Effective dynamics of an aircraft

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Abstract

To plan optimal aircraft routes we can not completely solve and describe 3D fluid structure interaction. This would be computationally very intense and unnecessary. Instead, we need to capture the main features related to fuel consumption, ability to turn, and affects from the atmosphere such as drag and lift. To achieve this we develop effective dynamical equations, together with a simple numerical scheme to solve these equations. The method is simple enough to using nonlinear optimisation to plan routes with minimal fuel, or distance, or time.

1 Equations of motion

The main forces acting on the aircraft are due to thrust, drag, lift, and a turning force. More accurately: an aircraft turns by rolling and then using lift, however we will not model these details, and instead have a turning force which is similar to a lift force.

Figure 1: A sketch showing that the aircraft velocity vector is given by $\boldsymbol{v}$ (relative to the ground), and the velocity of the wind (relative to the ground) is given by $\boldsymbol{u}$. We approximate that the aircraft always points in the direction of travel.

As it is impractical to solve for even the rigid body dynamics of the aircraft, which would allow for rotations, we assume that the aircraft always points in the direction of travel. See fig. 1 for a sketch.

To describe the aircraft dynamics it is useful to use a local coordinate system with basis vectors:

$$\boldsymbol{e}_v \quad \text{(the direction of travel)} \tag{1}$$

$$\boldsymbol{e}_r \quad \text{(radial direction from earth centre to aircraft)} \tag{2}$$

$$\boldsymbol{e}_p = \boldsymbol{e}_r \times \boldsymbol{e}_v \quad \text{(perpendicular to travel direction)} \tag{3}$$

where we will only model the 2D dynamics and assume the altitude is fixed, so that $\boldsymbol{e}_v$ is always orthogonal to $\boldsymbol{e}_r$. Note that changes of altitude based on balance of lift with gravity can easily be accomodated without modelling the motion that leads to that change.

There are two main forces on the aircraft, those that can be controlled by the pilot $\boldsymbol{f}$ and those that external $\boldsymbol{w}$. The forces that can be controlled are:

$$\boldsymbol{f} = T(\dot{m})\boldsymbol{e}_v + L_p(\boldsymbol{v} - \boldsymbol{u}, \alpha)\boldsymbol{e}_p + L_r(\boldsymbol{v} - \boldsymbol{u}, \beta)\boldsymbol{e}_r, \tag{4}$$

where $\dot{m}$ is the rate of change of mass in time (from using fuel), $v = |\boldsymbol{v}|$, $T(\dot{m}, t)$ is the force from thrust, $L_p(v, \alpha)$ is a force which leads to turns in the $\boldsymbol{e}_p direction$, where α captures the amount the pilot tries to turn, and $L_r(v, \beta)$ are the lift forces, where β is the amount to pilot tries to lift.

A simple and effective choice for thrust is

$$T(\dot{m}) = -\dot{m} C_T \quad \text{and} \quad L_p(v, \alpha) = v^2 \alpha, \tag{5}$$

where C_T is a constant which describes how efficiently fuel burn $\dot{m}$ is converted into a thrust. More accurately,

$$C_T = (\text{exhaust velocity}) - (\text{relative airspeed}),$$

for subsonic flight [1, Chapter 4], where exhaust velocity is the speed of exhaust relative to the aircraft, and relative airspeed equals $|\boldsymbol{v} - \boldsymbol{u}|$ is the speed of air relative to the aircraft.

A simple and effective formula for the turning force is

$$L_p(v, \alpha) = |(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{e}_v|^2 \alpha. \tag{6}$$

In practive, turning is due to rolling and then lift. The forces that lead to rolling and lift are due to wind drag, which is proportional to the relative airspeed in the direction of travel. The variable α is bounded: $\alpha \in [-C_\alpha, C_\alpha]$, where the positive constant C_α depends on the type of aircraft.

The dynamics of the aircraft are now governed by balance of momentum:

$$\begin{aligned} &\frac{d}{dt}(m(t)\boldsymbol{v}(t)) = \boldsymbol{f} + \boldsymbol{w} \implies \\ &\dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}} = T(\dot{m})\boldsymbol{e}_v + L_p(\boldsymbol{v} - \boldsymbol{u}, \alpha)\boldsymbol{e}_p + L_r(\boldsymbol{v} - \boldsymbol{u}, \beta)\boldsymbol{e}_r + \boldsymbol{w}, \end{aligned} \tag{7}$$

where $\boldsymbol{w}$ are the forces due to external factors, such as the wind and altitude. It is generally a function of the relative airspeed $\boldsymbol{v} - \boldsymbol{u}$, however we do not need to give explicit forms for this forces, nor for the lift L_r to simplify the equations of motion.

To simplify (7) we need to write all expressions in terms of the local coordinate basis $\boldsymbol{e}_v, \boldsymbol{e}_p, \boldsymbol{e}_r$. First we assume that the external forces can be decomposed in the form:

$$\boldsymbol{w} = w_v \boldsymbol{e}_v + w_p \boldsymbol{e}_p + w_r \boldsymbol{e}_r.$$

Next we rewrite the expression:

$$\dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}} = \dot{m} v \boldsymbol{e}_v + m \dot{v} \boldsymbol{e}_v + m v \dot{\boldsymbol{e}}_v. \tag{8}$$

To rewrite $\dot{\boldsymbol{e}}_v$ we use the standard identities for basis vectors $\frac{d}{dt}(\boldsymbol{e}_v \cdot \boldsymbol{e}_v) = 0$ to conclude that $\dot{\boldsymbol{e}}_v$ is orthogonal to $\boldsymbol{e}_v$ and can be expanded in the form:

$$\dot{\boldsymbol{e}}_v = (\dot{\boldsymbol{e}}_v \cdot \boldsymbol{e}_p)\boldsymbol{e}_p + (\dot{\boldsymbol{e}}_v \cdot \boldsymbol{e}_r)\boldsymbol{e}_r,$$

which combined with (8), and the separation of the external forces, leads us to rewrite the equations of motion (7) in the form

$$a_v \boldsymbol{e}_v + a_p \boldsymbol{e}_p + a_r \boldsymbol{e}_r = 0.$$

The above then implies that $a_v = 0$, $a_p = 0$, and $a_r = 0$, which written in full become:

$$\dot{m} v + m \dot{v} = w_v + T(\dot{m}), \tag{9}$$

$$m(t) v(t) (\boldsymbol{e}_p \cdot \dot{\boldsymbol{e}}_v) = w_p + L_p(\boldsymbol{v} - \boldsymbol{u}, \alpha), \tag{10}$$

$$m(t) v(t) (\boldsymbol{e}_r(t) \cdot \dot{\boldsymbol{e}}_v) = w_r + L_r(\boldsymbol{v} - \boldsymbol{u}, \beta). \tag{11}$$

1.1 Forward marching numerical method

Let us start by choosing a position $\boldsymbol{x}$ for the aircraft in a spherical coordinate system (r, θ, ϕ) with

$$\boldsymbol{x} = r[\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi].$$

Then the velocity vector of aircraft is given by $\dot{\boldsymbol{x}} = \boldsymbol{v}$, with $\boldsymbol{v}$ pointing in the direction of flight (relative to the ground). Rewriting in spherical coordinates we get

$$\boldsymbol{v} = r \sin\phi \boldsymbol{e}_\theta \dot{\theta} + r \boldsymbol{e}_\phi \dot{\phi},$$

by assuming that the altitude r is fixed. Then, from the above we deduce that

$$r \sin\phi \dot{\theta} = \boldsymbol{v} \cdot \boldsymbol{e}_\theta \quad \text{and} \quad r\dot{\phi} = \boldsymbol{v} \cdot \boldsymbol{e}_\phi. \tag{12}$$

From the above we can see that it is convenient to write the components $\boldsymbol{v}$ in terms of the local coordinate system $\boldsymbol{e}_\theta, \boldsymbol{e}_\phi, \boldsymbol{e}_r$ coordinate system. That is, in the code,

$$\boldsymbol{v}[1] = \boldsymbol{v} \cdot \boldsymbol{e}_\theta, \quad \boldsymbol{v}[2] = \boldsymbol{v} \cdot \boldsymbol{e}_\phi, \quad \boldsymbol{v}[3] = \boldsymbol{v} \cdot \boldsymbol{e}_r.$$

We can then use the equations (12) to calculate $\theta(t+h)$ and $\phi(t+h)$ by substituting

$$\dot{\theta} = \frac{\theta(t+h) - \theta(t)}{h} \quad \text{and} \quad \dot{\phi} = \frac{\phi(t+h) - \phi(t)}{h},$$

into (12) and solving for $\theta(t+h)$ and $\phi(t+h)$. For consistency, we have that the components of all vectors are given in terms of the local basis in the code.

The equations of motion (9 - 11) can now be turned into a forward marching numerical method to predict the trajectory of the aircraft, which we briefly summarise below.

Equation (9) can be used to update the speed $v(t)$ by substituting

$$\dot{v}(t) = \frac{v(t+h) - v(t)}{h},$$

and then solving for $v(t+h)$. From the second equation (10) we can update the direction $\boldsymbol{e}_v$ and as a consequence $\boldsymbol{e}_p$. To start

$$\dot{\boldsymbol{e}}_v = a\boldsymbol{e}_p + b\boldsymbol{e}_r,$$

because $\dot{\boldsymbol{e}}_v$ is orthogonal to $\boldsymbol{e}_v$. Substituting the above into (10) then leads to

$$a = \boldsymbol{e}_p \cdot \dot{\boldsymbol{e}}_v = (w_p(r, v, t) + L_p(\alpha(t), v(t))/(m(t)v(t)). \tag{13}$$

To obtain b we use

$$b = \dot{\boldsymbol{e}}_v \cdot \boldsymbol{e}_r = -\boldsymbol{e}_v \cdot \dot{\boldsymbol{e}}_r = -\boldsymbol{e}_v \cdot \boldsymbol{v}/r = -v/r, \tag{14}$$

where we also used the $\boldsymbol{e}_r = \boldsymbol{r}/r \implies \dot{\boldsymbol{e}}_r = \boldsymbol{v}/r$ for fixed r.

To summarise we can use the above to update the direction $\boldsymbol{e}_v$

$$\boldsymbol{e}_v(t+h) = \boldsymbol{e}_v(t) + ah\boldsymbol{e}_p(t) - \frac{vh}{r}\boldsymbol{e}_r(t) \tag{15}$$

where using the coordinate system (θ, ϕ, r) we always have that $\boldsymbol{e}_r = [0, 0, 1]$. As we have made first order approximations in h the norm of $\boldsymbol{e}_v(t+h)$ will only be approximately 1. It is better to correct this by taking $\boldsymbol{e}_v(t+h) \leftarrow \boldsymbol{e}_v(t+h)/|\boldsymbol{e}_v(t+h)|$. Finally, we can then update the perpendicular direction:

$$\boldsymbol{e}_p(t+h) = \boldsymbol{e}_r \times \boldsymbol{e}_v(t+h). \tag{16}$$

1.2 Optimal routes

The simplest way to find optimal routes if through the shooting method. That is, a few initial guesses are made, and then a nonlinear optimiser will try different values for the control variables thrust $\dot{m}$ and aircraft turning α to achieve some goal.

The most basic goal is to achieve a target destination $\boldsymbol{x}_B$. Let the start and end of the flight time be t_A and t_B, so that the last position of a simulated flight is $\boldsymbol{x}(t_B)$. Then we use a nonlinear optimiser to minimise:

$$\min_{\boldsymbol{\alpha}, \dot{\boldsymbol{m}}} |\boldsymbol{x}(t_B) - \boldsymbol{x}_B|,$$

where the control variables are

$$\boldsymbol{\alpha} = [\alpha(t_A), \alpha(t_A + h), \ldots, \alpha(t_B)] \text{ and } \dot{\boldsymbol{m}} = [\dot{m}(t_A), \dot{m}(t_A + h) \ldots, \dot{m}(t_B)].$$

In practice, instead of controlling $\alpha(t)$ and $\dot{m}(t)$ for every time step, we can instead represent each of these as piece-wise constant functions, each with about 10 different steps, and then minimise for the total of 20 steps. This allows us to refine simulations by making dt small while having a small number of control variables.

Naturally, we can also minimise the flight time as well as achieve the destination $\boldsymbol{x}_B$, which leads to the objective function:

$$\min_{\boldsymbol{\alpha}, \dot{\boldsymbol{m}}} c_t|t_B - t_A| + |\boldsymbol{x}(t_B) - \boldsymbol{x}_B|, \quad \text{(minimise flight time)}$$

where c_t is an appropriately chosen constant. To make the above robust, we should in fact non-dimensionalise each of the parts, and use a constant, such as c_t to give the appropriate weighting. For example, in the code the goal of minimising the total flight time 100 times less important than achieve the final destination.

1.3 Code and Numerical examples

Here we give a brief description of the code [2] and examples. The code used to generate the examples in this report is from AircraftRouteDynamics v0.0.1. Note the most recent version is here: AircraftRouteDynamics.jl, which may work differently then what is described below.

In the code AircraftRouteDynamics.jl a range of parameters of the aircraft and route can be given as shown in Figure 2, as well as running a simulation. A more detail description of the parameters and setting up a simulation are given in the landing page. Note the simulation result shown in Figure 3, which uses a more elaborate wind pattern which imitates a tornado, took 280 microseconds on a regular PC.

```
aircraft = Aircraft(
    altitude = 8,
    empty_weight = 4.0,
    drag_coefficient = 1.5,
    fuel = 10.0, # maximum fuel available
    fuel_burn_rate = 2.0 # typical fuel burn rate in time
)

setup = RouteSetup(
    aircraft = aircraft,
    iterations = 400,
    dt = 0.01,
    θφ_initial = [1.0,1.0],
    θφ_end = [2.5,1.0], # target end destination
    initial_velocity = [1.0,0.0],
    tol = 1e-2 # a tolerance for reaching the end destination
)
```

```
# fuel over time
fuel = LinRange(10,4,N);

# fuel over time
turns = 3.0 .* cos.(fuel)

wind_v = [1.0,1.0] .* 10.0
wind_speed(x,y) = wind_v

# calculate route
fuel = fuel .* 0;
turns = turns .* 0;
r = route(setup, fuel, turns, wind_speed)
```

Figure 2: An example of how to figure the package AircraftRouteDynamics.jl to run a simulation of one route. The parameters above are not realistic for a commercial aircraft.

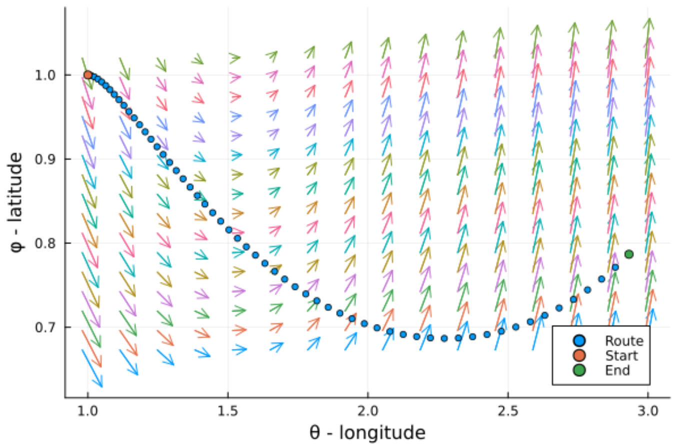


Figure 3: Example of the simulation of one route, with the start and end point highlighted. Note that the end point does not match the one given in the parameters shown in Figure 2. To achieve a specific end point you have to find an optimal route as specified below.

Finally, the package can find optimal routes

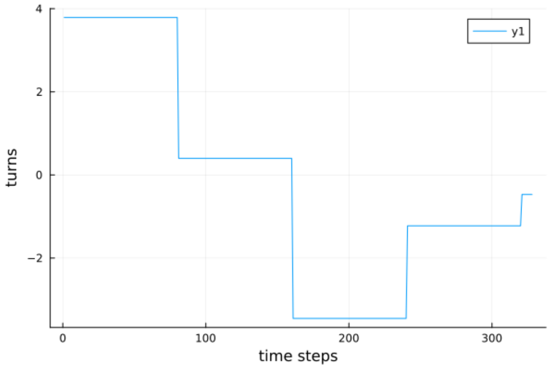
2 Calculating the Lift coefficient to keep the aircraft at a certain Height

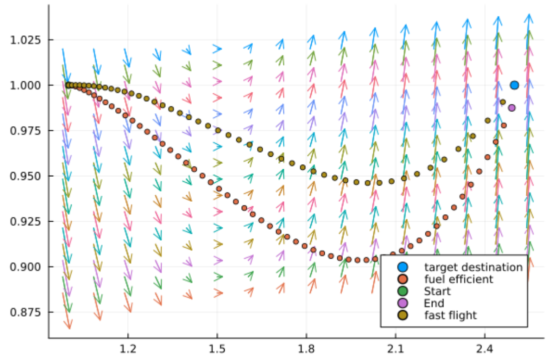
2.1 Lift coefficient above the tropopause

The dynamics model above does not include changes due to changes in altitude. These changes would affect drag for example. However, it is simple to add changes of altitude into the model, without consider the full three-dimensional trajectory of the aircraft. Below we give some relevant formulas, but did not have enough time to integrate them in the model above or the code.

The lift force is calculated as [3]

$$L_r = \frac{\beta}{2} \rho |(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{e}_v|^2 S \tag{17}$$





Where $|\boldsymbol{v}-\boldsymbol{u}|$ is the speed of the plane related to the wind, ρ is the air density, m is the mass of the plane, g is gravity , S is the area of the wing projected at the plane normal to $\boldsymbol{e}_r$, and β is the lift control parameter. From that we can see that the lift force is proportional to the air density. However the air density varies with the altitude of the aircraft. This dependece also can be approximatted by different formulas depending on the Altitude. For Heights above the tropopause [3] which is the altitude that separates the troposphere from the stratosphere ($\approx$ 11000m) the density varies as

$$\rho = \rho_{\text{trop}} e^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)}, \tag{18}$$

where, R is the gas constant and the sub-index 'trop' means the tropopause value of the density,temperature, and altitude provided in [3]. The pressure also follows the same behaviour.

$$P = P_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)}. \tag{19}$$

The force balance in $\boldsymbol{e}_r$ direction gives equilibrium of the plane at a constant altitude H. This allow us to determine the lift control variable β if we wish to keep a constant altitude H. So the total external force on the direction $\boldsymbol{e}_r$ is

$$w_r = -mg - PA, \tag{20}$$

where A is the projected area of the whole plane. Then the equation of motion in the $\boldsymbol{e}_r$ direction become

$$-m\frac{v^2}{r} = -mg - PA + L_r \tag{21}$$

$$-m(t)\frac{v(t)^2}{r} = -mg - P_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} A + \frac{\beta}{2}\rho_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} |(\boldsymbol{v}-\boldsymbol{u})\cdot \boldsymbol{e}_v|^2 S \tag{22}$$

notice that $r = R_{\text{earth}} + H$ where $R_{\text{earth}} \approx 6378$km. Then the assumption that $R_{\text{earth}} >> H$ is reasonable to do, so we approximate $r \approx R_{\text{earth}}$ on the above

$$-m\frac{v^2}{R_{\text{earth}}} = -mg - P_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} A + \frac{\beta}{2}\rho_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} |(\boldsymbol{v}-\boldsymbol{u})\cdot \boldsymbol{e}_v|^2 S \tag{23}$$

Isolating H we have that

$$H = -\frac{RT_{\text{trop}}}{g}\log\left(\frac{-m\frac{v^2}{R_{\text{earth}}} + mg}{\left(-P_{\text{trop}}A + \frac{\beta}{2}\rho_{\text{trop}}|(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{e}_v|^2 S\right)}\right) + h_{\text{trop}}, \tag{24}$$

or isolating β that

$$\beta = \frac{2}{\rho_{\text{trop}}|(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{e}_v|^2 S}\left(\mathrm{e}^{\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)}\left(-\frac{mv^2}{R_{\text{earth}}} + mg\right) + P_{\text{trop}}A\right) \tag{25}$$

Depending on which parameter we wish to control.

An important point to remember is that when the plane banks, a component of the lift contributes to the turn. However, this effect is not accounted for in our model due to a lack of available information.

2.2 Lift coefficient below below the tropopause

Another formula can be used to approximate density and pressure below the tropopause as explained in the previous section. The air density at a certain altitude can be approximated as [3]

$$\rho = \rho_0 \left(\frac{T_0 - \frac{6.5H}{1000}}{T_0} \right)^{-\frac{g}{k_T R} - 1} \quad \text{and} \quad P = P_0 \left(\frac{T_0 - \frac{6.5H}{1000}}{T_0} \right)^{-\frac{g}{k_T R}}, \tag{26}$$

where P_0, ρ_0 and T_0 are the values of pressure, temperature and densities of the atmosphere at the sea level. Also k_T is the boltzman constant and R is the real gas constant. So as in the previous section, the total external force on the direction $\boldsymbol{e}_r$ is

$$w_r = -mg - PA \tag{27}$$

where A is again the projected area of the whole plane.

$$-m(t)\frac{v(t)^2}{r} = -mg + L_r - PA \tag{28}$$

where again we can use the approximation

$$\Rightarrow -m\frac{v^2}{R_{\text{earth}}} = -mg - P_0 \left(\frac{T_0 - \frac{6.5H}{1000}}{T_0} \right)^{-\frac{g}{k_T R}} A + \frac{\beta}{2}\rho_0 \left(\frac{T_0 - \frac{6.5H}{1000}}{T_0} \right)^{-\frac{g}{k_T R} - 1} |(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{e}_v|^2 S \tag{29}$$

$$\beta = \frac{2}{\rho_0 \left(\frac{T_0 - \frac{6.5H}{1000}}{T_0} \right)^{-\frac{g}{k_T R} - 1} |(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{e}_v|^2 S} \left(-m\frac{v^2}{R_{\text{earth}}} + mg + P_0 \left(\frac{T_0 - \frac{6.5H}{1000}}{T_0} \right)^{-\frac{g}{k_T R}} A \right) \tag{30}$$

With that we can obtain the value of the lift coefficient to keep the plane at a certain H below the tropopause.

References

[1] John David Anderson and Mary L Bowden. *Introduction to flight*. Vol. 582. McGraw-Hill Higher Education New York, NY, USA, 2005.

[2] Artur L Gower. *AircraftRouteDynamics.jl*. Version v0.0.1. 2025. URL: `github.com/arturgower/AircraftRouteDynamics.jl`.

[3] A. Nuic. *User Manual for the Base of Aircraft Data (BADA) - Revision 3.6*. EEC Note 10/04. EUROCONTROL Agency. Brétigny-sur-Orge, France: EUROCONTROL Experimental Centre, 2004.