Brief on the package MultipleScattering2D

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Abstract

Written in Mathematica 10.2, this package calculates the scattered scalar wave, in 2D, from any configurations of cylinders. To see how to to do multiple scattering, and make pretty pictures, go to the folder `examples`. Inside the folder this is `examples/TwoBodyScattering.nb` for scattering from two cylinders, `examples/OneCylinder.nb` for a simpler example of scattering from one cylinder and `examples/Source.nb` for some details on sources. At the moment the cylinders all have to be the same and we consider no wave transmission through the cylinders.

Keywords: fish

1 Introduction

Below is a brief about the functions `AcousticScattering`, `AcousticImpulse` and `PlotWaves`. The next sections contains some of theory behind this package.

`scatteredWave = AcousticScattering[`$r_S, \boldsymbol{r}_I, \{t_0, t_{max}, \delta t\}, \{R_{max}, \delta R, \delta\theta\}$`,options];`

returns the scattered wave from one cylinder at $\{0, 0\}$, where

- each element of `Flatten[scatteredWave, 2]` is of the form $\{t, \theta, r, wave[t, \theta, r]\}$.

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- r_S is the radius of the scatterer, $\boldsymbol{r}_I = \{x_I, y_I\}$ is the centre of the impulse that generates the incident wave. The scatterer is always placed at $\{0,0\}$.

- The list $\{t_0,\ t_{max},\ \delta t\}$ specifies the range for the time considered, so $\{t_0, t_{max}, \delta t\}$ $= \{1,\ 2,\ 0.5\}$ means that the scattered wave will be calculated for $t = 1, 1.5$ and 2. The list $\{t_0, t_{max}, \delta t\}$ can also be replaced by just one time $\{t_0, t_{max}, \delta t\} = t_0$ and then the solution will be given just for $t = t_0$, which is useful if you want to calculate something that might crash the memory.

- The list $\{R_{max}, \delta R, \delta\theta\}$ specifies the discretization of r and θ (in cylindrical coordinates), so for $\{R_{max}, \delta R, \delta\theta\} = 2, 0.5, \pi/4$ and $r_S = 0.5$ means that the solution will be calculated for $r = 0.5, 1., 1.5, 2$ and $\theta = 0, \pi/4, 2\pi/4, \pi$.

- The `options` argument is optional but it can be of the form `options = Sequence["Impulse" -> b, "ImpulsePeriod" -> 2., "PrintChecks" -> True, "Boundary" -> "Neumann", "FrequencyModes" -> 20, "NAngularModes" -> 7];` , which would specify the body forcea $B(x,t) = \delta(\boldsymbol{x})b(t)$ where $b(t) = 0$ for $t < 0$ or $t > 2.$. The rest is better explained below.

- The function `incidentWave= AcousticImpulse[`$t, \{r_{max}, \delta r\}$`, options]` generates a list `incidentWave =`$\{\{0, \varphi^I(0,t)\}, \{\delta r, \varphi^I(\delta r, t)\}, \ldots, \{r_{max}, \varphi^I(r_{max}, t)\}\}$`;` .

- To plot the results just run `PlotWaves[scatteredWave]` or `PlotWaves[incidentWave]`.

1.1 Incident wave

We look to solve the 3D wave equation

$$\mathcal{L}\{\varphi\}(\boldsymbol{x},t) = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}(\boldsymbol{x},t) - \nabla^2\varphi(\boldsymbol{x},t) = \frac{1}{c^2}B(\boldsymbol{x},t), \tag{1}$$

with the conditions

$$\varphi(\boldsymbol{x}, 0^-) = 0, \quad \dot{\varphi}(\boldsymbol{x}, 0^-) = 0 \quad \text{and} \quad \lim_{\|\boldsymbol{x}\| \to 0} \varphi(\boldsymbol{x}, t) = 0, \tag{2}$$

where B is the body force*. To solve this we use the Delta Dirac $\delta(\boldsymbol{x}) = \delta(x)\delta(y)\delta(z) = \delta(r)/(4\pi r^2)$ if r is the radius of a spherical coordinate system, and first solve the wave equation in spherical coordinates

$$\frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g}{\partial r}\right) = \frac{\delta(r)}{4\pi r^2}\delta(t) \tag{3}$$

with $g(r, t) = 0$ for $t < 0$. This is not trivially solved, as when differentiating r^{-1} a distribution appears on the origin. The solution can be found in p. 92 Achenbach (1973)†

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$$g(r, t) = \frac{1}{4\pi r}\delta(t - r/c). \tag{4}$$

If we let $B(\boldsymbol{x}, t) = \delta(\boldsymbol{x})b(t)$, then the solution to Eq. (1) without a scatterer, i.e. for the incident wave, becomes

$$\varphi^I(\boldsymbol{x}, t) = \int g(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)\frac{1}{c^2}B(\boldsymbol{\xi}, \tau)d\boldsymbol{\xi}d\tau = \\ \int g(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)\frac{1}{c^2}\delta(\boldsymbol{\xi})b(\tau)d\boldsymbol{\xi}d\tau = \int \delta(t - \tau - r/c)\frac{b(\tau)}{4c^2\pi r}d\tau = \frac{b(t - r/c)}{4c^2\pi r}. \tag{5}$$

which is the solution to Eq. (1) because

$$\mathcal{L}\{\varphi^I\}(\boldsymbol{x}, t) = \int \mathcal{L}\{g\}(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)\frac{1}{c^2}B(\boldsymbol{\xi}, \tau)d\xi d\tau = \int \delta(\boldsymbol{x} - \boldsymbol{\xi}, t - \tau)\frac{1}{c^2}B(\boldsymbol{\xi}, \tau)d\xi d\tau = \frac{1}{c^2}B(\boldsymbol{x}, t).$$

*This B is technically only a body force for an elastic SH-wave. The interpretation of B depends on the physical interpretation of φ.

†There he changes to spherical coordinates, substitutes $\varphi(r, t) = \Phi(r, t)/r$, and with witchcraft solves the resulting scalar wave equation, picking only the outgoing wave

Let us adopt the Fourier transform convention:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega t} d\omega \ \text{ and } \ \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t} dt.$$

If we took one frequency $g(r,t) = \hat{g}(r,\omega)\mathrm{e}^{-\mathrm{i}\omega t}$ and solved for $\hat{g}$ with $B = c^2\mathrm{e}^{-\mathrm{i}\omega t}\delta(\boldsymbol{x})$, one frequency of $c^2\delta(\boldsymbol{x})\delta(t)$, the solution using only outgoing waves would be

$$\hat{g}(r,\omega) = \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi r}, \tag{6}$$

with $k = \omega/c$, which after a Fourier transform would give the causal 3D Greens function (4) as expected.

For the frequency decomposition of the 2D Greens function $\hat{g}_2$, we imagine that all functions will be independent of the z coordinate. So to use $\hat{g}$ in a convolution we need to first integrate over z:

$$\hat{g}_2 = \int_{-\infty}^{\infty} \hat{g}(r,\omega) dz = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}k\sqrt{r^2+z^2}}}{4\pi\sqrt{r^2+z^2}} dz = \frac{\mathrm{i}}{4} H_0^{(1)}(kr), \tag{7}$$

where $r^2 = x^2 + y^2$ and $\hat{g}_2$ is an outgoing wave solution to

$$k^2\hat{g}_2 + \nabla_2^2\hat{g}_2 = -\delta(x)\delta(y), \tag{8}$$

where $\hat{B}(\boldsymbol{x},\omega) = c^2\delta(\boldsymbol{x})$ and ∇_2 is the gradient in x and y. Note that Graf's and Gegenbauer's addition formulas are very useful to rewrite any bessel or hankel function.

To calculate g_2 we can take the 3D Greens (4) substitute $r \to \sqrt{r^2+z^2}$ and integrate in z to get

$$g_2 = \int_{-\infty}^{\infty} \frac{\delta(ct - \sqrt{r^2+z^2})}{4\pi\sqrt{r^2+z^2}} c dz = \frac{c}{2\pi} \frac{H_s(t - |r|/c)}{\sqrt{c^2t^2 - r^2}}, \tag{9}$$

where H_s is the Heavside step function, so that $H_s(t - |r|/c)$ is zero for $r > ct$.

We can now calculate the 2D incident wave for $B(\boldsymbol{x},t) = \delta(\boldsymbol{x})b(t)$ by using the procedure (5) for g_2 to find

$$\varphi^I(\boldsymbol{x},t) = \int_{-\infty}^{\infty} g_2(r,t-\tau)\frac{b(\tau)}{c^2}d\tau = \int_{|r|/c}^{\max\{t,\frac{|r|}{c}\}} \frac{1}{2\pi c}\frac{b(t-\tau)}{\sqrt{c^2\tau^2-r^2}}d\tau, \tag{10}$$

where we changed variables $\tau \leftarrow t-\tau$ so that we can differentiate the above expression in t more easily (specially numerically) and assumed that $b(-t)=0$ for $t>0$. Alternatively, Eq. (10) can also be written in terms of the Fourier transforms $\hat{g}_2(r,\omega)$ and $\hat{b}(\omega)$ as

$$\varphi^I(\boldsymbol{x},t) = \frac{1}{c^2}\int_{-\infty}^{\infty} g_2(r,\tau)b(t-\tau)d\tau = \frac{1}{2\pi c^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\hat{g}_2(r,\omega)\hat{b}(\omega)d\omega. \tag{11}$$

Assuming that $b(t)=0$ for $t \notin [0,T]$, then we turn into a numerical method by approximating $b(t)$ with its truncated Fourier series $b_N(t)$ so that

$$b_N(t) = \frac{1}{T}\sum_{n=-N}^{N} \hat{b}_N^n \mathrm{e}^{-\mathrm{i}t\frac{2\pi n}{T}}, \ \text{ where } \ \hat{b}_N^n := \hat{b}_N\left(\frac{2\pi n}{T}\right) = \int_0^T b(t)\mathrm{e}^{\mathrm{i}\frac{2\pi n t}{T}}dt, \tag{12}$$

We can turn this into the discrete Fourier transform by using only the points

$$b_N^m := b_N\left(\frac{mT}{2N+1}\right) = \frac{1}{T}\sum_{n=-N}^{N} \hat{b}_N^n \mathrm{e}^{-2\pi\mathrm{i}\frac{mn}{2N+1}} = \frac{1}{T}\sum_{n=0}^{2N} \hat{b}_N^{n-N} \mathrm{e}^{-2\pi\mathrm{i}\frac{mn}{2N}} \mathrm{e}^{2\pi\mathrm{i}m\frac{N}{2N+1}}, \tag{13}$$

for $m = 0,1,\ldots,2N$. We can now apply some linear algebra to extract the coefficients $T^{-1}\hat{b}_N^{n-N}$ of the vectors

$$(\boldsymbol{v}_n)^m := \mathrm{e}^{-2\pi\mathrm{i}\frac{mn}{2N+1}}\mathrm{e}^{2\pi\mathrm{i}m\frac{N}{2N+1}}, \text{ where } \ \boldsymbol{v}_n \cdot \bar{\boldsymbol{v}}_j = (2N+1)\delta_{nj}, \tag{14}$$

to reach that

$$\hat{b}_N^n = \frac{T}{(2N+1)}\sum_{m=0}^{2N} b_N^m \mathrm{e}^{2\pi\mathrm{i}\frac{mn}{2N+1}}, \tag{15}$$

which is the definition of the discrete Fourier transform.

The convolution formula (11) can now be approximated by

$$\varphi^I(\boldsymbol{x}, t) \approx \frac{1}{Tc^2} \sum_{n=-N}^{N} \mathrm{e}^{-\mathrm{i}\frac{2\pi n}{T}t} \hat{g}_2(r, 2\pi n/T) \hat{b}_N^n. \tag{16}$$

To choose $N = 20$ for the package add `"FrequencyModes"->20` to the function call. One possible issue is that $\hat{g}_2$ has a singularity at $\omega = 0$. More generally, every Hankel function of the first type has a singularity at $\omega = 0$, which we will deal with carefully in the next section.

1.2 Scattered wave

The Fourier transform of the outgoing waves from a cylinder can be anything in the form

$$\hat{\varphi}^S(r, \omega) = \sum_{n=-\infty}^{\infty} a_n H_n^{(1)}(\omega r/c) \mathrm{e}^{\mathrm{i}n\theta} \approx \sum_{n=-Na}^{Na} a_n H_n^{(1)}(\omega r/c) \mathrm{e}^{\mathrm{i}n\theta} \tag{17}$$

where we call Na the number of angular modes. To choose $Na = 6$ for the package add `"NAngularModes"->6` to the function call.

To calculate the scattered wave in time φ^S, we do the same operation on $\hat{\varphi}_S$ that we did on $\hat{g}_2$ in Eq.(16)

$$\begin{aligned} \varphi^S(r, t) \approx \frac{1}{2\pi c^2} \int_{-2\pi N/T}^{2\pi N/T} \mathrm{e}^{-\mathrm{i}\omega t} \hat{\varphi}^S(r, \omega) \hat{b}_N(\omega) d\omega \\ = \frac{1}{2\pi c} \sum_{n=-\infty}^{\infty} \int_{-2\pi N/T}^{2\pi N/T} \mathrm{e}^{-\mathrm{i}ckt + \mathrm{i}n\theta} a_n(k) H_n^{(1)}(kr) \hat{b}_N(ck) dk. \end{aligned} \tag{18}$$

The coefficients a_n will be determined by the boundary conditions.

1.2.1 Boundary condition

To compare the incident and scattered wave we use Graf's addition formula ‡ applied to

$$H_0^{(1),(2)}(k\|\boldsymbol{r}-\boldsymbol{r}_I\|) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n(\theta-\theta_1)} J_n(kr) H_n^{(1),(2)}(kr_I) \quad \text{if } \; r < r_I, \tag{19}$$

where the body force originates at $\boldsymbol{r}_I = r_I(\cos\theta_I, \sin\theta_I)$ and $\boldsymbol{r} = r(\cos\theta, \sin\theta)$. If $\hat{\varphi}^I = \hat{g}_2(\|\boldsymbol{r} - \boldsymbol{r}_I\|)$ and $\hat{\varphi}^S = a_n H_n^{(1)}(kr)\mathrm{e}^{\mathrm{i}n\theta}$ then the boundary condition $\hat{\varphi}_I + \hat{\varphi}_S = 0$ on $r = r_S < r_I$ implies

$$\frac{\mathrm{i}}{4}\mathrm{e}^{-\mathrm{i}n\theta_I} J_n(kr_S) H_n^{(1)}(kr_I) + a_n H_n^{(1)}(kr_S) = 0 \implies \tag{20}$$

$$a_n = -\frac{\mathrm{i}}{4}\mathrm{e}^{-\mathrm{i}n\theta_I} \frac{J_n(kr_S)}{H_n^{(1)}(kr_S)} H_n^{(1)}(kr_I). \tag{21}$$

while the boundary condition $\partial\hat{\varphi}^I/\partial r + \partial\hat{\varphi}^S/\partial r = 0$ on $r = r_S < r_I$ implies that

$$a_n = -\frac{i}{4}\mathrm{e}^{-\mathrm{i}n\theta_I} \frac{J_n'(kr_S)}{H_n'^{(1)}(kr_S)} H_n^{(1)}(kr_I). \tag{22}$$

1.3 Wave frequency to time response

This module is used for more complicated scattering, such as MST. Any outgoing wave from a scatterer can be expanded as

$$\hat{\psi}^S(\boldsymbol{r}, k) \approx \sum_{n=-Na}^{Na} a_n H_n(kr)\mathrm{e}^{\mathrm{i}n\theta},$$

for a cylindrical coordinate system with origin at the scatterer, where $H_n := H_n^{(1)}$ a Hankel function of the first kind. To recover the wave in time we need to approximate the inverse

‡see `http://dlmf.nist.gov/10.23#E7`

Fourier transform:

$$\psi^S(\boldsymbol{r},t) = \frac{c}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}^S(\boldsymbol{r},k)\mathrm{e}^{-\mathrm{i}ckt}dk \approx \frac{c}{2\pi}\sum_{n=-Na}^{Na} \mathrm{e}^{\mathrm{i}n\theta}\int_{-2\pi N/T}^{2\pi N/T} a_n H_n(kr)\mathrm{e}^{-\mathrm{i}ckt}dk,$$

where T is the period of the incident wave and N is the number of frequency modes.

Assuming that the scatterers are small compared with the wavelength $kr_S << 1$, where r_S is the radius of the scatterer, then the most general form for outgoing waves from the j-th scatterer is

$$\psi^S(\boldsymbol{r}) = aH_0(kr) + (c\cos\theta + s\sin\theta)H_1(kr), \tag{23}$$

which means that $a_0 = a$, $a_1 = (s+c)/4$ and $a_{-1} = (s-c)/4$.

This package was used in the talk:

Gower, Artur L., et al. *"Characterizing composites with acoustic backscattering: Combining data driven and analytical methods."* The Journal of the Acoustical Society of America 141.5 (2017): 3810-3810.