CSP FAST TRACK

ARTURO

ABSTRACT. In this note

1. INTRODUCTION

Let R be a set of relation symbols. Let $\mathcal{A} = (A, P)$ be a relational structure over R. Let X be a countable set of variables. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathcal{A})$[1] we mean the following decision problem: given a finite set Σ of atomic formulas over R, decide whether there is an assignment $(-)^{\mathcal{A}} : X \to A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x, y, x_1, \ldots, x_n \in X$

$$r(x_1, \ldots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}} \tag{1}$$

$$x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}} \tag{2}$$

Clearly, it is enough to find an assignment only for those variables that appear in Σ.

Important: if a set of relation symbols is not fixed, a relational structure $\mathcal{A}$ will be always thought as a structure over the set of its relations.

Starting point: consider the case when A is finite.

Let F be a set of function symbols. Let $\mathbf{A} = (A, \Phi)$ be an algebra over F. Let $\mathcal{A} := (A, \mathrm{Inv}(\Phi))$. By $\mathrm{CSP}(\mathbf{A})$ we mean the decision problem $\mathrm{CSP}(\mathcal{A})$.

Definition 1.1. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

Theorem 1.2. *Let $\mathbf{A}$ be a finite idempotent algebra. Then the following are equivalent:*

1. $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable;*
2. $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation;*
3. *for every* $\mathbf{B} \in HS(\mathbf{A})$, $\mathrm{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\mathrm{CSP}(\mathbf{A})$ *is* NP-*complete.*

[1] More often denoted by $\mathrm{CSP}(P)$.

2. Kinds of Operations

Definition 2.1. An operation $\varphi : A^n \to A$ is called

(1) **essentially unary** if there is an index i and a function $\psi : A \to A$ such that
$$\varphi(a_1, \ldots, a_n) = \psi(a_i)$$
for all $a_1, \ldots, a_n \in A$.

(2) **idempotent** if $\varphi(a, \ldots, a) = a$ for all $a \in A$.

3. Relational Clones

Definition 3.1. Let R be a set of relation symbols and $\mathcal{A}$ be a relational structure over R. We denote by $\mathrm{Clo}(\mathcal{A})$ the smallest set containing
$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$
and closed under

(1) **permutaion**: if $\rho \in \mathrm{Clo}(\mathcal{A})$, then also
$$\{(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) : \sigma \in S_n, (a_1, \ldots, a_n) \in \rho\} \in \mathrm{Clo}\,\mathcal{A}$$

(2) **extension**: if $\rho \in \mathrm{Clo}(\mathcal{A})$, then also
$$\{(a_1, \ldots, a_n, a_{n+1}) : (a_1, \ldots, a_n) \in \rho, a_{n+1} \in A\} \in \mathrm{Clo}\,\mathcal{A}$$

(3) **truncation**: if $\rho \in \mathrm{Clo}(\mathcal{A})$, then also
$$\{(a_1, \ldots, a_{n-1}) : (a_1, \ldots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \mathrm{Clo}\,\mathcal{A}$$

(4) intersection.

Remark 3.2. Observe that $\mathrm{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \ldots, x_n)$ such that
$$\mathcal{A} \models \varphi(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in \rho$$

Theorem 3.3. *For any pair of relational structures $\mathcal{A} = (A, \Gamma)$ and $\mathcal{B} = (A, H)$ such that H is finite and $H \subseteq \mathrm{Clo}(\mathcal{A})$, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$.*

Proof. Let Σ be a set of atomic formulas over H. Let $\eta(x_1, \ldots, x_n) \in \Sigma$. For every $a_1, \ldots, a_n \in A$
$$\mathcal{B} \models \eta(a_1, \ldots, a_n) \iff \mathcal{A} \models \varphi(a_1, \ldots, a_n) \tag{3}$$
for some $\varphi(x_1, \ldots, x_n)$ of the form
$$\exists y_1, \ldots, y_m \, (\gamma_1(z^1_1, \ldots, z^1_{n_1}) \wedge \cdots \wedge \gamma_k(z^k_1, \ldots, z^k_{n_k}))$$
where $\gamma_1, \ldots, \gamma_k \in \Gamma$ and $z^i_j \in \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$. We can assume (up to renaming of variables) that $y_1, \ldots, y_m$ do not appear in any formula of Σ.

Now, for each $\eta(x_1, \ldots, x_n) \in \Sigma$ perform the following steps:

(1) add $\{\gamma_1(z_1^1, \ldots, z_{n_1}^1), \ldots, \gamma_k(z_1^k, \ldots, z_{n_k}^k)\}$ to Σ;

(2) remove $\eta(x_1, \ldots, x_n)$ from Σ.

At the the end we obtain a set of equations T over G. This is a polynomial-time reduction. By (3) it is clear that we can find an assignment $X \to A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. □

Corollary 3.4. *Let* $\mathcal{A} = (A, P)$ *and* $\mathcal{B} = (A, \mathrm{Clo}(\mathcal{A}))$. *Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

(2) $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

Theorem 3.5 ([1])**.** *Let* $\mathcal{A}$ *be a relational structure. If* $\mathrm{Pol}(\mathcal{A})$ *contains essentially unary operations only,* $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete.*

4. Surjective Algebras

Remark 4.1. Let $\mathbf{A}$ be an algebra. Every element of $\mathrm{Clo}(\mathbf{A})$ is surjective iff every element of $\mathrm{Clo}_1(\mathbf{A})$ is. In this case $\mathrm{Clo}_1(\mathbf{A})$ is a group.

Lemma 4.2. *Let* $A = \{a_1, \ldots, a_k\}$ *be a finite set. Let* $\mathbf{A} = (A, \Phi)$ *be an algebra. Then the relation*

$$\sigma := \{(\psi(a_1), \ldots, \psi(a_k)) : \psi \in \mathrm{Clo}_1(\mathbf{A})\} \tag{4}$$

belongs to $\mathrm{Inv}(\Phi)$.

Proof. We show that for every $f \in F_n$ and for every matrix M

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_k^1 & \cdots & a_n^k \end{bmatrix}$$

such that $(a_1^1, \ldots, a_1^k) \in \sigma, \ldots, (a_1^n, \ldots, a_n^k) \in \sigma$ we have

$$(f^{\mathbf{A}}(a_1^1, \ldots, a_n^1), \ldots, f^{\mathbf{A}}(a_k^1, \ldots, a_n^k)) \in \sigma$$

By hypothesis we can write M as

$$\begin{bmatrix} \psi_1(a_1) & \cdots & \psi_n(a_1) \\ \vdots & \ddots & \vdots \\ \psi_1(a_k) & \cdots & \psi_n(a_k) \end{bmatrix}$$

but then

$$\begin{aligned} &(f^{\mathbf{A}}(a_1^1, \ldots, a_n^1), \ldots, f^{\mathbf{A}}(a_k^1, \ldots, a_n^k)) \\ &\qquad = (f^{\mathbf{A}}(\psi_1(a_1), \ldots, \psi_n(a_1)), \ldots, f^{\mathbf{A}}(\psi_1(a_k), \ldots, \psi_n(a_k))) \\ &\qquad\qquad = (f^{\mathbf{A}}[\psi_1, \ldots, \psi_n](a_1), \ldots, f^{\mathbf{A}}[\psi_1, \ldots, \psi_n](a_k)) \end{aligned}$$

and we conclude since $f^{\mathbf{A}}[\psi_1, \ldots, \psi_n] \in \mathrm{Clo}_1(\mathbf{A})$. □

Definition 4.3. Let $\mathbf{A}$ be an algebra. Let $\mathrm{Id}(A)$ be the set of idempotent operations on A. We define $\mathrm{Clo}_{\mathrm{Id}}(\mathbf{A}) := \mathrm{Clo}(\mathbf{A}) \cap \mathrm{Id}(A)$.

Theorem 4.4. *Let F be a set of function symbols and let $\mathbf{A} = (A, \Phi)$ be a finite surjective algebra over F. Let $\mathbf{B} := (A, \mathrm{Clo}_{\mathrm{Id}}(\mathbf{A}))$. Then*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

(2) $\mathrm{CSP}(\mathbf{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

Proof. Let $A = \{a_1, \ldots, a_k\}$ and let $\Gamma := \{\gamma_1, \ldots, \gamma_k\}$ where $\gamma_i := \{a_i\}$. Let $\mathcal{A} := (A, \mathrm{Inv}(\Phi))$ and $\mathcal{B} := (A, \mathrm{Inv}(\Phi) \cup \Gamma)$.

By definition and by Remark CSP($\mathbf{B}$) is polynomial-time equivalent to CSP($\mathbf{A}$) iff CSP($\mathcal{B}$) is polynomial-time equivalent to CSP($\mathcal{A}$).

That CSP($\mathcal{A}$) is polynomial-time reducible to CSP($\mathcal{B}$) is obvious. Let Σ be a set of atomic formulas over $\mathrm{Inv}(\Phi) \cup \Gamma$ and let $\{x_1, \ldots, x_k\}$ be variables that do not appear in Σ. By Remark 4.1, since $\mathbf{A}$ is surjective, $\mathrm{Clo}_1(\mathbf{A})$ forms a group. Moreover, the relation σ of Lemma 4.2 belongs to $\mathrm{Inv}(\Phi)$. Now, perform the following steps:

(1) replace every formula $\gamma_i(x)$ with $x \equiv x_i$;

(2) add the formula $\sigma(x_1, \ldots, x_k)$.

At the the end we obtain a set of equations T over R. This is a polynomial-time reduction. We finally show that we can find an assignment such that $\mathcal{A} \models T$ iff we can find an assignment such that $\mathcal{B} \models \Sigma$. Let $(-)^{\mathcal{B}} : X \to A$ be an assignment such that $\mathcal{B} \models \Sigma$. Consider the assignment

$$(x)^{\mathcal{A}} = \begin{cases} (x)^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then $(-)^{\mathcal{A}}$ is such that $\mathcal{A} \models T$. Conversely, assume that there is an assignment $(-)^{\mathcal{A}}$ such that $\mathcal{A} \models T$. There is $\psi \in \mathrm{Clo}_1(\mathbf{A})$ such that $x_i^{\mathcal{A}} = \psi(a_i)$ for all i. Consider $(-)'^{\mathcal{A}} := \psi^{-1}(-)^{\mathcal{A}}$. Every relation in $\mathrm{Inv}(\Phi)$ is invariant under ψ^{-1}, hence defining $(x)^{\mathcal{B}} := (x)'^{\mathcal{A}}$ is enough to have $\mathcal{B} \models \Sigma$. □

5. Subalgebras and Images

Theorem 5.1. *Let $\mathbf{A}$ be a finite algebra.*

(1) *if* $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable, so is* $\mathrm{CSP}(\mathbf{B})$ *for every* $\mathbf{B} \leq \mathbf{A}$*;*

(2) *if there is* $\mathbf{B} \leq \mathbf{A}$ *such that* $\mathrm{CSP}(\mathbf{B})$ *is* NP*-complete, so is* $\mathrm{CSP}(\mathbf{A})$.

Proof. Let $\mathbf{A} = (A, \Phi)$. Observe that $\mathrm{Inv}(\Phi|B) \subseteq \mathrm{Inv}(\Phi)$. □

Theorem 5.2. *Let $\mathbf{A}, \mathbf{B}$ be two algebras.*

(1) *if* $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable, so is* $\mathrm{CSP}(\mathbf{B})$ *for every surjective homomorphism* $\alpha : \mathbf{A} \to \mathbf{B}$*;*

(2) *if there is a surjective homomorphism* $\alpha : \mathbf{A} \to \mathbf{B}$ *such that* $\mathrm{CSP}(\mathbf{B})$ *is* NP*-complete, so is* $\mathrm{CSP}(\mathbf{A})$.

Proof. Let $\mathbf{A} = (A, \Phi_A)$, $\mathbf{B} = (B, \Phi_B)$ over the same function symbols F.

□

References

[1] Jeavons, P. (1998). On the algebraic structure of combinatorial problems, *Theoretical Computer Science* 200, 185–204.