Numerical Study of Pure Neutron Stars

Course: Theoretical and Numerical Aspects of Nuclear Physics
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Agenda

- Theoretical Introduction:
  - Structure Equations
  - Equation of State
- Numerical Approach of the Problem
  - Runge Kutta 4th Order Model
  - Flow Chart
  - Units
  - Initial Conditions
- Analysis of Results

Introduction

- Neutron star is the final stage of evolution of ordinary stars with $M \gtrsim 8M_{\odot}$
- Final neutron star originate from a iron core-collapse supernova

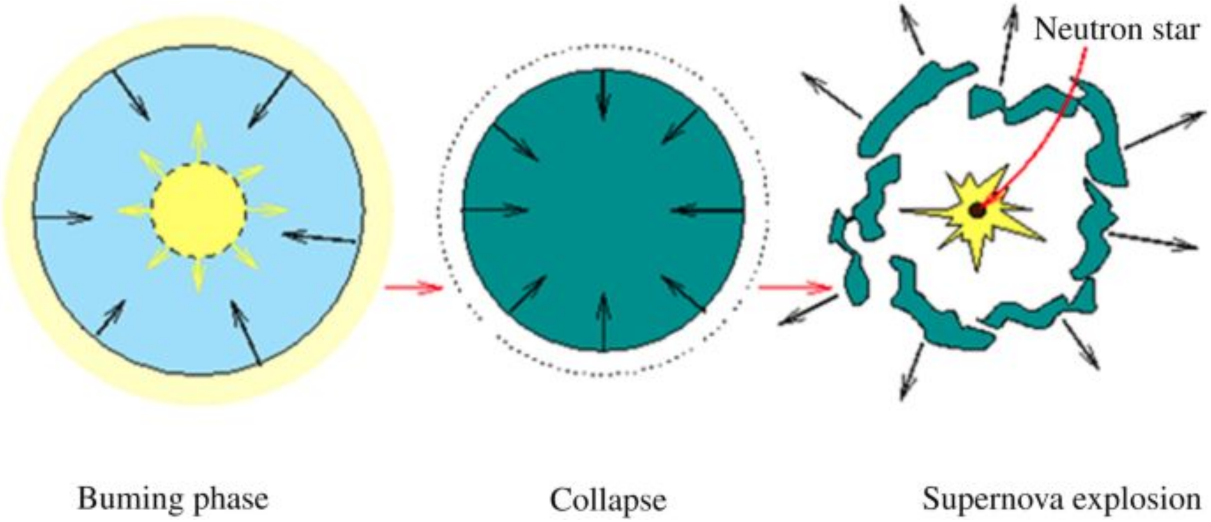


Image from: Tedila, H.M. Magnetic dipole interaction with multipole magnetic field lines of neutron stars. *Indian J Phys* **96,** 689–695 (2022)

Structure Equations

$$dp = \frac{dF}{A} = \frac{dF}{4\pi r^2}$$ where $$dF = \frac{-Gm(r)dm}{r^2}, \qquad dm = \rho(r)dV = \rho(r)4\pi r^2 dr$$

Mass density in terms of energy density: $\rho(r) = \varepsilon(r)/c^2$

$$\frac{dp}{dr} = -\frac{G\rho(r)m(r)}{r^2} \rightarrow \boxed{\frac{dp}{dr} = -\frac{G\varepsilon(r)m(r)}{c^2 r^2}}$$

$$\frac{dm}{dr} = \rho(r)4\pi r^2 \rightarrow \boxed{\frac{dm}{dr} = \frac{\varepsilon(r)4\pi r^2}{c^2}}$$

Two coupled differential equations

Must find $\varepsilon(p)$

Initial conditions: $m(r=0) = 0, \quad p(r=0) = p_0$

Radius of the star $R : p(R) = 0$, Mass of the star: $M = m(R)$

General Relativity corrections of structure equations

For very compact stars, one must take into account general relativity

Corrections become important when $R \approx R_0 = \frac{2GM}{c^2}$ (Schwarzschild radius)

Einstein equations: $G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$

Solving for an isotropic, general relativistic, static, ideal fluid sphere in hydrostatic equilibrium:

TOV equation: $$\frac{dp}{dr} = -\frac{G\varepsilon(r)m(r)}{c^2r^2}\left[1+\frac{p(r)}{\varepsilon(r)}\right]\left[1+\frac{4\pi r^3 p(r)}{m(r)c^2}\right]\left[1-\frac{2Gm(r)}{c^2 r}\right]^{-1}$$

Equation of State

Energy density of a degenerative neutron gas:

$$\varepsilon(k_F) = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F} E(k)k^2dk = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}\sqrt{k^2+m_n^2}k^2dk \equiv \varepsilon_0\int_0^{k_F/m_nc}(u^2+1)^{1/2}u^2du \qquad \varepsilon_0 = \frac{m_n^4c^5}{\pi^2\hbar^3}$$

The pressure of a system with an isotropic distribution of momenta is given by

$$p = \frac{1}{3}\frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F} kvk^2dk = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F}(k^2+m_n^2)^{-1/2}c^2k^4dk \equiv \varepsilon_0\int_0^{k_F/m_nc}(u^2+1)^{-1/2}u^4du$$

And therefore we get:

$$p(k_F) = \frac{\varepsilon_0}{24}[(2x^3-3x)(1+x^2)^{1/2}+3\sinh^{-1}(x)] \qquad [p]=[\varepsilon]=\mathrm{ergs/cm^3}$$

$$\varepsilon(k_F) = \frac{\varepsilon_0}{8}[(2x^3+x)(1+x^2)^{1/2}-\sinh^{-1}(x)] \qquad x = k_F/m_nc$$

We want $\varepsilon \equiv \varepsilon(p)$ as an EoS for the system -> Impossible Analytical -> Analyze various regimes

Equation of State: Polytropic Form

In the non-relativistic regime $x = k_F/m_n c \ll 1$

$$\varepsilon_0 = \frac{m_n^4 c^5}{\pi^2 \hbar^3}$$

$$p(x) = \frac{\varepsilon_0}{24}[(2x^3 - 3x)(1+x^2)^{1/2} + 3\sinh^{-1}(x)] \approx \frac{\varepsilon_0}{15}x^5$$

$$\varepsilon(x) = \frac{\varepsilon_0}{8}[(2x^3 + x)(1+x^2)^{1/2} - \sinh^{-1}(x)] \approx \frac{\varepsilon_0}{3}x^3 = n_n m_n c^2 = \rho c^2$$

$$\boxed{p(\varepsilon) = K_{\text{non-rel}}\varepsilon^{5/3}}$$

$$K_{\text{non-rel}} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_n c^2}\right)^{5/3}$$

$$[K_{\text{non-rel}}] = \text{cm}^2/\text{ergs}^{2/3}$$

In the ultra-relativistic regime $x = k_F/m_n c \gg 1$

$$p(x) = \frac{\varepsilon_0}{24}[(2x^3 - 3x)(1+x^2)^{1/2} + 3\sinh^{-1}(x)] \approx \frac{\varepsilon_0}{12}x^4$$

$$\varepsilon(x) = \frac{\varepsilon_0}{8}[(2x^3 + x)(1+x^2)^{1/2} - \sinh^{-1}(x)] \approx \frac{\varepsilon_0}{4}x^4$$

$$\boxed{p(\varepsilon) = K_{\text{rel}}\varepsilon}$$

$$K_{\text{rel}} = \frac{1}{3}$$

In both regimes the EoS can be expressed in a polytropic form: $p(\varepsilon) = K\varepsilon^{\gamma}$

Equation of State: General Form

Analytical solution of an EoS is impossible without approximations.

However, since both $\varepsilon(x)$ and $p(x)$ depend on the Fermi's momentum, find for given pressure the corresponding Fermi momentum and compute the respective energy density

Fit results into ansatz: $\varepsilon(p) = A_{\text{non-rel}} p^{3/5} + A_{\text{rel}} p$

For low pressures the nonrelativistic first term dominates over the second

Numerical Approach: Runge Kutta 4th Order Model (RK4)

Use Runge Kutta 4th order model to compute solutions of coupled differential equations

Let $\frac{dp(r)}{dr} = f(p(r), m(r), r), \quad p(0) = p_0$ and $\frac{dm}{dr} = g(p(r), r), \quad m(0) = 0$

Both p(r) and m(r) are unknown functions, we only know their rate of change as a function of the radius

Picking a "step" of size Δr > 0, we get:

$$\boxed{\begin{aligned} p(r_{n+1} = r_n + \Delta r) &= p(r_n) + \frac{\Delta r}{6}(\kappa_1 + 2\kappa_2 + 2\kappa_3 + \kappa_4) \\ m(r_{n+1} = r_n + \Delta r) &= m(r_n) + \frac{\Delta r}{6}(\kappa_1' + 2\kappa_2' + 2\kappa_3' + \kappa_4') \end{aligned}}$$

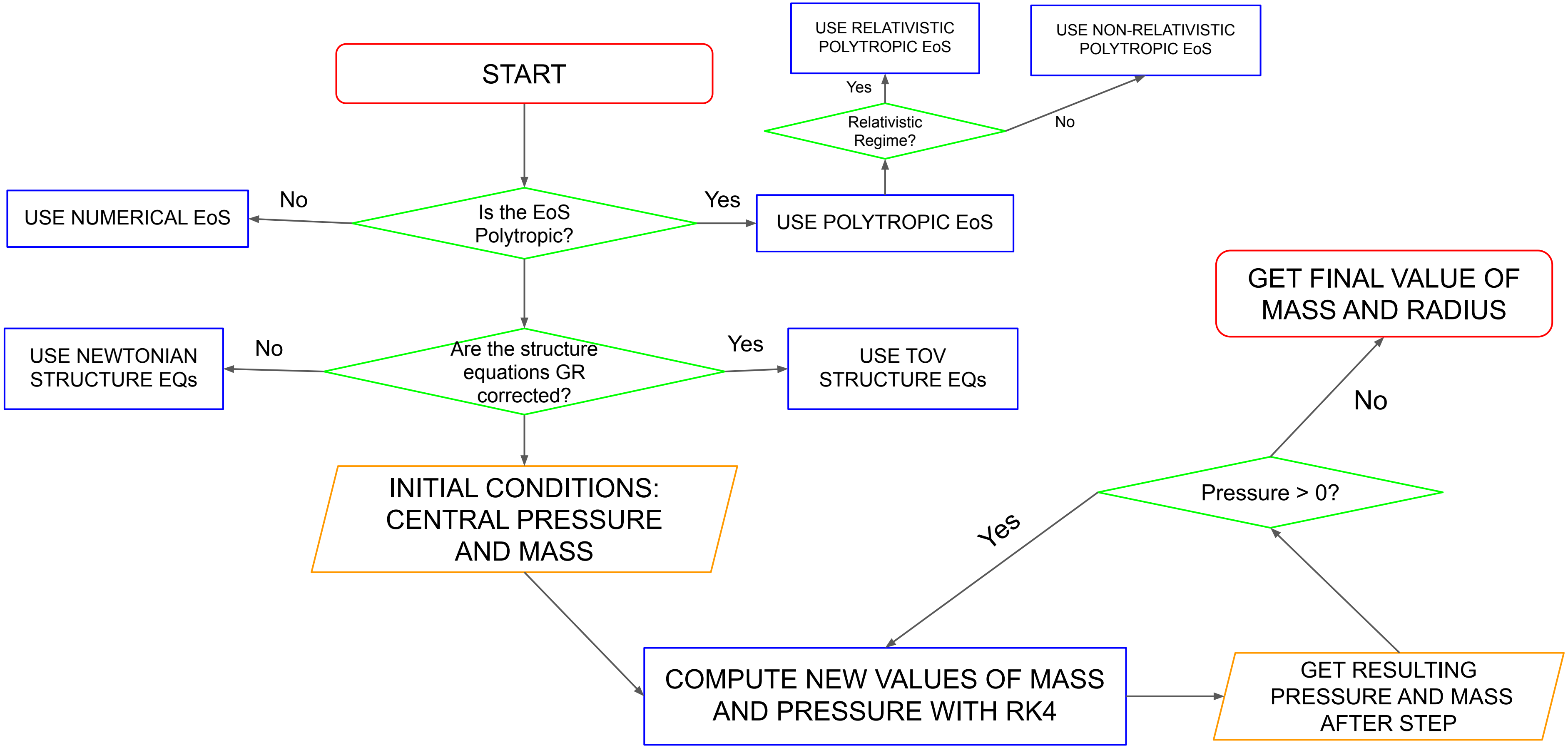
Global error: $O(\Delta r^4)$

With:

$$\kappa_1 = f(p(r_n), m(r_n), r_n)$$
$$\kappa_2 = f\left(p(r_n) + \Delta r\frac{\kappa_1}{2}, m(r_n) + \Delta r\frac{\kappa_1}{2}, r_n + \frac{\Delta r}{2}\right)$$
$$\kappa_3 = f\left(p(r_n) + \Delta r\frac{\kappa_2}{2}, m(r_n) + \Delta r\frac{\kappa_2}{2}, r_n + \frac{\Delta r}{2}\right)$$
$$\kappa_4 = f(p(r_n) + \Delta r\kappa_3, m(r_n) + \Delta r\kappa_3 r_n + \Delta r)$$

$$\kappa_1' = g(p(r_n), r_n)$$
$$\kappa_2' = g\left(p(r_n) + \Delta r\frac{\kappa_1'}{2}, r_n + \frac{\Delta r}{2}\right)$$
$$\kappa_3' = g\left(p(r_n) + \Delta r\frac{\kappa_2'}{2}, r_n + \frac{\Delta r}{2}\right)$$
$$\kappa_4' = g(p(r_n) + \Delta r\kappa_3', r_n + \Delta r)$$

Numerical Approach: Flow Chart



Numerical Approach: Units

We want to express the final radius and mass of the star in terms of km and $M_\odot$, thus, one first defines:

$$m = M_\odot \overline{m}$$

Also, ε and p carry dimensions of dyne/cm2. To simplify numerical computations, we define the dimensionless energy density and pressure as:

$$\varepsilon = \kappa \bar{\varepsilon} \qquad p = \kappa \bar{p} \qquad\qquad [\kappa] = dyne/cm^2 = ergs/cm^3$$

Inputting those in the structure equations:

$$\frac{dm}{dr} = \frac{\varepsilon(r) 4\pi r^2}{c^2} \rightarrow \boxed{\frac{d\bar{m}}{dr} = \frac{4\pi r^2 \kappa \bar{\varepsilon}(\bar{p})}{M_\odot c^2} \equiv \beta r^2 \bar{\varepsilon}(\bar{p})} \qquad \beta = \frac{4\pi\kappa}{M_\odot c^2}$$

$$\frac{dp}{dr} = -\frac{G\varepsilon(r) m(r)}{c^2 r^2} \rightarrow \boxed{\frac{d\bar{p}}{dr} = -\frac{G\bar{\varepsilon}(\bar{p})\bar{m}(r)}{M_\odot c^2 r^2} \equiv -R_0 \frac{\bar{\varepsilon}(\bar{p})\bar{m}(r)}{r^2}} \qquad R_0 = \frac{GM_\odot}{c^2}$$

TOV:

$$\boxed{\frac{d\bar{p}}{dr} = -\frac{R_0 \bar{\varepsilon}(\bar{p})\bar{m}(r)}{r^2}\left[1 + \frac{\bar{p}(r)}{\bar{\varepsilon}(\bar{p})}\right]\left[1 + \frac{\beta r^3 \bar{p}(r)}{\bar{m}(r)}\right]\left[1 - \frac{2R_0 \bar{m}(r)}{r}\right]^{-1}}$$

Numerical Approach: Units for a Polytropic EoS

Polytropic equation of state: $\bar{p} = K\kappa^{\gamma-1}\bar{\varepsilon}^{\gamma} \equiv \bar{K}\bar{\varepsilon}^{\gamma} \rightarrow \bar{\varepsilon}(\bar{p}) = \left(\frac{\bar{p}}{\bar{K}}\right)^{1/\gamma}$

$$\frac{d\bar{m}}{dr} = \frac{\beta}{\bar{K}^{1/\gamma}} r^2 \bar{p}^{1/\gamma} \equiv \delta r^2 \bar{p}^{1/\gamma} \qquad \delta = \frac{\beta}{\bar{K}^{1/\gamma}} \qquad [\delta] = \mathrm{km}^{-3}$$

$$\frac{d\bar{p}}{dr} = -\frac{R_0}{\bar{K}^{1/\gamma}} \frac{\bar{m}(r)\bar{p}^{1/\gamma}}{r^2} \equiv -\alpha \frac{\bar{m}(r)\bar{p}^{1/\gamma}}{r^2} \qquad \alpha = \frac{R_0}{\bar{K}^{1/\gamma}} \qquad [\alpha] = \mathrm{km}$$

And so we can fix κ by a suitable choice of α:

$$\bar{K} = \left(\frac{R_0}{\alpha}\right)^{\gamma} \rightarrow K\kappa^{\gamma-1} = \left(\frac{R_0}{\alpha}\right)^{\gamma} \rightarrow \boxed{\kappa = \left(\frac{1}{K}\left(\frac{R_0}{\alpha}\right)^{\gamma}\right)^{1/\gamma-1}}$$

How to choose α?

We want $\alpha \approx \delta \approx 1$

Choice of α will fix κ, which fixes $\delta = \frac{\beta}{\bar{K}^{1/\gamma}} = \frac{4\pi\kappa}{M_{\odot}c^2} \frac{1}{K\kappa^{\gamma-1}}$

Numerical Approach: Initial Conditions

$$\frac{dp(r)}{dr} = f(p(r), m(r), r), \quad p(0) = p_0 \qquad \frac{dm}{dr} = g(p(r), r), \quad m(0) = 0$$

Problem! GR corrected $f(p(r), m(r), r)$ is singular when m = 0 -> RK4 breaks (division by 0)

Must write it in different form:

$$\frac{d\bar{p}}{dr} = -\frac{R_0\bar{\varepsilon}(\bar{p})\bar{m}(r)}{r^2}\left[1+\frac{\bar{p}(r)}{\bar{\varepsilon}(\bar{p})}\right]\left[1+\frac{\beta r^3\bar{p}(r)}{\bar{m}(r)}\right]\left[1-\frac{2R_0\bar{m}(r)}{r}\right]^{-1}$$

$$= -R_0\bar{\varepsilon}(\bar{p})\left[\frac{\bar{m}(r)}{r^2}+\beta r\bar{p}(r)\right]\left[1+\frac{\bar{p}(r)}{\bar{\varepsilon}(\bar{p})}\right]\left[1-\frac{2R_0\bar{m}(r)}{r}\right]^{-1}$$

Singularity “solved” at m(0) = 0, but f and g are still singular at r = 0 -> choose very small initial radius

$$r_0 = 10^{-5}$$

Results: Non-relativistic regime with polytropic EoS

$$p(\varepsilon) = K_{\text{non-rel}}\varepsilon^{5/3} \rightarrow \gamma = 5/3$$

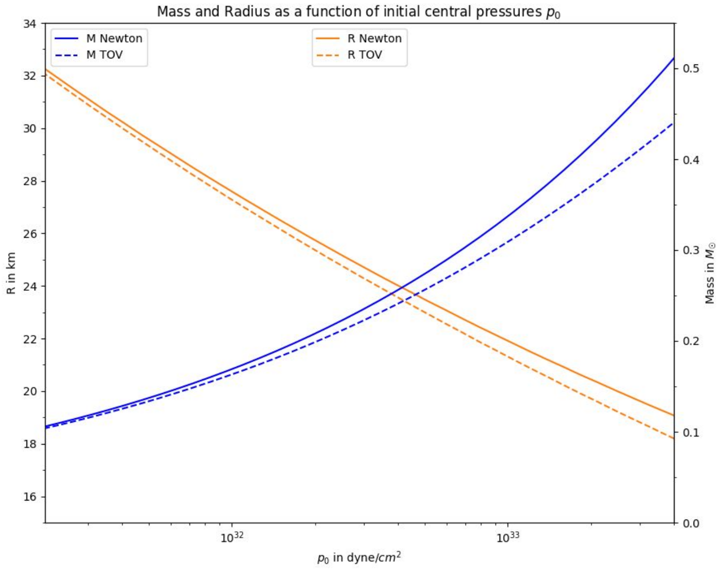
Since $K_{\text{non-rel}} = \frac{\hbar^2}{15\pi^2 m_n}\left(\frac{3\pi^2}{m_n c^2}\right)^{5/3}$, one can set: $\alpha = 1\text{km}$

$$\kappa = \left(\frac{1}{K_{non-rel}}\left(\frac{R_0}{\alpha}\right)^{\gamma}\right)^{1/\gamma-1} = \left(\frac{1}{K_{non-rel}}\left(\frac{R_0}{\text{km}}\right)^{5/3}\right)^{3/2} = 1.623\times10^{38}\text{ergs/cm}^3$$

$$\text{ergs/cm}^3 = 5.623\times10^{-40}\frac{M_\odot c^2}{\text{km}^3} \rightarrow \kappa = 0.0912\frac{M_\odot c^2}{\text{km}^3}$$

$$\delta = \frac{\beta}{\bar{K}_{\text{non-rel}}^{1/\gamma}} = \frac{4\pi\kappa}{M_\odot c^2}\frac{1}{K_{\text{non-rel}}^{3/5}\kappa^{2/3}} = \frac{1.1465}{1.4759}\text{km}^{-3} \approx 0.7767\text{km}^{-3}$$

$$\delta \approx \alpha \approx 1$$



The mass M and the radius R of neutron stars in the non-relativistic case as a function of the central pressure p0. The results from the TOV equation are compared to the Newtonian limit.

Results: General EoS for all regimes

$$\varepsilon(p) = A_{\text{non-rel}} p^{3/5} + A_{\text{rel}} p \quad \xrightarrow[p=\kappa\bar{p}]{\varepsilon=\kappa\bar{\varepsilon}} \quad \bar{\varepsilon}(\bar{p}) = \bar{A}_{\text{non-rel}} \bar{p}^{3/5} + \bar{A}_{\text{rel}} \bar{p}$$

$$\kappa = \varepsilon_0/3 = \frac{m_n^4 c^5}{3\pi^2\hbar^3} = 5.4885 \times 10^{36} \text{ergs/cm}^3 = 0.0031 M_\odot c^2/\text{km}^3$$

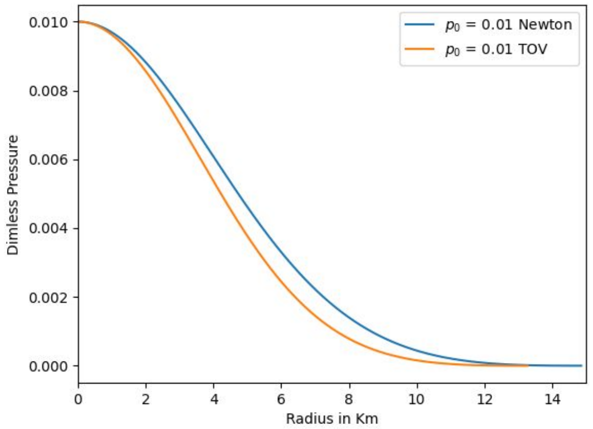
$$\beta = \frac{4\pi\kappa}{M_\odot c^2} \approx 0.039\text{km}^{-3} \qquad R_0 = \frac{GM_\odot}{c^2} \approx 1.476\text{km}$$

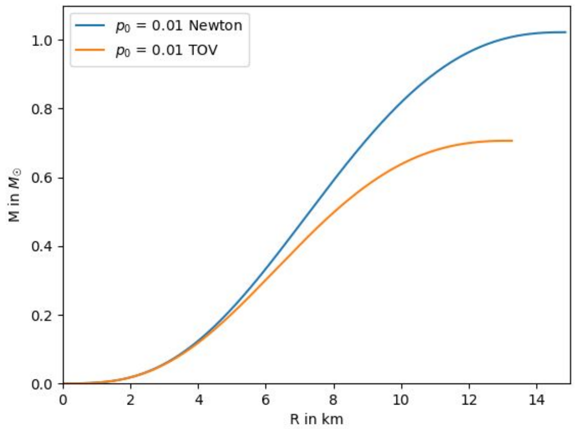
$$\bar{A}_{\text{non-rel}} = 2.4216 \qquad \bar{A}_{\text{rel}} = 2.8663$$

Results for a starting value of $\bar{p}_0 = 0.01$

$R = 14.9\text{km} \quad M = 1.022 M_\odot$ (Newtonian equations)

$R = 13.3\text{km} \quad M = 0.706 M_\odot$ (TOV equation)

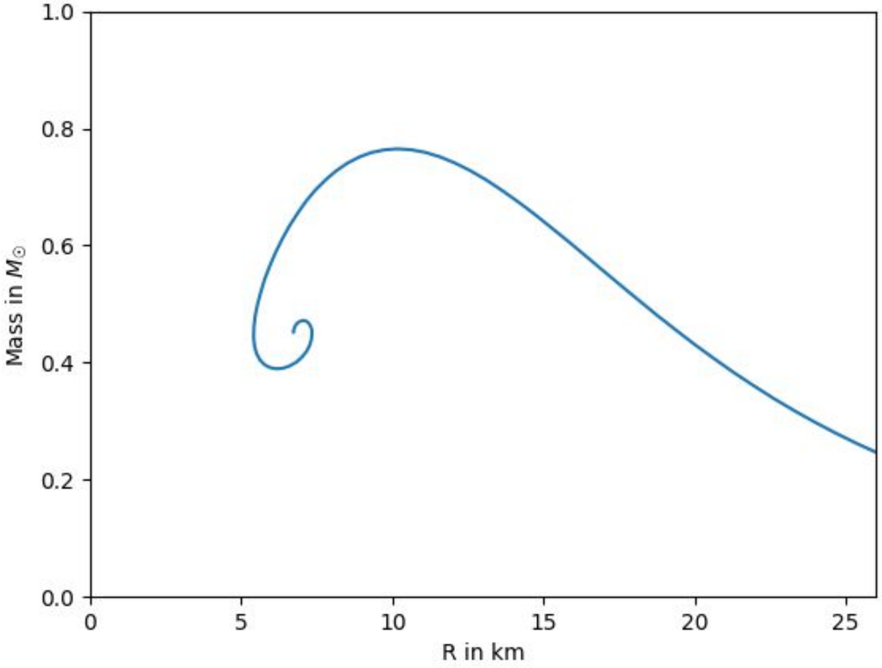




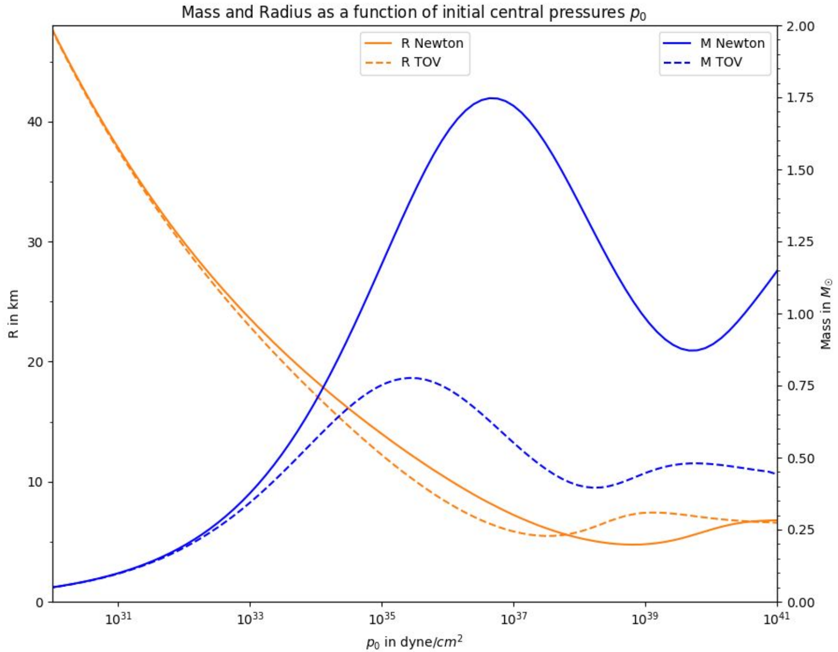
The dimensionless pressure (up) and mass (down) as a function of the radius of a pure neutron star with central pressure p0 = 0.01, using the general EoS

Results: General EoS for all regimes

Heaviest stable pure neutron star: $R = 10.2\text{km} \quad M = 0.765 M_{\odot}\,,\, p_0 = 2.783 \times 10^{35} \text{dyne/cm}^2$



Parametric graphic of the mass M for neutron stars as a function of radius R for various values of the central pressure p0 using TOV structure equation



The mass M and the radius R of neutron stars for a general EoS as a function of the central pressure p0. The results from the TOV equation are compared to the Newtonian limit.

References

1. Irina Sagert *et al* 2006 *Eur. J. Phys.* **27** 577
2. R. R. Silbar, S. Reddy *American Journal of Physics* **72**, 892 (2004)
3. https://lpsa.swarthmore.edu/NumInt/NumIntFourth.html