# The constitutive relations of initially stressed incompressible Mooney-Rivlin materials

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**Abstract**

Initial stresses originate in soft materials by the occurrence of misfits in the undeformed microstructure. Since the reference configuration is not stress-free, the effects of initial stresses on the hyperelastic behavior must be constitutively addressed. Notably, the free energy of an initially stressed material may not possess the same symmetry group as the one of the same material deforming from a naturally unstressed configuration. This work assumes that the hyperelastic strain energy density is characterized only by the deformation gradient and the initial stress tensor, using an explicit functional dependence on their independent invariants. Within this theoretical framework, a constitutive equation is derived for an initially stressed body that naturally behaves as an incompressible Mooney-Rivlin material. The strain energy densities for initially stressed neo-Hookean and Mooney materials are derived as special sub–cases. By assuming the existence of a virtual state that is naturally stress-free, the resulting strain energy functions are proved to fulfill the required frame–independence constraints for this special class of constitutive models. In the case of plane strain condition, great simplifications arise in the expression of the constitutive relations. Finally, the resulting constitutive relations prove useful guidelines for designing non-destructive methods for the quantification of the underlying initial stresses in naturally isotropic materials.

*Keywords:* Initial stress; residual stress; constitutive equations; Mooney-Rivlin material.

## 1. Introduction

By *initial stress* we mean any internal stress of a reference configuration of an elastic material. For instance, initial stress can be formed by applying surface tractions or body forces to the reference configuration, but can also be present in their absence, in which case we call it *residual stress*. Residual stresses can arise in solids due to *geometric incompatibility* of the material's microstructure [1, 2]. For instance, they can form after thermal expansion in inert matter [3, 4] or growth and remodeling in living materials [5, 6, 7].

The presence of initial stresses can be difficult to ignore, as they can greatly affect the elastic response [8, 9]. For instance, they can induce inhomogeneous and anisotropic responses [10] even when the material is homogeneous and structurally isotropic.

The mechanical response of hyperelastic materials can be completely determined from their free energy density function $W$. When there is no initial stress, $W$ can be written solely in terms of $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, where $\mathbf{F}$ is the elastic deformation gradient from the stress-free reference configuration to the current configuration [11]. For isotropic materials, the strain energy can be written as a scalar-value tensor function of the invariants of $\mathbf{C}$ [12]. Theoretically, the unknown mechanical parameters in the expression of $W$ can then be determined by performing the same number of independent mechanical tests as the number of independent invariants. However, it is useful to derive constitutive laws with minimal covariance amongst the response terms in the Cauchy stress associated to each invariant, since it can be shown [**?** ] that such a covariance correlates with the error in determining $W$ that is inherent in experimental tests on real materials.

Even when the material's reference configuration is initially stressed, we can use a similar approach to describe $W$ by simply assuming there exists a virtual stress-free state $\mathcal{B}_0$. We can then define $W$ in terms of $\mathbf{C}_2 = \mathbf{F}_2^T\mathbf{F}_2$, where $\mathbf{F}_2$ is the deformation gradient from $\mathcal{B}_0$ to the current configuration [13]. The challenge in this approach is that this virtual state in not known *a priori*, thus it is not clear how to choose $\mathbf{F}_2$.

A direct way of accounting for initial stresses $\boldsymbol{\tau}$ is to assume that the hyperelastic strain energy density depends on both the deformation gradient and the initial stress tensor, so that $W := \hat{W}(\mathbf{C}, \boldsymbol{\tau})$ [9, 14]. A general theoretical framework for this approach has recently been developed [15], which is independent of how the initial stress $\boldsymbol{\tau}$ is formed. Using a form $\hat{W}(\mathbf{C}, \boldsymbol{\tau})$ can be simpler because it is often easier to choose/postulate an initial stress $\boldsymbol{\tau}$ rather than $\mathbf{F}_2$.

The drawback of this approach is that it is generally not straightforward to choose a physically viable form for $W(\mathbf{C}, \boldsymbol{\tau})$. A simplifying assumption is to require that $W$ only depends on $\mathbf{C}$ and $\boldsymbol{\tau}$. Thus, $W$ has to satisfy a form of frame–independence [16, 17], one example being:

$$\hat{W}(\mathbf{C}, \boldsymbol{\tau}) = \hat{W}(\mathbf{I}, \boldsymbol{\sigma}), \quad \text{for every} \;\; \boldsymbol{\tau}, \; \mathbf{F}, \tag{1}$$

for incompressible materials. Accordingly, if the current stress $\boldsymbol{\sigma}$ is known, for the assumed frame-independence $\boldsymbol{\tau}$ can be also expressed as [16]

$$\boldsymbol{\tau} = 2\mathbf{F}^{-1}\frac{\partial \hat{W}(\mathbf{C}^{-1},\boldsymbol{\sigma})}{\partial \mathbf{C}^{-1}}\mathbf{F}^{-T} - \bar{p}_0\mathbf{I},$$

where $\bar{p}_0$ is an unknown that appears when the material is incompressible.

It is important to underline that Equation 1 defines a very specific class of constitutive responses. More restrictively than in [9], the mechanical parameters defining the functional dependence on the set of invariants are assumed in this model as constants, i.e. independent on the choice of the reference configuration.

To our knowledge there are three free energy density functions that satisfy this constitutive assumption, as discussed in [17]. However, these strain energies do not take into account for the invariants $J_2^{-1}$, $J_3$ or $J_4^{-1}$ (defined by equation (23)), which can be essential even in simple deformations [18]. In this work we aim at deducing strain energy functions that include a functional dependence on all the invariants, and satisfy the assumption of frame–independence(1).

An explicit expression for the free energy function is derived by assuming that there exists a stress–free state $\mathcal{B}_0$, and that the material behaves like a classical Mooney-Rivlin material after any deformation of $\mathcal{B}_0$, see figure 1. Specifically, let $\mathbf{F}_2$ be the deformation gradient from $\mathcal{B}_0$ to the current configuration $\mathcal{B}_2$, we deduce $\hat{W}$ such that $\hat{W}(\mathbf{C},\boldsymbol{\tau}) = W(\mathbf{C}_2)$, where $W(\mathbf{C}_2)$ has the classical functional dependence of a Mooney-Rivlin material. Accordingly, $\hat{W}$ describes an initially stressed Mooney-Rivlin material whose strain energy function automatically fulfills the frame independence constraints [16, 17].

The article is organized as follows. In Section 2 we summarize the main mathematical assumptions behind the definition of a virtual stress-free state. In Section 3 we derive the strain energy density and the constitutive relation for the Cauchy stress for an initially stressed, incompressible, Mooney-Rivlin material as a function of both $\mathbf{F}$ and $\boldsymbol{\tau}$. In Section 4 we simplify the Mooney-Rivlin model to an initially stresses Neo-Hookean and Mooney materials. We also present the Mooney-Rivlin model under planar initial stress and elastic strain in Section 5. The results are discussed in Section 6 together with few concluding remarks.

## 2. The virtual stress-free state

A constitutive relation for residually stressed bodies is classically derived by assuming that there is a virtual unstressed state whose properties can be measured through standard destructive experiments, such as material cutting [13]. The basic idea is sketched in Figure 1.

Let us introduce a virtual stress free state $\mathcal{B}_0$, an initially stressed configuration $\mathcal{B}_1$ and current configuration $\mathcal{B}_2$. The spherical neighborhood $\mathcal{P}_{\mathbf{X},\epsilon}$ with radius $\epsilon$, of the point $\mathbf{X}$ in $\mathcal{B}_1$, deforms

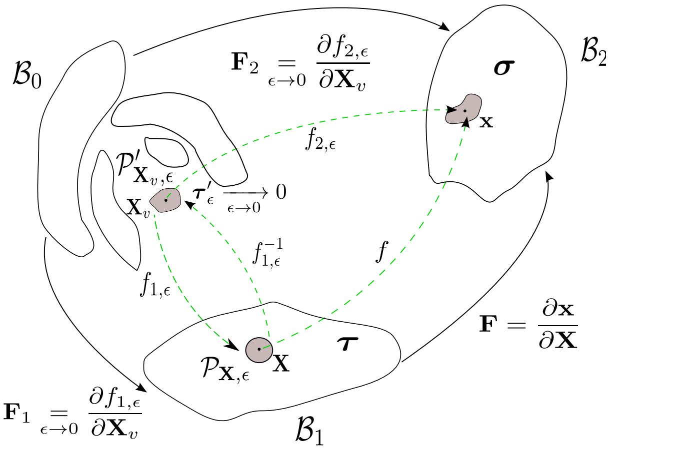


Figure 1: Virtual stress-free state $\mathcal{B}_0$, initially stressed configuration $\mathcal{B}_1$ with internal stress $\boldsymbol{\tau}$ and spatial configuration $\mathcal{B}_2$ with Cauchy stress $\boldsymbol{\sigma}$. The spherical neighborhood $\mathcal{P}_{\mathbf{X},\epsilon}$, with radius $\epsilon$, of the point $\mathbf{X}$ in $\mathcal{B}_1$ deforms elastically into the configuration $\mathcal{P}'_{\mathbf{X}_v,\epsilon}$ around the point $\mathbf{X}_v$, through a smooth invertible deformation $f^{-1}_{1,\epsilon}$. The residual stress $\boldsymbol{\tau}'_\epsilon$ in $\mathcal{P}'_{\mathbf{X}_v,\epsilon}$ vanishes as $\epsilon$ approaches zero. The tangent maps from $\mathcal{B}_0$ to $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by the tensors $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively, where $\mathbf{F}_1(\mathbf{X}_v) = \partial f_{1,\epsilon}/\partial \mathbf{X}_v$ and $\mathbf{F}_2(\mathbf{X}_v) = \partial f_{2,\epsilon}/\partial \mathbf{X}_v$ for $\epsilon$ tending to zero. $\mathbf{F} = \partial \mathbf{x}/\partial \mathbf{X}$ is the deformation gradient of the isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$.

elastically, after the removal of the tractions at its boundary from the surrounding body, into the configuration $\mathcal{P}'_{\mathbf{X}_v,\epsilon}$ around the point $\mathbf{X}_v$, assuming the existence of a smooth invertible deformation $f^{-1}_{1,\epsilon}$. The residual stress $\boldsymbol{\tau}'_\epsilon$ in $\mathcal{P}'_{\mathbf{X}_v,\epsilon}$ vanishes as $\epsilon$ approaches zero. The stressed configurations $\mathcal{B}_1$ and $\mathcal{B}_2$ are characterized by the internal stress $\boldsymbol{\tau}$ and Cauchy stress $\boldsymbol{\sigma}$, respectively. The infinitesimal deformations of the material from $\mathcal{B}_0$ to $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined by the maps $\mathbf{F}_1$ and $\mathbf{F}_2$ respectively, where $\mathbf{F}_1(\mathbf{X}_v) = \partial f_{1,\epsilon}/\partial \mathbf{X}_v$ and $\mathbf{F}_2(\mathbf{X}_v) = \partial f_{2,\epsilon}/\partial \mathbf{X}_v$ for $\epsilon$ tending to zero. If the response of the material is in the elastic regime, the Cauchy stress can be expressed in terms of the position in the configuration $\mathcal{B}_2$ and of the deformation gradient $\mathbf{F}_2$ from the virtual stress free state to the current configuration, which can be written as a multiplicative decomposition $\mathbf{F}_2 = \mathbf{F}\mathbf{F}_1$. In this decomposition, $\mathbf{F}_1$ represents a fixed initial strain which gives the change of metrics from $\mathcal{B}_0$ to $\mathcal{B}_1$ and determines the internal stress $\boldsymbol{\tau}$, whereas $\mathbf{F}$ represents the dynamical deformation. If the internal stress-strain equation which relates $\boldsymbol{\tau}$ and $\mathbf{F}_1$ is invertible, the Cauchy stress can be finally expressed in terms of $\mathbf{F}$ and $\boldsymbol{\tau}$ [10]. This procedure is valid if the residual stress field and the structural properties of the material locally vary in a sufficiently smooth manner. However, although this method is technically sound, it rarely leads to analytic results for soft materials, due to the presence of constitutive nonlinearities and the need to identify the natural state of the material using destructive techniques.

## 3. Derivation of the strain energy function for an initially stressed, incompressible Mooney–Rivlin material

In this section we derive the free energy density and the Cauchy stress tensor for an incompressible solid made of a Mooney-Rivlin material and subjected to initial stresses. In order to ensure the frame–independence assumption, we consider elastic deformations between a virtual stress-free state $\mathcal{B}_0$, a residually stressed configuration $\mathcal{B}_1$ and the current stressed configuration $\mathcal{B}_2$ (see Figure 1).

Before proceeding, we introduce some preliminary definitions and results. The principal invariants of a tensor $\mathbf{A}$ are defined as

$$I_A = \mathrm{tr}\mathbf{A}, \quad II_A = \frac{1}{2}\Big((\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}\mathbf{A}^2\Big), \quad III_A = \mathrm{det}\mathbf{A}.$$

The Cayley-Hamilton theorem states that every tensor satisfies its own characteristic equation [19, 20], i.e.

$$\mathbf{A}^3 - I_A\mathbf{A}^2 + II_A\mathbf{A} - II_A\mathbf{I} = \mathbf{0}. \tag{2}$$

Taking $\mathcal{B}_0$ as the reference configuration, the free energy density $W$ in the current configuration can be written as a function of $\mathbf{F}_2$ in the following way:

$$W = c_1(I_{C_2} - 3) + c_2(II_{C_2} - 3), \tag{3}$$

where $c_1, c_2$ are two positive constants and $\mathbf{C}_2 = \mathbf{F}_2^T\mathbf{F}_2$ is the right Cauchy-Green tensor. Note that the expression (3) is invariant under coordinate transformations. The Cauchy stress tensor has the form [21]

$$\boldsymbol{\sigma} = 2\mathbf{F}_2\frac{DW(\mathbf{C}_2)}{D\mathbf{C}_2}\mathbf{F}_2^T - \bar{p}\mathbf{I} = 2c_1\mathbf{B}_2 - 2c_2\mathbf{B}_2^{-1} - p\mathbf{I}, \tag{4}$$

where $\mathbf{B}_2$ is the left Cauchy-Green tensor $\mathbf{B}_2 = \mathbf{F}_2\mathbf{F}_2^T$, $\bar{p}$ is the Lagrange multiplier associated with the incompressibility constraint, $p = \bar{p} - 2c_2 II_{C_2}$ (see [**?** ] for a detailed discussion of its physical meaning) and $DW/D\mathbf{C}_2$ is the Frechét derivative of the scalar field $W(\mathbf{C}_2)$. At the same time, taking $\mathcal{B}_0$ as the reference configuration and $\mathcal{B}_1$ as the actual configuration, we have that

$$\boldsymbol{\tau} = 2\mathbf{F}_1\frac{DW(\mathbf{C}_1)}{D\mathbf{C}_1}\mathbf{F}_1^T - \bar{p}_0\mathbf{I} = 2c_1\mathbf{B}_1 - 2c_2\mathbf{B}_1^{-1} - p_0\mathbf{I}, \tag{5}$$

where $\bar{p}_0$ is the Lagrange multiplier for the incompressibility constraint on the infinitesimal deformation $\mathbf{F}_1$, which is determined by the boundary conditions in the configuration $\mathcal{B}_1$, and $p_0 = \bar{p}_0 - 2c_1 II_{C_1}$. Taking the multiplicative decomposition $\mathbf{F}_2 = \mathbf{F}\mathbf{F}_1$ in (4), we get

$$\boldsymbol{\sigma} = 2c_1\mathbf{F}\mathbf{B}_1\mathbf{F}^T - 2c_2\mathbf{F}^{-T}\mathbf{B}_1^{-1}\mathbf{F}^{-1} - p\mathbf{I}. \tag{6}$$

Considering now $\mathcal{B}_1$ as the reference configuration, we want to express $W$ and $\boldsymbol{\sigma}$ only as a function of the initial stress $\boldsymbol{\tau}$ and of the deformation gradient $\mathbf{F}$. As described in the introduction, the final expression will be generally valid independently on how the initial stress $\boldsymbol{\tau}$ is formed. The Cauchy stress tensor in this case has the form [21]

$$\boldsymbol{\sigma} = 2\mathbf{F}\frac{D\hat{W}(\mathbf{C}, \boldsymbol{\tau})}{D\mathbf{C}}\mathbf{F}^T - \hat{p}\mathbf{I}, \tag{7}$$

where, comparing with (4), we have that $W(\mathbf{C}_2) = \hat{W}(\mathbf{C}, \boldsymbol{\tau})$. It is useful to remark that Equation (7) is a more general constitutive statement than Equation (4), since the variation of the strain energy is performed on different tensor fields. Indeed, Equation (4) explicitly considers a tangent map defining a change of metrics induced by the initial strain $\mathbf{F}_1 = \mathbf{F}^{-1}\mathbf{F}_2$, whilst Equation (7) refers to a variation performed on the deformation gradient $\mathbf{F}$ at fixed initial stresses. In the following we will omit for simplicity to write $\hat{W}$ with the hat sign, using the symbol $W$ for indicating a general free energy density function. As a first step, let us express $p_0$ in (5) as a function of $\boldsymbol{\tau}$. Imposing directly the incompressibility constraint $\mathrm{det}\mathbf{F}_1 = 1$ to (5), we get that

$$\begin{aligned}\mathrm{det}(\boldsymbol{\tau} + p_0\mathbf{I}) =& \mathrm{det}\Big(2c_1\mathbf{B}_1 - 2c_2\mathbf{B}_1^{-1}\Big) = -8c_2^3\mathrm{det}\Big(\mathbf{I} - \frac{c_1}{c_2}\mathbf{B}_1^2\Big) = \\ & - 8c_2^3\Big(1 - \frac{c_1}{c_2}\mathrm{tr}\mathbf{B}_1^2 + \frac{1}{2}\frac{c_1^2}{c_2^2}[(\mathrm{tr}\mathbf{B}_1^2)^2 - \mathrm{tr}\mathbf{B}_1^4] - \frac{c_1^3}{c_2^3}\Big).\end{aligned} \tag{8}$$

From repeated use of the Cayley-Hamilton theorem (2), we have that

$$\mathbf{B}_1^3 = I_{B_1}\mathbf{B}_1^2 - II_{B_1}\mathbf{B}_1 + \mathbf{I}, \tag{9}$$

$$\mathbf{B}_1^4 = I_{B_1}^2\mathbf{B}_1^2 - I_{B_1}II_{B_1}\mathbf{B}_1 + I_{B_1} - II_{B_1}\mathbf{B}_1^2 + \mathbf{B}_1. \tag{10}$$

Taking the trace of (10), using the identity $\mathrm{tr}\mathbf{B}_1^2 = I_{B_1}^2 - 2II_{B_1}$ and substituting in (8), considering moreover that the first term on the left hand side of (8) is the characteristic polynomial of $\boldsymbol{\tau}$, we get that

$$\begin{aligned}&p_0^3 + I_\tau p_0^2 + II_\tau p_0 + III_\tau + 8c_2^3 - 8c_1c_2^2(I_{B_1}^2 - 2II_{B_1}) + \\ &8c_1^2c_2(II_{B_1}^2 - 2I_{B_1}) - 8c_1^3 = 0.\end{aligned} \tag{11}$$

In order to express $p_0$ as a function of the linear invariants of $\boldsymbol{\tau}$ alone, we need to write $I_{B_1}$ and $II_{B_1}$ in (11) as functions of $I_\tau$, $II_\tau$ and $p_0$. This can be done by solving the following system of nonlinear equations, which can be derived from (5) and (9),

$$\begin{cases} I_\tau = & -3p_0 + 2c_1I_{B_1} - 2c_2II_{B_1}, \\ II_\tau = & 3p_0^2 + 4c_1^2II_{B_1} + 4c_2^2I_{B_1} - 4p_0c_1I_{B_1} + \\ & 4p_0c_2II_{B_1} - 4c_1c_2(I_{B_1}II_{B_1} - 3). \end{cases} \tag{12}$$

From the first equation of (12) we get

$$I_{B_1} = \frac{1}{2c_1}(I_\tau + 3p_0 + 2c_2II_{B_1}), \tag{13}$$

which, substituted in the second equation of (12), gives

$$L\,II_{B_1}^2 + M(I_\tau, p_0)II_{B_1} + N(I_\tau, II_\tau, p_0) = 0, \tag{14}$$

where

$$\begin{aligned}&L = 4c_1c_2^2, \quad M(I_\tau, p_0) = 2c_1c_2I_\tau + 6c_1c_2p_0 - 4c_1^3 - 4c_2^3, \\ &N(I_\tau, II_\tau, p_0) = -12c_1^2c_2 - 2c_2^2I_\tau + c_1II_\tau - 6c_2^2p_0 + \\ &\qquad 2c_1I_\tau p_0 + 3c_1p_0^2.\end{aligned} \tag{15}$$

The quadratic equation (14) admits two solutions $II_{B_1}^{\pm}$, given by

$$II_{B_1}^{\pm}(I_\tau, II_\tau, p_0) = \frac{-M(I_\tau, p_0) \pm \sqrt{M(I_\tau, p_0)^2 - 4LN(I_\tau, II_\tau, p_0)}}{2L}. \tag{16}$$

It's easy to show that only one of the two solutions (16) is physically correct. Indeed, since $\mathbf{F}_1$ is arbitrary, we can choose $\mathbf{F}_1 = \mathbf{I}$, which corresponds to $\boldsymbol{\tau} = \mathbf{0}$ and $II_{B_1} = 3$. By choosing these specific values for $\mathbf{F}_1$ and $\boldsymbol{\tau}$, we get from (5) that $p_0 = 2c_1 - 2c_2$, and from (16) we get that $M^2 - 4LN$ is a perfect square,

$$\sqrt{M(I_\tau, p_0)^2 - 4LN(I_\tau, II_\tau, p_0)} = 4\left|c_1^3 - 3c_2c_1^2 - 3c_2^2c_1 + c_2^3\right| = 4(c_1 + c_2)\left|(c_1 - c_2)^2 - 2c_1c_2\right|,$$

with

$$II^-_{\mathbf{B}_1} = \begin{cases} 3 & c_1^2 + c_2^2 \geq 4c_1c_2, \\ \dfrac{c_1^3 - 3c_2c_1^2 + c_2^3}{c_1c_2^2} & \text{otherwise,} \end{cases} \tag{17}$$

and

$$II^+_{\mathbf{B}_1} = \begin{cases} 3 & c_1^2 + c_2^2 < 4c_1c_2, \\ \dfrac{c_1^3 - 3c_2c_1^2 + c_2^3}{c_1c_2^2} & \text{otherwise.} \end{cases} \tag{18}$$

Since $c_1$ and $c_2$ can take arbitrary positive values, and we expect that $II_{B_1}$ is a continuous function of the invariants of $\boldsymbol{\tau}$, with $II_{B_1} = 3$ for $\mathbf{F}_1 = \mathbf{I}$, we get that the physically viable solution is

$$II_{\mathbf{B}_1} = \frac{-M(I_\tau, p_0) + s\sqrt{M(I_\tau, p_0)^2 - 4LN(I_\tau, II_\tau, p_0)}}{2L}, \tag{19}$$

where,

$$s = \text{sign}(2c_1c_2 - (c_1 - c_2)^2), \tag{20}$$

with

$$sign(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

From (13) we obtain

$$\begin{cases} I_{B_1}(I_\tau, II_\tau, p_0) = \dfrac{I_\tau}{2c_1} + \dfrac{3p_0}{2c_1} - \dfrac{c_2}{2c_1L}\Big(M(I_\tau, p_0) \\ \qquad -s\sqrt{M(I_\tau, p_0)^2 - 4LN(I_\tau, II_\tau, p_0)}\Big), \\ II_{B_1}(I_\tau, II_\tau, p_0) = \dfrac{-M(I_\tau, p_0) + s\sqrt{M(I_\tau, p_0)^2 - 4LN(I_\tau, II_\tau, p_0)}}{2L}. \end{cases} \tag{21}$$

Finally, substituting (21) in (11), we obtain that $p_0(I_\tau, II_\tau, III_\tau)$ is a solution of the following equation:

$$\begin{aligned} p_0^3 + I_\tau p_0^2 + II_\tau p_0 + III_\tau + 8c_2^3 - 8c_1^3 = \\ 8c_1c_2^2[I_{B_1}(I_\tau, II_\tau, p_0)^2 - 2II_{B_1}(I_\tau, II_\tau, p_0)] - \\ 8c_1^2c_2[II_{B_1}(I_\tau, II_\tau, p_0)^2 - 2I_{B_1}(I_\tau, II_\tau, p_0)]. \end{aligned} \tag{22}$$

By studying the intersections of the cubic algebraic curve on the left hand side of (22) with the transcendental curve on the right hand side, we could show that there are at most three solutions of (22), depending on the values of $c_1$, $c_2$ and $\boldsymbol{\tau}$. In Section 5 we will find the physical solution of (22) in the simplified case of a planar initial strain.

In order to express $W$ as a function of the initial stress $\boldsymbol{\tau}$ and of the deformation gradient $\mathbf{F}$, let us assume that the free energy density is a smooth function of the following set of invariants of the strain measure and the residual stress, which constitute an integrity basis for isotropic scalar-valued functions:

$$\begin{aligned} &I_C = \text{tr}\mathbf{C}, \quad II_C = \frac{1}{2}\Big((\text{tr}\mathbf{C})^2 - \text{tr}\mathbf{C}^2\Big), \\ &I_\tau = \text{tr}\boldsymbol{\tau}, \quad II_\tau = \frac{1}{2}\Big((\text{tr}\boldsymbol{\tau})^2 - \text{tr}\boldsymbol{\tau}^2\Big), \quad III_\tau = \det\boldsymbol{\tau}, \\ &J_1 = \text{tr}(\boldsymbol{\tau}\mathbf{C}), \quad J_2^{-1} = \text{tr}(\boldsymbol{\tau}\mathbf{C}^{-1}), \quad J_3 = \text{tr}(\boldsymbol{\tau}^2\mathbf{C}), \quad J_4^{-1} = \text{tr}(\boldsymbol{\tau}^2\mathbf{C}^{-1}). \end{aligned} \tag{23}$$

Note that the set of invariants (23) is not the classical one introduced in the literature (see, e.g., [22], [14], [16]. The standard set of invariants for the isotropic case usually includes $J_2 = \text{tr}(\boldsymbol{\tau}\mathbf{C}^2)$ and $J_4 = \text{tr}(\boldsymbol{\tau}^2\mathbf{C}^2)$ instead of $J_2^{-1}$ and $J_4^{-1}$. Using Cayley-Hamilton theorem (2) with $\mathbf{A} = \mathbf{C}$, then multiplying both sides by either $\boldsymbol{\tau}$ or $\boldsymbol{\tau}^2$, we get

$$\begin{aligned} J_2^{-1} &= \text{tr}(\boldsymbol{\tau}\mathbf{C}^2) - I_C\text{tr}(\boldsymbol{\tau}\mathbf{C}) + II_C\text{tr}\boldsymbol{\tau} = J_2 - I_CJ_1 + II_CI_\tau, \\ J_4^{-1} &= \text{tr}(\boldsymbol{\tau}^2\mathbf{C}^2) - I_C\text{tr}(\boldsymbol{\tau}^2\mathbf{C}) + II_C\text{tr}\boldsymbol{\tau}^2 = J_4 - I_CJ_3 + II_C(I_\tau^2 - 2II_\tau). \end{aligned} \tag{24}$$

In [23] an alternative representation for the free energy function considering a set of spectral invariants defined in terms of the principal variables of the right Cauchy-Green tensor is introduced. By expressing the ten classical invariants in the compressible case in terms of the spectral invariants and using implicit relations between the spectral invariants, it has been shown that only nine of the classical invariants are functionally independent (see Appendices A and B in [23]), even if no explicit global relation of one of them in terms of the others is currently available.

Noting that

$$\frac{DJ_2^{-1}}{D\mathbf{C}} = -\mathbf{C}^{-1}\boldsymbol{\tau}\mathbf{C}^{-1}, \quad \frac{DJ_4^{-1}}{D\mathbf{C}} = -\mathbf{C}^{-1}\boldsymbol{\tau}^2\mathbf{C}^{-1},$$

from equation (7) we get

$$\begin{aligned} \boldsymbol{\sigma} = 2\mathbf{F}\frac{DW(\mathbf{C}, \boldsymbol{\tau})}{D\mathbf{C}}\mathbf{F}^T - \hat{p}\mathbf{I} = 2W_{,I_C}\,\mathbf{B} - 2W_{,II_C}\,\mathbf{B}^{-1} + \\ 2W_{,J_1}\,\mathbf{F}\boldsymbol{\tau}\mathbf{F}^T - 2W_{,J_2^{-1}}\,\mathbf{F}^{-T}\boldsymbol{\tau}\mathbf{F}^{-1} + 2W_{,J_3}\,\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^T - \\ 2W_{,J_4^{-1}}\,\mathbf{F}^{-T}\boldsymbol{\tau}^2\mathbf{F}^{-1} - \hat{p}\mathbf{I}, \end{aligned} \tag{25}$$

where the notation $W_{,(\cdot)}$ means derivative of $W$ with respect to the argument $(\cdot)$. We now substitute in (25) equation (5) and the relation

$$\boldsymbol{\tau}^2 = (p_0^2 - 8c_1c_2)\mathbf{I} - 4p_0c_1\mathbf{B}_1 + 4p_0c_2\mathbf{B}_1^{-1} + 4c_1^2\mathbf{B}_1^2 + 4c_2^2\mathbf{B}_1^{-2}, \tag{26}$$

expressing $\mathbf{B}_1^2$ and $\mathbf{B}_1^{-2}$ by means of the Cayley-Hamilton theorem as $\mathbf{B}_1^2 = I_{B_1}\mathbf{B}_1 - II_{B_1} + \mathbf{B}_1^{-1}$ and $\mathbf{B}_1^{-2} = \mathbf{B}_1 - I_{B_1}\mathbf{I} + II_{B_1}\mathbf{B}_1^{-1}$, where $I_{B_1} = I_{B_1}(I_\tau, II_\tau, p_0)$ and $II_{B_1} = II_{B_1}(I_\tau, II_\tau, p_0)$ according to (21). We get

$$
\begin{aligned}
&\boldsymbol{\sigma} = \\
&(2W_{,I_C} - 2p_0 W_{J_1} + 2[p_0^2 - 8c_1c_2 - 4(c_1^2 II_{B_1} + c_2^2 I_{B_1})]W_{J_3})\mathbf{B} \\
&- (2W_{,II_C} - 2p_0 W_{J_2^{-1}} + 2[p_0^2 - 8c_1c_2 - 4(c_1^2 II_{B_1} + c_2^2 I_{B_1})]W_{J_4^{-1}})\mathbf{B}^{-1} \\
&+ (4c_1 W_{,J_1} - 8[p_0c_1 - c_1^2 I_{B_1} - c_2^2]W_{J_3})\mathbf{F}\mathbf{B}_1\mathbf{F}^T \\
&- (4c_2 W_{,J_1} - 8[p_0c_2 + c_1^2 + c_2^2 II_{B_1}]W_{J_3})\mathbf{F}\mathbf{B}_1^{-1}\mathbf{F}^T \\
&- (4c_1 W_{,J_2^{-1}} - 8[p_0c_1 - c_1^2 I_{B_1} - c_2^2]W_{J_4^{-1}})\mathbf{F}^{-T}\mathbf{B}_1\mathbf{F}^{-1} \\
&+ (4c_2 W_{,J_2^{-1}} - 8[p_0c_2 + c_1^2 + c_2^2 II_{B_1}]W_{J_4^{-1}})\mathbf{F}^{-T}\mathbf{B}_1^{-1}\mathbf{F}^{-1} - \hat{p}\mathbf{I}.
\end{aligned} \tag{27}
$$

By equating (27) with (6) we obtain the following $7 \times 7$ linear system:

$$
\begin{cases}
2W_{,I_C} - 2p_0 W_{J_1} + 2[p_0^2 - 8c_1c_2 - 4(c_1^2 II_{B_1} + c_2^2 I_{B_1})]W_{J_3} = 0, \\
2W_{,II_C} - 2p_0 W_{J_2^{-1}} + 2[p_0^2 - 8c_1c_2 - 4(c_1^2 II_{B_1} + c_2^2 I_{B_1})]W_{J_4^{-1}} = 0, \\
4c_1 W_{,J_1} - 8[p_0c_1 - c_1^2 I_{B_1} - c_2^2]W_{J_3} = 2c_1, \\
4c_2 W_{,J_1} - 8[p_0c_2 + c_1^2 + c_2^2 II_{B_1}]W_{J_3} = 0, \\
4c_1 W_{,J_2^{-1}} - 8[p_0c_1 - c_1^2 I_{B_1} - c_2^2]W_{J_4^{-1}} = 0, \\
4c_2 W_{,J_2^{-1}} - 8[p_0c_2 + c_1^2 + c_2^2 II_{B_1}]W_{J_4^{-1}} = -2c_2, \\
p = \hat{p}.
\end{cases} \tag{28}
$$

Note that the last identity $p = \hat{p}$ does not give any information about the functional form of $W$. For ease of notation, we introduce the following definitions:

$$E = E(I_\tau, II_\tau, p_0) := p_0^2 - 8c_1c_2 - 4(c_1^2 II_{B_1} + c_2^2 I_{B_1}), \tag{29}$$

$$G = G(I_\tau, II_\tau, p_0) := p_0c_1 - c_1^2 I_{B_1} - c_2^2, \tag{30}$$

$$H = H(I_\tau, II_\tau, p_0) := p_0c_2 + c_1^2 + c_2^2 II_{B_1}, \tag{31}$$

$$\Delta := c_1H - c_2G = c_1^3 + c_2^3 + c_1c_2^2 II_{B_1} + c_2c_1^2 I_{B_1} > 0. \tag{32}$$

The solution of (28) is

$$
\begin{aligned}
&W_{,I_C} = \frac{c_1p_0H}{2\Delta} - \frac{c_1c_2E}{4\Delta}, W_{,II_C} = \frac{c_2p_0G}{2\Delta} - \frac{c_1c_2E}{4\Delta}, \\
&W_{,J_1} = \frac{c_1H}{2\Delta}, \quad W_{,J_2^{-1}} = \frac{c_2G}{2\Delta}, W_{,J_3} = \frac{c_1c_2}{4\Delta}, \quad W_{,J_4^{-1}} = \frac{c_1c_2}{4\Delta}.
\end{aligned} \tag{33}
$$

From (33) and (24) we finally get

$$
\begin{aligned}
W(\mathbf{F}, \boldsymbol{\tau}) = &\Big[\frac{c_1p_0H}{2\Delta} - \frac{c_1c_2E}{4\Delta}\Big]I_C + \Big[\frac{c_2p_0G}{2\Delta} - \frac{c_1c_2E}{4\Delta}\Big]II_C + \\
&\frac{c_1H}{2\Delta}J_1 + \frac{c_2G}{2\Delta}(J_2 - I_CJ_1 + II_CI_\tau) + \frac{c_1c_2}{4\Delta}J_3 + \\
&\frac{c_1c_2}{4\Delta}(J_4 - I_CJ_3 + II_C(I_\tau^2 - 2II_\tau)) + f(I_\tau, II_\tau, III_\tau),
\end{aligned} \tag{34}
$$

where $f(I_\tau, II_\tau, III_\tau)$ is a function of $\boldsymbol{\tau}$ which will be determined later. (34) is the strain energy function for an initially stressed, incompressible Mooney-Rivlin material. From (25) and (33) we get

$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = &\Big[\frac{c_1p_0H}{\Delta} - \frac{c_1c_2E}{2\Delta}\Big]\mathbf{B} - \Big[\frac{c_2p_0G}{\Delta} - \frac{c_1c_2E}{2\Delta}\Big]\mathbf{B}^{-1} + \\
&\frac{c_1H}{\Delta}\mathbf{F}\boldsymbol{\tau}\mathbf{F}^T - \frac{c_2G}{\Delta}\mathbf{F}^{-T}\boldsymbol{\tau}\mathbf{F}^{-1} + \frac{c_1c_2}{2\Delta}\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^T - \frac{c_1c_2}{2\Delta}\mathbf{F}^{-T}\boldsymbol{\tau}^2\mathbf{F}^{-1} - \hat{p}\mathbf{I}.
\end{aligned} \tag{35}
$$

It is now required the proof that, by using the functional form for $W(\mathbf{F}, \boldsymbol{\tau})$ expressed in (34), the last identity of (28) is verified. From (5) and (6) we get that

$$p_0\mathbf{I} = 2c_1\mathbf{B}_1 - 2c_2\mathbf{B}_1^{-1} - \boldsymbol{\tau}, \tag{36}$$

$$p\mathbf{I} = 2c_1\mathbf{F}\mathbf{B}_1\mathbf{F}^T - 2c_2\mathbf{F}^{-T}\mathbf{B}_1^{-1}\mathbf{F}^{-1} - \boldsymbol{\sigma}. \tag{37}$$

Using now (34) in (25) and substituting (25) in (37), expressing terms containing the factors $p_0\mathbf{B}$ and $p_0\mathbf{B}^{-1}$ by means of (36), and terms containing the factors $E\mathbf{B}$ and $E\mathbf{B}^{-1}$ by means of (29) and (26), we obtain after some algebra the following identity

$$
\begin{aligned}
p\mathbf{I} = &\Big(2c_1 - \frac{2c_1^2H}{\Delta} + \frac{2p_0c_1^2c_2}{\Delta} - \frac{2c_1^3c_2I_{B_1}}{\Delta} - \frac{2c_1c_2^3}{\Delta}\Big)\mathbf{F}\mathbf{B}_1\mathbf{F}^T + \\
&\Big(-2c_2 - \frac{2c_2^2G}{\Delta} + \frac{2p_0c_1c_2^2}{\Delta} + \frac{2c_1^3c_2}{\Delta} + \frac{2c_1c_2^3II_{B_1}}{\Delta}\Big)\mathbf{F}^{-T}\mathbf{B}_1^{-1}\mathbf{F}^{-1} + \\
&\Big(\frac{2c_1c_2H}{\Delta} - \frac{2p_0c_1c_2^2}{\Delta} - \frac{2c_1^3c_2}{\Delta} - \frac{2c_1c_2^3II_{B_1}}{\Delta}\Big)\mathbf{F}\mathbf{B}_1^{-1}\mathbf{F}^T + \\
&\Big(\frac{2c_1c_2G}{\Delta} - \frac{2p_0c_1^2c_2}{\Delta} + \frac{2c_1^3c_2I_{B_1}}{\Delta} + \frac{2c_1c_2^3}{\Delta}\Big)\mathbf{F}^{-T}\mathbf{B}_1\mathbf{F}^{-1} + \\
&\Big(\frac{c_1H}{\Delta} - \frac{c_1H}{\Delta}\Big)\mathbf{F}\boldsymbol{\tau}\mathbf{F}^T + \Big(\frac{c_2G}{\Delta} - \frac{c_2G}{\Delta}\Big)\mathbf{F}^{-T}\boldsymbol{\tau}\mathbf{F}^{-1} + \\
&\Big(\frac{c_1c_2}{\Delta} - \frac{c_1c_2}{\Delta}\Big)\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^T + \Big(\frac{c_1c_2}{\Delta} - \frac{c_1c_2}{\Delta}\Big)\mathbf{F}^{-T}\boldsymbol{\tau}^2\mathbf{F}^{-1} + \hat{p}\mathbf{I}.
\end{aligned} \tag{38}
$$

After lengthy but standard manipulations, the terms in the brackets in (38) can be shown to be all equal to zero, and hence we get that

$$p\mathbf{I} = \hat{p}\mathbf{I} \rightarrow p = \hat{p},$$

which is the last identity of (28).

We finally determine the form of $f(I_\tau, II_\tau, III_\tau)$ in (34). In order to do it, we impose the *Initial Stress Reference Independence* constraint expressed by (1). Note that the conditions (33) specify only the terms in (34) which depend on $\mathbf{F}$ and on combinations of $\mathbf{F}$ and $\boldsymbol{\tau}$, with $f(I_\tau, II_\tau, III_\tau)$ left undetermined as a generic constant of integration which depends on $\boldsymbol{\tau}$ only. We are thus left with the freedom to choose a form for $f(I_\tau, II_\tau, III_\tau)$ for which (1) is satisfied. Using (34) in (1), we get

$$
\begin{aligned}
&\Big[\Big(\frac{c_1p_0H}{2\Delta} - \frac{c_1c_2E}{4\Delta}\Big)I_C + \Big(\frac{c_2p_0G}{2\Delta} - \frac{c_1c_2E}{4\Delta}\Big)II_C + \frac{c_1H}{2\Delta}J_1 + \\
&\frac{c_2G}{2\Delta}J_2^{-1} + \frac{c_1c_2}{4\Delta}J_3 + \frac{c_1c_2}{4\Delta}J_4^{-1}\Big] + f(I_\tau, II_\tau, III_\tau) = \\
&\Big[3\Big(\frac{c_1p\hat{H}}{2\hat{\Delta}} - \frac{c_1c_2\hat{E}}{4\hat{\Delta}} + \frac{c_2p\hat{G}}{2\hat{\Delta}} - \frac{c_1c_2\hat{E}}{4\hat{\Delta}}\Big) + \Big(\frac{c_1\hat{H}}{2\hat{\Delta}} + \frac{c_2\hat{G}}{2\hat{\Delta}}\Big)\mathrm{tr}\boldsymbol{\sigma} + \\
&\frac{c_1c_2}{2\hat{\Delta}}\mathrm{tr}(\boldsymbol{\sigma}^2)\Big] + f(I_\sigma, II_\sigma, III_\sigma),
\end{aligned} \tag{39}
$$

where $\hat{E}, \hat{G}, \hat{H}, \hat{\Delta}$ are the corresponding factors to (29)-(32) depending on $p, I_\sigma, II_\sigma$ and $\boldsymbol{\sigma}$ is expressed in (35). By substituting (35) in (39), it is possible to show that the term in the square brackets in the left hand side of (39) is equal to the term in the square brackets in the right hand side. In [17] this fact is shown in a more general framework; indeed, it is shown there that, if the initial stress comes from an elastic deformation from a virtual state, the terms in the functional form of $W(\mathbf{F}, \boldsymbol{\tau})$ which depend on $\mathbf{F}$ and on combinations of $\mathbf{F}$ and $\boldsymbol{\tau}$ always satisfy (1). Thus, we are left with the following constraint

$$f(I_\tau, II_\tau, III_\tau) = f(I_\sigma, II_\sigma, III_\sigma),$$

which can be satisfied only if the function $f$ is a constant, since it can be shown starting from (35) that the principal invariants of $\boldsymbol{\sigma}$ depend on the set of invariants $(I_C, II_C, J_1, J_2^{-1}, J_3, J_4^{-1})$, which is functionally independent from the set of invariants $(I_\tau, II_\tau, III_\tau)$.

We finally choose the constant value of $f(I_\tau, II_\tau, III_\tau)$ in such a way that $W = 0$ when $\boldsymbol{\tau} = 0$ and $\mathbf{F} = \mathbf{I}$. From (11) we easily get that $p_0(\boldsymbol{\tau} = 0, \mathbf{F} = \mathbf{I}) = 2c_1 - 2c_2$, which, substituted in (34), gives that $W(\mathbf{I}, \mathbf{0}) = 3c_1 + 3c_2 + f(0,0,0)$. Hence we choose

$$f(I_\tau, II_\tau, III_\tau) = f(0,0,0) = -3(c_1 + c_2) = -\frac{3}{2}\mu, \tag{40}$$

where $\mu = 2(c_1 + c_2)$ is the shear modulus of the material.

In the next sections we specialize the free energy (34) to the cases of Neo-Hookean and Mooney material, and we consider the simplified case in which the initial strain has only planar components.

## 4. Initially stressed Neo-Hookean and Mooney materials

Let us consider some simpler specific cases $c_1 > 0, c_2 = 0$ (Neo-Hookean), and $c_1 = 0, c_2 > 0$ (Mooney).

Taking $c_1 > 0$ and $c_2 = 0$ in (34),(11) and (12), we get

$$W_{NH}(\mathbf{F}, \boldsymbol{\tau}) = \frac{p_0}{2} I_C + \frac{1}{2} J_1 - 3c_1, \tag{41}$$

where $p_0$ is the real solution of the equation

$$p_0^3 + I_\tau p_0^2 + II_\tau p_0 + III_\tau - 8c_1^3 = 0, \tag{42}$$

and $W_{NH}$ is the strain energy function for an initially stressed, incompressible Neo-Hookean material. Note that the expressions (41) and (42) are the same as those derived in [16], (see formulas (3.5) and (3.11) therein, where now $\mu = 2c_1$).

Taking $c_1 = 0$ and $c_2 > 0$ in (34), (11) and (12), we get

$$W_M(\mathbf{F}, \boldsymbol{\tau}) = -\frac{p_0}{2} II_C - \frac{1}{2}(J_2 - I_C J_1 + II_C I_\tau) - 3c_2, \tag{43}$$

where $p_0$ is the real solution of the equation

$$p_0^3 + I_\tau p_0^2 + II_\tau p_0 + III_\tau + 8c_2^3 = 0, \tag{44}$$

and $W_M$ is the strain energy function for an initially stressed, incompressible Mooney material. Note that $W_M(\mathbf{I}, \mathbf{0}) = 0$ and $W_M(\mathbf{I}, \boldsymbol{\tau}) = c_2 II_{C_1} - 3c_2 > 0$. It's possible to show that the feasible solution of (44) has the same form of the solution $(3.6)_1$ in [16], with the parameter $\mu$ therein substituted by $-2c_2$.

## 5. Initially stressed Mooney-Rivlin materials under plane strain

Considering the case of planar initial strains is useful when modeling tubular structures [16], implying that $(\mathbf{B}_1)_{13} = (\mathbf{B}_1)_{23} = 0$ and $(\mathbf{B}_1)_{33} = 1$. From (5), we have that $(\boldsymbol{\tau})_{13} = (\boldsymbol{\tau})_{23} = 0$ and $(\boldsymbol{\tau})_{33} = 2c_1 - 2c_2 - p_0$. We use an overline to restrict a $3D$ tensor to the $(x_1, x_2)$ plane. Equation (5) becomes

$$\bar{\boldsymbol{\tau}} = 2c_1 \bar{\mathbf{B}}_1 - 2c_2 \bar{\mathbf{B}}_1^{-1} - p_0 \bar{\mathbf{I}}, \tag{45}$$

which combined with the Cayley-Hamilton theorem $(\bar{\mathbf{B}}_1)^2 - I_{\bar{B}_1} \bar{\mathbf{B}}_1 + \bar{\mathbf{I}} = 0$, gives us

$$I_{\bar{B}_1} = \frac{I_{\bar{\tau}} + 2p_0}{2c_1 - 2c_2} \quad \text{and} \quad \operatorname{tr} \mathbf{B}_1^2 = I_{\bar{B}_1}^2 - 2. \tag{46}$$

Taking the determinant on either side of (45) and using (46), we obtain

$$p_0^2 + I_{\bar{\tau}} p_0 + III_{\bar{\tau}} - (2c_1 + 2c_2)^2 + 4c_1 c_2 \left( \frac{I_{\bar{\tau}} + 2p_0}{2c_1 - 2c_2} \right)^2 = 0. \tag{47}$$

The two solutions $p_0^\pm$ of the quadratic equation (47) are

$$p_0^\pm = -\frac{I_{\bar{\tau}}}{2} \pm \frac{1}{2} \frac{c_1 - c_2}{c_1 + c_2} \Gamma, \tag{48}$$

with

$$\Gamma = \sqrt{4^2 (c_1 + c_2)^2 + I_{\bar{\tau}}^2 - 4 III_{\bar{\tau}}}, \tag{49}$$

where $p_0 = -I_{\bar{\tau}}/2$ is the correct limit for $c_1 \to c_2$. In terms of the eigenvalues of $\boldsymbol{\tau}$,

$$\Gamma^2 = 4^2 (c_1 + c_2)^2 + (\tau_1 - \tau_2)^2 > 0. \tag{50}$$

We note that for $c_2 = 0$ the solutions (48) reduce to the Neo-Hookean material in [16] (see formula (3.18) therein).

The only physically relevant solution is $p_0 = p_0^+$, because when substituting $p_0 = p_0^-$ in (46) leads to $I_{\bar{B}_1} < 1$, while $p_0 = p_0^+$ in (46) leads to

$$I_{\bar{B}_1} = \frac{\Gamma}{2(c_1 + c_2)} > 2. \tag{51}$$

From the above, (45) and $\bar{\mathbf{B}}_1^{-1} = -\bar{\mathbf{B}}_1 + I_{\bar{B}_1} \bar{\mathbf{I}}$ we get

$$\bar{\mathbf{B}}_1 = \frac{\bar{\boldsymbol{\tau}}}{2(c_1 + c_2)} + \frac{\Gamma - I_{\bar{\tau}}}{4(c_1 + c_2)} \mathbf{I}, \tag{52}$$

$$\bar{\mathbf{B}}_1^{-1} = -\frac{\bar{\boldsymbol{\tau}}}{2(c_1 + c_2)} + \frac{\Gamma + I_{\bar{\tau}}}{4(c_1 + c_2)} \mathbf{I}. \tag{53}$$

Finally, using these equations in the stress(6), and supposing for simplicity that $(\mathbf{F})_{13} = (\mathbf{F})_{23} = 0$, we get that

$$\begin{aligned} \bar{\boldsymbol{\sigma}} = & \frac{1}{2} \frac{c_1}{c_1 + c_2} \bar{\mathbf{B}} (\Gamma - I_{\bar{\tau}}) - \frac{1}{2} \frac{c_2}{c_1 + c_2} \bar{\mathbf{B}}^{-1} (\Gamma + I_{\bar{\tau}}) \\ & + \frac{c_1}{c_1 + c_2} \bar{\mathbf{F}} \bar{\boldsymbol{\tau}} \bar{\mathbf{F}}^T + \frac{c_2}{c_1 + c_2} \bar{\mathbf{F}}^{-T} \bar{\boldsymbol{\tau}} \bar{\mathbf{F}}^{-1} - p \bar{\mathbf{I}}, \end{aligned} \tag{54}$$

and $\sigma_{33} = 2c_1 - 2c_2 - p$. We can see that the above recovers well known limits: for $c_1 > 0$ and $c_2 = 0$ we recover the Neo-Hookean case, whereas when $c_1 = 0$ and $c_2 > 0$ we recover the Mooney case. Setting $\boldsymbol{\tau} = 0$ we get $\bar{\boldsymbol{\sigma}} = 2c_1 \bar{\mathbf{B}} - 2c_2 (\bar{\mathbf{B}})^{-1} - p\bar{\mathbf{I}}$, whereas setting $\bar{\mathbf{F}} = \bar{\mathbf{I}}$, together with (48), we get $\boldsymbol{\sigma} = \boldsymbol{\tau}$ and $p_0 = p$.

## 6. Discussion and conclusion

In this paper we derived the free-energy $W$ and Cauchy stress tensor $\boldsymbol{\sigma}$ for an initially stressed, incompressible Mooney-Rivlin material. Neglecting body forces, we assumed that the strain energy function constitutively depends only on the combined invariants of the deformation gradient $\mathbf{F}$ and the initial stress $\boldsymbol{\tau}$. Accordingly, the functional dependence is derived in Section 4 by imposing that the resulting $W$ describes a classical Mooney-Rivlin material from a stress-free state, so that it automatically fulfills the assumed frame independence (1), as discussed in [16, 17]. In particular, $W(\mathbf{F}, \boldsymbol{\tau})$ is given by (34) and $\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$ by (35).

In Section 4 we specialized the free energy (34) to Neo-Hookean and Mooney materials, reporting that the Neo-Hookean case the strain energy function has the same expression as the one derived in [16].

In Section 5 we studied the simplified case in which the initial stress has only planar components, finding an explicit analytical solution of (22) and the corresponding form of the constitutive equation for the Cauchy stress with respect to the initially stressed configuration.

We note that the constitutive equation for the Cauchy stress (35) is similar to the one derived in [13] for the case of a residually stressed Mooney-Rivlin material (see in particular Section 4 therein). The advantage of the present formulation is that the derivation of the constitutive equations (34) and (35) is obtained by starting from an expression of the free energy $W$ in terms of the combined invariants of the deformation gradient and the initial stress, whereas in [13] the constitutive equation for the Cauchy stress only is derived by inverting the stress-strain relation of the material in the virtual stress free state. Thus, our approach allows us to obtain an explicit form of the strain energy density, whilst the previous method only leads to implicit constitutive equations. Notwithstanding, the case of a generic isotropic initially stressed material requires further consideration, and it is left for future endeavor.

One limitation of the proposed constitutive relation is that the solutions of equation (22) need to be calculated numerically. Alternatively, we could also impose equation (22) as a new constraint thus introducing a corresponding Lagrange multiplier without explicitly solving it. The details of these considerations will be treated in a forthcoming paper.

A final remark is to highlight the usefulness of explicit expressions of $W(\mathbf{C}, \boldsymbol{\tau})$. Since $W$ does not explicitly depend on the initial deformation gradient from a stress-free state, there is no need to identify the virtual stress-free configuration. Thus, the choice of a functional form of $W(\mathbf{C}, \boldsymbol{\tau})$ can also provide useful guidelines for designing non-destructive experiments to determine the initial stress. This can be done, for example, by using elastic waves[24, 25] or through mechanical tests. Finally, the derived model has also important advantages when implementing finite element codes for solving nonlinear elastic boundary value problems. In morpho–elasticity, for example, when the growth strain is very large there is the need to use a very fine mesh in order to avoid the occurrence of locking. However, this can be avoided by specifying the initial stress instead on a fixed mesh, which leads to more stable numerical schemes [26].


## Acknowledgement

The authors are grateful to Maurizio Vianello (Politecnico di Milano) for helpful discussions during the development of this work. P.C. acknowledges funding through the AIRC grant MFAG 17412. A.L.G gratefully acknowledges funding provided by the EPSRC (no. EP/M026205/1).


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