

MINIMAL ALGEBRAS

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1. PRELIMINARIES

Definition 1.1. Let F be a set of function symbols and \mathbf{A} be an algebra over F . We denote by $\text{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition. The elements of $\text{Clo}(\mathbf{A})$ are called **term** operations. We say that two algebras \mathbf{A} and \mathbf{B} on the same carrier are **term equivalent** if $\text{Clo}(\mathbf{A}) \simeq \text{Clo}(\mathbf{B})$.

Definition 1.2. Let F be a set of function symbols and \mathbf{A} be an algebra over F . We denote by $\text{Pol}(\mathbf{A})$ the smallest set containing

- (1) $\{f^{\mathbf{A}} : f \in F\}$;
- (2) $\{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$;
- (3) the constant 0-ary operations

and closed under composition. The elements of $\text{Pol}(\mathbf{A})$ are called **polynomial** operations. We say that two algebras \mathbf{A} and \mathbf{B} on the same carrier are **polynomially equivalent** if $\text{Pol}(\mathbf{A}) \simeq \text{Pol}(\mathbf{B})$.

Example 1.3. If $\varphi \in \text{Clo}_{m+n}(\mathbf{A})$ and $(a_1, \dots, a_m) \in A^m$, then

$$\psi : A^n \rightarrow A \quad (b_1, \dots, b_n) \mapsto \varphi(a_1, \dots, a_m, b_1, \dots, b_n)$$

is a polynomial operation.

Remark 1.4. Let \mathbf{A} be an algebra. An equivalence relation α is a congruence of \mathbf{A} iff $\varphi(\alpha) \subseteq \alpha$ for every $\varphi \in \text{Pol}_1(\mathbf{A})$.

Let \mathbf{A} be a finite algebra. We adopt the following convention, concerning the restriction $(-|U)$ operation, for $U \subseteq A$:

- if $\theta \in \text{Con}(\mathbf{A})$, $\theta|U := \theta \cap U^2$;
- if $\varphi : A^n \rightarrow A$, $\varphi|U$ is the function $U^n \rightarrow A$, $(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_n)$;
- $\text{Pol}(\mathbf{A})|U := \{\psi|U : \psi \in \text{Pol}(\mathbf{A}) \text{ and } \psi[U^n] \subseteq U\}$;
- $\mathbf{A}|U := (U, \text{Pol}(\mathbf{A})|U)$.

2. FINITE MINIMAL ALGEBRAS

Definition 2.1. A nontrivial finite algebra \mathbf{A} is **minimal** iff every nonconstant element of $\text{Pol}_1(\mathbf{A})$ is bijective.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

Example 2.2. The following are examples of minimal algebras.

- (1) any algebra with carrier 2;
- (2) a nontrivial finite vector space \mathbf{A} over a finite field \mathbf{k} : every $\pi \in \text{Pol}_1(\mathbf{A})$ is of the form $\pi(v) = av + b$ for some $a \in k, b \in A$;
- (3) a group of permutations acting on a finite set. If \mathbf{G} is a group acting on a set A each $g \in G$ induces an operation $\varphi_g : A \rightarrow A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. A \mathbf{G} -set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$.

We shall see that, up to polynomial equivalence, there are no other finite minimal algebras.

Lemma 2.3. *Let \mathbf{A} be a minimal algebra. If every element of $\text{Pol}(\mathbf{A})$ is essentially unary, then \mathbf{A} is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ where \mathbf{G} is a finite group acting on A .*

Proof. Since \mathbf{A} is minimal, $\text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$ is a subgroup of $\text{Sym}(A)$. Let $\mathbf{G} := \text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$. If $\psi \in \text{Pol}(\mathbf{A})$, either ψ is constant or ψ is essentially unary, hence $(A, \Phi_{\mathbf{G}})$ is polynomially equivalent to \mathbf{A} . \square

Theorem 2.4 ([3]). *Let \mathbf{A} be a minimal algebra with $|A| > 2$. If $\text{Pol}(\mathbf{A})$ contains an operation which is not essentially unary, then \mathbf{A} is polynomially equivalent to a \mathbf{k} -vector space for a finite field \mathbf{k} .*

Theorem 2.5. *Every algebra \mathbf{A} with carrier 2 is polynomially equivalent to one of the following:*

- (1) $\mathbf{E}_0 = (2, \emptyset)$;
- (2) $\mathbf{E}_1 = (2, \neg)$;
- (3) $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$;
- (4) $\mathbf{E}_4 = (2, \wedge, \vee)$;
- (5) $\mathbf{E}_5 = (2, \vee)$;
- (6) $\mathbf{E}_6 = (2, \wedge)$.

Each of them is not polynomially equivalent to the other¹.

¹A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem 2.5 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

Remark 2.6. Up to isomorphism, $\mathbf{E}_5 (\simeq \mathbf{E}_6)$ is the only semilattice with two elements, while \mathbf{E}_3 and \mathbf{E}_4 are the only Boolean algebra and lattice, respectively, with two elements.

Definition 2.7. Let \mathbf{A} be a minimal algebra. We say that \mathbf{A} is of

- (1) **type 1** (or **unary**) if \mathbf{A} is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ for some $\mathbf{G} \leq \text{Sym}(A)$;
- (2) **type 2** (or **affine**) if \mathbf{A} is polynomially equivalent to a vector space over a finite field \mathbf{k} ;
- (3) **type 3** (or **Boolean**) if \mathbf{A} is polynomially equivalent to \mathbf{E}_3 ;
- (4) **type 4** (or **lattice**) if \mathbf{A} is polynomially equivalent to \mathbf{E}_4 ;
- (5) **type 5** (or **semilattice**) if \mathbf{A} is polynomially equivalent to \mathbf{E}_5 .

3. MINIMAL ALGEBRAS RELATIVE TO A CONGRUENCE

Definition 3.1. Let \mathbf{A} be a finite algebra and let $\theta \in \text{Con}(\mathbf{A})$, $\Delta_{\mathbf{A}} \neq \theta$. We say that \mathbf{A} is θ -**minimal** if for all $\varepsilon \in \text{Pol}_1(\mathbf{A})$ either ε is bijective or ε is constant on the θ -equivalence classes.

Remark 3.2. Observe that \mathbf{A} is minimal iff \mathbf{A} is $\nabla_{\mathbf{A}}$ -minimal.

Definition 3.3. Let \mathbf{A} be a θ -minimal algebra. A θ -**trace** of \mathbf{A} is a nontrivial θ -equivalence class.

Lemma 3.4. Let \mathbf{A} be a finite θ -minimal algebra and let N be a θ -trace. Then the algebra $\mathbf{A}||N$ is minimal.

Proof. We need to show that for every

$$\psi \in \text{Pol}_1(\mathbf{A}||N) = \{\varphi|N : \varphi \in \text{Pol}_1(\mathbf{A}), \varphi[N] \subseteq N\}$$

either ψ is bijective, or ψ is constant. Let $\psi = \varphi|N$. Since \mathbf{A} is θ -minimal, either φ is bijective or φ is constant on the θ -equivalence classes. Clearly, in the first case ψ is bijective, in the second ψ is constant. \square

Definition 3.5. Let \mathbf{A} be a finite algebra and $B, C \subseteq A$. We say that B, C are **polynomially isomorphic** ($B \sim C$) if there are $\varphi, \psi \in \text{Pol}_1(\mathbf{A})$ such that $\varphi[B] = C$, $\psi[C] = B$ and $\psi\varphi|B = 1_B$, $\varphi\psi|C = 1_C$.

Remark 3.6. If $B, C \subseteq A$ are polynomial isomorphic in \mathbf{A} , then $\mathbf{A}||B \simeq \mathbf{A}||C$. Let $\pi := \varphi|B$, so that $\pi^{-1} = \psi|C$. Of course, $\pi : B \rightarrow C$ is a bijection. We show that π is a homomorphism. Let $f \in \text{Pol}_n(\mathbf{A})$ such that $f[B^n] \subseteq B$. Then $g(-, \dots, -) := \pi f(\pi^{-1}(-), \dots, \pi^{-1}(-)) \in \text{Pol}_n(\mathbf{A})$ too, $g[C^n] \subseteq C$ and

$$\pi f(b_1, \dots, b_n) = g(\pi(b_1), \dots, \pi(b_n))$$

for all $b_1, \dots, b_n \in B^n$.

Lemma 3.7. Let \mathbf{A} be a θ -minimal algebra and N be a θ -trace. Then

- (1) $(-|N) : [\Delta_A, \theta] \rightarrow \text{Con}(\mathbf{A}||N)$ is a surjective lattice homomorphism;
- (2) if any two θ -traces are polynomially isomorphic, then it is an isomorphism.

Proof. Clearly, the map is well defined and preserves meets. We show that

$$(\alpha \vee \beta) \cap N^2 = (\alpha \cap N^2) \vee (\beta \cap N^2).$$

Let $(x, y) \in (\alpha \vee \beta) \cap N^2$. Then there are $x = x_0, \dots, x_{n+1} = y$ such that either $(x_i, x_{i+1}) \in \alpha$ or $(x_i, x_{i+1}) \in \beta$. We show that for each i , $(x_i, x_{i+1}) \in N^2$. Inductively, if $x_i \in N$, and, say, $(x_i, x_{i+1}) \in \alpha \subseteq \theta$, then $x_{i+1} \in N$. Now, for $\beta \in \text{Con}(\mathbf{A}||N)$, let $\hat{\beta}$ be

$$\{(x, y) \in \theta : (\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta \quad \forall \psi \in \text{Pol}_1(\mathbf{A})\}$$

Then $\hat{\beta}$ is a congruence. We show that $\hat{\beta} \cap N^2 = \beta$, proving surjectivity. If $(x, y) \in \hat{\beta}$, then $(\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta$ for all $\psi \in \text{Pol}_1(\mathbf{A})$; if $(x, y) \in N^2$, then $(\psi(x), \psi(y)) \in N^2$. Therefore $(\psi(x), \psi(y)) \in \beta$ for all $\psi \in \text{Pol}_1(\mathbf{A})$, and, taking $\psi(x) = x$, $(x, y) \in \beta$. Conversely, let $(x, y) \in \beta$. As $\beta \subseteq N^2 \subseteq \theta$, $(x, y) \in \theta$. Let $\psi \in \text{Pol}_1(\mathbf{A})$. If $(\psi(x), \psi(y)) \in N^2$, then $\psi \in \text{Pol}_1(\mathbf{A}||N)$. Since $\beta \in \text{Con}(\mathbf{A}||N)$, $(\psi(x), \psi(y)) \in \beta$.

We need to prove injectivity. Let $\alpha < \beta \leq \theta$. Let $(x, y) \in \beta - \alpha$. Then $(x, y) \in \theta$ and $P := x/\theta$ is a θ -trace. Since P is polynomially isomorphic to N , there is $\psi \in \text{Pol}_1(\mathbf{A})$ such that

$$\psi[P] = \psi[x/\theta] = \{\psi(z) : (x, z) \in \theta\} = N.$$

Then, $(\psi(x), \psi(y)) \in \theta \cap N^2$. Also, $(\psi(x), \psi(y)) \in \beta - \alpha$, so that $(\psi(x), \psi(y)) \in \beta|N - \alpha|N$ and $\alpha|N < \beta|N$. \square

Lemma 3.8. *Let \mathbf{A} be a θ -minimal algebra with $\Delta_A \prec \theta$. Let N, K be two θ -traces. Then N and K are polynomially isomorphic.*

Proof. Since $\Delta_A \prec \theta$, $\theta_{\mathbf{A}}(N) = \theta$. But $\theta_{\mathbf{A}}(N)$ is the transitive closure of the relation $\{(\psi(x), \psi(y)) : (x, y) \in N^2, \psi \in \text{Pol}_1(\mathbf{A})\}$. Then, as K is a θ -class, there is $\varphi \in \text{Pol}_1(\mathbf{A})$ such that $\varphi[N] \cap K \neq \emptyset$ and φ is not constant on N . This implies that $\varphi \in \text{Sym}(A)$, so that $\varphi[N] = K$. Similarly, there is $\psi \in \text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$ such that $\psi[N] = K$. Now, $\psi\varphi \in \text{Sym}(A)$, hence $(\psi\varphi)^k = 1_A$ for some $k > 0$. The polynomials $\varphi(\psi\varphi)^{k-1}$ and ψ witness that $N \sim K$. \square

4. MINIMAL ALGEBRAS RELATIVE TO A PAIR

Definition 4.1. Let \mathbf{A} be a finite algebra and let $\delta < \theta \in \text{Con}(\mathbf{A})$. We say that \mathbf{A} is (δ, θ) -minimal if for all $\varepsilon \in \text{Pol}_1(\mathbf{A})$ either ε is bijective or $\varepsilon(\theta) \subseteq \delta$.

Remark 4.2. Observe that \mathbf{A} is minimal iff \mathbf{A} is (Δ, ∇) -minimal.

Definition 4.3. Let \mathbf{A} be a (δ, θ) -minimal algebra. An (α, β) -trace of \mathbf{A} is a β -equivalence class which contains at least two α -equivalence classes.

Lemma 4.4. *Let \mathbf{A} be a finite (δ, θ) -minimal algebra and let N be a (δ, θ) -trace. Then the algebra $(\mathbf{A}||N)/(\delta|N)$ is minimal.*

Proof. We need to show that for every

$$\psi \in \text{Pol}_1((\mathbf{A}||N)/(\delta|N)) = \{(\varphi|N)/(\delta|N) : \varphi \in \text{Pol}_1(\mathbf{A}), \varphi[N] \subseteq N\}$$

either ψ is bijective, or ψ is constant. Let $\psi = (\varphi|N)/(\delta|N)$. Since \mathbf{A} is (δ, θ) -minimal, either φ is bijective or $\varphi(\theta) \subseteq \delta$. Clearly, if φ is bijective, ψ is bijective. If $\varphi(\theta) \subseteq \delta$, ψ is constant: if $(x, y) \in N^2 \subseteq \theta$, then $(\psi(x), \psi(y)) \in \delta$ so that $\psi(x) = \psi(y)$ in $(\mathbf{A}||N)/(\delta|N)$. \square

Therefore, with an abuse of language, we shall refer unambiguously to the type of N as the type of $(\mathbf{A}||N)/(\delta|N)$.

Theorem 4.5. *Let \mathbf{A} be a (δ, θ) -minimal algebra. Then all (δ, θ) -traces of \mathbf{A} have the same type.*

Definition 4.6. Let \mathbf{A} be a finite (δ, θ) -minimal algebra. We say that \mathbf{A} is of type \mathbf{i} relative to (δ, θ) if each (δ, θ) -trace of \mathbf{A} is of type \mathbf{i} .

Lemma 4.7. *Let \mathbf{A} be a (δ, θ) -minimal algebra and N be a (δ, θ) -trace. Then*

- (1) *there is a surjective lattice homomorphism $[\delta, \theta] \rightarrow \text{Con}((\mathbf{A}||N)/(\delta|N))$;*
- (2) *if any two (δ, θ) -traces are polynomially isomorphic, then it is an isomorphism.*

Proof. The first part immediately follows from Lemma 3.7. The second is very similar. Let $\delta \leq \alpha < \beta \leq \theta$. Let $(x, y) \in \beta - \alpha$. Then $(x, y) \in \theta - \delta$ and $P := x/\theta$ is a (δ, θ) -trace. Since P is polynomially isomorphic to N , there is $\psi \in \text{Pol}_1(\mathbf{A})$ such that

$$\psi[P] = \psi[x/\theta] = \{\psi(z) : (x, z) \in \theta\} = N.$$

Then, $(\psi(x), \psi(y)) \in (\theta - \delta) \cap N^2$. Also, $(\psi(x), \psi(y)) \in \beta - \alpha$, so that $(\psi(x), \psi(y)) \in \beta|N - \alpha|N$ and $\alpha|N/\delta|N < \beta|N/\delta|N$. \square

The following is an example of a sufficient condition that guarantees that the lattices $[\delta, \theta]$ and $\text{Con}((\mathbf{A}||N)/(\delta|N))$ are isomorphic.

Lemma 4.8. *Let \mathbf{A} be a (δ, θ) -minimal algebra with $\delta \prec \theta$. Let N, K be two (δ, θ) -traces. Then N and K are polynomially isomorphic.*

Proof. Since $\delta \prec \theta$, $\delta \vee \theta_{\mathbf{A}}(N) = \theta$. Then there is $\varphi \in \text{Pol}_1(\mathbf{A})$ such that $\varphi[N] \cap K \neq \emptyset$ and $\varphi[N]^2 \not\subseteq \delta$. This implies that $\varphi \in \text{Sym}(A)$, so that $\varphi[N] = K$ and $N \sim K$. \square

5. TAME CONGRUENCES

Definition 5.1. Let \mathbf{A} be a finite algebra and $\alpha, \beta \in \text{Con}(\mathbf{A})$ with $\alpha < \beta$. We denote by

- (1) $\text{E}(\mathbf{A})$ the set of $\varepsilon \in \text{Pol}_1(\mathbf{A})$ such that $\varepsilon^2 = \varepsilon$;
- (2) $\text{U}_{\mathbf{A}}(\alpha, \beta)$ the set $\{\varphi[A] : \varphi \in \text{Pol}_1(\mathbf{A}), \varphi(\beta) \not\subseteq \alpha\}$;

- (3) $M_{\mathbf{A}}(\alpha, \beta)$ the set of minimal (with respect to the inclusion relation) elements of $U_{\mathbf{A}}(\alpha, \beta)$.

Remark 5.2. A finite algebra \mathbf{A} is (α, β) -minimal iff $A \in M_{\mathbf{A}}(\alpha, \beta)$. If \mathbf{A} is (α, β) -minimal, every $\psi \in M_{\mathbf{A}}(\alpha, \beta)$ such that $\psi(\alpha) \not\subseteq \beta$ is bijective. Clearly $A \in U_{\mathbf{A}}(\alpha, \beta)$, as witnessed by $\varepsilon(x) = x$. If there were $\psi \in \text{Pol}_1(\mathbf{A})$ with $\psi(\alpha) \not\subseteq \beta$ and $\psi[A] \subset A$ we would contradict (α, β) -minimality. Conversely, let $A \in M_{\mathbf{A}}(\alpha, \beta)$ and $\psi \in \text{Pol}_1(\mathbf{A})$ with $\psi(\alpha) \not\subseteq \beta$; we show that $\psi \in \text{Sym}(A)$. But by definition of $M_{\mathbf{A}}(\alpha, \beta)$, there is no $\psi \in \text{Pol}_1(\mathbf{A})$ with $\psi(\alpha) \not\subseteq \beta$ and $\psi[A] \subset A$.

Lemma 5.3. *Let $\varepsilon \in E(\mathbf{A})$, $U := \varepsilon[A]$, and $\emptyset \neq N \subseteq U$. Then $\mathbf{A}||N = (\mathbf{A}||U)||N$.*

Proof. That $(\text{Pol}(\mathbf{A})|U)|N \subseteq \text{Pol}(\mathbf{A})|N$ is obvious. Conversely, let $\psi = \varphi|N$ for some $\varphi \in \text{Pol}(\mathbf{A})$ with $\varphi[N^k] \subseteq N$. Clearly, $\varepsilon\varphi[U^k] \subseteq U$. If $(a_1, \dots, a_k) \in N$, $\varphi(a_1, \dots, a_k) \in N \subseteq U$, hence $\varphi(a_1, \dots, a_k) = \varepsilon(a)$ for some $a \in A$. Hence $\varepsilon\varphi(a_1, \dots, a_k) = \varepsilon^2(a) = \varepsilon(a) \in N$ so that $(\varepsilon\varphi|U)|N = \varphi|N$. We have shown that ψ is an operation of $(\mathbf{A}||U)||N$. \square

Definition 5.4. Let \mathbf{A} be a finite algebra and $\alpha, \beta \in \text{Con}(\mathbf{A})$ with $\alpha < \beta$. The pair (α, β) is a pair of **tame** congruences if there is $V \in M_{\mathbf{A}}(\alpha, \beta)$, $\varepsilon \in E(\mathbf{A})$ such that $\varepsilon[A] = V$ and $(-|V) : [\alpha, \beta] \rightarrow [\alpha|V, \beta|V]$ is 0, 1-separating

A lattice homomorphism $\alpha : \mathbf{L} \rightarrow \mathbf{N}$ is **0, 1-separating** if $\alpha^{-1}[\{\alpha(i)\}] = i$ for $i = 0, 1$.

Theorem 5.5. *Let (α, β) be a tame pair of congruences of a finite algebra \mathbf{A} . For every $U \in M_{\mathbf{A}}(\alpha, \beta)$,*

- (1) *there is $\varepsilon \in E(\mathbf{A})$ such that $\varepsilon[A] = U$;*
- (2) *$(-|U) : [\alpha, \beta] \rightarrow \text{Con}(\mathbf{A}||U)$ is a surjective lattice homomorphism which is 0, 1-separating;*
- (3) *$\mathbf{A}||U$ is $(\alpha|U, \beta|U)$ -minimal.*
- (4) *Moreover, any two (α, β) -minimal sets are polynomially isomorphic.*

Proof. (1) Since (α, β) is tame, there is $V_0 \in M_{\mathbf{A}}(\alpha, \beta)$ and $\varepsilon_0 \in E(\mathbf{A})$ such that $V_0 = \varepsilon_0[A]$ and $(-|V_0)$ is 0, 1-separating.

Claim 1. *If $(x, y) \in \beta - \alpha$, there is $\varphi \in \text{Pol}_1(\mathbf{A})$ with $\varphi[A] = V_0$ and such that $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$.*

Proof. Let $\theta := \{(x, y) \in \beta : (\varepsilon_0\varphi(x), \varepsilon_0\varphi(y)) \in \alpha \text{ for all } \varphi \in \text{Pol}_1(\mathbf{A})\}$. Now, $\theta \in [\alpha, \beta]$ and $\alpha|V_0 = \theta|V_0$; that $\alpha|V_0 \subseteq \theta|V_0$ is obvious, for the converse:

$$\begin{aligned}
 \theta|V_0 &= \{(a, b) \in \beta \cap V_0^2 : (\varepsilon_0\varphi(a), \varepsilon_0\varphi(b)) \in \alpha \quad \forall \varphi \in \text{Pol}_1(\mathbf{A})\} \\
 &= \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0\varphi\varepsilon_0(x), \varepsilon_0\varphi\varepsilon_0(y)) \in \alpha \quad \forall \varphi \in \text{Pol}_1(\mathbf{A})\} \\
 &\subseteq \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0(x), \varepsilon_0(y)) \in \alpha\} \\
 &= \{(a, b) \in \beta : (a, b) \in \alpha \cap V_0^2\} \\
 &= \alpha|V_0
 \end{aligned}$$

This implies that $\theta = \alpha$, since $(-|V_0)$ is 0-separating. Thus $(x, y) \in \beta - \alpha$ implies $(x, y) \in \beta - \theta$. By definition of θ , there is $\psi \in \text{Pol}_1(\mathbf{A})$ such that $(\varepsilon_0\psi(x), \varepsilon_0\psi(y)) \notin \alpha$. Thus $\varphi := \varepsilon_0\psi$ satisfies the conditions $\varphi[A] \subseteq V_0$ and $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$. Hence $\varphi[A] = V_0$ by (α, β) -minimality. \square

Claim 2. *The relation β is the transitive closure of*

$$\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \text{Pol}_1(\mathbf{A})\}.$$

Proof. The transitive closure of $\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \text{Pol}_1(\mathbf{A})\}$ is $\alpha \vee \theta_{\mathbf{A}}(\beta|V_0)$. But $\alpha \vee \theta_{\mathbf{A}}(\beta|V_0) \in [\alpha, \beta]$, and therefore, since $(-|V_0)$ is 1-separating, $\beta = \alpha \vee \theta_{\mathbf{A}}(\beta|V_0)$. \square

Assume that $U \in M_{\mathbf{A}}(\alpha, \beta)$. Then, by definition, there is $\mu \in \text{Pol}_1(\mathbf{A})$ such that $\mu[A] = U$ and $\mu(\beta) \not\subseteq \alpha$. This implies that the equivalence relation

$$\mu^{-1}(\alpha) = \{(a, b) : (\mu(a), \mu(b)) \in \alpha\}$$

is such that $\beta \not\subseteq \mu^{-1}(\alpha)$. Then by Claim 2 there are $a, b \in V_0$ and $\psi \in \text{Pol}_1(\mathbf{A})$ such that $(a, b) \in \beta$ and $(\mu\psi(a), \mu\psi(b)) \notin \alpha$. The function $\mu_1 := \mu\psi\varepsilon_0$ satisfies $\mu_1[A] \subseteq U$ and $\mu_1(\beta) \not\subseteq \alpha$: there are $x, y \in A$ such that $(a, b) = (\varepsilon_0(x), \varepsilon_0(y)) \in \beta$ but $(\mu_1(a), \mu_1(b)) = (\mu\psi\varepsilon_0(x), \mu\psi\varepsilon_0(y)) \notin \alpha$. Thus $\mu_1[A] = U$ by (α, β) -minimality. Observe that $\mu_1[V_0] = \mu_1\varepsilon_0[A] = \mu\psi\varepsilon_0^2[A] = \mu_1[A]$, so that $\mu_1[V_0] = U$. Apply Claim 1 to the pair $(\mu_1(a), \mu_1(b))$ to get $\nu \in \text{Pol}_1(\mathbf{A})$ such that $\nu[A] = V_0$ and $(\nu\mu_1(a), \nu\mu_1(b)) \notin \alpha$. Now, since $\mu_1\nu[A] = \mu_1[V_0] = U$, $\mu_1\nu|U$ is bijective; since U is finite, there is $k > 1$ such that $(\mu_1\nu|U)^k = (\mu_1\nu)^k|U = 1_U$. Let $\varepsilon := (\mu_1\nu)^k$. We have: $\varepsilon[A] = (\mu_1\nu)^k[A] = (\mu_1\nu)^{k-1}[U] = U$ and, consequently, for all $a \in A$, $\varepsilon^2(a) = \varepsilon(a)$ since $\varepsilon(a) \in U$. Therefore $\varepsilon \in E(\mathbf{A})$.

(2) Clearly, the map is well defined and preserves meets. For $\theta \in [\alpha|U, \beta|U]$, let

$$\hat{\theta} = \{(x, y) \in \beta : (\varepsilon\varphi(x), \varepsilon\varphi(y)) \in \theta \text{ for all } \varphi \in \text{Pol}_1(\mathbf{A})\}$$

The relation $\hat{\theta}$ is an equivalence relation. If $(x, y) \in \hat{\theta}$ and $\psi \in \text{Pol}_1(\mathbf{A})$, then $(\psi(x), \psi(y)) \in \hat{\theta}$ so that $\hat{\theta} \in [\alpha, \beta]$. missing

(4) In the above notation, let $\varphi := \nu$, $\psi := (\mu_1\nu)^{k-1}\mu_1$. Then $\varphi[U] = V_0$, $\psi[V_0] = U$ and $\psi\varphi|U = 1_U$, $\varphi\psi|V_0 = 1_{V_0}$. Then fact that any (α, β) -minimal set is polynomially isomorphic to V_0 implies that any two (α, β) -minimal sets are polynomially isomorphic.

(3) Let $\varphi \in \text{Pol}_1(\mathbf{A}||U) = \{\psi|U : \psi \in \text{Pol}_1(\mathbf{A}), \psi[U] \subseteq U\}$. We need to show that if $\varphi(\beta|U) \not\subseteq \alpha|U$, then $\varphi \in \text{Sym}(U)$. If $\varphi(\beta|U) \not\subseteq \alpha|U$, then in particular $\psi(\beta) \not\subseteq \alpha$, so that, by (α, β) -minimality, $U \subseteq \psi[A]$. missing \square

Definition 5.6. Let (α, β) be tame in a finite algebra \mathbf{A} . An (α, β) -**trace*** of \mathbf{A} is $N \subseteq A$ such that for some $U \in M_{\mathbf{A}}(\alpha, \beta)$ and $x \in U$, $N \subseteq U$ and $N = x/(\beta|U) \neq x/(\alpha|U)$. That is, N is an $(\alpha|U, \beta|U)$ -trace of the minimal algebra $\mathbf{A}||U$.

Definition 5.7. Let (α, β) be tame in a finite algebra \mathbf{A} . Let $\delta, \theta \in \text{Con}(\mathbf{A})$. Let K be a class of algebras. We define

- (1) the **type** of (α, β) , written $\text{typ}(\alpha, \beta)$, to be the type of $\mathbf{A}||U$ relative to $(\alpha|U, \beta|U)$;
- (2) $\text{typ}\{\delta, \theta\} := \{\text{typ}(\alpha, \beta) : \delta \leq \alpha \prec \beta \leq \theta\}$;
- (3) $\text{typ}\{\mathbf{A}\} := \text{typ}\{\Delta_A, \nabla_A\}$;
- (4) $\text{typ}\{K\} := \cup\{\text{typ}\{\mathbf{A}\} : \mathbf{A} \in K_{\text{fin}}\}$.

Remark 5.8. If (Δ, ∇) in $\text{Con}(\mathbf{A})$ is tame, then $\text{typ}(\Delta, \nabla)$ coincides with $\text{typ}(\mathbf{A})$ of Definition 2.7.

Lemma 5.9. *Let (α, β) be tame in a finite algebra \mathbf{A} . For every (α, β) -trace* of \mathbf{A} , the algebra $(\mathbf{A}||N)/(\alpha|N)$ is minimal and $\text{typ}(\alpha, \beta) = \text{typ}((\mathbf{A}||N)/(\alpha|N))$.*

Proof. Let U be any (α, β) -minimal set and N be an $(\alpha|U, \beta|U)$ -trace. The algebra $\mathbf{A}||U$ is minimal relative to $(\alpha|U, \beta|U)$. By Lemma 5.3 $\mathbf{A}||N = (\mathbf{A}||U)||N$ and consequently $(\mathbf{A}||N)/(\alpha|N) = ((\mathbf{A}||U)||N)/((\alpha|U)|N)$. The type of $\mathbf{A}||U$ relative to $(\alpha|U, \beta|U)$ is, by definition, the type of the minimal algebra $\mathbf{M} := ((\mathbf{A}||U)||N)/((\alpha|U)|N)$; but M is the only (Δ_M, ∇_M) -trace of \mathbf{M} , hence this is $\text{typ}(\mathbf{M})$. □

6. OMITTING TYPES

Definition 6.1. Let \mathbf{V} be a variety. An algebra $\mathbf{A} \in \mathbf{V}$ is called

- (1) **free** if there is an isomorphism $\mathbf{A} \simeq \mathbf{F}_{\mathbf{V}}(\kappa)$ for some cardinal κ ;
- (2) **finitely generated** if there is a surjective homomorphism $\mathbf{F}_{\mathbf{V}}(n) \rightarrow \mathbf{A}$ for some $n \in \omega$.

Definition 6.2. A variety \mathbf{V} is called

- (1) **locally finite** if all its finitely generated algebras are finite;
- (2) **finitely presented** if \mathbf{V} has a finite set of function symbols and $\mathbf{V} = \text{Alg}(\Sigma)$ for a finite set of equations Σ ;
- (3) **finitely generated** if $\mathbf{V} = V(\mathbf{A}_1, \dots, \mathbf{A}_n)$ for $\mathbf{A}_1, \dots, \mathbf{A}_n$ finite similar algebras.

Remark 6.3. Observe that $V(\mathbf{A}_1, \dots, \mathbf{A}_n) = V(\mathbf{A}_1 \times \dots \times \mathbf{A}_n)$.

Lemma 6.4. *Let \mathbf{V} be a variety. If \mathbf{V} is finitely generated then it is locally finite.*

Proof. By the previous remark, we can assume that $\mathbf{V} = V(\mathbf{A})$ for some \mathbf{A} finite. Let $n < \omega$. We prove that $\mathbf{F}_{\mathbf{V}}(n)$ is finite. Consider the homomorphism

$$\mathbf{F}_{\mathbf{V}}(n) \rightarrow \mathbf{A}^{A^n}, \quad t(x_1, \dots, x_n) \mapsto t^{\mathbf{A}}$$

This homomorphism is injective: if $t^{\mathbf{A}} = s^{\mathbf{A}}$, then $\mathbf{A} \models t \equiv s$, i.e. $t = s$ in $\mathbf{F}_{\mathbf{V}}(n)$. Thus $\mathbf{F}_{\mathbf{V}}(n)$ is finite. □

Definition 6.5. Let V and W be two varieties. We say that V is **interpretable** into W ($V \leq W$) if there is a clone homomorphism $\text{Clo}(V) \rightarrow \text{Clo}(W)$.

Remark 6.6. Let W, V be two varieties. We unravel what $V \leq W$ means in a simple case, that is when V is finitely presented. Let F be a finite set of function symbols. Let V be a variety of algebras over F , defined by the equations

$$(1) \quad s_1 \equiv t_1, \dots, s_k \equiv t_k.$$

Assume that for each $f \in F_n$ there is $t \in \text{Clo}_n(W)$ such that the interpretation of the t 's satisfy the equations (1). Then the assignment $f \mapsto t$ extends to a clone homomorphism $\text{Clo}(V) \rightarrow \text{Clo}(W)$. Of course, the converse also holds; thus this is equivalent to $V \leq W$.

Lemma 6.7. Let W, V be two varieties such that $W \leq V$. Assume that W is

- idempotent;
- finitely presented;
- the finite set of equations defining W contains at most one function symbol per side.

Let $\mathbf{A} \in V$, $\varepsilon \in E(\mathbf{A})$, $U := \varepsilon[A]$, $\beta \in \text{Con}(\mathbf{A})$ and $N := a/\beta \cap U$ for $a \in U$. Then $W \leq V(\mathbf{A}|N)$. Moreover, if $\mathbf{1} \in \text{typ}\{V\}$, then $W \leq \text{Set}$.

Proof. By assumption W can be described by a finite set of equations of the form

$$(2) \quad f_i(x_{i_1}, \dots, x_{i_h}) \equiv f_j(x_{j_1}, \dots, x_{j_k})$$

where f_i and f_j are members of a finite set F of function symbols. Since $W \leq V$, there is an assignment $f \mapsto t$ extending to a clone homomorphism. We need to find a clone homomorphism $\text{Clo}(W) \rightarrow \text{Clo}(\mathbf{A}|N)$. Consider $f \mapsto \varphi := \varepsilon t_i^{\mathbf{A}}|N$. Firstly, it is well defined: if $(a_1, \dots, a_n) \in S$, $(\varphi(a_1, \dots, a_n), \varphi(a, \dots, a)) \in \beta$ but

$$\varphi(a, \dots, a) = \varepsilon t^{\mathbf{A}}(a, \dots, a) = \varepsilon(a) \in U$$

so that $\varphi(a_1, \dots, a_n) \in S$ and therefore $\varphi \in \text{Pol}(\mathbf{A})|N$. Finally, using that $W \leq V$

$$\begin{aligned} \varphi_i(a_{i_1}, \dots, a_{i_h}) &= \varepsilon t_i^{\mathbf{A}}(a_{i_1}, \dots, a_{i_h}) \\ &= \varepsilon t_j^{\mathbf{A}}(a_{j_1}, \dots, a_{j_k}) \\ &= \varphi_j(a_{j_1}, \dots, a_{j_k}). \end{aligned}$$

If $\mathbf{1} \in \text{typ}\{V\}$. Then there are $\alpha \prec \beta \in \text{Con}(\mathbf{A})$ such that $\text{typ}(\alpha, \beta) = \mathbf{1}$. Without loss of generality we can assume that $\alpha = \Delta_A$. Let N be a (Δ_A, β) -trace*. Then there are $\varepsilon \in E(\mathbf{A})$, $U := \varepsilon[A]$ such that $N = a/\beta \cap U$ for some $a \in U$. Thus $W \leq V(\mathbf{A}|N)$.

Claim 3. Let \mathbf{M} be finite minimal algebra of unary type. Let $\mathbf{A} = (M, \text{Pol}(\mathbf{M}))$. Then $V(\mathbf{A})$ contains a finite algebra \mathbf{S} all of whose polynomials are constant or projections.

Proof. Let $\mathbf{G} := \text{Sym}(M) \cap \text{Pol}_1(\mathbf{M})$, subgroup of $\text{Sym}(M)$. Let $u, v \in M$, $u \neq v$ and let

$$D := \{(\sigma(u), \sigma(v)) : \sigma \in \mathbf{G}\} \cup \{(\sigma(v), \sigma(u)) : \sigma \in \mathbf{G}\}$$

Since \mathbf{M} is minimal with $\text{typ}(\mathbf{M}) = \mathbf{1}$, every polynomial ψ is constant or there is i , $\sigma \in \mathbf{G}$ such that

$$\psi(a_1, \dots, a_n) = \sigma(a_i).$$

This implies that \mathbf{D} is a subalgebra of \mathbf{A}^2 . Let

$$((x_1, x_2), (y_1, y_2)) \in \theta \iff \sigma(x_i) = y_i \text{ for some } \sigma \in \mathbf{G}$$

We show that every term operation of \mathbf{D}/θ is either constant or a projection. Let $\psi \in \text{Pol}_n(\mathbf{M})$ non constant and $(a_i, b_i) \in D$ for $i = 1, \dots, n$. Then there is $\tau \in \mathbf{G}$ such that

$$\begin{aligned} \psi((a_1, b_1)/\theta, \dots, (a_n, b_n)/\theta) &= \psi((a_1, b_1), \dots, (a_n, b_n))/\theta \\ &= (\psi(a_1, \dots, a_n), \psi(b_1, \dots, b_n))/\theta \\ &= (\tau(a_i), \tau(b_i))/\theta \\ &= (a_i, b_i)/\theta. \end{aligned} \quad \square$$

Let $\mathbf{S} \in V(\mathbf{A}||N)$ be such that every term operation of \mathbf{S} is constant or a projection. Then there is a clone homomorphism $\text{Clo}(\mathbf{A}||N) \rightarrow \text{Clo}(\mathbf{S})$. Since $\mathbf{W} \leq V(\mathbf{A}||N)$ there is a clone homomorphism $\text{Clo}(\mathbf{W}) \rightarrow \text{Clo}(\mathbf{A}||N)$. Thus we get a clone homomorphism $\text{Clo}(\mathbf{W}) \rightarrow \text{Clo}(\mathbf{S})$. But for every $f \in F$, \mathbf{W} satisfies $f(x, \dots, x) \equiv x$, hence the image of f through this clone homomorphism cannot be but a projection. This implies that $\mathbf{W} \leq \text{Set}$. \square

Theorem 6.8. *Let \mathbf{V} be a locally finite variety. The following are equivalent:*

- (1) $\mathbf{1} \notin \text{typ}\{\mathbf{V}\}$;
- (2) *there is an idempotent variety \mathbf{W} such that $\mathbf{W} \leq \mathbf{V}$ and $\mathbf{W} \not\leq \text{Set}$.*

Corollary 6.9. *Let \mathbf{A} be a finite idempotent algebra. There is $\mathbf{B} \in HS(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}) \simeq \mathbf{N}$ iff $\mathbf{1} \in \text{typ}\{HS(\mathbf{A})\}$.*

Proof. If $\mathbf{1} \in \text{typ}\{HS(\mathbf{A})\}$, then $\mathbf{1} \in \text{typ}\{V(\mathbf{A})\}$. Since \mathbf{A} is finite, then, by Lemma 6.4 $V(\mathbf{A})$ is locally finite, and therefore, by Theorem 6.8, for every idempotent variety \mathbf{W} , either $\mathbf{W} \not\leq V(\mathbf{A})$ or $\mathbf{W} \leq \text{Set}$. In particular, since \mathbf{A} is idempotent, $V(\mathbf{A}) \leq \text{Set}$. This means that there is a clone homomorphism $\text{Clo}(\mathbf{A}) \rightarrow \mathbf{N}$. Therefore, there is $\mathbf{S} \in \text{Set}$ such that $\mathbf{S} \in V(\mathbf{A})$.

Conversely, let $\mathbf{B} \in HS(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}) \simeq \mathbf{N}$; this means that $\mathbf{B} \in \text{Set}$ and therefore \mathbf{B} is minimal of type $\mathbf{1}$. \square

Theorem 6.10 ([2]). *Let \mathbf{A} be a finite idempotent algebra. Then $\text{Clo}(\mathbf{A})$ contains a weak near-unanimity operation iff $\mathbf{1} \notin \text{typ}\{HS(\mathbf{A})\}$.*

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