

CSP FAST TRACK

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ABSTRACT. In this note

1. INTRODUCTION

Let R be a set of relation symbols. Let \mathcal{A} be a relational structure over R . Let X be a countable set of variables. By the **constraint satisfaction problem** $\text{CSP}(\mathcal{A})$ ¹ we mean the following decision problem: given a **finite** set Σ of atomic formulas over R , decide whether there is an assignment $(-)^{\mathcal{A}} : X \rightarrow A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x, y, x_1, \dots, x_n \in X$

$$(1) \quad r(x_1, \dots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}}$$

$$(2) \quad x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}}$$

Clearly, it is enough to find an assignment only for those variables that appear in Σ .

Important: if a set of relation symbols is not fixed, a relational structure \mathcal{A} will be always thought as a structure over the set of its relations.

Starting point: consider the case when A is finite.

Definition 1.1. Let $\rho \in A^k$, and $\varphi : A^n \rightarrow A$. We say that φ **preserves** ρ if given a matrix

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

with $(a_1^1, \dots, a_n^1) \in \rho, \dots, (a_1^k, \dots, a_n^k) \in \rho$,

$$(\varphi(a_1^1, \dots, a_n^1), \dots, \varphi(a_1^k, \dots, a_n^k)) \in \rho$$

Definition 1.2. If Γ is a set of relations on A and Φ is a set of operations on A we denote by

- (1) $\text{Inv}(\Phi)$ the set of relations on A that are preserved by all the elements of Φ ;
- (2) $\text{Pol}(\Gamma)$ the set of operations on A that preserve all the elements of Γ .

Moreover, if \mathbf{A} is an algebra and \mathcal{A} a relational structure on the same set A :

- (1) $\text{Inv}(\mathbf{A})$ is the set of relations that are preserved by all the $f^{\mathbf{A}}$;

¹More often denoted by $\text{CSP}(P)$.

- (2) $\text{Pol}(\mathcal{A})$ is the set of operations that preserve all the $r^{\mathcal{A}}$.

Let \mathbf{A} be an algebra. Let $\mathcal{A} := (A, \text{Inv}(\mathbf{A}))$. By $\text{CSP}(\mathbf{A})$ we mean the decision problem $\text{CSP}(\mathcal{A})$.

Definition 1.3. Let F be a set of function symbols and \mathbf{A} be an algebra over F . We denote by $\text{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

Theorem 1.4. *Let \mathbf{A} be a finite algebra. Then the following are equivalent:*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable;
- (2) $\text{Clo}(\mathbf{A})$ contains a weak near-unanimity operation;
- (3) for every $\mathbf{B} \in \text{HS}(\mathbf{A})$, $\text{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\text{CSP}(\mathbf{A})$ is NP-complete.

2. KINDS OF OPERATIONS

Definition 2.1. An operation $\varphi : A^n \rightarrow A$ is called

- (1) **essentially unary** if there is an index i and a function $\psi : A \rightarrow A$ such that

$$\varphi(a_1, \dots, a_n) = \psi(a_i)$$

for all $a_1, \dots, a_n \in A$.

- (2) **idempotent** if $\varphi(a, \dots, a) = a$ for all $a \in A$.

3. RELATIONAL CLONES

Definition 3.1. Let R be a set of relation symbols and \mathcal{A} be a relational structure over R . We denote by $\text{Clo}(\mathcal{A})$ the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under

- (1) **permutation**: if $\rho \in \text{Clo}(\mathcal{A})$, then also

$$\{(a_{\sigma(1)}, \dots, a_{\sigma(n)}) : \sigma \in S_n, (a_1, \dots, a_n) \in \rho\} \in \text{Clo} \mathcal{A}$$

- (2) **extension**: if $\rho \in \text{Clo}(\mathcal{A})$, then also

$$\{(a_1, \dots, a_n, a_{n+1}) : (a_1, \dots, a_n) \in \rho, a_{n+1} \in A\} \in \text{Clo} \mathcal{A}$$

- (3) **truncation**: if $\rho \in \text{Clo}(\mathcal{A})$, then also

$$\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \text{Clo} \mathcal{A}$$

- (4) intersection.

Remark 3.2. Observe that $\text{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

Theorem 3.3 ([2]). *For any pair of relational structures $\mathcal{A} = (A, \Gamma)$ and $\mathcal{B} = (A, H)$ such that H is finite and $H \subseteq \text{Clo}(\mathcal{A})$, $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$.*

Proof. Let Σ be a set of atomic formulas over H . Let $\eta(x_1, \dots, x_n) \in \Sigma$. For every $a_1, \dots, a_n \in A$

$$(3) \quad \mathcal{B} \models \eta(a_1, \dots, a_n) \iff \mathcal{A} \models \varphi(a_1, \dots, a_n)$$

for some $\varphi(x_1, \dots, x_n)$ of the form

$$\exists y_1, \dots, y_m (\gamma_1(z_1^1, \dots, z_{n_1}^1) \wedge \dots \wedge \gamma_k(z_1^k, \dots, z_{n_k}^k))$$

where $\gamma_1, \dots, \gamma_k \in \Gamma$ and $z_j^i \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$. We can assume (up to renaming of variables) that y_1, \dots, y_m do not appear in any formula of Σ .

Now, for each $\eta(x_1, \dots, x_n) \in \Sigma$ perform the following steps:

- (1) add $\{\gamma_1(z_1^1, \dots, z_{n_1}^1), \dots, \gamma_k(z_1^k, \dots, z_{n_k}^k)\}$ to Σ ;
- (2) remove $\eta(x_1, \dots, x_n)$ from Σ .

At the the end we obtain a set of equations T over G . **This is a polynomial-time reduction.** By (3) it is clear that we can find an assignment $X \rightarrow A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. \square

Corollary 3.4. *Let \mathcal{A} be a relational structure and $\mathcal{B} = (A, \text{Clo}(\mathcal{A}))$. Then*

- (1) $\text{CSP}(\mathcal{A})$ is polynomial-time decidable iff $\text{CSP}(\mathcal{B})$ is.
- (2) $\text{CSP}(\mathcal{A})$ is NP-complete iff $\text{CSP}(\mathcal{B})$ is.

4. SURJECTIVE ALGEBRAS

Definition 4.1. An algebra \mathbf{A} is **surjective** if all the element of $\text{Clo}(\mathbf{A})$ are surjective.

Remark 4.2. Let \mathbf{A} be an algebra. Every element of $\text{Clo}(\mathbf{A})$ is surjective iff every element of $\text{Clo}_1(\mathbf{A})$ is. In this case $\text{Clo}_1(\mathbf{A})$ is a group.

Let \mathbf{A} be an algebra and let B be a subset of A . Let

$$\text{Clo}(\mathbf{A})|B := \{\varphi|B : \varphi \in \text{Clo}(\mathbf{A}), \nabla_B \in \text{Inv}(\varphi)\}$$

We denote by $\mathbf{A}|B$ the algebra $(B, \text{Clo}(\mathbf{A})|B)$.

Lemma 4.3 ([2]). *Let $\mathcal{A} = (A, P)$ be a relational structure. Let $\varphi \in \text{Pol}(\mathcal{A})_1$. For $\rho \in P_n$ let*

$$\varphi(\rho) := \{(\varphi(a_1), \dots, \varphi(a_n)) : (a_1, \dots, a_n) \in \rho\}$$

Let $\mathcal{B} := (A, \varphi(P))$ where $\varphi(P) := \{\varphi(\rho) : \rho \in P\}$. Then

- (1) $\text{CSP}(\mathcal{A})$ is polynomial-time decidable iff $\text{CSP}(\mathcal{B})$ is;
- (2) $\text{CSP}(\mathcal{A})$ is NP-complete iff $\text{CSP}(\mathcal{B})$ is.

Proof. We show that $\text{CSP}(\mathcal{A})$ is polynomial-time equivalent to $\text{CSP}(\mathcal{B})$. Let Σ be a set of atomic formulas over P . Replace every occurrence of $\rho(x_1, \dots, x_n)$ with $\varphi(\rho)(x_1, \dots, x_n)$, obtaining a set of formulas T over $\varphi(P)$. Given any assignment $(-)^{\mathcal{A}} : X \rightarrow A$ such that $\mathcal{A} \models \Sigma$, define $(x)^{\mathcal{B}} := (x)^{\mathcal{A}}$; then $\mathcal{B} \models T$. Conversely, given any assignment $(-)^{\mathcal{A}} : X \rightarrow A$ such that $\mathcal{A} \models \Sigma$, defining $(x)^{\mathcal{B}} := \varphi(x^{\mathcal{A}})$ is enough to have $\mathcal{B} \models T$. \square

Theorem 4.4. *Let \mathbf{A} be a finite algebra. Then there is $B \subseteq A$ such that $\mathbf{A}|B$ is surjective and*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable iff $\text{CSP}(\mathbf{A}|B)$ is;
- (2) $\text{CSP}(\mathbf{A})$ is NP-complete iff $\text{CSP}(\mathbf{A}|B)$.

Proof. Assume that \mathbf{A} is not surjective. Then by Remark 4.2 there is $\psi \in \text{Clo}_1(\mathbf{A})$ not surjective. Let $\varphi \in \text{Clo}_1(\mathbf{A})$ such that φ is not surjective and $\varphi[A]$ has minimal cardinality. Define $B := \varphi[A]$. We show that $\mathbf{A}|B$ is surjective. Let $\psi \in \text{Clo}_1(\mathbf{A})|B$; if, towards a contradiction, $\psi[B] \subset B$, then $\psi\varphi[A] \subset \varphi[A] \subset A$ contradicting the minimality. The last part of the statement follows immediately from Lemma 4.3. \square

Definition 4.5. Let \mathbf{A} be an algebra. Let $\text{id}(A)$ be the set of idempotent operations on A . We define $\text{Clo}_{\text{id}}(\mathbf{A}) := \text{Clo}(\mathbf{A}) \cap \text{id}(A)$.

Remark 4.6. Let $\mathbf{A} = (A, \Phi)$. Observe that an operation $\varphi \in \text{Clo}(\mathbf{A})$ is idempotent iff it preserves the relations in $\Delta^{(1)} = \{\{a\} : a \in A\}$. Hence $\text{Inv}(\text{Clo}_{\text{id}}(\mathbf{A})) = \text{Clo}(\mathcal{A})$ where $\mathcal{A} = (A, \text{Inv}(\mathbf{A}) \cup \Delta^{(1)})$.

Lemma 4.7. *Let $A = \{a_1, \dots, a_k\}$ be a finite set. Let $\mathbf{A} = (A, \Phi)$ be an algebra. Then the relation*

$$(4) \quad \sigma := \{(\psi(a_1), \dots, \psi(a_k)) : \psi \in \text{Clo}_1(\mathbf{A})\}$$

belongs to $\text{Inv}(\mathbf{A})$.

Proof. We show that for every $f \in F_n$ and for every matrix M

$$\begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_n^k \end{bmatrix}$$

such that $(a_1^1, \dots, a_n^1) \in \sigma, \dots, (a_1^k, \dots, a_n^k) \in \sigma$ we have

$$(f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \in \sigma$$

By hypothesis we can write M as

$$\begin{bmatrix} \psi_1(a_1) & \cdots & \psi_n(a_1) \\ \vdots & \ddots & \vdots \\ \psi_1(a_k) & \cdots & \psi_n(a_k) \end{bmatrix}$$

but then

$$\begin{aligned} & (f^{\mathbf{A}}(a_1^1, \dots, a_n^1), \dots, f^{\mathbf{A}}(a_1^k, \dots, a_n^k)) \\ &= (f^{\mathbf{A}}(\psi_1(a_1), \dots, \psi_n(a_1)), \dots, f^{\mathbf{A}}(\psi_1(a_k), \dots, \psi_n(a_k))) \\ &= (f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_1), \dots, f^{\mathbf{A}}[\psi_1, \dots, \psi_n](a_k)) \end{aligned}$$

and we conclude since $f^{\mathbf{A}}[\psi_1, \dots, \psi_n] \in \text{Clo}_1(\mathbf{A})$. \square

Theorem 4.8. *Let F be a set of function symbols and let $\mathbf{A} = (A, \Phi)$ be a finite surjective algebra over F . Let $\mathbf{B} := (A, \text{Clo}_{\text{id}}(\mathbf{A}))$. Then*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable iff $\text{CSP}(\mathbf{B})$ is.
- (2) $\text{CSP}(\mathbf{A})$ is NP-complete iff $\text{CSP}(\mathbf{B})$ is.

Proof. Let $A = \{a_1, \dots, a_k\}$ and let $\Gamma := \Delta^{(1)}$. Let $\mathcal{A} := (A, \text{Inv}(\mathbf{A}))$ and $\mathcal{B} := (A, \text{Inv}(\mathbf{A}) \cup \Gamma)$.

By definition and by Remark 4.6 $\text{CSP}(\mathbf{B})$ is polynomial-time equivalent to $\text{CSP}(\mathbf{A})$ iff $\text{CSP}(\mathcal{B})$ is polynomial-time equivalent to $\text{CSP}(\mathcal{A})$.

That $\text{CSP}(\mathcal{A})$ is polynomial-time reducible to $\text{CSP}(\mathcal{B})$ is obvious. Let Σ be a set of atomic formulas over $\text{Inv}(\mathbf{A}) \cup \Gamma$ and let $\{x_1, \dots, x_k\}$ be variables that do not appear in Σ . By Remark 4.2, since \mathbf{A} is surjective, $\text{Clo}_1(\mathbf{A})$ forms a group. Moreover, the relation σ of Lemma 4.7 belongs to $\text{Inv}(\mathbf{A})$. Now, perform the following steps:

- (1) replace every formula $\gamma_i(x)$ with $x \equiv x_i$;
- (2) add the formula $\sigma(x_1, \dots, x_k)$.

At the the end we obtain a set of equations T over R . **This is a polynomial-time reduction.** We finally show that we can find an assignment such that $\mathcal{A} \models T$ iff we can find an assignment such that $\mathcal{B} \models \Sigma$. Let $(-)^{\mathcal{B}} : X \rightarrow A$ be an assignment such that $\mathcal{B} \models \Sigma$. Consider the assignment

$$(x)^{\mathcal{A}} = \begin{cases} (x)^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then $(-)^{\mathcal{A}}$ is such that $\mathcal{A} \models T$. Conversely, assume that there is an assignment $(-)^{\mathcal{A}}$ such that $\mathcal{A} \models T$. There is $\psi \in \text{Clo}_1(\mathbf{A})$ such that $x_i^{\mathcal{A}} = \psi(a_i)$ for all i . Consider $(-)^{\mathcal{A}'} := \psi^{-1}(-)^{\mathcal{A}}$. Every relation in $\text{Inv}(\mathbf{A})$ is invariant under ψ^{-1} , hence defining $(x)^{\mathcal{B}} := (x)^{\mathcal{A}'}$ is enough to have $\mathcal{B} \models \Sigma$. \square

5. SUBALGEBRAS AND IMAGES

Theorem 5.1. *Let \mathbf{A} be a finite algebra.*

- (1) *if $\text{CSP}(\mathbf{A})$ is polynomial-time decidable, so is $\text{CSP}(\mathbf{B})$ for every $\mathbf{B} \leq \mathbf{A}$;*
- (2) *if there is $\mathbf{B} \leq \mathbf{A}$ such that $\text{CSP}(\mathbf{B})$ is NP-complete, so is $\text{CSP}(\mathbf{A})$.*

Proof. Let $\mathbf{A} = (A, \Phi)$. Observe that $\text{Inv}(\Phi|B) \subseteq \text{Inv}(\Phi)$. \square

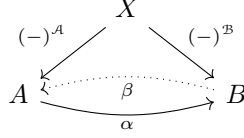
Theorem 5.2. *Let \mathbf{A}, \mathbf{B} be two finite algebras of the same type.*

- (1) if $\text{CSP}(\mathbf{A})$ is polynomial-time decidable, so is $\text{CSP}(\mathbf{B})$ for every surjective homomorphism $\alpha : \mathbf{A} \rightarrow \mathbf{B}$;
- (2) if there is a surjective homomorphism $\alpha : \mathbf{A} \rightarrow \mathbf{B}$ such that $\text{CSP}(\mathbf{B})$ is NP-complete, so is $\text{CSP}(\mathbf{A})$.

Proof. Let $\mathcal{B} = (B, \text{Inv}(\mathbf{B}))$. We show that there is $\Gamma \subseteq \text{Inv}(\mathbf{B})$ such that, defining $\mathcal{A} = (A, \Gamma)$, $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$. For every $\rho \in \text{Inv}(\mathbf{B})_n$ let

$$\alpha^{-1}(\rho) := \{(a_1, \dots, a_n) \in A^n : (\alpha(a_1), \dots, \alpha(a_n)) \in \rho\}$$

Clearly, $\alpha^{-1}(\rho) \in \text{Inv}(\mathbf{A})_n$ and therefore, letting $\Gamma := \{\alpha^{-1}(\rho) : \rho \in \Gamma\}$, $\Gamma \subseteq \text{Inv}(\mathbf{A})$. Let Σ be a set of atomic formulas over $\text{Inv}(\mathbf{B})$. Replace every formula $\rho(x_1, \dots, x_n)$ with $\alpha^{-1}(\rho)(x_1, \dots, x_n)$. We obtain a set of equations T over Γ by a polynomial-time reduction. Let β be a section of α .



Referring to the assignments in the picture, each defined in terms of the other so that the diagram commute, it is clear that $\mathcal{A} \models T$ iff $\mathcal{B} \models \Sigma$. \square

6. SIMPLE AND STRICTLY SIMPLE ALGEBRAS

Definition 6.1. An algebra \mathbf{A} is called **simple** if the lattice of congruences on \mathbf{A} is $\{\Delta, \nabla\}$. An algebra \mathbf{A} is **strictly simple** if it is simple and has no nontrivial subalgebras.

Remark 6.2. Since the lattice of congruence of \mathbf{A} is in bijection with the set of surjective homomorphisms from \mathbf{A} , \mathbf{A} is simple if whenever there is a surjective homomorphism $\mathbf{A} \rightarrow \mathbf{B}$ such that $|B| < |A|$, then \mathbf{B} is the trivial algebra.

Definition 6.3. Let \mathbf{G} be a group acting on a set A . We say that \mathbf{G} is (or, more accurately, that the action is) **primitive** if $\{A\}$ and $\{\{a\} : a \in A\}$ are the only partitions invariant under \mathbf{G} .

Remark 6.4. Let \mathbf{G} be a group acting on A . Each $g \in G$ induces an operation $\varphi_g : A \rightarrow A$ given by $\varphi_g(a) = g \cdot a$. Let $\Phi_{\mathbf{G}} := \{\varphi_g : g \in G\}$. Thus a \mathbf{G} -set can be seen as an algebra $(A, \Phi_{\mathbf{G}})$. The action is primitive iff this algebra is simple.

Theorem 6.5 (Classification of strictly simple finite algebras, [3]). *Let \mathbf{A} be a finite strictly simple surjective algebra. If \mathbf{A} has trivial subalgebras, \mathbf{A} is term-equivalent to one of the following:*

- (1) j
- (2) $kind$
- (3) $(A, \Phi_{\mathbf{G}})$ for some primitive group \mathbf{G} .

If \mathbf{A} has nontrivial subalgebras, \mathbf{A} is term-equivalent to one of the following:

(1)

Lemma 6.6 ([2]). *Let \mathcal{A} be a relational structure. If $\text{Pol}(\mathcal{A})$ contains essentially unary operations only, $\text{CSP}(\mathcal{A})$ is NP-complete.*

Theorem 6.7. *Let \mathbf{A} be a finite strictly simple surjective algebra. If all the elements of $\text{Clo}(\mathbf{A})$ are essentially unary, then $\text{CSP}(\mathbf{A})$ is NP-complete.*

Proof.

□

Corollary 6.8. *Let \mathbf{A} be a finite algebra. If there is $\mathbf{B} \in HS(\mathbf{A})$ such that $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$, then $\text{CSP}(\mathbf{A})$ is NP-complete.*

Proof. We can assume without loss of generality that \mathbf{B} is surjective by Theorem 4.8. **Why we can dispose of strictly simple?** If \mathbf{B} is such that $\text{Clo}(\mathbf{B}) = \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$, then $\text{CSP}(\mathbf{B})$ is NP-complete by Theorem 6.7. We conclude using the second cluases of Theorems 5.1 and 5.2. □

REFERENCES

- [1] Bulatov, A. A., Jeavons, P., Krokhin, A. A. (2005). Classifying the complexity of constraints using finite algebras. *SIAM Journal of Computation* 34(3), 720–742.
- [2] Jeavons, P. (1998). On the algebraic structure of combinatorial problems, *Theoretical Computer Science* 200, 185–204.
- [3] Szendrei, Á. (1990). Simple surjective algebras having no proper subalgebras. *Journal of the Australian Mathematical Society*, 48(3), 434–454.