

CSP FAST TRACK

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ABSTRACT. In this note

1. INTRODUCTION

Let R be a set of relation symbols. Let $\mathcal{A} = (A, P)$ be a relational structure over R . By the **constraint satisfaction problem** $\text{CSP}(\mathcal{A})$ ¹ we mean the following decision problem: given a set X of variables and a set Σ of atomic formulas² over R , decide whether there is an assignment $(-)^{\mathcal{A}} : X \rightarrow A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x_1, \dots, x_n \in X$

$$(1) \quad r(x_1, \dots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \dots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}}$$

Starting point: consider the case when A is finite.

We shall say that $\text{CSP}(\mathcal{A})$ is decidable if there is a uniform (unique) algorithm deciding $\text{CSP}(\mathcal{A})$ for every X and Σ over R .

Let F be a set of function symbols. Let $\mathbf{A} = (A, \Phi)$ be an algebra over F . By the **constraint satisfaction problem** $\text{CSP}(\mathbf{A})$ we mean the following decision problem: decide uniformly, that is by a unique algorithm, every $\text{CSP}((A, P))$ such that $P \subseteq \text{Inv}(\Phi)$.

Definition 1.1. Let F be a set of function symbols and \mathbf{A} be an algebra over F . We denote by $\text{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \rightarrow A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

Theorem 1.2. *Let \mathbf{A} be a finite idempotent algebra. Then the following are equivalent:*

- (1) $\text{CSP}(\mathbf{A})$ is polynomial-time decidable;
- (2) $\text{Clo}(\mathbf{A})$ contains a weak near-unanimity operation;
- (3) for every $\mathbf{B} \in \text{HS}(\mathbf{A})$, $\text{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\text{CSP}(\mathbf{A})$ is NP-complete.

¹More often denoted by $\text{CSP}(P)$.

²Without equality.

2. RELATIONAL CLONES

Definition 2.1. Let R be a set of relation symbols and \mathcal{A} be a relational structure over R . We denote by $\text{Clo}(\mathcal{A})$ the smallest set containing

$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$

and closed under intersection and truncation³.

Remark 2.2. Observe that $\text{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable⁴ by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

Theorem 2.3. For any $\mathcal{A} = (A, P)$ and $\mathcal{B} = (A, \Gamma)$ ⁵ with $\Gamma \subseteq \text{Clo}(\mathcal{A})$ and with Γ finite, $\text{CSP}(\mathcal{B})$ is polynomial-time reducible to $\text{CSP}(\mathcal{A})$.

Corollary 2.4. Let $\mathcal{A} = (A, P)$ and $\mathcal{B} = (A, \text{Clo}(\mathcal{A}))$. Then

- (1) $\text{CSP}(\mathcal{A})$ is polynomial-time decidable iff $\text{CSP}(\mathcal{B})$ is.
- (2) $\text{CSP}(\mathcal{A})$ is NP-complete iff $\text{CSP}(\mathcal{B})$ is.

³If $\rho \in \text{Clo}(\mathcal{A})$, then also $\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \text{Clo}(\mathcal{A})$.

⁴Using the symbols in R and allowing equality.

⁵Not necessarily over the same R !