

# MINIMAL ALGEBRAS

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## 1. PRELIMINARIES

**Definition 1.1.** Let  $F$  be a set of function symbols and  $\mathbf{A}$  be an algebra over  $F$ . We denote by  $\text{Clo}(\mathbf{A})$  the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition. The elements of  $\text{Clo}(\mathbf{A})$  are called **term** operations. We say that two algebras  $\mathbf{A}$  and  $\mathbf{B}$  on the same carrier are **term equivalent** if  $\text{Clo}(\mathbf{A}) \simeq \text{Clo}(\mathbf{B})$ .

**Definition 1.2.** Let  $F$  be a set of function symbols and  $\mathbf{A}$  be an algebra over  $F$ . We denote by  $\text{Pol}(\mathbf{A})$  the smallest set containing

- (1)  $\{f^{\mathbf{A}} : f \in F\}$ ;
- (2)  $\{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$ ;
- (3) the constant 0-ary operations

and closed under composition. The elements of  $\text{Pol}(\mathbf{A})$  are called **polynomial** operations. We say that two algebras  $\mathbf{A}$  and  $\mathbf{B}$  on the same carrier are **polynomially equivalent** if  $\text{Pol}(\mathbf{A}) \simeq \text{Pol}(\mathbf{B})$ .

*Example 1.3.* If  $\varphi \in \text{Clo}_{m+n}(\mathbf{A})$  and  $(a_1, \dots, a_m) \in A^m$ , then

$$\psi : A^n \rightarrow A \quad (b_1, \dots, b_n) \mapsto \varphi(a_1, \dots, a_m, b_1, \dots, b_n)$$

is a polynomial operation.

*Remark 1.4.* Let  $\mathbf{A}$  be an algebra. An equivalence relation  $\alpha$  is a congruence of  $\mathbf{A}$  iff  $\varphi(\alpha) \subseteq \alpha$  for every  $\varphi \in \text{Pol}_1(\mathbf{A})$ .

Let  $\mathbf{A}$  be a finite algebra. We adopt the following convention, concerning the restriction  $(-|U)$  operation, for  $U \subseteq A$ :

- if  $\theta \in \text{Con}(\mathbf{A})$ ,  $\theta|U := \theta \cap U^2$ ;
- if  $\varphi : A^n \rightarrow A$ ,  $\varphi|U$  is the function  $U^n \rightarrow A$ ,  $(u_1, \dots, u_n) \mapsto \varphi(u_1, \dots, u_n)$ ;
- $\text{Pol}(\mathbf{A})|U := \{\psi|U : \psi \in \text{Pol}(\mathbf{A}) \text{ and } \psi[U^n] \subseteq U\}$ ;
- $\mathbf{A}||U := (U, \text{Pol}(\mathbf{A})|U)$ .

## 2. FINITE MINIMAL ALGEBRAS

**Definition 2.1.** A nontrivial finite algebra  $\mathbf{A}$  is **minimal** iff every nonconstant element of  $\text{Pol}_1(\mathbf{A})$  is bijective.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

*Example 2.2.* The following are examples of minimal algebras.

- (1) any algebra with carrier 2;
- (2) a nontrivial finite vector space  $\mathbf{A}$  over a finite field  $\mathbf{k}$ : every  $\pi \in \text{Pol}_1(\mathbf{A})$  is of the form  $\pi(v) = av + b$  for some  $a \in k, b \in A$ ;
- (3) a group of permutations acting on a finite set. If  $\mathbf{G}$  is a group acting on a set  $A$  each  $g \in G$  induces an operation  $\varphi_g : A \rightarrow A$  given by  $\varphi_g(a) = g \cdot a$ . Let  $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$ . A  $\mathbf{G}$ -set can be seen as an algebra  $(A, \Phi_{\mathbf{G}})$ .

We shall prove that, up to polynomial equivalence, there are no other finite minimal algebras.

**Lemma 2.3.** *Let  $\mathbf{A}$  be a minimal algebra. If every element of  $\text{Pol}(\mathbf{A})$  is essentially unary, then  $\mathbf{A}$  is polynomially equivalent to  $(A, \Phi_{\mathbf{G}})$  where  $\mathbf{G}$  is a finite group acting on  $A$ .*

*Proof.* Since  $\mathbf{A}$  is minimal,  $\text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$  is a subgroup of  $\text{Sym}(A)$ . Let  $\mathbf{G} := \text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$ . If  $\psi \in \text{Pol}(\mathbf{A})$ , either  $\psi$  is constant or  $\psi$  is essentially unary, hence  $(A, \Phi_{\mathbf{G}})$  is polynomially equivalent to  $\mathbf{A}$ .  $\square$

**Theorem 2.4** ([3]). *Let  $\mathbf{A}$  be a minimal algebra with  $|A| > 2$ . If  $\text{Pol}(\mathbf{A})$  contains an operation which is not essentially unary, then  $\mathbf{A}$  is polynomially equivalent to a  $\mathbf{k}$ -vector space for a finite field  $\mathbf{k}$ .*

**Theorem 2.5.** *Every algebra  $\mathbf{A}$  with carrier 2 is polynomially equivalent to one of the following:*

- (1)  $\mathbf{E}_0 = (2, \emptyset)$ ;
- (2)  $\mathbf{E}_1 = (2, \neg)$ ;
- (3)  $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$ ;
- (4)  $\mathbf{E}_4 = (2, \wedge, \vee)$ ;
- (5)  $\mathbf{E}_5 = (2, \vee)$ ;
- (6)  $\mathbf{E}_6 = (2, \wedge)$ .

*Each of them is not polynomially equivalent to the other<sup>1</sup>.*

<sup>1</sup>A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem 2.5 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

*Remark 2.6.* Up to isomorphism,  $\mathbf{E}_5 (\simeq \mathbf{E}_6)$  is the only semilattice with two elements, while  $\mathbf{E}_3$  and  $\mathbf{E}_4$  are the only Boolean algebra and lattice, respectively, with two elements.

**Definition 2.7.** Let  $\mathbf{A}$  be a minimal algebra. We say that  $\mathbf{A}$  is of

- (1) **type 1** (or **unary**) if  $\mathbf{A}$  is polynomially equivalent to  $(A, \Phi_{\mathbf{G}})$  for some  $\mathbf{G} \leq \text{Sym}(A)$ ;
- (2) **type 2** (or **affine**) if  $\mathbf{A}$  is polynomially equivalent to a vector space over a finite field  $\mathbf{k}$ ;
- (3) **type 3** (or **Boolean**) if  $\mathbf{A}$  is polynomially equivalent to  $\mathbf{E}_3$ ;
- (4) **type 4** (or **lattice**) if  $\mathbf{A}$  is polynomially equivalent to  $\mathbf{E}_4$ ;
- (5) **type 5** (or **semilattice**) if  $\mathbf{A}$  is polynomially equivalent to  $\mathbf{E}_5$ .

### 3. RELATIVE MINIMAL ALGEBRAS

**Definition 3.1.** Let  $\mathbf{A}$  be a finite algebra and let  $\delta < \theta \in \text{Con}(\mathbf{A})$ . We say that  $\mathbf{A}$  is  $(\delta, \theta)$ -**minimal** if for all  $\varepsilon \in \text{Pol}_1(\mathbf{A})$  either  $\varepsilon$  is bijective or  $\varepsilon(\theta) \subseteq \delta$ .

*Remark 3.2.* Observe that  $\mathbf{A}$  is minimal iff  $\mathbf{A}$  is  $(\Delta, \nabla)$ -minimal.

**Definition 3.3.** Let  $\mathbf{A}$  be a  $(\alpha, \beta)$ -minimal algebra. An  $(\alpha, \beta)$ -**trace** of  $\mathbf{A}$  is a  $\beta$ -equivalence class which contains at least two  $\alpha$ -equivalence classes.

**Lemma 3.4.** Let  $\mathbf{A}$  be a finite  $(\delta, \theta)$ -minimal algebra and let  $N$  be a  $(\delta, \theta)$ -trace. Then the algebra  $(\mathbf{A}||N)/(\delta|N)$  is minimal.

*Proof.* We need to show that for every

$$\psi \in \text{Pol}_1((\mathbf{A}||N)/(\delta|N)) = \{(\varphi|N)/(\delta|N) : \varphi \in \text{Pol}_1(\mathbf{A}), \varphi[N] \subseteq N\}$$

either  $\psi$  is bijective, or  $\psi$  is constant. Let  $\psi = (\varphi|N)/(\delta|N)$ . Since  $\mathbf{A}$  is  $(\delta, \theta)$ -minimal, either  $\varphi$  is bijective or  $\varphi(\theta) \subseteq \delta$ . Clearly, if  $\varphi$  is bijective,  $\psi$  is bijective. If  $\varphi(\theta) \subseteq \delta$ ,  $\psi$  is constant: if  $(x, y) \in N^2 \subseteq \theta$ , then  $(\psi(x), \psi(y)) \in \delta$  so that  $\psi(x) = \psi(y)$  in  $(\mathbf{A}||N)/(\delta|N)$ .  $\square$

Therefore, with an abuse of language, we shall refer unambiguously to the type of  $N$  as the type of  $(\mathbf{A}||N)/(\delta|N)$ .

**Theorem 3.5.** Let  $\mathbf{A}$  be a  $(\delta, \theta)$ -minimal algebra. Then all  $(\delta, \theta)$ -traces of  $\mathbf{A}$  have the same type.

**Definition 3.6.** Let  $\mathbf{A}$  be a finite  $(\delta, \theta)$ -minimal algebra. We say that  $\mathbf{A}$  is of type  $\mathbf{i}$  relative to  $(\delta, \theta)$  if each  $(\delta, \theta)$ -trace of  $\mathbf{A}$  is of type  $\mathbf{i}$ .

**Definition 3.7.** Let  $\mathbf{A}$  be a finite algebra and  $B, C \subseteq A$ . We say that  $B, C$  are **polynomially isomorphic** ( $B \sim C$ ) if there are  $\varphi, \psi \in \text{Pol}_1(\mathbf{A})$  such that  $\varphi[B] = C$ ,  $\psi[C] = B$  and  $\psi\varphi|B = 1_B$ ,  $\varphi\psi|C = 1_C$ .

*Remark 3.8.* If  $B, C \subseteq A$  are polynomial isomorphic in  $\mathbf{A}$ , then  $\mathbf{A}||B \simeq \mathbf{A}||C$ .

**Lemma 3.9.** *Let  $\mathbf{A}$  be a  $(\delta, \theta)$ -minimal algebra and  $N$  be a  $(\delta, \theta)$ -trace. Then*

- (1)  $(-|N) : [\Delta_A, \theta] \rightarrow \text{Con}(\mathbf{A}||N)$  is a surjective lattice homomorphism;
- (2) quotienting out by  $\delta$ , there is a surjective lattice homomorphism  $[\delta, \theta] \rightarrow \text{Con}((\mathbf{A}||N)/(\delta|N))$ ;
- (3) if any two  $(\delta, \theta)$ -traces are polynomially isomorphic, then the map of the second item is an isomorphism.

*Proof.* Clearly, the map is well defined and preserves meets. We show that

$$(\alpha \vee \beta) \cap N^2 = (\alpha \cap N^2) \vee (\beta \cap N^2).$$

Let  $(x, y) \in (\alpha \vee \beta) \cap N^2$ . Then there are  $x = x_0, \dots, x_{n+1} = y$  such that either  $(x_i, x_{i+1}) \in \alpha$  or  $(x_i, x_{i+1}) \in \beta$ . We show that for each  $i$ ,  $(x_i, x_{i+1}) \in N^2$ . Inductively, if  $x_i \in N$ , and, say,  $(x_i, x_{i+1}) \in \alpha \subseteq \theta$ , then  $x_{i+1} \in N$ . Now, for  $\beta \in \text{Con}(\mathbf{A}||N)$ , let  $\hat{\beta}$  be

$$\{(x, y) \in \theta : (\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta \quad \forall \psi \in \text{Pol}_1(\mathbf{A})\}$$

Then  $\hat{\beta}$  is a congruence. We show that  $\hat{\beta} \cap N^2 = \beta$ , proving surjectivity. If  $(x, y) \in \hat{\beta}$ , then  $(\psi(x), \psi(y)) \in N^2 \implies (\psi(x), \psi(y)) \in \beta$  for all  $\psi \in \text{Pol}_1(\mathbf{A})$ ; if  $(x, y) \in N^2$ , then  $(\psi(x), \psi(y)) \in N^2$ . Therefore  $(\psi(x), \psi(y)) \in \beta$  for all  $\psi \in \text{Pol}_1(\mathbf{A})$ , and, taking  $\psi(x) = x$ ,  $(x, y) \in \beta$ . Conversely, let  $(x, y) \in \beta$ . As  $\beta \subseteq N^2 \subseteq \theta$ ,  $(x, y) \in \theta$ . Let  $\psi \in \text{Pol}_1(\mathbf{A})$ . If  $(\psi(x), \psi(y)) \in N^2$ , then  $\psi \in \text{Pol}_1(\mathbf{A}||N)$ . Since  $\beta \in \text{Con}(\mathbf{A}||N)$ ,  $(\psi(x), \psi(y)) \in \beta$ .

The second item immediately follows. As to the third, we need to prove injectivity. Let  $\delta \leq \alpha < \beta \leq \theta$ . Let  $(x, y) \in \beta - \alpha$ . Then  $(x, y) \in \theta - \delta$  and  $P := x/\theta$  is a  $(\delta, \theta)$ -trace. Since  $P$  is polynomially isomorphic to  $N$ , there is  $\psi \in \text{Pol}_1(\mathbf{A})$  such that

$$\psi[P] = \psi[x/\theta] = \{\psi(z) : (x, z) \in \theta\} = N.$$

Then,  $(\psi(x), \psi(y)) \in (\theta - \delta) \cap N^2$ . Also,  $(\psi(x), \psi(y)) \in \beta - \alpha$ , so that  $(\psi(x), \psi(y)) \in \beta|N - \alpha|N$  and  $\alpha|N/\delta|N < \beta|N/\delta|N$ .  $\square$

The following is an example of a sufficient condition that guarantees that the lattices  $[\delta, \theta]$  and  $\text{Con}((\mathbf{A}||N)/(\delta|N))$  are isomorphic.

**Lemma 3.10.** *Let  $\mathbf{A}$  be a  $(\delta, \theta)$ -minimal algebra with  $\delta \prec \theta$ . Let  $N, K$  be two  $(\delta, \theta)$ -traces. Then  $N$  and  $K$  are polynomially isomorphic.*

*Proof.* [to complete](#)  $\square$

#### 4. TAME CONGRUENCES

**Definition 4.1.** Let  $\mathbf{A}$  be a finite algebra and  $\alpha, \beta \in \text{Con}(\mathbf{A})$  with  $\alpha < \beta$ . We denote by

- (1)  $\text{E}(\mathbf{A})$  the set of  $\varepsilon \in \text{Pol}_1(\mathbf{A})$  such that  $\varepsilon^2 = \varepsilon$ ;
- (2)  $\text{U}_{\mathbf{A}}(\alpha, \beta)$  the set  $\{\varphi[A] : \varphi \in \text{Pol}_1(\mathbf{A}), \varphi(\beta) \not\subseteq \alpha\}$ ;

- (3)  $M_{\mathbf{A}}(\alpha, \beta)$  the set of minimal (with respect to the inclusion relation) elements of  $U_{\mathbf{A}}(\alpha, \beta)$ .

*Remark 4.2.* A finite algebra  $\mathbf{A}$  is  $(\delta, \theta)$ -minimal iff  $A \in M_{\mathbf{A}}(\delta, \theta)$ . If  $\mathbf{A}$  is  $(\delta, \theta)$ -minimal, every  $\psi \in M_{\mathbf{A}}(\delta, \theta)$  such that  $\psi(\delta) \not\subseteq \theta$  is bijective. Clearly  $A \in U_{\mathbf{A}}(\delta, \theta)$ , as witnessed by  $\varepsilon(x) = x$ . If there were  $\psi \in \text{Pol}_1(\mathbf{A})$  with  $\psi(\delta) \not\subseteq \theta$  and  $\psi[A] \subset A$  we would contradict  $(\delta, \theta)$ -minimality. Conversely, let  $A \in M_{\mathbf{A}}(\delta, \theta)$  and  $\psi \in \text{Pol}_1(\mathbf{A})$  with  $\psi(\delta) \not\subseteq \theta$ ; we show that  $\psi \in \text{Sym}(A)$ . But by definition of  $M_{\mathbf{A}}(\delta, \theta)$ , there is no  $\psi \in \text{Pol}_1(\mathbf{A})$  with  $\psi(\delta) \not\subseteq \theta$  and  $\psi[A] \subset A$ .

**Lemma 4.3.** *Let  $\varepsilon \in E(\mathbf{A})$ ,  $U := \varepsilon[A]$ , and  $\emptyset \neq N \subseteq U$ . Then  $\mathbf{A}||N = (\mathbf{A}||U)||N$ .*

*Proof.* That  $(\text{Pol}(\mathbf{A})|U)|N \subseteq \text{Pol}(\mathbf{A})|N$  is obvious. Conversely, let  $\psi = \varphi|N$  for some  $\varphi \in \text{Pol}(\mathbf{A})$  with  $\varphi[N^k] \subseteq N$ . Clearly,  $\varepsilon\varphi[U^k] \subseteq U$ . If  $(a_1, \dots, a_k) \in N$ ,  $\varphi(a_1, \dots, a_k) \in N \subseteq U$ , hence  $\varphi(a_1, \dots, a_k) = \varepsilon(a)$  for some  $a \in A$ . Hence  $\varepsilon\varphi(a_1, \dots, a_k) = \varepsilon^2(a) = \varepsilon(a) \in N$  so that  $(\varepsilon\varphi|U)|N = \varphi|N$ . We have shown that  $\psi$  is an operation of  $(\mathbf{A}||U)||N$ .  $\square$

**Definition 4.4.** Let  $\mathbf{A}$  be a finite algebra and  $\alpha, \beta \in \text{Con}(\mathbf{A})$  with  $\alpha < \beta$ . The pair  $(\alpha, \beta)$  is a pair of **tame** congruences if there is  $V \in M_{\mathbf{A}}(\alpha, \beta)$ ,  $\varepsilon \in E(\mathbf{A})$  such that  $\varepsilon[A] = V$  and  $(-|V) : [\alpha, \beta] \rightarrow [\alpha|V, \beta|V]$  is 0, 1-separating

A  $(\wedge, \vee)$ -homomorphism  $\alpha : \mathbf{L} \rightarrow \mathbf{N}$  is **0, 1-separating** if  $\alpha^{-1}[\{\alpha(i)\}] = i$  for  $i = 0, 1$ .

**Theorem 4.5.** *Let  $(\alpha, \beta)$  be a tame pair of congruences of a finite algebra  $\mathbf{A}$ . For every  $U \in M_{\mathbf{A}}(\alpha, \beta)$ ,*

- (1)  $\mathbf{A}||U$  is  $(\alpha|U, \beta|U)$ -minimal;
- (2) there is  $\varepsilon \in E(\mathbf{A})$  such that  $\varepsilon[A] = U$ ;
- (3)  $(-|U) : [\delta, \theta] \rightarrow \text{Con}(\mathbf{A}||U)$  is a surjective lattice homomorphism which is 0, 1-separating.

*Moreover, any two  $(\alpha, \beta)$ -minimal sets are polynomially isomorphic.*

*Proof.* The first item does not depend on  $(\alpha, \beta)$  being tame. Let  $\varphi \in \text{Pol}_1(\mathbf{A})$  with  $\varphi[U] \subseteq U$  and  $\varphi(\beta) \not\subseteq \alpha$ . Then  $\varphi[U] = U$  by  $(\alpha, \beta)$ -minimality, so that  $\varphi \in \text{Sym}(A)$ .

As to the second: since  $(\alpha, \beta)$  is tame, there is  $V_0 \in M_{\mathbf{A}}(\alpha, \beta)$  and  $\varepsilon_0 \in E(\mathbf{A})$  such that  $V_0 = \varepsilon_0[A]$  and  $(-|V_0)$  is 0, 1-separating.

**Claim 1.** *If  $(x, y) \in \beta - \alpha$ , there is  $\varphi \in \text{Pol}_1(\mathbf{A})$  with  $\varphi[A] = V_0$  and such that  $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$ .*

*Proof.* Let  $\theta := \{(x, y) \in \beta : (\varepsilon_0\varphi(x), \varepsilon_0\varphi(y)) \in \alpha \text{ for all } \varphi \in \text{Pol}_1(\mathbf{A})\}$ . Now,  $\theta \in [\alpha, \beta]$  and  $\alpha|V_0 = \theta|V_0$ ; that  $\alpha|V_0 \subseteq \theta|V_0$  is obvious, for the converse:

$$\begin{aligned} \theta|V_0 &= \{(a, b) \in \beta \cap V_0^2 : (\varepsilon_0\varphi(a), \varepsilon_0\varphi(b)) \in \alpha \quad \forall \varphi \in \text{Pol}_1(\mathbf{A})\} \\ &= \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0\varphi\varepsilon_0(x), \varepsilon_0\varphi\varepsilon_0(y)) \in \alpha \quad \forall \varphi \in \text{Pol}_1(\mathbf{A})\} \\ &\subseteq \{(\varepsilon_0(x), \varepsilon_0(y)) \in \beta : (\varepsilon_0(x), \varepsilon_0(y)) \in \alpha\} \\ &= \{(a, b) \in \beta : (a, b) \in \alpha \cap V_0^2\} \\ &= \alpha|V_0 \end{aligned}$$

This implies that  $\theta = \alpha$ , since  $(-|V_0)$  is 0-separating. Thus  $(x, y) \in \beta - \alpha$  implies  $(x, y) \in \beta - \theta$ . By definition of  $\theta$ , there is  $\psi \in \text{Pol}_1(\mathbf{A})$  such that  $(\varepsilon_0\psi(x), \varepsilon_0\psi(y)) \notin \alpha$ . Thus  $\varphi := \varepsilon_0\psi$  satisfies the conditions  $\varphi[A] \subseteq V_0$  and  $(\varphi(x), \varphi(y)) \in \beta|V_0 - \alpha|V_0$ . Hence  $\varphi[A] = V_0$  by  $(\alpha, \beta)$ -minimality.  $\square$

**Claim 2.** *The relation  $\beta$  is the transitive closure of*

$$\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \text{Pol}_1(\mathbf{A})\}.$$

*Proof.* The transitive closure of  $\alpha \cup \{(\psi(x), \psi(y)) : (x, y) \in \beta|V_0, \psi \in \text{Pol}_1(\mathbf{A})\}$  is  $\alpha \vee \theta(\beta|V_0)$ . But  $\alpha \vee \theta(\beta|V_0) \in [\alpha, \beta]$ , and therefore, since  $(-|V_0)$  is 1-separating,  $\beta = \alpha \vee \theta(\beta|V_0)$ .  $\square$

Assume that  $U \in M_{\mathbf{A}}(\alpha, \beta)$ . Then, by definition, there is  $\mu \in \text{Pol}_1(\mathbf{A})$  such that  $\mu[A] = U$  and  $\mu(\beta) \not\subseteq \alpha$ . This implies that the equivalence relation

$$\mu^{-1}(\alpha) = \{(a, b) : (\mu(a), \mu(b)) \in \alpha\}$$

is such that  $\beta \not\subseteq \mu^{-1}(\alpha)$ . Then by Claim 2 there are  $a, b \in V_0$  and  $\psi \in \text{Pol}_1(\mathbf{A})$  such that  $(a, b) \in \beta$  and  $(\mu\psi(a), \mu\psi(b)) \notin \alpha$ . The function  $\mu_1 := \mu\psi\varepsilon_0$  satisfies  $\mu_1[A] \subseteq U$  and  $\mu_1(\beta) \not\subseteq \alpha$ : there are  $x, y \in A$  such that  $(a, b) = (\varepsilon_0(x), \varepsilon_0(y)) \in \beta$  but  $(\mu_1(a), \mu_1(b)) = (\mu\psi\varepsilon_0(x), \mu\psi\varepsilon_0(y)) \notin \alpha$ . Thus  $\mu_1[A] = U$  by  $(\alpha, \beta)$ -minimality. Observe that  $\mu_1[V_0] = \mu_1\varepsilon_0[A] = \mu\psi\varepsilon_0^2[A] = \mu_1[A]$ , so that  $\mu_1[V_0] = U$ . Apply Claim 1 to the pair  $(\mu_1(a), \mu_1(b))$  to get  $\nu \in \text{Pol}_1(\mathbf{A})$  such that  $\nu[A] = V_0$  and  $(\nu\mu_1(a), \nu\mu_1(b)) \notin \alpha$ . Now, since  $\mu_1\nu[A] = \mu_1[V_0] = U$ ,  $\mu_1\nu|U$  is bijective; since  $U$  is finite, there is  $k > 1$  such that  $(\mu_1\nu|U)^k = (\mu_1\nu)^k|U = 1_U$ . Let  $\varepsilon := (\mu_1\nu)^k$ . We have:  $\varepsilon[A] = (\mu_1\nu)^k[A] = (\mu_1\nu)^{k-1}[U] = U$  and, consequently, for all  $a \in A$ ,  $\varepsilon^2(a) = \varepsilon(a)$  since  $\varepsilon(a) \in U$ . Therefore  $\varepsilon \in E(\mathbf{A})$ .

We prove the third statement. Clearly, the map is well defined and preserves meets. For  $\theta \in [\alpha|U, \beta|U]$ , [to complete](#)

Finally, in the above notation, let  $\varphi := \nu$ ,  $\psi := (\mu_1\nu)^{k-1}\mu_1$ . Then  $\varphi[U] = V_0$ ,  $\psi[V_0] = U$  and  $\psi\varphi|U = 1_U$ ,  $\varphi\psi|V_0 = 1_{V_0}$ . Then fact that any  $(\alpha, \beta)$ -minimal set is polynomially isomorphic to  $V_0$  implies that any two  $(\alpha, \beta)$ -minimal sets are polynomially isomorphic.  $\square$

**Definition 4.6.** Let  $(\alpha, \beta)$  be tame in a finite algebra  $\mathbf{A}$ . An  $(\alpha, \beta)$ -**trace**<sup>\*</sup> of  $\mathbf{A}$  is  $N \subseteq A$  such that for some  $U \in M_{\mathbf{A}}(\alpha, \beta)$  and  $x \in U$ ,  $N \subseteq U$  and  $N = x/(\beta|U) \neq x/(\alpha|U)$ . That is,  $N$  is an  $(\alpha|U, \beta|U)$ -trace of the minimal algebra  $\mathbf{A}||U$ .

**Definition 4.7.** Let  $(\alpha, \beta)$  be tame in a finite algebra  $\mathbf{A}$ . Let  $\delta, \theta \in \text{Con}(\mathbf{A})$ . Let  $K$  be a class of algebras. We define

- (1) the **type** of  $(\alpha, \beta)$ , written  $\text{typ}(\alpha, \beta)$ , to be the type of  $\mathbf{A}||U$  relative to  $(\alpha|U, \beta|U)$ ;
- (2)  $\text{typ}\{\delta, \theta\} := \{\text{typ}(\alpha, \beta) : \delta \leq \alpha \prec \beta \leq \theta\}$ ;
- (3)  $\text{typ}\{\mathbf{A}\} := \text{typ}\{\Delta_A, \nabla_A\}$ ;
- (4)  $\text{typ}\{K\} := \cup\{\text{typ}\{\mathbf{A}\} : \mathbf{A} \in K_{\text{fin}}\}$ .

*Remark 4.8.* If  $(\Delta, \nabla)$  in  $\text{Con}(\mathbf{A})$  is tame, then  $\text{typ}(\Delta, \nabla)$  coincides with  $\text{typ}(\mathbf{A})$  of Definition 2.7.

**Lemma 4.9.** Let  $(\alpha, \beta)$  be tame in a finite algebra  $\mathbf{A}$ . For every  $(\alpha, \beta)$ -trace\* of  $\mathbf{A}$ , the algebra  $(\mathbf{A}||N)/(\alpha|N)$  is minimal and  $\text{typ}(\alpha, \beta) = \text{typ}((\mathbf{A}||N)/(\alpha|N))$ .

*Proof.* Let  $U$  be any  $(\alpha, \beta)$ -minimal set and  $N$  be an  $(\alpha|U, \beta|U)$ -trace. The algebra  $\mathbf{A}||U$  is minimal relative to  $(\alpha|U, \beta|U)$ . By Lemma 4.3  $\mathbf{A}||N = (\mathbf{A}||U)||N$  and consequently  $(\mathbf{A}||N)/(\alpha|N) = ((\mathbf{A}||U)||N)/((\alpha|U)|N)$ . The type of  $\mathbf{A}||U$  relative to  $(\alpha|U, \beta|U)$  is, by definition, the type of the minimal algebra  $\mathbf{M} := ((\mathbf{A}||U)||N)/((\alpha|U)|N)$ ; but  $M$  is the only  $(\Delta_M, \nabla_M)$ -trace of  $\mathbf{M}$ , hence this is  $\text{typ}(\mathbf{M})$ . □

## 5. OMITTING TYPES

**Definition 5.1.** Let  $\mathbf{V}$  be a variety. An algebra  $\mathbf{A} \in \mathbf{V}$  is called

- (1) **free** if there is an isomorphism  $\mathbf{A} \simeq \mathbf{F}_{\mathbf{V}}(\kappa)$  for some cardinal  $\kappa$ ;
- (2) **finitely generated** if there is a surjective homomorphism  $\mathbf{F}_{\mathbf{V}}(n) \rightarrow \mathbf{A}$  for some  $n \in \omega$ .

**Definition 5.2.** A variety  $\mathbf{V}$  is called

- (1) **locally finite** if all its finitely generated algebras are finite;
- (2) **finitely presented** if  $\mathbf{V}$  has a finite set of function symbols and  $\mathbf{V} = \text{Alg}(\Sigma)$  for a finite set of equations  $\Sigma$ ;
- (3) **finitely generated** if  $\mathbf{V} = V(\mathbf{A}_1, \dots, \mathbf{A}_n)$  for  $\mathbf{A}_1, \dots, \mathbf{A}_n$  finite similar algebras.

*Remark 5.3.* Observe that  $V(\mathbf{A}_1, \dots, \mathbf{A}_n) = V(\mathbf{A}_1 \times \dots \times \mathbf{A}_n)$ .

**Lemma 5.4.** Let  $\mathbf{V}$  be a variety. If  $\mathbf{V}$  is finitely generated then it is locally finite.

*Proof.* By the previous remark, we can assume that  $\mathbf{V} = V(\mathbf{A})$  for some  $\mathbf{A}$  finite. Let  $n < \omega$ . We prove that  $\mathbf{F}_{\mathbf{V}}(n)$  is finite. Consider the homomorphism

$$\mathbf{F}_{\mathbf{V}}(n) \rightarrow \mathbf{A}^{A^n}, \quad t(x_1, \dots, x_n) \mapsto t^{\mathbf{A}}$$

This homomorphism is injective: if  $t^{\mathbf{A}} = s^{\mathbf{A}}$ , then  $\mathbf{A} \models t \equiv s$ , i.e.  $t = s$  in  $\mathbf{F}_V(n)$ . Thus  $\mathbf{F}_V(n)$  is finite.  $\square$

**Definition 5.5.** Let  $V$  and  $W$  be two varieties. We say that  $V$  is **interpretable** into  $W$  ( $V \leq W$ ) if there is a clone homomorphism  $\text{Clo}(V) \rightarrow \text{Clo}(W)$ .

*Remark 5.6.* Let  $W, V$  be two varieties. We unravel what  $V \leq W$  means in a simple case, that is when  $V$  is finitely presented. Let  $F$  be a finite set of function symbols. Let  $V$  be a variety of algebras over  $F$ , defined by the equations

$$(1) \quad s_1 \equiv t_1, \dots, s_k \equiv t_k.$$

Assume that for each  $f \in F_n$  there is  $t \in \text{Clo}_n(W)$  such that the interpretation of the  $t$ 's satisfy the equations (1). Then the assignment  $f \mapsto t$  extends to a clone homomorphism  $\text{Clo}(V) \rightarrow \text{Clo}(W)$ . Of course, the converse also holds; thus this is equivalent to  $V \leq W$ .

**Lemma 5.7.** Let  $W, V$  be two varieties such that  $W \leq V$ . Assume that  $W$  is

- idempotent;
- finitely presented;
- the finite set of equations defining  $W$  contains at most one function symbol per side.

Let  $\mathbf{A} \in V$ ,  $\varepsilon \in E(\mathbf{A})$ ,  $U := \varepsilon[A]$ ,  $\beta \in \text{Con}(\mathbf{A})$  and  $N := a/\beta \cap U$  for  $a \in U$ . Then  $W \leq V(\mathbf{A}||N)$ . Moreover, if  $\mathbf{1} \in \text{typ}\{V\}$ , then  $W \leq \text{Set}$ .

*Proof.* By assumption  $W$  can be described by a finite set of equations of the form

$$(2) \quad f_i(x_{i_1}, \dots, x_{i_h}) \equiv f_j(x_{j_1}, \dots, x_{j_k})$$

where  $f_i$  and  $f_j$  are members of a finite set  $F$  of function symbols. Since  $W \leq V$ , there is an assignment  $f \mapsto t$  extending to a clone homomorphism. We need to find a clone homomorphism  $\text{Clo}(W) \rightarrow \text{Clo}(\mathbf{A}||N)$ . Consider  $f \mapsto \varphi := \varepsilon t_i^{\mathbf{A}}|N$ . Firstly, it is well defined: if  $(a_1, \dots, a_n) \in S$ ,  $(\varphi(a_1, \dots, a_n), \varphi(a, \dots, a)) \in \beta$  but

$$\varphi(a, \dots, a) = \varepsilon t^{\mathbf{A}}(a, \dots, a) = \varepsilon(a) \in U$$

so that  $\varphi(a_1, \dots, a_n) \in S$  and therefore  $\varphi \in \text{Pol}(\mathbf{A})|N$ . Finally, using that  $W \leq V$

$$\begin{aligned} \varphi_i(a_{i_1}, \dots, a_{i_h}) &= \varepsilon t_i^{\mathbf{A}}(a_{i_1}, \dots, a_{i_h}) \\ &= \varepsilon t_j^{\mathbf{A}}(a_{j_1}, \dots, a_{j_k}) \\ &= \varphi_j(a_{j_1}, \dots, a_{j_k}). \end{aligned}$$

If  $\mathbf{1} \in \text{typ}\{V\}$ . Then there are  $\alpha \prec \beta \in \text{Con}(\mathbf{A})$  such that  $\text{typ}(\alpha, \beta) = \mathbf{1}$ . Without loss of generality we can assume that  $\alpha = \Delta_A$ . Let  $N$  be a  $(\Delta_A, \beta)$ -trace\*. Then there are  $\varepsilon \in E(\mathbf{A})$ ,  $U := \varepsilon[A]$  such that  $N = a/\beta \cap U$  for some  $a \in U$ . Thus  $W \leq V(\mathbf{A}||N)$ .

**Claim 3.** Let  $\mathbf{M}$  be finite minimal algebra of unary type. Let  $\mathbf{A} = (M, \text{Pol}(\mathbf{M}))$ . Then  $V(\mathbf{A})$  contains a finite algebra  $\mathbf{S}$  all of whose polynomials are constant or projections.



*Proof.* Let  $\mathbf{G} := \text{Sym}(M) \cap \text{Pol}_1(\mathbf{M})$ , subgroup of  $\text{Sym}(M)$ . Let  $u, v \in M$ ,  $u \neq v$  and let

$$D := \{(\sigma(u), \sigma(v)) : \sigma \in \mathbf{G}\} \cup \{(\sigma(v), \sigma(u)) : \sigma \in \mathbf{G}\}$$

Since  $\mathbf{M}$  is minimal with  $\text{typ}(\mathbf{M}) = \mathbf{1}$ , every polynomial  $\psi$  is constant or there is  $i$ ,  $\sigma \in \mathbf{G}$  such that

$$\psi(a_1, \dots, a_n) = \sigma(a_i).$$

This implies that  $\mathbf{D}$  is a subalgebra of  $\mathbf{A}^2$ . Let

$$((x_1, x_2), (y_1, y_2)) \in \theta \iff \sigma(x_i) = y_i \text{ for some } \sigma \in \mathbf{G}$$

We show that every term operation of  $\mathbf{D}/\theta$  is either constant or a projection. Let  $\psi \in \text{Pol}_n(\mathbf{M})$  non constant and  $(a_i, b_i) \in D$  for  $i = 1, \dots, n$ . Then there is  $\tau \in \mathbf{G}$  such that

$$\begin{aligned} \psi((a_1, b_1)/\theta, \dots, (a_n, b_n)/\theta) &= \psi((a_1, b_1), \dots, (a_n, b_n))/\theta \\ &= (\psi(a_1, \dots, a_n), \psi(b_1, \dots, b_n))/\theta \\ &= (\tau(a_i), \tau(b_i))/\theta \\ &= (a_i, b_i)/\theta. \end{aligned} \quad \square$$

Let  $\mathbf{S} \in V(\mathbf{A}||N)$  be such that every term operation of  $\mathbf{S}$  is constant or a projection. Then there is a clone homomorphism  $\text{Clo}(\mathbf{A}||N) \rightarrow \text{Clo}(\mathbf{S})$ . Since  $\mathbf{W} \leq V(\mathbf{A}||N)$  there is a clone homomorphism  $\text{Clo}(\mathbf{W}) \rightarrow \text{Clo}(\mathbf{A}||N)$ . Thus we get a clone homomorphism  $\text{Clo}(\mathbf{W}) \rightarrow \text{Clo}(\mathbf{S})$ . But for every  $f \in F$ ,  $\mathbf{W}$  satisfies  $f(x, \dots, x) \equiv x$ , hence the image of  $f$  through this clone homomorphism cannot be but a projection. This implies that  $\mathbf{W} \leq \text{Set}$ .  $\square$

**Theorem 5.8.** *Let  $\mathbf{V}$  be a locally finite variety. The following are equivalent:*

- (1)  $\mathbf{1} \notin \text{typ}\{\mathbf{V}\}$ ;
- (2) *there is an idempotent variety  $\mathbf{W}$  such that  $\mathbf{W} \leq \mathbf{V}$  and  $\mathbf{W} \not\leq \text{Set}$ .*

**Corollary 5.9.** *Let  $\mathbf{A}$  be a finite idempotent algebra. There is  $\mathbf{B} \in HS(\mathbf{A})$  such that  $\text{Clo}(\mathbf{B}) \simeq \mathbf{N}$  iff  $\mathbf{1} \in \text{typ}\{HS(\mathbf{A})\}$ .*

*Proof.* If  $\mathbf{1} \in \text{typ}\{HS(\mathbf{A})\}$ , then  $\mathbf{1} \in \text{typ}\{HSP(\mathbf{A})\}$ . Since  $\mathbf{A}$  is finite, then, by Lemma 5.4  $HSP(\mathbf{A})$  is locally finite, and therefore, by Theorem 5.8, for every idempotent variety  $\mathbf{W}$ , either  $\mathbf{W} \not\leq HSP(\mathbf{A})$  or  $\mathbf{W} \leq \text{Set}$ . In particular, since  $\mathbf{A}$  is idempotent,  $HSP(\mathbf{A}) \leq \text{Set}$ . This means that there is a clone homomorphism  $\text{Clo}(\mathbf{A}) \rightarrow \mathbf{N}$ . Equivalently, every term operation of  $\mathbf{A}$  is a projection. Hence,  $\text{Clo}(\mathbf{A}) \simeq \mathbf{N}$ .  $\square$

**Theorem 5.10** ([2]). *Let  $\mathbf{A}$  be a finite idempotent algebra. Then  $\text{Clo}(\mathbf{A})$  contains a weak near-unanimity operation iff  $\mathbf{1} \notin \text{typ}\{HS(\mathbf{A})\}$ .*

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