MINIMAL ALGEBRAS

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1. Preliminaries

Definition 1.1. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$$

and closed under composition. The elements of $\mathrm{Clo}(\mathbf{A})$ are called **term** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **term equivalent** if $\mathrm{Clo}(\mathbf{A}) \simeq \mathrm{Clo}(\mathbf{B})$.

Definition 1.2. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Pol}(\mathbf{A})$ the smallest set containing

(1) $\{f^{\mathbf{A}} : f \in F\}$;

(2) $\{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$;

(3) the constant 0-ary operations

and closed under composition. The elements of $\mathrm{Pol}(\mathbf{A})$ are called **polynomial** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **polynomially equivalent** if $\mathrm{Pol}(\mathbf{A}) \simeq \mathrm{Pol}(\mathbf{B})$.

Example 1.3. If $\varphi \in \mathrm{Clo}_{m+n}(\mathbf{A})$ and $(a_1, \dots, a_m) \in A^m$, then

$$\psi : A^n \to A \quad (b_1, \dots, b_n) \mapsto \varphi(a_1, \dots, a_m, b_1, \dots, b_n)$$

is a polynomial operation.

Remark 1.4. Let $\mathbf{A}$ be an algebra. An equivalence relation α is a congruence of $\mathbf{A}$ iff $\varphi(\alpha) \subseteq \alpha$ for every $\varphi \in \mathrm{Pol}_1(\mathbf{A})$.

Let $\mathbf{A}$ be a finite algebra. We adopt the following convention, concerning the restriction $(-|U)$ operation, for $U \subseteq A$:

- if $\theta \in \mathrm{Con}(\mathbf{A})$, $\theta|U := \theta \cap U^2$;
- if $\varphi : A^n \to A$, $\varphi|U$ is the function $U^n \to A$, $(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_n)$;
- $\mathrm{Pol}(\mathbf{A})|U := \{\psi|U : \psi \in \mathrm{Pol}(\mathbf{A}) \text{ and } \psi[U^n] \subseteq U\}$;
- $\mathbf{A}||U := (U, \mathrm{Pol}(\mathbf{A})|U)$.

2. Finite Minimal Algebras

Definition 2.1. A nontrivial finite algebra $\mathbf{A}$ is **minimal** iff every noncostant element of $\mathrm{Pol}_1(\mathbf{A})$ is bijective.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

Example 2.2. The following are examples of minimal algebras.

(1) any algebra with carrier 2;

(2) a nontrivial finite vector space $\mathbf{A}$ over a finite field $\mathbf{k}$: every $\pi \in \mathrm{Pol}_1(\mathbf{A})$ is of the form $\pi(v) = av + b$ for some $a \in k$, $b \in A$;

(3) a group of permutations acting on a finite set. If $\mathbf{G}$ is a group acting on a set A each $g \in G$ induces an operation $\varphi_g : A \to A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. A $\mathbf{G}$-set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$.

We shall prove that, up to polynomial equivalence, there are no other finite minimal algebras.

Lemma 2.3. *Let $\mathbf{A}$ be a minimal algebra. If every element of $\mathrm{Pol}(\mathbf{A})$ is essentially unary, then $\mathbf{A}$ is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ where $\mathbf{G}$ is a finite group acting on A.*

Proof. Since $\mathbf{A}$ is minimal, $\mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$ is a subgroup of $\mathrm{Sym}(A)$. Let $\mathbf{G} := \mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$. If $\psi \in \mathrm{Pol}(\mathbf{A})$, either ψ is constant or ψ is essentially unary, hence $(A, \Phi_{\mathbf{G}})$ is polynomially equivalent to $\mathbf{A}$. $\square$

Theorem 2.4 ([3])**.** *Let $\mathbf{A}$ be a minimal algebra with $|A| > 2$. If $\mathrm{Pol}(\mathbf{A})$ contains an operation which is not essentially unary, then $\mathbf{A}$ is polynomially equivalent to a $\mathbf{k}$-vector space for a finite field $\mathbf{k}$.*

Theorem 2.5. *Every algebra $\mathbf{A}$ with carrier 2 is polynomially equivalent to one of the following:*

(1) $\mathbf{E}_0 = (2, \varnothing)$;

(2) $\mathbf{E}_1 = (2, \neg)$;

(3) $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$;

(4) $\mathbf{E}_4 = (2, \wedge, \vee)$;

(5) $\mathbf{E}_5 = (2, \vee)$;

(6) $\mathbf{E}_6 = (2, \wedge)$.

Each of them is not polynomially equivalent to the other[1].

[1] A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem 2.5 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

Remark 2.6. Up to isomorphism, $\mathbf{E}_5(\simeq \mathbf{E}_6)$ is the only semilattice with two elements, while $\mathbf{E}_3$ and $\mathbf{E}_4$ are the only Boolean algebra and lattice, respectively, with two elements.

Definition 2.7. Let $\mathbf{A}$ be a minimal algebra. We say that $\mathbf{A}$ is of

(1) **type 1** (or **unary**) if $\mathbf{A}$ is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ for some $\mathbf{G} \leq \mathrm{Sym}(A)$;

(2) **type 2** (or **affine**) if $\mathbf{A}$ is polynomially equivalent to a vector space over a finite field $\mathbf{k}$;

(3) **type 3** (or **Boolean**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_3$;

(4) **type 4** (or **lattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_4$;

(5) **type 5** (or **semilattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_5$.

3. Relative Minimal Algebras

Definition 3.1. Let $\mathbf{A}$ be a finite algebra and let $\delta < \theta \in \mathrm{Con}(\mathbf{A})$. We say that $\mathbf{A}$ is (δ, θ)-**minimal** if for all $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ either ε is bijective or $\varepsilon(\theta) \subseteq \delta$.

Remark 3.2. Observe that $\mathbf{A}$ is minimal iff $\mathbf{A}$ is (Δ, ∇)-minimal.

Definition 3.3. Let $\mathbf{A}$ be a (α, β)-minimal algebra. An (α, β)-**trace** of $\mathbf{A}$ is a β-equivalence class which contains at least two α-equivalence classes.

Lemma 3.4. *Let $\mathbf{A}$ be a finite (δ, θ)-minimal algebra and let N be a (δ, θ)-trace. Then the algebra $(\mathbf{A}||N)/(\delta|N)$ is minimal.*

Proof. We need to show that for every

$$\psi \in \mathrm{Pol}_1((\mathbf{A}||N)/(\delta|N)) = \{(\varphi|N)/(\delta|N) : \varphi \in \mathrm{Pol}_1(\mathbf{A}), \varphi[N] \subseteq N\}$$

either ψ is bijective, or ψ is constant. Let $\psi = (\varphi|N)/(\delta|N)$. Since $\mathbf{A}$ is (δ, θ)-minimal, either φ is bijective or $\varphi(\theta) \subseteq \delta$. Clearly, if φ is bijective, ψ is bijective. If $\varphi(\theta) \subseteq \delta$, ψ is constant: if $(x, y) \in N^2 \subseteq \theta$, then $(\psi(x), \psi(y)) \in \delta$ so that $\psi(x) = \psi(y)$ in $(\mathbf{A}||N)/(\delta|N)$. $\square$

Therefore, with an abuse of language, we shall refer unambiguoulsy to the type of N as the type of $(\mathbf{A}||N)/(\delta|N)$.

Theorem 3.5. *Let $\mathbf{A}$ be a (δ, θ)-minimal algebra. Then all (δ, θ)-traces of $\mathbf{A}$ have the same type.*

Definition 3.6. Let $\mathbf{A}$ be a finite (δ, θ)-minimal algebra. We say that $\mathbf{A}$ is of type $\mathbf{i}$ relative to (δ, θ) if each (δ, θ)-trace of $\mathbf{A}$ is of type $\mathbf{i}$.

Definition 3.7. Let $\mathbf{A}$ be a finite algebra and $B, C \subseteq A$. We say that B, C are **polynomially isomorphic** $(B \sim C)$ if there are $\varphi, \psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varphi[B] = C, \psi[C] = B$ and $\psi\varphi|B = 1_B, \varphi\psi|C = 1_C$.

Remark 3.8. If $B, C \subseteq A$ are polynomial isomorphic in $\mathbf{A}$, then $\mathbf{A}||B \simeq \mathbf{A}||C$.

Lemma 3.9. *Let $\mathbf{A}$ be a (δ,θ)-minimal algebra and N be a (δ,θ)-trace. Then*

(1) $(-|N) : [\Delta_A, \theta] \to \mathrm{Con}(\mathbf{A}||N)$ *is a surjective lattice homomorphism;*

(2) *quotienting out by δ, there is a surjective lattice homomorphism $[\delta,\theta] \to \mathrm{Con}((\mathbf{A}||N)/(\delta|N))$;*

(3) *if any two (δ,θ)-traces are polynomially isomorphic, then the map of the second item is an isomorphism.*

Proof. Clearly, the map is well defined and preserves meets. We show that

$$(\alpha \vee \beta) \cap N^2 = (\alpha \cap N^2) \vee (\beta \cap N^2).$$

Let $(x,y) \in (\alpha \vee \beta) \cap N^2$. Then there are $x = x_0, \dots, x_{n+1} = y$ such that either $(x_i, x_{i+1}) \in \alpha$ or $(x_i, x_{i+1}) \in \beta$. We show that for each i, $(x_i, x_{i+1}) \in N^2$. Inductively, if $x_i \in N$, and, say, $(x_i, x_{i+1}) \in \alpha \subseteq \theta$, then $x_{i+1} \in N$. Now, for $\beta \in \mathrm{Con}(\mathbf{A}||N)$, let $\hat{\beta}$ be

$$\{(x,y) \in \theta : (\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta \quad \forall \psi \in \mathrm{Pol}_1(\mathbf{A})\}$$

Then $\hat{\beta}$ is a congruence. We show that $\hat{\beta} \cap N^2 = \beta$, proving surjectivity. If $(x,y) \in \hat{\beta}$, then $(\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta$ for all $\psi \in \mathrm{Pol}_1(\mathbf{A})$; if $(x,y) \in N^2$, then $(\psi(x), \psi(y)) \in N^2$. Therefore $(\psi(x), \psi(y)) \in \beta$ for all $\psi \in \mathrm{Pol}_1(\mathbf{A})$, and, taking $\psi(x) = x$, $(x,y) \in \beta$. Conversely, let $(x,y) \in \beta$. As $\beta \subseteq N^2 \subseteq \theta$, $(x,y) \in \theta$. Let $\psi \in \mathrm{Pol}_1(\mathbf{A})$. If $(\psi(x), \psi(y)) \in N^2$, then $\psi \in \mathrm{Pol}_1(\mathbf{A}||N)$. Since $\beta \in \mathrm{Con}(\mathbf{A}||N)$, $(\psi(x), \psi(y)) \in \beta$.

The second item immediately follows. As to the third, we need to prove injectivity. Let $\delta \leq \alpha < \beta \leq \theta$. Let $(x,y) \in \beta - \alpha$. Then $(x,y) \in \theta - \delta$ and $P := x/\theta$ is a (δ,θ)-trace. Since P is polynomially isomorphic to N, there is $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that

$$\psi[P] = \psi[x/\theta] = \{\psi(z) : (x,z) \in \theta\} = N.$$

Then, $(\psi(x), \psi(y)) \in (\theta - \delta) \cap N^2$. Also, $(\psi(x), \psi(y)) \in \beta - \alpha$, so that $(\psi(x), \psi(y)) \in \beta|N - \alpha|N$ and $\alpha|N/\delta|N < \beta|N/\delta|N$. □

The following is an example of a sufficient condition that guarantees that the lattices $[\delta,\theta]$ and $\mathrm{Con}((\mathbf{A}||N)/(\delta|N))$ are isomorphic.

Lemma 3.10. *Let $\mathbf{A}$ be a (δ,θ)-minimal algebra with $\delta \prec \theta$. Let N, K be two (δ,θ)-traces. Then N and K are polynomially isomorphic.*

Proof. □

4. Tame Congruences

Definition 4.1. Let $\mathbf{A}$ be a finite algebra and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha < \beta$. We denote by

(1) $\mathrm{E}(\mathbf{A})$ the set of $\varepsilon \in \mathrm{Pol}_1(\mathbf{A})$ such that $\varepsilon^2 = \varepsilon$;

(2) $\mathrm{U}_{\mathbf{A}}(\alpha,\beta)$ the set $\{\varphi[A] : \varphi \in \mathrm{Pol}_1(\mathbf{A}), \varphi(\beta) \nsubseteq \alpha\}$;

(3) $\mathrm{M}_{\mathbf{A}}(\alpha,\beta)$ the set of minimal (with respect to the inclusion relation) elements of $\mathrm{U}_{\mathbf{A}}(\alpha,\beta)$.

Remark 4.2. A finite algebra $\mathbf{A}$ is (δ,θ)-minimal iff $A \in \mathrm{M}_{\mathbf{A}}(\delta,\theta)$. If $\mathbf{A}$ is (δ,θ)-minimal, every $\psi \in \mathrm{M}_{\mathbf{A}}(\delta,\theta)$ such that $\psi(\delta) \nsubseteq \theta$ is bijective. Clearly $A \in \mathrm{U}_{\mathbf{A}}(\delta,\theta)$, as witnessed by $\varepsilon(x) = x$. If there were $\psi \in \mathrm{Pol}_1(\mathbf{A})$ with $\psi(\delta) \nsubseteq \theta$ and $\psi[A] \subset A$ we would contradict (δ,θ)-minimality. Conversely, let $A \in \mathrm{M}_{\mathbf{A}}(\delta,\theta)$ and $\psi \in \mathrm{Pol}_1(\mathbf{A})$ with $\psi(\delta) \nsubseteq \theta$; we show that $\psi \in \mathrm{Sym}(A)$. But by definition of $\mathrm{M}_{\mathbf{A}}(\delta,\theta)$, there is no $\psi \in \mathrm{Pol}_1(\mathbf{A})$ with $\psi(\delta) \nsubseteq \theta$ and $\psi[A] \subset A$.

Lemma 4.3. *Let $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$, and $\varnothing \neq N \subseteq U$. Then $\mathbf{A}||N = (\mathbf{A}||U)||N$.*

Proof. That $(\mathrm{Pol}(\mathbf{A})|U)|N \subseteq \mathrm{Pol}(\mathbf{A})|N$ is obvious. Conversely, let $\psi = \varphi|N$ for some $\varphi \in \mathrm{Pol}(\mathbf{A})$ with $\varphi[N^k] \subseteq N$. Clearly, $\varepsilon\varphi[U^k] \subseteq U$. If $(a_1,\ldots,a_k) \in N$, $\varphi(a_1,\ldots,a_k) \in N \subseteq U$, hence $\varphi(a_1,\ldots,a_k) = \varepsilon(a)$ for some $a \in A$. Hence $\varepsilon\varphi(a_1,\ldots,a_k) = \varepsilon^2(a) = \varepsilon(a) \in N$ so that $(\varepsilon\varphi|U)|N = \varphi|N$. We have shown that ψ is an operation of $(\mathbf{A}||U)||N$. $\square$

Definition 4.4. Let $\mathbf{A}$ be a finite algebra and $\alpha,\beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha < \beta$. The pair (α,β) is a pair of **tame** congruences if there is $V \in \mathrm{M}_{\mathbf{A}}(\alpha,\beta)$, $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = V$ and $(-|V) : [\alpha,\beta] \to [\alpha|V,\beta|V]$ is $0,1$-separating

A $(\wedge,\vee)$-homomorphism $\alpha : \mathbf{L} \to \mathbf{N}$ is $0,1$-**separating** if $\alpha^{-1}[\{\alpha(i)\}] = i$ for $i = 0,1$.

Theorem 4.5. *Let (α,β) be a tame pair of congruences of a finite algebra $\mathbf{A}$. For every $U \in \mathrm{M}_{\mathbf{A}}(\alpha,\beta)$,*

(1) *$\mathbf{A}||U$ is $(\alpha|U,\beta|U)$-minimal;*

(2) *there is $\varepsilon \in \mathrm{E}(\mathbf{A})$ such that $\varepsilon[A] = U$;*

(3) *$(-|U) : [\delta,\theta] \to \mathrm{Con}(\mathbf{A}||U)$ is a surjective lattice homomorphism which is $0,1$-separating.*

Moreover, any two (α,β)-minimal sets are polynomially isomorphic.

Proof. Since (α,β) is tame, there is $V_0 \in \mathrm{M}_{\mathbf{A}}(\alpha,\beta)$ and $\varepsilon_0 \in \mathrm{E}(\mathbf{A})$ such that $V_0 = \varepsilon_0[A]$ and $(-|V_0)$ is $0,1$-separating.

Claim 1. *If $(x,y) \in \beta - \alpha$, there is $\varphi \in \mathrm{Pol}_1(\mathbf{A})$ with $\varphi[A] = V_0$ and such that $(\varphi(x),\varphi(y)) \in \beta|V_0 - \alpha|V_0$.*

Proof. Let $\theta := \{(x,y) \in \beta : (\varepsilon_0\varphi(x),\varepsilon_0\varphi(y)) \in \alpha \text{ for all } \varphi \in \mathrm{Pol}_1(\mathbf{A})\}$. Now, $\theta \in [\alpha,\beta]$ and $\alpha|V_0 = \theta|V_0$; that $\alpha|V_0 \subseteq \theta|V_0$ is obvious, for the converse:

$$\begin{aligned}
\theta|V_0 &= \{(a,b) \in \beta \cap V_0^2 : (\varepsilon_0\varphi(a),\varepsilon_0\varphi(b)) \in \alpha \quad \forall\varphi \in \mathrm{Pol}_1(\mathbf{A})\} \\
&= \{(\varepsilon_0(x),\varepsilon_0(y)) \in \beta : (\varepsilon_0\varphi\varepsilon_0(x),\varepsilon_0\varphi\varepsilon_0(y)) \in \alpha \quad \forall\varphi \in \mathrm{Pol}_1(\mathbf{A})\} \\
&\subseteq \{(\varepsilon_0(x),\varepsilon_0(y)) \in \beta : (\varepsilon_0(x),\varepsilon_0(y)) \in \alpha\} \\
&= \{(a,b) \in \beta : (a,b) \in \alpha \cap V_0^2\} \\
&= \alpha|V_0
\end{aligned}$$

This implies that $\theta = \alpha$, since $(-|V_0)$ is 0-separating. Thus $(x, y) \in \beta - \alpha$ implies $(x, y) \in \beta - \theta$. By definition of θ, there is $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $(\varepsilon_0\psi(x), \varepsilon_0\psi(y)) \notin \alpha$. Thus $\varphi := \varepsilon_0\psi$ satisfies the conditions $\varphi[A] \subseteq V_0$ and $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$. Hence $\varphi[A] = V_0$ by (α, β)-minimality. □

Claim 2. *The relation β is the transitive closure of*

$$\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \mathrm{Pol}_1(\mathbf{A})\}.$$

Proof. The transitive closure of $\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \mathrm{Pol}_1(\mathbf{A})\}$ is $\alpha \vee \theta(\beta|V_0)$. But $\alpha \vee \theta(\beta|V_0) \in [\alpha, \beta]$, and therefore, since $(-|V_0)$ is 1-seprating, $\beta = \alpha \vee \theta(\beta|V_0)$. □

Assume that $U \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$. Then, by definition, there is $\mu \in \mathrm{Pol}_1(\mathbf{A})$ such that $\mu[A] = U$ and $\mu(\beta) \nsubseteq \alpha$. This implies that the equivalence relation

$$\mu^{-1}(\alpha) = \{(a, b) : (\mu(a), \mu(b)) \in \alpha\}$$

is such that $\beta \nsubseteq \mu^{-1}(\alpha)$. Then by Claim 2 there are $a, b \in V_0$ and $\psi \in \mathrm{Pol}_1(\mathbf{A})$ such that $(a, b) \in \beta$ and $(\mu\psi(a), \mu\psi(b)) \notin \alpha$. The function $\mu_1 := \mu\psi\varepsilon_0$ satisfies $\mu_1[A] \subseteq U$ and $\mu_1(\beta) \nsubseteq \alpha$: there are $x, y \in A$ such that $(a, b) = (\varepsilon_0(x), \varepsilon_0(y)) \in \beta$ but $(\mu_1(a), \mu_1(b)) = (\mu\psi\varepsilon_0(x), \mu\psi\varepsilon_0(y)) \notin \alpha$. Thus $\mu_1[A] = U$ by (α, β)-minimality. Observe that $\mu_1[V_0] = \mu_1\varepsilon_0[A] = \mu\psi\varepsilon_0^2[A] = \mu_1[A]$, so that $\mu_1[V_0] = U$. Apply Claim 1 to the pair $(\mu_1(a), \mu_1(b))$ to get $\nu \in \mathrm{Pol}_1(\mathbf{A})$ such that $\nu[A] = V_0$ and $(\nu\mu_1(a), \nu\mu_1(b)) \notin \alpha$. Now, since $\mu_1\nu[A] = \mu_1[V_0] = U$, $\mu_1\nu|U$ is bijective; since U is finite, there is $k > 1$ such that $(\mu_1\nu|U)^k = (\mu_1\nu)^k|U = 1_U$. Let $\varepsilon := (\mu_1\nu)^k$. We have: $\varepsilon[A] = (\mu_1\nu)^k[A] = (\mu_1\nu)^{k-1}[U] = U$ and, consequently, for all $a \in A$, $\varepsilon^2(a) = \varepsilon(a)$ since $\varepsilon(a) \in U$. Therefore $\varepsilon \in \mathrm{E}(\mathbf{A})$. □

Definition 4.6. Let (α, β) be tame in a finite algebra $\mathbf{A}$. An (α, β)-**trace** of $\mathbf{A}$ is $N \subseteq A$ such that for some $U \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$ and $x \in U$, $N \subseteq U$ and $N = x/(\beta|U) \neq x/(\alpha|U)$. That is, N is an $(\alpha|U, \beta|U)$-trace of the minimal algebra $\mathbf{A}||U$.

Definition 4.7. Let (α, β) be tame in a finite algebra $\mathbf{A}$. Let $\delta, \theta \in \mathrm{Con}(\mathbf{A})$. Let K be a class of finite algebras. We define

(1) the **type** of (α, β), written $\mathrm{typ}(\alpha, \beta)$, to be the type of $\mathbf{A}||U$ relative to $(\alpha|U, \beta|U)$;

(2) $\mathrm{typ}\{\delta, \theta\} := \{\mathrm{typ}(\alpha, \beta) : \delta \leq \alpha \prec \beta \leq \beta\}$;

(3) $\mathrm{typ}\{\mathbf{A}\} := \mathrm{typ}\{\Delta_A, \nabla_A\}$;

(4) $\mathrm{typ}\{K\} := \cup\{\mathrm{typ}\{\mathbf{A}\} : \mathbf{A} \in K\}$.

Remark 4.8. If (Δ, ∇) in $\mathrm{Con}(\mathbf{A})$ is tame, then $\mathrm{typ}(\Delta, \nabla)$ coincides with $\mathrm{typ}(\mathbf{A})$ of Definition 2.7.

Lemma 4.9. *Let (α, β) be tame in a finite algebra $\mathbf{A}$. For every (α, β)-trace of $\mathbf{A}$, the algebra $(\mathbf{A}||N)/(\alpha|N)$ is minimal and* $\mathrm{typ}(\alpha, \beta) = \mathrm{typ}((\mathbf{A}||N)/(\alpha|N))$.

5. Omitting Types

Definition 5.1. Let V be a variety. An algebra $\mathbf{A} \in \mathsf{V}$ is called

(1) **free** if there is an isomorphism $\mathbf{A} \simeq \mathbf{F}_{\mathsf{V}}(\kappa)$ for some cardinal κ;

(2) **finitely generated** if there is a surjective homomorphism $\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}$ for some $n \in \omega$.

Definition 5.2. A variety V is called

(1) **locally finite** if all its finitely generated algebras are finite;

(2) **finitely presented** if V has a finite set of function symbols and $\mathsf{V} = \mathrm{Alg}(\Sigma)$ for a finite set of equations Σ;

(3) **finitely generated** if $\mathsf{V} = V(\mathbf{A}_1, \dots, \mathbf{A}_n)$ for $\mathbf{A}_1, \dots, \mathbf{A}_n$ finite similar algebras.

Remark 5.3. Observe that $V(\mathbf{A}_1, \dots, \mathbf{A}_n) = V(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.

Lemma 5.4. *Let V be a variety. If V is finitely generated then it is locally finite.*

Proof. By the previuos remark, we can assume that $\mathsf{V} = V(\mathbf{A})$ for some $\mathbf{A}$ finite. Let $n < \omega$. We prove that $\mathbf{F}_{\mathsf{V}}(n)$ is finite. Consider the homomorphism

$$\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}^{A^n}, \quad t(x_1, \dots, x_n) \mapsto t^{\mathbf{A}}$$

This homomorphism is injective: if $t^{\mathbf{A}} = s^{\mathbf{A}}$, then $\mathbf{A} \models t \equiv s$, i.e. $t = s$ in $\mathbf{F}_{\mathsf{V}}(n)$. Thus $\mathbf{F}_{\mathsf{V}}(n)$ is finite. □

Definition 5.5. Let V and W be two varieties. We say that V is **interpretable** into W ($\mathsf{V} \leq \mathsf{W}$) if there is a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$.

Remark 5.6. Let W, V be two varieties. We unravel what $\mathsf{W} \leq \mathsf{V}$ means in a simple case. Let F be a finite set of function symbols. Let V be a variety of algebras over F, defined by the equations

$$s_1 \equiv t_1, \dots, s_k \equiv t_k. \tag{1}$$

Assume that for each $f \in F_n$ there is $t \in \mathrm{Clo}_n(\mathsf{W})$ such that the interpetation of the t's satisfy the equations (1). Then the assignment $f \mapsto t$ extends to a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$. Of course, the converse also holds; thus this is equivalent to $\mathsf{W} \leq \mathsf{V}$.

Lemma 5.7. *Let W, V be two varieties such that $\mathsf{W} \leq \mathsf{V}$. Assume that W is*

- *idempotent;*
- *finitely presented;*
- *the finite set of equations defining W contains at most one function symbol per side.*

Let $\mathbf{A} \in \mathsf{V}$, $\varepsilon \in \mathrm{E}(\mathbf{A})$, $U := \varepsilon[A]$, $\beta \in \mathrm{Con}(\mathbf{A})$ and $S := a/\beta \cap U$ for $a \in U$. Then $\mathsf{W} \leq V(\mathbf{A}||S)$.

Proof. By assumption W can be described by a finite set of equations of the form

$$f_i(x_{i_1}, \dots, x_{i_h}) \equiv f_j(x_{j_1}, \dots, x_{j_k}) \tag{2}$$

where f_i and f_j are members of a finite set F of function symbols. Since $\mathsf{W} \leq \mathsf{V}$, there is an assignment $f \mapsto t$ extending to a clone homomorphism. We need to find a clone homomorphism $\mathrm{Clo}(\mathsf{W}) \to \mathrm{Clo}(\mathbf{A}||S)$. Consider $f \mapsto \varphi := \varepsilon t_i^{\mathbf{A}}|S$. Firstly, it is well defined: if $(a_1, \dots, a_n) \in S$, $(\varphi(a_1, \dots, a_n), \varphi(a, \dots, a)) \in \beta$ but

$$\varphi(a, \dots, a) = \varepsilon t^{\mathbf{A}}(a, \dots, a) = \varepsilon(a) \in U$$

so that $\varphi(a_1, \dots, a_n) \in S$ and therefore $\varphi \in \mathrm{Pol}(\mathbf{A})|S$. Finally, using that $\mathsf{W} \leq \mathsf{V}$

$$\begin{aligned} \varphi_i(a_{i_1}, \dots, a_{i_h}) &= \varepsilon t_i^{\mathbf{A}}(a_{i_1}, \dots, a_{i_h}) \\ &= \varepsilon t_j^{\mathbf{A}}(a_{j_1}, \dots, a_{j_k}) \\ &= \varphi_j(a_{j_1}, \dots, a_{j_k}). \end{aligned}$$

□

Theorem 5.8. *Let V be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) *there is an idempotent variety W such that $\mathsf{W} \leq \mathsf{V}$ and $\mathsf{W} \nleq \mathsf{Set}$.*

Corollary 5.9. *Let $\mathbf{A}$ be a finite idempotent algebra. There is $\mathbf{B} \in HS(\mathbf{A})$ such that $\mathrm{Clo}(\mathbf{B}) \simeq \mathbf{N}$ iff $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$.*

Proof. If $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$, then $\mathbf{1} \in \mathrm{typ}\{HSP(\mathbf{A})\}$. Since $\mathbf{A}$ is finite, then, by Lemma 5.4 $HSP(\mathbf{A})$ is locally finite, and therefore, by Theorem 5.8, for every idempotent variety W, either $\mathsf{W} \nleq HSP(\mathbf{A})$ or $\mathsf{W} \leq \mathsf{Set}$. In particular, since $\mathbf{A}$ is idempotent, $HSP(\mathbf{A}) \leq \mathsf{Set}$. This means that there is a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathbf{N}$. Equivalently, every term operation of $\mathbf{A}$ is a projection. Hence, $\mathrm{Clo}(\mathbf{A}) \simeq \mathbf{N}$.

□

Theorem 5.10 ([2])**.** *Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathrm{Clo}(\mathbf{A})$ contains a weak near-unanimity operation iff $\mathbf{1} \notin \mathrm{typ}\{HS(\mathbf{A})\}$.*

References

[1] Hobby, D., McKenzie, R. (1988). *The structure of Finite Algebras.* Contemporary Mathematics 76, American Mathematical Society.

[2] Maróti, M., McKenzie, R. (2008). Existence theorems for weakly symmetric operations. *Algebra Universalis* 59, 463-489.

[3] Pálfy, P. P. (1984). Unary polynomials in algebras I. *Algebra Universalis* 18, 262-273.