CSP FAST TRACK

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ABSTRACT. In this note

1. INTRODUCTION

Let R be a set of relation symbols. Let $\mathcal{A} = (A, P)$ be a relational structure over R. Let X be a countable set of variables. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathcal{A})$[1] we mean the following decision problem: given a set Σ of atomic formulas over R, decide whether there is an assignment $(-)^{\mathcal{A}} : X \to A$ and an interpretation of the symbols of R as relations of the appropriate arity such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x, y, x_1, \ldots, x_n \in X$

$$(1) \qquad r(x_1, \ldots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}}$$

$$(2) \qquad x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}}$$

Clearly, it is enough to find an assignment only for those variables that appear in Σ.

Starting point: consider the case when A is finite. The set of relations is finite?

We shall say that $\mathrm{CSP}(\mathcal{A})$ is decidable if there is a uniform (unique) algorithm deciding $\mathrm{CSP}(\mathcal{A})$ for every Σ over R.

Let F be a set of function symbols. Let $\mathbf{A} = (A, \Phi)$ be an algebra over F. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathbf{A})$ we mean the following decision problem: decide uniformly, that is by a unique algorithm, every $\mathrm{CSP}((A, P))$ such that $P \subseteq \mathrm{Inv}(\Phi)$.

Definition 1.1. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$$

and closed under composition.

Goal: prove

Theorem 1.2. *Let $\mathbf{A}$ be a finite idempotent algebra. Then the following are equivalent:*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable;*

(2) $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation;*

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[1]More often denoted by $\mathrm{CSP}(P)$.

(3) *for every* $\mathbf{B} \in HS(\mathbf{A})$, $\mathrm{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\mathrm{CSP}(\mathbf{A})$ *is* NP-*complete.*

2. KINDS OF OPERATIONS

Definition 2.1. An operation $\varphi : A^n \to A$ is called

(1) **essentially unary** if there is a function $\psi : A \to A$ such that
$$\varphi(a_1, \ldots, a_n) = \psi(a_i)$$
for all $a_1, \ldots, a_n \in A$.

(2) **idempotent** if $\varphi(a, \ldots, a) = a$ for all $a \in A$.

3. RELATIONAL CLONES

Definition 3.1. Let R be a set of relation symbols and $\mathcal{A}$ be a relational structure over R. We denote by $\mathrm{Clo}(\mathcal{A})$ the smallest set containing
$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$
and closed under intersection and truncation[2].

Remark 3.2. Observe that $\mathrm{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \ldots, x_n)$ such that
$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in \rho$$

Theorem 3.3. *Let G be a set and D be a finite set of relation symbols. For any $\mathcal{A} = (A, \Gamma)$ over G and $\mathcal{B} = (A, \Delta)$ over D with $\Delta \subseteq \mathrm{Clo}(\mathcal{A})$, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$.*

Proof. Let Σ be a set of atomic formulas over D. Let $d(x_1, \ldots, x_n) \in \Sigma$. For every $a_1, \ldots, a_n \in A$

(3) $\quad \mathcal{B} \models d(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in \delta \iff \mathcal{A} \models \varphi(a_1, \ldots, a_n)$

for some $\varphi(x_1, \ldots, x_n)$ of the form
$$\exists y_1, \ldots, y_m \, (g_1(z_1^1, \ldots, z_{n_1}^1) \wedge \cdots \wedge g_k(z_1^k, \ldots, z_{n_k}^k))$$
where $g_1, \ldots, g_k \in G$ and $z_j^i \in \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. We can assume (up to renaming of variables) that $y_1, \ldots, y_m$ do not appear in any formula of Σ.

Now, for each $d(x_1, \ldots, x_n) \in \Sigma$ perform the following steps:

(1) add $\{g_1(z_1^1, \ldots, z_{n_1}^1), \ldots, g_k(z_1^k, \ldots, z_{n_k}^k)\}$ to Σ;

(2) remove $d(x_1, \ldots, x_n)$ from Σ.

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[2]If $\rho \in \mathrm{Clo}(\mathcal{A})$, then also $\{(a_1, \ldots, a_{n-1}) : (a_1, \ldots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \mathrm{Clo}\,\mathcal{A}$.

At the the end we obtain a set of equations T over G. This is a polynomial-time reduction. (It's reasonable but for me this kind of stuff is like a leap of faith). By (3) it is clear that we can find an assignment $X \to A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. $\square$

Corollary 3.4. *Let* $\mathcal{A} = (A, P)$ *and* $\mathcal{B} = (A, \mathrm{Clo}(\mathcal{A}))$. *Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

(2) $\mathrm{CSP}(\mathcal{A})$ *is NP-complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

Theorem 3.5 ([1])**.** *Let* $\mathcal{A}$ *be a relational structure. If* $\mathrm{Pol}(\mathcal{A})$ *contains essentially unary operations only,* $\mathrm{CSP}(\mathcal{A})$ *is NP-complete.*

4. Surjective Algebras

Remark 4.1. Let $\mathbf{A}$ be an algebra. Every element of $\mathrm{Clo}(\mathbf{A})$ is surjective iff every element of $\mathrm{Clo}_1(\mathbf{A})$ is. In this case $\mathrm{Clo}_1(\mathbf{A})$ is a group.

Lemma 4.2. *Let* $A = \{a_1, \ldots, a_k\}$ *be a finite set. Let* $\mathbf{A} = (A, \Phi)$ *be an algebra. such that* $\mathrm{Clo}_1(\mathbf{A})$ *is a group – where do we need it? Then the relation*

$$(4) \qquad \sigma := \{(\psi(a_1), \ldots, \psi(a_k)) : \psi \in \mathrm{Clo}_1(\mathbf{A})\}$$

belongs to $\mathrm{Inv}(\Phi)$.

Proof. We show that for every $f \in F_n$ and for every matrix M

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_k^1 & \cdots & a_n^k \end{bmatrix}$$

such that $(a_1^1, \ldots, a_1^k) \in \sigma, \ldots, (a_1^n, \ldots, a_n^k) \in \sigma$ we have

$$(f^{\mathbf{A}}(a_1^1, \ldots, a_n^1), \ldots, f^{\mathbf{A}}(a_k^1, \ldots, a_n^k)) \in \sigma$$

By hypothesis we can write M as

$$\begin{bmatrix} \psi_1(a_1) & \cdots & \psi_n(a_1) \\ \vdots & \ddots & \vdots \\ \psi_1(a_k) & \cdots & \psi_n(a_k) \end{bmatrix}$$

but then

$$(f^{\mathbf{A}}(a_1^1, \ldots, a_n^1), \ldots, f^{\mathbf{A}}(a_k^1, \ldots, a_n^k))$$
$$= (f^{\mathbf{A}}(\psi_1(a_1), \ldots, \psi_n(a_1)), \ldots, f^{\mathbf{A}}(\psi_1(a_k), \ldots, \psi_n(a_k)))$$
$$= (f^{\mathbf{A}}[\psi_1, \ldots, \psi_n](a_1), \ldots, f^{\mathbf{A}}[\psi_1, \ldots, \psi_n](a_k))$$

and we conclude since $f^{\mathbf{A}}[\psi_1, \ldots, \psi_n] \in \mathrm{Clo}_1(\mathbf{A})$. $\square$

Theorem 4.3. *Let* F *be a set of function symbols and let* $\mathbf{A} = (A, \Phi)$ *be a finite surjective algebra over* F. *Let* $\mathbf{B} := (A, \mathrm{Clo}_{\mathrm{Id}}(\mathbf{A}))$. *Then*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

(2) $\mathrm{CSP}(\mathbf{A})$ *is* NP-*complete iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

Proof. Let $A = \{a_1, \ldots, a_k\}$ and let $\Delta := \{\{a_1\}, \ldots, \{a_k\}\}$. For every $\rho \in \mathrm{Inv}(\Phi)$, let r be a relation symbol of the same arity and let $R := \{r : \rho \in \mathrm{Inv}(\Phi)\}$. For every $\delta_i := \{a_i\} \in \Delta$, let d_i be a relation symbol of the same arity and let $G := R \cup \{d_i : \delta_i \in \Delta\}$. Let $\mathcal{A} := (A, \mathrm{Inv}(\Phi))$ and $\mathcal{B} := (A, \mathrm{Inv}(\Phi) \cup \Delta)$.

By definition and by Remark $\mathrm{CSP}(\mathbf{B})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathbf{A})$ iff $\mathrm{CSP}(\mathcal{B})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathcal{A})$.

That $\mathrm{CSP}(\mathcal{A})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{B})$ is obvious. Let Σ be a set of atomic formulas over G and let $\{x_1, \ldots, x_k\}$ be variables that do not appear in Σ. By Remark 4.1, since $\mathbf{A}$ is surjective, $\mathrm{Clo}_1(\mathbf{A})$ forms a group. Moreover, the relation σ of Lemma 4.2 belongs to $\mathrm{Inv}(\Phi)$. Now, perform the following steps:

(1) replace every formula $d_i(x)$ with $x \equiv x_i$;

(2) add the formula $s(x_1, \ldots, x_k)$.

At the the end we obtain a set of equations T over R. This is a polynomial-time reduction. We finally show that we can find an assignment such that $\mathcal{A} \models T$ iff we can find an assignment such that $\mathcal{B} \models \Sigma$. Let $(-)^{\mathcal{B}} : X \to A$ be an assignment such that $\mathcal{B} \models \Sigma$. Consider the assignment

$$(x)^{\mathcal{A}} = \begin{cases} (x)^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then $(-)^{\mathcal{A}}$ is such that $\mathcal{A} \models T$. Conversely, assume that there is an assignment $()^{\mathcal{A}}$ such that $\mathcal{A} \models T$. There is $\psi \in \mathrm{Clo}_1(\mathbf{A})$ such that $x_i^{\mathcal{A}} = \psi(a_i)$ for all i. Consider $(-)'^{\mathcal{A}} := \psi^{-1}(-)^{\mathcal{A}}$. Every relation in $\mathrm{Inv}(\Phi)$ is invariant under ψ^{-1}, hence defining $(x)^{\mathcal{B}} := (x)'^{\mathcal{A}}$ is enough to have $\mathcal{B} \models \Sigma$. $\square$

5. Subalgebras and Images

Theorem 5.1. *Let* $\mathbf{A}$ *be a finite algebra.*

(1) *if* $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable, so is* $\mathrm{CSP}(\mathbf{B})$ *for every* $\mathbf{B} \leq \mathbf{A}$*;*

(2) *if there is* $\mathbf{B} \leq \mathbf{A}$ *such that* $\mathrm{CSP}(\mathbf{B})$ *is* NP-*complete, so is* $\mathrm{CSP}(\mathbf{A})$*.*

Proof. Let $\mathbf{A} = (A, \Phi)$. Observe that $\mathrm{Inv}(\Phi|B) \subseteq \mathrm{Inv}(\Phi)$. $\square$

Theorem 5.2. *Let* $\mathbf{A}, \mathbf{B}$ *be two algebras.*

(1) *if* $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable, so is* $\mathrm{CSP}(\mathbf{B})$ *for every surjective homomorphism* $\alpha : \mathbf{A} \to \mathbf{B}$*;*

(2) *if there is a surjective homomorphism* $\alpha : \mathbf{A} \to \mathbf{B}$ *such that* $\mathrm{CSP}(\mathbf{B})$ *is* NP-*complete, so is* $\mathrm{CSP}(\mathbf{A})$*.*

Proof. Let $\mathbf{A} = (A, \Phi_A)$, $\mathbf{B} = (B, \Phi_B)$ over the same function symbols F.

$\square$

References

[1] Jeavons, P. (1998). On the algebraic structure of combinatorial problems, *Theoretical Computer Science* 200, 185–204.