# CSP FAST TRACK

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Abstract. In this note

## 1. Introduction

Let $R$ be a set of relation symbols. Let $\mathcal{A} = (A, P)$ be a relational structure over $R$. Let $X$ be a countable set of variables. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathcal{A})$[1] we mean the following decision problem: given a set $\Sigma$ of atomic formulas over $R$, decide whether there is an assignment $(-)^{\mathcal{A}} : X \to A$ and an interpretation of the symbols of $R$ as relations of the appropriate arity such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x, y, x_1, \dots, x_n \in X$

$$r(x_1, \dots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}} \tag{1}$$

$$x \equiv y \in \Sigma \implies x^{\mathcal{A}} = y^{\mathcal{A}} \tag{2}$$

Clearly, it is enough to find an assignment only for those variables that appear in $\Sigma$.

Starting point: consider the case when $A$ is finite.

We shall say that $\mathrm{CSP}(\mathcal{A})$ is decidable if there is a uniform (unique) algorithm deciding $\mathrm{CSP}(\mathcal{A})$ for every $\Sigma$ over $R$.

Let $F$ be a set of function symbols. Let $\mathbf{A} = (A, \Phi)$ be an algebra over $F$. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathbf{A})$ we mean the following decision problem: decide uniformly, that is by a unique algorithm, every $\mathrm{CSP}((A, P))$ such that $P \subseteq \mathrm{Inv}(\Phi)$.

**Definition 1.1.** Let $F$ be a set of function symbols and $\mathbf{A}$ be an algebra over $F$. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \le i \le n, n \in \omega\}$$

and closed under composition.

Goal: prove

**Theorem 1.2.** *Let $\mathbf{A}$ be a finite idempotent algebra. Then the following are equivalent:*

1. $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable;*
2. $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation;*

[1]More often denoted by $\mathrm{CSP}(P)$.

(3) *for every* $\mathbf{B} \in HS(\mathbf{A})$, $\mathrm{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

*Otherwise,* $\mathrm{CSP}(\mathbf{A})$ *is* NP-*complete.*

## 2. Kinds of Operations

**Definition 2.1.** An operation $\varphi : A^n \to A$ is called

(1) **essentially unary** if there is a function $\psi : A \to A$ such that

$$\varphi(a_1, \dots, a_n) = \psi(a_i)$$

for all $a_1, \dots, a_n \in A$.

(2) **idempotent** if $\varphi(a, \dots, a) = a$ for all $a \in A$.

## 3. Relational Clones

**Definition 3.1.** Let $R$ be a set of relation symbols and $\mathcal{A}$ be a relational structure over $R$. We denote by $\mathrm{Clo}(\mathcal{A})$ the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under intersection and truncation[2].

*Remark* 3.2. Observe that $\mathrm{Clo}(\mathcal{A})$ is given by all the relations $\rho$ of $A$ definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

**Theorem 3.3.** *Let $G$ be a set and $D$ be a finite set of relation symbols. For any $\mathcal{A} = (A, \Gamma)$ over $G$ and $\mathcal{B} = (A, \Delta)$ over $D$ with $\Delta \subseteq \mathrm{Clo}(\mathcal{A})$, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$.*

*Proof.* Let $\Sigma$ be a set of atomic formulas over $D$. Let $d(x_1, \dots, x_n) \in \Sigma$. For every $a_1, \dots, a_n \in A$

$$\mathcal{B} \models d(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \delta \iff \mathcal{A} \models \varphi(a_1, \dots, a_n) \tag{3}$$

for some $\varphi(x_1, \dots, x_n)$ of the form

$$\exists y_1, \dots, y_m \, (g_1(z_1^1, \dots, z_{n_1}^1) \wedge \dots \wedge g_k(z_1^k, \dots, z_{n_k}^k))$$

where $g_1, \dots, g_k \in G$ and $z_j^i \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$. We can assume (up to renaming of variables) that $y_1, \dots, y_m$ do not appear in any formula of $\Sigma$.

Now, for each $d(x_1, \dots, x_n) \in \Sigma$ perform the following steps:

(1) add $\{g_1(z_1^1, \dots, z_{n_1}^1), \dots, g_k(z_1^k, \dots, z_{n_k}^k)\}$ to $\Sigma$;

(2) remove $d(x_1, \dots, x_n)$ from $\Sigma$.

[2]If $\rho \in \mathrm{Clo}(\mathcal{A})$, then also $\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \mathrm{Clo}\,\mathcal{A}$.

At the the end we obtain a set of equations $T$ over $G$. This is a polynomial-time reduction. (It's reasonable but for me this kind of stuff is like a leap of faith). By (3) it is clear that we can find an assignment $X \to A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. $\square$

**Corollary 3.4.** *Let* $\mathcal{A} = (A, P)$ *and* $\mathcal{B} = (A, \mathrm{Clo}(\mathcal{A}))$. *Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

(2) $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

**Theorem 3.5** ([1])**.** *Let* $\mathcal{A}$ *be a relational structure. If* $\mathrm{Pol}(\mathcal{A})$ *contains essentially unary operations only,* $\mathrm{CSP}(\mathcal{A})$ *is* NP*-complete.*

## 4. Surjective Algebras

*Remark* 4.1. Let $\mathbf{A}$ be an algebra. Every element of $\mathrm{Clo}(\mathbf{A})$ is surjective iff every element of $\mathrm{Clo}_1(\mathbf{A})$ is. In this case $\mathrm{Clo}_1(\mathbf{A})$ is a group.

**Theorem 4.2.** *Let* $F$ *be a set of function symbols and let* $\mathbf{A} = (A, \Phi)$ *be a finite surjective algebra over* $F$. *Let* $\mathbf{B} := (A, \mathrm{Clo}_{\mathrm{Id}}(\mathbf{A}))$. *Then*

(1) $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

(2) $\mathrm{CSP}(\mathbf{A})$ *is* NP*-complete iff* $\mathrm{CSP}(\mathbf{B})$ *is.*

*Proof.* Let $A = \{a_1, \dots, a_k\}$ and let $\Delta := \{\{a_1\}, \dots, \{a_k\}\}$. For every $\rho \in \mathrm{Inv}(\Phi)$, let $r$ be a relation symbol of the same arity and let $R := \{r : \rho \in \mathrm{Inv}(\Phi)\}$. For every $\delta_i := \{a_i\} \in \Delta$, let $d_i$ be a relation symbol of the same arity and let $G := R \cup \{d_i : \delta_i \in \Delta\}$. Let $\mathcal{A} := (A, \mathrm{Inv}(\Phi))$ and $\mathcal{B} := (A, \mathrm{Inv}(\Phi) \cup \Delta)$.

By definition and by Remark $\mathrm{CSP}(\mathbf{B})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathbf{A})$ iff $\mathrm{CSP}(\mathcal{B})$ is polynomial-time equivalent to $\mathrm{CSP}(\mathcal{A})$.

That $\mathrm{CSP}(\mathcal{A})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{B})$ is obvious. Let $\Sigma$ be a set of atomic formulas over $G$ and let $\{x_1, \dots, x_k\}$ be variables that do not appear in $\Sigma$. By Remark 4.1, since $\mathbf{A}$ is surjective, $\mathrm{Clo}_1(\mathbf{A})$ forms a group. Hence, the relation $\sigma$ of Lemma belongs to $\mathrm{Inv}(\Phi)$. Now, perform the following steps:

(1) replace every formula $d_i(x)$ with $x \equiv x_i$;

(2) add the formula $s(x_1, \dots, x_k)$.

At the the end we obtain a set of equations $T$ over $R$. This is a polynomial-time reduction. We finally show that we can find an assignment such that $\mathcal{A} \models T$ iff we can find an assignment such that $\mathcal{B} \models \Sigma$. Let $(-)^{\mathcal{B}} : X \to A$ be an assignment such that $\mathcal{B} \models \Sigma$. Consider the assignment

$$(x)^{\mathcal{A}} = \begin{cases} (x)^{\mathcal{B}} & \text{if } x \neq x_i \\ a_i & \text{if } x = x_i \end{cases}$$

Then $(-)^{\mathcal{A}}$ is such that $\mathcal{A} \models T$. Conversely, assume that there is an assignment $()^{\mathcal{A}}$ such that $\mathcal{A} \models T$. There is $\psi \in \mathrm{Clo}_1(\mathbf{A})$ such that $x_i^{\mathcal{A}} = \psi(a_i)$ for all $i$. Consider $(-)'^{\mathcal{A}} := \psi^{-1}(-)^{\mathcal{A}}$. Every relation in $\mathrm{Inv}(\Phi)$ is invariant under $\psi^{-1}$, hence defining $(x)^{\mathcal{B}} := (x)'^{\mathcal{A}}$ is enough to have $\mathcal{B} \models \Sigma$. $\square$

## References

[1] Jeavons, P. (1998). On the algebraic structure of combinatorial problems, *Theoretical Computer Science* 200, 185–204.