Assignment II: The Bouncing Mass Problem

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Problem Specification

In the following, the motion of a mass mounted on top of a vertical, massless spring dropped from an initial height of 1.2 meters will be analyzed. The mass is released from rest. Air resistance is neglected, and only the effects of gravity and the spring force are considered. The goal is to model the motion of the mass, derive the governing equations of motion, and evaluate how the mechanical energy evolves over time.

Parameter	**Symbol**	**Value**	**Unit**
Mass	m	1.0	kg
Initial drop height	y_0	1.2	m
Initial velocity	v_0	0	m/s
Gravitational acceleration	g	9.81	m/s^2
Spring stiffness	k	100	N/m
Spring length	l	1.0	m

Table 1: System Parameters

Figure 1 shows the two states of the mass–spring system. In the **flight phase**, the mass is falling freely under gravity, and the spring has not yet been compressed, so no spring force acts on the system. Once the mass reaches the spring's unstretched length, it enters the **bouncing phase**, during which the spring is compressed and applies a force upward. The motion is constrained to the vertical y-axis.

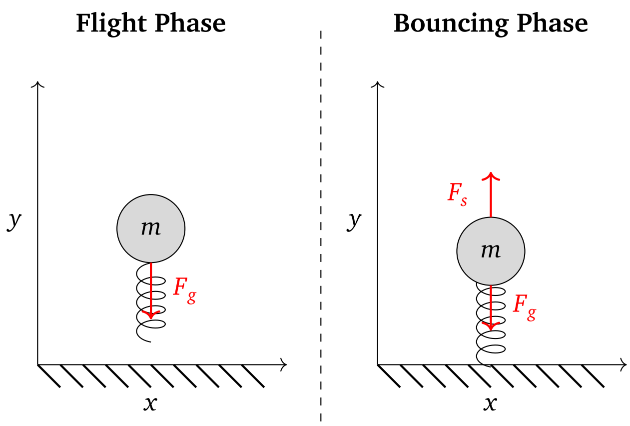

Figure 1: Free-Body Diagrams for Flight and Bouncing Phase.

EoM : Flight Phase

Based of Newton's 2nd law of motion.

$$\sum F = m\ddot{y} = -mg \tag{1}$$

Defining acceleration, velocity, and position of a mass. Velocity and position are obtained through integration, using initial conditions as integration constants.

$$\ddot{y}(t) = -g \tag{2}$$

$$\dot{y}(t) = -gt + v_{0,flying} \tag{3}$$

$$y(t) = -\frac{1}{2}gt^2 + v_{0,flying}t + y_{0,flying} \tag{4}$$

EoM : Bouncing Phase

Based of Newton's 2nd law of motion:

$$\sum F = m\ddot{y} = -mg + F_s = -mg + k(l-y) \tag{5}$$

Defining the linear second-order equation:

$$\ddot{y}(t) + \frac{k}{m}y(t) = -g + \frac{k}{m}l \tag{6}$$

Solving the homogeneous equation, where $\omega = \sqrt{\frac{k}{m}}$ is the natural angular frequency.

$$y_h(t) = A\cos(\omega t) + B\sin(\omega t) \tag{7}$$

Find a particular solution, where $y(t)_p) = C$

$$0 + \frac{k}{m}C = \frac{k}{m}l - g \quad \Rightarrow \quad C = y(t)_p = l - \frac{mg}{k} \tag{8}$$

Therefore a general solution takes the form:

$$y(t) = y_h(t) + y_p(t) = A\cos(\omega t) + B\sin(\omega t) + l - \frac{mg}{k} \tag{9}$$

Finding the constants A and B, taking into account the initial conditions for the bouncing phase .

$$y(0) = A\cos(0) + B\sin(0) + C = A + C \Rightarrow A = y_{0,bouncing} - C = y_{0,bouncing} - \left(l - \frac{mg}{k}\right) \tag{10}$$

$$\dot{y}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \Rightarrow \dot{y}(0) = B\omega \Rightarrow B = \frac{v_{0,bouncing}}{\omega} = \frac{v_{0,bouncing}}{\sqrt{k/m}} \tag{11}$$

Final expression for the position during bouncing phase.

$$y(t) = \left(y_{0,\text{bouncing}} - l + \frac{mg}{k}\right)\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{v_{0,\text{bouncing}}}{\sqrt{\frac{k}{m}}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) + l - \frac{mg}{k} \tag{12}$$

Numerical Simulation in MATLAB

The following pseudocode outlines the logic implemented to model the dynamics of the mass-spring system alternating between **flight phase** and **bouncing phase**. At each time step, the algorithm determines the current phase based on the mass's vertical position and applies the appropriate equations of motion. Transitions between phases are detected to reset the time and initial conditions relative to the start of the new phase.For a complete implementation of the simulation code, please refer to the MATLAB file (BouncingMass.mlx) provided in the attached ZIP file.

```
Initialize constants and initial conditions
for each time step do
    Compute time since last phase started
    if mass is in flight ( y > 1) then
        if phase just changed to flight then
            Recalculate initial conditions; save new phase start time
        end if
        Compute position and velocity using Eq. 4
        Mark phase as flight
    else
        if phase just changed to bounce then
            Recalculate initial conditions; save new phase start time
        end if
        Compute position and velocity using Eq. 12
        Mark phase as bounce
    end if
    Store current position and velocity
end for
```

Note: While the algorithm tracks phase transitions and recalculates initial conditions after each switch of phase, this complexity is not necessary in an energy-conserving system like the one modeled here. Initial conditions for both phases remain constant. However, this structure becomes essential in more realistic scenarios involving energy dissipation, where the initial conditions vary between transitions.

Results and Visualization

Figure 2 shows the time evolution of the mass's vertical position in the mass-spring system over a 3-second interval. The position curve (blue) and velocity curve (red) are plotted using separate y-axes for clarity. The position oscillates between approximately 0.6808 m and 1.2 m. Each cycle consists of a parabolic segment (flight) followed by a sinusoidal rebound due to the spring force. The velocity curve, being the derivative of position, exhibits a waveform that reaches zero whenever the position reaches a local maximum or minimum, validating the expected physical behavior of the system.

Figure 3 illustrates the time evolution of the mechanical energy components in the mass-spring system. At each time step, the kinetic energy, gravitational potential energy, and spring potential energy were computed based on the current state of the mass using Eq. 13. The total mechanical energy remains constant throughout the simulation, as expected in this energy-conserving system. This confirms the correct transfer of energy between kinetic and potential forms during both the **flight and bouncing phases** .

$$TME(t) = KE + GPE + SPE = \frac{1}{2}m\dot{y}^2(t) + mgy(t) + \begin{cases} \frac{1}{2}k(l - y(t))^2, & \text{if } y(t) \le l \\ 0, & \text{if } y(t) > l \end{cases} \tag{13}$$

Conclusion

This report presented the analytical modeling and numerical simulation of a mass-spring system undergoing alternating flight and bouncing phases. The governing equations of motion were derived for each phase, and a MATLAB-based simulation was implemented to evaluate the system's behavior over time. The results confirmed the expected dynamics of a conservative system, with periodic motion and constant total mechanical energy. The phase-dependent behavior, along with the clear energy exchange between kinetic, gravitational, and spring potential energy, validated both the physical model and the numerical implementation.

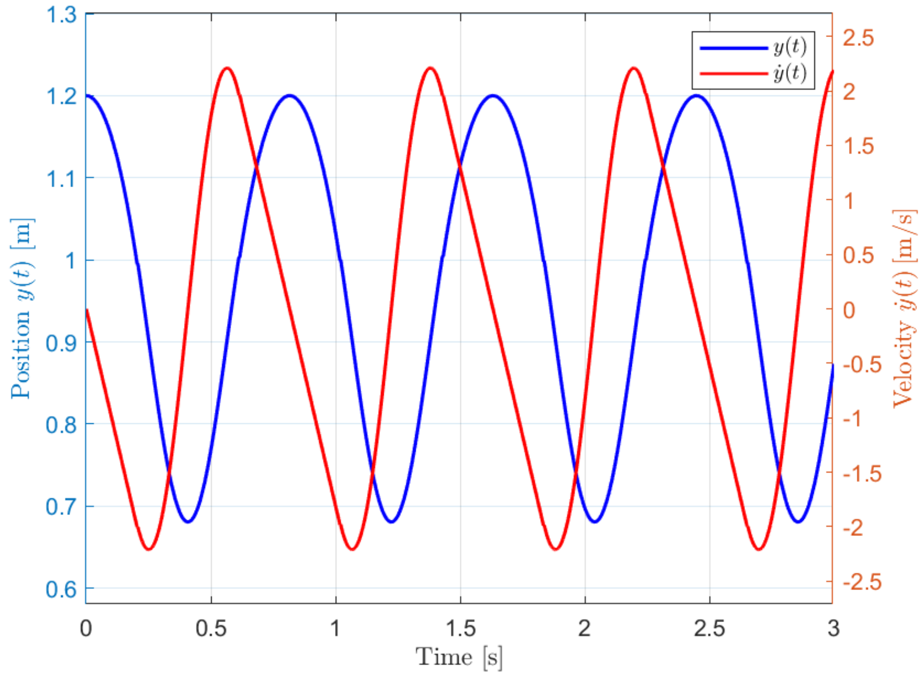

Figure 2: Flight and Bounce Phases in the Mass-Spring System: $y(t)$ and $\dot{y}(t)$

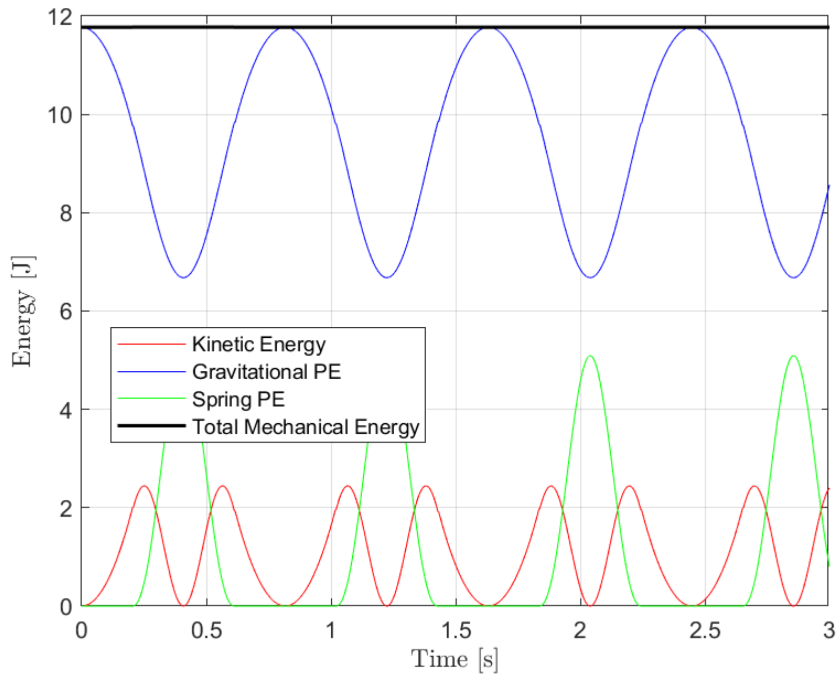

Figure 3: Energy Components in the Mass-Spring System

AI usage declaration

tools used

ChatGPT

comments

This includes support with generation of the TikZ free-body diagram (Impossible to get an accurate diagram directly, but gave me a base to start and modify manually). Some aspects of the MATLAB plottingwere improved with the assistance of AI-based tools . As well to improve the grammar, structure, and readability of the text.
