# CSP FAST TRACK

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ABSTRACT. In this note

## 1. INTRODUCTION

Let $R$ be a set of relation symbols. Let $\mathcal{A}$ be a relational structure over $R$. Let $X$ be a countable set of variables. By the **constraint satisfaction problem** CSP($\mathcal{A}$)[1] we mean the following decision problem: given a finite set $\Sigma$ of atomic formulas over $R$, decide whether there is an assignment $(-)^{\mathcal{A}} : X \to A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x, y, x_1, \ldots, x_n \in X$

$$r(x_1, \ldots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}} \tag{1}$$

$$x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}} \tag{2}$$

Clearly, it is enough to find an assignment only for those variables that appear in $\Sigma$.

*Remark* 1.1. Usually we deal with *indexed* relational structures, that is we fix a set of relation symbols $R$ and we consider a set $A$ with a set $P$ of relations on $A$ indexed by the elements of $R$. Sometimes is useful to deal with *non-indexed* relational structures $(A, P)$. In this case $P$ will serve as the index set as well. The same applies to algebraic structures (i.e. algebras) and function symbols.

**Definition 1.2.** Let $\rho \in A^k$, and $\varphi : A^n \to A$. We say that $\varphi$ **preserves** $\rho$ if given a matrix

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

with $(a_1^1, \ldots, a_1^k) \in \rho, \ldots, (a_1^n, \ldots, a_n^k) \in \rho$,

$$(\varphi(a_1^1, \ldots, a_n^1), \ldots, \varphi(a_1^k, \ldots, a_n^k)) \in \rho$$

**Definition 1.3.** If $\Gamma$ is a set of relations on $A$ and $\Phi$ is a set of operations on $A$ we denote by

(1) $\mathrm{Inv}(\Phi)$ the set of relations on $A$ that are preserved by all the elements of $\Phi$;

(2) $\mathrm{Pol}(\Gamma)$ the set of operations on $A$ that preserve all the elements of $\Gamma$.

Moreover, if $\mathbf{A}$ is an algebra and $\mathcal{A}$ a relational structure on the same set $A$:

[1] More often denoted by CSP($P$).

(1) $\mathrm{Inv}(\mathbf{A})$ is the set of relations that are preserved by all the $f^{\mathbf{A}}$;

(2) $\mathrm{Pol}(\mathcal{A})$ is the set of operations that preserve all the $r^{\mathcal{A}}$.

Let $\mathbf{A}$ be an algebra. Let $\mathcal{A} := (A, \mathrm{Inv}(\mathbf{A}))$. By $\mathrm{CSP}(\mathbf{A})$ we mean the decision problem $\mathrm{CSP}(\mathcal{A})$.

**Definition 1.4.** Let $F$ be a set of function symbols and $\mathbf{A}$ be an algebra over $F$. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition. The elements of $\mathrm{Clo}(\mathbf{A})$ are called **term** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **term equivalent** if $\mathrm{Clo}(\mathbf{A}) = \mathrm{Clo}(\mathbf{B})$.

Recall that a variety is a class of algebras closed under homomorphic images, subalgebras and products. If $\mathsf{V}$ is a variety, we define $\mathrm{Clo}(\mathsf{V})$ to be $\mathrm{Clo}(\mathbf{F}_{\mathsf{V}}(\omega))$, where $\mathbf{F}_{\mathsf{V}}(\omega)$ is the $\mathsf{V}$-free algebra generated by a countable number of generators. When $\mathsf{V}$ is $\mathsf{Set}$, the variety of sets, $\mathrm{Clo}(\mathsf{Set})$ is the clone of projections $\{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$, that we denote by $\mathbf{N}$.

Goal: prove

**Theorem 1.5.** *Let $\mathbf{A}$ be a finite idempotent algebra. Then the following are equivalent:*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable;*

(2) $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation;*

(3) *for every* $\mathbf{B} \in HS(\mathbf{A})$, $\mathrm{Clo}(\mathbf{B}) \neq \mathbf{N}$.

*Otherwise,* $\mathrm{CSP}(\mathbf{A})$ *is* NP*-complete.*

Observe that

## 2. Kinds of Operations

**Definition 2.1.** An operation $\varphi : A^n \to A$ is called

(1) **essentially unary** if there is an index $i$ and a non-constant function $\psi : A \to A$ such that

$$\varphi(a_1, \ldots, a_n) = \psi(a_i)$$

for all $a_1, \ldots, a_n \in A$.

(2) **idempotent** if $\varphi(a, \ldots, a) = a$ for all $a \in A$.

(3) a **near unanimity** operation if for all $a, b \in A$

$$\varphi(b, a, \ldots, a) = \varphi(a, b, \ldots, a) = \cdots = \varphi(a, \ldots, a, b) = a$$

*Example* 2.2. A ternary near unanimity operation is called a **majority** operation. For instance the ternary function defined as

$$\delta(a,b,c) = \begin{cases} b & \text{if } b = c \\ a & \text{otherwise} \end{cases}$$

is a majority operation called **dual discriminator**.

## 3. Relational Clones

**Definition 3.1.** Let $R$ be a set of relation symbols and $\mathcal{A}$ be a relational structure over $R$. We denote by $\mathrm{Clo}(\mathcal{A})$ the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under

(1) **permutation**: if $\rho \in \mathrm{Clo}(\mathcal{A})$, then also
$$\{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) : \sigma \in S_n, (a_1, \dots, a_n) \in \rho\} \in \mathrm{Clo}\,\mathcal{A}$$

(2) **extension**: if $\rho \in \mathrm{Clo}(\mathcal{A})$, then also
$$\{(a_1, \dots, a_n, a_{n+1}) : (a_1, \dots, a_n) \in \rho, a_{n+1} \in A\} \in \mathrm{Clo}\,\mathcal{A}$$

(3) **truncation**: if $\rho \in \mathrm{Clo}(\mathcal{A})$, then also
$$\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \mathrm{Clo}\,\mathcal{A}$$

(4) intersection.

*Remark* 3.2. Observe that $\mathrm{Clo}(\mathcal{A})$ is given by all the relations $\rho$ of $A$ definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

**Theorem 3.3** ([2])**.** *For any pair of relational structures $\mathcal{A} = (A, \Gamma)$ and $\mathcal{B} = (A, H)$ such that $H$ is finite and $H \subseteq \mathrm{Clo}(\mathcal{A})$, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$.*

*Proof.* Let $\Sigma$ be a set of atomic formulas over $H$. Let $\eta(x_1, \dots, x_n) \in \Sigma$. Then there is $\varphi(x_1, \dots, x_n)$ of the form

$$\exists y_1, \dots, y_m\, (\gamma_1(z^1_1, \dots, z^1_{n_1}) \wedge \dots \wedge \gamma_k(z^k_1, \dots, z^k_{n_k}))$$

(where $\gamma_1, \dots, \gamma_k \in \Gamma$ and $z^i_j \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$) such that for $a_1, \dots, a_n \in A$

$$\mathcal{B} \models \eta(a_1, \dots, a_n) \iff \mathcal{A} \models \varphi(a_1, \dots, a_n) \tag{3}$$

We can assume (up to renaming of variables) that $y_1, \dots, y_m$ do not appear in any formula of $\Sigma$.

Now, for each $\eta(x_1, \dots, x_n) \in \Sigma$ perform the following steps:

- add $\{\gamma_1(z^1_1, \dots, z^1_{n_1}), \dots, \gamma_k(z^k_1, \dots, z^k_{n_k})\}$ to $\Sigma$;

- remove $\eta(x_1, \ldots, x_n)$ from $\Sigma$.

At the the end we obtain a set of equations $T$ over $G$. This is a polynomial-time reduction. By (3) it is clear that we can find an assignment $X \to A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. $\square$

*Remark* 3.4. We observe that in the above result it is not necessary that $A$ is a finite set.

**Corollary 3.5.** *Let $\mathcal{A}$ be a relational structure and $\mathcal{B} = (A, \mathrm{Clo}(\mathcal{A}))$. Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

(2) $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

## 4. Surjective and Idempotent Algebras

**Definition 4.1.** An algebra $\mathbf{A}$ is **surjective** if all the element of $\mathrm{Clo}(\mathbf{A})$ are surjective.

*Remark* 4.2. Let $\mathbf{A}$ be a finite algebra over $F$. Every element of $\mathrm{Clo}(\mathbf{A})$ is surjective iff every element of $\mathrm{Clo}_1(\mathbf{A})$ is. In this case $\mathrm{Clo}_1(\mathbf{A})$ is a group. It is enough to show that for every $f \in F$, there is $\varphi \in \mathrm{Clo}_1(\mathbf{A})$ such that $f^{\mathbf{A}}\varphi = 1_{\mathbf{A}}$. Let $m := |A|$. Then $(f^{\mathbf{A}})^{m!} = 1_{\mathbf{A}}$. Let $n_f$ be the least $n$ such that $(f^{\mathbf{A}})^{n_f} = 1_{\mathbf{A}}$. Let $\varphi := (f^{\mathbf{A}})^{n_f - 1}$.

Let $\mathbf{A}$ be a finite algebra and let $B$ be a subset of $A$. Let

$$\mathrm{Clo}(\mathbf{A})|B := \{\varphi \in \mathrm{Clo}(\mathbf{A}) : \varphi[B^n] \subseteq B\}$$

We denote by $\mathbf{A}|B$ the algebra $(B, \mathrm{Clo}(\mathbf{A})|B)$.

**Lemma 4.3** ([2])**.** *Let $\mathcal{A} = (A, P)$ be a relational structure. Let $\varphi \in \mathrm{Pol}_1(\mathcal{A})$. For $\rho \in P_n$ let*

$$\varphi(\rho) := \{(\varphi(a_1), \ldots, \varphi(a_n)) : (a_1, \ldots, a_n) \in \rho\}$$

*Let $\mathcal{B} := (A, \varphi(P))$ where $\varphi(P) := \{\varphi(\rho) : \rho \in P\}$. Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is;*

(2) $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

*Proof.* We show that $\mathrm{CSP}(\mathcal{A})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathcal{B})$. Let $\Sigma$ be a finite set of atomic formulas over $P$. Replace every occurrence of $\rho(x_1, \ldots, x_n)$ with $\varphi(\rho)(x_1, \ldots, x_n)$, obtaining a set of formulas $T$ over $\varphi(P)$. Given any assignment $(-)^{\mathcal{B}} : X \to A$ such that $\mathcal{B} \models T$, define $(x)^{\mathcal{A}} := (x)^{\mathcal{B}}$; then $\mathcal{A} \models \Sigma$. Indeed, given $\rho(x_1, \ldots, x_n) \in \Sigma$, since $\varphi \in \mathrm{Pol}_1(\mathcal{A})$:

$$\begin{aligned}
\mathcal{B} \models \varphi(\rho)(x_1, \ldots, x_n) &\implies (x_1^{\mathcal{B}}, \ldots, x_n^{\mathcal{B}}) \in \varphi(\rho) \\
&\implies (x_1^{\mathcal{B}}, \ldots, x_n^{\mathcal{B}}) \in \rho \\
&\implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in \rho \\
&\implies \mathcal{A} \models \rho(x_1, \ldots, x_n)
\end{aligned}$$

Conversely, given any assignment $(-)^{\mathcal{A}} : X \to A$ such that $\mathcal{A} \models \Sigma$, defining $(x)^{\mathcal{B}} := \varphi(x^{\mathcal{A}})$ is enough to have $\mathcal{B} \models T$. Indeed, given $\varphi(\rho)(x_1, \ldots, x_n) \in T$, by definition of $\varphi(P)$:

$$\begin{aligned}
\mathcal{A} \models \rho(x_1, \ldots, x_n) &\implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in \rho \\
&\implies (\varphi(x_1^{\mathcal{A}}), \ldots, \varphi(x_n^{\mathcal{A}})) \in \varphi(\rho) \\
&\implies (x_1^{\mathcal{B}}, \ldots, x_n^{\mathcal{B}}) \in \varphi(\rho) \\
&\implies \mathcal{B} \models \varphi(\rho)(x_1, \ldots, x_n)
\end{aligned}$$ □

**Theorem 4.4.** *Let* $\mathbf{A}$ *be a finite algebra. Then there is* $B \subseteq A$ *such that* $\mathbf{A}|B$ *is surjective and*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathbf{A}|B)$ *is;*

(2) $\mathrm{CSP}(\mathbf{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathbf{A}|B)$.

*Proof.* Assume that $\mathbf{A}$ is not surjective. Then by Remark 4.2 there is $\psi \in \mathrm{Clo}_1(\mathbf{A})$ not surjective. Let $\varphi \in \mathrm{Clo}_1(\mathbf{A})$ such that $\varphi$ is not surjective and $\varphi[A]$ has minimal cardinality. Define $B := \varphi[A]$. We show that $\mathbf{A}|B$ is surjective. Let $\psi \in \mathrm{Clo}_1(\mathbf{A}|B) = \mathrm{Clo}_1(\mathbf{A})|B$; if, towards a contradiction, $\psi[B] \subset B$, then $\psi\varphi[A] \subset \varphi[A] \subset A$ contradicting the minimality. The last part of the statement follows immediately from Lemma 4.3. □

**Definition 4.5.** Let $\mathbf{A}$ be an algebra. Let $\mathrm{id}(A)$ be the set of idempotent operations on $A$. We define $\mathrm{Clo}_{\mathrm{id}}(\mathbf{A}) := \mathrm{Clo}(\mathbf{A}) \cap \mathrm{id}(A)$. We say that $\mathbf{A}$ is **idempotent** of all the elements of $\mathrm{Clo}(\mathbf{A})$ are.

*Remark* 4.6. Let $\mathbf{A}$ be an algebra. Observe that an operation $\varphi \in \mathrm{Clo}(\mathbf{A})$ is idempotent iff it preserves the relations in $\Delta^{(1)} = \{\{a\} : a \in A\}$. Hence $\mathrm{Inv}(\mathrm{Clo}_{\mathrm{id}}(\mathbf{A})) = \mathrm{Clo}(\mathcal{A})$ where $\mathcal{A} = (A, \mathrm{Inv}(\mathbf{A}) \cup \Delta^{(1)})$.

**Lemma 4.7.** *Let* $A = \{a_1, \ldots, a_k\}$ *be a finite set. Let* $\mathbf{A}$ *be an algebra over* $F$. *Then the relation*

$$\sigma := \{(\psi(a_1), \ldots, \psi(a_k)) : \psi \in \mathrm{Clo}_1(\mathbf{A})\} \tag{4}$$

*belongs to* $\mathrm{Inv}(\mathbf{A})$.

*Proof.* We show that for every $f \in F_n$ and for every matrix $M$

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

such that $(a_1^1, \ldots, a_1^k) \in \sigma, \ldots, (a_1^n, \ldots, a_n^k) \in \sigma$ we have

$$(f^{\mathbf{A}}(a_1^1, \ldots, a_n^1), \ldots, f^{\mathbf{A}}(a_1^k, \ldots, a_n^k)) \in \sigma$$

By hypothesis we can write $M$ as

$$\begin{bmatrix} \psi_1(a_1) & \cdots & \psi_n(a_1) \\ \vdots & \ddots & \vdots \\ \psi_1(a_k) & \cdots & \psi_n(a_k) \end{bmatrix}$$

but then

$$\begin{aligned}(f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \\ = (f^{\mathbf{A}}(\psi_1(a_1), \dots, \psi_n(a_1)), \dots, f^{\mathbf{A}}(\psi_1(a_k), \dots, \psi_n(a_k))) \\ = (f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_1), \dots, f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_k))\end{aligned}$$

and we conclude since $f^{\mathbf{A}}[\psi_1, \dots, \psi_n] \in \mathrm{Clo}_1(\mathbf{A})$. □

**Theorem 4.8.** *Let $\mathbf{A} = A$ be a finite surjective algebra. Let $\mathbf{B} := (A, \mathrm{Clo}_{\mathrm{id}}(\mathbf{A}))$. Then*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

(2) $\mathrm{CSP}(\mathbf{A})$ *is* NP-*complete iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

*Proof.* Let $A = \{a_1, \dots, a_k\}$ and let $\Gamma := \Delta^{(1)}$. Let $\mathcal{A} := (A, \mathrm{Inv}(\mathbf{A}))$ and $\mathcal{B} := (A, \mathrm{Inv}(\mathbf{A}) \cup \Gamma)$.

By definition and by Remark 4.6 $\mathrm{CSP}(\mathbf{B})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathbf{A})$ iff $\mathrm{CSP}(\mathcal{B})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathcal{A})$.

That $\mathrm{CSP}(\mathcal{A})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{B})$ is obvious. Let $\Sigma$ be a finite set of atomic formulas over $\mathrm{Inv}(\mathbf{A}) \cup \Gamma$ and let $\{x_1, \dots, x_k\}$ be variables that do not appear in $\Sigma$. By Remark 4.2, since $\mathbf{A}$ is surjective, $\mathrm{Clo}_1(\mathbf{A})$ forms a group. Moreover, the relation $\sigma$ of Lemma 4.7 belongs to $\mathrm{Inv}(\mathbf{A})$. Now, perform the following steps:

- replace every formula $\gamma_i(x)$ with $x \equiv x_i$;
- add the formula $\sigma(x_1, \dots, x_k)$.

At the the end we obtain a set of equations $T$ over $\mathrm{Inv}(\mathbf{A})$. This is a polynomial-time reduction. We finally show that we can find an assignment such that $\mathcal{A} \models T$ iff we can find an assignment such that $\mathcal{B} \models \Sigma$. Let $(-)^{\mathcal{B}} : X \to A$ be an assignment such that $\mathcal{B} \models \Sigma$. Consider the assignment

$$x^{\mathcal{A}} = \begin{cases} x^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then $(-)^{\mathcal{A}}$ is such that $\mathcal{A} \models T$. Conversely, assume that there is an assignment $(-)^{\mathcal{A}}$ such that $\mathcal{A} \models T$. By definition of $\sigma$, there is $\psi \in \mathrm{Clo}_1(\mathbf{A})$ such that $x_i^{\mathcal{A}} = \psi(a_i)$ for all $i$. Consider $(-)'^{\mathcal{A}} := \psi^{-1} \circ (-)^{\mathcal{A}}$. Define $(x)^{\mathcal{B}} := (x)'^{\mathcal{A}}$. Let

$\rho(y_1,\dots,y_n),\gamma_i(x)\in\Sigma$. Every relation in $\mathrm{Inv}(\mathbf{A})$ is invariant under $\psi^{-1}$, hence

$$\begin{aligned}
\mathcal{A}\models\rho(y_1,\dots,y_n) &\implies (y_1^{\mathcal{A}},\dots,y_n^{\mathcal{A}})\in\rho & \mathcal{A}\models x\equiv x_i &\implies x^{\mathcal{A}}=x_i^{\mathcal{A}}\\
&\implies (\psi^{-1}(y_1^{\mathcal{A}}),\dots,\psi^{-1}(y_n^{\mathcal{A}}))\in\rho & &\implies x^{\mathcal{A}}=\psi(a_i)\\
&\implies (y_1^{\mathcal{B}},\dots,y_n^{\mathcal{B}})\in\rho & &\implies \psi^{-1}(x^{\mathcal{A}})=a_i\\
&\implies \mathcal{B}\models\rho(y_1,\dots,y_n) & &\implies x^{\mathcal{B}}=a_i
\end{aligned}$$

Thus $\mathcal{B}\models\Sigma$. ☐

*Remark* 4.9. By the above result, given a finite algebra $\mathbf{A}$, to the purpose of the study of $\mathrm{CSP}(\mathbf{A})$, we can assume without loss of generality that $\mathbf{A}$ is idempotent.

## 5. Subalgebras and Images

**Theorem 5.1.** *Let $\mathbf{A}$ be a finite algebra.*

(1) *if $\mathrm{CSP}(\mathbf{A})$ is polynomial-time decidable, so is $\mathrm{CSP}(\mathbf{B})$ for every $\mathbf{B}\le\mathbf{A}$;*

(2) *if there is $\mathbf{B}\le\mathbf{A}$ such that $\mathrm{CSP}(\mathbf{B})$ is $\mathrm{NP}$-complete, so is $\mathrm{CSP}(\mathbf{A})$.*

*Proof.* Let $\mathcal{A}=(A,\mathrm{Inv}(\mathbf{A}))$ and $\mathcal{B}=(A,\mathrm{Inv}(\mathbf{B}))$. By definition $\mathrm{CSP}(\mathbf{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathbf{A})$ iff $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$. But that $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$ is obvious since $\mathrm{Inv}(\mathbf{B})\subseteq\mathrm{Inv}(\mathbf{A})$. ☐

**Theorem 5.2.** *Let $\mathbf{A},\mathbf{B}$ be two finite algebras of the same type.*

(1) *if $\mathrm{CSP}(\mathbf{A})$ is polynomial-time decidable, so is $\mathrm{CSP}(\mathbf{B})$ for every surjective homomorphism $\alpha:\mathbf{A}\to\mathbf{B}$;*

(2) *if there is a surjective homomorphism $\alpha:\mathbf{A}\to\mathbf{B}$ such that $\mathrm{CSP}(\mathbf{B})$ is $\mathrm{NP}$-complete, so is $\mathrm{CSP}(\mathbf{A})$.*

*Proof.* Let $\mathcal{B}=(B,\mathrm{Inv}(\mathbf{B}))$. We show that there is $\Gamma\subseteq\mathrm{Inv}(\mathbf{B})$ such that, defining $\mathcal{A}=(A,\Gamma)$, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$. For every $\rho\in\mathrm{Inv}(\mathbf{B})_n$ let

$$\alpha^{-1}(\rho):=\{(a_1,\dots,a_n)\in A^n:(\alpha(a_1),\dots,\alpha(a_n))\in\rho\}$$

Clearly, $\alpha^{-1}(\rho)\in\mathrm{Inv}(\mathbf{A})_n$ and therefore, letting $\Gamma:=\{\alpha^{-1}(\rho):\rho\in\Gamma\}$, $\Gamma\subseteq\mathrm{Inv}(\mathbf{A})$. Let $\Sigma$ be a set of atomic formulas over $\mathrm{Inv}(\mathbf{B})$. Replace every formula $\rho(x_1,\dots,x_n)$ with $\alpha^{-1}(\rho)(x_1,\dots,x_n)$. We obtain a set of equations $T$ over $\Gamma$ by a polynomial-time reduction. Let $\beta$ be a section of $\alpha$.

$$\begin{tikzcd}
 & X \arrow[dl, "(-)^{\mathcal{A}}"'] \arrow[dr, "(-)^{\mathcal{B}}"] & \\
A \arrow[rr, bend right, "\alpha"'] & & B \arrow[ll, dotted, bend right, "\beta"']
\end{tikzcd}$$

Referring to the assignments in the picture, each defined in terms of the other so that the diagram commute, it is clear that $\mathcal{A}\models T$ iff $\mathcal{B}\models\Sigma$. ☐

**Lemma 5.3** ([2])**.** *Let $\mathcal{A}$ be a relational structure. If* $\mathrm{Pol}(\mathcal{A})$ *contains essentially unary operations only,* $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete.*

**Corollary 5.4.** *Let* $\mathbf{A}$ *be a finite algebra. If there is* $\mathbf{B} \in HS(\mathbf{A})$ *such that* $\mathrm{Clo}(\mathbf{B}) = \mathbf{N}$*, then* $\mathrm{CSP}(\mathbf{A})$ *is* NP*-complete.*

*Proof.* If $\mathbf{B}$ is such that $\mathrm{Clo}(\mathbf{B}) = \mathbf{N}$, then $\mathrm{CSP}(\mathbf{B})$ is NP-complete by Lemma 5.3. We conclude using the second cluases of Theorems 5.1 and 5.2. □

## 6. Omitting Types and Complexity

Refer to the Appendices.

**Theorem 6.1.** *Let* $\mathbf{A}$ *be a finite idempotent algebra. If* $\mathrm{CSP}(\mathbf{A})$ *is decidable in polynomial-time, then* $\mathbf{1} \notin \mathrm{typ}\{\mathbf{A}\}$*.*

*Proof.* Assume for the sake of condtradiction that $\mathbf{1} \in \mathrm{typ}\{\mathbf{A}\}$. We show that $\mathrm{CSP}(\mathbf{A})$ is NP-complete. If $\mathbf{1} \in \mathrm{typ}\{\mathbf{A}\}$, then there are $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{typ}(\alpha, \beta) = \mathbf{1}$. By Theorem 5.2, up to replacing $\mathbf{A}$ by $\mathbf{A}/\alpha$, we can assume without loss of generality that $\alpha = \Delta_A$.

Therefore $\mathrm{Clo}(\mathbf{C}|V)$ contains essentially unary operations only. By Lemma 5.3 $\mathrm{CSP}(\mathbf{C}|V)$ is NP-complete. By Lemma 4.3 $\mathrm{CSP}(\mathbf{B}|U)$ and $\mathbf{B}$ are NP-complete. Since $\mathrm{Clo}(\mathbf{A}) \subseteq \mathrm{Clo}(\mathbf{B})$, $\mathbf{A}$ is NP-complete. □

## Appendix A. Classification of Finite Minimal Algebras

**Definition A.1.** Let $F$ be a set of function symbols and $\mathbf{A}$ be an algebra over $F$. We denote by $\mathrm{Pol}(\mathbf{A})$ the smallest set containing

(1) $\{f^{\mathbf{A}} : f \in F\}$;

(2) $\{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$;

(3) the constant 0-ary operations

and closed under composition. The elements of $\mathrm{Pol}(\mathbf{A})$ are called **polynomial** operations. We say that two algebras $\mathbf{A}$ and $\mathbf{B}$ on the same carrier are **polynomial equivalent** if $\mathrm{Pol}(\mathbf{A}) = \mathrm{Pol}(\mathbf{B})$.

*Example* A.2. If $\varphi \in \mathrm{Clo}_{m+n}(\mathbf{A})$ and $(a_1, \dots, a_m) \in A^m$, then

$$\psi : A^n \to A \quad (b_1, \dots, b_n) \mapsto \varphi(a_1, \dots, a_m, b_1, \dots, b_n)$$

is a polynomial operation.

**Definition A.3** (*)**.** A nontrivial finite algebra $\mathbf{A}$ is **minimal** iff every noncostant element of $\mathrm{Pol}_1(\mathbf{A})$ is bijective.

The goal is to classify, up to polynomial equivalence, all the finite minimal algebras.

*Example* A.4. The following are examples of minimal algebras.

(1) any algebra with carrier 2;

(2) a nontrivial finite vector space $\mathbf{A}$ over a finite field $\mathbf{k}$: every $\pi \in \mathrm{Pol}_1(\mathbf{A})$ is of the form $\pi(v) = av + b$ for some $a \in k$, $b \in A$;

(3) a group of permutations acting on a finite set[2].

We shall prove that, up to polynomial equivalence, there are no other finite minimal algebras.

**Lemma A.5.** *Let $\mathbf{A}$ be a minimal algebra. If every element of $\mathrm{Pol}(\mathbf{A})$ is essentially unary, then $\mathbf{A}$ is polynomial equivalent to $(A, \Phi_{\mathbf{G}})$ where $\mathbf{G}$ is a finite group acting on $A$.*

*Proof.* Since $\mathbf{A}$ is minimal, $\mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$ is a subgroup of $\mathrm{Sym}(A)$. Let $\mathbf{G} := \mathrm{Pol}_1(\mathbf{A}) \cap \mathrm{Sym}(A)$. If $\psi \in \mathrm{Pol}(\mathbf{A})$, either $\psi$ is constant or $\psi$ is essentially unary, hence $(A, \Phi_{\mathbf{G}})$ is polynomially equivalent to $\mathbf{A}$. □

**Theorem A.6** ([4])**.** *Let $\mathbf{A}$ be a minimal algebra with $|A| > 2$. If $\mathrm{Pol}(\mathbf{A})$ contains an operation which is not essentially unary, then $\mathbf{A}$ is polynomially equivalent to a $\mathbf{k}$-vector space for a finite field $\mathbf{k}$.*

**Theorem A.7.** *Every algebra $\mathbf{A}$ with carrier 2 is polynomially equivalent to one of the following:*

(1) $\mathbf{E}_0 = (2, \varnothing)$;

(2) $\mathbf{E}_1 = (2, \neg)$;

(3) $\mathbf{E}_3 = (2, \wedge, \vee, \neg)$;

(4) $\mathbf{E}_4 = (2, \wedge, \vee)$;

(5) $\mathbf{E}_5 = (2, \vee)$;

(6) $\mathbf{E}_6 = (2, \wedge)$.

*Each of them is not polynomially equivalent to the other*[3].

*Remark* A.8. Up to isomorphism, $\mathbf{E}_5 (\simeq \mathbf{E}_6)$ is the only semilattice with two elements, while $\mathbf{E}_3$ and $\mathbf{E}_4$ are the only Boolean algebra and lattice, respectively, with two elements.

**Definition A.9.** Let $\mathbf{A}$ be a minimal algebra. We say that $\mathbf{A}$ is of

(1) **type 1** (or **unary**) if $\mathbf{A}$ is polynomially equivalent to $(A, \Phi_{\mathbf{G}})$ for some $\mathbf{G} \leq \mathrm{Sym}(A)$;

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[2]Let $\mathbf{G}$ be a group acting on a set $A$. Each $g \in G$ induces an operation $\varphi_g : A \to A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. A $\mathbf{G}$-set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$.

[3]A classical theorem by Post states that the set of clones of operations on 2 is countable infinite. By Theorem A.7 among these there are exactly seven distinct clones containing the constant operations. However it has been proven that the set of clones on 3 containing the constant operations is uncountable.

(2) **type 2** (or **affine**) if $\mathbf{A}$ is polynomially equivalent to a vector space over a finite field $\mathbf{k}$;

(3) **type 3** (or **Boolean**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_3$;

(4) **type 4** (or **lattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_4$;

(5) **type 5** (or **semilattice**) if $\mathbf{A}$ is polynomially equivalent to $\mathbf{E}_5$.

**Definition A.10** (*)**.** Let $\mathbf{A}$ be a finite algebra and let $\delta < \theta \in \mathrm{Con}(\mathbf{A})$. We say that $\mathbf{A}$ is **$(\delta, \theta)$-minimal** if for all $e \in \mathrm{Pol}_1(\mathbf{A})$ either $e$ is bijective or $e(\theta) \subseteq \delta$.

*Remark* A.11. Observe that $\mathbf{A}$ is minimal iff $\mathbf{A}$ is $(\Delta, \nabla)$-minimal.

## Appendix B. Omitting Types

**Definition B.1.** Let $\mathsf{V}$ be a variety. An algebra $\mathbf{A} \in \mathsf{V}$ is called

(1) **free** if there is an isomorphism $\mathbf{A} \simeq \mathbf{F}_{\mathsf{V}}(\kappa)$ for some cardinal $\kappa$;

(2) **finitely generated** if there is a surjective homomorphism $\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}$ for some $n \in \omega$.

**Definition B.2.** A variety $\mathsf{V}$ is called

(1) **locally finite** if all its finitely generated algebras are finite;

(2) **finitely generated** if $\mathsf{V} = V(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ for $\mathbf{A}_1, \ldots, \mathbf{A}_n$ finite similar algebras.

*Remark* B.3. Observe that $V(\mathbf{A}_1, \ldots, \mathbf{A}_n) = V(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$.

**Lemma B.4.** *Let $\mathsf{V}$ be a variety. If $\mathsf{V}$ is finitely generated then it is locally finite.*

*Proof.* By the previuos remark, we can assume that $\mathsf{V} = V(\mathbf{A})$ for some $\mathbf{A}$ finite. Let $n < \omega$. We prove that $\mathbf{F}_{\mathsf{V}}(n)$ is finite. Consider the homomorphism

$$\mathbf{F}_{\mathsf{V}}(n) \to \mathbf{A}^{A^n}, \quad t(x_1, \ldots, x_n) \mapsto t^{\mathbf{A}}$$

This homomorphism is injective: if $t^{\mathbf{A}} = s^{\mathbf{A}}$, then $\mathbf{A} \models t \equiv s$, i.e. $t = s$ in $\mathbf{F}_{\mathsf{V}}(n)$. Thus $\mathbf{F}_{\mathsf{V}}(n)$ is finite. $\square$

**Definition B.5.** Let $\mathsf{V}$ and $\mathsf{W}$ be two varieties. We say that $\mathsf{V}$ is **interpretable** into $\mathsf{W}$ ($\mathsf{V} \leq \mathsf{W}$) if there is a clone homomorphism $\mathrm{Clo}(\mathsf{V}) \to \mathrm{Clo}(\mathsf{W})$.

**Theorem B.6.** *Let $\mathsf{V}$ be a locally finite variety. The following are equivalent:*

(1) $\mathbf{1} \notin \mathrm{typ}\{\mathsf{V}\}$*;*

(2) *there is an idempotent variety* $\mathsf{W}$ *such that* $\mathsf{W} \leq \mathsf{V}$ *and* $\mathsf{W} \nleq \mathsf{Set}$.

**Corollary B.7.** *Let $\mathbf{A}$ be a finite idempotent algebra. There is $\mathbf{B} \in HS(\mathbf{A})$ such that* $\mathrm{Clo}(\mathbf{B}) = \mathbf{N}$ *iff* $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$.

*Proof.* If $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$, then $\mathbf{1} \in \mathrm{typ}\{HSP(\mathbf{A})\}$. Since $\mathbf{A}$ is finite, then, by Lemma B.4 $HSP(\mathbf{A})$ is locally finite, and therefore, by Theorem B.6, for every idempotent variety $\mathsf{W}$, either $\mathsf{W} \not\leq HSP(\mathbf{A})$ or $\mathsf{W} \leq \mathsf{Set}$. In particular, since $\mathbf{A}$ is idempotent, $HSP(\mathbf{A}) \leq \mathsf{Set}$. This means that there is a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathbf{N}$. Equivalently, every term operation of $\mathbf{A}$ is a projection. Hence, $\mathrm{Clo}(\mathbf{A}) = \mathbf{N}$.

Conversely, let $\mathbf{B} \in HS(\mathbf{A})$ such that $\mathrm{Clo}(\mathbf{B}) = \mathbf{N}$; this means that $\mathbf{B}$ is term equivalent to a set. Hence $\mathbf{B}$ is polynomial equivalent to a set on which the trivial group acts. Then $\mathbf{1} \in \mathrm{typ}\{\mathbf{B}\}$, and therefore $\mathbf{1} \in \mathrm{typ}\{HS(\mathbf{A})\}$. □

**Theorem B.8** ([3])**.** *Let* $\mathbf{A}$ *be a finite idempotent algebra. Then* $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation iff* $\mathbf{1} \notin \mathrm{typ}\{HS(\mathbf{A})\}$*.*

## References


[1] Bulatov, A. A., Jeavons, P., Krokhin, A. A. (2005). Classifying the complexity of constraints using finite algebras. *SIAM Journal of Computation* 34(3), 720-742.

[2] Jeavons, P. (1998). On the algebraic structure of combinatorial problems, *Theoretical Computer Science* 200, 185-204.

[3] Maróti, M., McKenzie, R. (2008). Existence theorems for weakly symmetric operations. *Algebra Universalis* 59, 463-489.

[4] Pálfy, P. P. (1984). Unary polynomials in algebras I. *Algebra Universalis* 18, 262-273.

[5] Szendrei, Á. (1990). Simple surjective algebras having no proper subalgebras. *Journal of the Australian Mathematical Society*, 48(3), 434-454.