

CSP FAST TRACK

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ABSTRACT. In this note

1. INTRODUCTION

Let R be a set of relation symbols. Let \mathcal{A} be a relational structure over R . Let X be a countable set of variables. By the **constraint satisfaction problem** $\text{CSP}(\mathcal{A})$ ¹ we mean the following decision problem: given a finite set Σ of atomic formulas over R , decide whether there is an assignment $(-)^{\mathcal{A}} : X \rightarrow A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x, y, x_1, \dots, x_n \in X$

- (1) $r(x_1, \dots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}}$
- (2) $x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}}$

Clearly, it is enough to find an assignment only for those variables that appear in Σ .

Remark 1.1. Usually we deal with *indexed* relational structures, that is we fix a set of relation symbols R and we consider a set A with a set P of relations on A indexed by the elements of R . Sometimes is useful to deal with *non-indexed* relational structures (A, P) . In this case P will serve as the index set as well. The same applies to algebraic structures (i.e. algebras) and function symbols.

Definition 1.2. Let $\rho \in A^k$, and $\varphi : A^n \rightarrow A$. We say that φ **preserves** ρ if given a matrix

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

with $(a_1^1, \dots, a_n^1) \in \rho, \dots, (a_1^k, \dots, a_n^k) \in \rho$,

$$(\varphi(a_1^1, \dots, a_n^1), \dots, \varphi(a_1^k, \dots, a_n^k)) \in \rho$$

Definition 1.3. If Γ is a set of relations on A and Φ is a set of operations on A we denote by

- (1) $\text{Inv}(\Phi)$ the set of relations on A that are preserved by all the elements of Φ ;
- (2) $\text{Pol}(\Gamma)$ the set of operations on A that preserve all the elements of Γ .

Moreover, if \mathbf{A} is an algebra and \mathcal{A} a relational structure on the same set A :

¹More often denoted by $\text{CSP}(P)$.

- (1) $\text{Inv}(\mathbf{A})$ is the set of relations that are preserved by all the $f^{\mathbf{A}}$;
- (2) $\text{Pol}(\mathcal{A})$ is the set of operations that preserve all the $r^{\mathcal{A}}$.

Let \mathbf{A} be an algebra. Let $\mathcal{A} := (A, \text{Inv}(\mathbf{A}))$. By $\text{CSP}(\mathbf{A})$ we mean the decision problem $\text{CSP}(\mathcal{A})$.

Definition 1.4. Let F be a set of function symbols and \mathbf{A} be an algebra over F . We denote by $\text{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Recall that a variety is a class of algebras closed under homomorphic images, subalgebras and products. If \mathbf{V} is a variety, we define $\text{Clo}(\mathbf{V})$ to be $\text{Clo}(\mathbf{F}_{\mathbf{V}}(\omega))$, where $\mathbf{F}_{\mathbf{V}}(\omega)$ is the \mathbf{V} -free algebras generated by a countable number of generators.

Goal: prove

Theorem 1.5. *Let \mathbf{A} be a finite idempotent algebra. Then the following are equivalent:*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable;
- (2) $\text{Clo}(\mathbf{A})$ contains a weak near-unanimity operation;
- (3) for every $\mathbf{B} \in \text{HS}(\mathbf{A})$, $\text{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\text{CSP}(\mathbf{A})$ is NP-complete.

Observe that

2. KINDS OF OPERATIONS

Definition 2.1. An operation $\varphi : A^n \rightarrow A$ is called

- (1) **essentially unary** if there is an index i and a non-constant function $\psi : A \rightarrow A$ such that

$$\varphi(a_1, \dots, a_n) = \psi(a_i)$$

for all $a_1, \dots, a_n \in A$.

- (2) **idempotent** if $\varphi(a, \dots, a) = a$ for all $a \in A$.

- (3) a **near unanimity** operation if for all $a, b \in A$

$$\varphi(b, a, \dots, a) = \varphi(a, b, \dots, a) = \dots = \varphi(a, \dots, a, b) = a$$

Example 2.2. A ternary near unanimity operation is called a **majority** operation. For instance the ternary function defined as

$$\delta(a, b, c) = \begin{cases} b & \text{if } b = c \\ a & \text{otherwise} \end{cases}$$

is a majority operation called **dual discriminator**.

3. RELATIONAL CLONES

Definition 3.1. Let R be a set of relation symbols and \mathcal{A} be a relational structure over R . We denote by $\text{Clo}(\mathcal{A})$ the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under

- (1) **permutation:** if $\rho \in \text{Clo}(\mathcal{A})$, then also

$$\{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) : \sigma \in S_n, (a_1, \dots, a_n) \in \rho\} \in \text{Clo} \mathcal{A}$$

- (2) **extension:** if $\rho \in \text{Clo}(\mathcal{A})$, then also

$$\{(a_1, \dots, a_n, a_{n+1}) : (a_1, \dots, a_n) \in \rho, a_{n+1} \in A\} \in \text{Clo} \mathcal{A}$$

- (3) **truncation:** if $\rho \in \text{Clo}(\mathcal{A})$, then also

$$\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \text{Clo} \mathcal{A}$$

- (4) intersection.

Remark 3.2. Observe that $\text{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

Theorem 3.3 ([2]). *For any pair of relational structures $\mathcal{A} = (A, \Gamma)$ and $\mathcal{B} = (A, H)$ such that H is **finite** and $H \subseteq \text{Clo}(\mathcal{A})$, $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$.*

Proof. Let Σ be a set of atomic formulas over H . Let $\eta(x_1, \dots, x_n) \in \Sigma$. Then there is $\varphi(x_1, \dots, x_n)$ of the form

$$\exists y_1, \dots, y_m (\gamma_1(z_1^1, \dots, z_{n_1}^1) \wedge \dots \wedge \gamma_k(z_1^k, \dots, z_{n_k}^k))$$

(where $\gamma_1, \dots, \gamma_k \in \Gamma$ and $z_j^i \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$) such that for $a_1, \dots, a_n \in A$

$$(3) \quad \mathcal{B} \models \eta(a_1, \dots, a_n) \iff \mathcal{A} \models \varphi(a_1, \dots, a_n)$$

We can assume (up to renaming of variables) that y_1, \dots, y_m do not appear in any formula of Σ .

Now, for each $\eta(x_1, \dots, x_n) \in \Sigma$ perform the following steps:

- add $\{\gamma_1(z_1^1, \dots, z_{n_1}^1), \dots, \gamma_k(z_1^k, \dots, z_{n_k}^k)\}$ to Σ ;
- remove $\eta(x_1, \dots, x_n)$ from Σ .

At the the end we obtain a set of equations T over G . **This is a polynomial-time reduction.** By (3) it is clear that we can find an assignment $X \rightarrow A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. \square

Remark 3.4. We observe that in the above result it is not necessary that A is a finite set.

Corollary 3.5. *Let \mathcal{A} be a relational structure and $\mathcal{B} = (A, \text{Clo}(\mathcal{A}))$. Then*

- (1) $\text{CSP}(\mathcal{A})$ is polynomial-time decidable iff $\text{CSP}(\mathcal{B})$ is.
- (2) $\text{CSP}(\mathcal{A})$ is NP-complete iff $\text{CSP}(\mathcal{B})$ is.

4. SURJECTIVE AND IDEMPOTENT ALGEBRAS

Definition 4.1. An algebra \mathbf{A} is **surjective** if all the element of $\text{Clo}(\mathbf{A})$ are surjective.

Remark 4.2. Let \mathbf{A} be a finite algebra over F . Every element of $\text{Clo}(\mathbf{A})$ is surjective iff every element of $\text{Clo}_1(\mathbf{A})$ is. In this case $\text{Clo}_1(\mathbf{A})$ is a group. It is enough to show that for every $f \in F$, there is $\varphi \in \text{Clo}_1(\mathbf{A})$ such that $f^{\mathbf{A}}\varphi = 1_{\mathbf{A}}$. Let $m := |A|$. Then $(f^{\mathbf{A}})^m = 1_{\mathbf{A}}$. Let n_f be the least n such that $(f^{\mathbf{A}})^{n_f} = 1_{\mathbf{A}}$. Let $\varphi := (f^{\mathbf{A}})^{n_f-1}$.

Let \mathbf{A} be a finite algebra and let B be a subset of A . Let

$$\text{Clo}(\mathbf{A})|B := \{\varphi \in \text{Clo}(\mathbf{A}) : \varphi|B \in O_{\mathbf{B}}\}$$

We denote by $\mathbf{A}|B$ the algebra $(B, \text{Clo}(\mathbf{A})|B)$.

Lemma 4.3 ([2]). *Let $\mathcal{A} = (A, P)$ be a relational structure. Let $\varphi \in \text{Pol}_1(\mathcal{A})$. For $\rho \in P_n$ let*

$$\varphi(\rho) := \{(\varphi(a_1), \dots, \varphi(a_n)) : (a_1, \dots, a_n) \in \rho\}$$

Let $\mathcal{B} := (A, \varphi(P))$ where $\varphi(P) := \{\varphi(\rho) : \rho \in P\}$. Then

- (1) $\text{CSP}(\mathcal{A})$ is polynomial-time decidable iff $\text{CSP}(\mathcal{B})$ is;
- (2) $\text{CSP}(\mathcal{A})$ is NP-complete iff $\text{CSP}(\mathcal{B})$ is.

Proof. We show that $\text{CSP}(\mathcal{A})$ is polynomial-time equivalent to $\text{CSP}(\mathcal{B})$. Let Σ be a finite set of atomic formulas over P . Replace every occurrence of $\rho(x_1, \dots, x_n)$ with $\varphi(\rho)(x_1, \dots, x_n)$, obtaining a set of formulas T over $\varphi(P)$. Given any assignment $(-)^{\mathcal{B}} : X \rightarrow A$ such that $\mathcal{B} \models T$, define $(x)^{\mathcal{A}} := (x)^{\mathcal{B}}$; then $\mathcal{A} \models \Sigma$. Indeed, given $\rho(x_1, \dots, x_n) \in \Sigma$, since $\varphi \in \text{Pol}_1(\mathcal{A})$:

$$\begin{aligned} \mathcal{B} \models \varphi(\rho)(x_1, \dots, x_n) &\implies (x_1^{\mathcal{B}}, \dots, x_n^{\mathcal{B}}) \in \varphi(\rho) \\ &\implies (x_1^{\mathcal{B}}, \dots, x_n^{\mathcal{B}}) \in \rho \\ &\implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in \rho \\ &\implies \mathcal{A} \models \rho(x_1, \dots, x_n) \end{aligned}$$

Conversely, given any assignment $(-)^{\mathcal{A}} : X \rightarrow A$ such that $\mathcal{A} \models \Sigma$, defining $(x)^{\mathcal{B}} := \varphi(x^{\mathcal{A}})$ is enough to have $\mathcal{B} \models T$. Indeed, given $\varphi(\rho)(x_1, \dots, x_n) \in T$, by

definition of $\varphi(P)$:

$$\begin{aligned}
\mathcal{A} \models \rho(x_1, \dots, x_n) &\implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in \rho \\
&\implies (\varphi(x_1^{\mathcal{A}}), \dots, \varphi(x_n^{\mathcal{A}})) \in \varphi(\rho) \\
&\implies (x_1^{\mathcal{B}}, \dots, x_n^{\mathcal{B}}) \in \varphi(\rho) \\
&\implies \mathcal{B} \models \varphi(\rho)(x_1, \dots, x_n) \quad \square
\end{aligned}$$

Theorem 4.4. *Let \mathbf{A} be a finite algebra. Then there is $B \subseteq A$ such that $\mathbf{A}|B$ is surjective and*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable iff $\text{CSP}(\mathbf{A}|B)$ is;
- (2) $\text{CSP}(\mathbf{A})$ is NP-complete iff $\text{CSP}(\mathbf{A}|B)$.

Proof. Assume that \mathbf{A} is not surjective. Then by Remark 4.2 there is $\psi \in \text{Clo}_1(\mathbf{A})$ not surjective. Let $\varphi \in \text{Clo}_1(\mathbf{A})$ such that φ is not surjective and $\varphi[A]$ has minimal cardinality. Define $B := \varphi[A]$. We show that $\mathbf{A}|B$ is surjective. Let $\psi \in \text{Clo}_1(\mathbf{A}|B) = \text{Clo}_1(\mathbf{A})|B$; if, towards a contradiction, $\psi[B] \subset B$, then $\psi\varphi[A] \subset \varphi[A] \subset A$ contradicting the minimality. The last part of the statement follows immediately from Lemma 4.3. \square

Definition 4.5. Let \mathbf{A} be an algebra. Let $\text{id}(A)$ be the set of idempotent operations on A . We define $\text{Clo}_{\text{id}}(\mathbf{A}) := \text{Clo}(\mathbf{A}) \cap \text{id}(A)$. We say that \mathbf{A} is **idempotent** if all the elements of $\text{Clo}(\mathbf{A})$ are.

Remark 4.6. Let \mathbf{A} be an algebra. Observe that an operation $\varphi \in \text{Clo}(\mathbf{A})$ is idempotent iff it preserves the relations in $\Delta^{(1)} = \{\{a\} : a \in A\}$. Hence $\text{Inv}(\text{Clo}_{\text{id}}(\mathbf{A})) = \text{Clo}(\mathcal{A})$ where $\mathcal{A} = (A, \text{Inv}(\mathbf{A}) \cup \Delta^{(1)})$.

Lemma 4.7. *Let $A = \{a_1, \dots, a_k\}$ be a finite set. Let \mathbf{A} be an algebra over F . Then the relation*

$$(4) \quad \sigma := \{(\psi(a_1), \dots, \psi(a_k)) : \psi \in \text{Clo}_1(\mathbf{A})\}$$

belongs to $\text{Inv}(\mathbf{A})$.

Proof. We show that for every $f \in F_n$ and for every matrix M

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

such that $(a_1^1, \dots, a_n^1) \in \sigma, \dots, (a_1^k, \dots, a_n^k) \in \sigma$ we have

$$(f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \in \sigma$$

By hypothesis we can write M as

$$\begin{bmatrix} \psi_1(a_1) & \cdots & \psi_n(a_1) \\ \vdots & \ddots & \vdots \\ \psi_1(a_k) & \cdots & \psi_n(a_k) \end{bmatrix}$$

but then

$$\begin{aligned} & (f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \\ &= (f^{\mathbf{A}}(\psi_1(a_1), \dots, \psi_n(a_1)), \dots, f^{\mathbf{A}}(\psi_1(a_k), \dots, \psi_n(a_k))) \\ &= (f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_1), \dots, f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_k)) \end{aligned}$$

and we conclude since $f^{\mathbf{A}}[\psi_1, \dots, \psi_n] \in \text{Clo}_1(\mathbf{A})$. \square

Theorem 4.8. *Let $\mathbf{A} = A$ be a finite surjective algebra. Let $\mathbf{B} := (A, \text{Clo}_{\text{id}}(\mathbf{A}))$. Then*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable iff $\text{CSP}(\mathbf{B})$ is.
- (2) $\text{CSP}(\mathbf{A})$ is NP-complete iff $\text{CSP}(\mathbf{B})$ is.

Proof. Let $A = \{a_1, \dots, a_k\}$ and let $\Gamma := \Delta^{(1)}$. Let $\mathcal{A} := (A, \text{Inv}(\mathbf{A}))$ and $\mathcal{B} := (A, \text{Inv}(\mathbf{A}) \cup \Gamma)$.

By definition and by Remark 4.6 $\text{CSP}(\mathbf{B})$ is polynomial-time equivalent to $\text{CSP}(\mathbf{A})$ iff $\text{CSP}(\mathcal{B})$ is polynomial-time equivalent to $\text{CSP}(\mathcal{A})$.

That $\text{CSP}(\mathcal{A})$ is polynomial-time reducible to $\text{CSP}(\mathcal{B})$ is obvious. Let Σ be a finite set of atomic formulas over $\text{Inv}(\mathbf{A}) \cup \Gamma$ and let $\{x_1, \dots, x_k\}$ be variables that do not appear in Σ . By Remark 4.2, since \mathbf{A} is surjective, $\text{Clo}_1(\mathbf{A})$ forms a group. Moreover, the relation σ of Lemma 4.7 belongs to $\text{Inv}(\mathbf{A})$. Now, perform the following steps:

- replace every formula $\gamma_i(x)$ with $x \equiv x_i$;
- add the formula $\sigma(x_1, \dots, x_k)$.

At the end we obtain a set of equations T over $\text{Inv}(\mathbf{A})$. **This is a polynomial-time reduction.** We finally show that we can find an assignment such that $\mathcal{A} \models T$ iff we can find an assignment such that $\mathcal{B} \models \Sigma$. Let $(-)^{\mathcal{B}} : X \rightarrow A$ be an assignment such that $\mathcal{B} \models \Sigma$. Consider the assignment

$$x^{\mathcal{A}} = \begin{cases} x^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then $(-)^{\mathcal{A}}$ is such that $\mathcal{A} \models T$. Conversely, assume that there is an assignment $(-)^{\mathcal{A}}$ such that $\mathcal{A} \models T$. By definition of σ , there is $\psi \in \text{Clo}_1(\mathbf{A})$ such that $x_i^{\mathcal{A}} = \psi(a_i)$ for all i . Consider $(-)^{\mathcal{A}} := \psi^{-1} \circ (-)^{\mathcal{A}}$. Define $(x)^{\mathcal{B}} := (x)^{\mathcal{A}}$. Let $\rho(y_1, \dots, y_n), \gamma_i(x) \in \Sigma$. Every relation in $\text{Inv}(\mathbf{A})$ is invariant under ψ^{-1} , hence

$$\begin{aligned} \mathcal{A} \models \rho(y_1, \dots, y_n) &\implies (y_1^{\mathcal{A}}, \dots, y_n^{\mathcal{A}}) \in \rho & \mathcal{A} \models x \equiv x_i &\implies x^{\mathcal{A}} = x_i^{\mathcal{A}} \\ &\implies (\psi^{-1}(y_1^{\mathcal{A}}), \dots, \psi^{-1}(y_n^{\mathcal{A}})) \in \rho & &\implies x^{\mathcal{A}} = \psi(a_i) \\ &\implies (y_1^{\mathcal{B}}, \dots, y_n^{\mathcal{B}}) \in \rho & &\implies \psi^{-1}(x^{\mathcal{A}}) = a_i \\ &\implies \mathcal{B} \models \rho(y_1, \dots, y_n) & &\implies x^{\mathcal{B}} = a_i \end{aligned}$$

Thus $\mathcal{B} \models \Sigma$. \square

Remark 4.9. By the above result, given a finite algebra \mathbf{A} , to the purpose of the study of $\text{CSP}(\mathbf{A})$, we can assume without loss of generality that \mathbf{A} is idempotent.

5. SUBALGEBRAS AND IMAGES

Theorem 5.1. *Let \mathbf{A} be a finite algebra.*

- (1) *if $\text{CSP}(\mathbf{A})$ is polynomial-time decidable, so is $\text{CSP}(\mathbf{B})$ for every $\mathbf{B} \leq \mathbf{A}$;*
- (2) *if there is $\mathbf{B} \leq \mathbf{A}$ such that $\text{CSP}(\mathbf{B})$ is NP-complete, so is $\text{CSP}(\mathbf{A})$.*

Proof. Let $\mathcal{A} = (A, \text{Inv}(\mathbf{A}))$ and $\mathcal{B} = (A, \text{Inv}(\mathbf{B}))$. By definition $\text{CSP}(\mathbf{B})$ is polynomial-time reducible to $\text{CSP}(\mathbf{A})$ iff $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$. But that $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$ is obvious since $\text{Inv}(\mathbf{B}) \subseteq \text{Inv}(\mathbf{A})$. \square

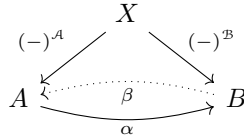
Theorem 5.2. *Let \mathbf{A}, \mathbf{B} be two finite algebras of the same type.*

- (1) *if $\text{CSP}(\mathbf{A})$ is polynomial-time decidable, so is $\text{CSP}(\mathbf{B})$ for every surjective homomorphism $\alpha : \mathbf{A} \rightarrow \mathbf{B}$;*
- (2) *if there is a surjective homomorphism $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ such that $\text{CSP}(\mathbf{B})$ is NP-complete, so is $\text{CSP}(\mathbf{A})$.*

Proof. Let $\mathcal{B} = (B, \text{Inv}(\mathbf{B}))$. We show that there is $\Gamma \subseteq \text{Inv}(\mathbf{B})$ such that, defining $\mathcal{A} = (A, \Gamma)$, $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$. For every $\rho \in \text{Inv}(\mathbf{B})_n$ let

$$\alpha^{-1}(\rho) := \{(a_1, \dots, a_n) \in A^n : (\alpha(a_1), \dots, \alpha(a_n)) \in \rho\}$$

Clearly, $\alpha^{-1}(\rho) \in \text{Inv}(\mathbf{A})_n$ and therefore, letting $\Gamma := \{\alpha^{-1}(\rho) : \rho \in \Gamma\}$, $\Gamma \subseteq \text{Inv}(\mathbf{A})$. Let Σ be a set of atomic formulas over $\text{Inv}(\mathbf{B})$. Replace every formula $\rho(x_1, \dots, x_n)$ with $\alpha^{-1}(\rho)(x_1, \dots, x_n)$. We obtain a set of equations T over Γ by a polynomial-time reduction. Let β be a section of α .



Referring to the assignments in the picture, each defined in terms of the other so that the diagram commute, it is clear that $\mathcal{A} \models T$ iff $\mathcal{B} \models \Sigma$. \square

Lemma 5.3 ([2]). *Let \mathcal{A} be a relational structure. If $\text{Pol}(\mathcal{A})$ contains essentially unary operations only, $\text{CSP}(\mathcal{A})$ is NP-complete.*

Corollary 5.4. *Let \mathbf{A} be a finite algebra. If there is $\mathbf{B} \in \text{HS}(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$, then $\text{CSP}(\mathbf{A})$ is NP-complete.*

Proof. If \mathbf{B} is such that $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$, then $\text{CSP}(\mathbf{B})$ is NP-complete by Lemma 5.3. We conclude using the second clauses of Theorems 5.1 and 5.2. \square

6. OMITTING TYPES

Refer to the Appendix.

Theorem 6.1. *Let \mathbf{A} be a finite idempotent algebra. If $\text{CSP}(\mathbf{A})$ is decidable in polynomial-time, then $\mathbf{1} \notin \text{typ}\{\mathbf{A}\}$. If $\mathbf{1} \in \text{typ}\{\mathbf{A}\}$, then $\text{CSP}(\mathbf{A})$ is NP-complete.*

APPENDIX A. CLASSIFICATION OF FINITE MINIMAL ALGEBRAS

Definition A.1. Let F be a set of function symbols and \mathbf{A} be an algebra over F . We denote by $\text{Pol}(\mathbf{A})$ the smallest set containing

- (1) $\{f^{\mathbf{A}} : f \in F\}$;
- (2) $\{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$;
- (3) the constant 0-ary operations

and closed under composition. The elements of $\text{Pol}(\mathbf{A})$ are called **polynomial operations**. We say that two algebras \mathbf{A} and \mathbf{B} on the same carrier are **polynomial equivalent** if $\text{Pol}(\mathbf{A}) = \text{Pol}(\mathbf{B})$.

Remark A.2. Explicitly, if $\varphi \in \text{Clo}_{m+n}(\mathbf{A})$ and $(a_1, \dots, a_m) \in A^m$, then

$$\psi : A^n \rightarrow A \quad (b_1, \dots, b_n) \mapsto \varphi(a_1, \dots, a_m, b_1, \dots, b_n)$$

is a polynomial operation and every polynomial operation arises in this way.

Definition A.3 (provisional). A nontrivial finite algebra \mathbf{A} is **minimal** iff every nonconstant element of $\text{Pol}_1(\mathbf{A})$ is a permutation.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

Example A.4. The following are examples of minimal algebras.

- (1) any algebra with carrier 2;
- (2) a nontrivial finite vector space \mathbf{A} over a finite field \mathbf{k} : every $\pi \in \text{Pol}_1(\mathbf{A})$ is of the form $\pi(v) = av + b$ for some $a \in \mathbf{k}$, $b \in A$;
- (3) a group of permutation acting on a finite set².

We shall prove that, up to polynomial equivalence, there are no other finite minimal algebras.

Lemma A.5. *Let \mathbf{A} be a minimal algebra such that every element of $\text{Pol}(\mathbf{A})$ is essentially unary is polynomial to $(A, \Phi_{\mathbf{G}})$ where \mathbf{G} is a finite group acting on A .*

Proof. Since \mathbf{A} is minimal, $\text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$ is a subgroup of $\text{Sym}(A)$. Let $\mathbf{G} := \text{Pol}_1(\mathbf{A}) \cap \text{Sym}(A)$. If $\psi \in \text{Pol}(\mathbf{A})$, either ψ is constant or ψ is essentially unary, hence $(A, \Phi_{\mathbf{G}})$ is polynomially equivalent to \mathbf{A} . \square

²Let \mathbf{G} be a group acting on a set A . Each $g \in G$ induces an operation $\varphi_g : A \rightarrow A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. A \mathbf{G} -set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$.

Theorem A.6 ([4]). *Let \mathbf{A} be a minimal algebra with $|A| > 2$. If $\text{Pol}(\mathbf{A})$ contains an operation which is not essentially unary, then \mathbf{A} is polynomially equivalent to a \mathbf{k} -vector space for a finite field \mathbf{k} .*

Theorem A.7. *Every algebra \mathbf{A} with carrier 2 is polynomially equivalent to one of the following:*

- (1) $\mathbf{E}_0 = (2, \emptyset)$;
- (2) $\mathbf{E}_1 = (2, \neg)$;
- (3) $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$;
- (4) $\mathbf{E}_4 = (2, \wedge, \vee)$;
- (5) $\mathbf{E}_5 = (2, \vee)$;
- (6) $\mathbf{E}_6 = (2, \wedge)$.

Each of them is not polynomially equivalent to the other³.

Remark A.8. Up to isomorphism, $\mathbf{E}_5 (\simeq \mathbf{E}_6)$ is the only semilattice with two elements, while \mathbf{E}_4 and \mathbf{E}_3 are the only Boolean algebra and lattice, respectively, with two elements.

Definition A.9. Let \mathbf{A} be a minimal algebra. We say that \mathbf{A} is of

- (1) **type 1** (or **unary**) if \mathbf{A} is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ for some $\mathbf{G} \leq \text{Sym}(A)$;
- (2) **type 2** (or **affine**) if \mathbf{A} is polynomially equivalent to a vector space over a finite field \mathbf{k} ;
- (3) **type 3** (or **Boolean**) if \mathbf{A} is polynomially equivalent to \mathbf{E}_3 ;
- (4) **type 4** (or **lattice**) if \mathbf{A} is polynomially equivalent to \mathbf{E}_4 ;
- (5) **type 5** (or **semilattice**) if \mathbf{A} is polynomially equivalent to \mathbf{E}_5 .

APPENDIX B. OMITTING TYPES

Definition B.1. A variety \mathbf{V} is called

- (1) **locally finite** if all its finitely generated algebras are finite;
- (2) **finitely presented** if \mathbf{V} has a finite set of function symbols and $\mathbf{V} = \text{Alg}(\Sigma)$ for a finite set of equations Σ ;
- (3) **finitely generated** if $\mathbf{V} = HSP(\mathbf{A}_1, \dots, \mathbf{A}_n)$ for $\mathbf{A}_1, \dots, \mathbf{A}_n$ finite similar algebras.

Remark B.2. Finitely generated implies locally finite (?)

³A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem A.7 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

Definition B.3. Let V and W be two variety. We say that V is **interpretable** into W ($V \leq W$) if there is a clone-homomorphism $\text{Clo}(V) \rightarrow \text{Clo}(W)$.

Theorem B.4. Let V be a locally finite variety. The following are equivalent:

- (1) $\mathbf{1} \notin \text{typ}\{V\}$;
- (2) there is an idempotent variety W such that $W \leq V$ and $W \not\leq \text{Set}$.

Corollary B.5. Let \mathbf{A} be a finite idempotent algebra. Then $\mathbf{1} \in \text{typ}\{HS(\mathbf{A})\}$ iff $HSP_{\text{fin}}(\mathbf{A})$ contains a set.

Proof. □

Theorem B.6 ([3]). Let \mathbf{A} be a finite idempotent algebra. Then $\text{Clo}(\mathbf{A})$ contains a weak near-unanimity operation iff $\mathbf{1} \notin \text{typ}\{HS(\mathbf{A})\}$.

Important: missing piece:

Theorem B.7. Let \mathbf{A} be a finite idempotent algebra. There is $\mathbf{B} \in HS(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$ iff $\mathbf{1} \in \text{typ}\{HS(\mathbf{A})\}$.

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