# MINIMAL ALGEBRAS

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Abstract. In this note

## 1. Preliminaries

**Definition 1.1.** Let $F$ be a set of function symbols and $\mathbf{A}$ be an algebra over $F$. We denote by $\mathrm{Pol}(\mathbf{A})$ the smallest set containing

(1) $\{f^{\mathbf{A}} : f \in F\}$;

(2) $\{\pi_i^n : A^n \to A, 1 \leq i \leq n, n \in \omega\}$;

(3) the constant 0-ary operations

and closed under composition. The elements of $\mathrm{Pol}(\mathbf{A})$ are called **polynomial** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **polynomial equivalent** if $\mathrm{Pol}(\mathbf{A}) = \mathrm{Pol}(\mathbf{B})$.

*Example* 1.2. If $\varphi \in \mathrm{Clo}_{m+n}(\mathbf{A})$ and $(a_1, \ldots, a_m) \in A^m$, then

$$\psi : A^n \to A \quad (b_1, \ldots, b_n) \mapsto \varphi(a_1, \ldots, a_m, b_1, \ldots, b_n)$$

is a polynomial operation.

## 2. Finite Minimal Algebras

**Definition 2.1.** A nontrivial finite algebra $\mathbf{A}$ is **minimal** iff every noncostant element of $\mathrm{Pol}_1(\mathbf{A})$ is bijective.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

*Example* 2.2. The following are examples of minimal algebras.

(1) any algebra with carrier 2;

(2) a nontrivial finite vector space $\mathbf{A}$ over a finite field $\mathbf{k}$: every $\pi \in \mathrm{Pol}_1(\mathbf{A})$ is of the form $\pi(v) = av + b$ for some $a \in k$, $b \in A$;

(3) a group of permutations acting on a finite set[1].

We shall prove that, up to polynomial equivalence, there are no other finite minimal algebras.

[1] Let $\mathbf{G}$ be a group acting on a set $A$. Each $g \in G$ induces an operation $\varphi_g : A \to A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. A $\mathbf{G}$-set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$.

**Lemma 2.3.** *Let* $\mathbf{A}$ *be a minimal algebra. If every element of* $\mathrm{Pol}(\mathbf{A})$ *is essentially unary, then* $\mathbf{A}$ *is polynomial equivalent to* $(A, \Phi_{\mathbf{G}})$ *where* $\mathbf{G}$ *is a finite group acting on* $A$*.*

*Proof.* Since $\mathbf{A}$ is minimal, $\mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$ is a subgroup of $\mathrm{Sym}(A)$. Let $\mathbf{G} := \mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$. If $\psi \in \mathrm{Pol}(\mathbf{A})$, either $\psi$ is constant or $\psi$ is essentially unary, hence $(A, \Phi_{\mathbf{G}})$ is polynomially equivalent to $\mathbf{A}$. □

**Theorem 2.4** ([3])**.** *Let* $\mathbf{A}$ *be a minimal algebra with* $|A| > 2$*. If* $\mathrm{Pol}(\mathbf{A})$ *contains an operation which is not essentially unary, then* $\mathbf{A}$ *is polynomially equivalent to a* $\mathbf{k}$*-vector space for a finite field* $\mathbf{k}$*.*

**Theorem 2.5.** *Every algebra* $\mathbf{A}$ *with carrier* $2$ *is polynomially equivalent to one of the following:*

(1) $\mathbf{E}_0 = (2, \varnothing)$;

(2) $\mathbf{E}_1 = (2, \neg)$;

(3) $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$;

(4) $\mathbf{E}_4 = (2, \wedge, \vee)$;

(5) $\mathbf{E}_5 = (2, \vee)$;

(6) $\mathbf{E}_6 = (2, \wedge)$.

*Each of them is not polynomially equivalent to the other*[2]*.*

*Remark* 2.6. Up to isomorphism, $\mathbf{E}_5 (\simeq \mathbf{E}_6)$ is the only semilattice with two elements, while $\mathbf{E}_3$ and $\mathbf{E}_4$ are the only Boolean algebra and lattice, respectively, with two elements.

**Definition 2.7.** Let $\mathbf{A}$ be a minimal algebra. We say that $\mathbf{A}$ is of

(1) **type 1** (or **unary**) if $\mathbf{A}$ is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ for some $\mathbf{G} \leq \mathrm{Sym}(A)$;

(2) **type 2** (or **affine**) if $\mathbf{A}$ is polynomially equivalent to a vector space over a finite field $\mathbf{k}$;

(3) **type 3** (or **Boolean**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_3$;

(4) **type 4** (or **lattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_4$;

(5) **type 5** (or **semilattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_5$.

[2]A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem 2.5 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

## 3. Relative Minimal Algebras

**Definition 3.1.** Let $\mathbf{A}$ be a finite algebra and let $\delta < \theta \in \mathrm{Con}(\mathbf{A})$. We say that $\mathbf{A}$ is $(\delta, \theta)$-**minimal** if for all $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ either $\varepsilon$ is bijective or $\varepsilon(\theta) \subseteq \delta$.

*Remark* 3.2. Observe that $\mathbf{A}$ is minimal iff $\mathbf{A}$ is $(\Delta, \nabla)$-minimal.

**Definition 3.3.** Let $\mathbf{A}$ be a $(\alpha, \beta)$-minimal algebra. An $(\alpha, \beta)$-**trace** of $\mathbf{A}$ is a $\beta$-equivalence class which contains at least two $\delta$-equivalence classes.

**Lemma 3.4.** *Let $\mathbf{A}$ be a finite $(\delta, \theta)$-minimal algebra and let $N$ be a $(\delta, \theta)$-trace. Then the algebra $(\mathbf{A}||N)/(\delta/N)$ is minimal.*

**Definition 3.5.** Let $\mathbf{A}$ be a finite algebra and $B, C \subseteq A$. We say that $B, C$ are **polynomial isomorphic** $(B \sim C)$

## 4. Tame Congruences

**Definition 4.1.** Let $\mathbf{A}$ be a finite algebra and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha < \beta$. We denote by

(1) $\mathrm{E}(\mathbf{A})$ the set of $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varepsilon^2 = \varepsilon$;

(2) $\mathrm{U}_{\mathbf{A}}(\alpha, \beta)$ the set $\{\varphi[A] : \varphi \in \mathrm{Pol}_1(\mathbf{A}), \varphi(\beta) \nsubseteq \alpha\}$;

(3) $\mathrm{M}_{\mathbf{A}}(\alpha, \beta)$ the set of minimal elements of $\mathrm{U}_{\mathbf{A}}(\alpha, \beta)$.

*Remark* 4.2. Let $\mathbf{A}$ be an algebra. An equivalence relation $\alpha$ is a congruence of $\mathbf{A}$ iff $\varphi(\alpha) \subseteq \alpha$ for every $\varphi \in \mathrm{Pol}_1(\mathbf{A})$.

Let $\mathbf{A}$ be a finite algebra. In this section we adopt the following convention, concerning the restriction $(-|U)$ operation, for $U \subseteq A$:

- if $\theta \in \mathrm{Con}(\mathbf{A})$, $\theta|U := \theta \cap U^2$;
- if $\varphi : A^n \to A$, $\varphi|U$ is the function $U^n \to A$, $(u_1, \ldots, u_n) \mapsto \varphi(u_1, \ldots, u_n)$;
- $\mathrm{Pol}(\mathbf{A})|U := \{\psi|U : \psi \in \mathrm{Pol}(\mathbf{A}) \text{ and } \psi[U^n] \subseteq U\}$;
- $\mathbf{A}||U := (U, \mathrm{Pol}(\mathbf{A})|U)$.

**Lemma 4.3.** *Let $\mathbf{A}$ be a finite algebra. Let $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U = \varepsilon[A]$. Then $(-|U) : \mathrm{Con}(\mathbf{A}) \to \mathrm{Con}(\mathbf{A}||U)$ is a surjective $(\wedge, \vee)$-homomorphism.*

**Lemma 4.4.** *Let $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$, and $\varnothing \neq N \subseteq U$. Then $\mathbf{A}||N = (\mathbf{A}||U)||N$.*

*Proof.* That $(\mathrm{Pol}(\mathbf{A})|U)|N \subseteq \mathrm{Pol}(\mathbf{A})|N$ is obvious. Conversely, let $\psi = \varphi|N$ for some $\varphi \in \mathrm{Pol}(\mathbf{A})$ with $\varphi[N^k] \subseteq N$. Clearly, $\varepsilon\varphi[U^k] \subseteq U$. If $(a_1, \ldots, a_k) \in N$, $\varphi(a_1, \ldots, a_k) \in N \subseteq U$, hence $\varphi(a_1, \ldots, a_k) = \varepsilon(a)$ for some $a \in A$. Hence $\varepsilon\varphi(a_1, \ldots, a_k) = \varepsilon^2(a) = \varepsilon(a) \in N$ so that $(\varepsilon\varphi|U)|N = \varphi|N$. We have shown that $\psi$ is an operation of $(\mathbf{A}||U)||N$. $\square$

**Definition 4.5.** Let $\mathbf{A}$ be a finite algebra and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha < \beta$. The pair $(\alpha, \beta)$ is a pair of **tame** congruences if there is $V \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$, $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = V$ and $(-|V) : [\alpha, \beta] \to [\alpha|V, \beta|V]$ is $0,1$-separating[3].

**Lemma 4.6.** *Let $(\alpha, \beta)$ be a tame pair of congruences of a finite algebra $\mathbf{A}$. For every $U \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$, there is $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = U$.*

*Proof.* Since $(\alpha, \beta)$ is tame, there is $V_0 \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$ and $\varepsilon_0 \in \mathrm{E}(\mathbf{A})$ such that $V_0 = \varepsilon_0[A]$ and $(-|V_0)$ is $0,1$-separating.

**Claim 1.** *If $(x, y) \in \beta - \alpha$, there is $\varphi \in \mathrm{Pol}_1(\mathbf{A})$ with $\varphi[A] = V_0$ and such that $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$.*

*Proof.* Let $\theta := \{(x, y) \in \beta : (\varepsilon_0\varphi(x), \varepsilon_0\varphi(y)) \in \alpha$ for all $\varphi \in \mathrm{Pol}_1(\mathbf{A})\}$. Now, $\theta \in [\alpha, \beta]$ and $\alpha|V_0 = \theta|V_0$; that $\alpha|V_0 \subseteq \theta|V_0$ is obvious, for the converse:

$$\begin{aligned}
\theta|V_0 &= \{(a,b) \in \beta \cap V_0^2 : (\varepsilon_0\varphi(a), \varepsilon_0\varphi(b)) \in \alpha \quad \forall \varphi \in \mathrm{Pol}_1(\mathbf{A})\} \\
&= \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0\varphi\varepsilon_0(x), \varepsilon_0\varphi\varepsilon_0(y)) \in \alpha \quad \forall \varphi \in \mathrm{Pol}_1(\mathbf{A})\} \\
&\subseteq \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0(x), \varepsilon_0(y)) \in \alpha\} \\
&= \{(a,b) \in \beta : (a,b) \in \alpha \cap V_0^2\} \\
&= \alpha|V_0
\end{aligned}$$

This implies that $\theta = \alpha$, since $(-|V_0)$ is 0-separating. Thus $(x, y) \in \beta - \alpha$ implies $(x, y) \in \beta - \theta$. By definition of $\theta$, there is $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $(\varepsilon_0\psi(x), \varepsilon_0\psi(y)) \notin \alpha$. Thus $\varphi := \varepsilon_0\psi$ satisfies the conditions $\varphi[A] \subseteq V_0$ and $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$. Hence $\varphi[A] = V_0$ by $(\alpha, \beta)$-minimality. ☐

**Claim 2.** *The relation $\beta$ is the transitive closure of*

$$\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \mathrm{Pol}_1(\mathbf{A})\}.$$

*Proof.* The transitive closure of $\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \mathrm{Pol}_1(\mathbf{A})\}$ is $\alpha \vee \theta(\beta|V_0)$. But $\alpha \vee \theta(\beta|V_0) \in [\alpha, \beta]$, and therefore, since $(-|V_0)$ is 1-seprating, $\beta = \alpha \vee \theta(\beta|V_0)$. ☐

Assume that $U \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$. Then, by definition, there is $\mu \in \mathrm{Pol}_1(\mathbf{A})$ such that $\mu[A] = U$ and $\mu(\beta) \nsubseteq \alpha$. This implies that the equivalence relation

$$\mu^{-1}(\alpha) = \{(a, b) : (\mu(a), \mu(b)) \in \alpha\}$$

is such that $\beta \nsubseteq \mu^{-1}(\alpha)$. Then by Claim 2 there are $a, b \in V_0$ and $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $(a, b) \in \beta$ and $(\mu\psi(a), \mu\psi(b)) \notin \alpha$. The function $\mu_1 := \mu\psi\varepsilon_0$ satisfies $\mu_1[A] \subseteq U$ and $\mu_1(\beta) \nsubseteq \alpha$: there are $x, y \in A$ such that $(a, b) = (\varepsilon_0(x), \varepsilon_0(y)) \in \beta$ but $(\mu_1(a), \mu_1(b)) = (\mu\psi\varepsilon_0(x), \mu\psi\varepsilon_0(y)) \notin \alpha$. Thus $\mu_1[A] = U$ by $(\alpha, \beta)$-minimality. Observe that $\mu_1[V_0] = \mu_1\varepsilon_0[A] = \mu\psi\varepsilon_0^2[A] = \mu_1[A]$, so that $\mu_1[V_0] = U$. Apply Claim 1 to the pair $(\mu_1(a), \mu_1(b))$ to get $\nu \in \mathrm{Pol}_1(\mathbf{A})$ such that $\nu[A] = V_0$ and $(\nu\mu_1(a), \nu\mu_1(b)) \notin \alpha$. Now, since $\mu_1\nu[A] = \mu_1[V_0] = U$, $\mu_1\nu|U$ is bijective; since $U$ is finite, there is $k > 1$ such that $(\mu_1\nu|U)^k = (\mu_1\nu)^k|U = 1_U$. Let $\varepsilon := (\mu_1\nu)^k$.

[3] Let $\mathbf{L}$, $\mathbf{N}$ be two lattices. A $(\wedge, \vee)$-homomorphism $\alpha : \mathbf{L} \to \mathbf{N}$ is $0,1$-**separating** if $\alpha^{-1}[\{\alpha(i)\}] = i$ for $i = 0, 1$.

We have: $\varepsilon[A] = (\mu_1\nu)^k[A] = (\mu_1\nu)^{k-1}[U] = U$ and, consequently, for all $a \in A$, $\varepsilon^2(a) = \varepsilon(a)$ since $\varepsilon(a) \in U$. Therefore $\varepsilon \in \mathrm{E}(\mathbf{A})$. $\square$

**Definition 4.7.** Let $\mathbf{A}$ be a finite algebra minimal with respect to $(\delta, \theta)$. We say that $\mathbf{A}$ has **type i** relative to $(\delta, \theta)$ if for every $(\delta, \theta)$-trace $N$, $(\mathbf{A}||N)/(\delta|N)$ is a minimal algebra of type **i**.

**Definition 4.8.** Let $(\alpha, \beta)$ be tame in a finite algebra $\mathbf{A}$. Let $N$ be a $(\alpha, \beta)$-trace. We define the **type** of $(\alpha, \beta)$, written $\mathrm{typ}(\alpha, \beta)$, to be the type of the minimal algebra $\mathbf{M} = (\mathbf{A}||N)/(\alpha|N)$.

Finally, for a finite algebra $\mathbf{A}$, we define $\mathrm{typ}\{\mathbf{A}\} := \{\mathrm{typ}(\alpha, \beta) : (\alpha, \beta) \text{ is tame}\}$ and for a class $K$ of finite algebras, $\mathrm{typ}\{K\} := \cup\{\mathrm{typ}\{\mathbf{A}\} : \mathbf{A} \in K\}$.

## 5. Omitting Types

**Definition 5.1.** Let $\mathsf{V}$ be a variety. An algebra $\mathbf{A} \in \mathsf{V}$ is called

(1) **free** if there is an isomorphism $\mathbf{A} \simeq \mathbf{F}_{\mathsf{V}}(\kappa)$ for some cardinal $\kappa$;

(2) **finitely generated** if there is a surjective homomorphism $\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}$ for some $n \in \omega$.

**Definition 5.2.** A variety $\mathsf{V}$ is called

(1) **locally finite** if all its finitely generated algebras are finite;

(2) **finitely presented** if $\mathsf{V}$ has a finite set of function symbols and $\mathsf{V} = \mathrm{Alg}(\Sigma)$ for a finite set of equations $\Sigma$;

(3) **finitely generated** if $\mathsf{V} = V(\mathbf{A}_1, \dots, \mathbf{A}_n)$ for $\mathbf{A}_1, \dots, \mathbf{A}_n$ finite similar algebras.

*Remark* 5.3. Observe that $V(\mathbf{A}_1, \dots, \mathbf{A}_n) = V(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.

**Lemma 5.4.** *Let $\mathsf{V}$ be a variety. If $\mathsf{V}$ is finitely generated then it is locally finite.*

*Proof.* By the previuos remark, we can assume that $\mathsf{V} = V(\mathbf{A})$ for some $\mathbf{A}$ finite. Let $n < \omega$. We prove that $\mathbf{F}_{\mathsf{V}}(n)$ is finite. Consider the homomorphism

$$\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}^{A^n}, \quad t(x_1, \dots, x_n) \mapsto t^{\mathbf{A}}$$

This homomorphism is injective: if $t^{\mathbf{A}} = s^{\mathbf{A}}$, then $\mathbf{A} \models t \equiv s$, i.e. $t = s$ in $\mathbf{F}_{\mathsf{V}}(n)$. Thus $\mathbf{F}_{\mathsf{V}}(n)$ is finite. $\square$

**Definition 5.5.** Let $\mathsf{V}$ and $\mathsf{W}$ be two varieties. We say that $\mathsf{V}$ is **interpretable** into $\mathsf{W}$ ($\mathsf{V} \leq \mathsf{W}$) if there is a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$.

*Remark* 5.6. Let $\mathsf{W}, \mathsf{V}$ be two varieties. We unravel what $\mathsf{W} \leq \mathsf{V}$ means in a simple case. Let $F$ be a finite set of function symbols. Let $\mathsf{V}$ be a variety of algebras over $F$, defined by the equations

$$s_1 \equiv t_1, \dots, s_k \equiv t_k. \tag{1}$$

Assume that for each $f \in F_n$ there is $t \in \mathrm{Clo}_n(\mathsf{W})$ such that the interpetation of the $t$'s satisfy the equations (1). Then the assignment $f \mapsto t$ extends to a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$. Of course, the converse also holds; thus this is equivalent to $\mathsf{W} \leq \mathsf{V}$.

**Lemma 5.7.** *Let* $\mathsf{W}, \mathsf{V}$ *be two varieties such that* $\mathsf{W} \leq \mathsf{V}$. *Assume that* $\mathsf{W}$ *is*

- *idempotent;*
- *finitely presented;*
- *the finite set of equations defining* $\mathsf{W}$ *contains at most one function symbol per side.*

*Let* $\mathbf{A} \in \mathsf{V}$, $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$, $\beta \in \mathrm{Con}(\mathbf{A})$ *and* $S := a/\beta \cap U$ *for* $a \in U$. *Then* $\mathsf{W} \leq V(\mathbf{A}||S)$.

*Proof.* By assumption $\mathsf{W}$ can be described by a finite set of equations of the form

$$f_i(x_{i_1}, \dots, x_{i_h}) \equiv f_j(x_{j_1}, \dots, x_{j_k}) \tag{2}$$

where $f_i$ and $f_j$ are members of a finite set $F$ of function symbols. Since $\mathsf{W} \leq \mathsf{V}$, there is an assignment $f \mapsto t$ extending to a clone homomorphism. We need to find a clone homomorphism $\mathrm{Clo}(\mathsf{W}) \to \mathrm{Clo}(\mathbf{A}||S)$. Consider $f \mapsto \varphi := \varepsilon t_i^{\mathbf{A}}|S$. Firstly, it is well defined: if $(a_1, \dots, a_n) \in S$, $(\varphi(a_1, \dots, a_n), \varphi(a, \dots, a)) \in \beta$ but

$$\varphi(a, \dots, a) = \varepsilon t^{\mathbf{A}}(a, \dots, a) = \varepsilon(a) \in U$$

so that $\varphi(a_1, \dots, a_n) \in S$ and therefore $\varphi \in \mathrm{Pol}(\mathbf{A})|S$. Finally, using that $\mathsf{W} \leq \mathsf{V}$

$$\begin{aligned} \varphi_i(a_{i_1}, \dots, a_{i_h}) &= \varepsilon t_i^{\mathbf{A}}(a_{i_1}, \dots, a_{i_h}) \\ &= \varepsilon t_j^{\mathbf{A}}(a_{j_1}, \dots, a_{j_k}) \\ &= \varphi_j(a_{j_1}, \dots, a_{j_k}). \end{aligned}$$

□

**Theorem 5.8.** *Let* $\mathsf{V}$ *be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) *there is an idempotent variety* $\mathsf{W}$ *such that* $\mathsf{W} \leq \mathsf{V}$ *and* $\mathsf{W} \not\leq \mathsf{Set}$.

**Corollary 5.9.** *Let* $\mathbf{A}$ *be a finite idempotent algebra. There is* $\mathbf{B} \in HS(\mathbf{A})$ *such that* $\mathrm{Clo}(\mathbf{B}) = \mathbf{N}$ *iff* $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$.

*Proof.* If $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$, then $\mathbf{1} \in \mathrm{typ}\{HSP(\mathbf{A})\}$. Since $\mathbf{A}$ is finite, then, by Lemma 5.4 $HSP(\mathbf{A})$ is locally finite, and therefore, by Theorem 5.8, for every idempotent variety $\mathsf{W}$, either $\mathsf{W} \not\leq HSP(\mathbf{A})$ or $\mathsf{W} \leq \mathsf{Set}$. In particular, since $\mathbf{A}$ is idempotent, $HSP(\mathbf{A}) \leq \mathsf{Set}$. This means that there is a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathbf{N}$. Equivalently, every term operation of $\mathbf{A}$ is a projection. Hence, $\mathrm{Clo}(\mathbf{A}) = \mathbf{N}$.

Conversely, let $\mathbf{B} \in HS(\mathbf{A})$ such that $\mathrm{Clo}(\mathbf{B}) = \mathbf{N}$; this means that $\mathbf{B}$ is term equivalent to a set. Hence $\mathbf{B}$ is polynomial equivalent to a set on which the trivial group acts. Then $\mathbf{1} \in \mathrm{typ}\{\mathbf{B}\}$, and therefore $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$. □

**Theorem 5.10** ([2])**.** *Let* $\mathbf{A}$ *be a finite idempotent algebra. Then* $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation iff* $\mathbf{1} \notin \mathrm{typ}\{HS(\mathbf{A})\}$*.*

## References


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