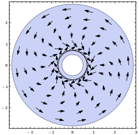
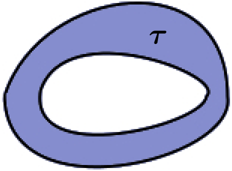


THE INITIAL STRESS SYMMETRY

Author:
Artur L. Gower

Co-Authors:
Dr. Pasquale Ciarletta
Prof. Michel Destrade

From initial stress to Cauchy stress



Initial stress $\boldsymbol{\tau}$

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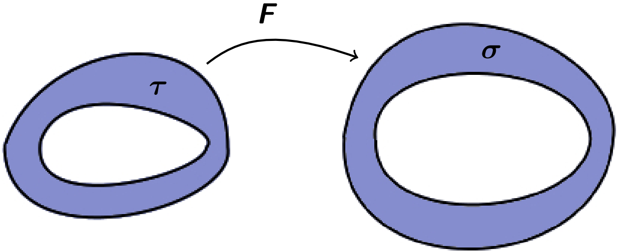

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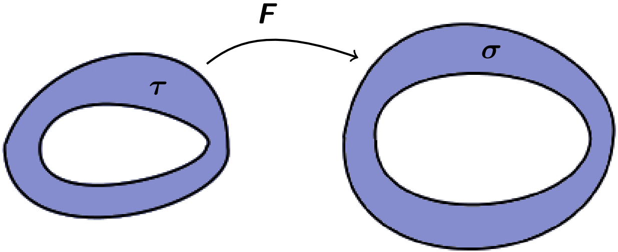

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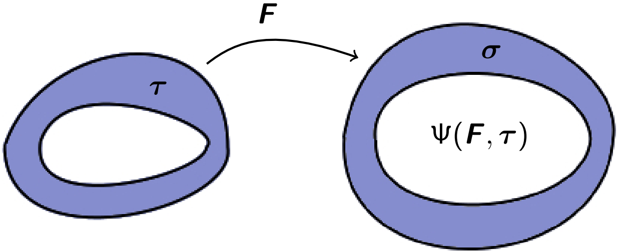

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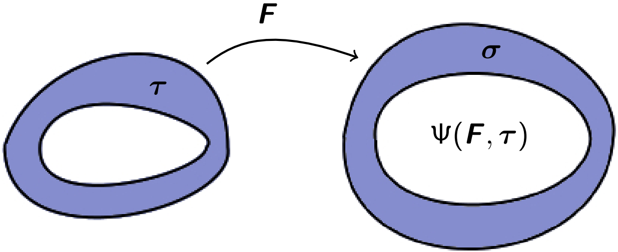

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$$\text{Something like: } \Psi(\boldsymbol{F}, \boldsymbol{\tau}) = C_1 \operatorname{tr}(\boldsymbol{F}^T \boldsymbol{F}) + C_2 \operatorname{tr} \boldsymbol{\tau}.$$

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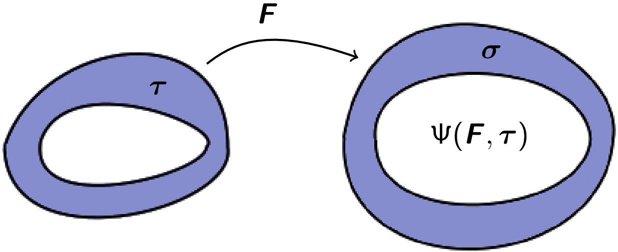

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(Great for in-vivo characterization!)

Exploding residual stress

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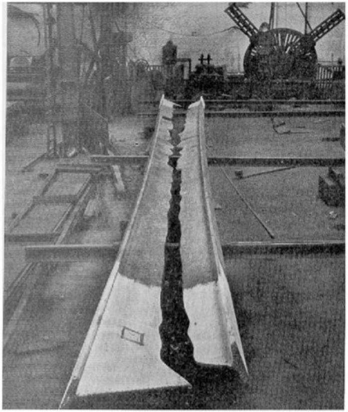

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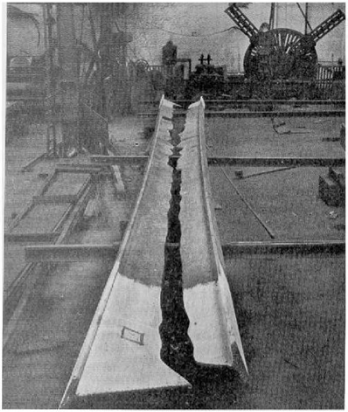

Typically measuring residual stress in metal matrix composites involves cutting the component in half or drilling small holes.

How to choose $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$?

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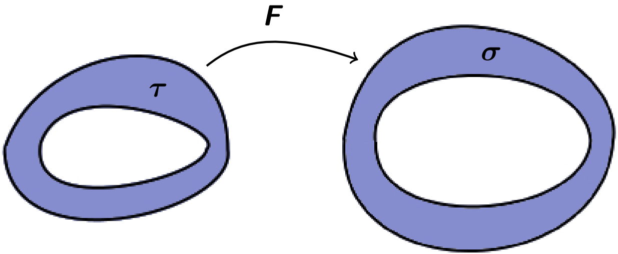
where $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$.
Shall we just drop a few invariants?

The Initial Stress Symmetry (ISS)



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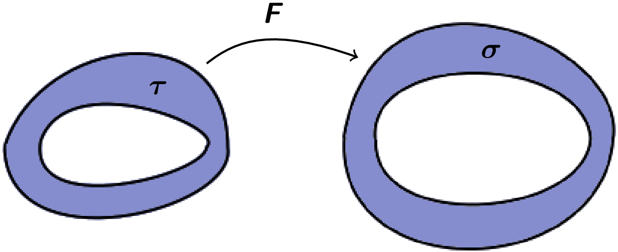

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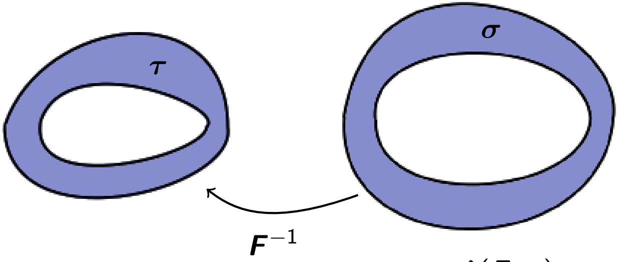

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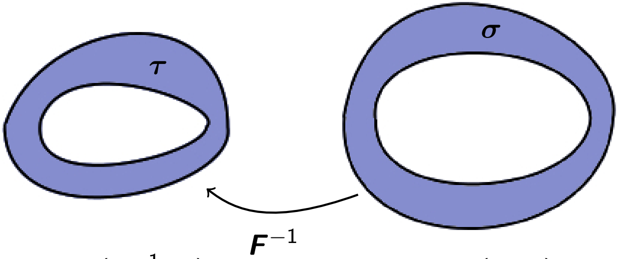

$\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma})$ $\qquad$ $\boldsymbol{\sigma} = \hat{\varsigma}(\boldsymbol{F}, \boldsymbol{\tau})$

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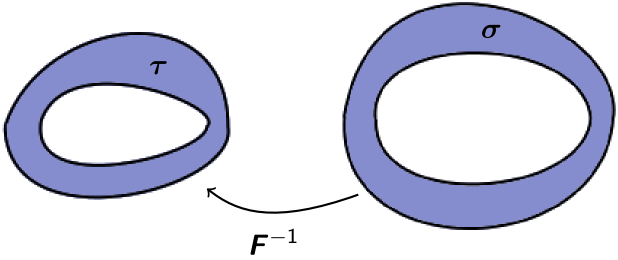

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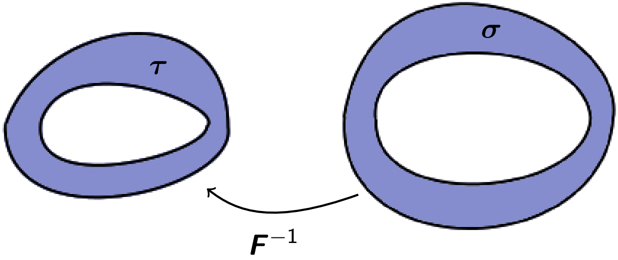

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Turns out that ISS is automatically satisfied if $\boldsymbol{\tau}$ is due to the deformation of a virtual stress-free configuration!

Elasticity

For a free energy density $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$, ISS can be written as 9 scalar equations involving the invariants of $\boldsymbol{F}$ and $\boldsymbol{\tau}$.

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For example, if $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$ is independent of $\mathrm{tr}(\boldsymbol{\tau}^2\boldsymbol{C})$ and $\mathrm{tr}(\boldsymbol{\tau}^2\boldsymbol{C}^2)$ then

$$4\Psi_{J_1}\Psi^{\boldsymbol{\sigma}}_{J_1} = 1, \quad \frac{\Psi^{\boldsymbol{\sigma}}_{I_2}}{\sqrt{\Psi^{\boldsymbol{\sigma}}_{J_1}}} + \frac{\Psi_{I_2}}{\sqrt{\Psi_{J_1}}} = 0, \quad p\Psi^{\boldsymbol{\sigma}}_{J_1} = \Psi^{\boldsymbol{\sigma}}_{I_1}, \quad p_{\boldsymbol{\tau}}\Psi_{J_1} = \Psi_{I_1},$$

where

$$\Psi_{I_k} = \Psi_{I_k}(\boldsymbol{F}, \boldsymbol{\tau}), \quad \Psi_{J_1} = \Psi_{J_1}(\boldsymbol{F}, \boldsymbol{\tau}), \tag{1}$$

$$\Psi^{\sigma}_{I_k} := \Psi_{I_k}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}), \; \Psi^{\sigma}_{J_m} := \Psi_{J_m}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}), \tag{2}$$

p and p_τ are Lagrange multipliers due to incompressibility.

Examples of $\Psi(\boldsymbol{F}, \boldsymbol{\tau})$

Examples that **do not** satisfy ISS:

$$\Psi = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}(J_1 - I_{\tau 1}),$$
$$\Psi = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{4}(J_2 - I_{\tau 1}),$$
$$\Psi = \frac{1}{2}\mu(I_1 - 3) + \frac{1}{2}\overline{\mu}(J_1 - I_{\tau 1})^2 + \frac{1}{2}(J_1 - I_{\tau 1}).$$

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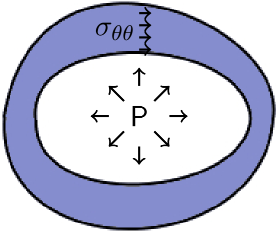
Examples that **do** satisfy ISS:

$$\Psi = \frac{1}{2}\left(p_\tau(\boldsymbol{\tau}, 1)I_1 + J_1 - 3\mu\right) \quad \text{(Incompressible neo-Hookean)},$$
$$\Psi = \frac{1}{2}I_3^{-1/3}\left(p_\tau(\boldsymbol{\tau}, I_3)I_1 + J_1 - f(\boldsymbol{\tau}, I_3)\right) \quad \text{(Compressible)}.$$

Choose any $\boldsymbol{\sigma}$

Suppose you want a homogeanious circumferential Cauchy stress:



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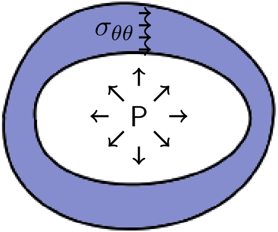

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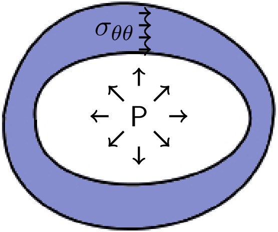

ISS in the form $\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma})$ suggests that for any choice of $\boldsymbol{\sigma}$, there will exist an initial stress $\boldsymbol{\tau}$ that supports $\boldsymbol{\sigma}$.

So first solve for your ideal $\boldsymbol{\sigma}$ then find $\boldsymbol{\tau}$!

Choose any $\boldsymbol{\sigma}$

So we can drop hyperelasticity and minimize

$$\frac{1}{b-a}\int_a^b \|\nabla \boldsymbol{\sigma}\|^2 dr,$$

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$$\sigma_{rr} = -P(1-\rho)(1-\rho\beta+\rho^2\beta^2) + O\left(\beta^3\right) P,$$
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$\sigma_{zz} = 0$, where

$$\rho = \frac{r-a}{b-a} \quad \text{and} \quad \beta = \frac{b-a}{a} \ (\approx 0.07).$$

Choose any $\boldsymbol{\sigma}$

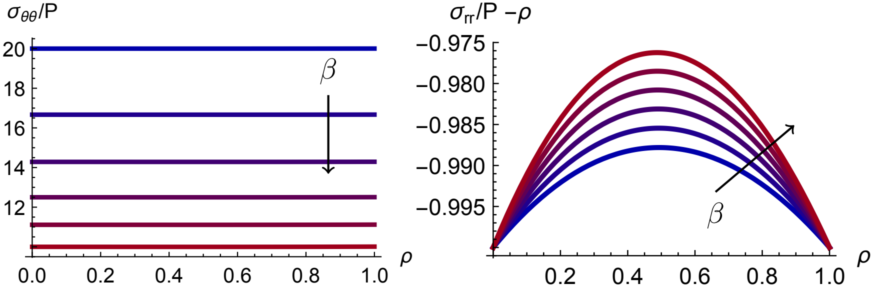

Figure: the ideal adimensional Cauchy stress against the adimensional radius ρ, where β goes from 0.05 to 0.1.

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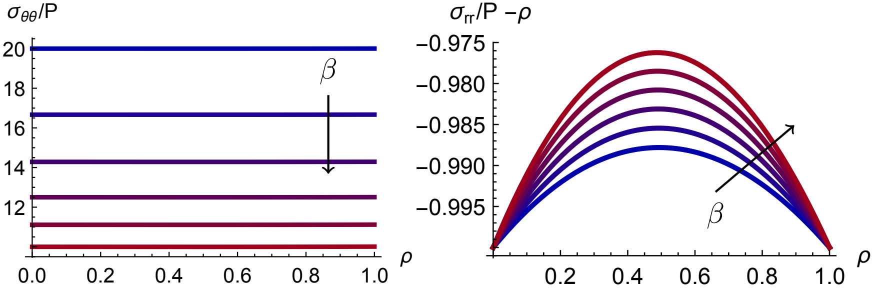

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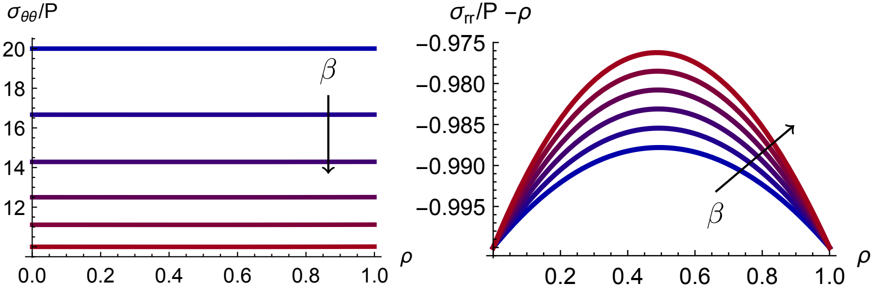

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$$\boldsymbol{\tau} = \hat{\varsigma}(\boldsymbol{F}^{-1}, \boldsymbol{\sigma}),$$

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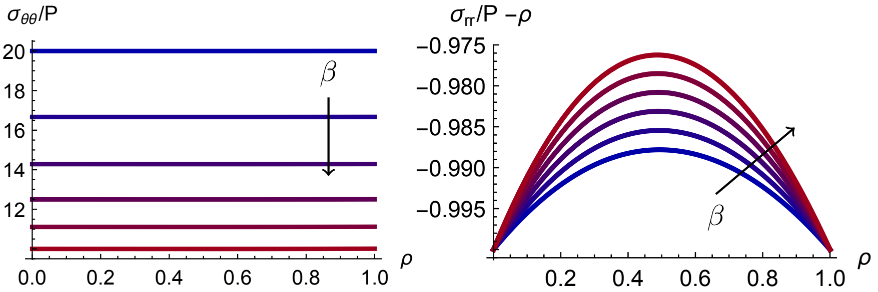

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and unloaded boundary conditions. Now on to predictions!

Predictions for ideal $\boldsymbol{\sigma}$

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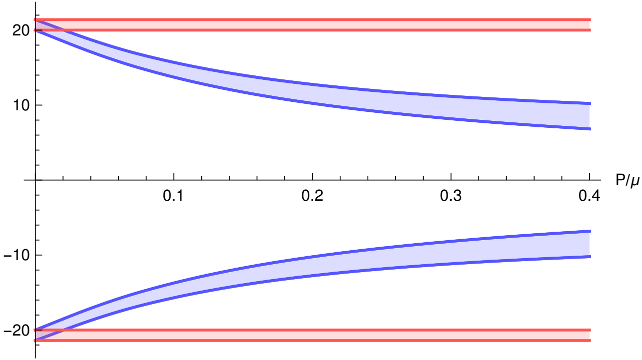

Figure: The loaded aorta wall is red and the unloaded aorta wall is blue. The parameters are for the descending thoracic aorta: loaded inner radius $= 20$mm, outer radius $= 21.4$mm and $\beta = 0.07$.

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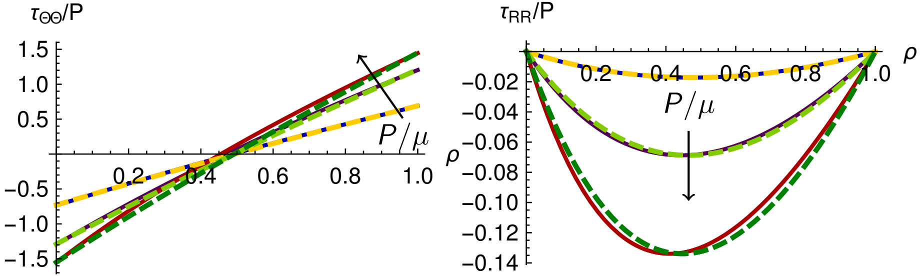

Figure: the adimensional residual stress against the adimensional radius ρ. The dashed curves are from the opening angle method while the solid curves are from the ideal stress $\boldsymbol{\sigma}$, both with $P/\mu = 0.05$, 0.2 or 0.35.

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Thanks for listening! Any questions?

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