Calculating scattering coefficients from a cylinder

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Abstract

Here we derive some of the theory used in the Mathematica package MultipleScattering2D. Specifically the function `ScatteringCoefficientOperator`, which calculates the scattering coefficients for any source/exciting function, is given by combing equations (26,23,19) for Dirichlet boundary conditions. Similar equations are given for Neumann too.

Keywords: Multi-pole method, multiple scattering, scattering operator

1 The General Setup

For a review on multiple scattering from a finite number of obstacles see [1].

Consider a homogeneous isotropic medium that can propagate wave according to the scalar Helmholtz equation

$$(\Delta + k^2)\psi(\boldsymbol{r}) = 0, \tag{1}$$

where k is real for acoustics and possibly imaginary for fluids. Let there be N scatterers, with the centre located at $\boldsymbol{r}_1$, $\boldsymbol{r}_1$, ..., $\boldsymbol{r}_N$. If we excite the scatterers by sending an incident wave $\psi^I(\boldsymbol{r})$, we can then write the total wave field $\psi(\boldsymbol{r}|\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N) = \psi^I(\boldsymbol{r}) + \sum_{j=1}^N \psi^S(\boldsymbol{r};\boldsymbol{r}_j|\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$, where $\psi^S_j(\boldsymbol{r}) := \psi^S(\boldsymbol{r};\boldsymbol{r}_j|\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$ is the outward going wave* being emitted from the boundary of the j-th scatterer. The term

*We are only concerned with what the scatterer is emitting, and not waves that might bounce around inside.

$\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N$ indicates the dependence of ψ and ψ_j^S's on the position of all the scatterers. Each scattered wave ψ_j^S is excited by

$$\psi_j^E(\boldsymbol{r}) = \psi^I(\boldsymbol{r}) + \sum_{i \neq j} \psi_i^S(\boldsymbol{r}), \tag{2}$$

which we can relate to ψ_j^S through the boundary condition on the j-th scatterer to give

$$\psi_j^S(\boldsymbol{r}) = \mathcal{T}_j \left\{ \psi_j^E \right\}(\boldsymbol{r}), \tag{3}$$

where $\mathcal{T}_j\{\circ\}(\boldsymbol{r})$ is the linear *scattering operator* which acts on the whole function $\circ$. Dirchlett and Neumann boundary condtions are examples of when $\mathcal{T}_j$ is a linear operator. By expanding ψ_j^S as in Eq.(3), then substituting ψ_j^E from Eq.(2) and then repeating the process by expanding ψ_i^S with eq.(3) again we obtain

$$\psi_j^S(\boldsymbol{r}) = \mathcal{T}_j\{\psi^I\}(\boldsymbol{r}) + \sum_{i \neq j} \mathcal{T}_j \circ \mathcal{T}_i \left\{ \psi^I \right\}(\boldsymbol{r}) + \sum_{i \neq j} \sum_{n \neq i} \mathcal{T}_j \circ \mathcal{T}_i \circ \mathcal{T}_n \left\{ \psi^I \right\}(\boldsymbol{r}) + \ldots. \tag{4}$$

For two scatterers this becomes,

$$\psi_1^S(\boldsymbol{r}) = \mathcal{T}_1\{\psi^I\}(\boldsymbol{r}) + \mathcal{T}_1 \circ \mathcal{T}_2 \left\{ \psi^I \right\}(\boldsymbol{r}) + \mathcal{T}_1 \circ \mathcal{T}_2 \circ \mathcal{T}_1 \left\{ \psi^I \right\}(\boldsymbol{r}) + \ldots. \tag{5}$$

If on the other hand we stopped expanding the series at

$$\begin{aligned} \psi_1^S(\boldsymbol{r}) &= \mathcal{T}_1 \left\{ \psi^I \right\}(\boldsymbol{r}) + \mathcal{T}_1 \left\{ \psi_2^S \right\}(\boldsymbol{r}), \\ \psi_2^S(\boldsymbol{r}) &= \mathcal{T}_2 \left\{ \psi^I \right\}(\boldsymbol{r}) + \mathcal{T}_2 \left\{ \psi_1^S \right\}(\boldsymbol{r}). \end{aligned} \tag{6}$$

If we could expand each ψ_j^S into a general outgoing wave

$$\psi_j^S(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} s_{jn} \mathcal{C}_n(r) \mathcal{A}_n(\theta, \phi),$$

then we could solve for the coefficients s_{jn} by substituting into Eqs.(6).

2 Cylindrical Scatterers

The outgoing wave from a scatterer at $\boldsymbol{r}_j$ be expanded as

$$\psi^{S_j}(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} a_{jn} H_n(k\|\boldsymbol{r} - \boldsymbol{r}_j\|) \mathrm{e}^{\mathrm{i}n\alpha_j}, \tag{7}$$

where $H_n := H_n^{(1)}$ is a Hankel function of the first kind and α_j is the angle between $\boldsymbol{r} - \boldsymbol{r}_j$ and the x–axis.

We can expand the outgoing waves (7) by using Graf's addition theorem[†]:

$$\mathcal{C}_\nu(k\|\boldsymbol{r}-\boldsymbol{r}_1\|)\mathrm{e}^{\mathrm{i}\nu(\alpha_1-\theta_{12})} = \sum_{m=-\infty}^{\infty} \mathcal{C}_{\nu+m}(k\|\boldsymbol{r}_1-\boldsymbol{r}_2\|)J_m(k\|\boldsymbol{r}-\boldsymbol{r}_2\|)\mathrm{e}^{\mathrm{i}m(\pi+\theta_{12}-\alpha_2)}, \quad (8)$$

provided $\|\boldsymbol{r}-\boldsymbol{r}_2\| < \|\boldsymbol{r}_1-\boldsymbol{r}_2\|$, where θ_{12} and α_2 are respectively the angular cylindrical coordinate of $\boldsymbol{r}_1-\boldsymbol{r}_2$ and $\boldsymbol{r}-\boldsymbol{r}_2$. In the addition theorem $\mathcal{C}_\nu$ can be substituted with any of the Bessel functions $J_\nu, Y_\nu, H_\nu^{(1)}$ and $H_\nu^{(2)}$, or linear combinations of them.

Using Eq.(8) we write ψ^{S_j} in terms of an origin centered at the i-th scatterer by substituting

$$H_n(k\|\boldsymbol{r}-\boldsymbol{r}_j\|)\mathrm{e}^{\mathrm{i}n\alpha_j} = \sum_{m=-\infty}^{\infty} H_{n-m}(k\|\boldsymbol{r}_j-\boldsymbol{r}_i\|)J_m(k\|\boldsymbol{r}-\boldsymbol{r}_i\|)\mathrm{e}^{\mathrm{i}m\alpha_i+\mathrm{i}(n-m)\theta_{ij}},$$

into the ψ^{S_j} to arrive at

$$\psi^{S_j}(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_{jn}H_{n-m}(k\|\boldsymbol{r}_j-\boldsymbol{r}_i\|)J_m(k\|\boldsymbol{r}-\boldsymbol{r}_i\|)\mathrm{e}^{\mathrm{i}m\alpha_i+\mathrm{i}(n-m)\theta_{ij}}. \quad (9)$$

In Section 2.1 we derive some explicit forms for the scattering operator $\mathcal{T}$, given by Eqs.(26) and (27). Assuming that the scatterers are small compared with the wavelength $kr_S << 1$, where r_S is the radius of the scatterer, then the most general form for outgoing waves from the j-th scatterer is

$$\psi^{S_j}(\boldsymbol{r}) = a_j H_0(k\|\boldsymbol{r}-\boldsymbol{r}_j\|) + (c_j\cos(\boldsymbol{r}-\boldsymbol{r}_j) + s_j\sin(\boldsymbol{r}-\boldsymbol{r}_j))H_1(k\|\boldsymbol{r}-\boldsymbol{r}_j\|), \quad (10)$$

with $c_j = s_j = 0$ for Dirichlet boundary conditions, where $\cos(\boldsymbol{r}-\boldsymbol{r}_j) = \cos\vartheta$ and ϑ is the angle between $\boldsymbol{r}-\boldsymbol{r}_j$ and the x-axis. In which case, using the results in Section 2.1, the scattering operator for Neumann boundary conditions can be written as

$$\begin{aligned}\mathcal{T}_j : \psi^E(r,\theta) \to & \frac{\mathrm{i}\pi r_S^2}{4}(\psi^E_{\boldsymbol{r}_j,xx}+\psi^E_{\boldsymbol{r}_j,yy})H_0(k\|\boldsymbol{r}-\boldsymbol{r}_j\|) \\ & + \frac{i\pi r_S^2}{2}k(\psi^E_{\boldsymbol{r}_j,x}\cos(\boldsymbol{r}-\boldsymbol{r}_j)+\psi^E_{\boldsymbol{r}_j,y}\sin(\boldsymbol{r}-\boldsymbol{r}_j))H_1(k\|\boldsymbol{r}-\boldsymbol{r}_j\|), \quad (11)\end{aligned}$$

accurate up to order r_S^3 in the scatterer radius r_S, where the subscript $\boldsymbol{r}_j$ on $\psi^E_{\boldsymbol{r}_j}$ means that ψ^E and its derivatives are evaluated at $\boldsymbol{r}=\boldsymbol{r}_j$ after differentiation. This

[†]I double checked this with both `http://dlmf.nist.gov/10.23` and `http://www.wikiwaves.org/Graf's_Addition_Theorem`.

implies that to solve Eqns. (6) for every $\boldsymbol{r}$ we need to equate the coefficients of $H_0(k\|\boldsymbol{r}-\boldsymbol{r}_j\|)$, $\cos(\boldsymbol{r}-\boldsymbol{r}_j)H_1(k\|\boldsymbol{r}-\boldsymbol{r}_j\|)$ and $\sin(\boldsymbol{r}-\boldsymbol{r}_j)H_1(k\|\boldsymbol{r}-\boldsymbol{r}_j\|)$ each to be zero, from which we get

$$a_i = \frac{\mathrm{i}\pi r_S^2}{4}(\psi^I_{\boldsymbol{r}_i,xx} + \psi^I_{\boldsymbol{r}_i,yy} + \psi^{S_j}_{\boldsymbol{r}_i,xx} + \psi^{S_j}_{\boldsymbol{r}_i,yy}), \tag{12}$$

$$c_i = \frac{i\pi r_S^2}{2}k(\psi^I_{\boldsymbol{r}_i,x} + \psi^{S_j}_{\boldsymbol{r}_i,x}), \quad s_i = \frac{i\pi r_S^2}{2}k(\psi^I_{\boldsymbol{r}_i,y} + \psi^{S_j}_{\boldsymbol{r}_i,y}). \tag{13}$$

To simplify we choose, without loss of generality, $x_2 = x_1$ and $-y_1 = y_2 = Y/2$ which we use to reduce Eqns. (6) to

$$\frac{k^2}{2}(H_0 - H_2)a_1 + \frac{k^2}{4}(3H_1 - H_3)s_1 - \frac{4\mathrm{i}}{\pi r_S^2}a_2 = \psi^I_{\boldsymbol{r}_2,xx} + \psi^I_{\boldsymbol{r}_2,yy}, \tag{14}$$

$$\frac{H_1}{Y}c_1 + \frac{2i}{\pi k r_S^2}c_2 = -\psi^I_{\boldsymbol{r}_2,x}, \quad kH_1 a_1 - \frac{2i}{\pi k r_S^2}s_2 - \frac{k}{2}(H_0 - H_2)s_1 = \psi^I_{\boldsymbol{r}_2,y} \tag{15}$$

where the Hankel functions H_0, H_1, H_2 and H_3 are evaluated at kY, with $Y = \|\boldsymbol{r}_1 - \boldsymbol{r}_2\|$, and to reach the above equations we have used some recurrence relations to calculate the derivatives of H_0 and H_1.

Apply $\mathcal{T}_1$ on ψ_2^S with the centre of scatterer 1 as the origin we get

$$\mathcal{T}_1\psi_2^S(\boldsymbol{r}) = a_2 H_0(k\|\boldsymbol{r}-\boldsymbol{r}_2\|) + (c_2\cos(\boldsymbol{r}-\boldsymbol{r}_2) + s_2\sin(\boldsymbol{r}-\boldsymbol{r}_2))H_1(k\|\boldsymbol{r}-\boldsymbol{r}_2\|), \tag{16}$$

The following formulas will be useful, for $f(\|\boldsymbol{r}-\boldsymbol{r}_j\|)$ we have

$$f_{,x}(kr_j) = \frac{kx_j}{r_j}f'(kr_j), \quad f_{,y}(kr_j) = \frac{ky_j}{r_j}f'(kr_j), \tag{17}$$

$$f_{,yy}(kr_j) + f_{,xx}(kr_j) = \frac{k}{r_j}f'(kr_j) + k^2 f''(kr_j). \tag{18}$$

2.1 Boundary conditions and point scatterers

Here we develop expressions for the scattering operator $\mathcal{T}$ and establish how point scatterers behave for different boundary conditions. Given an incident wave ψ^E and outgoing cylindrical wave $\psi^S = a_n H_n(kr)\mathrm{e}^{\mathrm{i}n\theta}$, together $\psi^E + \psi^S$ must satisfy the boundary condition on the cylinder with radius $r = r_S$. To simplify we choose the origin of our coordinate system to be the centre of the scatterer and will use $\bar{r} := rk$.

For a Dirichlet boundary conditions we have

$$\psi^E(\bar{r}_S,\theta) + \psi^S(\bar{r}_S,\theta) = 0 \quad \text{for} \;\; 0 \le \theta < 2\pi \implies a_n = -\frac{H_n(\bar{r}_S)^{-1}}{2\pi}\int_0^{2\pi}\psi^E(\bar{r}_S,\theta)\mathrm{e}^{-\mathrm{i}n\theta}d\theta, \tag{19}$$

noting that $\mathrm{e}^{in\theta}$ forms a complete basis for functions on $\theta \in [0, 2\pi]$.

For a Neumann boundary conditions we have

$$\frac{\partial \psi^E}{\partial \bar{r}}(\bar{r}_S, \theta) + \frac{\partial \psi^S}{\partial \bar{r}}(\bar{r}_S, \theta) = 0 \quad \text{for} \quad 0 \leq \theta < 2\pi \implies \tag{20}$$

$$a_n = -\frac{\partial_{\bar{r}} H_n(\bar{r}_S)^{-1}}{2\pi} \int_0^{2\pi} \partial_{\bar{r}} \psi^E(\bar{r}_S, \theta) \mathrm{e}^{-\mathrm{i}n\theta} d\theta, \tag{21}$$

Our main interest is to model point scatterers for which $\bar{r}_S \to 0$. In this limit the incident wave should converge in an open ball to its Taylor series at r_S (as the wave field should be perfectly smooth), so we can write

$$\begin{aligned} \psi^E(\bar{r}_S, \vartheta) &= \sum_{k=0}^{\infty} \sum_{m=0}^{p} \frac{\partial^p \psi_0^E}{\partial_x^m \partial_y^{p-m}} \frac{r_S^p (\cos\vartheta)^m (\sin\vartheta)^{p-m}}{m!(p-m)!} \\ &= \sum_{p=0}^{\infty} \sum_{m=0}^{p} \frac{\partial^p \psi_0^E}{\partial_x^m \partial_y^{p-m}} \frac{r_S^p}{m!(p-m)!} \sum_{m_1=0}^{m} \sum_{p_1=0}^{p-m} \frac{\mathrm{e}^{\mathrm{i}(p-2m_1-2p_1)\vartheta}}{(-1)^{p_1} 2^p \mathrm{i}^{p-m}} \binom{m}{m_1} \binom{p-m}{p_1}, \end{aligned} \tag{22}$$

where the subscript 0 on ψ_0^E and its derivatives means that the expression was evaluated at $x = y = 0$ after differentiation. When substituting this expression into the integral in Eq.(19), only terms with $p - 2m_1 - 2p_1 = n$ will have a non-zero contribution, because every other term after multiplying with $\mathrm{e}^{-\mathrm{i}n\vartheta}$ and integrating in ϑ over 0 to 2π will be zero. Likewise, only terms with $p - 2m_1 - 2p_1 = n$ will have a non-zero contribution to Eq.(21). So letting $p_1 = j - n/2 - m_1$ and $p = 2j$, so that $p - 2m_1 - 2p_1 = n$, and integrating over ϑ we conclude that

$$\mathcal{P}_n\{\psi^E\} = \frac{1}{2\pi} \int_0^{2\pi} \psi^E(\bar{r}_s, \vartheta) \mathrm{e}^{-\mathrm{i}n\vartheta} d\vartheta = \sum_{j=|n|/2}^{\infty} \left(\frac{r_S}{2}\right)^{2j} \sum_{m=0}^{2j} \frac{\partial^{2j} \psi_0^E}{\partial_x^m \partial_y^{2j-m}} \frac{\mathrm{i}^{m-2j} C_{n,j}^m}{m!(2j-m)!}, \tag{23}$$

where

$$C_{n,j}^m = \sum_{m_1 = \max\{0, m-j-n/2\}}^{\min\{m, j-n/2\}} (-1)^{-n/2+j-m_1} \binom{m}{m_1} \binom{2j-m}{j-n/2-m_1}, \tag{24}$$

where j (same as all the dummy indexes) increases in increments of 1, and as p_1 was substituted for $j - n/2 - m_1$, then from $0 \leq p_1 \leq p - m$ we conclude that $m_1 \leq j - n/2$ and $-j - n/2 + m \leq m_1$, which combined with the restriction $0 \leq m_1 \leq m$ implies that $\max\{0, m - j - n/2\} \leq m_1 \leq \min\{m, j - n/2\}$.

For the Neumann boundary condition we need

$$\frac{1}{2\pi} \int_0^{2\pi} \partial_{\bar{r}} \psi^E(\bar{r}_s, \vartheta) \mathrm{e}^{-\mathrm{i}n\vartheta} d\vartheta = k^{-1} \partial_{r_S} \mathcal{P}_n\{\psi^E\}. \tag{25}$$

The terms (23) suggest that the series of the scattering operators converge absolutely if $r_S \leq 2$, assuming the derivatives of ψ^E are uniformly bounded for every n and $\bar{r} > \bar{r}_S$ and $k > \delta > 0$ (as w^E will often have a singularity at $k = 0$). So for $r_S \leq 2$ we can truncate the series (23) at $j = N_j/2$ for every n, which we denote by $\mathcal{P}_n^{N_j} := \mathcal{P}_n + \mathcal{O}(r_s^{N_j+2})$.

Eq. (23) together with Eq. (19) imply that the scattering operator becomes

$$\mathcal{T} : \psi^E(\bar{r}, \theta) \to \psi^S(\bar{r}, \theta) = -\sum_{n=-\infty}^{\infty} \frac{H_n(\bar{r})}{H_n(\bar{r}_S)} \mathcal{P}_n\{\psi^E\} \mathrm{e}^{\mathrm{i}n\theta}, \tag{26}$$

which is indeed linear in ψ^E. Similarly for the Neumann boundary condition the scattering operator becomes

$$\mathcal{T} : \psi^E(\bar{r}, \theta) \to \psi^S(\bar{r}, \theta) = -\sum_{n=-\infty}^{\infty} \frac{H_n(\bar{r})}{\partial_{\bar{r}} H_n(\bar{r}_S)} k^{-1} \partial_{r_S} \mathcal{P}_n\{\psi^E\} \mathrm{e}^{\mathrm{i}n\theta}, \tag{27}$$

which is also linear in ψ^E.

To examine the limit where the particles radius r_S is small in comparison to the wavelength $kr_S \to 0$, we first note that

$$\frac{H_n(\bar{r})}{H_n(\bar{r}_S)} = H_n(\bar{r}) \begin{cases} \frac{\mathrm{i}\pi (kr_S)^n}{2^n (n-1)!} + \mathcal{O}(kr_S)^{n+2}, & n > 0, \\ \frac{\pi}{\pi + 2\mathrm{i}(\gamma_0 + \log(kr_S/2))} + \mathcal{O}(kr_S)^2, & n = 0, \end{cases} \tag{28}$$

and

$$\frac{H_n(\bar{r})}{\partial_{\bar{r}_S} H_n(\bar{r}_S)} = H_n(\bar{r}) \begin{cases} -\frac{\pi \mathrm{i} (kr_S)^{n+1}}{2^n n!} + \mathcal{O}(kr_S)^{n+3}, & n > 0, \\ -\frac{\pi \mathrm{i}}{2} kr_S + \mathcal{O}(kr_S)^3, & n = 0, \end{cases} \tag{29}$$

here $\gamma_0 = 0.5772$, the term $H_n(\bar{r})$ has not been asymptotically expanded as $\bar{r}$ can be of any size, and note that for $n < 0$ we have $H_n(\bar{r}_S) = (-1)^n H_{|n|}(\bar{r}_S)$. So, for example, if we want to asymptotically expand the scattered wave w^S up too $\mathcal{O}(\bar{r}_S^2)$ for the Dirchlett boundary condition, we need to expand

$$\psi^S(\bar{r}, \theta) = -\frac{H_1(\bar{r})}{H_1(\bar{r}_S)} \mathcal{P}_{-1}\{\psi^E\} \mathrm{e}^{-\mathrm{i}\theta} - \frac{H_0(\bar{r})}{H_0(\bar{r}_S)} \mathcal{P}_0\{\psi^E\} - \frac{H_1(\bar{r})}{H_1(\bar{r}_S)} \mathcal{P}_1\{\psi^E\} \mathrm{e}^{\mathrm{i}\theta} + \mathcal{O}(\bar{r}_S^2) \tag{30}$$

$$= -H_0(kr) \frac{\pi \psi_0^E}{2\mathrm{i} \log(\bar{r}_S/2) + \pi + 2\mathrm{i}\gamma_0} + \mathcal{O}(\bar{r}_S^2), \tag{31}$$

while to expand w^S up too $\mathcal{O}(\bar{r}_S^3)$ for the Neumann boundary condition, we need

$$\begin{aligned}\psi^S(\bar{r},\theta) =& -\frac{k^{-1}H_1(\bar{r})}{\partial_{\bar{r}} H_1(\bar{r}_S)}\partial_{r_S}\mathcal{P}_1\{\psi^E\}\mathrm{e}^{\mathrm{i}\theta} - \frac{k^{-1}H_0(\bar{r})}{\partial_{\bar{r}} H_0(\bar{r}_S)}\partial_{r_S}\mathcal{P}_0\{\psi^E\} \\ & - \frac{k^{-1}H_1(\bar{r})}{\partial_{\bar{r}} H_1(\bar{r}_S)}\partial_{r_S}\mathcal{P}_{-1}\{\psi^E\}\mathrm{e}^{-\mathrm{i}\theta} + \mathcal{O}(\bar{r}_S^3) \\ =& \frac{i\pi r_S^2}{2}k(\partial_x\psi_0^E\cos\theta + \partial_y\psi_0^E\sin\theta)H_1(kr) \\ & + \frac{\mathrm{i}\pi r_S^2}{4}(\partial_x^2\psi_0^E + \partial_y^2\psi_0^E)H_0(kr) + \mathcal{O}(\bar{r}_S^3). \end{aligned} \tag{32}$$

If we are interested in lower frequencies for k, then we must be careful with the dependence that ψ^E has on the wavenumber k. In general ψ^E will be a sum of terms of the form

$$\psi_q^E = H_q(k\|\boldsymbol{r} - \boldsymbol{r}_E\|)\mathrm{e}^{\mathrm{i}q\arctan(\boldsymbol{r}-\boldsymbol{r}_E)}, \tag{33}$$

where $\arctan(x,y) = \arctan(y/x)$, $\boldsymbol{r}_E$ is a constant vector, c_q is determined by boundary conditions and we assume they are uniformly bounded for every k. If we are to approximate $\mathcal{P}_n\{\psi_q^E\}(k\partial_{r_S}\mathcal{P}_n\{\psi_q^E\})$ up too $\mathcal{O}(r_s^{N_j+2})(\mathcal{O}(r_s^{N_j+1}))$ by truncating at $j = N_j/2$, then we should investigate the term left out $j = N_j/2+1$. Using (33) and expanding the term $j = N_j/2+1$ in $\mathcal{P}_n\{\psi_q^E\}$ in a series of powers of k, the lowest order term will be

$$\mathcal{P}_n\{\psi_q^E\} - \mathcal{P}_n^{N_j}\{\psi_q^E\} = k^{-|q|}r_S^{N_j+2}\mathcal{O}(1) + \mathcal{O}(r_S^{N_j+2}k^{2-|q|}), \tag{34}$$

which is multiplied with Eq. (28) to get the scattering operator. For the Neumann boundary condition,

$$k^{-1}\left(\partial_{r_S}\mathcal{P}_n\{\psi_q^E\} - \partial_{r_S}\mathcal{P}_n^{N_j}\{\psi_q^E\}\right) = k^{-q-1}r_S^{N_j+1}\mathcal{O}(1) + \mathcal{O}(r_S^{N_j+1})\mathcal{O}(k^{1-q}), \tag{35}$$

which is multiplied with Eq. (29).

References

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