

# CSP FAST TRACK

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ABSTRACT. In this note

## 1. INTRODUCTION

Let  $R$  be a set of relation symbols. Let  $\mathcal{A}$  be a relational structure over  $R$ . Let  $X$  be a countable set of variables. By the **constraint satisfaction problem**  $\text{CSP}(\mathcal{A})$ <sup>1</sup> we mean the following decision problem: given a finite set  $\Sigma$  of atomic formulas over  $R$ , decide whether there is an assignment  $(-)^{\mathcal{A}} : X \rightarrow A$  such that  $\mathcal{A} \models \Sigma$ ; i.e. for all  $r \in R_n$  and for all  $x, y, x_1, \dots, x_n \in X$

- (1)  $r(x_1, \dots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}}$
- (2)  $x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}}$

Clearly, it is enough to find an assignment only for those variables that appear in  $\Sigma$ .

*Remark 1.1.* Usually we deal with *indexed* relational structures, that is we fix a set of relation symbols  $R$  and we consider a set  $A$  with a set  $P$  of relations on  $A$  indexed by the elements of  $R$ . Sometimes is useful to deal with *non-indexed* relational structures  $(A, P)$ . In this case  $P$  will serve as the index set as well. The same applies to algebraic structures (i.e. algebras) and function symbols.

**Definition 1.2.** Let  $\rho \in A^k$ , and  $\varphi : A^n \rightarrow A$ . We say that  $\varphi$  **preserves**  $\rho$  if given a matrix

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

with  $(a_1^1, \dots, a_n^1) \in \rho, \dots, (a_1^k, \dots, a_n^k) \in \rho$ ,

$$(\varphi(a_1^1, \dots, a_n^1), \dots, \varphi(a_1^k, \dots, a_n^k)) \in \rho$$

**Definition 1.3.** If  $\Gamma$  is a set of relations on  $A$  and  $\Phi$  is a set of operations on  $A$  we denote by

- (1)  $\text{Inv}(\Phi)$  the set of relations on  $A$  that are preserved by all the elements of  $\Phi$ ;
- (2)  $\text{Pol}(\Gamma)$  the set of operations on  $A$  that preserve all the elements of  $\Gamma$ .

Moreover, if  $\mathbf{A}$  is an algebra and  $\mathcal{A}$  a relational structure on the same set  $A$ :

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<sup>1</sup>More often denoted by  $\text{CSP}(P)$ .

- (1)  $\text{Inv}(\mathbf{A})$  is the set of relations that are preserved by all the  $f^{\mathbf{A}}$ ;
- (2)  $\text{Pol}(\mathcal{A})$  is the set of operations that preserve all the  $r^{\mathcal{A}}$ .

Let  $\mathbf{A}$  be an algebra. Let  $\mathcal{A} := (A, \text{Inv}(\mathbf{A}))$ . By  $\text{CSP}(\mathbf{A})$  we mean the decision problem  $\text{CSP}(\mathcal{A})$ .

**Definition 1.4.** Let  $F$  be a set of function symbols and  $\mathbf{A}$  be an algebra over  $F$ . We denote by  $\text{Clo}(\mathbf{A})$  the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

**Theorem 1.5.** *Let  $\mathbf{A}$  be a finite algebra. Then the following are equivalent:*

- (1)  $\text{CSP}(\mathbf{A})$  is polynomial-time decidable;
- (2)  $\text{Clo}(\mathbf{A})$  contains a weak near-unanimity operation;
- (3) for every  $\mathbf{B} \in \text{HS}(\mathbf{A})$ ,  $\text{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$ .

Otherwise,  $\text{CSP}(\mathbf{A})$  is NP-complete.

## 2. KINDS OF OPERATIONS

**Definition 2.1.** An operation  $\varphi : A^n \rightarrow A$  is called

- (1) **essentially unary** if there is an index  $i$  and a non-constant function  $\psi : A \rightarrow A$  such that

$$\varphi(a_1, \dots, a_n) = \psi(a_i)$$

for all  $a_1, \dots, a_n \in A$ .

- (2) **idempotent** if  $\varphi(a, \dots, a) = a$  for all  $a \in A$ .

- (3) a **near unanimity** operation if for all  $a, b \in A$

$$\varphi(b, a, \dots, a) = \varphi(a, b, \dots, a) = \dots = \varphi(a, \dots, a, b) = b$$

*Example 2.2.* A ternary near unanimity operation is called a **majority** operation. For instance the ternary function defined as

$$\delta(a, b, c) = \begin{cases} b & \text{if } b = c \\ a & \text{otherwise} \end{cases}$$

is a majority operation called **dual discriminator**.

## 3. RELATIONAL CLONES

**Definition 3.1.** Let  $R$  be a set of relation symbols and  $\mathcal{A}$  be a relational structure over  $R$ . We denote by  $\text{Clo}(\mathcal{A})$  the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under

- (1) **permutation:** if  $\rho \in \text{Clo}(\mathcal{A})$ , then also

$$\{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) : \sigma \in S_n, (a_1, \dots, a_n) \in \rho\} \in \text{Clo} \mathcal{A}$$

- (2) **extension:** if  $\rho \in \text{Clo}(\mathcal{A})$ , then also

$$\{(a_1, \dots, a_n, a_{n+1}) : (a_1, \dots, a_n) \in \rho, a_{n+1} \in A\} \in \text{Clo} \mathcal{A}$$

- (3) **truncation:** if  $\rho \in \text{Clo}(\mathcal{A})$ , then also

$$\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \text{Clo} \mathcal{A}$$

- (4) intersection.

*Remark 3.2.* Observe that  $\text{Clo}(\mathcal{A})$  is given by all the relations  $\rho$  of  $A$  definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that  $\rho \subseteq A^n$  is definable if there is a formula  $\varphi(x_1, \dots, x_n)$  such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

**Theorem 3.3** ([2]). *For any pair of relational structures  $\mathcal{A} = (A, \Gamma)$  and  $\mathcal{B} = (A, H)$  such that  $H$  is finite and  $H \subseteq \text{Clo}(\mathcal{A})$ ,  $\text{CSP}(\mathcal{B})$  is polynomial-time reducible to  $\text{CSP}(\mathcal{A})$ .*

*Proof.* Let  $\Sigma$  be a set of atomic formulas over  $H$ . Let  $\eta(x_1, \dots, x_n) \in \Sigma$ . Then there is  $\varphi(x_1, \dots, x_n)$  of the form

$$\exists y_1, \dots, y_m (\gamma_1(z_1^1, \dots, z_{n_1}^1) \wedge \dots \wedge \gamma_k(z_1^k, \dots, z_{n_k}^k))$$

(where  $\gamma_1, \dots, \gamma_k \in \Gamma$  and  $z_j^i \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$ ) such that for  $a_1, \dots, a_n \in A$

$$(3) \quad \mathcal{B} \models \eta(a_1, \dots, a_n) \iff \mathcal{A} \models \varphi(a_1, \dots, a_n)$$

We can assume (up to renaming of variables) that  $y_1, \dots, y_m$  do not appear in any formula of  $\Sigma$ .

Now, for each  $\eta(x_1, \dots, x_n) \in \Sigma$  perform the following steps:

- (1) add  $\{\gamma_1(z_1^1, \dots, z_{n_1}^1), \dots, \gamma_k(z_1^k, \dots, z_{n_k}^k)\}$  to  $\Sigma$ ;
- (2) remove  $\eta(x_1, \dots, x_n)$  from  $\Sigma$ .

At the the end we obtain a set of equations  $T$  over  $G$ . **This is a polynomial-time reduction.** By (3) it is clear that we can find an assignment  $X \rightarrow A$  such that  $\mathcal{B} \models \Sigma$  iff we can find an assignment such that  $\mathcal{A} \models T$ .  $\square$

**Corollary 3.4.** *Let  $\mathcal{A}$  be a relational structure and  $\mathcal{B} = (A, \text{Clo}(\mathcal{A}))$ . Then*

- (1)  $\text{CSP}(\mathcal{A})$  is polynomial-time decidable iff  $\text{CSP}(\mathcal{B})$  is.
- (2)  $\text{CSP}(\mathcal{A})$  is NP-complete iff  $\text{CSP}(\mathcal{B})$  is.

*Proof.* We prove that  $\text{CSP}(\mathcal{A})$  is polynomial-time equivalent to  $\text{CSP}(\mathcal{B})$ . □

#### 4. SURJECTIVE ALGEBRAS

**Definition 4.1.** An algebra  $\mathbf{A}$  is **surjective** if all the element of  $\text{Clo}(\mathbf{A})$  are surjective.

*Remark 4.2.* Let  $\mathbf{A}$  be a finite algebra over  $F$ . Every element of  $\text{Clo}(\mathbf{A})$  is surjective iff every element of  $\text{Clo}_1(\mathbf{A})$  is. In this case  $\text{Clo}_1(\mathbf{A})$  is a group. It is enough to show that for every  $f \in F$ , there is  $\varphi \in \text{Clo}_1(\mathbf{A})$  such that  $f^{\mathbf{A}}\varphi = 1_{\mathbf{A}}$ . Let  $m := |A|$ . Then  $(f^{\mathbf{A}})^{m!} = 1_{\mathbf{A}}$ . Let  $n_f$  be the least  $n$  such that  $(f^{\mathbf{A}})^{n_f} = 1_{\mathbf{A}}$ . Let  $\varphi := (f^{\mathbf{A}})^{n_f-1}$ .

Let  $\mathbf{A}$  be a finite algebra and let  $B$  be a subset of  $A$ . Let

$$\text{Clo}(\mathbf{A})|B := \{\varphi \in \text{Clo}(\mathbf{A}) : \varphi|B \in O_{\mathbf{B}}\}$$

We denote by  $\mathbf{A}|B$  the algebra  $(B, \text{Clo}(\mathbf{A})|B)$ .

**Lemma 4.3** ([2]). Let  $\mathcal{A} = (A, P)$  be a relational structure. Let  $\varphi \in \text{Pol}(\mathcal{A})_1$ . For  $\rho \in P_n$  let

$$\varphi(\rho) := \{(\varphi(a_1), \dots, \varphi(a_n)) : (a_1, \dots, a_n) \in \rho\}$$

Let  $\mathcal{B} := (A, \varphi(P))$  where  $\varphi(P) := \{\varphi(\rho) : \rho \in P\}$ . Then

- (1)  $\text{CSP}(\mathcal{A})$  is polynomial-time decidable iff  $\text{CSP}(\mathcal{B})$  is;
- (2)  $\text{CSP}(\mathcal{A})$  is NP-complete iff  $\text{CSP}(\mathcal{B})$  is.

*Proof.* We show that  $\text{CSP}(\mathcal{A})$  is polynomial-time equivalent to  $\text{CSP}(\mathcal{B})$ . Let  $\Sigma$  be a set of atomic formulas over  $P$ . Replace every occurrence of  $\rho(x_1, \dots, x_n)$  with  $\varphi(\rho)(x_1, \dots, x_n)$ , obtaining a set of formulas  $T$  over  $\varphi(P)$ . Given any assignment  $(-)^{\mathcal{B}} : X \rightarrow B$  such that  $\mathcal{B} \models T$ , define  $(x)^{\mathcal{A}} := (x)^{\mathcal{B}}$ ; then  $\mathcal{A} \models \Sigma$ . Conversely, given any assignment  $(-)^{\mathcal{A}} : X \rightarrow A$  such that  $\mathcal{A} \models \Sigma$ , defining  $(x)^{\mathcal{B}} := \varphi(x^{\mathcal{A}})$  is enough to have  $\mathcal{B} \models T$ . □

**Theorem 4.4.** Let  $\mathbf{A}$  be a finite algebra. Then there is  $B \subseteq A$  such that  $\mathbf{A}|B$  is surjective and

- (1)  $\text{CSP}(\mathbf{A})$  is polynomial-time decidable iff  $\text{CSP}(\mathbf{A}|B)$  is;
- (2)  $\text{CSP}(\mathbf{A})$  is NP-complete iff  $\text{CSP}(\mathbf{A}|B)$ .

*Proof.* Assume that  $\mathbf{A}$  is not surjective. Then by Remark 4.2 there is  $\psi \in \text{Clo}_1(\mathbf{A})$  not surjective. Let  $\varphi \in \text{Clo}_1(\mathbf{A})$  such that  $\varphi$  is not surjective and  $\varphi[A]$  has minimal cardinality. Define  $B := \varphi[A]$ . We show that  $\mathbf{A}|B$  is surjective. Let  $\psi \in \text{Clo}_1(\mathbf{A}|B) = \text{Clo}_1(\mathbf{A})|B$ ; if, towards a contradiction,  $\psi[B] \subset B$ , then  $\psi\varphi[A] \subset$

$\varphi[A] \subset A$  contradicting the minimality. The last part of the statement follows immediately from Lemma 4.3.  $\square$

**Definition 4.5.** Let  $\mathbf{A}$  be an algebra. Let  $\text{id}(A)$  be the set of idempotent operations on  $A$ . We define  $\text{Clo}_{\text{id}}(\mathbf{A}) := \text{Clo}(\mathbf{A}) \cap \text{id}(A)$ .

*Remark 4.6.* Let  $\mathbf{A}$  be an algebra. Observe that an operation  $\varphi \in \text{Clo}(\mathbf{A})$  is idempotent iff it preserves the relations in  $\Delta^{(1)} = \{\{a\} : a \in A\}$ . Hence  $\text{Inv}(\text{Clo}_{\text{id}}(\mathbf{A})) = \text{Clo}(\mathcal{A})$  where  $\mathcal{A} = (A, \text{Inv}(\mathbf{A}) \cup \Delta^{(1)})$ .

**Lemma 4.7.** Let  $A = \{a_1, \dots, a_k\}$  be a finite set. Let  $\mathbf{A}$  be an algebra over  $F$ . Then the relation

$$(4) \quad \sigma := \{(\psi(a_1), \dots, \psi(a_k)) : \psi \in \text{Clo}_1(\mathbf{A})\}$$

belongs to  $\text{Inv}(\mathbf{A})$ .

*Proof.* We show that for every  $f \in F_n$  and for every matrix  $M$

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

such that  $(a_1^1, \dots, a_n^1) \in \sigma, \dots, (a_1^k, \dots, a_n^k) \in \sigma$  we have

$$(f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \in \sigma$$

By hypothesis we can write  $M$  as

$$\begin{bmatrix} \psi_1(a_1) & \cdots & \psi_n(a_1) \\ \vdots & \ddots & \vdots \\ \psi_1(a_k) & \cdots & \psi_n(a_k) \end{bmatrix}$$

but then

$$\begin{aligned} & (f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \\ &= (f^{\mathbf{A}}(\psi_1(a_1), \dots, \psi_n(a_1)), \dots, f^{\mathbf{A}}(\psi_1(a_k), \dots, \psi_n(a_k))) \\ &= (f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_1), \dots, f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_k)) \end{aligned}$$

and we conclude since  $f^{\mathbf{A}}[\psi_1, \dots, \psi_n] \in \text{Clo}_1(\mathbf{A})$ .  $\square$

**Theorem 4.8.** Let  $\mathbf{A} = A$  be a finite surjective algebra. Let  $\mathbf{B} := (A, \text{Clo}_{\text{id}}(\mathbf{A}))$ . Then

- (1)  $\text{CSP}(\mathbf{A})$  is polynomial-time decidable iff  $\text{CSP}(\mathbf{B})$  is.
- (2)  $\text{CSP}(\mathbf{A})$  is NP-complete iff  $\text{CSP}(\mathbf{B})$  is.

*Proof.* Let  $A = \{a_1, \dots, a_k\}$  and let  $\Gamma := \Delta^{(1)}$ . Let  $\mathcal{A} := (A, \text{Inv}(\mathbf{A}))$  and  $\mathcal{B} := (A, \text{Inv}(\mathbf{A}) \cup \Gamma)$ .

By definition and by Remark 4.6  $\text{CSP}(\mathbf{B})$  is polynomial-time equivalent to  $\text{CSP}(\mathbf{A})$  iff  $\text{CSP}(\mathcal{B})$  is polynomial-time equivalent to  $\text{CSP}(\mathcal{A})$ .

That  $\text{CSP}(\mathcal{A})$  is polynomial-time reducible to  $\text{CSP}(\mathcal{B})$  is obvious. Let  $\Sigma$  be a set of atomic formulas over  $\text{Inv}(\mathbf{A}) \cup \Gamma$  and let  $\{x_1, \dots, x_k\}$  be variables that do not appear in  $\Sigma$ . By Remark 4.2, since  $\mathbf{A}$  is surjective,  $\text{Clo}_1(\mathbf{A})$  forms a group. Moreover, the relation  $\sigma$  of Lemma 4.7 belongs to  $\text{Inv}(\mathbf{A})$ . Now, perform the following steps:

- (1) replace every formula  $\gamma_i(x)$  with  $x \equiv x_i$ ;
- (2) add the formula  $\sigma(x_1, \dots, x_k)$ .

At the end we obtain a set of equations  $T$  over  $\text{Inv}(\mathbf{A})$ . **This is a polynomial-time reduction.** We finally show that we can find an assignment such that  $\mathcal{A} \models T$  iff we can find an assignment such that  $\mathcal{B} \models \Sigma$ . Let  $(-)^{\mathcal{B}} : X \rightarrow A$  be an assignment such that  $\mathcal{B} \models \Sigma$ . Consider the assignment

$$(x)^{\mathcal{A}} = \begin{cases} (x)^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then  $(-)^{\mathcal{A}}$  is such that  $\mathcal{A} \models T$ . Conversely, assume that there is an assignment  $(-)^{\mathcal{A}}$  such that  $\mathcal{A} \models T$ . By definition of  $\sigma$ , there is  $\psi \in \text{Clo}_1(\mathbf{A})$  such that  $x_i^{\mathcal{A}} = \psi(a_i)$  for all  $i$ . Consider  $(-)^{\mathcal{A}'} := \psi^{-1} \circ (-)^{\mathcal{A}}$ . Every relation in  $\text{Inv}(\mathbf{A})$  is invariant under  $\psi^{-1}$ , hence defining  $(x)^{\mathcal{B}} := (x)^{\mathcal{A}'}$  is enough to have  $\mathcal{B} \models \Sigma$ .  $\square$

## 5. SUBALGEBRAS AND IMAGES

**Theorem 5.1.** *Let  $\mathbf{A}$  be a finite algebra.*

- (1) *if  $\text{CSP}(\mathbf{A})$  is polynomial-time decidable, so is  $\text{CSP}(\mathbf{B})$  for every  $\mathbf{B} \leq \mathbf{A}$ ;*
- (2) *if there is  $\mathbf{B} \leq \mathbf{A}$  such that  $\text{CSP}(\mathbf{B})$  is NP-complete, so is  $\text{CSP}(\mathbf{A})$ .*

*Proof.* Observe that  $\text{Inv}(\mathbf{B}) \subseteq \text{Inv}(\mathbf{A})$ .  $\square$

**Theorem 5.2.** *Let  $\mathbf{A}, \mathbf{B}$  be two finite algebras of the same type.*

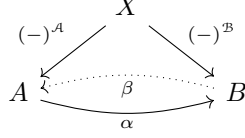
- (1) *if  $\text{CSP}(\mathbf{A})$  is polynomial-time decidable, so is  $\text{CSP}(\mathbf{B})$  for every surjective homomorphism  $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ ;*
- (2) *if there is a surjective homomorphism  $\alpha : \mathbf{A} \rightarrow \mathbf{B}$  such that  $\text{CSP}(\mathbf{B})$  is NP-complete, so is  $\text{CSP}(\mathbf{A})$ .*

*Proof.* Let  $\mathcal{B} = (B, \text{Inv}(\mathbf{B}))$ . We show that there is  $\Gamma \subseteq \text{Inv}(\mathbf{B})$  such that, defining  $\mathcal{A} = (A, \Gamma)$ ,  $\text{CSP}(\mathcal{B})$  is polynomial-time reducible to  $\text{CSP}(\mathcal{A})$ . For every  $\rho \in \text{Inv}(\mathbf{B})_n$  let

$$\alpha^{-1}(\rho) := \{(a_1, \dots, a_n) \in A^n : (\alpha(a_1), \dots, \alpha(a_n)) \in \rho\}$$

Clearly,  $\alpha^{-1}(\rho) \in \text{Inv}(\mathbf{A})_n$  and therefore, letting  $\Gamma := \{\alpha^{-1}(\rho) : \rho \in \Gamma\}$ ,  $\Gamma \subseteq \text{Inv}(\mathbf{A})$ . Let  $\Sigma$  be a set of atomic formulas over  $\text{Inv}(\mathbf{B})$ . Replace every formula  $\rho(x_1, \dots, x_n)$  with  $\alpha^{-1}(\rho)(x_1, \dots, x_n)$ . We obtain a set of equations  $T$  over  $\Gamma$  by a

polynomial-time reduction. Let  $\beta$  be a section of  $\alpha$ .



Referring to the assignments in the picture, each defined in terms of the other so that the diagram commute, it is clear that  $\mathcal{A} \models T$  iff  $\mathcal{B} \models \Sigma$ .  $\square$

## 6. SIMPLE AND STRICTLY SIMPLE ALGEBRAS

**Definition 6.1.** An algebra  $\mathbf{A}$  is called **simple** if the lattice of congruences on  $\mathbf{A}$  is  $\{\Delta, \nabla\}$ . An algebra  $\mathbf{A}$  is **strictly simple** if it is simple and has no nontrivial subalgebras.

*Remark 6.2.* Since the lattice of congruence of  $\mathbf{A}$  is in bijection with the set of surjective homomorphisms from  $\mathbf{A}$ ,  $\mathbf{A}$  is simple if whenever there is a surjective homomorphism  $\mathbf{A} \rightarrow \mathbf{B}$  such that  $|B| < |A|$ , then  $\mathbf{B}$  is the trivial algebra.

**6.1. Group actions.** Let  $\mathbf{G}$  be a group acting on a set  $A$ .

There is a well-defined function

$$G \times A \rightarrow A \times A, (g, a) \mapsto (a, g \cdot a)$$

**Definition 6.3.** The action is called

- (1) **transitive** if the above map is surjective;
- (2) **free** if the above map is injective;
- (3) **regular** if the above map is bijective.

We say that the group  $\mathbf{G}$  is **regular** if the action is.

**Definition 6.4.** We say that  $\mathbf{G}$  is (or, more accurately, that the action is) **primitive** if  $\{A\}$  and  $\{\{a\} : a \in A\}$  are the only partitions invariant under  $\mathbf{G}$ .

*Remark 6.5.* Each  $g \in G$  induces an operation  $\varphi_g : A \rightarrow A$  given by  $\varphi_g(a) = g \cdot a$ . Let  $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$ . Thus a  $\mathbf{G}$ -set can be seen as an algebra  $(A, \Phi_{\mathbf{G}})$ . The action is primitive iff this algebra is simple.

## 6.2. The Classification Theorem.

**Theorem 6.6 ([3]).** Let  $\mathbf{A}$  be a finite strictly simple surjective algebra. If  $\mathbf{A}$  has trivial subalgebras,  $\mathbf{A}$  is term-equivalent to one of the following:

- (1)  $(A, \mathcal{R}(\mathbf{G}))$  for some regular group  $\mathbf{G}$ ;
- (2)  $(A, \Phi_{\mathbf{G}})$  for some primitive group  $\mathbf{G}$ ;

or

- (3) *there is a finite field  $k$  and a  $k$ -vector space  $\mathbf{E}$  with carrier  $A$  such that letting  $\mathbf{E}'$  be the  $\text{End}_k(\mathbf{E})$ -module with natural action*

$$\text{End}_k(\mathbf{E}) \times A \rightarrow A, (\eta, a) \rightarrow \eta(a)$$

*$\mathbf{A}$  is term-equivalent to  $(A, \text{Clo}_{\text{id}}(\mathbf{E}') \cup T(\mathbf{E}))$ .*

*If  $\mathbf{A}$  has nontrivial subalgebras,  $\mathbf{A}$  is term-equivalent to one of the following:*

(1)

**Lemma 6.7** ([2]). *Let  $\mathcal{A}$  be a relational structure. If  $\text{Pol}(\mathcal{A})$  contains essentially unary operations only,  $\text{CSP}(\mathcal{A})$  is NP-complete.*

**Lemma 6.8.** *Let  $\mathcal{A}$  be a relational structure. If  $\text{Pol}(\mathcal{A})$  contains a majority operation,  $\text{CSP}(\mathcal{A})$  is polynomial-time decidable.*

**Theorem 6.9.** *Let  $\mathbf{A}$  be a finite strictly simple surjective algebra. If all the elements of  $\text{Clo}(\mathbf{A})$  are essentially unary, then  $\text{CSP}(\mathbf{A})$  is NP-complete.*

*Proof.* If  $\mathbf{A}$  is of kind (1), we prove that the dual discriminator operation  $\delta$  of Example 2.2 belongs to  $\text{Clo}(\mathbf{A})$ . But  $\delta \in \text{Clo}(\mathbf{A})$  if, by definition of  $\mathcal{R}(\mathbf{G})$ ,

□

**Corollary 6.10.** *Let  $\mathbf{A}$  be a finite algebra. If there is  $\mathbf{B} \in HS(\mathbf{A})$  such that  $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$ , then  $\text{CSP}(\mathbf{A})$  is NP-complete.*

*Proof.* We can assume without loss of generality that  $\mathbf{B}$  is surjective by Theorem 4.8. Then  $\text{Clo}_1(\mathbf{B})$  forms a group. This implies that either  $\mathbf{B}$  is idempotent or has no nontrivial nonproper subalgebras. **Why we can dispose of strictly simple?** If  $\mathbf{B}$  is such that  $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$ , then  $\text{CSP}(\mathbf{B})$  is NP-complete by Theorem 6.9. We conclude using the second clauses of Theorems 5.1 and 5.2. □

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