Effective dynamics of an aircraft

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Abstract

To plan optimal aircraft routes we can not completely solve and describe 3D fluid structure interaction. This would be computationally very intense and unnecessary. Instead, we need to capture the main features related to fuel consumption, ability to turn, and affects from the atmosphere such as drag and lift. To achieve this we develop effective dynamical equations, together with a simple numerical scheme to solve these equations. The method is simple enough to using nonlinear optimisation to plan routes with minimal fuel, or distance, or time.

1 Equations of motion

The main forces acting on the aircraft are due to thrust, drag, lift, and a turning force. More accurately: an aircraft turns by rolling and then using lift, however we will not model these details, and instead have a turning force which is similar to a lift force.See fig. 1 for a sketch.

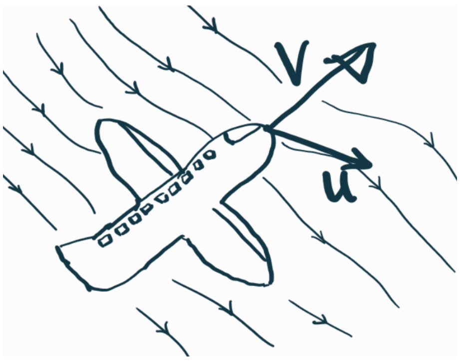

Figure 1: A sketch showing that the aircraft velocity vector is given by $\boldsymbol{v}$ (relative to the ground), and the velocity of the wind (relative to the ground) is given by $\boldsymbol{u}$.

To describe the aircraft dynamics it is useful to use a local coordinate system with basis vectors:

$$\boldsymbol{e}_a \quad \text{(aligned with the direction of travel)} \tag{1}$$

$$\boldsymbol{e}_r \quad \text{(radial direction from earth centre to aircraft)} \tag{2}$$

$$\boldsymbol{e}_p = \boldsymbol{e}_r \times \boldsymbol{e}_v \quad \text{(perpendicular to travel direction)} \tag{3}$$

where we will only model the 2D dynamics and assume the altitude is fixed, so that $\boldsymbol{e}_a$ is always orthogonal to $\boldsymbol{e}_r$. Note that changes of altitude based on balance of lift with gravity can easily be accommodated without modelling the motion that leads to that change.

There are two main forces on the aircraft, those that can be controlled by the pilot $\boldsymbol{f}$ and those that external $\boldsymbol{w}$. The forces that can be controlled are:

$$\boldsymbol{f} = T(\dot{m})\boldsymbol{e}_a + L_p(\boldsymbol{v} - \boldsymbol{u}, \alpha)\boldsymbol{e}_p + L_r(\boldsymbol{v} - \boldsymbol{u}, \beta)\boldsymbol{e}_r, \tag{4}$$

where $\dot{m}$ is the rate of change of mass in time (from using fuel), $v = |\boldsymbol{v}|$, $T(\dot{m})$ is the force from thrust, $L_p(v, \alpha)$ is a force which leads to turns in the $\boldsymbol{e}_p$ direction, where α captures the amount the pilot tries to turn, and $L_r(v, \beta)$ are the lift forces, where β is the amount to pilot tries to lift.

A simple and effective choice for thrust is

$$T(\dot{m}) = -\dot{m}C_T, \tag{5}$$

where C_T is a constant which describes how efficiently burning fuel $\dot{m}$ is converted into a thrust. More accurately,

$$C_T = (\text{exhaust velocity}) - (\text{relative airspeed}),$$

for subsonic flight [1, Chapter 4], where exhaust velocity is the speed of exhaust relative to the aircraft, and relative airspeed is equal to $|\boldsymbol{v} - \boldsymbol{u}|$.

A simple and effective formula for the turning force is

$$L_p(v, \alpha) = |(\boldsymbol{v} - \boldsymbol{u}) \cdot \boldsymbol{e}_a|^2 \alpha. \tag{6}$$

In practive, turning is due to rolling and then lift. The forces that lead to rolling and lift are due to wind drag, which is proportional to the relative airspeed in the direction of travel. The variable α is bounded: $\alpha \in [-C_\alpha, C_\alpha]$, where the positive constant C_α depends on the type of aircraft.

The dynamics of the aircraft are now governed by balance of momentum:

$$\begin{aligned} &\frac{d}{dt}(m(t)\boldsymbol{v}(t)) = \boldsymbol{f} + \boldsymbol{w} \implies \\ &\dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}} = T(\dot{m})\boldsymbol{e}_a + L_p(\boldsymbol{v} - \boldsymbol{u}, \alpha)\boldsymbol{e}_p + L_r(\boldsymbol{v} - \boldsymbol{u}, \beta)\boldsymbol{e}_r + \boldsymbol{w}, \end{aligned} \tag{7}$$

where $\boldsymbol{w}$ are the forces due to external factors, such as the wind and altitude. It is generally a function of the relative airspeed $\boldsymbol{v} - \boldsymbol{u}$, however we do not need to give explicit forms for this forces.

For the lift force L_r, we assume it is enough to balance the external vertical, or radial, forces. That is, by taking dot product of $\boldsymbol{e}_r$ on either side of (7), and using $\boldsymbol{w} \cdot \boldsymbol{e}_r = -L_r(\boldsymbol{v} - \boldsymbol{u}, \beta)$ we obtain:

$$m\dot{\boldsymbol{v}} \cdot \boldsymbol{e}_r = -\dot{m}\boldsymbol{v} \cdot \boldsymbol{e}_r. \tag{8}$$

nor for the lift L_r to simplify the equations of motion.

Equation (7) can be used to update the velocity vector $\boldsymbol{v}$ in time, which we show in more detail in the next section.

Turning the plane also changes its orientation, which as a consequence changes the basis vectors $\boldsymbol{e}_a$ and $\boldsymbol{e}_p$. To capture this we can use:

To simplify (7) we need to write all expressions in terms of the local coordinate basis $\boldsymbol{e}_a, \boldsymbol{e}_p, \boldsymbol{e}_r$. First we assume that the external forces can be decomposed in the form:

$$\boldsymbol{w} = w_a\boldsymbol{e}_a + w_p\boldsymbol{e}_p + w_r\boldsymbol{e}_r.$$

Next we use $\boldsymbol{v} = v_a\boldsymbol{e}_a + v_p\boldsymbol{e}_p$ to rewrite the expression:

$$\dot{m}\boldsymbol{v} + m\dot{\boldsymbol{v}} = \dot{v}_a\boldsymbol{e}_a + \dot{v}_p\boldsymbol{e}_p + \dot{v}_r\boldsymbol{e}_r, \tag{9}$$

where we also used

$$\dot{\boldsymbol{e}}_a = (\dot{\boldsymbol{e}}_a \cdot \boldsymbol{e}_p)\boldsymbol{e}_p + (\dot{\boldsymbol{e}}_a \cdot \boldsymbol{e}_r)\boldsymbol{e}_r \quad \dot{\boldsymbol{e}}_p = (\dot{\boldsymbol{e}}_p \cdot \boldsymbol{e}_a)\boldsymbol{e}_a + (\dot{\boldsymbol{e}}_p \cdot \boldsymbol{e}_r)\boldsymbol{e}_r.$$

To rewrite $\dot{\boldsymbol{e}}_v$ we use the standard identities for basis vectors $\frac{d}{dt}(\boldsymbol{e}_v \cdot \boldsymbol{e}_v) = 0$ to conclude that $\dot{\boldsymbol{e}}_v$ is orthogonal to $\boldsymbol{e}_v$ and can be expanded in the form:

$$\dot{\boldsymbol{e}}_v = (\dot{\boldsymbol{e}}_v \cdot \boldsymbol{e}_p)\boldsymbol{e}_p + (\dot{\boldsymbol{e}}_v \cdot \boldsymbol{e}_r)\boldsymbol{e}_r,$$

which combined with (9), and the separation of the external forces, leads us to rewrite the equations of motion (7) in the form

$$a_v\boldsymbol{e}_v + a_p\boldsymbol{e}_p + a_r\boldsymbol{e}_r = 0.$$

The above then implies that $a_v = 0$, $a_p = 0$, and $a_r = 0$, which written in full become:

$$\dot{m}v + m\dot{v} = w_r + T(\dot{m}, v), \tag{10}$$

$$m(t)v(t)(\boldsymbol{e}_p \cdot \dot{\boldsymbol{e}}_v) = w_n(r, v, t) + L_c(\alpha(t), v(t)), \tag{11}$$

$$m(t)v(t)(\boldsymbol{e}_r(t) \cdot \dot{\boldsymbol{e}}_v) = w_r(r, v, t) + L_u(\beta(t), v(t)). \tag{12}$$

1.1 Forward marching numerical method

Let us start by choosing a position $\boldsymbol{x}$ for the aircraft in a spherical coordinate system (r, θ, ϕ) with

$$\boldsymbol{x} = r[\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi].$$

Then the velocity vector of the aircraft is given by $\dot{\boldsymbol{x}} = \boldsymbol{v}$, with $\boldsymbol{v}$. Rewriting in spherical coordinates we get

$$\boldsymbol{v} = r\sin\phi \boldsymbol{e}_\theta \dot{\theta} + r\boldsymbol{e}_\phi \dot{\phi},$$

by assuming that the height r is fixed. Then, from the above we deduce that

$$r\sin\phi\dot{\theta} = \boldsymbol{v}\cdot\boldsymbol{e}_\theta \quad \text{and} \quad r\dot{\phi} = \boldsymbol{v}\cdot\boldsymbol{e}_\phi. \tag{13}$$

From the above we can see that it is convenient to write the components $\boldsymbol{v}$ in terms of the local coordinate system $\boldsymbol{e}_\theta, \boldsymbol{e}_\phi, \boldsymbol{e}_r$ coordinate system. That is, in the code,

$$\boldsymbol{v}[1] = \boldsymbol{v}\cdot\boldsymbol{e}_\theta, \quad \boldsymbol{v}[2] = \boldsymbol{v}\cdot\boldsymbol{e}_\phi, \quad \boldsymbol{v}[3] = \boldsymbol{v}\cdot\boldsymbol{e}_r.$$

We can then use the equations (13) to calculate $\theta(t+h)$ and $\phi(t+h)$ by substituting

$$\dot{\theta} = \frac{\theta(t+h) - \theta(t)}{h} \quad \text{and} \quad \dot{\phi} = \frac{\phi(t+h) - \phi(t)}{h},$$

into (13) and solving for $\theta(t+h)$ and $\phi(t+h)$. For consistency, we have that the components of all vectors are given in terms of the local basis in the code.

The equations of motion (10 - 12) can now be turned into a forward marching numerical method to predict the trajectory of the aircraft, which we briefly summarise below.

Equation (10) can be used to update the speed $v(t)$ by substituting

$$\dot{v}(t) = \frac{v(t+h) - v(t)}{h},$$

and then solving for $v(t+h)$. From the second equation (11) we can update the direction $\boldsymbol{e}_v$ and as a consequence $\boldsymbol{e}_p$. To start

$$\dot{\boldsymbol{e}}_v = a\boldsymbol{e}_p + b\boldsymbol{e}_r,$$

because $\dot{\boldsymbol{e}}_v$ is orthogonal to $\boldsymbol{e}_v$. Substituting the above into (11) then leads to

$$a = \boldsymbol{e}_p\cdot\dot{\boldsymbol{e}}_v = (w_n(r,v,t) + L_c(\alpha(t), v(t))/(m(t)v(t)). \tag{14}$$

To obtain b we use

$$b = \dot{\boldsymbol{e}}_v\cdot\boldsymbol{e}_r = -\boldsymbol{e}_v\cdot\dot{\boldsymbol{e}}_r = -\boldsymbol{e}_v\cdot\boldsymbol{v}/r = -v/r, \tag{15}$$

where we also used the $\boldsymbol{e}_r = \boldsymbol{r}/r \implies \dot{\boldsymbol{e}}_r = \boldsymbol{v}/r$ for fixed r.

To summarise we can use the above to update the direction $\boldsymbol{e}_v$

$$\boldsymbol{e}_v(t+h) = \boldsymbol{e}_v(t) + ah\boldsymbol{e}_p(t) - \frac{vh}{r}\boldsymbol{e}_r(t) \tag{16}$$

where using the coordinate system (θ, ϕ, r) we always have that $\boldsymbol{e}_r = [0,0,1]$. As we have made first order approximations in h the norm of $\boldsymbol{e}_v(t+h)$ will only be approximately 1. It is better to correct this by taking $\boldsymbol{e}_v(t+h) \leftarrow \boldsymbol{e}_v(t+h)/|\boldsymbol{e}_v(t+h)|$. Finally, we can then update the perpendicular direction:

$$\boldsymbol{e}_p(t+h) = \boldsymbol{e}_r \times \boldsymbol{e}_v(t+h). \tag{17}$$

2 Finding the right altitude to keep a certain speed above the tropopause

The tropopause is the height that separates the troposphere and the stratosphere

The force balance in normal direction gives equilibrium of the plane at a constant altitude H. The air density at a certain height can be approximated as

$$\rho = \rho_{\text{trop}} e^{-\left(\frac{g}{RT_{\text{trop}}}(H - h_{\text{trop}})\right)}, \tag{18}$$

where the sub-index 'trop' means the tropopause value of the density,temperature, and altitude provided in the NATS guide. The pressure also follows the same behaviour.

$$P = P_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} \tag{19}$$

The lift force is calculated as

$$L_u = \frac{\beta C_L}{2} \rho u^2 S \tag{20}$$

Where $u = |\boldsymbol{v} - \boldsymbol{W}|$ is the speed of the plane related to the wind, m is the mass of the plane, g is gravity , S is the area of the wing projected at the plane normal to $\boldsymbol{e}_r$, and C_L is the lift coefficient with β being its control parameter. From that we can see that the lift force is proportional to the density. So the Lift force is

$$L_u = \frac{\beta C_L}{2} \rho_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} u^2 S \tag{21}$$

So the total external force on the direction $\boldsymbol{e}_r$ is

$$w_r = -mg - PA \tag{22}$$

where A is the projected area of the whole plane.

$$-m(t)\frac{v(t)^2}{r} = -mg - PA + L_u \tag{23}$$

$$-m(t)\frac{v(t)^2}{r} = -mg - P_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} A + \frac{\beta C_L}{2} \rho_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} u^2 S \tag{24}$$

notice that $r = R_{\text{earth}} + H$ and $R_{\text{earth}} >> H$, so we approximate $r \approx R_{\text{earth}}$

$$-m\frac{v^2}{R_{\text{earth}}} = -mg - P_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} A + \frac{\beta C_L}{2} \rho_{\text{trop}} \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} u^2 S \tag{25}$$

$$\Rightarrow -m\frac{v^2}{R_{\text{earth}}} + mg = \left(-P_{\text{trop}} A + \frac{\beta C_L}{2} \rho_{\text{trop}} u^2 S\right) \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} \tag{26}$$

$$\Rightarrow \frac{-m\frac{v^2}{R_{\text{earth}}} + mg}{\left(-P_{\text{trop}} A + \frac{\beta C_L}{2} \rho_{\text{trop}} u^2 S\right)} = \mathrm{e}^{-\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right)} \tag{27}$$

$$\Rightarrow \log\left(\frac{-m\frac{v^2}{R_{\text{earth}}} + mg}{\left(-P_{\text{trop}} A + \frac{\beta C_L}{2} \rho_{\text{trop}} u^2 S\right)}\right) = -\left(\frac{g}{RT_{\text{trop}}}(H-h_{\text{trop}})\right) \tag{28}$$

Isolating H and substituting u we have that

$$H = -\frac{RT_{\text{trop}}}{g} \log\left(\frac{-m\frac{v^2}{R_{\text{earth}}} + mg}{\left(-P_{\text{trop}} A + \frac{\beta C_L}{2} \rho_{\text{trop}} |\boldsymbol{v} - \boldsymbol{W}|^2 S\right)}\right) + h_{trop} \tag{29}$$

But C_L in literature is given by

$$C_L = \frac{2mg}{\rho_{\text{trop}} |\boldsymbol{v} - \boldsymbol{W}|^2 S} \tag{30}$$

$$\Rightarrow H = -\frac{RT_{\text{trop}}}{g} \log\left(\frac{-m\frac{v^2}{R_{\text{earth}}} + mg}{-P_{\text{trop}} A + \beta mg}\right) + h_{trop} \tag{31}$$

3 Finding the right altitude to keep a certain speed below the tropopause

The air density at a certain height can be approximated as

$$\rho = \rho_0 \left(\frac{T_0 - \frac{6.5h}{1000}}{T_0} \right)^{-\frac{g}{k_T R} - 1} \tag{32}$$

and the pressure is

$$P = P_0 \left(\frac{T_0 - \frac{6.5h}{1000}}{T_0} \right)^{-\frac{g}{k_T R}} \tag{33}$$

So the total external force on the direction $\boldsymbol{e}_r$ is

$$w_r = -mg - PA \tag{34}$$

where A is the projected area of the whole plane.

$$-m(t)\frac{v(t)^2}{r} = -mg + L_u \tag{35}$$

were we dropped the pressure term just to have an estimate height

$$\Rightarrow \frac{-m\frac{v^2}{R_{\text{earth}}} + mg}{\frac{\beta C_L}{2}\rho_0 u^2 S} = \left(\frac{T_0 - \frac{6.5h}{1000}}{T_0} \right)^{-\frac{g}{k_T R} - 1} \tag{36}$$

References

[1] John David Anderson and Mary L Bowden. *Introduction to flight*. Vol. 582. McGraw-Hill Higher Education New York, NY, USA, 2005.