CSP FAST TRACK

ARTURO

ABSTRACT. In this note

1. Introduction

Let R be a set of relation symbols. Let $\mathcal{A} = (A, P)$ be a relational structure over R. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathcal{A})$[1] we mean the following decision problem: given a set X of variables and a set Σ of atomic formulas[2] over R, decide whether there is an assignment $(-)^{\mathcal{A}} : X \to A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x_1, \ldots, x_n \in X$

$$r(x_1, \ldots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}} \tag{1}$$

Starting point: consider the case when A is finite.

We shall say that $\mathrm{CSP}(\mathcal{A})$ is decidable if there is a uniform (unique) algorithm deciding $\mathrm{CSP}(\mathcal{A})$ for every X and Σ over R.

Let F be a set of function symbols. Let $\mathbf{A} = (A, \Phi)$ be an algebra over F. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathbf{A})$ we mean the following decision problem: decide uniformly, that is by a unique algorithm, every $\mathrm{CSP}((A, P))$ such that $P \subseteq \mathrm{Inv}(\Phi)$.

Definition 1.1. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

Theorem 1.2. *Let $\mathbf{A}$ be a finite idempotent algebra. Then the following are equivalent:*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable;*

(2) $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation;*

(3) *for every* $\mathbf{B} \in HS(\mathbf{A})$, $\mathrm{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\mathrm{CSP}(\mathbf{A})$ *is* NP*-complete.*

[1] More often denoted by $\mathrm{CSP}(P)$.
[2] Without equality.

2. Relational Clones

Definition 2.1. Let R be a set of relation symbols and $\mathcal{A}$ be a relational structure over R. We denote by $\mathrm{Clo}(\mathcal{A})$ the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under intersection and truncation[3].

Remark 2.2. Observe that $\mathrm{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable[4] by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

Theorem 2.3. *For any $\mathcal{A} = (A, P)$ and $\mathcal{B} = (A, \Gamma)$[5] with $\Gamma \subseteq \mathrm{Clo}(\mathcal{A})$ and with Γ finite, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$.*

Proof. □

Corollary 2.4. *Let $\mathcal{A} = (A, P)$ and $\mathcal{B} = (A, \mathrm{Clo}(\mathcal{A}))$. Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

(2) $\mathrm{CSP}(\mathcal{A})$ *is* NP-*complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

[3] If $\rho \in \mathrm{Clo}(\mathcal{A})$, then also $\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \mathrm{Clo}\,\mathcal{A}$.

[4] Using the symbols in R and allowing equality.

[5] Not necessarily over the same R!