

# CSP FAST TRACK

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ABSTRACT. In this note

## 1. INTRODUCTION

Let  $R$  be a set of relation symbols. Let  $\mathcal{A} = (A, P)$  be a relational structure over  $R$ . By the **constraint satisfaction problem**  $\text{CSP}(\mathcal{A})$ <sup>1</sup> we mean the following decision problem: given a set  $X$  of variables and a set  $\Sigma$  of atomic formulas<sup>2</sup> over  $R$ , decide whether there is an assignment  $(-)^{\mathcal{A}} : X \rightarrow A$  such that  $\mathcal{A} \models \Sigma$ ; i.e. for all  $r \in R_n$  and for all  $x_1, \dots, x_n \in X$

$$(1) \quad r(x_1, \dots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}}$$

Starting point: consider the case when  $A$  is finite.

We shall say that  $\text{CSP}(\mathcal{A})$  is decidable if there is a uniform (unique) algorithm deciding  $\text{CSP}(\mathcal{A})$  for every  $X$  and  $\Sigma$  over  $R$ .

Let  $F$  be a set of function symbols. Let  $\mathbf{A} = (A, \Phi)$  be an algebra over  $F$ . By the **constraint satisfaction problem**  $\text{CSP}(\mathbf{A})$  we mean the following decision problem: decide uniformly, that is by a unique algorithm, every  $\text{CSP}((A, P))$  such that  $P \subseteq \text{Inv}(\Phi)$ .

**Definition 1.1.** Let  $F$  be a set of function symbols and  $\mathbf{A}$  be an algebra over  $F$ . We denote by  $\text{Clo}(\mathbf{A})$  the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

**Theorem 1.2.** *Let  $\mathbf{A}$  be a finite idempotent algebra. Then the following are equivalent:*

- (1)  $\text{CSP}(\mathbf{A})$  is polynomial-time decidable;
- (2)  $\text{Clo}(\mathbf{A})$  contains a weak near-unanimity operation;
- (3) for every  $\mathbf{B} \in \text{HS}(\mathbf{A})$ ,  $\text{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$ .

Otherwise,  $\text{CSP}(\mathbf{A})$  is NP-complete.

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<sup>1</sup>More often denoted by  $\text{CSP}(P)$ .

<sup>2</sup>Without equality.

## 2. RELATIONAL CLONES

**Definition 2.1.** Let  $R$  be a set of relation symbols and  $\mathcal{A}$  be a relational structure over  $R$ . We denote by  $\text{Clo}(\mathcal{A})$  the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under intersection and truncation<sup>3</sup>.

*Remark 2.2.* Observe that  $\text{Clo}(\mathcal{A})$  is given by all the relations  $\rho$  of  $A$  definable<sup>4</sup> by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that  $\rho \subseteq A^n$  is definable if there is a formula  $\varphi(x_1, \dots, x_n)$  such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

**Theorem 2.3.** For any  $\mathcal{A} = (A, P)$  and  $\mathcal{B} = (A, \Gamma)$ <sup>5</sup> with  $\Gamma \subseteq \text{Clo}(\mathcal{A})$  and with  $\Gamma$  finite,  $\text{CSP}(\mathcal{B})$  is polynomial-time reducible to  $\text{CSP}(\mathcal{A})$ .

**Corollary 2.4.** Let  $\mathcal{A} = (A, P)$  and  $\mathcal{B} = (A, \text{Clo}(\mathcal{A}))$ . Then

- (1)  $\text{CSP}(\mathcal{A})$  is polynomial-time decidable iff  $\text{CSP}(\mathcal{B})$  is.
- (2)  $\text{CSP}(\mathcal{A})$  is NP-complete iff  $\text{CSP}(\mathcal{B})$  is.

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<sup>3</sup>If  $\rho \in \text{Clo}(\mathcal{A})$ , then also  $\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \text{Clo}(\mathcal{A})$ .

<sup>4</sup>Using the symbols in  $R$  and allowing equality.

<sup>5</sup>Not necessarily over the same  $R$ !