MINIMAL ALGEBRAS

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1. Preliminaries

Definition 1.1. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$$

and closed under composition. The elements of $\mathrm{Clo}(\mathbf{A})$ are called **term** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **term equivalent** if $\mathrm{Clo}(\mathbf{A}) \simeq \mathrm{Clo}(\mathbf{B})$.

Definition 1.2. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Pol}(\mathbf{A})$ the smallest set containing

(1) $\{f^{\mathbf{A}} : f \in F\}$;

(2) $\{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$;

(3) the constant 0-ary operations

and closed under composition. The elements of $\mathrm{Pol}(\mathbf{A})$ are called **polynomial** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **polynomially equivalent** if $\mathrm{Pol}(\mathbf{A}) \simeq \mathrm{Pol}(\mathbf{B})$.

Example 1.3. If $\varphi \in \mathrm{Clo}_{m+n}(\mathbf{A})$ and $(a_1, \dots, a_m) \in A^m$, then

$$\psi : A^n \to A \quad (b_1, \dots, b_n) \mapsto \varphi(a_1, \dots, a_m, b_1, \dots, b_n)$$

is a polynomial operation.

Remark 1.4. Let $\mathbf{A}$ be an algebra. An equivalence relation α is a congruence of $\mathbf{A}$ iff $\varphi(\alpha) \subseteq \alpha$ for every $\varphi \in \mathrm{Pol}_1(\mathbf{A})$.

Let $\mathbf{A}$ be a finite algebra. We adopt the following convention, concerning the restriction $(-|U)$ operation, for $U \subseteq A$:

- if $\theta \in \mathrm{Con}(\mathbf{A})$, $\theta|U := \theta \cap U^2$;
- if $\varphi : A^n \to A$, $\varphi|U$ is the function $U^n \to A$, $(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_n)$;
- $\mathrm{Pol}(\mathbf{A})|U := \{\psi|U : \psi \in \mathrm{Pol}(\mathbf{A}) \text{ and } \psi[U^n] \subseteq U\}$;
- $\mathbf{A}||U := (U, \mathrm{Pol}(\mathbf{A})|U)$.

2. Finite Minimal Algebras

Definition 2.1. A nontrivial finite algebra $\mathbf{A}$ is **minimal** iff every noncostant element of $\mathrm{Pol}_1(\mathbf{A})$ is bijective.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

Example 2.2. The following are examples of minimal algebras.

(1) any algebra with carrier 2;

(2) a nontrivial finite vector space $\mathbf{A}$ over a finite field $\mathbf{k}$: every $\pi \in \mathrm{Pol}_1(\mathbf{A})$ is of the form $\pi(v) = av + b$ for some $a \in k$, $b \in A$;

(3) a group of permutations acting on a finite set. If $\mathbf{G}$ is a group acting on a set A each $g \in G$ induces an operation $\varphi_g : A \to A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. A $\mathbf{G}$-set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$.

We shall see that, up to polynomial equivalence, there are no other finite minimal algebras.

Lemma 2.3. *Let $\mathbf{A}$ be a minimal algebra. If every element of $\mathrm{Pol}(\mathbf{A})$ is essentially unary, then $\mathbf{A}$ is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ where $\mathbf{G}$ is a finite group acting on A.*

Proof. Since $\mathbf{A}$ is minimal, $\mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$ is a subgroup of $\mathrm{Sym}(A)$. Let $\mathbf{G} := \mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$. If $\psi \in \mathrm{Pol}(\mathbf{A})$, either ψ is constant or ψ is essentially unary, hence $(A, \Phi_{\mathbf{G}})$ is polynomially equivalent to $\mathbf{A}$. $\square$

Theorem 2.4 ([3])**.** *Let $\mathbf{A}$ be a minimal algebra with $|A| > 2$. If $\mathrm{Pol}(\mathbf{A})$ contains an operation which is not essentially unary, then $\mathbf{A}$ is polynomially equivalent to a $\mathbf{k}$-vector space for a finite field $\mathbf{k}$.*

Theorem 2.5. *Every algebra $\mathbf{A}$ with carrier 2 is polynomially equivalent to one of the following:*

(1) $\mathbf{E}_0 = (2, \varnothing)$;

(2) $\mathbf{E}_1 = (2, \neg)$;

(3) $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$;

(4) $\mathbf{E}_4 = (2, \wedge, \vee)$;

(5) $\mathbf{E}_5 = (2, \vee)$;

(6) $\mathbf{E}_6 = (2, \wedge)$.

Each of them is not polynomially equivalent to the other[1].

[1] A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem 2.5 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

Remark 2.6. Up to isomorphism, $\mathbf{E}_5(\simeq \mathbf{E}_6)$ is the only semilattice with two elements, while $\mathbf{E}_3$ and $\mathbf{E}_4$ are the only Boolean algebra and lattice, respectively, with two elements.

Definition 2.7. Let $\mathbf{A}$ be a minimal algebra. We say that $\mathbf{A}$ is of

(1) **type 1** (or **unary**) if $\mathbf{A}$ is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ for some $\mathbf{G} \leq \mathrm{Sym}(A)$;

(2) **type 2** (or **affine**) if $\mathbf{A}$ is polynomially equivalent to a vector space over a finite field $\mathbf{k}$;

(3) **type 3** (or **Boolean**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_3$;

(4) **type 4** (or **lattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_4$;

(5) **type 5** (or **semilattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_5$.

3. Minimal Algebras Relative to a Congruence

Definition 3.1. Let $\mathbf{A}$ be a finite algebra and let $\theta \in \mathrm{Con}(\mathbf{A}), \Delta_A \neq \theta$. We say that $\mathbf{A}$ is θ-**minimal** if for all $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ either ε is constant on the θ-equivalence classes or ε is bijective.

Remark 3.2. Observe that $\mathbf{A}$ is minimal iff $\mathbf{A}$ is ∇_A-minimal.

Definition 3.3. Let $\mathbf{A}$ be a θ-minimal algebra. A θ-**trace** of $\mathbf{A}$ is a nontrivial θ-equivalence class.

Lemma 3.4. *Let $\mathbf{A}$ be a finite θ-minimal algebra and let N be a θ-trace. Then the algebra $\mathbf{A}||N$ is minimal.*

Proof. We need to show that for every

$$\psi \in \mathrm{Pol}_1(\mathbf{A}||N) = \{\varphi|N : \varphi \in \mathrm{Pol}_1(\mathbf{A}), \varphi[N] \subseteq N\}$$

either ψ is bijective, or ψ is constant. Let $\psi = \varphi|N$. Since $\mathbf{A}$ is θ-minimal, either φ is bijective or φ is constant on the θ-equivalence classes. Clearly, in the first case ψ is bijective, in the second ψ is constant. □

Definition 3.5. Let $\mathbf{A}$ be a finite algebra and $B, C \subseteq A$. We say that B, C are **polynomially isomorphic** ($B \sim C$) if there are $\varphi, \psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varphi[B] = C, \psi[C] = B$ and $\psi\varphi|B = 1_B, \varphi\psi|C = 1_C$.

Remark 3.6. If $B, C \subseteq A$ are polynomial isomorphic in $\mathbf{A}$, then $\mathbf{A}||B \simeq \mathbf{A}||C$. Let $\pi := \varphi|B$, so that $\pi^{-1} = \psi|C$. Of course, $\pi : B \to C$ is a bijection. We show that π is a homomorphism. Let $f \in \mathrm{Pol}_n(\mathbf{A})$ such that $f[B^n] \subseteq B$. Then $g(-, \dots, -) := \pi f(\pi^{-1}(-), \dots, \pi^{-1}(-)) \in \mathrm{Pol}_n(\mathbf{A})$ too, $g[C^n] \subseteq C$ and

$$\pi f(b_1, \dots, b_n) = g(\pi(b_1, \dots, b_n))$$

for all $b_1, \dots, b_n \in B^n$.

Lemma 3.7. *Let $\mathbf{A}$ be a θ-minimal algebra and N be a θ-trace. Then*

(1) $(-|N) : [\Delta_A, \theta] \to \mathrm{Con}(\mathbf{A}||N)$ *is a surjective lattice homomorphism;*

(2) *if any two θ-traces are polynomially isomorphic, then $(-|N)$ is an isomorphism.*

Proof. Clearly, the map is well defined and preserves meets. We show that

$$(\alpha \vee \beta) \cap N^2 = (\alpha \cap N^2) \vee (\beta \cap N^2).$$

Let $(x, y) \in (\alpha \vee \beta) \cap N^2$. Then there are $x = x_0, \ldots, x_{n+1} = y$ such that either $(x_i, x_{i+1}) \in \alpha$ or $(x_i, x_{i+1}) \in \beta$. We show that for each i, $(x_i, x_{i+1}) \in N^2$. Inductively, if $x_i \in N$, and, say, $(x_i, x_{i+1}) \in \alpha \subseteq \theta$, then $x_{i+1} \in N$. Now, for $\beta \in \mathrm{Con}(\mathbf{A}||N)$, let $\hat{\beta}$ be

$$\{(x, y) \in \theta : (\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta \quad \forall \psi \in \mathrm{Pol}_1(\mathbf{A})\}$$

Then $\hat{\beta}$ is a congruence. We show that $\hat{\beta} \cap N^2 = \beta$, proving surjectivity. If $(x, y) \in \hat{\beta}$, then $(\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta$ for all $\psi \in \mathrm{Pol}_1(\mathbf{A})$; if $(x, y) \in N^2$, then $(\psi(x), \psi(y)) \in N^2$. Therefore $(\psi(x), \psi(y)) \in \beta$ for all $\psi \in \mathrm{Pol}_1(\mathbf{A})$, and, taking $\psi(x) = x$, $(x, y) \in \beta$. Conversely, let $(x, y) \in \beta$. As $\beta \subseteq N^2 \subseteq \theta$, $(x, y) \in \theta$. Let $\psi \in \mathrm{Pol}_1(\mathbf{A})$. If $(\psi(x), \psi(y)) \in N^2$, then $\psi \in \mathrm{Pol}_1(\mathbf{A}||N)$. Since $\beta \in \mathrm{Con}(\mathbf{A}||N)$, $(\psi(x), \psi(y)) \in \beta$.

We need to prove injectivity. Let $\alpha < \beta \leq \theta$. Let $(x, y) \in \beta - \alpha$. Then $(x, y) \in \theta$ and $P := x/\theta$ is a θ-trace. Since P is polynomially isomorphic to N, there is $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that

$$\psi[P] = \psi[x/\theta] = \{\psi(z) : (x, z) \in \theta\} = N.$$

Then, $(\psi(x), \psi(y)) \in \theta \cap N^2$. Also, $(\psi(x), \psi(y)) \in \beta - \alpha$, so that $(\psi(x), \psi(y)) \in \beta|N - \alpha|N$ and $\alpha|N/ < \beta|N$. □

Lemma 3.8. *Let $\mathbf{A}$ be a θ-minimal algebra with $\Delta_A \prec \theta$. Let N, K be two θ-traces. Then N and K are polynomially isomorphic.*

Proof. Since $\Delta_A \prec \theta$, $\theta_{\mathbf{A}}(N) = \theta$. But $\theta_{\mathbf{A}}(N)$ is the transitive closure of the relation $\{(\psi(x), \psi(y)) : (x, y) \in N^2, \psi \in \mathrm{Pol}_1(\mathbf{A})\}$. Then, as K is a θ-class, there is $\varphi \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varphi[N] \cap K \neq \varnothing$ and φ is not constant on N. This implies that $\varphi \in \mathrm{Sym}(A)$ and $\varphi[N] \subseteq K$. Similarly, there is $\psi \in \mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$ such that $\psi[K] \subseteq N$. But then $\varphi[N] = K$ and $\psi[K] = N$. Now, $\psi\varphi \in \mathrm{Sym}(A)$, hence $(\psi\varphi)^k = 1_A$ for some $k > 0$. The polynomials $\varphi(\psi\varphi)^{k-1}$ and ψ witness that $N \sim K$. □

4. Minimal Algebras Relative to a Pair

Definition 4.1. Let $\mathbf{A}$ be a finite algebra and let $\delta < \theta \in \mathrm{Con}(\mathbf{A})$. We say that $\mathbf{A}$ is (δ, θ)-**minimal** if for all $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ either ε is bijective or $\varepsilon(\theta) \subseteq \delta$.

Remark 4.2. Observe that $\mathbf{A}$ is minimal iff $\mathbf{A}$ is (Δ, ∇)-minimal.

Definition 4.3. Let $\mathbf{A}$ be a (δ, θ)-minimal algebra. An (δ, θ)-**trace** of $\mathbf{A}$ is a θ-equivalence class which contains at least two δ-equivalence classes.

Lemma 4.4. *Let* $\mathbf{A}$ *be a finite* (δ,θ)*-minimal algebra and let* N *be a* (δ,θ)*-trace. Then the algebra* $(\mathbf{A}||N)/(\delta|N)$ *is minimal.*

Proof. We need to show that for every

$$\psi \in \mathrm{Pol}_1((\mathbf{A}||N)/(\delta|N)) = \{(\varphi|N)/(\delta|N) : \varphi \in \mathrm{Pol}_1(\mathbf{A}), \varphi[N] \subseteq N\}$$

either ψ is bijective, or ψ is constant. Let $\psi = (\varphi|N)/(\delta|N)$. Since $\mathbf{A}$ is (δ,θ)-minimal, either φ is bijective or $\varphi(\theta) \subseteq \delta$. Clearly, if φ is bijective, ψ is bijective. If $\varphi(\theta) \subseteq \delta$, ψ is constant: if $(x,y) \in N^2 \subseteq \theta$, then $(\psi(x),\psi(y)) \in \delta$ so that $\psi(x) = \psi(y)$ in $(\mathbf{A}||N)/(\delta|N)$. □

Therefore, with an abuse of language, we shall refer unambiguoulsy to the type of N as the type of $(\mathbf{A}||N)/(\delta|N)$.

Theorem 4.5. *Let* $\mathbf{A}$ *be a* (δ,θ)*-minimal algebra. Then all* (δ,θ)*-traces of* $\mathbf{A}$ *have the same type.*

Definition 4.6. Let $\mathbf{A}$ be a finite (δ,θ)-minimal algebra. We say that $\mathbf{A}$ is of type $\mathbf{i}$ relative to (δ,θ) if each (δ,θ)-trace of $\mathbf{A}$ is of type $\mathbf{i}$.

Lemma 4.7. *Let* $\mathbf{A}$ *be a* (δ,θ)*-minimal algebra and* N *be a* (δ,θ)*-trace. Then*

(1) *there is a surjective lattice homomorphism* $[\delta,\theta] \to \mathrm{Con}((\mathbf{A}||N)/(\delta|N))$*;*

(2) *if any two* (δ,θ)*-traces are polynomially isomorphic, then the above map is an isomorphism.*

Proof. The first part immediately follows from Lemma 3.7. The second is very similar. Let $\delta \leq \alpha < \beta \leq \theta$. Let $(x,y) \in \beta - \alpha$. Then $(x,y) \in \theta - \delta$ and $P := x/\theta$ is a (δ,θ)-trace. Since P is polynomially isomorphic to N, there is $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that

$$\psi[P] = \psi[x/\theta] = \{\psi(z) : (x,z) \in \theta\} = N.$$

Then, $(\psi(x),\psi(y)) \in (\theta-\delta)\cap N^2$. Also, $(\psi(x),\psi(y)) \in \beta-\alpha$, so that $(\psi(x),\psi(y)) \in \beta|N - \alpha|N$ and $\alpha|N/\delta|N < \beta|N/\delta|N$. □

The following is an example of a sufficient condition that guarantees that the lattices $[\delta,\theta]$ and $\mathrm{Con}((\mathbf{A}||N)/(\delta|N))$ are isomorphic.

Lemma 4.8. *Let* $\mathbf{A}$ *be a* (δ,θ)*-minimal algebra with* $\delta \prec \theta$*. Let* N, K *be two* (δ,θ)*-traces. Then* N *and* K *are polynomially isomorphic.*

Proof. Since $\delta \prec \theta$, $\delta \vee \theta_{\mathbf{A}}(N) = \theta$. Then there is $\varphi \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varphi[N] \cap K \neq \varnothing$ and $\varphi[N]^2 \nsubseteq \delta$. This implies that $\varphi \in \mathrm{Sym}(A)$, so that $\varphi[N] = K$ and $N \sim K$. □

5. Tame Congruences

Definition 5.1. Let $\mathbf{A}$ be a finite algebra and $\delta, \theta \in \mathrm{Con}(\mathbf{A})$ with $\delta < \theta$. We denote by

(1) $\mathrm{E}(\mathbf{A})$ the set of $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varepsilon^2 = \varepsilon$;

(2) $\mathrm{U}_{\mathbf{A}}(\delta, \theta)$ the set $\{\varphi[A] : \varphi \in \mathrm{Pol}_1(\mathbf{A}), \varphi(\theta) \nsubseteq \delta\}$;

(3) $\mathrm{M}_{\mathbf{A}}(\delta, \theta)$ the set of minimal (with respect to the inclusion relation) elements of $\mathrm{U}_{\mathbf{A}}(\delta, \theta)$.

Remark 5.2. A finite algebra $\mathbf{A}$ is (δ, θ)-minimal iff $A \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$. If $\mathbf{A}$ is (δ, θ)-minimal, every $\psi \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$ such that $\psi(\delta) \nsubseteq \theta$ is bijective. Clearly $A \in \mathrm{U}_{\mathbf{A}}(\delta, \theta)$, as witnessed by $\varepsilon(x) = x$. If there were $\psi \in \mathrm{Pol}_1(\mathbf{A})$ with $\psi(\delta) \nsubseteq \theta$ and $\psi[A] \subset A$ we would contradict (δ, θ)-minimality. Conversely, let $A \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$ and $\psi \in \mathrm{Pol}_1(\mathbf{A})$ with $\psi(\delta) \nsubseteq \theta$; we show that $\psi \in \mathrm{Sym}(A)$. But by definition of $\mathrm{M}_{\mathbf{A}}(\delta, \theta)$, there is no $\psi \in \mathrm{Pol}_1(\mathbf{A})$ with $\psi(\delta) \nsubseteq \theta$ and $\psi[A] \subset A$.

Lemma 5.3. *Let $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$, and $\varnothing \neq N \subseteq U$. Then $\mathbf{A}||N = (\mathbf{A}||U)||N$.*

Proof. That $(\mathrm{Pol}(\mathbf{A})|U)|N \subseteq \mathrm{Pol}(\mathbf{A})|N$ is obvious. Conversely, let $\psi = \varphi|N$ for some $\varphi \in \mathrm{Pol}(\mathbf{A})$ with $\varphi[N^k] \subseteq N$. Clearly, $\varepsilon\varphi[U^k] \subseteq U$. If $(a_1, \dots, a_k) \in N$, $\varphi(a_1, \dots, a_k) \in N \subseteq U$, hence $\varphi(a_1, \dots, a_k) = \varepsilon(a)$ for some $a \in A$. Hence $\varepsilon\varphi(a_1, \dots, a_k) = \varepsilon^2(a) = \varepsilon(a) \in N$ so that $(\varepsilon\varphi|U)|N = \varphi|N$. We have shown that ψ is an operation of $(\mathbf{A}||U)||N$. $\square$

Definition 5.4. Let $\mathbf{A}$ be a finite algebra and $\delta, \theta \in \mathrm{Con}(\mathbf{A})$ with $\delta < \theta$. The pair (δ, θ) is a pair of **tame** congruences if there is $V \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$, $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = V$ and $(-|V) : [\delta, \theta] \to [\delta|V, \theta|V]$ is $0,1$-separating

A lattice homomorphism $f : \mathbf{L} \to \mathbf{N}$ is $0,1$-**separating** if $f^{-1}[\{\delta(i)\}] = i$ for $i = 0, 1$.

Theorem 5.5. *Let (δ, θ) be a tame pair of congruences of a finite algebra $\mathbf{A}$. For every $U \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$,*

(1) *there is $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = U$;*

(2) *$(-|U) : [\delta, \theta] \to \mathrm{Con}(\mathbf{A}||U)$ is a surjective lattice homomorphism which is $0,1$-separating;*

(3) *$\mathbf{A}||U$ is $(\delta|U, \theta|U)$-minimal.*

(4) *Moreover, any two (δ, θ)-minimal sets are polynomially isomorphic.*

Proof. (1) Since (δ, θ) is tame, there is $V_0 \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$ and $\varepsilon_0 \in \mathrm{E}(\mathbf{A})$ such that $V_0 = \varepsilon_0[A]$ and $(-|V_0)$ is $0,1$-separating.

Claim 1. *If $(x, y) \in \beta - \alpha$, there is $\varphi \in \mathrm{Pol}_1(\mathbf{A})$ with $\varphi[A] = V_0$ and such that $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$.*

Proof. Let $\theta := \{(x,y) \in \beta : (\varepsilon_0\varphi(x), \varepsilon_0\varphi(y)) \in \alpha \text{ for all } \varphi \in \mathrm{Pol}_1(\mathbf{A})\}$. Now, $\theta \in [\alpha,\beta]$ and $\alpha|V_0 = \theta|V_0$; that $\alpha|V_0 \subseteq \theta|V_0$ is obvious, for the converse:

$$\begin{aligned}
\theta|V_0 &= \{(a,b) \in \beta \cap V_0^2 : (\varepsilon_0\varphi(a), \varepsilon_0\varphi(b)) \in \alpha \quad \forall\varphi \in \mathrm{Pol}_1(\mathbf{A})\}\\
&= \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0\varphi\varepsilon_0(x), \varepsilon_0\varphi\varepsilon_0(y)) \in \alpha \quad \forall\varphi \in \mathrm{Pol}_1(\mathbf{A})\}\\
&\subseteq \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0(x), \varepsilon_0(y)) \in \alpha\}\\
&= \{(a,b) \in \beta : (a,b) \in \alpha \cap V_0^2\}\\
&= \alpha|V_0
\end{aligned}$$

This implies that $\theta = \alpha$, since $(-|V_0)$ is 0-separating. Thus $(x,y) \in \beta - \alpha$ implies $(x,y) \in \beta - \theta$. By definition of θ, there is $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $(\varepsilon_0\psi(x), \varepsilon_0\psi(y)) \notin \alpha$. Thus $\varphi := \varepsilon_0\psi$ satisfies the conditions $\varphi[A] \subseteq V_0$ and $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$. Hence $\varphi[A] = V_0$ by (α,β)-minimality. □

Claim 2. *The relation β is the transitive closure of*

$$\alpha \cup \{(\psi(x), \psi(y)) : (x,y) \in \beta|V_0, \psi \in \mathrm{Pol}_1(\mathbf{A})\}.$$

Proof. The transitive closure of $\alpha \cup \{(\psi(x), \psi(y)) : (x,y) \in \beta|V_0, \psi \in \mathrm{Pol}_1(\mathbf{A})\}$ is $\alpha \vee \theta_{\mathbf{A}}(\beta|V_0)$. But $\alpha \vee \theta_{\mathbf{A}}(\beta|V_0) \in [\alpha,\beta]$, and therefore, since $(-|V_0)$ is 1-separating, $\beta = \alpha \vee \theta_{\mathbf{A}}(\beta|V_0)$. □

Assume that $U \in \mathrm{M}_{\mathbf{A}}(\alpha,\beta)$. Then, by definition, there is $\mu \in \mathrm{Pol}_1(\mathbf{A})$ such that $\mu[A] = U$ and $\mu(\beta) \nsubseteq \alpha$. This implies that the equivalence relation

$$\mu^{-1}(\alpha) = \{(a,b) : (\mu(a), \mu(b)) \in \alpha\}$$

is such that $\beta \nsubseteq \mu^{-1}(\alpha)$. Then by Claim 2 there are $a, b \in V_0$ and $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $(a,b) \in \beta$ and $(\mu\psi(a), \mu\psi(b)) \notin \alpha$. The function $\mu_1 := \mu\psi\varepsilon_0$ satisfies $\mu_1[A] \subseteq U$ and $\mu_1(\beta) \nsubseteq \alpha$: there are $x, y \in A$ such that $(a,b) = (\varepsilon_0(x), \varepsilon_0(y)) \in \beta$ but $(\mu_1(a), \mu_1(b)) = (\mu\psi\varepsilon_0(x), \mu\psi\varepsilon_0(y)) \notin \alpha$. Thus $\mu_1[A] = U$ by (α,β)-minimality. Observe that $\mu_1[V_0] = \mu_1\varepsilon_0[A] = \mu\psi\varepsilon_0^2[A] = \mu_1[A]$, so that $\mu_1[V_0] = U$. Apply Claim 1 to the pair $(\mu_1(a), \mu_1(b))$ to get $\nu \in \mathrm{Pol}_1(\mathbf{A})$ such that $\nu[A] = V_0$ and $(\nu\mu_1(a), \nu\mu_1(b)) \notin \alpha$. Now, since $\mu_1\nu[A] = \mu_1[V_0] = U$, $\mu_1\nu|U$ is bijective; since U is finite, there is $k > 1$ such that $(\mu_1\nu|U)^k = (\mu_1\nu)^k|U = 1_U$. Let $\varepsilon := (\mu_1\nu)^k$. We have: $\varepsilon[A] = (\mu_1\nu)^k[A] = (\mu_1\nu)^{k-1}[U] = U$ and, consequently, for all $a \in A$, $\varepsilon^2(a) = \varepsilon(a)$ since $\varepsilon(a) \in U$. Therefore $\varepsilon \in \mathrm{E}(\mathbf{A})$.

(2) Clearly, the map is well defined and preserves meets. For $\theta \in [\alpha|U, \beta|U]$, let

$$\hat{\theta} = \{(x,y) \in \beta : (\varepsilon\varphi(x), \varepsilon\varphi(y)) \in \theta \text{ for all } \varphi \in \mathrm{Pol}_1(\mathbf{A})\}$$

The relation $\hat{\theta}$ is an equivalence relation. If $(x,y) \in \hat{\theta}$ and $\psi \in \mathrm{Pol}_1(\mathbf{A})$, then $(\psi(x), \psi(y)) \in \hat{\theta}$ so that $\hat{\theta} \in [\alpha,\beta]$. missing

(4) In the above notation, let $\varphi := \nu$, $\psi := (\mu_1\nu)^{k-1}\mu_1$. Then $\varphi[U] = V_0, \psi[V_0] = U$ and $\psi\varphi|U = 1_U, \varphi\psi|V_0 = 1_{V_0}$. Then fact that any (α,β)-minimal set is polynomially isomorphic to V_0 implies that any two (α,β)-minimal sets are polynomially isomorphic.

(3) Let $\varphi \in \mathrm{Pol}_1(\mathbf{A}||U) = \{\psi|U : \psi \in \mathrm{Pol}_1(\mathbf{A}), \psi[U] \subseteq U\}$. We need to show that if $\varphi(\beta|U) \nsubseteq \alpha|U$, then $\varphi \in \mathrm{Sym}(U)$. If $\varphi(\beta|U) \nsubseteq \alpha|U$, then in particular $\psi(\beta) \nsubseteq \alpha$, so that, by (α,β)-minimality, $U \subseteq \psi[A]$. Let $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = U$ Now,

$\varphi(\beta|U) \nsubseteq \alpha|U$ is equivalent $\psi\varepsilon(\beta) \nsubseteq \alpha$, and $\psi\varepsilon[A] \subseteq U$. Then by (α,β)-minimality, $U = \psi\varepsilon[A] = \psi[U]$. □

Definition 5.6. Let (δ,θ) be tame in a finite algebra $\mathbf{A}$. An (δ,θ)-**trace*** of $\mathbf{A}$ is $N \subseteq A$ such that for some $U \in \mathrm{M}_{\mathbf{A}}(\delta,\theta)$ and $x \in U$, $N \subseteq U$ and $N = x/(\theta|U) \neq x/(\delta|U)$. That is, N is an $(\delta|U,\theta|U)$-trace of the minimal algebra $\mathbf{A}||U$.

Definition 5.7. Let (δ,θ) be tame in a finite algebra $\mathbf{A}$ and $U \in \mathrm{M}_{\mathbf{A}}(\delta,\theta)$. Let $\alpha,\theta \in \mathrm{Con}(\mathbf{A})$. Let K be a class of algebras. We define

(1) the **type** of (δ,θ), written $\mathrm{typ}(\delta,\theta)$, to be the type of $\mathbf{A}||U$ relative to $(\delta|U,\theta|U)$;

(2) $\mathrm{typ}\{\alpha,\beta\} := \{\mathrm{typ}(\delta,\theta) : \alpha \leq \delta \prec \theta \leq \beta\}$;

(3) $\mathrm{typ}\{\mathbf{A}\} := \mathrm{typ}\{\Delta_A,\nabla_A\}$;

(4) $\mathrm{typ}\{K\} := \cup\{\mathrm{typ}\{\mathbf{A}\} : \mathbf{A} \in K_{\mathrm{fin}}\}$.

Remark 5.8. If (Δ,∇) in $\mathrm{Con}(\mathbf{A})$ is tame, then $\mathrm{typ}(\Delta,\nabla)$ coincides with $\mathrm{typ}(\mathbf{A})$ of Definition 2.7.

Lemma 5.9. *Let (δ,θ) be tame in a finite algebra $\mathbf{A}$. For every (δ,θ)-trace* of $\mathbf{A}$, the algebra $(\mathbf{A}||N)/(\delta|N)$ is minimal and* $\mathrm{typ}(\delta,\theta) = \mathrm{typ}((\mathbf{A}||N)/(\delta|N))$.

Proof. Let U be any (δ,θ)-minimal set and N be an $(\delta|U,\theta|U)$-trace. The algebra $\mathbf{A}||U$ is minimal relative to $(\delta|U,\theta|U)$. By Lemma 5.3 $\mathbf{A}||N = (\mathbf{A}||U)||N$ and consequently $(\mathbf{A}||N)/(\delta|N) = ((\mathbf{A}||U)||N)/((\delta|U)|N)$. The type of $\mathbf{A}||U$ relative to $(\delta|U,\theta|U)$ is, by definition, the type of the minimal algebra $\mathbf{M} := ((\mathbf{A}||U)||N)/((\delta|U)|N)$; but M is the only (Δ_M,∇_M)-trace of $\mathbf{M}$, hence this is $\mathrm{typ}(\mathbf{M})$. □

6. Type One and Solvability

Definition 6.1. Let $\mathbf{A}$ be a finite algebra and $\delta,\theta \in \mathrm{Con}(\mathbf{A})$ with $\delta < \theta$. We say that (δ,θ) is

(1) **abelian** if for all $\varphi \in \mathrm{Pol}_{n+1}(\mathbf{A})$, $(u,v),(a_1,b_1),\ldots,(a_n,b_n) \in \theta$

$$\varphi(u,a_1,\ldots,a_n)\delta\varphi(u,b_1,\ldots,b_n) \iff \varphi(v,a_1,\ldots,a_n)\delta\varphi(v,b_1,\ldots,b_n)$$

(2) **strongly abelian**[2] if for all $\varphi \in \mathrm{Pol}_{n+1}(\mathbf{A})$, $(a_0,b_0) \in \theta$, $a_i\theta b_i\theta c_i$

$$\varphi(a_0,a_1,\ldots,a_n)\delta\varphi(b_0,b_1,\ldots,b_n) \implies \varphi(a_0,c_1,\ldots,c_n)\delta\varphi(b_0,c_1,\ldots,c_n)$$

(3) **(strongly) solvable** if there is a finite chain $\delta = \gamma_0 \leq \cdots \leq \gamma_{n+1} = \theta$ with (γ_i,γ_{i+1}) (strongly) abelian.

[2] "Strongly abelian prime quotients do not occur in most 'normal' algebras. For example, they do not occur in groups, rings, modules, or lattices. [...] The concept is important in our theory, but primarily in the negative sense of a bad example we wish to exclude." [1, p. 44]

Remark 6.2. When (Δ_A, ∇_A) is abelian if for all $\varphi \in \mathrm{Pol}_{n+1}(\mathbf{A})$, $u, v, a_i, b_i \in A$

$$\varphi(u, a_1, \ldots, a_n) = \varphi(u, b_1, \ldots, b_n) \iff \varphi(v, a_1, \ldots, a_n) = \varphi(v, b_1, \ldots, b_n) \tag{1}$$

Let the **center** $\mathrm{Z}(\mathbf{A})$ of $\mathbf{A}$ be the set of pairs $(u, v) \in A^2$ such that Equation (1) holds. Then $\mathrm{Z}(\mathbf{A})$ is a congruence. Then we could say that $\mathbf{A}$ is abelian if (Δ_A, ∇_A) is abelian, or equivalently, if $\mathrm{Z}(\mathbf{A}) = \nabla_A$. By a classical result, $\mathbf{A}$ is abelian iff it is polynomially equivalent to a left $\mathbf{R}$-module, for some ring $\mathbf{R}$.

Remark 6.3. If (δ, θ) is strongly abelian, then it is abelian.

Theorem 6.4. *Let* $\mathbf{A}$ *be a finite algebra. A pair of tame congruences* (δ, θ) *has type* $\mathbf{1}$ *iff it is strongly abelian.*

Definition 6.5. Let $\mathbf{A}$ be a finite algebra. A 1-**snag** is a pair (a, b) of distinct elements of A such that for some $\varphi \in \mathrm{Pol}_2(\mathbf{A})$

$$\varphi(a, b) = \varphi(b, a) = a \quad \varphi(b, b) = b.$$

A 2-**snag** is a pair (a, b) of distinct elements of A such that for some $\varphi \in \mathrm{Pol}_2(\mathbf{A})$

$$\varphi(a, b) = \varphi(b, a) = \varphi(a, a) = a \quad \varphi(b, b) = b.$$

We denote by $\mathrm{Sn}_1(\mathbf{A})$ and $\mathrm{Sn}_2(\mathbf{A})$ the set of 1-snags and 2-snags, respectively.

Remark 6.6. If (a, b) is a 2-snag as witnessed by φ, then $\{a, b\}$ is closed under φ and it is a semilattice.

Lemma 6.7. *Let* $\mathbf{A}$ *be a finite algebra. The following are equivalent:*

1. (δ, θ) *is strongly solvable;*
2. $\mathrm{Sn}_1(\mathbf{A}) \cap (\theta - \delta) = \varnothing$*;*
3. *for all* $\delta \leq \alpha \prec \beta \leq \theta$*,* (α, β) *is strongly abelian;*
4. $\mathrm{typ}\{\delta, \theta\} = \mathbf{1}$.

Proof. The equivalence '(3) $\implies$ (4)' is the content of Theorem 6.4.

'(1) $\implies$ (2)': if (δ, θ) is strongly solvable, there is a chain □

7. Congruence Lattice Conditions for Omitting Type One

Definition 7.1. A lattice $\mathbf{L}$ is **meet semi-distributive** if it satisfies

$$a \wedge b = a \wedge c \implies a \wedge b = a \wedge (b \vee c) \tag{SD($\wedge$)}$$

for all $a, b, c \in L$. A lattice $\mathbf{L}$ is **join semi-distributive** if it satisfies $\mathrm{SD}(\vee)$.

The smallest lattice satisfying $\mathrm{SD}(\vee)$ but not $\mathrm{SD}(\wedge)$ is called $\mathbf{D}_1$ and it is depicted in Figure 7.

Lemma 7.2. *Let* $\mathbf{A}$ *be a finite algebra. Suppose that there are* $\delta_1, \delta_2, \delta_3 \in \mathrm{Con}(\mathbf{A})$ *such that* $\mathrm{Con}(\mathbf{A})$ *contains an isomorphic copy of* $\mathbf{D}_1$*, like the fugure below. If* $0_{\mathbf{D}_1} \prec \alpha \leq \delta_2$*, then* $\mathrm{typ}(0_{\mathbf{D}_1}, \alpha) = \mathbf{1}$.

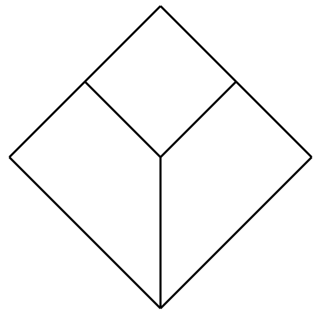
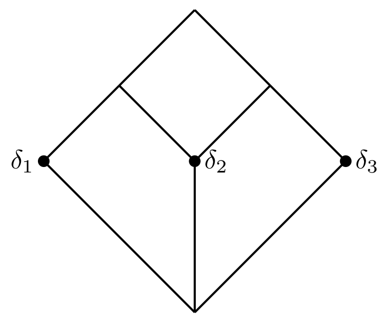


FIGURE 1. The lattice $\mathbf{D}_1$.



Definition 7.3. Let $\mathbf{A}$ be a finite algebra. For $\gamma, \delta \in \mathrm{Con}(\mathbf{A})$ we let

$$\gamma \approx \delta \iff \gamma \cap \mathrm{Sn}_1(\mathbf{A}) = \delta \cap \mathrm{Sn}_1(\mathbf{A})$$
$$\gamma \sim \delta \iff \gamma \cap \mathrm{Sn}_2(\mathbf{A}) = \delta \cap \mathrm{Sn}_2(\mathbf{A})$$

Theorem 7.4. *Let $\mathbf{A}$ be a finite algebra. The relations $\sim$ and $\approx$ are congruences of $\mathbf{L} := \mathrm{Con}(\mathbf{A})$. The quotient lattice $\mathbf{L}/\sim$ is meet semi-distributive.*

Definition 7.5. Let K be a class of algebras. We define $\mathrm{Con}(K) := \{\mathrm{Con}(\mathbf{A}) : \mathbf{A} \in K\}$.

Theorem 7.6. *Let V be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) $\mathbf{D}_1 \notin IS(\mathrm{Con}(\mathsf{V}))$*;*

(3) *for every $\mathbf{A} \in \mathsf{V}$ there is a congruence θ of $\mathbf{L} := \mathrm{Con}(\mathbf{A})$ such that $\mathbf{L}/\theta$ is meet semi-distributive and for all $a \in L$, a/θ is modular;*

(4) *for every $\mathbf{A} \in \mathsf{V}$, if $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ are such that $\alpha \sim \beta$, then $\alpha \circ \beta = \beta \circ \alpha$;*

(5) *there is an idempotent term t such that for every $\mathbf{A} \in \mathsf{V}$, if $\Delta_A \sim \theta$ in $\mathrm{Con}(\mathbf{A})$, then*
$$t^{\mathbf{A}}(a, b, b) = a \quad t^{\mathbf{A}}(a, a, b) = b$$
for all $(a, b) \in \theta$, i.e. t is a Mal'cev term on the θ-equivalence classes.

8. Syntactic Conditions for Omitting Type One

Definition 8.1. Let V be a variety. An algebra $\mathbf{A} \in \mathsf{V}$ is called

(1) **free** if there is an isomorphism $\mathbf{A} \simeq \mathbf{F}_{\mathsf{V}}(\kappa)$ for some cardinal κ;

(2) **finitely generated** if there is a surjective homomorphism $\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}$ for some $n \in \omega$.

Definition 8.2. A variety V is called

(1) **locally finite** if all its finitely generated algebras are finite;

(2) **finitely presented** if V has a finite set of function symbols and $\mathsf{V} = \mathrm{Alg}(\Sigma)$ for a finite set of equations Σ;

(3) **finitely generated** if $\mathsf{V} = V(\mathbf{A}_1, \dots, \mathbf{A}_n)$ for $\mathbf{A}_1, \dots, \mathbf{A}_n$ finite similar algebras;

(4) **linear** if there is a set of equations defining V containing at most one function symbol per side.

Remark 8.3. Observe that $V(\mathbf{A}_1, \dots, \mathbf{A}_n) = V(\mathbf{A}_1 \times \dots \times \mathbf{A}_n)$.

Lemma 8.4. *Let V be a variety. If V is finitely generated then it is locally finite.*

Proof. By the previuos remark, we can assume that $\mathsf{V} = V(\mathbf{A})$ for some $\mathbf{A}$ finite. Let $n < \omega$. We prove that $\mathbf{F}_{\mathsf{V}}(n)$ is finite. Consider the homomorphism

$$\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}^{A^n}, \quad t(x_1, \dots, x_n) \mapsto t^{\mathbf{A}}$$

This homomorphism is injective: if $t^{\mathbf{A}} = s^{\mathbf{A}}$, then $\mathbf{A} \models t \equiv s$, i.e. $t = s$ in $\mathbf{F}_{\mathsf{V}}(n)$. Thus $\mathbf{F}_{\mathsf{V}}(n)$ is finite. □

Definition 8.5. Let V and W be two varieties. We say that V is **interpretable** into W ($\mathsf{V} \leq \mathsf{W}$) if there is a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$.

Remark 8.6. Let W, V be two varieties. We unravel what $\mathsf{V} \leq \mathsf{W}$ means in a simple case, that is when V is finitely presented. Let F be a finite set of function symbols. Let V be a variety of algebras over F, defined by the equations

$$s_1 \equiv t_1, \dots, s_k \equiv t_k. \tag{2}$$

Assume that for each $f \in F_n$ there is $t \in \mathrm{Clo}_n(\mathsf{W})$ such that the interpetation of the t's satisfy the equations (2). Then the assignment $f \mapsto t$ extends to a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$. Of course, the converse also holds; thus this is equivalent to $\mathsf{V} \leq \mathsf{W}$.

Lemma 8.7. *Let $\mathbf{M}$ be finite minimal algebra of type $\mathbf{1}$. Let $\mathbf{A} = (M, \mathrm{Pol}(\mathbf{M}))$. Then $V(\mathbf{A})$ contains a finite algebra $\mathbf{S}$ all of whose polynomials are constant or projections.*

Proof. Let $\mathbf{G} := \mathrm{Sym}(M) \cap \mathrm{Pol}_1(\mathbf{M})$, subgroup of $\mathrm{Sym}(M)$. Let $u, v \in M$, $u \neq v$ and let

$$D := \{(\sigma(u), \sigma(v)) : \sigma \in \mathbf{G}\} \cup \{(\sigma(v), \sigma(u)) : \sigma \in \mathbf{G}\}$$

Since $\mathbf{M}$ is minimal with $\mathrm{typ}(\mathbf{M}) = \mathbf{1}$, every polynomial ψ is constant or there is i, $\sigma \in \mathbf{G}$ such that

$$\psi(a_1, \dots, a_n) = \sigma(a_i).$$

This implies that $\mathbf{D}$ is a subalgebra of $\mathbf{A}^2$. Let

$$((x_1, x_2), (y_1, y_2)) \in \theta \iff \sigma(x_i) = y_i \text{ for some } \sigma \in \mathbf{G}$$

We show that every term operation of $\mathbf{D}/\theta$ is either constant or a projection. Let $\psi \in \mathrm{Pol}_n(\mathbf{M})$ non constant and $(a_i, b_i) \in D$ for $i = 1, \dots, n$. Then there is $\tau \in \mathbf{G}$ such that

$$\begin{aligned} \psi((a_1, b_1)/\theta, \dots, (a_n, b_n)/\theta) &= \psi((a_1, b_1), \dots, (a_n, b_n))/\theta \\ &= (\psi(a_1, \dots, a_n), \psi(b_1, \dots, b_n))/\theta \\ &= (\tau(a_i), \tau(b_i))/\theta \\ &= (a_i, b_i)/\theta. \end{aligned}$$

□

Lemma 8.8. *Let* W, V *be two varieties such that* $\mathsf{W} \leq \mathsf{V}$. *Assume that* W *is*

- *idempotent;*
- *finitely presented;*
- *linear.*

Let $\mathbf{A} \in \mathsf{V}$, $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$, $\beta \in \mathrm{Con}(\mathbf{A})$ *and* $N := a/\beta \cap U$ *for* $a \in U$. *Then* $\mathsf{W} \leq V(\mathbf{A}||N)$. *Moreover, if* $\mathbf{1} \in \mathrm{typ}\{\mathsf{V}\}$, *then* $\mathsf{W} \leq \mathsf{Set}$.

Proof. By assumption W can be described by a finite set of equations of the form

$$f_i(x_{i_1}, \dots, x_{i_h}) \equiv f_j(x_{j_1}, \dots, x_{j_k}) \tag{3}$$

where f_i and f_j are members of a finite set F of function symbols. Since $\mathsf{W} \leq \mathsf{V}$, there is an assignment $f \mapsto t$ extending to a clone homomorphism. We need to find a clone homomorphism $\mathrm{Clo}(\mathsf{W}) \to \mathrm{Clo}(\mathbf{A}||N)$. Consider $f \mapsto \varphi := \varepsilon t^{\mathbf{A}}|N$. Firstly, it is well defined: if $(a_1, \dots, a_n) \in N$, $(\varphi(a_1, \dots, a_n), \varphi(a, \dots, a)) \in \beta$ but

$$\varphi(a, \dots, a) = \varepsilon t^{\mathbf{A}}(a, \dots, a) = \varepsilon(a) = a \in U$$

so that $\varphi(a_1, \dots, a_n) \in N$ and therefore $\varphi \in \mathrm{Pol}(\mathbf{A})|N$. Finally, using that $\mathsf{W} \leq \mathsf{V}$, for every $a_{i_1}, \dots, a_{i_h}, a_{j_1}, \dots, a_{j_k} \in N$

$$\begin{aligned} \varphi_i(a_{i_1}, \dots, a_{i_h}) &= \varepsilon t_i^{\mathbf{A}}(a_{i_1}, \dots, a_{i_h}) \\ &= \varepsilon t_j^{\mathbf{A}}(a_{j_1}, \dots, a_{j_k}) \\ &= \varphi_j(a_{j_1}, \dots, a_{j_k}). \end{aligned}$$

If $\mathbf{1} \in \mathrm{typ}\{\mathsf{V}\}$, then there is $\mathbf{A} \in \mathsf{V}$ and $\alpha \prec \beta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{typ}(\alpha, \beta) = \mathbf{1}$. Without loss of generality we can assume that $\alpha = \Delta_A$. Let N be a (Δ_A, β)-trace*. Then there are $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$ such that $N = a/\beta \cap U$ for some $a \in U$. Thus $\mathsf{W} \leq V(\mathbf{A}||N)$. The algebra $\mathbf{A}||N$ is minimal of type $\mathrm{typ}(\Delta_A, \beta) = \mathbf{1}$. Hence by

Lemma 8.7 there is $\mathbf{S} \in V(\mathbf{A}||N)$ such that every term operation of $\mathbf{S}$ is constant or a projection. Then there is a clone homomorphism $\mathrm{Clo}(\mathbf{A}||N) \to \mathrm{Clo}(\mathbf{S})$. Since $\mathsf{W} \leq V(\mathbf{A}||N)$ there is a clone homomorphism $\mathrm{Clo}(\mathsf{W}) \to \mathrm{Clo}(\mathbf{A}||N)$. Thus we get a clone homomorphism $\mathrm{Clo}(\mathsf{W}) \to \mathrm{Clo}(\mathbf{S})$. But for every $f \in F$, W satisfies $f(x, \ldots, x) \equiv x$, hence the image of f through this clone homomorphism cannot be but a projection. This implies that $\mathsf{W} \leq \mathsf{Set}$. ☐

Lemma 8.9. *Let* V *be an idempotent variety over the set of function symbols* F*. Then the following are equivalent:*

(1) $\mathsf{V} \not\leq \mathsf{Set}$*;*

(2) *there is an idempotent, finitely presented, linear variety* W *such that* $\mathsf{W} \leq \mathsf{V}$ *but* $\mathsf{W} \not\leq \mathsf{Set}$*;*

(3) *there is a term* $t(x_1, \ldots, x_n)$ *for* $n > 0$ *such that* V *satisfies the equations*

$$(\star)\qquad \begin{aligned} t(x_{11}, \ldots, x_{1n}) &\equiv t(y_{11}, \ldots, y_{1n}) \\ &\vdots \\ t(x_{n1}, \ldots, x_{nn}) &\equiv t(y_{n1}, \ldots, y_{nn}) \end{aligned}$$

with $x_{ii} \neq y_{ii}$.

Proof. Firstly, we prove that (3) implies (2). Let W be the variety over $\{t\}$ defined by the equations ($\star$). Then W is idempotent, finitely presented and linear. Clearly, $\mathsf{W} \leq \mathsf{V}$ but $\mathsf{W} \not\leq \mathsf{Set}$.

The implication "(2) $\Longrightarrow$ (1)" is obvious.

Finally, we prove that (1) implies (3). If $\mathsf{V} \not\leq \mathsf{Set}$, since $\mathrm{Clo}_n(\mathsf{Set})$ is finite for every n, we can assume without loss of generality that V is finitely presented. As shown in Example 8.10, there is $n > 0$ and $h \in \mathrm{Clo}_n(\mathsf{V})$ such that, defining

$$(4)\qquad l(x_1, \ldots, x_{n^2}) \equiv h(h(x_1, \ldots, x_n), \ldots, h(x_{n^2-n+1}, \ldots, x_{n^2}))$$

it allows to rewrite the equations defining V as linear equations in $\{l\}$. Let Σ be this set of equations plus

$$(5)\qquad l(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n) \equiv l(x_1, \ldots, x_n, \ldots, x_1, \ldots, x_n)$$

Let Σ' be $\Sigma \cup \{h(x, \ldots, x) \equiv x\}$ plus Equation (4). We claim the the term l and the equations of Σ give the thesis. If that were not true, there is $m < n^2$, $m = i \cdot n + (j + 1)$, such that in every equation of Σ, the same variable appears at the m-th place on both side. Since Equation 5 belongs to Σ, $i = j$. If we define $h(x_1, \ldots, x_n) := x_i$, $l(x_1, \ldots, x_{n^2}) := x_m$, then Set satisfies Σ'. But $\mathsf{V} \not\leq \mathsf{Set}$ and if W satisfies Σ', $\mathsf{V} \leq \mathsf{W}$. Contradiction. ☐

Example 8.10. Consider the variety of $\vee$-semilattices. It is defined by the equations

$$\begin{aligned} f(x, x) &\equiv x \\ f(x, y) &\equiv f(y, x) \\ f(x, f(y, z)) &\equiv f(f(x, y), z) \end{aligned}$$

We expand the set of function symbols $\{f\}$ to $\{f, p_1, p_2, p_3 f_1, f_2\}$ and the set of equations to

$$\begin{aligned} f(x,x) &\equiv x \\ f(x,y) &\equiv f(y,x) \\ f_1(x,y,z) &\equiv f(p_2(x,y,z), p_3(x,y,z)) \\ f_2(x,y,z) &\equiv f(p_1(x,y,z), p_2(x,y,z)) \end{aligned}$$

Let $k(x_1,\dots,x_6) := f(f_1(x_1,x_2,x_3), f_1(x_4,x_5,x_6))$ and

$$h(x_1,\dots,x_{18}) := f_2(k(x_1,\dots,x_6), k(x_7,\dots,x_{12}), k(x_{13},\dots,x_{18})).$$

It is easy to see that each element of $\{f, p_1, p_2, p_3 f_1, f_2\}$ can be obtained by h identifying variables (f is idempontent!):

$$\begin{aligned} h(x_1\dots,x_1,x_2,\dots,x_2,x_3\dots,x_3) &\equiv f_2(x_1,x_2,x_3) \\ h(x_1,\dots,x_6,\dots,x_1,\dots,x_6) &\equiv k(x_1,\dots,x_6) \\ k(x_1,x_2,x_3,x_1,x_2,x_3) &\equiv f_1(x_1,x_2,x_3) \\ k(x_1,x_1,x_1,x_2,x_2,x_2) &\equiv f(x_1,x_2). \end{aligned}$$

Theorem 8.11. *Let V be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) *there is an idempotent variety* W *such that* $\mathsf{W} \leq \mathsf{V}$ *and* $\mathsf{W} \not\leq \mathsf{Set}$.

(3) *there is $m > 0$ such that for every $\mathbf{A} \in \mathsf{V}$, $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$*

$$\alpha \wedge (\beta \circ \gamma) \leq \gamma_m \circ \beta_m$$

where

$$\begin{cases} (\beta_0, \gamma_0) = (\beta, \gamma) \\ (\beta_{n+1}, \gamma_{n+1}) = (\beta \vee (\alpha \wedge \gamma_n), \gamma \vee (\alpha \wedge \beta_n)) \end{cases}$$

Proof. (2) $\implies$ (1): if there is W idempontent such that $\mathsf{W} \not\leq \mathsf{Set}$, then there is W' idempotent, finitely presented, linear such that $\mathsf{W}' \leq \mathsf{W}$, $\mathsf{W}' \not\leq \mathsf{Set}$ by Lemma 8.9. Assume that $\mathbf{1} \in \mathrm{typ}\{\mathsf{V}\}$, then by Lemma 8.8, $\mathsf{W}' \leq \mathsf{Set}$. Absurd.

(1) $\implies$ (3): consider the algebra $\mathbf{F}_{\mathsf{V}}(x,y,z) \in \mathsf{V}$. Let $\alpha := \theta(x,z), \beta := \theta(x,y), \gamma := \theta(y,z)$ and $(\beta_n), (\gamma_n)$ as above. By induction, the two sequences $(\beta_n), (\gamma_n)$ are increasing. Since $\mathbf{F}_{\mathsf{V}}(x,y,z)$ is finite, there is $m > 0$ such that $\beta_m = \beta_{m+1}$, $\gamma_m = \gamma_{m+1}$. Then

$$\begin{aligned} \alpha \wedge \gamma_m \leq \beta \vee (\alpha \wedge \gamma_m) = \beta_m \\ \alpha \wedge \beta_m \leq \beta \vee (\alpha \wedge \beta_m) = \gamma_m \end{aligned}$$

so that $\alpha \wedge \beta_m = \alpha \wedge \gamma_m$. By Lemma 7.4, $\mathrm{Con}(\mathbf{F}_{\mathsf{V}}(x,y,x))/\sim$ is meet semi-distributive, so that

$$\alpha \wedge \beta_m \sim \alpha \wedge (\beta_m \vee \gamma_m).$$

Claim 3. $\gamma_m \sim \beta_m$

By Theorem 7.6, this implies that $\gamma_m \circ \beta_m = \beta_m \circ \gamma_m$. Since $(x, z) \in \beta \circ \gamma \leq \beta_m \circ \gamma_m$, $(x, z) \in \gamma_m \circ \beta_m$. Now, let $\mathbf{A} \in \mathsf{V}$. Let $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ and $(a, c) \in \alpha \wedge (\beta \circ \gamma)$. Let $b \in A$ such that $(a, b) \in \beta, (b, c) \in \gamma$. Let $f : \mathbf{F}_{\mathsf{V}}(x, y, z) \to \mathbf{A}$ be the homomorphism

$$x \mapsto a, y \mapsto b, z \mapsto c.$$

Now, the function $\theta \mapsto f^{-1}(\theta)$ is an isomorphism of lattices

$$\mathrm{Con}(\mathbf{A}) \simeq [f^{-1}(\Delta_A), \nabla_A].$$

Consequently, as $\theta(x, z) \subseteq f^{-1}(\alpha)$, $\theta(x, y) \subseteq f^{-1}(\beta)$, $\theta(y, z) \subseteq f^{-1}(\gamma)$, by induction $\theta(x, z)_m \subseteq f^{-1}(\alpha_m)$, $\theta(x, y)_m \subseteq f^{-1}(\beta_m)$, $\theta(y, z)_m \subseteq f^{-1}(\gamma_m)$. Then $f(\theta(y, z)_m \circ \theta(x, y)_m) \subseteq \gamma_m \circ \beta_m$ and therefore $(a, c) \in \gamma_m \circ \beta_m$.

(3) $\implies$ (2): assume that there is $m > 0$ such that if $\mathbf{A} = \mathbf{F}_{\mathsf{V}}(x, y, z)$, $\alpha = \theta(x, z), \beta = \theta(x, y), \gamma = \theta(y, z)$, and $(x, z) \in \alpha \wedge (\beta \circ \gamma)$, then $(x, z) \in \gamma_m \circ \beta_m$.

Claim 4. *That $(x, z) \in \gamma_m \circ \beta_m$ implies: there are terms*

$$t_1(x, y, z), \ldots, t_k(x, y, z) \in \mathbf{F}_{\mathsf{V}}(x, y, z)$$

such that V satisfies a finite number of equations of the kind $t_i \equiv t_j$ for some $i, j \leq k$.

Proof. We prove the claim when $m = 1$. Assume that $x, z \in \mathbf{F}_{\mathsf{V}}(x, y, z)$ are such that $(x, z) \in (\gamma \vee (\alpha \wedge \beta)) \circ (\beta \vee (\alpha \wedge \gamma))$. Then there are terms

$$d_1(x, y, z), \ldots, d_n(x, y, z), p(x, y, z), e_1(x, y, z), \ldots, e_n(x, y, z) \in \mathbf{F}_{\mathsf{V}}(x, y, z)$$

such that V satisfies

$$\begin{array}{rlrl}
x \equiv & d_1(x, y, z) & & \\
d_i(x, y, y) \equiv & d_{i+1}(x, y, y) & & \text{for } i \text{ odd} \\
d_i(x, y, x) \equiv & d_{i+1}(x, y, x) & d_i(x, x, y) \equiv d_{i+1}(x, x, y) & \text{for } i \text{ even} \\
d_n(x, y, y) \equiv & p(x, y, y) & & \\
p(x, x, y) \equiv & e_1(x, x, y) & & \\
e_i(x, y, x) \equiv & d_{i+1}(x, y, x) & e_i(x, y, y) \equiv e_{i+1}(x, y, y) & \text{for } i \text{ odd} \\
& & e_i(x, x, y) \equiv e_{i+1}(x, x, y) & \text{for } i \text{ even} \\
& & e_n(x, y, z) \equiv z & \square
\end{array}$$

Hence there is a idempontent, finitely presented, linear variety E such that $\mathsf{E} \leq \mathsf{V}$ and whenever $\mathsf{E} \leq \mathsf{W}$, W satisfies a set of equations similar to those above. However, Set does not satisfy equations of this kind; hence there is an idempontent variety E such that $\mathsf{E} \leq \mathsf{V}$ with $\mathsf{E} \not\leq \mathsf{Set}$. $\square$

Corollary 8.12. *Let $\mathbf{A}$ be a finite idempotent algebra. There is $\mathbf{B} \in HS(\mathbf{A})$ such that $\mathrm{Clo}(\mathbf{B}) \simeq \mathbf{N}$ iff $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$.*

Proof. If $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$, then $\mathbf{1} \in \mathrm{typ}\{V(\mathbf{A})\}$. Since $\mathbf{A}$ is finite, then, by Lemma 8.4 $V(\mathbf{A})$ is locally finite, and therefore, by Theorem 8.11, for every idempotent variety W, either $\mathsf{W} \not\leq V(\mathbf{A})$ or $\mathsf{W} \leq \mathsf{Set}$. In particular, since $\mathbf{A}$ is idempotent,

$V(\mathbf{A}) \leq \mathsf{Set}$. This means that there is a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathbf{N}$. Therefore, there is $\mathbf{S} \in \mathsf{Set}$ such that $\mathbf{S} \in V(\mathbf{A})$. missing

Conversely, let $\mathbf{B} \in HS(\mathbf{A})$ such that $\mathrm{Clo}(\mathbf{B}) \simeq \mathbf{N}$; this means that $\mathbf{B} \in \mathsf{Set}$ and therefore $\mathbf{B}$ is minimal of type $\mathbf{1}$. □

Definition 8.13. Let $\mathbf{A}$ be an algebra and V be a variety. Let $t = t(x_1, \ldots, x_n)$ with $n > 0$. We say that t is a

(1) **Taylor** term

(2) **weak near-unanimity** term

for $\mathbf{A}$ (or V) if $\mathbf{A}$ (or V) satisfies

(1) $t(x_1, \ldots, x_n) \equiv t(y_1, \ldots, y_n)$ with $x_i, y_i \in \{x, y\}$ and $x_i \neq y_i$;

(2) $t(y, x, \ldots, x) \equiv t(x, y, x, \ldots, x) \equiv \cdots \equiv t(x, \ldots, x, y)$

respectively.

Theorem 8.14. *Let V be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) V *has an n-ary Taylor idempotent term for some $n > 1$.*

Theorem 8.15 ([2])**.** *Let V be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) V *has an n-ary weak near-unanimity idempotent term for some $n > 1$.*

Corollary 8.16. *Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathrm{Clo}(\mathbf{A})$ contains a weak near-unanimity operation iff $\mathbf{1} \notin \mathrm{typ}\{HS(\mathbf{A})\}$.*

Theorem 8.17 ([4])**.** *Let V be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) V *has an idempotent 6-ary term t such that V satisfies*

$$t(x,x,x,x,y,y) \equiv t(x,y,x,y,x,x),\ t(y,y,x,x,x,x) \equiv t(x,x,y,x,y,x)$$

References

[1] Hobby, D., McKenzie, R. (1988). *The structure of Finite Algebras.* Contemporary Mathematics 76, American Mathematical Society.

[2] Maróti, M., McKenzie, R. (2008). Existence theorems for weakly symmetric operations. *Algebra Universalis* 59, 463-489.

[3] Pálfy, P. P. (1984). Unary polynomials in algebras I. *Algebra Universalis* 18, 262-273.

[4] Siggers, M. H. (2010). A strong Mal'cev condition for locally finite varieties omitting the unary type. *Algebra Universalis* 64, 15-20.