CSP FAST TRACK

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Abstract. In this note

1. Introduction

Let R be a set of relation symbols. Let $\mathcal{A} = (A, P)$ be a relational structure over R. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathcal{A})$[1] we mean the following decision problem: given a set X of variables and a set $\Sigma(X, R)$ of atomic formulas over R (and X), decide whether there is an assignment $(-)^{\mathcal{A}} : X \to A$ such that $\mathcal{A} \models \Sigma$; i.e. for all $r \in R_n$ and for all $x_1, \ldots, x_n \in X$

$$r(x_1, \ldots, x_n) \in \Sigma \implies (x_1^{\mathcal{A}}, \ldots, x_n^{\mathcal{A}}) \in r^{\mathcal{A}} \tag{1}$$

Starting point: consider the case when A is finite.

We shall say that $\mathrm{CSP}(\mathcal{A})$ is decidable if there is a uniform (unique) algorithm deciding $\mathrm{CSP}(\mathcal{A})$ for every X and Σ over R.

Let F be a set of function symbols. Let $\mathbf{A} = (A, \Phi)$ be an algebra over F. By the **constraint satisfaction problem** $\mathrm{CSP}(\mathbf{A})$ we mean the following decision problem: decide uniformly, that is by a unique algorithm, every $\mathrm{CSP}((A, P))$ such that $P \subseteq \mathrm{Inv}(\Phi)$.

Definition 1.1. Let F be a set of function symbols and $\mathbf{A}$ be an algebra over F. We denote by $\mathrm{Clo}(\mathbf{A})$ the smallest set containing

$$\{f^{\mathbf{A}} : f \in F\} \quad \text{and} \quad \{\pi_i^n : A^n \to A, 1 \leq i \leq n, n \in \omega\}$$

and closed under composition.

Goal: prove

Theorem 1.2. *Let $\mathbf{A}$ be a finite idempotent algebra. Then the following are equivalent:*

1. $\mathrm{CSP}(\mathbf{A})$ *is polynomial-time decidable;*
2. $\mathrm{Clo}(\mathbf{A})$ *contains a weak near-unanimity operation;*
3. *for every* $\mathbf{B} \in HS(\mathbf{A})$, $\mathrm{Clo}(\mathbf{B}) \neq \{\pi_i^n : 1 \leq i \leq n, n \in \omega\}$.

Otherwise, $\mathrm{CSP}(\mathbf{A})$ *is* NP-*complete.*

[1] More often denoted by $\mathrm{CSP}(P)$.

2. Kinds of Operations

Definition 2.1. An operation $\varphi : A^n \to A$ is called

(1) **essentially unary** if there is a function $\psi : A \to A$ such that
$$\varphi(a_1, \dots, a_n) = \psi(a_i)$$
for all $a_1, \dots, a_n \in A$.

(2) **idempotent** if $\varphi(a, \dots, a) = a$ for all $a \in A$.

3. Relational Clones

Definition 3.1. Let R be a set of relation symbols and $\mathcal{A}$ be a relational structure over R. We denote by $\mathrm{Clo}(\mathcal{A})$ the smallest set containing
$$\{r^{\mathcal{A}} : r \in R\} \quad \text{and} \quad \{\Delta^{(n)} : n \in \omega\}$$
and closed under intersection and truncation[2].

Remark 3.2. Observe that $\mathrm{Clo}(\mathcal{A})$ is given by all the relations ρ of A definable by a first-order primitive positive formula (that is, involving only conjunctions and existential quantifications). Recall that $\rho \subseteq A^n$ is definable if there is a formula $\varphi(x_1, \dots, x_n)$ such that
$$\mathcal{A} \models \varphi(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \rho$$

Theorem 3.3. *Let G be a set and D be a finite set of relation symbols. For any $\mathcal{A} = (A, \Gamma)$ over G and $\mathcal{B} = (A, \Delta)$ over D with $\Delta \subseteq \mathrm{Clo}(\mathcal{A})$, $\mathrm{CSP}(\mathcal{B})$ is polynomial-time reducible to $\mathrm{CSP}(\mathcal{A})$.*

Proof. Let $\Sigma(X, D)$ be a set of equations. Let $d(x_1, \dots, x_n) \in \Sigma$. For every $a_1, \dots, a_n \in A$
$$\mathcal{B} \models d(a_1, \dots, a_n) \iff (a_1, \dots, a_n) \in \delta \iff \mathcal{A} \models \varphi(a_1, \dots, a_n) \tag{2}$$
for some $\varphi(x_1, \dots, x_n)$ of the form
$$\exists y_1, \dots, y_m \,(g_1(z^1_1, \dots, z^1_{n_1}) \wedge \dots \wedge g_k(z^k_1, \dots, z^k_{n_k}))$$
where $g_1, \dots, g_k \in G$ and $z^i_j \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$.

Now, for each $d(x_1, \dots, x_n) \in \Sigma$ perfrom the following steps:

(1) add $\{g_1(z^1_1, \dots, z^1_{n_1}), \dots, g_k(z^k_1, \dots, z^k_{n_k})\}$ to Σ;

(2) remove $d(x_1, \dots, x_n)$ from Σ.

At the the end we obtain a set of equations $T(X, G)$ over G. This is a polynomial-time reduction. (It's reasonable but for me this kind of stuff is like a leap of faith). By (2) it is clear that we can find an assignment $X \to A$ such that $\mathcal{B} \models \Sigma$ iff we can find an assignment such that $\mathcal{A} \models T$. □

Corollary 3.4. *Let $\mathcal{A} = (A, P)$ and $\mathcal{B} = (A, \mathrm{Clo}(\mathcal{A}))$. Then*

(1) $\mathrm{CSP}(\mathcal{A})$ *is polynomial-time decidable iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

[2] If $\rho \in \mathrm{Clo}(\mathcal{A})$, then also $\{(a_1, \dots, a_{n-1}) : (a_1, \dots, a_{n-1}, a_n) \in \rho, \text{ for some } a_n \in A\} \in \mathrm{Clo}\,\mathcal{A}$.

(2) $\mathrm{CSP}(\mathcal{A})$ *is* NP-*complete iff* $\mathrm{CSP}(\mathcal{B})$ *is.*

Theorem 3.5 ([1])**.** *Let* $\mathcal{A}$ *be a relational structure. If* $\mathrm{Pol}(\mathcal{A})$ *contains essentially unary operations only,* $\mathrm{CSP}(\mathcal{A})$ *is* NP-*complete.*

References

[1] Jeavons, P. (1998). On the algebraic structure of combinatorial problems, *Theoretical Computer Science* 200, 185–204.