Ampere; other variables are of type *voltage* and expressed in Volt. gb refers to the controlled current source I_b and the rest is defined as before. Notes: v_4 refers to a special added node, which will be explained on 3; the 2 last values on the left table cannot be directly seen on the right table, but can be easily deduced from the ones presented.

2.2 Nodal analysis for $t = 0$, equivalent Resistance R_{eq}, Time Constant τ

Making $V_s = 0$ (node 1 also becomes GND), the $t = 0$ voltage levels for each node can be calculated, by also replacing the capacitor with a voltage source $V_x = V_6 - V_8$, with V_6 and V_8 being the voltages in 1. This can be done because $V_6 - V_8$ remains the same for $t < 0$ and $t = 0$ (because the capacitor does not discharge instantly). The new circuit is, again, analysed with the Nodal Method, as suggested by the following system (results on table **??**):

$$\begin{cases} V_2G_1 + (V_2 - V_3)G_2 + (V_2 - V_5)G_3 = 0 \\ -K_b(V_2 - V_5) + (V_3 - V_2)G_2 = 0 \\ V_5G_4 + (V_5 - V_6)G_5 + (V_5 - V_2)G_3 + (V_8 - V_7)G_7 - [-K_b(V_2 - V_5) + (V_5 - V_6)G_5] = 0 \\ V_6 - V_8 = V_x \\ V_7G_6 + (V_7 - V_8)G_7 = 0 \\ V_8 - V_5 - K_dV_7G_6 = 0 \end{cases}$$

Again, some substitutions were made, such as:

$$I_b = K_b(V_2 - V_5) \tag{3}$$

$$V_d = -K_dV_7G_6 \tag{4}$$

$$I_x = -I_b + (V_5 - V_6)G_5 \tag{5}$$

with I_x being the current that flows downwards in V_x (positive sign up in V_x).

This circuit can be seen as a simple RC circuit, with the voltage source V_x and the current that flows in the mesh being $I_x = -K_b(V_2 - V_5) + (V_5 - V_6)G_5$. Of course, all the remaining circuit, as seen by the capacitor's terminals, works as an equivalent resistor. Its equivalent resistance can be calculated with the expression

$$R_{eq} = -\frac{V_x}{I_x} \tag{6}$$

with R_{eq} being the equivalent resistance as seen by the capacitor's terminals.

The minus sign results from the fact that, in the equivalent circuit, I_x flows through the equivalent resistor from the negative to the positive terminal (in opposition to the convention). Thus, the time constant for the equivalent circuit τ can be computed as $R_{eq}C$.

By solving the refered system using Octave, the following table presents the voltage in every node, as well as the equivalent resistance and the time constant (once again, one can imediately consult the simulation values as well).

Variable name	Value [A or V or Ω]
I_b	-0.000000e+00
I_{R_1}	0.000000e+00
I_{R_2}	0.000000e+00
I_{R_3}	-0.000000e+00
I_{R_4}	0.000000e+00
I_{R_5}	-2.749362e-03
I_{R_6}	0.000000e+00
I_{R_7}	0.000000e+00
V_2	-0.000000e+00
V_3	0.000000e+00
V_5	0.000000e+00
V_6	8.535665e+00
V_7	-0.000000e+00
V_8	-0.000000e+00
I_x	-2.749362e-03
R_{eq}	3.104598e+03
τ	3.224319e-03

Variable name	Value [A or V]
@gb[i]	0.000000e+00
@r1[i]	0.000000e+00
@r2[i]	0.000000e+00
@r3[i]	0.000000e+00
@r4[i]	0.000000e+00
@r5[i]	-2.74936e-03
@r6[i]	0.000000e+00
@r7[i]	0.000000e+00
v(2)	0.000000e+00
v(3)	0.000000e+00
v(4)	0.000000e+00
v(5)	0.000000e+00
v(6)	8.535665e+00
v(7)	0.000000e+00
v(8)	0.000000e+00

Table 2: On the left, nodal analysis, current values, equivalent resistance and time constant for $t = 0$ and $v_s(t) = 0$ by Octave. On the right, operating point analysis for $t = 0$ by Ngspice: a variable preceded by @ is of type *current* and expressed in Ampere; other variables are of type *voltage* and expressed in Volt. gb refers to the controlled current source I_b and the rest is defined as before. Notes: v_4 refers to a special added node, which will be explained on 3; the 2 last values on the left table cannot be directly seen on the right table, but can be easily deduced from the ones presented.

2.3 Natural Solution $v_{6n}(t)$

Let's consider the situation for $t \geq 0$. The circuit's solution can be described as the superposition of two components. The natural solution, which is the system's behaviour with no external excitation (it leads to the capacitor's discharge), and the forced solution, which will allow an oscillating steady-state.

To compute the natural solution, particularly, for node 6 (v_{6n}), the circuit is considered to be the previously refered simple RC circuit with the capacitor C and the equivalent resistor R_{eq}. The following deduction applies KVL to the mesh, uses the capacitor's law to replace the mesh's current ($i_c(t) = C\frac{dv_c(t)}{dt}$), and solves the resulting differential equation:

$$v_c + R_{eq}i = 0 \Leftrightarrow v_c + R_{eq}C\frac{dv_c}{dt} = 0 \Leftrightarrow v_c(t) = Ae^{-\frac{t}{R_{eq}C}} = \tag{7}$$

The constant A is determined by the initial condition as being the capacitor's voltage $v_c(0) = V_x = V_6 - V_8$ at $t < 0$. Considering the values obtained in the previous section for voltages in 2, the natural solution for V_6 becomes

$$v_{6n}(t) = V_x e^{-\frac{t}{R_{eq}C}} = 8.53567e^{-\frac{t}{0.22432\times10^{-3}}}[V] \tag{8}$$

The following figure represents its plot in the interval $[0, 20]$ms:

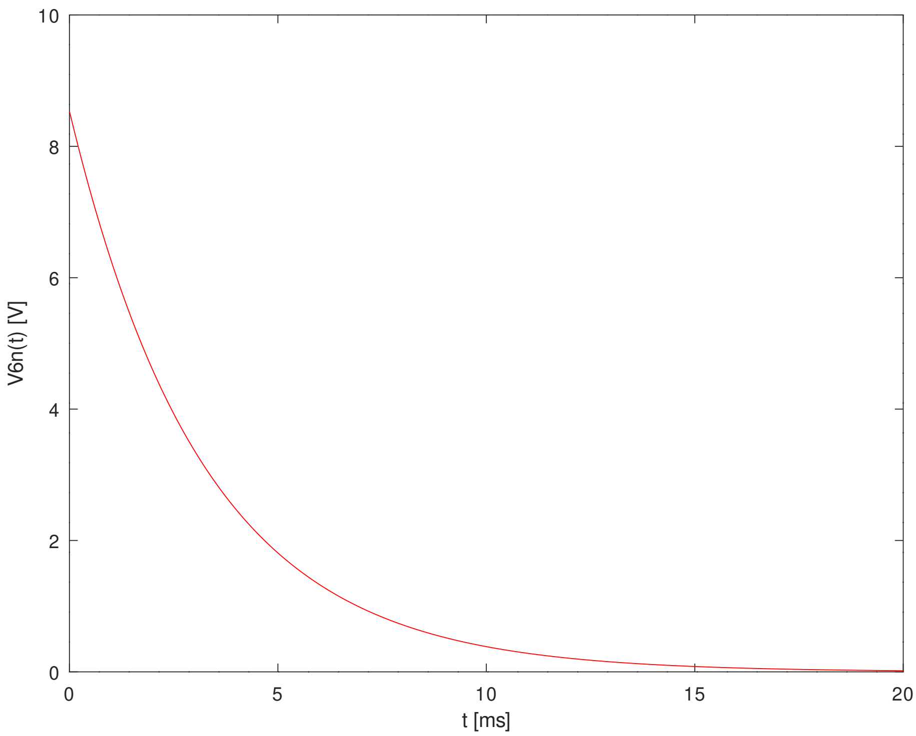

Figure 2: $v_{6n}(t)$ in the interval $[0, 20]$ ms.

2.4 Forced Solution $v_{6f}(t)$

In order to compute the forced solution for $t \geq 0$, consequence of the sinusoidal excitacion due to the source V_1, in $t \geq 0$, it is considered the frequency $f = 1$kHz and a phasor voltage

$\widetilde{V_s} = 1$. The circuit in figure 1 is analysed using the Nodal Method (applied to phasors), as suggested by the following system, with Y_x being the admittances of the components (G_x for resistors and $2\pi fCj$ for the capacitor)

$$\begin{cases} \widetilde{V_1} = \widetilde{V_s} \\ (\widetilde{V_2} - \widetilde{V_1})Y_1 + (\widetilde{V_2} - \widetilde{V_3})Y_2 + (\widetilde{V_2} - \widetilde{V_5})Y_3 = 0 \\ (\widetilde{V_3} - \widetilde{V_2})Y_2 - K_b(\widetilde{V_2} - \widetilde{V_5}) = 0 \\ (\widetilde{V_5} - \widetilde{V_2})Y_3 + \widetilde{V_5}Y_4 + (\widetilde{V_5} - \widetilde{V_6})Y_5 + (\widetilde{V_8} - \widetilde{V_7})Y_7 + (\widetilde{V_8} - \widetilde{V_6})Y_C = 0 \\ (\widetilde{V_6} - \widetilde{V_5})Y_5 + (\widetilde{V_6} - \widetilde{V_8})Y_C + K_b(\widetilde{V_2} - \widetilde{V_5}) = 0 \\ \widetilde{V_7}Y_6 + (\widetilde{V_7} - \widetilde{V_8})Y_7 = 0 \\ (\widetilde{V_8} - \widetilde{V_5}) - K_d\widetilde{V_7}Y_6 = 0 \end{cases}$$

Note that, once again, some substitutions were made, such as:

$$\widetilde{I_b} = K_b(\widetilde{V_2} - \widetilde{V_5}) \tag{9}$$

$$\widetilde{V_d} = -K_d\widetilde{V_7}Y_6 \tag{10}$$

By solving the system using Octave, the following table presents the complex amplitude (phasor) in every node:

Complex amplitude	Value
$\widetilde{V_1}$	1.000000+0.000000j
$\widetilde{V_2}$	0.951793-0.000000j
$\widetilde{V_3}$	0.847430-0.000000j
$\widetilde{V_5}$	0.958717-0.000000j
$\widetilde{V_6}$	-0.572401-0.083292j
$\widetilde{V_7}$	-0.383408+0.000000j
$\widetilde{V_8}$	-0.576512+0.000000j

Table 3: Complex amplitude values for every node (forced solution, $t \geq 0$.

The expression for the real forced solution in $\widetilde{V_6}$ can be calculated the following way (considering $\widetilde{V_1} = 1$):

$$\frac{\widetilde{V_6}}{\widetilde{V_1}} = a + jb = |a + jb|e^{-j(-\phi)}, \quad a, b \in \mathbb{R}, \quad \phi \quad \text{the angle of} \quad a + jb \tag{11}$$

One then has:

$$\frac{V_6}{V_1} = |a + jb| \tag{12}$$

$$\phi_6 - \phi_1 = \phi \tag{13}$$

Using the values of table 3, one obtains:

$$v_{6f}(t) = 0.57843\sin(2\pi \times 1000t - 2.9971)[V] \tag{14}$$

Note that this process is usually made when the input voltage source is in the form of a cosine. By doing some algebraic manipulation, one can come to the conclusion that the use of sines is totally analogue.

2.5 Total Solution $v_6(t)$

As previously said, the total solution consists on the superposition of both natural and forced solutions. Hence,

$$v_6(t) = v_{6n}(t) + v_{6f}(t) = 8.53567e^{-\frac{t}{0.22432\times 10^{-3}}} + 0.57843\sin(2\pi \times 1000t - 2.9971)[V] \tag{15}$$

The following figure represents the plot of both $v_s(t)$ and $v_6(t)$, in the interval $[-5, 20]$ms. Note that this time range includes the transition at $t = 0$. Therefore, it is normal to observe a discontinuity at $t = 0$, from the constant to the oscillating solution.

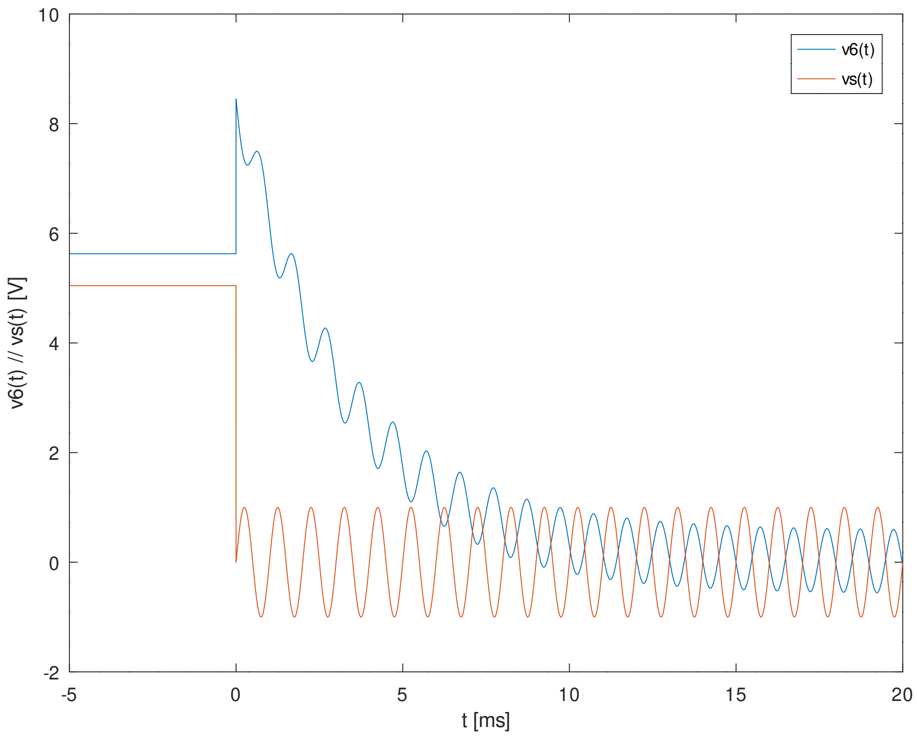

Figure 3: $v_s(t)$ and $v_6(t)$ in the interval $[-5, 20]$ ms.

It can be observed the effect of the dissipative solution (natural), which leads to a voltage decrease, until it vanishes, only remaining the forced solution, which shows a steady-state harmonic oscillation.

2.6 Frequency analysis

It is possible to study the effect of the input ($v_s(t)$) frequency f in the output signal $v_c(t) = v_6(t) - v_8(t)$ or in any of the system nodes. This way, it is displayed the plot for both the magnitude in dB (Figure 4) and phase in degrees (Figure 5) of the signals v_c and v_6 as function of f and also of v_s (for comparison). The type of scale chosen for the frequency axis $-$ logscale $-$ is convenient to make big plot ranges fit in small figures.

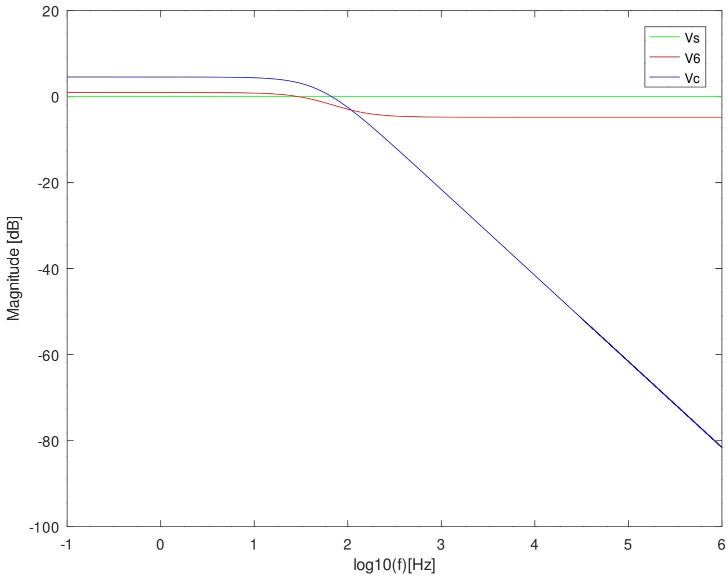

Figure 4: Magnitude of v_s and magnitude responses of v_6 and v_c.

An RC circuit behaves like a low-pass filter (blocks high frequencies and passes low frequencies). That is, at low frequencies, there's a very small magnitude response of the output frequency v_c. One can look at the plot and see that, for values of f close to 0, v_c and v_6 both have an amplitude of approximately $0\ dB$, which translates to $1V$ (the same as v_s $-$ no response).

On the other side, for high frequencies (as soon as it reaches cut-off frequency $\approx -3dB$), the magnitude (dB) of the output becomes more and more negative, with a slope of $-20dB/decade$ (this values can be obtained through the system's transfer function). This means the output voltage value in volt becomes lower and lower (high signal attenuation). The impedance of a capacitor decreases with as the input frequency increases, which means it will each time offers less resistance to the current flow between its terminals, so one can almost decribe it as a short-circuit (zero output).

Keep in mind that the circuit does not have any amplifying elements (*e.g.*, transistors), so the output level is almost always less/equal to the input one.

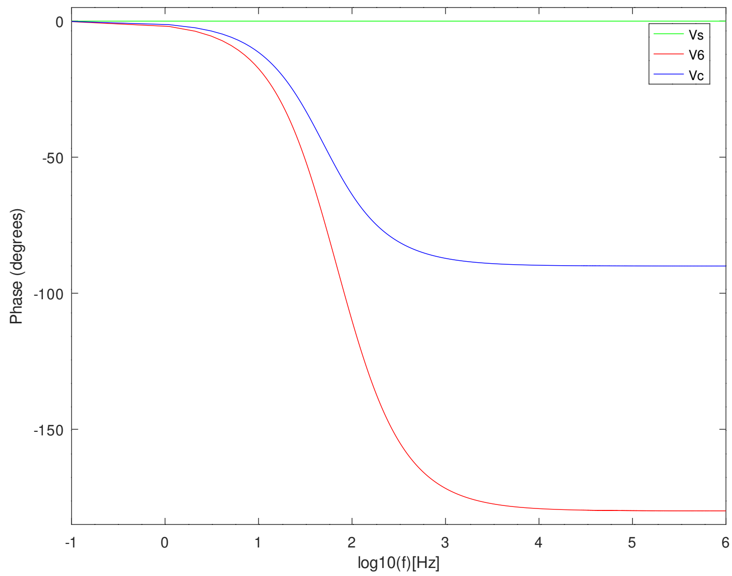

Figure 5: Phase of v_s and phase responses of v_6 and v_c.

As it can be seen, for high frequencies (from about $10^3 Hz$), the voltage value registered in node 6 presents a phase close to $-\pi$, which means it is in phase opposition with the input voltage source v_s. That can be easily confirmed by the observation of Figure 3, in which the plot is made precisely for $1\ KHz$. Considering v_c, it lags behind the source as well (in $\pi/2$). This happens because the excitation is changing too fast for the output to be able to keep up with it (it takes time to charge the capacitor's plates while the input voltage is changing). The sudden drop happens for two decades, arround the time $\omega = 2\pi f$ achieves the value of $\frac{1}{R_{eq}C}$ (which is analogue to the ressonance/natural frequency of a spring-mass system). That would be pretty evident, once again, by looking at the transfer function for the equivalent circuit previously described (which, in this case, represents the value of $\widetilde{V_c}$, since $\widetilde{V_s} = 1$). Its angle is given by $-\arctan(\omega R_{eq}C)$, so for $\omega = \frac{1}{R_{eq}C}$ we get a value of $-\frac{\pi}{4}$, which is the central value of the phase drop of v_c.

3 Simulation Analysis

3.1 Operating Point Analysis for $t < 0$

For $t < 0$, $v_s(t) = V_s$ and the system behaves as explained in 2.1.

The simulated operating point results (obtained from Ngspice) for the circuit under analysis, which for $t < 0$ behaves as explained in subsection 2.1, are shown in Table 1. Once again, the positive and negative terminals of the components (and, accordingly, the current directions) were defined as seen on Figure 1.

Compared to the theoretical analysis results, one notices a few differences.

Focusing on the obtained nodal voltage levels, an almost exact resemblance is shown, except for the fact that Octave's used result precision is of 7 significant digits (excluding the minus sign), whereas Ngspice always presents 7 significant digits, but counting the minus sign. Even though the numerical roundings of the values present some diminute differences, they are mostly correct. It's worth noting that node 4 (introduced exclusively on the simulation analysis) was created so a zero valued voltage source (working as an ammeter) could be placed in series with resistor 6, allowing the algorithm to measure the current between GND and 7. The same issue takes place with the current values.

Other than that, we could say there's a perfect match on the results obtained from both analysis methods.

3.2 Operating Point Analysis for $v_s = 0$ and V_x as a replacement for the capacitor

Considering $v_s(t) = 0$ (short-circuit) and replacing the capacitor with a voltage source $V_x = V(6) - V(8)$ (where V(6) and V(8) are the voltages in nodes 6 and 8 obtained in the previous subsection), we obtain the results on Table 2 (subsection 2.2).

The same considerations can be made for the comparison between the theoretical/simulation results, for both voltages and currents, and so one obtains the same value for R_{eq} as shown on table 2.

The reason behing this type of analysis has been described before. Once again, the values obtained for $V(6)$ and $V(8)$ will be used to describe the initial conditions ($t = 0$) for the voltage levels on the capacitor terminals.

3.3 Transient analysis - natural solution

Using the boundary conditions $V(6)$ and $V(8)$ as obtained in 3.2 and *Ngpice*'s transient analysis mode, the natural response of the circuit was simulated (setting $v_s(t) = 0$ - short-circuit). Figure6 represents the plot for $v_{6n}(t)$ in the interval $[0, 20]\ ms$. One can easily compare it to the Octave's plot and certify that they are equivalent.

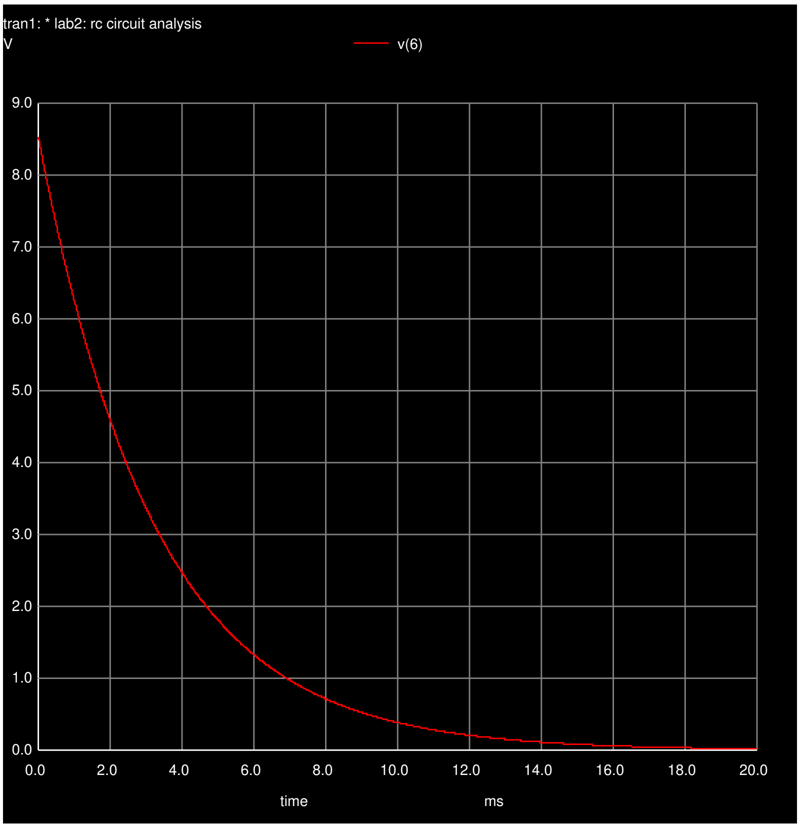

Figure 6: $v_{6n}(t)$ in the interval $[0, 20]$ ms.

3.4 Transient analysis - total solution

By reproducing the previous step with $v_s(t) = \sin(2\pi ft)$ ($f = 1kHz$), the total response of the circuit (natural + forced solution) was simulated. Figure 7 represents the plot for both $v_s(t) = v_1(t)$ (stimulus) and $v_6(t)$ (response) in the interval $[0, 20]$ ms. Once again, we can be certain that Figures 3 and 7 represent the same voltage behaviour through time.

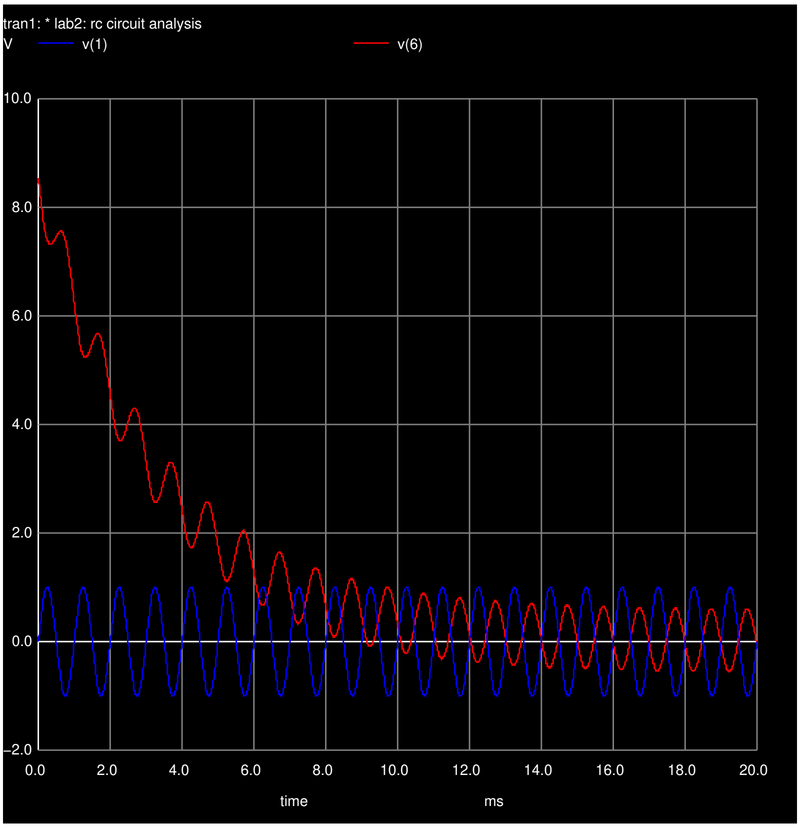

Figure 7: $v_6(t)$ and $v_s(t)$ in the interval $[0, 20]$ ms

3.5 Frequency analysis

Using *Ngpice*'s frequency analysis mode, the dependency of $v_s = v_1$, v_6 and v_c on the frequency $f(Hz)$ of the input was studied. Figures 8 and 9 represent, respectively, the magnitude in dB $db(v_1)$, $db(v_6)$ and $db(v_c) = db(v_6, v_8)$ and the phase in degrees $ph(v_1)$, $ph(v_6)$ and $ph(v_c) = ph(v_6, v_8)$ as a function of f. Note that it was chosen a logarithmic scale for the frequency axis (the space between each two consecutive vertical white lines represents a decade) and that both plots were made for a frequency range of 0.1 Hz to 1 MHz.

Consult subsection 2.6 the conclusions related to the several plots (how and why the differ).

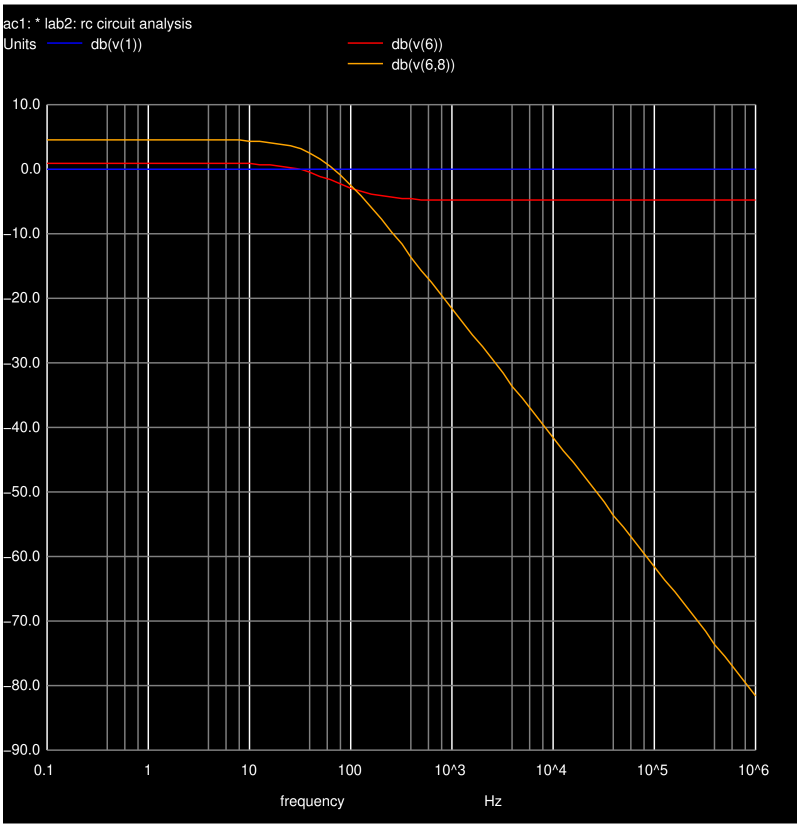

Figure 8: Magnitude in dB $db(v_1)$, $db(v_6)$ and $db(v_c) = db(v_6, v_8)$ as a function of f.

4 Conclusion

In this laboratory assignment, the goal of analysing the circuit has been achieved. Theoretically, static analysis has been performed along with the study of time and frequency responses (studying both magnitude and phase dependence of frequency). The results were obtained using the Octave maths tool. Besides that, a circuit simulation, using the Ngspice tool, was also executed. The simulation results allowed us to verify our theoretical conclusions, since they matched almost precisely, as explained in Section 3.

In conclusion, we consider that this was a very important circuit to study, since it involves multiple essential components, such as the capacitor. It was also quite interesting to apply what we have learned in class and confirm it by observing the obtained results.

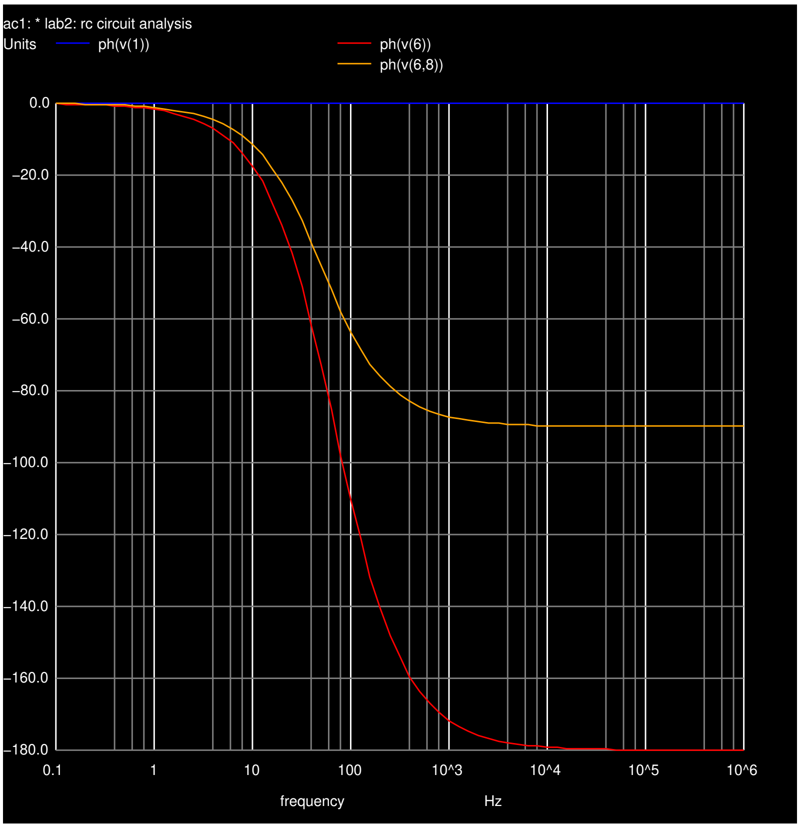

Figure 9: Phase in degrees $ph(v_1)$, $ph(v_6)$ and $ph(v_c) = ph(v_6, v_8)$ as a function of f.